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ON THE SECOND CONFORMAL EIGENVALUE OF THE STANDARD SPHERE*

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Abstract. In this paper, we answer a natural question about the maximum of the second eigenvalue of the Laplacian, see [11], for metrics conformal to the round one on spheres of dimensions $n \geq 3$.

Key words. Eigenvalues, Laplacian, conformal class.

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We consider (M, g) a smooth compact Riemannian manifold of dimension $n \geq 2$. We let Δ_g be the Laplace-Beltrami operator given by $\Delta_g = -div_g(\nabla\,.)$. It is well-known that its spectrum is given by a discrete sequence of eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p \leq \ldots$$

where each eigenvalue is counted with multiplicity one, the sequence converges to $+\infty$, and $\lambda_0 = 0$ is the trivial eigenvalue with eigenspace associated restricted to constants. Getting bounds on these eigenvalues under some geometric assumptions has been the subject of an intensive study.

In this paper, we consider the so-called conformal eigenvalue problem. Given (M, g) a smooth compact Riemannian manifold of dimension $n \geq 2$ and $k \geq 1$ some integer, we set

$$\Lambda_k(M, [g]) = \sup_{\tilde{g} \in [g]} \lambda_k(\tilde{g})\, Vol_{\tilde{g}}(M)^{\frac{2}{n}} \ . \tag{0.1}$$

The quantity in the supremum is scale invariant. In other words, we are interested in maximizing the k-th eigenvalue of the Laplacian among the metrics conformal to a given one with fixed volume. Here, $[g]$ denotes the conformal class of the metric g, that is

$$[g] = \left\{\tilde{g} = e^{2u} g,\ u \in C^\infty(M)\right\} \ .$$

Korevaar [18] proved that the supremum is always finite. Note that, if not restricted to a given conformal class, the supremum is always infinite, except in dimension 2 (see [2, 22]). Note also that the infimum of any eigenvalue in a given conformal class (with fixed volume) is 0. One can arrange the conformal factor in such a way that any of the k first eigenvalues is as small as we want (see [3]). These remarks make the problem rather natural to look at. For surfaces, this subject has been recently intensively studied, see for instance [5, 14, 15, 16, 17, 19]. In higher dimensions, much less is known and we refer to [3, 7, 8]. Let us also mention recent works on Steklov eigenvalues, a problem somewhat related to the above one : [9, 10, 12].

Note that one can easily prove that

$$\Lambda_k(M, [g]) \geq \Lambda_k(\mathbb{S}^n, [h])$$

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for any compact n-dimensional Riemannian manifold (M,g) where $(\mathbb{S}^n,h)$ is the standard sphere (see [3]). It is of course natural to think that the case of equality is achieved only for the standard sphere. It is thus rather natural to first investigate the case of the round sphere.

In this context, the first result concerns the first conformal eigenvalue and is an extension of the celebrated theorem of Hersch [13] : we have that

$$\Lambda_1\left(\mathbb{S}^n,[h]\right)=n\omega_n^{\frac{2}{n}} \tag{0.2}$$

and the supremum in (0.1) is achieved only by the round sphere. Here, ω_n denotes the volume of the standard unit n-sphere. This result was proved by Hersch [13] in dimension 2 and extended to higher dimensions by El Soufi and Ilias [6]. Remember that, in dimension 2, there is only one conformal class on the sphere.

A Hersch-type result was proved for the second eigenvalue in 2-d by Nadirashvili [20] : namely, we have that, for any metric g on the 2-sphere $\mathbb{S}^2$,

$$\lambda_2(g)Vol_g\left(\mathbb{S}^2\right)<16\pi\ ,$$

the case of equality, never achieved, being asymptotically approached by the disjoint union of two spheres of same volume. In particular, on deduces that

$$\Lambda_2\left(\mathbb{S}^2,[h]\right)=16\pi\ . \tag{0.3}$$

It is then natural to ask, as was done in [11], if the same holds true in higher dimensions, namely if

$$\Lambda_2\left(\mathbb{S}^n,[h]\right)=n\left(2\omega_n\right)^{\frac{2}{n}}\ , \tag{0.4}$$

the supremum being approached by two disjoint spheres of same volume. A big step toward this conjecture was done by Girouard-Nadirashvili-Polterovich [11] since they gave, in odd dimensions, an upper-bound on $\Lambda_2\left(\mathbb{S}^n,[h]\right)$, really close to (0.4) (see theorem 1 below). In this paper [11], the authors also investigate a problem close to the above one and they prove that the second Neumann eigenvalue of domains in the plane of fixed volume is always bounded from above by the second Neumann eigenvalue of two attached disks. Petrides [21] recently extended their result on the second conformal eigenvalue of the standard sphere to all dimensions, unifying by the way the 2-dimensional proof of Nadirashvili [20] and the odd-dimensional proof of Girouard-Nadirashvili-Polterovich [11] :

THEOREM 1 ([11, 20, 21]). *Let $(\mathbb{S}^n,h)$ be the standard unit n-sphere. For any metric $g=e^{2u}h$ conformal to the round metric h, we have that*

$$\lambda_2(g)Vol_g\left(\mathbb{S}^n\right)^{\frac{2}{n}}<K_n n\left(2\omega_n\right)^{\frac{2}{n}}$$

where K_n is some universal constant depending only on the dimension. Moreover, we have that $K_2=1$, $1<K_n\le 1.04$ for all $n\ge 3$ and $K_n\to 1$ as $n\to+\infty$.

This result is extremely close to (0.4). However, as surprising as it may be, we prove in this paper that (0.4) is false :

THEOREM 2. *Let $(\mathbb{S}^n,h)$ be the standard unit n-sphere, $n\ge 3$. There exists a metric $g=e^{2u}h$ conformal to the round metric h such that*

$$\lambda_2(g)Vol_g\left(\mathbb{S}^n\right)^{\frac{2}{n}}>n\left(2\omega_n\right)^{\frac{2}{n}}\ .$$

In particular, we have that

$$\Lambda_2(\mathbb{S}^n,[h]) > n(2\omega_n)^{\frac{2}{n}}$$

for all $n \geq 3$.

Note that this theorem immediately leads to the following corollary :

Corollary 1. *Let* $(\mathbb{S}^n, h)$ *be the standard unit* n*-sphere,* $n \geq 3$*. Then, for any* $k \geq 2$*,*

$$\Lambda_k(\mathbb{S}^n,[h]) > n(k\omega_n)^{\frac{2}{n}} \ .$$

This proves that the k-th eigenvalue of the Laplacian for metrics conformal to the standard one on the sphere is not maximized by a union of k disconnected spheres as soon as $k \geq 2$. This corollary is a consequence of theorem 2 and of the following fact, proved in [3] :

$$\Lambda_{k+1}(M,[g])^{\frac{n}{2}} \geq \Lambda_k(M,[g])^{\frac{n}{2}} + n^{\frac{n}{2}}\omega_n$$

for all $k \geq 1$ and all smooth compact Riemannian manifold (M,g).

In the rest of the paper, we prove theorem 2. In section 1, we set up some notations and introduce a family of metrics g_β conformal to h. In section 2, we give some preliminary results on the two first eigenvalues of Δ_{g_β}. We prove in particular that $\lambda_2(g_\beta) Vol_{g_\beta}(\mathbb{S}^n)^{\frac{2}{n}} \to n(2\omega_n)^{\frac{2}{n}}$ as $\beta \to 1$. Section 3 is devoted to a fine asymptotic study of $\lambda_2(g_\beta)$, proving that the previous limit is achieved from above as $\beta \to 1$. At last, we recall and improve in section 4 some known results used throughout the paper.

1. Notations and preliminaries. We let $(\mathbb{S}^n, h)$ be the unit sphere

$$\mathbb{S}^n = \left\{(x_0,\dots,x_n) \in \mathbb{R}^{n+1} \text{ s.t. } x_0^2 + x_1^2 + \dots + x_n^2 = 1\right\}$$

with the round metric h, which is the metric induced from the Euclidean one in $\mathbb{R}^{n+1}$. In the following, $n \geq 3$.

1.1. Stereographic projections. We define in the following $\pi_N : \mathbb{R}^n \mapsto \mathbb{S}^n \setminus \{N\}$ and $\pi_S : \mathbb{R}^n \mapsto \mathbb{S}^n \setminus \{S\}$ where $N = (1,0,\dots,0)$ and $S = (-1,0,\dots,0)$ are the north and south poles by

$$\pi_N(x_1,\dots,x_n) = \left(\frac{|x|^2-1}{|x|^2+1}, \frac{2x_1}{1+|x|^2}, \dots, \frac{2x_n}{1+|x|^2}\right) \tag{1.1}$$

and

$$\pi_S(x_1,\dots,x_n) = \left(\frac{1-|x|^2}{1+|x|^2}, \frac{2x_1}{1+|x|^2}, \dots, \frac{2x_n}{1+|x|^2}\right) \tag{1.2}$$

where $|x|^2 = x_1^2 + \dots + x_n^2$. It is well-known that these stereographic projections π_N and π_S are conformal maps and that

$$\pi_N^\star h = \pi_S^\star h = U^{\frac{4}{n-2}}\xi \tag{1.3}$$

where ξ is the Euclidean metric and

$$U(x)=\left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}} . \tag{1.4}$$

In the following, given a function $u:\mathbb{S}^n\mapsto\mathbb{R}$, we let u^N and u^S be defined on $\mathbb{R}^n$ by

$$u^N(x)=U(x)u\circ\pi_N(x) \text{ and } u^S(x)=U(x)u\circ\pi_S(x) . \tag{1.5}$$

Note that, thanks to (1.3), we have that

$$\int_{\mathbb{S}^n} u\,dv_h=\int_{\mathbb{R}^n}(u\circ\pi_N)\,U^{\frac{2n}{n-2}}\,dx=\int_{\mathbb{R}^n}(u\circ\pi_S)\,U^{\frac{2n}{n-2}}\,dx. . \tag{1.6}$$

1.2. The Green function of the conformal Laplacian. The conformal Laplacian of a Riemannian manifold (M,g) is in dimensions $n\geq 3$ the operator

$$L_g=\Delta_g+\frac{n-2}{4(n-1)}S_g$$

where S_g is the scalar curvature of (M,g) and $\Delta_g=-div_g\,(\nabla\,.)$ is the Laplace-Beltrami operator. On the standard sphere $(\mathbb{S}^n,h)$, we have $S_h\equiv n(n-1)$ so that

$$L_h=\Delta_h+\frac{n(n-2)}{4}$$

while, on the Euclidean space $(\mathbb{R}^n,\xi)$, $L_\xi=\Delta_\xi$. The conformal Laplacian is conformally invariant in the following sense : if $\tilde g=u^{\frac{4}{n-2}}g$ for some smooth positive function u, we have that

$$L_{\tilde g}\varphi=u^{-\frac{n+2}{n-2}}L_g\,(u\varphi) \tag{1.7}$$

for all $\varphi\in C^2\,(M)$. In particular, for any function $u\in C^2\,(\mathbb{S}^n)$, (1.3), (1.5) and (1.7) give that

$$\Delta_\xi u^N=U^{\frac{n+2}{n-2}}L_h u\circ\pi_N \text{ and } \Delta_\xi u^S=U^{\frac{n+2}{n-2}}L_h u\circ\pi_S. \tag{1.8}$$

We let $\mathcal{G}$ be the Green function of L_h on $(\mathbb{S}^n,h)$. It is defined on $\mathbb{S}^n\times\mathbb{S}^n$ minus the diagonal by

$$\mathcal{G}\,(x,y)=\frac{1}{(n-2)\omega_{n-1}}\left(\frac{1}{2-2\langle x,y\rangle}\right)^{\frac{n-2}{2}}=\frac{1}{(n-2)\omega_{n-1}}|x-y|^{2-n} \tag{1.9}$$

where $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the Euclidean scalar product and norm in $\mathbb{R}^{n+1}$. For any $x\in\mathbb{S}^n$ and any function $u\in C^2\,(\mathbb{S}^n)$, we have that

$$u(x)=\int_{\mathbb{S}^n}\mathcal{G}\,(x,y)\,L_h u(y)\,dv_h(y). \tag{1.10}$$

Noting that

$$\begin{aligned}\mathcal{G}\,(\pi_N(x),\pi_N(y))&=\mathcal{G}\,(\pi_S(x),\pi_S(y))\\&=\frac{1}{(n-2)\omega_{n-1}}U(x)^{-1}U(y)^{-1}\,|x-y|^{2-n}\,,\end{aligned} \tag{1.11}$$

we get with (1.6), (1.8) and (1.10) that

$$u^N(x)=\frac{1}{(n-2)\omega_{n-1}}\int_{\mathbb{R}^n}|x-y|^{2-n}\Delta_\xi u^N(y)\,dy \tag{1.12}$$

and the same holds with respect to the south pole.

1.3. The fundamental functions. We let $\beta > 1$ and we set

$$\mu_\beta = \sqrt{\frac{\beta - 1}{\beta + 1}} . \tag{1.13}$$

We shall in the following consider the following functions defined on $\mathbb{S}^n$:

$$\begin{aligned} &U_\beta = \left(\beta^2 - 1\right)^{\frac{n-2}{4}} \left(\beta + x_0\right)^{1-\frac{n}{2}} , \quad V_\beta = \left(\beta^2 - 1\right)^{\frac{n-2}{4}} \left(\beta - x_0\right)^{1-\frac{n}{2}} , \\ &U_\beta^0 = \left(\beta^2 - 1\right)^{-\frac{1}{2}} U_\beta^{\frac{n}{n-2}} \left(1 + \beta x_0\right) , \quad V_\beta^0 = \left(\beta^2 - 1\right)^{-\frac{1}{2}} V_\beta^{\frac{n}{n-2}} \left(1 - \beta x_0\right) , \\ &U_\beta^i = x_i U_\beta^{\frac{n}{n-2}} \text{ and } V_\beta^i = x_i V_\beta^{\frac{n}{n-2}} \end{aligned}$$

for $i = 1, \dots, n$. We have that

$$\begin{aligned} U_\beta^N = V_\beta^S = \mu_\beta^{1-\frac{n}{2}} U\left(\frac{x}{\mu_\beta}\right) &, \quad V_\beta^N = U_\beta^S = \mu_\beta^{\frac{n}{2}-1} U\left(\mu_\beta x\right) , \\ \left(U_\beta^0\right)^N = \left(V_\beta^0\right)^S = \mu_\beta^{1-\frac{n}{2}} U^0\left(\frac{x}{\mu_\beta}\right) &, \quad \left(U_\beta^0\right)^S = \left(V_\beta^0\right)^N = -\mu_\beta^{\frac{n}{2}-1} U^0\left(\mu_\beta x\right) , \\ \left(U_\beta^i\right)^N = \left(V_\beta^i\right)^S = \mu_\beta^{1-\frac{n}{2}} U^i\left(\frac{x}{\mu_\beta}\right) &\text{ and } \left(U_\beta^i\right)^S = \left(V_\beta^i\right)^N = \mu_\beta^{\frac{n}{2}-1} U^i\left(\mu_\beta x\right) \end{aligned}$$

where

$$U^0(x) = U(x)^{\frac{n}{n-2}} \frac{|x|^2 - 1}{2} \tag{1.14}$$

and

$$U^i(x) = U(x)^{\frac{n}{n-2}} x_i \tag{1.15}$$

for $i = 1, \dots, n$. Note that

$$\Delta_\xi U = \frac{n(n-2)}{4} U^{\frac{n+2}{n-2}} \tag{1.16}$$

and

$$\Delta_\xi U^i = \frac{n(n+2)}{4} U^{\frac{4}{n-2}} U^i \tag{1.17}$$

for $i = 0, \dots, n$ so that, by (1.8), for any $\beta > 1$,

$$L_h U_\beta = \frac{n(n-2)}{4} U_\beta^{\frac{n+2}{n-2}}, \; L_h V_\beta = \frac{n(n-2)}{4} V_\beta^{\frac{n+2}{n-2}} \tag{1.18}$$

and

$$L_h U_\beta^i = \frac{n(n+2)}{4} U_\beta^{\frac{4}{n-2}} U_\beta^i, \; L_h V_\beta^i = \frac{n(n+2)}{4} V_\beta^{\frac{4}{n-2}} U_\beta^i \tag{1.19}$$

for $i = 0, \dots, n$.

1.4. The metric g_β and its volume. In the following, we consider the metric $g_\beta \in [h]$ defined for $\beta > 1$ by

$$g_\beta = u_\beta^{\frac{4}{n-2}} h \tag{1.20}$$

where

$$u_\beta = U_\beta + V_\beta \ . \tag{1.21}$$

Note that

$$u_\beta \left(-x_0, x_1, \dots, x_n\right) = u_\beta \left(x_0, x_1, \dots, x_n\right) \ .$$

The manifold $(\mathbb{S}^n, g_\beta)$ looks geometrically like two spheres attached by a small neck.

Let us compute an expansion of the volume of $(\mathbb{S}^n, g_\beta)$:

LEMMA 1.1. *We have that*

$$Vol_{g_\beta}\left(\mathbb{S}^n\right) = 2\omega_n + \frac{2^{n+2}}{n-2}\omega_{n-1}\mu_\beta^{n-2} + o\left(\mu_\beta^{n-2}\right)$$

as $\beta \to 1$.

Proof. We write, by symmetry, that

$$Vol_{g_\beta}\left(\mathbb{S}^n\right) = 2\int_{\mathbb{S}^n_-} u_\beta^{\frac{2n}{n-2}}\, dv_h$$

where $\mathbb{S}^n_- = \{(x_0, \dots, x_n) \in \mathbb{S}^n \text{ s.t. } x_0 < 0\}$. Using the stereographic projection π_N, we get thanks to (1.5) and (1.6) that

$$Vol_{g_\beta}\left(\mathbb{S}^n\right) = 2\int_{B_0(1)} \left(u_\beta^N\right)(x)^{\frac{2n}{n-2}}\, dx \ . \tag{1.22}$$

We have that

$$\begin{aligned} u_\beta^N &= U_\beta^N + V_\beta^N \\ &= \mu_\beta^{-\frac{n-2}{2}} U\left(\frac{x}{\mu_\beta}\right) + \mu_\beta^{\frac{n-2}{2}} U\left(\mu_\beta x\right) \\ &= \mu_\beta^{-\frac{n-2}{2}} U\left(\frac{x}{\mu_\beta}\right) + \left(2\mu_\beta\right)^{\frac{n-2}{2}} + o\left(\mu_\beta^{\frac{n-2}{2}}\right) \end{aligned}$$

in $B_0(1)$. Noting that

$$\mu_\beta^{-\frac{n-2}{2}} U\left(\frac{x}{\mu_\beta}\right) \geq \left(2\mu_\beta\right)^{\frac{n-2}{2}} \left(1 + \mu_\beta^2\right)^{-\frac{n-2}{2}}$$

in $B_0(1)$, it is straightforward to make an asymptotic expansion of $Vol_{g_\beta}\left(S^n\right)$, using lemma 4.1 of section 4, to get the result stated in the lemma. ◻

1.5. Eigenvalues and eigenfunctions of Δ_{g_β}. Consider λ_β an eigenvalue of Δ_{g_β} with its associated eigenfunction ψ_β. Then we have that

$$\Delta_{g_\beta}\psi_\beta = \lambda_\beta\psi_\beta \ .$$

Using (1.7), we can write that

$$\Delta_{g_\beta}\psi_\beta = L_{g_\beta}\psi_\beta - L_{g_\beta}(1)\psi_\beta = u_\beta^{-\frac{n+2}{n-2}}\left(L_h\left(u_\beta\psi_\beta\right) - \psi_\beta L_h u_\beta\right)$$

so that, setting $\varphi_\beta = u_\beta\psi_\beta$,

$$L_h\varphi_\beta = \left(\lambda_\beta u_\beta^{\frac{4}{n-2}} + \frac{L_h u_\beta}{u_\beta}\right)\varphi_\beta \ .$$

We shall rewrite this as

$$L_h\varphi_\beta = \left(\left(\frac{n(n-2)}{4} + \lambda_\beta\right)u_\beta^{\frac{4}{n-2}} + A_\beta\right)\varphi_\beta \tag{1.23}$$

where

$$A_\beta = \frac{L_h u_\beta}{u_\beta} - \frac{n(n-2)}{4}u_\beta^{\frac{4}{n-2}} \ . \tag{1.24}$$

In the following, we shall also assume that ψ_β is normalized such that

$$\int_{\mathbb{S}^n} u_\beta^{\frac{4}{n-2}}\varphi_\beta^2\, dv_h = 2\omega_n \ . \tag{1.25}$$

Note also that a direct consequence of the fact that $\int_{\mathbb{S}^n}\psi_\beta\, dv_{g_\beta} = 0$ is that

$$\int_{\mathbb{S}^n} u_\beta^{\frac{n+2}{n-2}}\varphi_\beta\, dv_h = 0 \ . \tag{1.26}$$

We shall need in the following some estimates on A_β. For that purpose, let us write with (1.18) that

$$L_h u_\beta = \frac{n(n-2)}{4}\left(U_\beta^{\frac{n+2}{n-2}} + V_\beta^{\frac{n+2}{n-2}}\right)$$

so that

$$A_\beta = \frac{n(n-2)}{4}\left(\frac{U_\beta^{\frac{n+2}{n-2}} + V_\beta^{\frac{n+2}{n-2}}}{U_\beta + V_\beta} - \left(U_\beta + V_\beta\right)^{\frac{4}{n-2}}\right) \ .$$

In $\mathbb{S}^n_-$, we have that $U_\beta \geq V_\beta$ so that

$$\begin{aligned}\frac{4}{n(n-2)}A_\beta &= \left(U_\beta^{\frac{n+2}{n-2}} + V_\beta^{\frac{n+2}{n-2}}\right)U_\beta^{-1}\left(1 - \frac{V_\beta}{U_\beta} + O\left(\frac{V_\beta^2}{U_\beta^2}\right)\right)\\ &\quad - U_\beta^{\frac{4}{n-2}}\left(1 + \frac{4}{n-2}\frac{V_\beta}{U_\beta} + O\left(\frac{V_\beta^2}{U_\beta^2}\right)\right)\\ &= U_\beta^{\frac{4}{n-2}}\left(-\frac{n+2}{n-2}\frac{V_\beta}{U_\beta} + O\left(\frac{V_\beta^2}{U_\beta^2}\right)\right) + O\left(V_\beta^{\frac{n+2}{n-2}}U_\beta^{-1}\right)\end{aligned}$$

in $\mathbb{S}^n_-$. By symmetry, we obtain in particular that

$$|A_\beta| \le C\mu_\beta^{\frac{n-2}{2}}\left(U_\beta^{\frac{6-n}{n-2}} + V_\beta^{\frac{6-n}{n-2}}\right) \le C\left(U_\beta^{\frac{4}{n-2}} + V_\beta^{\frac{4}{n-2}}\right) \tag{1.27}$$

in $\mathbb{S}^n$ for some $C > 0$ and that

$$\mu_\beta^{4-n} A_\beta\left(\pi\left(\mu_\beta x\right)\right) \to -n(n+2)2^{\frac{n}{2}-5}U^{\frac{6-n}{n-2}} \tag{1.28}$$

in $C^0_{loc}\left(\mathbb{R}^n\right)$ as $\beta \to 1$ where π stands for π_N or π_S.

2. First properties of the eigenvalues of Δ_{g_β}. We first prove an estimate which holds for any sequence of eigenfunctions of Δ_{g_β} of bounded associated eigenvalues :

PROPOSITION 2.1. *Let $(\lambda_\beta, \varphi_\beta)$ satisfying (1.23), (1.25) and (1.26) be such that $\lambda_\beta = O(1)$. Then there exists $C > 0$ such that*

$$|\varphi_\beta| \le C\left(U_\beta + V_\beta\right)$$

on $\mathbb{S}^n$.

Proof. We use the Green representation formula (1.10) to write that

$$\varphi_\beta(x) = \int_{\mathbb{S}^n} \mathcal{G}\left(x,y\right) L_h \varphi_\beta(y)\, dv_h(y)\ .$$

Thanks to (1.23), this leads to

$$\varphi_\beta(x) = \left(\frac{n(n-2)}{4} + \lambda_\beta\right)\int_{\mathbb{S}^n} \mathcal{G}\left(x,y\right) u_\beta(y)^{\frac{4}{n-2}}\varphi_\beta(y)\, dv_h(y)$$
$$+ \int_{\mathbb{S}^n} \mathcal{G}\left(x,y\right) A_\beta(y)\varphi_\beta(y)\, dv_h(y)$$

where A_β is given by (1.24). Thanks to the fact that $\lambda_\beta = O(1)$ and to (1.27), we know that there exists $C > 0$ such that

$$|\varphi_\beta(x)| \le C\int_{\mathbb{S}^n} \mathcal{G}\left(x,y\right)\left(U_\beta(y)^{\frac{4}{n-2}} + V_\beta(y)^{\frac{4}{n-2}}\right)|\varphi_\beta(y)|\, dv_h(y)\ . \tag{2.1}$$

Let us consider the following inequality : there exists $D_{p,n} > 0$ such that

$$|\varphi_\beta(x)| \le D_{p,n}\left\|\varphi_\beta\right\|_\infty \mu_\beta^{\frac{n-2}{2}p}\left(U_\beta(x)^p + V_\beta(x)^p\right)\ . \tag{2.2}$$

It is clear that the above inequality holds for $p = 0$ with $D_{0,n} = 1$. Assume that it holds for some $p \ge 0$. We can then use (2.1) to write that

$$|\varphi_\beta(x)| \le CD_{p,n}\mu_\beta^{\frac{n-2}{2}p}\left\|\varphi_\beta\right\|_\infty$$
$$\times\int_{\mathbb{S}^n} \mathcal{G}\left(x,y\right)\left(U_\beta(y)^{\frac{4}{n-2}} + V_\beta(y)^{\frac{4}{n-2}}\right)\left(U_\beta(y)^p + V_\beta(y)^p\right)\, dv_h(y)$$

which leads to

$$|\varphi_\beta(x)| \le \tilde{D}_{p,n}\mu_\beta^{\frac{n-2}{2}p}\left\|\varphi_\beta\right\|_\infty\int_{\mathbb{S}^n} \mathcal{G}\left(x,y\right)\left(U_\beta(y)^{\frac{4}{n-2}+p} + V_\beta(y)^{\frac{4}{n-2}+p}\right)\, dv_h(y)$$

for some $\tilde{D}_{p,n}$ depending only on p and n. We can then apply lemma 4.2 of section 4.2 to get that

$$(2.2)_p \Longrightarrow \begin{cases} (2.2)_1 & \text{if } p > \dfrac{n-4}{n-2} \\ (2.2)_q \text{ for all } 0 < q < 1 & \text{if } p = \dfrac{n-4}{n-2} \\ (2.2)_{p+\frac{2}{n-2}} & \text{if } p < \dfrac{n-4}{n-2} \end{cases}$$

It is then easy to deduce by induction that (2.2) holds for $p = 1$ in all dimensions. In other words, there exists $C > 0$ such that

$$|\varphi_\beta| \le C\mu_\beta^{\frac{n-2}{2}} \|\varphi_\beta\|_\infty (U_\beta + V_\beta) \ . \tag{2.3}$$

It remains to prove that $\|\varphi_\beta\|_\infty = O\left(\mu_\beta^{1-\frac{n}{2}}\right)$. We let $x_\beta \in \mathbb{S}^n$ be such that $\varphi_\beta(x_\beta) = \|\varphi_\beta\|_\infty$. Thanks to (2.3), we have that

$$1 - \langle x_\beta, N\rangle = O\left(\mu_\beta^2\right) \text{ or } 1 - \langle x_\beta, S\rangle = O\left(\mu_\beta^2\right) \ .$$

We can assume, without loss of generality, that the second possibility occurs, which means that

$$|z_\beta| = O(\mu_\beta)$$

where $x_\beta = \pi_N(z_\beta)$. We set

$$\tilde{\varphi}_\beta = \|\varphi_\beta\|_\infty^{-1} \varphi_\beta^N(\mu_\beta x) \ .$$

Thanks to (2.3), we know that

$$|\tilde{\varphi}_\beta(x)| \le 2CU(x)$$

for all $x \in B_0\left(\mu_\beta^{-1}\right)$. Moreover we have that

$$\left|\tilde{\varphi}_\beta\left(\frac{z_\beta}{\mu_\beta}\right)\right| = U(z_\beta) \to 2^{\frac{n-2}{2}} \text{ as } \beta \to 1 \ .$$

Thanks to (1.8) and (1.23), we have that

$$\Delta_\xi \tilde{\varphi}_\beta = \left(\frac{n(n-2)}{4} + \lambda_\beta\right) \mu_\beta^2 \left(u_\beta^N\right)^{\frac{4}{n-2}} \tilde{\varphi}_\beta + \mu_\beta^2 U(\mu_\beta x)^{\frac{4}{n-2}} A_\beta(\pi_N(\mu_\beta x)) \tilde{\varphi}_\beta \ .$$

Standard elliptic theory then clearly gives that $(\tilde{\varphi}_\beta)$ is uniformly bounded in $C^1(K)$ for all K compact subset of $\mathbb{R}^n$ so that, after the extraction of a subsequence,

$$\tilde{\varphi}_\beta \to \varphi_0 \text{ in } C^0_{loc}(\mathbb{R}^n)$$

where $\varphi_0 \not\equiv 0$ since $\varphi_0(z_0) = 2^{\frac{n-2}{2}}$ where $z_0 = \lim_{\beta\to 1} \mu_\beta^{-1} z_\beta$. Then we easily get with (1.6) that

$$\int_{\mathbb{S}^n} u_\beta^{\frac{4}{n-2}} \varphi_\beta^2 \, dv_h \ge \int_{B_0(1)} \left(u_\beta^N\right)^{\frac{4}{n-2}} \left(\varphi_\beta^N\right)^2 dx$$

which gives thanks to (1.25) and the above convergence that

$$2\omega_n \geq \|\varphi_\beta\|_\infty^2 \mu_\beta^{n-2} \left(\int_{B_0(1)} U^{\frac{4}{n-2}} \varphi_0^2 \, dx + o(1) \right) .$$

We deduce that $\|\varphi_\beta\|_\infty = O\left(\mu_\beta^{1-\frac{n}{2}}\right)$ which concludes the proof of the proposition with (2.3). □

We are now ready to get estimates on the two first eigenvalues of Δ_{g_β} that we denote by $\lambda_1(\beta)$ and $\lambda_2(\beta)$:

PROPOSITION 2.2. *We have that $\lambda_1(\beta) \to 0$ and that $\lambda_2(\beta) \to n$ as $\beta \to 1$.*

Proof. We start by proving that $\lambda_1(\beta) \to 0$ as $\beta \to 1$ and that $\limsup_{\beta\to 1} \lambda_2(\beta) \leq n$. In order to prove it, we let $u \in C^\infty(\mathbb{S}^n)$ be defined by $u(x) = x_0$ and $v_\beta \in C^\infty(\mathbb{S}^n)$ be defined by

$$v_\beta = \sqrt{\beta^2 - 1} \frac{x_1}{\beta + x_0} .$$

For symmetry reasons, it is clear that

$$\int_{\mathbb{S}^n} u \, dv_{g_\beta} = \int_{\mathbb{S}^n} v_\beta \, dv_{g_\beta} = \int_{\mathbb{S}^n} u v_\beta \, dv_{g_\beta} = 0 . \tag{2.4}$$

We claim that

$$\frac{\int_{\mathbb{S}^n} |\nabla u|_{g_\beta}^2 \, dv_{g_\beta}}{\int_{\mathbb{S}^n} u^2 \, dv_{g_\beta}} \to 0 \text{ as } \beta \to 1 \tag{2.5}$$

and that

$$\frac{\int_{\mathbb{S}^n} |\nabla v_\beta|_{g_\beta}^2 \, dv_{g_\beta}}{\int_{\mathbb{S}^n} v_\beta^2 \, dv_{g_\beta}} \to n \text{ as } \beta \to 1. \tag{2.6}$$

This will clearly prove that $\lambda_1(\beta) \to 0$ as $\beta \to 1$ and that $\limsup_{\beta\to 1} \lambda_2(\beta) \leq n$ by the variational characterization of the two first eigenvalues :

$$\lambda_1(g_\beta) = \inf_{u \in H_2^1(M),\, \int_M u \, dv_{g_\beta} = 0,\, u \not\equiv 0} \frac{\int_M |\nabla u|_{g_\beta}^2 \, dv_{g_\beta}}{\int_M u^2 \, dv_{g_\beta}}$$

and

$$\lambda_2(g_\beta) = \inf_{E \subset H_1^2(M)} \sup_{u \in E \setminus \{0\}} \frac{\int_M |\nabla u|_{g_\beta}^2 \, dv_{g_\beta}}{\int_M u^2 \, dv_{g_\beta}}$$

where the infimum is taken over vector subspaces E of dimension 2 of functions in $H_1^2(M)$ with mean value, w.r.t. g_β, 0.

Using the expression of u and v_β and stereographic projections, it is easily checked that

$$\lim_{\beta\to 1}\int_{\mathbb{S}^n} u^2\,dv_{g_\beta} = 2\int_{\mathbb{R}^n} U^{\frac{2n}{n-2}}\,dx\ ,$$
$$\lim_{\beta\to 1}\int_{\mathbb{S}^n} |\nabla u|^2_{g_\beta}\ dv_{g_\beta} = 0\ ,$$
$$\lim_{\beta\to 1}\int_{\mathbb{S}^n} v_\beta^2\,dv_{g_\beta} = \frac{4}{n}\int_{\mathbb{R}^n} U(x)^{\frac{2n}{n-2}}|x|^2\left(1+|x|^2\right)^{-2}\,dx \text{ and}$$
$$\lim_{\beta\to 1}\int_{\mathbb{S}^n} |\nabla v_\beta|^2_{g_\beta}\ dv_{g_\beta} = \int_{\mathbb{R}^n} U(x)^{\frac{2n}{n-2}}\left[1-\frac{4}{n}|x|^2\left(1+|x|^2\right)^{-2}\right]\,dx\ .$$

Lemma 4.1 of section 4 permits to conclude to (2.5) and (2.6) which, as already said, proves that

$$\lim_{\beta\to 1}\lambda_1(\beta) = 0 \text{ and } \limsup_{\beta\to 1}\lambda_2(\beta) \le n\ . \tag{2.7}$$

Consider now an eigenfunction φ_β associated to some eigenvalue λ_β normalised by (1.25) which satisfies

$$\limsup_{\beta\to 1}\lambda_\beta \le n\ . \tag{2.8}$$

Using proposition 2.1 and the argument at the end of the proof of it, we get, that, up to the extraction of a subsequence, $\lambda_\beta \to \lambda_0$ as $\beta\to 1$ with $0\le\lambda_0\le n$ and

$$\mu_\beta^{\frac{n}{2}-1}\varphi_\beta^N \to \varphi^N \text{ in } C^0_{loc}(\mathbb{R}^n) \text{ as } \beta\to 1 \tag{2.9}$$

and

$$\mu_\beta^{\frac{n}{2}-1}\varphi_\beta^S \to \varphi^S \text{ in } C^0_{loc}(\mathbb{R}^n) \text{ as } \beta\to 1 \tag{2.10}$$

where φ^S and φ^N are solutions in $\mathbb{R}^n$ of

$$\Delta_\xi\varphi = \left(\frac{n(n-2)}{4}+\lambda_0\right)U^{\frac{4}{n-2}}\varphi \tag{2.11}$$

which satisfy

$$\varphi \le CU \text{ in } \mathbb{R}^n\ ,$$

one of which, at least, being nonzero thanks to (1.25). Thanks to lemma 4.3, section 4.3, this implies that $\lambda_0 = 0$ or $\lambda_0 = n$. Moreover, in the case $\lambda_0 = 0$, we necessarily have that

$$\varphi^N = a_N U \text{ and } \varphi^S = a_S U$$

for some real numbers a_N and a_S. Then, using again proposition 2.1, it is easily checked thanks to (1.25) that

$$2\omega_n = \int_{\mathbb{S}^n} u_\beta^{\frac{4}{n-2}}\varphi_\beta^2\,dv_h = \omega_n\left(a_{\mathbb{S}}^2 + a_N^2\right)$$

and thanks to (1.26) that

$$0 = \int_{\mathbb{S}^n} u_\beta^{\frac{n+2}{n-2}} \varphi_\beta \, dv_h(y) = \omega_n \left(a_S + a_N\right) \ .$$

We deduce that necessarily, $a_N = -a_S$ and $|a_N| = 1$.

Let us assume now that $\lambda_2(\beta) \not\to n$ as $\beta \to 1$. This means that, up to a subsequence, there exists an eigenfunction φ_β^2 associated to $\lambda_2(\beta)$ such that $\lambda_2(\beta) \to 0$ as $\beta \to 1$ since, as already said, the only possible accumulation values for $(\lambda_2(\beta))$ are 0 and n. Let then b_S and b_N be the coefficients associated to $\left(\varphi_\beta^2\right)$ in the above limit while a_S and a_N will denote those associated to an eigenfunction φ_β^1 associated to $\lambda_1(\beta)$. Since we can choose φ_β^1 and φ_β^2 such that

$$\int_{\mathbb{S}^n} u_\beta^{\frac{4}{n-2}} \varphi_\beta^2 \varphi_\beta^1 \, dv_h = 0$$

and since we can apply proposition 2.1 to both eigenfunctions, one can check that

$$a_S b_S + a_N b_N = 0 \ .$$

Since $a_S = -a_N$, $b_S = -b_N$ and $|a_N| = |b_N| = 1$, this clearly leads to a contradiction. Thus we have proved that $\lambda_2(\beta) \to n$ as $\beta \to 1$. This ends the proof of this proposition. □

3. Proof of theorem 2. In order to prove the theorem, we let $(\lambda_\beta, \varphi_\beta)$ satisfying (1.23), (1.25) and (1.26) where $\lambda_\beta = \lambda_2\left(\Delta_{g_\beta}\right)$. Then

$$\lambda_\beta \to n \text{ as } \beta \to 1$$

thanks to proposition 2.2. The aim is to get an expansion of λ_β as $\beta \to 1$. We shall write in the following

$$\lambda_\beta = n + \varepsilon_\beta \text{ with } \varepsilon_\beta \to 0 \text{ as } \beta \to 1 \ . \tag{3.1}$$

3.1. Fine pointwise estimates on the second eigenfunction. We set

$$\Phi_\beta = \varphi_\beta - \sum_{i=0}^{n} \left(X_\beta^N\right)_i U_\beta^i - \sum_{i=0}^{n} \left(X_\beta^S\right)_i V_\beta^i \tag{3.2}$$

where $\left(X_\beta^N\right)_i$ and $\left(X_\beta^S\right)_i$, $i = 0, \dots, n$, are chosen such that

$$\Phi_\beta(N) = \Phi_\beta(S) = 0 \text{ and } \nabla\Phi_\beta(N) = \nabla\Phi_\beta(S) = 0 \ . \tag{3.3}$$

It is not difficult to check that such $\left(X_\beta^N\right)_i$ and $\left(X_\beta^S\right)_i$ do exist. In fact, they are given by

$$\left(X_\beta^N\right)_0 = \frac{1}{\mu_\beta^{2-n} - \mu_\beta^{n-2}} \left(\mu_\beta^{-\frac{n-2}{2}} \varphi_\beta(S) - \mu_\beta^{\frac{n-2}{2}} \varphi_\beta(N)\right) \ ,$$

$$\left(X_\beta^S\right)_0 = \frac{1}{\mu_\beta^{2-n} - \mu_\beta^{n-2}} \left(\mu_\beta^{-\frac{n-2}{2}} \varphi_\beta(N) - \mu_\beta^{\frac{n-2}{2}} \varphi_\beta(S)\right) \ ,$$

$$\left(X_\beta^N\right)_i = \frac{2^{-\frac{n}{2}}}{\mu_\beta^{n} - \mu_\beta^{n}} \left(\mu_\beta^{-\frac{n}{2}} \partial_i \varphi_\beta^N(0) - \mu_\beta^{\frac{n}{2}} \partial_i \varphi_\beta^S(0)\right) \ ,$$

$$\left(X_\beta^S\right)_i = \frac{2^{-\frac{n}{2}}}{\mu_\beta^{n} - \mu_\beta^{n}} \left(\mu_\beta^{-\frac{n}{2}} \partial_i \varphi_\beta^S(0) - \mu_\beta^{\frac{n}{2}} \partial_i \varphi_\beta^N(0)\right) \ .$$

Thanks to proposition 2.1, we know that

$$\mu_\beta^{\frac{n-2}{2}} \varphi_\beta^N (\mu_\beta x) \to \varphi_0^N \text{ in } C^1_{loc}(\mathbb{R}^n) \text{ as } \beta \to 1 \tag{3.4}$$

and

$$\mu_\beta^{\frac{n-2}{2}} \varphi_\beta^S (\mu_\beta x) \to \varphi_0^S \text{ in } C^1_{loc}(\mathbb{R}^n) \text{ as } \beta \to 1 \tag{3.5}$$

where φ_0^N and φ_0^S are solutions of

$$\Delta_\xi \varphi = \frac{n(n+2)}{4} U^{\frac{4}{n-2}} \varphi$$

with $\varphi \le CU$ in $\mathbb{R}^n$. Thanks to lemma 4.3, section 4.3, we know that

$$\varphi_0^N = \sum_{i=0}^n X_i^N U^i \tag{3.6}$$

and that

$$\varphi_0^S = \sum_{i=0}^n X_i^S U^i \tag{3.7}$$

where the U^i's were defined in (1.14) and (1.15). It is clear that

$$\lim_{\beta\to 1} \left(X_\beta^N\right)_i = X_i^N \text{ and } \lim_{\beta\to 1} \left(X_\beta^S\right)_i = X_i^S . \tag{3.8}$$

And we also have that

$$\mu_\beta^{\frac{n-2}{2}} \Phi_\beta^N (\mu_\beta x) \to 0 \text{ and } \mu_\beta^{\frac{n-2}{2}} \Phi_\beta^S (\mu_\beta x) \to 0 \text{ in } C^1_{loc}(\mathbb{R}^n) \tag{3.9}$$

as $\beta \to 1$. Thanks to proposition 2.1, (3.4), (3.5), (3.6) and (3.7), we have that

$$\int_{\mathbb{S}^n} u_\beta^{\frac{4}{n-2}} \varphi_\beta^2 \, dv_h = \sum_{i=0}^n \left(\left(X_i^N\right)^2 + \left(X_i^S\right)^2 \right) \int_{\mathbb{R}^n} U^{\frac{4}{n-2}} U_i^2 \, dx + o(1) .$$

Using lemma 4.1 of section 4.1, we get that

$$\int_{\mathbb{R}^n} U^{\frac{4}{n-2}} U_i^2 \, dx = \frac{\omega_n}{n+1}$$

for all $i = 0, \ldots, n$. Thus, thanks to (1.25), we obtain that

$$\left|X^N\right|^2 + \left|X^S\right|^2 = 2(n+1) . \tag{3.10}$$

Using (1.19), (1.23) and (3.1), we can write that

$$\begin{aligned} L_h \Phi_\beta = {} & \frac{n(n+2)}{4} u_\beta^{\frac{4}{n-2}} \Phi_\beta + \left(\varepsilon_\beta u_\beta^{\frac{4}{n-2}} + A_\beta \right) \varphi_\beta \\ & + \frac{n(n+2)}{4} \sum_{i=0}^N \left(X_\beta^N\right)_i \left(u_\beta^{\frac{4}{n-2}} - U_\beta^{\frac{4}{n-2}} \right) U_\beta^i \\ & + \frac{n(n+2)}{4} \sum_{i=0}^N \left(X_\beta^S\right)_i \left(u_\beta^{\frac{4}{n-2}} - V_\beta^{\frac{4}{n-2}} \right) V_\beta^i . \end{aligned} \tag{3.11}$$

We know moreover thanks to proposition 2.1 that there exists $C > 0$ such that

$$|\varphi_\beta| \le C\left(U_\beta + V_\beta\right)\ . \tag{3.12}$$

Let us write thanks to the Green representation formula (1.10) that

$$\Phi_\beta(x) = I_1^\beta(x) + I_2^\beta(x) + I_3^\beta(x) + I_4^\beta(x) + I_5^\beta(x) \tag{3.13}$$

where

$$I_1^\beta(x) = \frac{n(n+2)}{4}\int_{\mathbb{S}^n} \mathcal{G}\left(x,y\right) u_\beta(y)^{\frac{4}{n-2}}\Phi_\beta(y)\, dv_h(y)\ ,$$

$$I_2^\beta(x) = \varepsilon_\beta \int_{\mathbb{S}^n} \mathcal{G}\left(x,y\right) u_\beta(y)^{\frac{4}{n-2}}\varphi_\beta(y)\, dv_h(y)\ ,$$

$$I_3^\beta(x) = \int_{\mathbb{S}^n} \mathcal{G}\left(x,y\right) A_\beta(y)\varphi_\beta(y)\, dv_h(y)\ ,$$

$$I_4^\beta(x) = \frac{n(n+2)}{4}\sum_{i=0}^N \left(X_\beta^N\right)_i \int_{\mathbb{S}^n} \mathcal{G}\left(x,y\right)\left(u_\beta(y)^{\frac{4}{n-2}} - U_\beta(y)^{\frac{4}{n-2}}\right) U_\beta^i(y)\, dv_h(y) \text{ and}$$

$$I_5^\beta(x) = \frac{n(n+2)}{4}\sum_{i=0}^N \left(X_\beta^S\right)_i \int_{\mathbb{S}^n} \mathcal{G}\left(x,y\right)\left(u_\beta(y)^{\frac{4}{n-2}} - V_\beta(y)^{\frac{4}{n-2}}\right) V_\beta^i(y)\, dv_h(y)\ .$$

Using lemma 4.2 of appendix B, section 4.2, we have that

$$\left|I_1^\beta(x)\right| \le C\left\|\Phi_\beta\right\|_\infty \begin{cases} \mu_\beta^{\frac{1}{2}}\left(U_\beta(x) + V_\beta(x)\right) & \text{if } n=3 \\ \mu_\beta U_\beta(x)\ln\left(1 + \dfrac{1}{\mu_\beta U_\beta(x)}\right) & \\ \qquad + \mu_\beta V_\beta(x)\ln\left(1 + \dfrac{1}{\mu_\beta V_\beta(x)}\right) & \text{if } n=4 \\ \mu_\beta\left(U_\beta(x)^{\frac{2}{n-2}} + V_\beta(x)^{\frac{2}{n-2}}\right) & \text{if } n>4 \end{cases} \tag{3.14}$$

for some $C > 0$ independent of x and β. Using (3.12) and lemma 4.2 of section 4.2, we have that

$$\left|I_2^\beta(x)\right| \le C\varepsilon_\beta\left(U_\beta(x) + V_\beta(x)\right) \tag{3.15}$$

for some $C > 0$ independent of x and β. By (1.27) and (3.12), we know that

$$A_\beta\varphi_\beta \le C\mu_\beta^{\frac{n-2}{2}}\left(U_\beta^{\frac{4}{n-2}} + V_\beta^{\frac{4}{n-2}}\right)$$

for some $C > 0$ independent of x and β. Using once again lemma 4.2 of section 4.2, we deduce that

$$\left|I_3^\beta(x)\right| \le C \begin{cases} \mu_\beta\left(U_\beta(x) + V_\beta(x)\right) & \text{if } n=3 \\ \mu_\beta^2 U_\beta(x)\ln\left(1 + \dfrac{1}{\mu_\beta U_\beta(x)}\right) + \mu_\beta^2 V_\beta(x)\ln\left(1 + \dfrac{1}{\mu_\beta V_\beta(x)}\right) & \text{if } n=4 \\ \mu_\beta^{\frac{n}{2}}\left(U_\beta(x)^{\frac{2}{n-2}} + V_\beta(x)^{\frac{2}{n-2}}\right) & \text{if } n>4 \end{cases} \tag{3.16}$$

It is clear that there exists $C > 0$ such that

$$\left|u_\beta(x)^{\frac{4}{n-2}} - U_\beta(x)^{\frac{4}{n-2}}\right|\left|U_i^\beta(x)\right| \le C\mu_\beta^{\frac{n-2}{2}} u_\beta(x)^{\frac{4}{n-2}}$$

for all $x \in \mathbb{S}^n$ so that, with lemma 4.2 of section 4.2 and by symmetry,

$$\left|I_4^\beta(x)\right| + \left|I_5^\beta(x)\right| \le C \begin{cases} \mu_\beta\left(U_\beta(x) + V_\beta(x)\right) & \text{if } n = 3 \\ \mu_\beta^2 U_\beta(x)\ln\left(1 + \frac{1}{\mu_\beta U_\beta(x)}\right) & \\ \qquad + \mu_\beta^2 V_\beta(x)\ln\left(1 + \frac{1}{\mu_\beta V_\beta(x)}\right) & \text{if } n = 4 \\ \mu_\beta^{\frac{n}{2}}\left(U_\beta(x)^{\frac{2}{n-2}} + V_\beta(x)^{\frac{2}{n-2}}\right) & \text{if } n > 4 \end{cases} \tag{3.17}$$

Combining (3.14)-(3.17) to (3.13), we get the existence of some $C > 0$ such that

$$|\Phi_\beta(x)| \le C\gamma_\beta \begin{cases} \mu_\beta^{\frac{1}{2}}\left(U_\beta(x) + V_\beta(x)\right) & \text{if } n = 3 \\ \mu_\beta U_\beta(x)\ln\left(1 + \frac{1}{\mu_\beta U_\beta(x)}\right) + \mu_\beta V_\beta(x)\ln\left(1 + \frac{1}{\mu_\beta V_\beta(x)}\right) & \text{if } n = 4 \\ \mu_\beta\left(U_\beta(x)^{\frac{2}{n-2}} + V_\beta(x)^{\frac{2}{n-2}}\right) & \text{if } n > 4 \end{cases} \tag{3.18}$$

for all $x \in \mathbb{S}^n$ where

$$\gamma_\beta = \varepsilon_\beta \mu_\beta^{-\frac{n-2}{2}} + \mu_\beta^{\frac{n-2}{2}} + \|\Phi_\beta\|_\infty \ . \tag{3.19}$$

Up to the extraction of a subsequence, we have that

$$\frac{\varepsilon_\beta \mu_\beta^{-\frac{n-2}{2}}}{\gamma_\beta} \to \varepsilon_0 \text{ as } \beta \to 1 \tag{3.20}$$

where $0 \le \varepsilon_0 \le 1$ and

$$\frac{\mu_\beta^{\frac{n-2}{2}}}{\gamma_\beta} \to \mu_0 \text{ as } \beta \to 1 \tag{3.21}$$

where $0 \le \mu_0 \le 1$. We let $x_\beta \in \mathbb{S}^n$ be such that $|\Phi_\beta(x_\beta)| = \|\Phi_\beta\|_\infty$. Without loss of generality and by symmetry, we can assume that $x_\beta \in \mathbb{S}^n_-$. Then, using the fact that

$$U_\beta(x_\beta) = o\left(\mu_\beta^{1-\frac{n}{2}}\right) \text{ and } V_\beta(x_\beta) = o\left(\mu_\beta^{1-\frac{n}{2}}\right)$$

if $\frac{\left|\pi_N^{-1}(x_\beta)\right|}{\mu_\beta} \to +\infty$ and $x_\beta \in \mathbb{S}^n_-$, we obtain thanks to (3.18) and (3.19) that

$$\varepsilon_0 = \mu_0 = 0 \Longrightarrow \left|\pi_N^{-1}(x_\beta)\right| = O(\mu_\beta) \ . \tag{3.22}$$

We set now, for $x \in B_0\left(\mu_\beta^{-1}\right)$,

$$\tilde{\Phi}_\beta(x) = \gamma_\beta^{-1}\Phi_\beta^N(\mu_\beta x) \ .$$

Then (3.18) gives that

$$\left|\tilde{\Phi}_\beta(x)\right| \le C \begin{cases} U(x) & \text{if } n=3 \\ U(x)\ln\left(2+|x|\right) & \text{if } n=4 \\ U(x)^{\frac{2}{n-2}} & \text{if } n>4 \end{cases} \tag{3.23}$$

for all $x \in B_0\left(\mu_\beta^{-1}\right)$. Thanks to (1.8) and (3.11), we have that

$$\begin{aligned} &\Delta_\xi \tilde{\Phi}_\beta(x) \\ &= \frac{n(n+2)}{4}\left(U(x)+\mu_\beta^{n-2}U\left(\mu_\beta^2 x\right)\right)^{\frac{4}{n-2}}\tilde{\Phi}_\beta(x) \\ &+\frac{\varepsilon_\beta}{\gamma_\beta \mu_\beta^{\frac{n-2}{2}}}\left(U(x)+\mu_\beta^{n-2}U\left(\mu_\beta^2 x\right)\right)^{\frac{4}{n-2}}\mu_\beta^{\frac{n-2}{2}}\varphi_\beta^N\left(\mu_\beta x\right) \\ &+\frac{\mu_\beta^{\frac{n-2}{2}}}{\gamma_\beta}\mu_\beta^{\frac{n-2}{2}}\varphi_\beta^N\left(\mu_\beta x\right)\mu_\beta^{4-n}A_\beta\left(\pi_N\left(\mu_\beta x\right)\right)U\left(\mu_\beta x\right)^{\frac{4}{n-2}} \\ &+\frac{n(n+2)}{4}\frac{1}{\gamma_\beta \mu_\beta^{\frac{n-2}{2}}}\sum_{i=0}^N\left(X_\beta^N\right)_i\left[\left(U(x)+\mu_\beta^{n-2}U\left(\mu_\beta^2 x\right)\right)^{\frac{4}{n-2}}-U(x)^{\frac{4}{n-2}}\right]U^i(x) \\ &+\frac{n(n+2)}{4}\frac{\mu_\beta^{\frac{n-2}{2}}}{\gamma_\beta}\sum_{i=0}^N\left(X_\beta^S\right)_i\left[\left(U(x)+\mu_\beta^{n-2}U\left(\mu_\beta^2 x\right)\right)^{\frac{4}{n-2}}-\mu_\beta^4 U\left(\mu_\beta^2 x\right)^{\frac{4}{n-2}}\right]U^i\left(\mu_\beta^2 x\right) . \end{aligned} \tag{3.24}$$

Thanks to (3.4) and (3.20), we get that

$$\frac{\varepsilon_\beta}{\gamma_\beta \mu_\beta^{\frac{n-2}{2}}}\left(U(x)+\mu_\beta^{n-2}U\left(\mu_\beta^2 x\right)\right)^{\frac{4}{n-2}}\mu_\beta^{\frac{n-2}{2}}\varphi_\beta^N\left(\mu_\beta x\right) \to \varepsilon_0 U^{\frac{4}{n-2}}\varphi_0^N \tag{3.25}$$

in $C^0_{loc}\left(\mathbb{R}^n\right)$ as $\beta \to 1$. We also have thanks to (1.28), (3.4) and (3.21) that

$$\begin{aligned} &\frac{\mu_\beta^{\frac{n-2}{2}}}{\gamma_\beta}\mu_\beta^{\frac{n-2}{2}}\varphi_\beta^N\left(\mu_\beta x\right)\mu_\beta^{4-n}A_\beta\left(\pi_N\left(\mu_\beta x\right)\right)U\left(\mu_\beta x\right)^{\frac{4}{n-2}} \\ &\qquad \to -\mu_0 n(n+2)2^{\frac{n}{2}-3}U^{\frac{6-n}{n-2}}\varphi_0^N \end{aligned} \tag{3.26}$$

in $C^0_{loc}\left(\mathbb{R}^n\right)$ as $\beta \to 1$. At last, using (3.6), (3.8) and (3.21), we can write that

$$\begin{aligned} &\frac{n(n+2)}{4}\frac{1}{\gamma_\beta \mu_\beta^{\frac{n-2}{2}}}\sum_{i=0}^N\left(X_\beta^N\right)_i\left[\left(U(x)+\mu_\beta^{n-2}U\left(\mu_\beta^2 x\right)\right)^{\frac{4}{n-2}}-U(x)^{\frac{4}{n-2}}\right]U^i(x) \\ &\to \frac{n(n+2)}{n-2}2^{\frac{n}{2}-1}\mu_0 U^{\frac{6-n}{n-2}}\varphi_0^N \end{aligned} \tag{3.27}$$

and that

$$\begin{aligned} &\frac{n(n+2)}{4}\frac{\mu_\beta^{\frac{n-2}{2}}}{\gamma_\beta}\sum_{i=0}^N\left(X_\beta^S\right)_i\left[\left(U(x)+\mu_\beta^{n-2}U\left(\mu_\beta^2 x\right)\right)^{\frac{4}{n-2}}-\mu_\beta^4 U\left(\mu_\beta^2 x\right)^{\frac{4}{n-2}}\right]U^i\left(\mu_\beta^2 x\right) \\ &\to -n(n+2)2^{\frac{n}{2}-3}\mu_0 X_0^S U^{\frac{4}{n-2}} \end{aligned} \tag{3.28}$$

in $C^0_{loc}(\mathbb{R}^n)$ as $\beta \to 1$.

We deduce from (3.24)-(3.28) that $\left(\Delta_\xi \tilde{\Phi}_\beta\right)$ is uniformly bounded in any compact subset of $\mathbb{R}^n$. Thus, standard elliptic theory gives that, up to the extraction of a subsequence,

$$\tilde{\Phi}_\beta \to \Phi_0 \text{ in } C^1_{loc}(\mathbb{R}^n) \text{ as } \beta \to 1 . \tag{3.29}$$

Passing to the limit in the estimate (3.23) and in the equation (3.24) thanks to (3.6), (3.8), (3.20), (3.21) and (3.25)-(3.28), we get that

$$|\Phi_0(x)| \le C \begin{cases} U(x) & \text{if } n = 3 \\ U(x) \ln(2 + |x|) & \text{if } n = 4 \\ U(x)^{\frac{2}{n-2}} & \text{if } n > 4 \end{cases} \tag{3.30}$$

for all $x \in \mathbb{R}^n$ and that

$$\begin{aligned} \Delta_\xi \Phi_0 = {} & \frac{n(n+2)}{4} U^{\frac{4}{n-2}} \Phi_0 - n(n+2) 2^{\frac{n}{2}-3} \mu_0 X_0^S U^{\frac{4}{n-2}} + \varepsilon_0 U^{\frac{4}{n-2}} \varphi_0^N \\ & - \frac{n(n+2)(n-6)}{n-2} 2^{\frac{n}{2}-3} \mu_0 U^{\frac{6-n}{n-2}} \varphi_0^N . \end{aligned} \tag{3.31}$$

Moreover, (3.22) tells us that

$$\varepsilon_0 = \mu_0 = 0 \Longrightarrow \Phi_0 \not\equiv 0 . \tag{3.32}$$

At last, (3.3) gives that

$$\Phi_0(0) = 0 \text{ and } \nabla \Phi_0(0) = 0 . \tag{3.33}$$

Since there are no nonzero solution of

$$\Delta_\xi \Phi_0 = \frac{n(n+2)}{4} U^{\frac{4}{n-2}} \Phi_0$$

in $\mathbb{R}^n$ satisfying (3.30) and (3.33), see section 4.3, we deduce from (3.31) and (3.32) that

$$\varepsilon_0 \neq 0 \text{ or } \mu_0 \neq 0 . \tag{3.34}$$

We use (3.6) to remark that

$$\left| \Delta_\xi \Phi_0 - \frac{n(n+2)}{4} U^{\frac{4}{n-2}} \Phi_0 \right| \le C (1 + |x|)^{-4} .$$

Thus we can use lemma 4.4 of section 4.3 to write that

$$\begin{aligned} & \mu_0 \left(X_0^S \int_{\mathbb{R}^n} U^{\frac{4}{n-2}} U^i \, dx + \frac{n-6}{n-2} \int_{\mathbb{R}^n} U^{\frac{6-n}{n-2}} \varphi_0^N U^i \, dx \right) \\ & = \frac{2^{3-\frac{n}{2}}}{n(n+2)} \varepsilon_0 \int_{\mathbb{R}^n} U^{\frac{4}{n-2}} \varphi_0^N U^i \, dx \end{aligned} \tag{3.35}$$

for $i = 0, \dots, n$. Simple computations using lemma 4.1 of section 4.1 and (3.6) show that

$$\int_{\mathbb{R}^n} U^{\frac{4}{n-2}} U^0 = \frac{2^{\frac{n}{2}+1}(n-2)}{n(n+2)} \omega_{n-1} ,$$
$$\int_{\mathbb{R}^n} U^{\frac{6-n}{n-2}} \varphi_0^N U^0 \, dx = \frac{2^{\frac{n}{2}+1}\left(n^2-2n+8\right)}{n(n+2)(n+4)} \omega_{n-1} X_0^N ,$$
$$\int_{\mathbb{R}^n} U^{\frac{4}{n-2}} \varphi_0^N U^0 \, dx = \frac{\omega_n}{n+1} X_0^N ,$$
$$\int_{\mathbb{R}^n} U^{\frac{4}{n-2}} U^i = 0 ,$$
$$\int_{\mathbb{R}^n} U^{\frac{6-n}{n-2}} \varphi_0^N U^i \, dx = \frac{2^{\frac{n}{2}+4}}{n(n+2)(n+4)} \omega_{n-1} X_i^N \text{ and}$$
$$\int_{\mathbb{R}^n} U^{\frac{4}{n-2}} \varphi_0^N U^i \, dx = \frac{\omega_n}{n+1} X_i^N$$

for $i = 1, \dots, n$. Coming back to (3.35) with these results, we obtain that

$$\varepsilon_0 X_0^N = \frac{\omega_{n-1}}{\omega_n} 2^{n-2}(n+1)\left((n-2)X_0^S + \frac{(n-6)\left(n^2-2n+8\right)}{(n-2)(n+4)} X_0^N\right)\mu_0 \tag{3.36}$$

and that

$$X_i^N \left(\varepsilon_0 - \frac{(n-6)(n+1)}{(n+4)(n-2)} \frac{\omega_{n-1}}{\omega_n} 2^{n+1} \mu_0\right) = 0 \tag{3.37}$$

for $i = 1, \dots, N$. Of course, the same holds, by symmetry, exchanging N and S.

3.2. Conclusion of the proof. The aim is to prove that

$$\lambda_\beta Vol_{g_\beta}(\mathbb{S}^n)^{\frac{2}{n}} > n(2\omega_n)^{\frac{2}{n}} \tag{3.38}$$

for β close enough to 1.

Case 1 - There exists $i \in \{1, \dots, n\}$ such that $X_i^N \neq 0$ or $X_i^S \neq 0$. Then we can use (3.37) for N or S to write that

$$\varepsilon_0 = \frac{(n-6)(n+1)}{(n+4)(n-2)} \frac{\omega_{n-1}}{\omega_n} 2^{n+1} \mu_0$$

which gives that

$$\lim_{\beta \to 1} \frac{\varepsilon_\beta}{\mu_\beta^{n-2}} = \frac{(n-6)(n+1)}{(n+4)(n-2)} \frac{\omega_{n-1}}{\omega_n} 2^{n+1}$$

thanks to (3.20) and (3.21). This implies with lemma 1.1 and (3.1) that

$$\begin{aligned}
\lambda_\beta Vol_{g_\beta}(\mathbb{S}^n)^{\frac{2}{n}} &= (n+\varepsilon_\beta)\left(2\omega_n + \frac{2^{n+2}}{n-2}\omega_{n-1}\mu_\beta^{n-2} + o\left(\mu_\beta^{n-2}\right)\right)^{\frac{2}{n}} \\
&= n(2\omega_n)^{\frac{2}{n}}\left(1 + \frac{\varepsilon_\beta}{n} + \frac{2^{n+2}}{n(n-2)}\frac{\omega_{n-1}}{\omega_n}\mu_\beta^{n-2} + o\left(\mu_\beta^{n-2}\right) + o(\varepsilon_\beta)\right) \\
&= n(2\omega_n)^{\frac{2}{n}}\left(1 + \frac{\omega_{n-1}}{\omega_n}2^{n+1}\frac{n-1}{n(n+4)}\mu_\beta^{n-2} + o\left(\mu_\beta^{n-2}\right)\right) \\
&> n(2\omega_n)^{\frac{2}{n}}
\end{aligned}$$

for β close enough to 1. This proves (3.38) in this first case.

Case 2 - We assume that $X_i^N = X_i^S = 0$ for all $i = 1, \dots, n$. We claim first that

$$X_0^S = X_0^N \ . \tag{3.39}$$

Thanks to (3.10) and to (3.36) for N and S, we already know that $|X_0^N| = |X_0^S| = \sqrt{2(n+1)}$. It remains to prove that they have same sign. But (3.4), (3.5), (3.6) and (3.7) together with the assumptions of this case give that

$$sign\,(\varphi_\beta) = \begin{cases} -sign\left(X_0^N\right) & \text{for } |\pi_N(x)| \le \frac{1}{2}\mu_\beta \\ sign\left(X_0^N\right) & \text{for } 2\mu_\beta \le |\pi_N(x)| \le 4\mu_\beta \\ sign\left(X_0^S\right) & \text{for } 2\mu_\beta \le |\pi_S(x)| \le 4\mu_\beta \\ -sign\left(X_0^S\right) & \text{for } |\pi_S(x)| \le \frac{1}{2}\mu_\beta \end{cases}$$

for β close to 1. If X_0^N and X_0^S have opposite signs, this implies that φ_β has at least four nodal domains, which is impossible by Courant's celebrated theorem (see [4]) since φ_β is the second eigenfunction of Δ_{g_β} and can thus possess at most three nodal domains. Thus (3.39) is proved.

Now (3.36) gives that

$$\varepsilon_0 = \frac{\omega_{n-1}}{\omega_n} 2^{n-1} \frac{(n+1)(n-4)\left(n^2+4\right)}{(n-2)(n+4)} \mu_0$$

so that

$$\lim_{\beta \to 1} \frac{\varepsilon_\beta}{\mu_\beta^{n-2}} = \frac{\omega_{n-1}}{\omega_n} 2^{n-1} \frac{(n+1)(n-4)\left(n^2+4\right)}{(n-2)(n+4)}$$

thanks to (3.20) and (3.21). This implies with lemma 1.1 and (3.1) that

$$\begin{aligned} &\lambda_\beta Vol_{g_\beta}\left(\mathbb{S}^n\right)^{\frac{2}{n}} \\ &= (n+\varepsilon_\beta)\left(2\omega_n + \frac{2^{n+2}}{n-2}\omega_{n-1}\mu_\beta^{n-2} + o\left(\mu_\beta^{n-2}\right)\right)^{\frac{2}{n}} \\ &= n\left(2\omega_n\right)^{\frac{2}{n}}\left(1 + \frac{\varepsilon_\beta}{n} + \frac{2^{n+2}}{n(n-2)}\frac{\omega_{n-1}}{\omega_n}\mu_\beta^{n-2} + o\left(\mu_\beta^{n-2}\right) + o\left(\varepsilon_\beta\right)\right) \\ &= n\left(2\omega_n\right)^{\frac{2}{n}}\left(1 + \frac{\omega_{n-1}}{\omega_n}\frac{2^{n-1}}{n(n-2)(n+4)}\left(n^4 - 3n^3 - 4n + 16\right)\mu_\beta^{n-2} + O\left(\mu_\beta^{n-2}\right)\right) \\ &> n\left(2\omega_n\right)^{\frac{2}{n}} \end{aligned}$$

for β close enough to 1 since $n^4 - 3n^3 - 4n + 16 > 0$ for $n \ge 3$. This proves (3.38) in this second case.

The study of these two cases ends the proof of the theorem. □

4. Appendices. We prove in this section some results used during the proof of the theorem.

4.1. Computations of some integrals. We compute in this section some integrals that were used in the paper :

LEMMA 4.1. *We have that*

$$I_p = \int_{\mathbb{R}^n} U^{\frac{n+2+2p}{n-2}}\,dx = \frac{2^{\frac{n+2}{2}}}{n}\omega_{n-1}\left(\prod_{i=1}^{p}\frac{4i}{n+2i}\right) \quad \textit{and}$$

$$J_p = \int_{\mathbb{R}^n} U^{\frac{2(n+p)}{n-2}}\,dx = \omega_n\left(\prod_{i=1}^{p}\frac{n-2+2i}{n-1+i}\right)$$

for $p \in \mathbb{N}$ *where*

$$U(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}} .$$

Proof. For $p = 0$, we have by (1.6) that

$$J_0 = \int_{\mathbb{R}^n} U^{\frac{2n}{n-2}}\,dx = \int_{\mathbb{S}^n} 1\,dv_h = \omega_n$$

while, using equation (1.16), we easily get that

$$I_0 = \frac{2^{\frac{n+2}{2}}}{n}\omega_{n-1} .$$

For $p \geq 1$, noting that

$$|\nabla U|^2 = \frac{(n-2)^2}{4}|x|^2\left(\frac{2}{1+|x|^2}\right)^n = \frac{(n-2)^2}{4}U^{\frac{2n}{n-2}}\left(2U^{-\frac{2}{n-2}}-1\right) ,$$

we get by integrations by parts and equation (1.16) that

$$I_p = \frac{4p}{n+2p}I_{p-1}$$

and that

$$J_p = \frac{2p+n-2}{p+n-1}J_{p-1} .$$

The lemma clearly follows. □

4.2. Estimates on solutions of $L_h\varphi = U_\beta^{\frac{4}{n-2}+p}$ on $\mathbb{S}^n$. We prove in this section the following estimates :

LEMMA 4.2. *Let $p \geq 0$ be some non-negative real number. There exists $C_{p,n}$ such that*

$$\int_{\mathbb{S}^n} \mathcal{G}(x,y)\,U_\beta(y)^{\frac{4}{n-2}+p}\,dv_h(y) \leq C_{p,n}\begin{cases} \mu_\beta^{\frac{n-2}{2}(1-p)}U_\beta(x) & \textit{if } p > \frac{n-4}{n-2} \\ \mu_\beta U_\beta(x)\ln\left(1+\frac{1}{\mu_\beta U_\beta(x)^{\frac{2}{n-2}}}\right) & \textit{if } p = \frac{n-4}{n-2} \\ \mu_\beta U_\beta(x)^{p+\frac{2}{n-2}} & \textit{if } p < \frac{n-4}{n-2} \end{cases}$$

where $\mathcal{G}$ is the Green function of L_h, see (1.9), U_β is as in section 1.3 and μ_β is as in (1.13). Of course, the same holds true, replacing U_β by V_β everywhere.

Proof. During this proof, C will denote a constant independent of β which may change from line to line. Let $x_\beta \in \mathbb{S}^n$. We set

$$C_\beta = \begin{cases} \mu_\beta^{\frac{n-2}{2}(1-p)} U_\beta(x_\beta) & \text{if } p > \frac{n-4}{n-2} \\ \mu_\beta U_\beta(x_\beta) \ln\left(1 + \frac{1}{\mu_\beta U_\beta(x_\beta)^{\frac{2}{n-2}}}\right) & \text{if } p = \frac{n-4}{n-2} \\ \mu_\beta U_\beta(x_\beta)^{p+\frac{2}{n-2}} & \text{if } p < \frac{n-4}{n-2} \end{cases} \tag{4.1}$$

We remark thanks to (1.10) that

$$1 = \int_{\mathbb{S}^n} \mathcal{G}(x,y)\, L_h 1\, dv_h(y) = \frac{n(n-2)}{4} \int_{\mathbb{S}^n} \mathcal{G}(x,y)\, dv_h(y)$$

so that, since

$$U_\beta(y) \le (2\mu_\beta)^{\frac{n-2}{2}}$$

in $\mathbb{S}^n_+$, we can write that

$$\begin{aligned} &\int_{\mathbb{S}^n} \mathcal{G}(x_\beta,y)\, U_\beta(y)^{\frac{4}{n-2}+p}\, dv_h(y) \\ &\quad \le \int_{\mathbb{S}^n_-} \mathcal{G}(x_\beta,y)\, U_\beta(y)^{\frac{4}{n-2}+p}\, dv_h(y) + C\mu_\beta^{2+p\frac{n-2}{2}}\,. \end{aligned} \tag{4.2}$$

Let us write now thanks to (1.5), (1.6) and (1.11) that

$$\begin{aligned} &\int_{\mathbb{S}^n_-} \mathcal{G}(x_\beta,y)\, U_\beta(y)^{\frac{4}{n-2}+p}\, dv_h(y) \\ &= \begin{cases} \frac{1}{(n-2)\omega_{n-1}} U(z_\beta)^{-1} \int_{B_0(1)} U(y)^{1-p} |z_\beta - y|^{2-n}\, U_\beta^N(y)^{\frac{4}{n-2}+p}\, dy & \text{if } x_\beta \ne N \\ \frac{1}{2^{\frac{n-2}{2}}(n-2)\omega_{n-1}} \int_{B_0(1)} U(y)^{1-p} U_\beta^N(y)^{\frac{4}{n-2}+p}\, dy & \text{if } x_\beta = N \end{cases} \end{aligned}$$

where $x_\beta = \pi_n(z_\beta)$ if $x_\beta \ne N$.

Case 1 - We assume that $x_\beta = N$. We write that

$$\begin{aligned} \int_{B_0(1)} U(y)^{1-p} U_\beta^N(y)^{\frac{4}{n-2}+p}\, dy &\le C\mu_\beta^{n-2-p\frac{n-2}{2}} \int_{B_0\left(\mu_\beta^{-1}\right)} \left(1+|y|^2\right)^{-2-p\frac{n-2}{2}}\, dy \\ &\le C \begin{cases} \mu_\beta^{n-2-p\frac{n-2}{2}} & \text{if } p > \frac{n-4}{n-2} \\ \mu_\beta^{\frac{n}{2}} \ln\frac{1}{\mu_\beta} & \text{if } p = \frac{n-4}{n-2} \\ \mu_\beta^{2+p\frac{n-2}{2}} & \text{if } p < \frac{n-4}{n-2} \end{cases} \end{aligned}$$

Thus we get thanks to (4.2) that

$$\int_{\mathbb{S}^n} \mathcal{G}(x_\beta, y)\, U_\beta(y)^{\frac{4}{n-2}+p}\, dv_h(y) \le C \begin{cases} \mu_\beta^{n-2-p\frac{n-2}{2}} & \text{if } p > \dfrac{n-4}{n-2} \\ \mu_\beta^{\frac{n}{2}} \ln \dfrac{1}{\mu_\beta} & \text{if } p = \dfrac{n-4}{n-2} \\ \mu_\beta^{2+p\frac{n-2}{2}} & \text{if } p < \dfrac{n-4}{n-2} \end{cases}$$

Note also that, in this case, $U_\beta(x_\beta) = U_\beta(N) = \mu_\beta^{\frac{n-2}{2}}$ so that

$$C_\beta = \begin{cases} \mu_\beta^{n-2-p\frac{n-2}{2}} & \text{if } p > \dfrac{n-4}{n-2} \\ \mu_\beta^{\frac{n}{2}} \ln \left(1 + \dfrac{1}{\mu_\beta^2}\right) & \text{if } p = \dfrac{n-4}{n-2} \\ \mu_\beta^{2+p\frac{n-2}{2}} & \text{if } p < \dfrac{n-4}{n-2} \end{cases}$$

Thus we get in this case that

$$\int_{\mathbb{S}^n} \mathcal{G}(x_\beta, y)\, U_\beta(y)^{\frac{4}{n-2}+p}\, dv_h(y) = O(C_\beta) \; .$$

From now on, we assume that $x_\beta \neq N$. We write then that

$$\begin{aligned} &\int_{\mathbb{S}^n_-} \mathcal{G}(x_\beta, y)\, U_\beta(y)^{\frac{4}{n-2}+p}\, dv_h(y) \\ &\quad \le C \mu_\beta^{2+p\frac{n-2}{2}} U(z_\beta)^{-1} \int_{B_0(1)} |z_\beta - y|^{2-n} \left(\mu_\beta^2 + |y|^2\right)^{-2-p\frac{n-2}{2}} dy \; . \end{aligned} \tag{4.3}$$

In the following, we let

$$\theta_\beta^2 = \frac{\mu_\beta^2 + |z_\beta|^2}{1 + |z_\beta|^2} \tag{4.4}$$

so that

$$U_\beta(x_\beta) = \mu_\beta^{\frac{n-2}{2}} \theta_\beta^{2-n} \; . \tag{4.5}$$

Note that $0 < \theta_\beta < 1$.

Using the change of variables $y = \theta_\beta x$, we write that

$$\begin{aligned} &\int_{\mathbb{S}^n_-} \mathcal{G}(x_\beta, y)\, U_\beta(y)^{\frac{4}{n-2}+p}\, dv_h(y) \\ &\le C \frac{\mu_\beta^{2+p\frac{n-2}{2}}}{\theta_\beta^{2+p(n-2)}} U(z_\beta)^{-1} \int_{B_0(\theta_\beta^{-1})} \left| \frac{z_\beta}{\theta_\beta} - y \right|^{2-n} \left(\frac{\mu_\beta^2}{\theta_\beta^2} + |y|^2 \right)^{-2-p\frac{n-2}{2}} dy \; . \end{aligned} \tag{4.6}$$

We can also rewrite (4.1) as

$$C_\beta = \begin{cases} \mu_\beta^{n-2-\frac{n-2}{2}p}\theta_\beta^{2-n} & \text{if } p > \dfrac{n-4}{n-2} \\ \mu_\beta^{\frac{n}{2}}\theta_\beta^{2-n}\ln\left(1+\dfrac{\theta_\beta^2}{\mu_\beta^2}\right) & \text{if } p = \dfrac{n-4}{n-2} \\ \mu_\beta^{2+\frac{n-2}{2}p}\theta_\beta^{-2-p(n-2)} & \text{if } p < \dfrac{n-4}{n-2} \end{cases} \tag{4.7}$$

Case 2 - We assume that $\theta_\beta \to \theta_0$ as $\beta \to 1$ with $\theta_0 > 0$. Then, up to a subsequence, $z_\beta \to z_0$ as $\beta \to 1$ with $\frac{|z_0|^2}{1+|z_0|^2} = \theta_0$. Note that $|z_0| = +\infty$ if $\theta_0 = 1$. It is then easily checked that

$$U(z_\beta)^{-1}\int_{B_0\left(\theta_\beta^{-1}\right)}\left|\frac{z_\beta}{\theta_\beta}-y\right|^{2-n}\left(\frac{\mu_\beta^2}{\theta_\beta^2}+|y|^2\right)^{-2-p\frac{n-2}{2}}dy$$
$$\leq C\begin{cases} \mu_\beta^{n-4-p(n-2)} & \text{if } p > \dfrac{n-4}{n-2} \\ \ln\dfrac{1}{\mu_\beta} & \text{if } p = \dfrac{n-4}{n-2} \\ 1 & \text{if } p < \dfrac{n-4}{n-2} \end{cases}$$

Thanks to (4.2), (4.6) and (4.7), we then conclude that

$$\int_{\mathbb{S}^n}\mathcal{G}(x_\beta,y)\,U_\beta(y)^{\frac{4}{n-2}+p}\,dv_h(y) = O(C_\beta)$$

in this second case.

Case 3 - We assume that $\theta_\beta \to 0$ as $\beta \to 1$. Then we have that $z_\beta \to 0$ and that

$$\frac{\mu_\beta^2+|z_\beta|^2}{\theta_\beta^2} \to 1 \text{ as } \beta \to 1\,.$$

This implies that the two potential singularities of the integral below can not be both at 0 so that one can check that

$$\int_{B_0\left(\theta_\beta^{-1}\right)}\left|\frac{z_\beta}{\theta_\beta}-y\right|^{2-n}\left(\frac{\mu_\beta^2}{\theta_\beta^2}+|y|^2\right)^{-2-p\frac{n-2}{2}}dy$$
$$\leq C\begin{cases} \left(1+\dfrac{\theta_\beta}{\mu_\beta}\right)^{4-n+p(n-2)} & \text{if } p > \frac{n-4}{n-2} \\ \ln\left(2+\dfrac{\theta_\beta}{\mu_\beta}\right) & \text{if } p = \frac{n-4}{n-2} \\ 1 & \text{if } p < \frac{n-4}{n-2} \end{cases}$$

Using again (4.2), (4.6) and (4.7), we then conclude that

$$\int_{\mathbb{S}^n} \mathcal{G}(x_\beta, y)\, U_\beta(y)^{\frac{4}{n-2}+p}\, dv_h(y) = O(C_\beta)$$

in this third case, remembering that $\mu_\beta \le \theta_\beta$.

The study of these three cases ends the proof of the lemma. □

4.3. Study of the equation $\Delta_\xi \varphi = \frac{n(n+2)}{4} U^{\frac{4}{n-2}} \varphi + f$ in $\mathbb{R}^n$. In this subsection, we let $\varphi \in C^2(\mathbb{R}^n)$ be a solution of

$$\Delta_\xi \varphi = \frac{n(n+2)}{4} U^{\frac{4}{n-2}} \varphi + f \tag{4.8}$$

in $\mathbb{R}^n$ where $f \in C^1(\mathbb{R}^n)$ and

$$U(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}} .$$

In the case $f \equiv 0$, this equation was studied by Bianchi-Egnell [1] and we have that :

LEMMA 4.3. *Any solution $\varphi \in C^2(\mathbb{R}^n)$ of (4.8) with $f \equiv 0$ such that $\varphi(x) \to 0$ as $|x| \to +\infty$ is of the form*

$$\varphi = \sum_{i=0}^{n} \lambda_i U^i$$

where the U^i's are given by (1.14) and (1.15) and the λ_i's are real numbers. In particular, if $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$, then $\varphi \equiv 0$.

Proof. This result was proved by Bianchi-Egnell [1] under the assumption that

$$\int_{\mathbb{R}^n} U^{\frac{4}{n-2}} \varphi^2\, dx < +\infty . \tag{4.9}$$

We shall prove that this holds under the assumptions of the lemma. We let $\varphi \in C^2(\mathbb{R}^n)$ be a solution of (4.8) such that $\varphi(x) \to 0$ as $|x| \to +\infty$. We write with the Green representation formula that

$$\varphi(x) = \frac{1}{\omega_{n-1} R^{n-1}} \int_{\partial B_x(R)} \varphi\, d\sigma + \frac{n(n+2)}{4(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} H_{x,R}(y) U(y)^{\frac{4}{n-2}} \varphi(y)\, dy$$

for any $x \in \mathbb{R}^n$ and any $R > 0$ where

$$H_{x,R}(y) = \begin{cases} |x-y|^{2-n} - R^{2-n} & \text{for } y \in B_x(R) \\ 0 & \text{for } y \in \mathbb{R}^n \setminus B_x(R) \end{cases}$$

Since $\varphi(x) \to 0$ as $|x| \to +\infty$, we have that

$$\frac{1}{\omega_{n-1} R^{n-1}} \int_{\partial B_x(R)} \varphi\, d\sigma \to 0 \text{ as } R \to +\infty$$

while the dominated convergence theorem ensures that

$$\int_{\mathbb{R}^n} H_{x,R}(y)U(y)^{\frac{4}{n-2}}\varphi(y)\,dy \to \int_{\mathbb{R}^n} |y-x|^{2-n} U(y)^{\frac{4}{n-2}}\varphi(y)\,dy \text{ as } R\to+\infty\ .$$

Thus we have obtained that

$$\varphi(x) = \frac{n(n+2)}{(n-2)\omega_{n-1}}\int_{\mathbb{R}^n} |y-x|^{2-n}\left(1+|y|^2\right)^{-2}\varphi(y)\,dy\ .$$

Assume now that

$$|\varphi(x)| \le C\,(1+|x|)^{-\alpha}$$

for some $C>0$ and some $\alpha\ge 0$. This is already the case for $\alpha=0$ by assumption. Then we write that

$$\begin{aligned}|\varphi(x)| &\le \frac{n(n+2)}{(n-2)\omega_{n-1}}\int_{\mathbb{R}^n} |y-x|^{2-n}\left(1+|y|^2\right)^{-2}|\varphi(y)|\,dy\\ &\le \frac{n(n+2)}{(n-2)\omega_{n-1}}C\int_{\mathbb{R}^n} |y-x|^{2-n}\left(1+|y|^2\right)^{-2}(1+|y|)^{-\alpha}\,dy\\ &\le D\begin{cases}(1+|x|)^{-2-\alpha} & \text{if } \alpha<n-4\\ (1+|x|)^{2-n}\ln(2+|x|) & \text{if } \alpha=n-4\\ (1+|x|)^{2-n} & \text{if } \alpha>n-4\end{cases}\end{aligned}$$

By induction, we thus get the existence of some $C>0$ such that

$$|\varphi(x)| \le C\,(1+|x|)^{2-n}$$

which clearly proves that (4.9) holds. As already said, this ends the proof of the lemma. □

We now study the case where $f\not\equiv 0$ but decays at infinity. We then have the following result :

LEMMA 4.4. *Let $f\in C^1(\mathbb{R}^n)$ be such that*

$$f(x)\le C\,(1+|x|)^{-\alpha}$$

for some $C>0$ and some $\alpha>2$. If there exists a solution $\varphi\in C^2(\mathbb{R}^n)$ of (4.8) such that $\varphi(x)\to 0$ as $|x|\to+\infty$, then

$$\int_{\mathbb{R}^n} fU^i\,dx = 0 \text{ for } i=0,\dots,n$$

where the U^i's are given by (1.14) and (1.15).

Proof. We multiply equation (4.8) by

$$U^0-\eta(R)$$

where

$$\eta(R) = \left(\frac{2}{1+R^2}\right)^{\frac{n}{2}}\frac{R^2-1}{2}$$

and integrate over $B_0(R)$. Since $U^0 - \eta(R) = 0$ on $\partial B_0(R)$ and $\Delta_\xi U^0 = \frac{n(n+2)}{4} U^{\frac{4}{n-2}} U^0$, we obtain that

$$\int_{B_0(R)} f U^0 \, dx = \eta(R) \int_{B_0(R)} f \, dx + \frac{n(n+2)}{4} \eta(R) \int_{B_0(R)} U^{\frac{4}{n-2}} \varphi \, dx + \int_{\partial B_0(R)} \varphi \partial_\nu U^0 \, d\sigma \ .$$

Since $|f| \le C(1+|x|)^{-\alpha}$ for some $\alpha > 2$, we get that

$$\int_{B_0(R)} f U^0 \, dx \to \int_{\mathbb{R}^n} f U^0 \, dx \text{ as } R \to +\infty$$

and that

$$\eta(R) \int_{B_0(R)} f \, dx \to 0 \text{ as } R \to +\infty \ .$$

Since $\varphi(x) \to 0$ as $|x| \to +\infty$, we also have that

$$\eta(R) \int_{B_0(R)} U^{\frac{4}{n-2}} \varphi \, dx \to 0 \text{ as } R \to +\infty$$

and that

$$\int_{\partial B_0(R)} \varphi \partial_\nu U^0 \, d\sigma \to 0 \text{ as } R \to +\infty \ .$$

Thus

$$\int_{\mathbb{R}^n} f U^0 \, dx = 0 \ .$$

We multiply now equation (4.8) by

$$U^i - \varepsilon(R) x^i$$

where

$$\varepsilon(R) = \left(\frac{2}{1+R^2} \right)^{\frac{n}{2}}$$

and integrate over $B_0(R)$. Since $U^i - \varepsilon(R) x^i = 0$ on $\partial B_0(R)$ and $\Delta_\xi U^i = \frac{n(n+2)}{4} U^{\frac{4}{n-2}} U^i$, we obtain that

$$\int_{B_0(R)} f U^i \, dx = \varepsilon(R) \int_{B_0(R)} x^i f \, dx + \frac{n(n+2)}{4} \varepsilon(R) \int_{B_0(R)} U^{\frac{4}{n-2}} x^i \varphi \, dx + \int_{\partial B_0(R)} \varphi \partial_\nu \left(U^i - \varepsilon(R) x^i \right) \, d\sigma \ .$$

Since $|f| \le C(1+|x|)^{-\alpha}$ for some $\alpha > 2$ and $\varphi(x) \to 0$ as $|x| \to +\infty$, we can pass to the limit as above to obtain that

$$\int_{\mathbb{R}^n} f U^i \, dx = 0 \ .$$

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