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# CURVATURE ESTIMATES FOR WEAKLY STABLE SURFACES OF CONSTANT NULL EXPANSIONS*

ZHUOBIN LIANG†

**Abstract.** In this paper, we prove by blow-up arguments the curvature estimates for weakly stable constant null expansion surfaces in Cauchy data set for the Einstein equation. This improves the sup-estimate by Andersson-Metzger [AM10] for the curvature of stable marginally outer trapped surfaces.

**Key words.** Curvature estimates, Null expansion, Stability, Convergence, blow-up.

**Mathematics Subject Classification.** Primary 53C42; Secondary 53C80.

**1. Introduction.** Let $(M^3, g, K)$ be the Cauchy data set for the Einstein equation, where $(M^3, g)$ is a Riemannian 3-manifold and $K$ is a symmetric 2-tensor on $M$. Assume that there is a closed Riemannian 2-surface $(\Sigma^2, h)$ and an isometric immersion $f : (\Sigma, h) \to (M, g)$ of trivial normal bundle. Let $\nu$ be one of the global unit normal vector field on $\Sigma$. We identify the tangent vector $X \in T_p\Sigma$ with $f_*X \in T_{f(p)}M$, and sometimes just say $\Sigma$ is a surface in $M$ ($\Sigma \looparrowright M$) if there is no ambiguity. The second fundamental form $\Pi$ of $\Sigma$ is computed as $\Pi(X, Y) = g(D_X\nu, Y)$, $\forall\, X, Y \in \mathfrak{X}(\Sigma)$, where $D$ is the metric connection of $g$. The mean curvature $H = H(\Sigma)$ of $\Sigma$ is computed as $H = tr_h\Pi$. The null expansion $\theta(\Sigma)$ of $\Sigma$ is then defined as $\theta(\Sigma) = H(\Sigma) + tr_hK$. We say $\Sigma$ is a surface of constant null expansion if $\theta(\Sigma)$ is a constant.

In the context of general relativity, it turns out that these surfaces of constant null expansion are deeply related to the physical properties of the spacetime. A surface of vanishing null expansion, i.e., $\theta(\Sigma) = 0$, is called marginally outer trapped surface (MOTS). The existence of MOTS indicates the formation of black hole under suitable circumstances (cf. [Wal84]). For time symmetric Cauchy data set which is asymptotic to Schwarzschild, Huisken and Yau [HY96] prove the existence of the foliation of the asymptotic region by constant mean curvature surfaces (which are in particular of constant null expansions), and as a result they are able to define the center of mass. Also surfaces of constant null expansion appear in an interesting way in the positive mass theorem: Recently, Luo, Xie and Zhang [LXZ10] establish the positive mass theorem for positive cosmological constant $\Lambda$. Singularities are allowed if they are surrounded by some kind of apparent horizons with null expansion equal to $2\sqrt{\Lambda/3}$.

In this paper we exploit surfaces of constant null expansion in general data set. Precisely, we define the notion of weak stability (see definition 9) for such surfaces, and then derive the resulting curvature estimates.

Before stating our main results we specify some basic settings throughout this paper. For simplicity we assume all the geometric objects, for examples the $M$, $g$, $K$, $\Sigma$, and $f$ as above, are all $C^\infty$ unless otherwise stated. Denote by the $d_g$ distance function on $M$ associated to metric $g$. Similarly we have $d_h$ for the surface $(\Sigma, h)$. It is

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†Department of Mathematics, Jinan University, Guangzhou, 510632, P. R. China (tzbliang@jnu.edu.cn). This work is a part of author's dissertation at Chinese Academy of Sciences, P. R. China. The author would like to thank his advisor, Professor Xiao Zhang, for orientation throughout this work and continuous encouragement. The author is partially supported by National Science Foundation of China (grant 11171328, 11571345).

assumed that $\Sigma$ is compact. Although the interesting case is that $\Sigma$ is boundaryless, we also exploit at the same time the case $\partial\Sigma \neq \emptyset$ for later reference. In the case $\partial\Sigma \neq \emptyset$, we do not need any smoothness assumption on the boundary $\partial\Sigma$. This is because our curvature estimates are the interior estimates. Constants are always denoted by $C$, and by $C(a, b, \dots)$ we mean a constant depends only on the quantities $a, b, \dots$ in such a way that $C$ deteriorates as any of these quantities diverges to infinity. We obtain the following two theorems:

THEOREM 1. *Let $(M^3, g, K)$ be a three dimensional Cauchy data set, $(\Sigma^2, h)$ a compact connected 2-surface and $f : (\Sigma, h) \to (M, g)$ an isometric immersion of trivial normal bundle. Suppose that $\Sigma$ is of constant null expansion, i.e. $\theta(\Sigma) \equiv \theta_0$ for some constant $\theta_0$, and that $\Sigma$ is weakly stable. Moreover, assume that there is an open subset $\Omega$ of $M$ and positive constants $\sigma$, $i_0$, $\Lambda_0$ and $\Lambda_1$ so that $f(\Sigma) \subset \Omega$ and*

$$\| Ric_g \|_{C^0(\Omega_\sigma, g)} \leq \Lambda_0, \quad \inf_{P \in \Omega_\sigma} inj_{(M, g)}(P) \geq i_0 > 0, \quad \| K \|_{C^1(\Omega, g)} \leq \Lambda_1$$

*where $\Omega_\sigma = \{P \in M : d_g(P, \Omega) < \sigma\}$.*

*Then there is a constant $C = C(\sigma^{-1}, i_0^{-1}, \Lambda_0, \Lambda_1)$, which does* not *depend on $\theta_0$, so that the following holds:*

(1) *If $\partial\Sigma \neq \emptyset$, then*

$$|\Pi(p)|_h \min\{d_h(p, \partial\Sigma), r_H\} \leq C, \qquad \forall p \in \Sigma \backslash \partial\Sigma.$$

(2) *If $\partial\Sigma = \emptyset$, then*

$$|\Pi(p)|_h \min\{\frac{1}{2} diam(\Sigma), r_H\} \leq C, \qquad \forall p \in \Sigma.$$

*Here $r_H$ is the infimum of the harmonic radius of $(M, g)$ taken over $\Omega$, and it turns out that $r_H > 0$ and $r_H$ depends only on $\sigma^{-1}$, $i_0^{-1}$ and $\Lambda_0$.*

THEOREM 2. *Assume the conditions of Theorem 1, and assume $\partial\Sigma = \emptyset$. It holds that*

$$\sup_{p \in \Sigma} |\Pi(p)|_h \leq C(\sigma^{-1}, i_0^{-1}, \Lambda_0, \Lambda_1, |\theta_0|).$$

Theorem 1 is analogous to the Main Theorem of [RST10] (also Corollary 1.1 and Theorem 2.5 therein) by Rosenberg, Souam and Toubian, shifting from strongly stable constant mean curvature surfaces to weakly stable constant null expansion surfaces. Precisely, they show that there is a universal constant $C$ so that for any Riemannian 3-manifold $(M^3, g)$ with |sectional curvature|$\leq \Lambda$, and for any strongly stable constant mean curvature surface $(\Sigma^2, h) \looparrowright (M^3, g)$ with $\partial\Sigma \neq \emptyset$, the following curvature estimate is valid:

$$|\Pi(p)|_h \min\{d_h(p, \partial\Sigma), \frac{\pi}{2\sqrt{\Lambda}}\} \leq C, \quad \forall\, p \in \Sigma.$$

One of the remarkable features of their result is that the curvature estimate does not involves the injectivity radius of $(M, g)$. They could achieve this by passing the proof to the universal covering of $M$. However this technique no longer works here: strong stability lifts to the covering while weak stability fails even in the symmetric case

[EM12, P. 88]. By contrast, the upper bound $C$ in our Theorem 1 depends on the injectivity radius.

The proof of Theorem 1 is based on the blow up argument of [RST10] (also see that of [EM12, Proposition 2.3]), which remains valid to our case. This blow up argument is to assume toward a contradiction that such curvature estimate is wrong so that there exists a sequence of surfaces whose the norms of second fundamental forms diverge to infinity. After suitably rescaling these surfaces, one would obtain a smooth limiting surface in $\mathbb{R}^3$ that is boundaryless, orientable, complete, weakly stable and of constant mean curvature. The existence of such a limiting surface could be inferred by the argument by Breuning [Bre14] with some extra effort. For the sake of completeness, we present the proof in detail in the last section. On the other hand, it is by the rigidity theorem [BD84, Pal86, DS87, LR89] that this kind of limiting surface must be exactly either a sphere or a plane. However this is impossible by detecting our derivation in the proof. Hence the desired curvature estimate is inferred. Since the rigidity theorem is only known to be available in dimension three, this limit Theorem 1 to the situation of dim $M = 3$.

Theorem 2 is a consequence of Theorem 1 by estimating the lower bound of the diameter of $\Sigma$. For the case $\Sigma$ is a strongly stable (see definition 5) MOTS, Andersson and Metzger [AM10] use the iteration technique, initiated by Simons, Schoen and Yau [SSY75], to obtain the sup-estimate of $|\Pi|$ depending on the $C^0$ norm of the full Riemannian curvature tensor of $(M, g)$, the injectivity radius of $(M, g)$ and the $C^1$ norm of $K$. This estimate plays an important role in solving Jang's equation and in proving the smoothness of the boundary of trapped region [AM09]. As we will see in section 2 that strong stability implies weak stability, our Theorem 2 is an improvement of Andersson and Metzger's curvature estimate. It should be pointed out that their curvature estimate, although proven originally in the case dim $M = 3$, is actually valid for $3 \leq \dim M \leq 6$ [AEM11]. This dimensional restriction is the same as that celebrated curvature estimate by [SSY75].

In time symmetric case, i.e. $K \equiv 0$, null expansion reduces to mean curvature and the notion weak stability reduces to volume preserving stability. In this case Theorem 2 reduces essentially to [EM12, Proposition 2.2] (also see [Ye96]), which states that the norm of the second fundamental form of a volume preserving constant mean curvature surface $(\Sigma^2, h) \looparrowright (M^3, g)$ is bounded by a constant depending only on the $C^0$ norm of the Ricci curvature of $g$, the injectivity radius of $(M, g)$ and the absolute value of the mean curvature of $\Sigma$.

We point out that Theorem 1 and 2 require neither energy condition nor constraint equations on $(g, K)$.

In the Euclidean 3-space, the harmonic radius $r_H$ is infinity. Thus as a direct consequence of Theorem 1 (1), we also obtain a generalization of a curvature estimate of Schoen [Sch83, Corollary 4], shifting from stable minimal surfaces to volume preserving stable constant mean curvature surfaces. Precisely, we have the following corollary:

Corollary 3. *There is a universal constant $C$ so that if*

(1) $(\Sigma, h) \looparrowright \mathbb{R}^3$ *is a compact connected orientable isometrically immersed 2-surface with $\partial\Sigma \neq \emptyset$; and*

(2) $\Sigma$ *is of constant mean curvature and volume preserving stable,*

*then*

$$|\Pi(p)|_h \leq C d_h(p, \partial\Sigma)^{-1}, \qquad \forall\, p \in \Sigma\backslash\partial\Sigma.$$

## 2. Notions and Preliminaries.

**2.1. Stabilities.** In order to establishing notions of stabilities, we first compute the linearization of the null expansion.

Denote by $\nabla$ and $\Delta$ the metric connection and the nonpositive Beltrami-Laplacian of $h$, respectively. Latin indices $i, j \ldots$ run from 1 to 2, and $a, b \ldots$ run from 1 to 3. The Einstein summation convention of summing on repeated indices is used.

Suppose $F : \Sigma \times (-\epsilon, \epsilon) \to M$, $(p, t) \mapsto F(p, t)$ is a normal variation so that $F(\cdot, 0) = f$ and $\partial_t F(\cdot, 0) = \varphi\nu$ for some $\varphi \in C^\infty(\Sigma)$. We have

Lemma 4. *The linearization of the null expansion in normal direction is*

$$L_\Sigma \varphi := \frac{d}{dt}\theta(F(\Sigma, t))|_{t=0} = -\Delta\varphi + 2K(\nu, \nabla\varphi) + (-|\Pi|_h^2 - Ric_g(\nu, \nu) + tr_h(D_\nu K))\varphi.$$

This variational formula could be found in [Met07, AEM11]. The proof is included here for completeness.

*Proof.* Note that $\frac{d}{dt}\theta(t)|_{t=0} = \frac{d}{dt}H(F(\Sigma, t))|_{t=0} + \frac{d}{dt}tr_{h_t}(K)|_{t=0}$ where $h_t$ is the pulled back metric of $g$ to $\Sigma$ by $F(\cdot, t)$. It is well known that

$$\frac{d}{dt}H(F(\Sigma, t))|_{t=0} = -\Delta\varphi - (|\Pi|^2 + Ric_g(\nu, \nu))\varphi.$$

Thus it remains to calculate $\frac{d}{dt}tr_{h_t}(K)|_{t=0}$. For any $p \in \Sigma$, let $\{z^1, z^2\}$ be an $h$-normal coordinate system centered at $p$. The following computation is then valid as evaluated at $p$:

$$\begin{aligned}\frac{d}{dt}tr_{h_t}(K)|_{t=0} &= \frac{d}{dt}(K_{ij}(h_t)^{ij})|_{t=0} \\ &= (D_{\partial_t}K)(\partial_i, \partial_i) + 2K(D_{\partial_i}\partial_t, \partial_i) - 2K_{ij}g(D_{\partial_i}\partial_t, \partial_j) \\ &= tr_h(D_\nu K)\varphi + 2K(\nabla\varphi, \nu).\end{aligned}$$

This completes the proof. □

In general, the elliptic operator $L_\Sigma$ is not symmetric due to the presence of the 2-tensor $K$. However, as pointed out in [AM10] (and references therein), the principal eigenvalue $\lambda_1(L_\Sigma)$ of $L_\Sigma$, being defined as the eigenvalue of least real part, must be real. Moreover the corresponding eigenfunction does not change sign in the interior of $\Sigma$. The following definition is a straightforward generalization of the stable MOTS in [AM10].

Definition 5. An immersed surface $\Sigma \looparrowright M$ of constant null expansion is strongly stable if the principal eigenvalue $\lambda_1(L_\Sigma)$ of $L_\Sigma$ is nonnegative.

To proceed further people symmetrize $L_\Sigma$ [GS06, AM10] (also see the proof of Lemma 8 below) and obtain the following symmetric elliptic operator:

$$L_\Sigma^{sym}\varphi := -\Delta\varphi + (-|\Pi|^2 + Z)\varphi \tag{2.1}$$

where

$$Z = -div_h S + |S|_h^2 - Ric_g(\nu, \nu) + tr_h(D_\nu K) \tag{2.2}$$

and $S \in \mathfrak{X}(\Sigma)$ is the dual vector field of the 1-form $f^*(K(\nu, \cdot)) \in \Lambda^1(\Sigma)$ with respect to metric $h$.

LEMMA 6. *The following inequality holds pointwise on* $\Sigma$*:*

$$|Z|_h \leq |K|_g(|\Pi|_h + |H(\Sigma)|) + 2|DK|_g + |K|_g^2 + |Ric_g|_g.$$

*Proof.* Let $p \in \Sigma$, and let $\{z^1, z^2\}$ be an $h$-normal coordinate system centered at $p$. Then the following holds as evaluated at $p$:

$$\begin{aligned} div_h S &= \nabla_{\partial_i}(S(\partial_i)) = D_{\partial_i}(K(\nu, \partial_i)) \\ &= (D_{\partial_i}K)(\nu, \partial_i) + K(D_{\partial_i}\nu, \partial_i) + K(\nu, D_{\partial_i}\partial_i) \\ &= (D_{\partial_i}K)(\nu, \partial_i) + K_{ij}\Pi_{ij} - H(\Sigma)K(\nu, \nu). \end{aligned}$$

Substituting this into (2.2) and applying Cauchy inequality, the lemma is then inferred. □

DEFINITION 7. An immersed surface $\Sigma \looparrowright M$ of constant null expansion is symmetrized stable if the principal eigenvalue $\lambda_1(L_\Sigma^{sym})$ of $L_\Sigma^{sym}$ is nonnegative.

This definition is a direct generalization of symmetrized stable MOTS (cf. [GM08]) to surface of constant null expansion. The following lemma (cf. [GM08, Proposition 2.1] for MOTS) is inferred by the argument in [GS06]. We include the proof here and show that how to obtain $L_\Sigma^{sym}$ from $L_\Sigma$.

LEMMA 8. *Strong stability implies symmetrized stability.*

*Proof.* Suppose $\lambda := \lambda_1(L_\Sigma) \geq 0$ and let $\varphi$ be a corresponding eigenfunction that is positive in the interior of $\Sigma$. Then

$$\begin{aligned} 0 \leq \lambda_1 &= \varphi^{-1}L_\Sigma\varphi \\ &= -\varphi^{-1}\Delta\varphi + 2h(S, \nabla\varphi)\varphi^{-1} + (-|\Pi|^2 - Ric_g(\nu, \nu) + tr_h(D_\nu K)) \\ &= div_h(S - \nabla\log\varphi) - |S - \nabla\log\varphi|_h^2 - |\Pi|^2 + Z. \end{aligned}$$

Partially integrating the above inequality against $\psi^2$, with $\psi$ being any function of $C_c^\infty(\Sigma\backslash\partial\Sigma)$, and applying the Cauchy inequality, we then obtain

$$\begin{aligned} 0 &\leq \int_\Sigma -h(S - \nabla\log\varphi, 2\psi\nabla\psi) - |S - \nabla\log\varphi|_h^2\psi^2 + (-|\Pi|^2 + Z)\psi^2\, d\mu \\ &\leq \int_\Sigma |\nabla\psi|^2 + (-|\Pi|^2 + Z)\psi^2\, d\mu. \end{aligned}$$

This implies $\lambda_1(L_\Sigma^{sym}) \geq 0$ by the Rayleigh formula. □

DEFINITION 9. An immersed surface $\Sigma \looparrowright M$ of constant null expansion is weakly stable if

$$I_\Sigma(\varphi) := \int_\Sigma |\nabla\varphi|^2 + (-|\Pi|^2 + Z)\varphi^2\, d\mu \geq 0 \tag{2.3}$$

for all $\varphi \in C_c^\infty(\Sigma\backslash\partial\Sigma)$ with $\int_\Sigma \varphi\, d\mu = 0$.

It is obvious that symmetrized stability implies weak stability.

Let us consider the above notion of stabilities in the special case that $K = cg$ with $c$ being a real number. In this case, it holds that $\theta(\Sigma) = H(\Sigma) + 2c$. Thus surfaces of constant null expansion reduces to surfaces of constant mean curvature, and $L_\Sigma$

reduces to the linearization of the mean curvature. Moreover, it holds $L_\Sigma = L_\Sigma^{sym}$. Thus strong stability reduces to symmetrized stability, and weak stability reduces to the volume preserving stability for surfaces of constant mean curvature. Here is an example relevant to this case:

Example. Consider the Schwarzschild-de Sitter metric in the Mc Vittie form (cf. [LXZ10]) as follow

$$\tilde{g} = -\frac{(1-\frac{m}{2Ar})^2}{(1+\frac{m}{2Ar})^2}dt^2 + A^2(1+\frac{m}{2Ar})^4\delta$$

where $\delta = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ is the standard Euclidean metric, $r = (\sum_{a=1}^3 (x^a)^2)^{1/2}$, $A = e^{t/\lambda}$, and $\lambda$ and $m$ are positive numbers. This metric $\tilde{g}$ satisfies the vacuum Einstein equation with positive cosmological constant $3/\lambda^2$.

We consider the stability of 2-spheres $S_r = \{|x|_\delta = r\}$ in time slices $\{t = \text{constant}\}$. (Also compare with [EM12, P. 85] for spheres in Schwarzschild.) Each time slice is $\mathbb{R}^3/\{0\}$ with induced metric $g := \phi^4\delta$, where $\phi := A^{1/2}(1+\frac{m}{2Ar})$, and with second fundamental form $K = \frac{1}{\lambda}g$. Applying the following conformal transformation formula for Ricci curvature

$$\begin{aligned} Ric_g(\partial_a, \partial_b) =& Ric_\delta(\partial_a, \partial_b) - (\Delta_\delta \log\phi^2)\delta_{ab} - \partial_a\partial_b \log\phi^2 \\ & - |\partial \log\phi^2|_\delta^2 \delta_{ab} + \partial_a \log\phi^2 \partial_b \log\phi^2, \end{aligned}$$

we infer that

$$Ric_g(\partial_a, \partial_b) = (\delta_{ab} - \frac{3x^a x^b}{r^2})\frac{m}{r^3}\phi^{-2}.$$

We claim that each time slice is symmetric along the 2-sphere $S_{\frac{m}{2A}}$. To see this, just note that the differentiable mapping $F : \mathbb{R}^3/\{0\} \to \mathbb{R}^3/\{0\}$ defined by $F(x) = (\frac{m}{2A})^2\frac{x}{|x|^2}$ is a diffeomorphism satisfying $F|_{S_{\frac{m}{2A}}} = Id_{S_{\frac{m}{2A}}}$, $F^*g = g$ as well as $F^*K = K$. Thus we could restrict ourselves to consider the 2-sphere $S_r$ only for $r \geq \frac{m}{2A}$. Let $(S^2, g_{S^2})$ be standard unit sphere of center 0 in $\mathbb{R}^3$. Then $S_r$ has induced metric $h_r = \phi^4 r^2 g_{S^2}$, and unit outer normal $\nu = \phi^{-2}\partial_r$ with respect to $g$. The second fundamental form of $S_r$ is computed as

$$\Pi = \frac{1}{2}\mathcal{L}_{\phi^{-2}\partial_r} g = \frac{1}{2}\phi^{-2}\partial_r(\phi^4 r^2)g_{S^2} = A^{1/2}\phi^{-3}r^{-1}(1-\frac{m}{2Ar})h_r,$$

where $\mathcal{L}$ is the Lie derivative. Thus $S_r$ is of constant null expansion $\theta(S_r) = H(S_r) + tr_{h_r}(K) = 2A^{1/2}\phi^{-3}(1-\frac{m}{2Ar})r^{-1} + \frac{2}{\lambda}$, and it holds $|\Pi|_{h_r}^2 = 2A\phi^{-6}(1-\frac{m}{2Ar})^2 r^{-2}$. Also note that

$$Ric_g(\nu, \nu) = Ric_g(\partial_r, \partial_r)\phi^{-4} = \frac{x^a}{r}\frac{x^b}{r}Ric_g(\partial_a, \partial_b)\phi^{-4} = -\frac{2m}{r^3}\phi^{-6}.$$

Thus we have

$$L_{S_r}^{sym}\varphi = L_{S_r}\varphi = -r^{-2}\phi^{-4}\Delta_{g_{S^2}}\varphi + \frac{-4A^2r^2 + 8Amr - m^2}{2Ar^4\phi^6}\varphi.$$

Since the first two eigenvalues of $-\Delta_{g_{S^2}}$ are $\lambda_1(-\Delta_{g_{S^2}}) = 0$ and $\lambda_2(-\Delta_{g_{S^2}}) = 2$, it follows that

$$\lambda_1(L_{S_r}) = \frac{2A}{r^4\phi^6}(\frac{2+\sqrt{3}}{2A}m - r)(r + \frac{\sqrt{3}-2}{2A}m) \quad \text{and} \quad \lambda_2(L_{S_r}) = \frac{6m}{r^3\phi^6} \geq 0.$$

This implies that $S_r$ is strongly stable (or equivalently, symmetrized stable) if and only if $\frac{m}{2A} \leq r \leq \frac{2+\sqrt{3}}{2A}m$, and that $S_r$ is weakly stable for all $r \geq \frac{m}{2A}$.

**2.2. Harmonic coordinates and rescaling formulae.** Let $\Omega$ be as given in Theorem 1, and $P_0 \in \Omega$. There is a harmonic coordinate system $(y^1, y^2, y^3)$ defined on the $g$-geodesic ball $B_g(P_0, r_H)$ with uniform radius $r_H$. More precisely, the following holds (cf. [Heb96, Theorem 1.3]).

THEOREM 10. *Let $\beta \in (0, 1)$ and $\sigma > 0$ be two given numbers. Let $(M^3, g)$ be a smooth Riemannian 3-manifold, and $\Omega$ an open subset of $M$. Set*

$$\Omega_\sigma = \{P \in M : d_g(P, \Omega) < \sigma\}$$

*where $d_g$ is the distance function associated to metric $g$. Suppose that there are positive constants $\Lambda_0$ and $i_0$ so that*

$$\| Ric_g \|_{C^0(\Omega_\sigma, g)} \leq \Lambda_0 \quad \textit{and} \quad \inf_{P\in\Omega_\sigma} inj_{(M, g)}(P) \geq i_0 > 0.$$

*Then there exists positive numbers $Q_0 \geq 1$ and $r_H \in (0, i_0]$, both of which depend only on $\beta$, $\sigma$, $\Lambda_0$ and $i_0$, so that for each $P_0 \in \Omega$ there is a harmonic coordinate system $(y^1, y^2, y^3)$ defined on the $g$-geodesic ball $B_g(P_0, r_H)$ with $(y^1, y^2, y^3)(P_0) = 0$ so that the metric coefficients $g_{y^a y^b} = g(\partial_{y^a}, \partial_{y^b})$ satisfy the following:*

(1) $g_{y^a y^b}(0) = \delta_{ab}$*;*

(2) $Q_0^{-2}\delta_{ab} \leq g_{y^a y^b} \leq Q_0^2\delta_{ab}$ *(as quadratic forms);*

(3) $\sum_{c=1}^{3} \sup |\partial_{y^c} g_{y^a y^b}| + \sum_{c=1}^{3} [\partial_{y^c} g_{y^a y^b}]_\beta \leq Q_0$*, with norms taken in the $y^a$ coordinates.*

In this paper, all the strength to apply the harmonic coordinates is that the sizes of the coordinate neighborhood, i.e. $r_H$, and the $C^{1,\beta}$-norms of $g_{y^a y^b}$ could be estimated. The property $y^a$'s being harmonic functions is not made use of.

Now, we fix an arbitrary number $\lambda \geq 1$ and call $(x^1, x^2, x^3) := \lambda(y^1, y^2, y^3)$ the $\lambda$*-coordinates* on $B_g(P_0, r_H)$. We identify $B_g(P_0, r_H)$ with its image $\tilde{U}$ in $\mathbb{R}^3$ via $\{x^a\}$, and view $g$ as being defined in $\tilde{U}$. It follows from Theorem 10 (2) that

$$B_\delta(0, \lambda r_H/Q_0) \subset \tilde{U} \subset B_\delta(0, \lambda r_H Q_0). \tag{2.4}$$

(Hereafter $B_\delta(0, \rho)$ is the Euclidean 3-ball centered at the origin and of radius $\rho > 0$.) Set

$$\tilde{g} := \lambda^2 g.$$

Note that since the coefficients $\tilde{g}_{x^a x^b}(x)$ of $\tilde{g}$ in the $\lambda$-coordinates $\{x^a\}$ satisfy

$$\tilde{g}_{x^a x^b}(x) = g_{y^a y^b}(x/\lambda) \quad \text{and} \quad \partial_{x^c}\tilde{g}_{x^a x^b}(x) = \frac{1}{\lambda}\partial_{y^c} g_{y^a y^b}(x/\lambda),$$

it follows that:

(A1) $\tilde{g}_{x^a x^b}(0) = \delta_{ab}$;

(A2) $Q_0^{-2}\delta_{ab} \leq \tilde{g}_{x^a x^b}(x) \leq Q_0^2\delta_{ab}$, $\forall\, x \in \tilde{U}$, as quadratic forms;

(A3) $\|\tilde{g}_{x^a x^b} - \delta_{ab}\|_{C^{1,\beta}(B_\delta(o,\rho))} \leq \frac{Q_0}{\lambda}(1+\rho)$, $\forall\, \rho \in (0, \lambda r_H/Q_0]$,

where

$$\|\tilde{g}_{x^a x^b} - \delta_{ab}\|_{C^{1,\beta}(B_\delta(o,\rho))} := \sup_{B_\delta(o,\rho)} |\tilde{g}_{x^a x^b} - \delta_{ab}| + \sum_{c=1}^{3} \sup_{B_\delta(o,\rho)} |\partial_{x^c}\tilde{g}_{x^a x^b}| + \sum_{c=1}^{3} [\partial_{x^c} g_{x^a x^b}]_{\beta;\, B_\delta(o,\rho)}.$$

Denote by $\tilde{h}$ the pulled back metric $f^*\tilde{g} = \lambda^2 h$. Obviously $f : (\Sigma, \tilde{h}) \to (M, \tilde{g})$ is isometrically immersed with global unit normal $\tilde{\nu} := \frac{1}{\lambda}\nu$. Denote by $\tilde{H}$ and $\tilde{\Pi}$ the mean curvature and second fundamental form with respect to $\tilde{\nu}$, and by $\tilde{\nabla}$ and $\tilde{D}$ the metric connection of $\tilde{h}$ and $\tilde{g}$, respectively. Set

$$\tilde{K} := \lambda^2 K.$$

It is easy to check that:

LEMMA 11. *With notion as above, the following rescaling formulae hold:*
(1) $\tilde{H} = \frac{1}{\lambda} H$, $\tilde{\Pi} = \lambda \Pi$, $|\tilde{\Pi}|_{\tilde{h}} = \frac{1}{\lambda}|\Pi|_h$;
(2) $\tilde{\nabla}\varphi = \lambda^{-2}\nabla\varphi$, $|\tilde{\nabla}\varphi|_{\tilde{h}} = \lambda^{-1}|\nabla\varphi|_h$ *for all differentiable function* $\varphi$ *on* $\Sigma$;
(3) $d\mu_{\tilde{h}} = \lambda^2 d\mu_h$ *where* $d\mu_{\tilde{h}}$ *and* $d\mu_h$ *are the volume form of* $\tilde{h}$ *and* $h$ *respectively;*
(4) $tr_{\tilde{h}}(\tilde{K}) = tr_h(K)$, $|\tilde{K}|_{\tilde{g}} = |K|_g$ *and* $|\tilde{D}\tilde{K}|_{\tilde{g}} = \lambda^{-1}|DK|_g$. *In particular, we have* $|\tilde{K}|_{C^1,\tilde{g}} \le |K|_{C^1,g}$.

With the above rescaling, the equation $H(\Sigma) + tr_h(K) = \theta_0$ then becomes

$$\tilde{H} + \frac{1}{\lambda} tr_{\tilde{h}}(\tilde{K}) = \frac{\theta_0}{\lambda}, \tag{2.5}$$

and the integral $I(\varphi)$ in definition 9 could be rewritten as

$$I(\varphi) = \int_\Sigma |\tilde{\nabla}\varphi|^2_{\tilde{h}} + (-|\tilde{\Pi}|^2_{\tilde{h}} + \lambda^{-2}Z)\varphi^2 \, d\mu_{\tilde{h}}. \tag{2.6}$$

**2.3. Local computations.** Continued with the above $\lambda$-coordinates $\{x^1, x^2, x^3\}$, we assume that $\Sigma$ is contained in $\tilde{U}$ and that $\Sigma$ is the graph of a differentiable function $u$ which is defined on some open subset of $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. We will give some explicit expressions of geometric quantities of $\Sigma$ in terms of $u$ and $\tilde{g}_{ab}$. Write $u = u(x')$ with $x' = (x^1, x^2)$, and set $s = s(x) = x^3 - u(x')$. Clearly $\Sigma$ is the zero level set of $s$. Putting (A2) into $|\tilde{D}s|^2_{\tilde{g}} = \partial_a(x^3 - u)\partial_b(x^3 - u)\tilde{g}^{ab}$, it follows that

$$Q_0^{-2}(1 + |\partial u|^2_\delta) \le |\tilde{D}s|^2_{\tilde{g}} \le Q_0^2(1 + |\partial u|^2_\delta).$$

Let $\tilde{\mu} = \tilde{\mu}(x') = |\tilde{D}s|^{-1}_{\tilde{g}}(x', u(x'))$. Then

$$\frac{Q_0^{-2}}{1 + |\partial u|^2_\delta} \le \tilde{\mu}^2 \le \frac{Q_0^2}{1 + |\partial u|^2_\delta}. \tag{2.7}$$

Assume $\tilde{\nu}$ equals the upward unit normal of graph $u$; namely, $\tilde{\nu} = \tilde{\nu}(x') = \tilde{\mu}\tilde{D}s(x', u(x'))$. Note that

$$\tilde{h}_{ij}(x') := \tilde{g}(\partial_i + u_i\partial_3, \partial_j + u_j\partial_3)(x', u(x')), \quad 1 \le i, j \le 2,$$

are metric coefficients of $\tilde{h}$ relative to the frame $\{e_i := \partial_i + u_i\partial_3 : i = 1, 2\}$ on graph $u$. Set

$$\tilde{h}^{ab} = \tilde{h}^{ab}(x') = \tilde{g}^{ab}(x', u(x')) - \tilde{\nu}^a(x')\tilde{\nu}^b(x')$$

for $1 \le a, b \le 3$. It is well known that $(\tilde{h}^{ij})$ is the inverse metric of $(\tilde{h}_{ij})$. We next estimate the eigenvalues of $(\tilde{h}_{ij})$. Denote by $\lambda_{\max}(\tilde{h}_{ij})$ and $\lambda_{\min}(\tilde{h}_{ij})$ the maximal and minimal eigenvalues of $(\tilde{h}_{ij})$ respectively. Then we have

$$\begin{aligned}\lambda_{\max}(\tilde{h}_{ij}) &= \max_{\substack{w=(w^1,\, w^2)\in\mathbb{R}^2\\ |w|_\delta=1}} \tilde{h}_{ij}w^iw^j = \max_{\substack{w\in\mathbb{R}^2\\ |w|_\delta=1}} \tilde{g}(w^ie_i,\, w^je_j)\\ &\le Q_0^2 \max_{\substack{w\in\mathbb{R}^2\\ |w|_\delta=1}} |w^ie_i|_\delta^2 \le Q_0^2(1+|\partial u|_\delta^2),\end{aligned}$$

and similarly

$$\begin{aligned}\lambda_{\min}(\tilde{h}_{ij}) &= \min_{\substack{w=(w^1,\, w^2)\in\mathbb{R}^2\\ |w|_\delta=1}} \tilde{h}_{ij}w^iw^j = \min_{\substack{w\in\mathbb{R}^2\\ |w|_\delta=1}} \tilde{g}(w^ie_i,\, w^je_j)\\ &\ge Q_0^{-2} \min_{\substack{w\in\mathbb{R}^2\\ |w|_\delta=1}} |w^ie_i|_\delta^2 \ge Q_0^{-2}.\end{aligned}$$

Hence, there holds that

$$Q_0^{-2}\delta_{ij} \le \tilde{h}_{ij} \le Q_0^2(1+|\partial u|_\delta^2)\delta_{ij}$$

and

$$Q_0^{-2}(1+|\partial u|_\delta^2)^{-1}\delta_{ij} \le \tilde{h}^{ij} \le Q_0^2\delta_{ij} \tag{2.8}$$

as quadratic forms. Recall that $\tilde{H}$ and $\tilde{\Pi}$ are the mean curvature and second fundamental form of $(\Sigma,\, \tilde{h})$ in $(\tilde{U},\, \tilde{g})$ respectively. We have

Lemma 12. *With notion as above, the following holds:*
(1) $\tilde{H} = -\tilde{\mu}\tilde{h}^{ij}\partial_i\partial_j u + \tilde{\mu}\tilde{h}^{ab}\tilde{\Gamma}^i_{ab}\partial_i u - \tilde{\mu}\tilde{h}^{ab}\tilde{\Gamma}^3_{ab}$;
(2) $tr_{\tilde{h}}(\tilde{K}) = (\tilde{K}_{ij} + \tilde{K}_{33}\partial_i u\partial_j u + 2\tilde{K}_{3j}\partial_i u)\tilde{h}^{ij}$;
(3) $\tilde{\Pi}_{ij} := \tilde{\Pi}(\partial_i + u_i\partial_3,\, \partial_j + u_j\partial_3) = -\tilde{\mu}\partial_i\partial_j u - \tilde{\mu}\tilde{\Gamma}^a_{j3}\partial_a s\partial_i u - \tilde{\mu}\tilde{\Gamma}^a_{i3}\partial_a s\partial_j u - \tilde{\mu}\tilde{\Gamma}^a_{33}\partial_a s\partial_i u\partial_j u + \tilde{\mu}\tilde{\Gamma}^k_{ij}\partial_k u - \tilde{\mu}\tilde{\Gamma}^3_{ij}$.
*where $\tilde{\Gamma}^a_{bc}$'s are the Christoffel symbols of metric $\tilde{g}$ in $x^a$ coordinates.*

*Proof.* Note that

$$\begin{aligned}\tilde{H} &= div_{\tilde{h}}(\tilde{\mu}\tilde{D}s) = div_{\tilde{g}}(\tilde{\mu}\tilde{D}s) = \tilde{g}^{ab}\tilde{g}(\tilde{D}_{\partial_a}(\tilde{\mu}\tilde{D}s),\, \partial_b)\\ &= \tilde{\mu}(\tilde{g}^{ab} - \tilde{\nu}^a\tilde{\nu}^b)\tilde{D}^2_{a,\,b}s = \tilde{\mu}\tilde{h}^{ab}(\partial_a\partial_b s - \tilde{\Gamma}^c_{ab}\partial_c s).\end{aligned}$$

This together with $s = x^3 - u$ implies (1). (2) follows directly from $tr_{\tilde{h}}(\tilde{K}) = \tilde{K}(\partial_i + u_i\partial_3,\, \partial_j + u_j\partial_3)\tilde{h}^{ij}$. As for (3), observe that

$$\tilde{\Pi}_{ij} = -\tilde{g}(\tilde{D}_{\partial_i+u_i\partial_3}\partial_j + u_j\partial_3,\, \tilde{\mu}\tilde{D}(x^3-u)).$$

Then the direct calculation of the right hand side implies (3). □

With (2.7) and (2.8), one sees from the above lemma that the equation (2.5) is not uniformly elliptic unless there is a priori estimate for the upper bound of $|\partial u|_\delta$.

Denote by $\nu_\delta$ the unit normal of $\Sigma$ in $(\tilde{U},\, \delta)$, and by $\Pi_\delta$ the relative second fundamental form. Then the difference between $|\tilde{\Pi}|^2_{\tilde{h}}$ and $|\Pi_\delta|^2_\delta$ could be estimated as follow ($\Sigma$ is not necessarily a graph):

Proposition 13. *Assume condition (A2). The following holds pointwise on $\Sigma$:*

$$||\tilde{\Pi}|^2_{\tilde{h}} - |\Pi_\delta|^2_\delta| \le C(Q_0)\left[(Q_0 - 1 + |\partial\tilde{g}|)|\tilde{\Pi}|^2_{\tilde{h}} + |\partial\tilde{g}|(1+|\partial\tilde{g}|^2)\right],$$

*where* $C(Q_0)$ *is a constant, and* $|\partial\tilde{g}|^2(x) := \sum_{a,\,b,\,c=1}^{3}(\partial_a \tilde{g}_{bc}(x))^2$.

*Proof.* Pick any $p \in \Sigma$. Observe that both $Q_0$ and $|\partial\tilde{g}|$ are invariant under coordinate transformation by translations and rotations, so we could assume without loss of generality that $\Sigma$ is the graph of a differentiable function $x^3 = u(x')$ near $p$ so that $p = (0,\, u(0))$ and $\partial u(0) = 0$. To the end of this proof, the calculation is carried out only at $p$. By (2.8) and Lemma 12, we have

$$Q_0^{-2}\delta_{ij} \le \tilde{h}^{ij} \le Q_0^2\delta_{ij}$$

and

$$\tilde{\Pi}_{ij} = -(\tilde{g}^{33})^{-1/2}(\partial_i\partial_j u + \tilde{\Gamma}^3_{ij}), \quad (\tilde{\Pi}_\delta)_{ij} = -\partial_i\partial_j u.$$

It follows that

$$\begin{aligned} ||\tilde{\Pi}|^2_{\tilde{h}} - |\Pi_\delta|^2_\delta| &= |\tilde{h}^{ik}\tilde{h}^{jl}(\tilde{g}^{33})^{-1}(\partial_i\partial_j u + \tilde{\Gamma}^3_{ij})(\partial_k\partial_l u + \tilde{\Gamma}^3_{kl}) - |\partial^2 u|^2_\delta| \\ &\le |(\tilde{g}^{33})^{-1}\tilde{h}^{ik}\tilde{h}^{jl}\partial_i\partial_j u\partial_k\partial_l u - |\partial^2 u|^2_\delta| \\ &\quad + |(\tilde{g}^{33})^{-1}\tilde{h}^{ik}\tilde{h}^{jl}(\tilde{\Gamma}^3_{kl}\partial_i\partial_j u + \tilde{\Gamma}^3_{ij}\partial_k\partial_l u + \tilde{\Gamma}^3_{ij}\tilde{\Gamma}^3_{kl})| \\ &\le (Q_0^6 - 1)|\partial^2 u|^2_\delta + Q_0^6\left(2|\partial^2 u|_\delta\left(\textstyle\sum_{i,\,j=1}^2(\tilde{\Gamma}^3_{ij})^2\right)^{1/2} + \textstyle\sum_{i,\,j=1}^2(\tilde{\Gamma}^3_{ij})^2\right). \end{aligned} \tag{2.9}$$

It also holds that

$$\begin{aligned} |\tilde{\Pi}|^2_{\tilde{h}} &= |\tilde{h}^{ik}\tilde{h}^{jl}(\tilde{g}^{33})^{-1}(\partial_i\partial_j u + \tilde{\Gamma}^3_{ij})(\partial_k\partial_l u + \tilde{\Gamma}^3_{kl})| \\ &\ge Q_0^{-6}\textstyle\sum_{i,\,j=1}^2(\partial_i\partial_j u + \tilde{\Gamma}^3_{ij})^2 \\ &\ge Q_0^{-6}\left(\frac{1}{2}|\partial^2 u|^2_\delta - \textstyle\sum_{i,\,j=1}^2(\tilde{\Gamma}^3_{ij})^2\right). \end{aligned}$$

Thus

$$|\partial^2 u|^2_\delta \le 2Q_0^6|\tilde{\Pi}|^2_{\tilde{h}} + 2\sum_{i,\,j=1}^{2}(\tilde{\Gamma}^3_{ij})^2. \tag{2.10}$$

Note that

$$\sum_{i,\,j=1}^{2}(\tilde{\Gamma}^3_{ij})^2 = \sum_{i,\,j=1}^{2}(\frac{1}{2}\tilde{g}^{3a}(\partial_i\tilde{g}_{ja} + \partial_j\tilde{g}_{ia} - \partial_a\tilde{g}_{ij}))^2 \le C(Q_0)|\partial\tilde{g}|^2. \tag{2.11}$$

Putting (2.11) and (2.10) into (2.9), we get the desired estimate. □

**2.4. Langer Charts.** Although it is true by Implicit Function Theorem that any 2-surface immersed in $\mathbb{R}^3$ could be represented locally as the graph of a differentiable function, we need more information for our purpose: To what extent this is true. Indeed, the size of the graph is controlled by the norm of the second fundamental form. For the sake of clarity and for latter reference, we first introduce some notation, following [Bre14] and [Lan85].

A mapping $A : \mathbb{R}^3 \to \mathbb{R}^3$ is called a Euclidean isometry if there is a rotation $R \in SO(3)$ and a translation $T \in \mathbb{R}^3$ such that $A(x) = Rx + T$ for all $x \in \mathbb{R}^3$. Let $f : \Sigma^2 \to \mathbb{R}^3$ be a 2-surface immersed in $\mathbb{R}^3$ with global unit normal $\nu_\delta$. For any given point $q \in \Sigma$, there is a (but not unique) Euclidean isometry $A_q(x) = R_q x + T_q$, which maps the origin to $f(q)$ and whose differential, i.e. $R_q$, takes $(0, 0, 1)$ to $\nu_\delta(q)$. Denote by $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ the standard projection onto the $x^1$, $x^2$-plane. Let $U_{r,q} \subset \Sigma$ be the $q$-component of the set $(\pi \circ A_q^{-1} \circ f)^{-1}(D_r)$, where $D_r := D_\delta(0, r)$ is the Euclidean 2-disk of center 0 and of radius $r$ in $\mathbb{R}^2$. Although $A_q$ is not uniquely determined the set $U_{r,q}$ does not depend on the choice of $A_q$.

DEFINITION 14. Let $r > 0$ and $\alpha > 0$. An immersion $f : \Sigma^2 \to \mathbb{R}^3$ is said to be $(r, \alpha)$-immersed at some point $q \in \Sigma$ if the set $A_q^{-1} \circ f(U_{r,q})$ is the graph of a differentiable function $u_q : D_r \to \mathbb{R}$ which satisfies $|\partial u_q(x')| < \alpha$ for all $x' = (x^1, x^2) \in D_r$. If $f$ is $(r, \alpha)$-immersed at each point of a subset $W$ of $\Sigma$, then $f$ is called $(r, \alpha)$-immersed in $W$.

If $f$ is $(r, \alpha)$-immersed at $q \in \Sigma$, then it is clear that the restriction

$$f_q := \pi \circ A_q^{-1} \circ f|_{U_{r,q}} : U_{r,q} \to D_r$$

is a diffeomorphism; that is, $(U_{r,q}, f_q)$ is a coordinate chart of $\Sigma$. We shall call $(U_{r,q}, f_q)$ the *Langer chart* centered at $q$, with defining Euclidean isometry $A_q$ and defining function $u_q$. Note that by the choice of $A_q$ it must holds that $u_q(0) = 0$ and $\partial u_q(0) = 0$. Moreover, $R_q^{-1}\nu_\delta$ is the upward unit normal of graph $u_q$; that is,

$$R_q\left(\frac{(-\partial u_q, 1)}{\sqrt{1 + |\partial u_q|_\delta^2}}(x')\right) = \nu_\delta(f_q^{-1}(x')), \quad \forall\, x' \in D_r. \tag{2.12}$$

The following lemma is analogous to [PR02, Lemma 4.35]. We modify the proof given therein.

LEMMA 15. *Let $\tilde{U}$ be an open subset of $\mathbb{R}^3$ and $\tilde{g}$ be a Riemannian metric defined in $\tilde{U}$ satisfying (A2) and*

$$\sup_{x \in \tilde{U}} |\partial \tilde{g}|^2(x) \le Q_0^2. \tag{2.13}$$

*Suppose that $f : (\Sigma^2, \tilde{h}) \to (\tilde{U}, \tilde{g})$ is an isometric immersion with bounded second fundamental form $\tilde{\Pi}$ so that*

$$\sup_\Sigma |\tilde{\Pi}|_{\tilde{h}}^2 \le \Lambda$$

*for some constant $\Lambda$.*

*Then $f$ is $(r, \alpha)$-immersed at any point $q \in \Sigma \backslash \partial\Sigma$ provided*

$$r \le \min\{Q_0^{-1}(1 + \alpha^2)^{-1/2} d_{\tilde{h}}(q, \partial\Sigma),\, C(Q_0, \Lambda)^{-1}(1 + \alpha^2)^{-3/2}\alpha\}$$

*for some positive constant $C(Q_0, \Lambda)$. Here $d_{\tilde{h}}(q, \partial\Sigma)$ would be defined as $\infty$ if $\partial\Sigma = \emptyset$.*

*Proof.* Since the value $Q_0$ in both (A2) and (2.13) is invariant under coordinate transformation by Euclidean isometries, we assume $A_q = id_{\mathbb{R}^3}$ for simplicity. Set

$$r_* = \sup\{r : f \text{ is } (r, \alpha)\text{-immersed at } q\}.$$

Clearly $r_* > 0$ and $f$ is $(r_*, \alpha)$-immersed at $q$. Let $u = u_q : D_{r_*} \to \mathbb{R}$ be the defining function. In particular, $f(U_{r_*,q})$ is the graph of $u$. By the choice of $r_*$, we infer that there is a sequence $\{p_i\} \subset U_{r_*,q}$ so that at least one of the following holds:

(a) $p_i \to \partial\Sigma$ as $i \to \infty$;
(b) $\lim_{i\to\infty} |\partial u(x_i')| = \alpha$, where $x_i' = \pi \circ f(p_i)$.
Assume (a) first. Note that

$$U_{r_*,\, q} \subset D_{\tilde{h}}(q,\, (1+\alpha^2)^{1/2} Q_0 r_*).$$

Thus we have

$$d_{\tilde{h}}(q,\, \partial\Sigma) \le \limsup_{i\to\infty} d_{\tilde{h}}(q,\, p_i) \le (1+\alpha^2)^{1/2} Q_0 r_*,$$

or equivalently,

$$r_* \ge (1+\alpha^2)^{-1/2} Q_0^{-1} d_{\tilde{h}}(q,\, \partial\Sigma). \tag{2.14}$$

Next we assume (b). For any $x' \in D_{r_*}$, we have $|\partial u(x')|_\delta < \alpha$. Hence by the Mean Value Theorem we infer

$$|\partial u(x')|_\delta^2 = |\partial u(x')|_\delta^2 - |\partial u(0)|_\delta^2 \le 2\alpha r_* \|\partial^2 u\|_{C^0(D_{r_*})}. \tag{2.15}$$

On the other hand, it holds that

$$\frac{|\partial^2 u|_\delta^2}{(1+|\partial u|_\delta^2)^3} \le |\Pi_\delta|_\delta^2 \le ||\tilde{\Pi}|_{\tilde{h}}^2 - |\Pi_\delta|_\delta^2| + |\tilde{\Pi}|_{\tilde{h}}^2 \le C(Q_0,\, \Lambda)$$

where we have used (2.7), (2.8) and Lemma 12 (3) in the first inequality and Proposition 13 in the last inequality. This gives rise to

$$\|\partial^2 u\|_{C^0(D_{r_*})} \le C(Q_0,\, \Lambda)(1+\alpha^2)^{3/2}.$$

Substituting this into (2.15), it yields

$$\alpha^2 = \lim_{i\to\infty} |\partial u(x_i')|^2 \le 2\alpha r_* C(Q_0,\, \Lambda)(1+\alpha^2)^{3/2},$$

that is

$$r_* \ge C^{-1}(Q_0,\, \Lambda)(1+\alpha^2)^{-3/2}\alpha. \tag{2.16}$$

Now the Lemma follows from (2.14) and (2.16). □

Finally we need the following convergence theorem.

THEOREM 16 (Convergence Theorem). *Let $f^n : (\Sigma_n,\, h_{\delta,\, n}) \to (\mathbb{R}^3,\, \delta)$, $n \in \mathbb{N}$, be a sequence of isometric immersions with trivial normal bundle and with marked points $p_n \in \Sigma_n$. Suppose that $\{f^n(p_n)\}_{n\in\mathbb{N}}$ is uniformly bounded in $\mathbb{R}^3$, and that there is a sequence $\{R_n\}_{n\in\mathbb{N}}$ of positive numbers increasing to infinity so that each $h_{\delta,\, n}$-geodesic disk $D_{h_{\delta,\, n}}(p_n,\, R_n)$ is relatively compact in $\Sigma_n \backslash \partial\Sigma_n$. Let $r > 0$, $\alpha > 0$ and $0 < \beta < 1$. We further assume that*

(∗) *$f^n$ is $(r,\, \alpha)$-immersed in $D_{h_{\delta,\, n}}(p_n,\, R_n)$ for each $n \in \mathbb{N}$, and for any $p \in D_{h_{\delta,\, n}}(p_n,\, R_n)$ the corresponding defining function $u_p^{(n)} : D_r \to \mathbb{R}$ has uniform $\|\cdot\|_{C^{2,\,\beta}(D_r)}$-norm with respect to $n$, i.e.*

$$\sup_{n\in\mathbb{N}} \sup_{p\in D_{h_{\delta,\, n}}(p_n,\, R_n)} \|u_p^{(n)}\|_{C^{2,\,\beta}(D_r)} < \infty. \tag{2.17}$$

*Then there is a $C^2$ boundaryless orientable 2-surface $\Sigma_\infty$ and a $C^2$ immersion*

$$f_\infty : \Sigma_\infty \to \mathbb{R}^3$$

*so that*

(1) *The pulled back metric $h_\infty = f_\infty^* \delta$ on $\Sigma_\infty$ is complete.*
(2) *$\Sigma_\infty$ admits an exhaustion $W_1 \Subset W_2 \Subset \cdots$ of relatively compact open subsets and a marked point $p_\infty \in W_1$ so that, passing to a subsequence of $\{n\}$ if necessary, there are $C^1$ embeddings $\Phi_n : W_n \to D_{h_{\delta,n}}(p_n, R_n)$ with $\Phi_n(p_\infty) = p_n$.*
(3) *$f^n \circ \Phi_n \to f_\infty$ on any compact subset of $\Sigma_\infty$ in $C^1$ topology .*
(4) *$|\Pi_{\delta,n}|_{h_{\delta,n}} \circ \Phi_n \to |\Pi_\infty|_{h_\infty}$ on compact subsets of $\Sigma_\infty$ in $C^0$ topology, where $\Pi_{\delta,n}$ and $\Pi_\infty$ are the second fundamental forms of $f^n$ and $f_\infty$, respectively.*

The condition $(*)$ in the above theorem is stronger than $\sup |\Pi_{\delta,n}|_{h_{\delta,n}} < \infty$, but a litter weaker than $\sup |\Pi_{\delta,n}|_{h_{\delta,n}} + |\nabla_{h_{\delta,n}} \Pi_{\delta,n}|_{h_{\delta,n}} < \infty$, where the supremum is taken over $D_{h_{\delta,n}}(p_n, R_n)$ with $n$ ranging over $\mathbb{N}$. This condition is suitable for our latter application.

The proof of Theorem 16 is essentially due to the main idea of [Bre14, Theorem 1.3]. The latter concerns the convergence of a sequence of immersed surfaces in Euclidean space with uniformly bounded second fundamental forms under slightly different conditions. The proof is presented in the final section, but let us here make some further remarks which are helpful for the application in the next section. The limiting immersion $f_\infty : \Sigma_\infty \to \mathbb{R}^3$ turns to have globally bounded second fundamental form, and hence be $(r_0, \alpha_0)$-immersed on the entire $\Sigma_\infty$ for some $r_0 \in (0, r]$ and $\alpha_0 > 0$. Given any Langer chart $(U^{(\infty)}_{r_0,q}, (f_\infty)_q)$ of $\Sigma_\infty$ with defining Euclidean isometry $A_q$ and defining function $u_q \in C^2(D_{r_0})$, there are Langer charts $(U^{(n)}_{r_0,q_n}, f^n_{q_n})$ of $\Sigma_n$ with defining Euclidean isometry $A_{q_n}$ and defining function $u_{q_n}$ so that $A_{q_n} \to A_q$ and $u_{q_n} \to u_q$ in $C^2(D_{r_0})$. Let us identify $A_q(\text{graph } u_q)$ and $A_{q_n}(\text{graph } u_{q_n})$ with the portion of $\Sigma_\infty$ and $\Sigma_n$ respectively. Intuitively one could project $A_q(\text{graph } u_q)$ along its normal direction into $A_{q_n}(\text{graph } u_{q_n})$. Indeed this gives the local construction of $\Phi_n$. Since this construction involves the normal direction of $\Sigma_\infty$, it is reasonable that $\Phi_n$ looses one derivative and it is only $C^1$. However, since $u_{q_n} \to u_q$ in $C^2(D_{r_0})$, one could check that $|\Pi_{\delta,n}|_{h_{\delta,n}}(x', u_{q_n}(x')) \to |\Pi_\infty|_{h_\infty}(x', u_q(x'))$. This implies the conclusion (4) of 16 at least in the sense of pointwise convergence. Also if each defining function $u_q$ is $C^\infty$, then both $\Sigma_\infty$ and $f_\infty$ are smooth.

Finally we point out that the limiting immersion $f_\infty : \Sigma_\infty \to \mathbb{R}^3$ is not necessarily proper. To see this, for each $n$ take $\Sigma_n = \mathbb{R}^2$ and $f^n = f$ to be the (non-proper) covering map $f : \mathbb{R}^2 \to T^2 \looparrowright \mathbb{R}^3$, where $T^2$ is a torus. Then $\Sigma_\infty = \mathbb{R}^2$, $f_\infty = f$ , and hence $f_\infty$ is not proper.

## 3. Proof of Theorem 1 and 2.

*Proof of Theorem 1.* The statement (2) is a direct consequence of (1). To see this, observe that the geodesic disk $D_h(p, \frac{1}{2}\text{diam}(\Sigma))$ is still weakly stable. Thus applying (1) with $\Sigma$ replaced by $D_h(p, \frac{1}{2}\text{diam}(\Sigma))$ concludes (2).

So it remains to prove the statement (1). Without loss of generality, we assume $|\Pi|_h$ is continuous up to $\partial\Sigma$. (Otherwise, we replace $\Sigma$ by $\Sigma_\sigma := \{p \in \Sigma : d_h(p, \partial\Sigma) \geq \sigma\}$ with $\sigma \to 0^+$.) Toward a contradiction, we suppose that for each $n \in \mathbb{N}$ there is an initial data set $(M^3_n, g_n, K_n)$ and an isometric immersion

$$f^n : (\Sigma^2_n, h_n) \to (M^3_n, g_n),$$

satisfying assumptions of Theorem 1 with constant null expansion $\theta_n$, so that

$$\sup_{\Sigma_n} \left\{ |\Pi_n(\cdot)|_{h_n} \min\{d_{h_n}(\cdot,\, \partial\Sigma_n),\, r_H\} \right\} \geq 2Q_0^2(n+1). \tag{3.1}$$

Here, the harmonic radius $r_H$ and the corresponding constant $Q_0 \geq 1$, as was shown by Theorem 10, depend only on $\delta^{-1}$, $i_0^{-1}$, and $\Lambda_0$, but not on $n$. Moreover, each $|\Pi_n(\cdot)|_{h_n}$ is continuous up to $\partial\Sigma_n$.

Then the compactness of $\Sigma_n$ implies that the supremum in the left hand side of (3.1) could be achieved by some point, say $p_n$, in $\Sigma_n \backslash \partial\Sigma_n$. Set

$$\rho_n = \frac{1}{2Q_0^2} \min\{d_{h_n}(p_n,\, \partial\Sigma_n),\, r_H\}$$

and $\lambda_n = |\Pi_n(p_n)|_{h_n}$. Then (3.1) implies that

$$\lambda_n \rho_n \geq n + 1 \to \infty \quad \text{as } n \to \infty,$$

and that

$$\lambda_n \geq 2Q_0^2 r_H^{-1}(n+1) \to \infty \quad \text{as } n \to \infty. \tag{3.2}$$

Denote by $D_n$ the $h_n$-geodesic disk $D_{h_n}(p_n,\, \rho_n)$ of center $p_n$ and radius $\rho_n$. Clearly, $D_n$ is relatively compact in $\Sigma_n \backslash \partial\Sigma_n$. For any $p \in D_n$, it is easy to check that

$$\min\{d_{h_n}(p,\, \partial\Sigma_n),\, r_H\} \geq \frac{1}{2} \min\{d_{h_n}(p_n,\, \partial\Sigma_n),\, r_H\} = Q_0^2 \rho_n.$$

Thus

$$\begin{aligned} Q_0^2 \rho_n |\Pi_n(p)|_{h_n} &\leq |\Pi_n(p)|_{h_n} \min\{d_{h_n}(p,\, \partial\Sigma_n),\, r_H\} \\ &\leq |\Pi_n(p_n)|_{h_n} \min\{d_{h_n}(p_n,\, \partial\Sigma_n),\, r_H\} \\ &= \lambda_n 2Q_0^2 \rho_n, \end{aligned}$$

that is

$$|\Pi_n(p)|_{h_n} \leq 2\lambda_n, \quad \forall p \in D_n. \tag{3.3}$$

Next we identify, in exactly the same way as is done in subsection 2.2, the $g_n$-geodesic ball $B_{g_n}(f^n(p_n),\, r_H)$ with its image $\tilde{U}_n$ in $\mathbb{R}^3$ via the $\lambda_n$-coordinates $\{x^a\}$. Set $\tilde{g}_n = \lambda_n^2 g_n$. Then $f^n(p_n) = 0$ and, by (2.4),

$$B_\delta(0,\, \lambda_n r_H/Q_0) \subset \tilde{U}_n. \tag{3.4}$$

Moreover, in view of (A2) and (A3), we obtain:

(B1) $Q_0^{-2}\delta_{ab} \leq (\tilde{g}_n)_{x^a x^b}(x) \leq Q_0^2 \delta_{ab}$, $\forall\, x \in B_\delta(0,\, \lambda_n r_H/Q_0)$, as quadratic forms;

(B2) $\|\tilde{g}_{x^a x^b}\|_{C^{1,\,\beta}(B_\delta(0,\,\lambda_n r_H/Q_0))} \leq 1 + 2r_H$;

(B3) $(\tilde{g}_n)_{x^a x^b} \to \delta_{ab}$ on any compact set of $\mathbb{R}^3$ in $C^{1,\,\beta}$ topology as $n \to \infty$,

where (3.2) is used in (B2) and both (3.2) and (3.4) are used in (B3).

Let $\tilde{h}_n = \lambda_n^2 h_n$ and let $\tilde{D}_n := D_{\tilde{h}_n}(p_n,\, \lambda_n \rho_n)$ be the $\tilde{h}_n$-geodesic disk of center $p_n$ and radius $\lambda_n \rho_n \geq n + 1$. Then $D_n = \tilde{D}_n$, and it is easy to check that

$$f^n(\tilde{D}_n) \subset B_\delta(0,\, Q_0 \lambda_n \rho_n) \subset B_\delta(0,\, \frac{1}{2Q_0} r_H \lambda_n) \subset \tilde{U}_n.$$

To abuse of notation, we still denote by $f^n$ the restriction $f^n|\tilde{D}_n \;:\; \tilde{D}_n \to B_\delta(0,\, \lambda_n r_H/Q_0)$. By Lemma 11, (2.5) and (2.6), it holds

(B4) $|\tilde{\Pi}_n(p_n)|_{\tilde{h}_n} = 1$, and $\sup_{\tilde{D}_n} |\tilde{\Pi}_n|_{\tilde{h}_n} \leq 2$;
(B5) $\tilde{H}_n + \frac{1}{\lambda_n} tr_{\tilde{h}_n}(\tilde{K}_n) = \frac{\theta_n}{\lambda_n}$;
(B6) $\sup_{B_\delta(0,\,\lambda_n r_H/Q_0)} |\tilde{K}_n|_{C^1,\,\tilde{g}_n} \leq \Lambda_1$;
(B7) $0 \leq \int_{\tilde{D}_n} |\tilde{\nabla}_{\tilde{h}_n}\varphi|^2_{\tilde{h}_n} + (-|\tilde{\Pi}_n|^2_{\tilde{h}_n} + \lambda_n^{-2} Z_n)\varphi^2 \, d\mu_{\tilde{h}_n}$ for all $\varphi \in C^1_c(\tilde{D}_n)$ with $\int_{\tilde{D}_n} \varphi \, d\mu_{\tilde{h}_n} = 0$.

Let $h_{\delta,\,n} = (f^n)^*\delta$ be the pulled back metric by $\delta$ on $\tilde{D}_n$, and denote by $\Pi_{\delta,\,n}$ the second fundamental form of $f^n : (\tilde{D}_n,\, h_{\delta,\,n}) \to (B_\delta(0,\, \lambda_n r_H/Q_0),\, \delta)$. Applying Lemma 13 and using (B1), (B2) and (B4), we infer

$$\sup_{\tilde{D}_n} |\Pi_{\delta,\,n}|_{h_{\delta,\,n}} \leq C(Q_0,\, r_H) = C(\delta^{-1},\, i_0^{-1},\, \Lambda_0).$$

From (B5) and (B6), we see that

$$|\frac{\theta_n}{\lambda_n}| \leq |\tilde{H}_n| + \lambda_n^{-1}|tr_{\tilde{h}_n}(\tilde{K}_n)| \leq \sqrt{2}(|\tilde{\Pi}_n|_{\tilde{h}_n} + \lambda_n^{-1}|\tilde{K}_n|_{\tilde{g}_n}) \leq C(\delta^{-1},\, i_0^{-1},\, \Lambda_0,\, \Lambda_1).$$

Thus by passing to a subsequence we could assume

$$\frac{\theta_n}{\lambda_n} \to \theta_\infty \tag{3.5}$$

for some constant $\theta_\infty \in \mathbb{R}$. By Lemma 6 and (3.2), the term $\lambda_n^{-2} Z_n$ in (B7) satisfies

$$\sup_{\tilde{D}_n} \lambda_n^{-2}|Z_n| \leq C(\Lambda_0,\, \Lambda_1)(1 + \lambda_n^{-1})\lambda_n^{-1} \to 0, \tag{3.6}$$

as $n \to \infty$. Fix any $\alpha \in (0,\,1)$. Combining (B4) and the fact $d_{\tilde{h}_n}(D_{\tilde{h}_n}(p_n,\,n),\, \partial\tilde{D}_n) \geq 1$, we assert from Lemma 15 that $f^n$ is $(2r,\,\alpha)$-immersed in $D_{\tilde{h}_n}(p_n,\,n)$, where $r$ is a constant depending only on $Q_0$, $r_H$ and $\alpha$. Pick any $q_n \in D_{\tilde{h}_n}(p_n,\,n)$ and let $v := u_{q_n}$ be the corresponding defining function as in definition 14. Since $v(0) = 0$ and $\|\partial v\|_{C^0(D_{2r})} \leq \alpha < 1$, we have $\|v\|_{C^1(D_{2r})} \leq 1 + 2r \leq C(\delta^{-1},\, i_0^{-1},\, \Lambda_0)$. Moreover, since it follows from Proposition 13 that

$$\frac{|\partial^2 v|^2_\delta}{(1 + |\partial v|^2_\delta)^3} \leq |\Pi_{\delta,\,n}|^2_{h_{\delta,\,n}} \leq ||\tilde{\Pi}_n|^2_{\tilde{h}_n} - |\Pi_{\delta,\,n}|^2_{h_{\delta,\,n}}| + |\tilde{\Pi}_n|^2_{\tilde{h}_n} \leq C(\delta^{-1},\, i_0^{-1},\, \Lambda_0),$$

so we obtain

$$\|v\|_{C^2(D_{2r})} \leq C(\delta^{-1},\, i_0^{-1},\, \Lambda_0). \tag{3.7}$$

Now using (3.7), (B2) and (B6) into Lemma 12, we see that the quasilinear elliptic equation (B5) could be viewed as a uniform linear elliptic equation, with uniform ellipticity constants and uniformly $C^\beta$-bounded coefficients with respect to $n$. Thus the standard Schauder interior estimate (cf. [GT98, Theorem 6.2]) gives that

$$\|v\|_{C^{2,\,\beta}(D_r)} \leq C(\delta^{-1},\, i_0^{-1},\, \Lambda_0,\, \Lambda_1).$$

Hence by Arzela-Ascoli's Lemma, there is a $u \in C^2(D_r)$ so that $u_{q_n} \to u$ in $C^2(D_r)$ for some subsequence. Moreover, if $\{d_{\tilde{h}_n}(p_n,\, q_n)\}$ is uniformly bounded, then the limit $u$ must satisfy the equation

$$H_\delta = \theta_\infty$$

where $H_\delta$ is the mean curvature of graph $u$ in $(\mathbb{R}^3, \delta)$. This just comes from (B5) by letting $n \to \infty$, with (3.2), (B3), (B6) and (3.5) in mind. Consequently, $u$ is $C^\infty$ by the regularity theorem (cf. [GT98, Theorem 6.17]). By Theorem 16 and the discussion after it, we then obtain a $C^\infty$ boundaryless orientable complete limiting Riemannian 2-surface $(\Sigma_\infty, h_\infty)$ with a marked point $p_\infty$ and an isometry immersion $f_\infty : \Sigma_\infty \to \mathbb{R}^3$ of constant mean curvature $\theta_\infty$ so that $f_\infty(p_\infty) = 0$.

Next we show that $f_\infty : \Sigma_\infty \to \mathbb{R}^3$ is weakly stable. By Theorem 16 again, there is an exhaustion $W_1 \Subset W_2 \Subset \cdots$ of $\Sigma_\infty$ and, passing to a subsequence of $\{n\}$ if necessary, a sequence of $C^1$ embeddings $\Phi_n : W_n \to \tilde{D}_n$ with $\Phi_n(p_\infty) = p_n$. Moreover, $f^n \circ \Phi_n$ converges to $f_\infty$ on any compact set of $\Sigma_\infty$ in $C^1$ topology. This together with (B3) implies that:

(B8) $\Phi_n^* \tilde{h}_n = (f^n \circ \Phi_n)^* \tilde{g}_n \to f_\infty^* \delta = h_\infty$ on compact sets of $\Sigma_\infty$ in $C^0$ topology.

Denote by $\Phi_n^{-1} : \Phi_n(W_n) \to W_n$ the inverse of $\Phi_n$. Clearly, $\Phi_n^{-1}$ is a $C^1$ diffeomorphism. Fix a $\psi \geq 0, \in C_c^\infty(\Sigma_\infty)$ with $\int_{\Sigma_\infty} \psi d\mu_{h_\infty} = 1$, and pick any $\varphi \in C_c^\infty(\Sigma_\infty)$ with $\int_{\Sigma_\infty} \varphi d\mu_{h_\infty} = 0$. For $n$ sufficiently large, we have $\mathrm{spt}\varphi \cup \mathrm{spt}\psi \subset W_n$. Then we set

$$c_n := \int_{\Phi_n(W_n)} \varphi \circ \Phi_n^{-1} d\mu_{\tilde{h}_n} = \int_{W_n} \varphi d\mu_{\Phi_n^* \tilde{h}_n},$$
$$d_n := \int_{\Phi_n(W_n)} \psi \circ \Phi_n^{-1} d\mu_{\tilde{h}_n} = \int_{W_n} \psi d\mu_{\Phi_n^* \tilde{h}_n}.$$

By (B8), we have $c_n \to \int_{\Sigma_\infty} \varphi d\mu_{h_\infty} = 0$ and $d_n \to \int_{\Sigma_\infty} \psi d\mu_{h_\infty} = 1$ as $n \to \infty$. Since

$$\int_{\Phi_n(W_n)} (\varphi - \frac{c_n}{d_n}\psi) \circ \Phi_n^{-1} d\mu_{\tilde{h}_n} = 0,$$

we apply (B7) to obtain that

$$\begin{aligned} 0 \leq \int_{\Phi_n(W_n)} & |\tilde{\nabla}_{\tilde{h}_n} (\varphi - \frac{c_n}{d_n}\psi) \circ \Phi_n^{-1}|^2_{\tilde{h}_n} \\ & + (-|\tilde{\Pi}_n|^2_{\tilde{h}_n} + \lambda_n^{-2} Z_n)(\varphi - \frac{c_n}{d_n}\psi)^2 \circ \Phi_n^{-1} \, d\mu_{\tilde{h}_n} \\ = \int_{\Sigma_\infty} & |\nabla_{\Phi_n^* \tilde{h}_n} (\varphi - \frac{c_n}{d_n}\psi)|^2_{\Phi_n^* \tilde{h}_n} \\ & + (-|\tilde{\Pi}_n|^2_{\tilde{h}_n} + \lambda_n^{-2} Z_n) \circ \Phi_n (\varphi - \frac{c_n}{d_n}\psi)^2 \, d\mu_{\Phi_n^* \tilde{h}_n}. \end{aligned} \tag{3.8}$$

We assert that:

(B9) $|\tilde{\Pi}_n|_{\tilde{h}_n} \circ \Phi_n - |\Pi_\infty|_{h_\infty} \to 0$ on compacts set of $\Sigma_\infty$ in $C^0$ topology.

To see this, let $F$ be any compact set of $\Sigma_\infty$. Then by Theorem 16 (4), it holds $|\Pi_{\delta,n}|_{h_{\delta,n}} \circ \Phi_n \to |\Pi_\infty|_{h_\infty}$ on $F$ in $C^0$ topology. It also follows from Theorem 16 (3) that $f^n \circ \Phi_n(F)$ is uniformly bounded. This together with Proposition 13 leads to $|\tilde{\Pi}_n|_{\tilde{h}_n} \circ \Phi_n - |\Pi_{\delta,n}|_{h_{\delta,n}} \circ \Phi_n \to 0$ on $F$ in $C^0$ topology. Hence (B9) is inferred.

Letting $n \to \infty$ in (3.8) and using (3.6), (B8) and (B9), we infer that

$$0 \leq \int_{\Sigma_\infty} |\nabla_{h_\infty} \varphi|^2_{h_\infty} - |\Pi_\infty|^2_{h_\infty} \varphi^2 \, d\mu_{h_\infty}.$$

This shows that the constant mean curvature immersion $f_\infty : \Sigma_\infty \to \mathbb{R}^3$ is weakly stable (i.e. volume preserving stable). Thank to the results of [BD84, Pal86, DS87, LR89], we know that $f_\infty(\Sigma_\infty)$ must be either a plane or a sphere. If $f_\infty(\Sigma_\infty)$ is a plane, then $\Pi_\infty = 0$, which contradicts which $|\Pi_\infty|_{h_\infty}(p_\infty) = \lim_{n\to\infty} |\tilde{\Pi}_n|_{\tilde{h}_n}(p_n) = 1$. If $f_\infty(\Sigma_\infty)$ is a sphere, then so is $\Sigma_\infty$. Thus it must be $W_n = \Sigma_\infty$ for any sufficiently large $n$. This leads to an embedding of a sphere into $\tilde{D}_n$, and hence into $\Sigma_n$, by $\Phi_n$. This is impossible since $\Sigma_n$ is connected and $\partial\Sigma_n \neq \emptyset$. Therefore the curvature estimate of Theorem 1 (1) must be true. This completes the proof. □

For Theorem 2, we need the following lemma.

LEMMA 17. *Let $U$ be an open subset of $\mathbb{R}^3$ , and let $r(x) = |x|_\delta$ be the Euclidean distance to the origin. Suppose that there is a Riemannian metric $g$ defined in $U$ and a constant $Q_0 \geq 1$ so that*

(1) $Q_0^{-2}\delta_{ab} \leq g_{ab}(x) \leq Q_0^2\delta_{ab}$ *as quadratic form for all $x \in U$;*

(2) $\sup_{x\in U} |\partial g|^2(x) \leq Q_0^2$.

*Let $S_r := \{x \in \mathbb{R}^3 : r(x) = r\}$ be any Euclidean 2-sphere contained in $U$, and denote by $H(S_r)$ the mean curvature of $S_r$ in $(U, g)$ computed as the tangential divergence of the outer unit normal vector $Dr/|Dr|_g$ , where $D$ is the metric connection of $g$. Then there is a constant $C = C(Q_0)$ so that*

$$H(S_r) \geq \frac{2}{Q_0^7 r} - C(Q_0).$$

*Proof.* Pick any $x \in S_r$. Observe that we could assume, applying a rotation if necessary, that $(g^{ab}(x))$ =diag$(\lambda_1, \lambda_2, \lambda_3)$. To the end of this proof, calculation is only carried out at $x$. Then

$$|Dr|_g^2 = \sum_{a,\,b=1}^{3} \partial_a r \partial_b r g^{ab} = \sum_{a=1}^{3} \frac{\lambda_a (x^a)^2}{r^2} \in [Q_0^{-2},\, Q_0^2].$$

Recall that $\partial_c(\ln\det g) = g^{ab}\partial_c g_{ab}$. Thus

$$\begin{aligned} H(S_r) &= div_g\Big(\frac{Dr}{|Dr|_g}\Big) = \frac{1}{\sqrt{\det g}}\partial_a\Big(\sqrt{\det g}\frac{D^a r}{|Dr|_g}\Big) \\ &= \frac{g^{aa}}{r|Dr|_g} - \frac{x^b g^{ba} x^c g^{ac}}{r^3|Dr|_g^3} + \frac{1}{2} g^{cd}\partial_a g_{cd}\frac{x^b g^{ab}}{r|Dr|_g} + \frac{x^b \partial_a g^{ab}}{r|Dr|_g} - \frac{x^b g^{ba} x^c x^d \partial_a g^{cd}}{2r^3|Dr|_g^3} \\ &\geq \frac{1}{r^3|Dr|_g^3}\left((\lambda_1+\lambda_2+\lambda_3)\sum_{a=1}^{3}\lambda_a (x^a)^2 - \sum_{a=1}^{3}\lambda_a^2 (x^a)^2\right) - C(Q_0) \\ &\geq \frac{2}{Q_0^7 r} - C(Q_0). \end{aligned}$$

Since $x$ is arbitrary, the lemma follows. □

*Proof of Theorem 2.* Pick any $p \in \Sigma$. By Theorem 10, there is a harmonic coordinates chart on the geodesic ball $B_g(f(p), r_H)$. We identify $B_g(f(p), r_H)$ with its image $U$ in $\mathbb{R}^3$ via this harmonic coordinates. There is a constant $Q_0 \geq 1$ depending only on $\delta$, $i_0$ and $\Lambda_0$ so that

(C1) $f(p) = 0 \in \mathbb{R}^3$, $B_\delta(0, r_H/Q_0) \subset U$;

(C2) $Q_0^{-2}\delta_{ab} \leq g_{ab}(x) \leq Q_0^2\delta_{ab}$ as quadratic form for all $x \in U$;

(C3) $\sup_{x\in U} |\partial g|^2(x) \leq Q_0^2$.

By Lemma 17, there is a positive number $r_0 < r_H/Q_0$, depending only on $r_H$ and $Q_0$, so that the 2-sphere $S_r = \{x \in \mathbb{R}^3 : |x|_\delta = r\}$ is mean convex with respect to $(U, g)$ for any $0 < r \le r_0$. Namely, it holds $H(S_r) > 0$, where $H(S_r)$ is computed as the tangential divergence of the outer unit normal of $S_r$. Assume $\mathrm{diam}(\Sigma) < r_0/Q_0$. It follows from (C1) and (C2) that

$$f(\Sigma) \subset \bar{B}_\delta(0, Q_0\mathrm{diam}(\Sigma)) \subset U.$$

Thus there is a $r_* \in (0, Q_0\mathrm{diam}(\Sigma)]$ so that the 2-sphere $S_{r_*}$ encloses $f(\Sigma)$ and is tangent to $f(\Sigma)$ at some point $x_0 = f(q)$ $(q \in \Sigma)$. By the maximal principle, this implies that

$$|H(\Sigma)(q)| \ge H(S_{r_*})(x_0) > 0. \tag{3.9}$$

On one hand, we have by Lemma 17 that

$$H(S_{r_*}) \ge 2Q_0^{-7}r_*^{-1} - C(Q_0) \ge 2Q_0^{-8}\mathrm{diam}(\Sigma)^{-1} - C(Q_0). \tag{3.10}$$

On the other hand, we have

$$|H(\Sigma)| \le |tr_h(K)| + |\theta_0| \le \sqrt{2}\Lambda_1 + |\theta_0|. \tag{3.11}$$

Putting (3.10) and (3.11) into (3.9), we then infer $\mathrm{diam}(\Sigma) \ge C(Q_0, \Lambda_1, |\theta_0|)^{-1}$. To summarize, we obtain

$$\mathrm{diam}(\Sigma) \ge \min\{r_0/Q_0, C(Q_0, \Lambda_1, |\theta_0|)^{-1}\} = C(\delta^{-1}, i_0^{-1}, \Lambda_0, \Lambda_1, |\theta_0|)^{-1}.$$

This together with Theorem 1 (2) concludes the result. □

**4. Proof of Theorem 16.** In this section, we prove the Theorem 16 following the systematic approach given by Breuning [Bre14]. However, somethings different should be noted. The conditions of Theorem 16 are slightly different from those of [Bre14, Theorem 1.3] so that the resulting limiting immersions in our case, as illustrated in the discussion after Theorem 16, is not necessarily proper.(By contrast, the limiting immersion in [Bre14, Theorem 1.3] is proper.) In addition, we have to show that the limiting Riemannian 2-surface $(\Sigma_\infty, h_\infty)$ is orientable and complete. Also, the regularity of the objects in construction should be carefully treated.

We begin by a few observations relevant to a single immersion $f : \Sigma^2 \to \mathbb{R}^3$ with global unit normal $\nu_\delta$. In the sequel, we denote by $B_s$ the Euclidean disk in $\mathbb{R}^2$ of center 0 and radius $s$, and set $x = (x^1, x^2)$. Beyond that, notion below would coincide with that of subsection 2.4, unless where stated.

LEMMA 18. *Suppose that $f : \Sigma^2 \to \mathbb{R}^3$ is $(r, \alpha)$-immersed at $q \in \Sigma$, and that the defining function $u_q : B_r \to \mathbb{R}$ satisfies $\sup_{B_r} |\partial^2 u_q| \le c_1$. Then $f$ is $(r', (2c_1\alpha r')^{\frac{1}{2}})$-immersed at $q$ for any $0 < r' < r$.*

*Proof.* Observe that $A_q^{-1} \circ f(U_{q, r'})$ is already the graph $\{(x, u_q(x)) : x \in B_{r'}\}$. Since $\partial u_q(0) = 0$, we have from the Mean Value Theorem that

$$|\partial u_q|^2(x) \le \sup_{B_{r'}} 2|\partial u_q||\partial^2 u_q||x| < 2c_1\alpha r'.$$

Hence the lemma is proved. □

Let $h_\delta = f^*\delta$ to be the pulled back metric on $\Sigma$. For any $p \in \Sigma$ and $r' > 0$, denote by $D_{h_\delta}(p, r')$, or just $D(p, r')$ when there is no possibility of confusion, the $h_\delta$-geodesic disk of $\Sigma$ of center $p$ and of radius $r'$.

LEMMA 19. *Suppose that $f : \Sigma^2 \to \mathbb{R}^3$ is $(r, \alpha)$-immersed at $p \in \Sigma$, then*

$$D(p, r) \subset U_{r,p} \subset D(p, (1+\alpha^2)^{\frac{1}{2}} r).$$

*Proof.* It follows by the fact that the Langer chart $(U_{r,p}, f_p)$ satisfies

$$(1+\alpha^2)^{-\frac{1}{2}} \leq \|df_p\| \leq 1.$$

□

The following lemma (cf. [Lan85, Lemma 3.1]) is crucial.

LEMMA 20. *Suppose that $f : \Sigma^2 \to \mathbb{R}^3$ is $(r, \alpha)$-immersed at $p, q \in \Sigma$. If $\alpha^2 < \frac{1}{3}$ and $U_{\frac{r}{4},p} \cap U_{\frac{r}{4},q} \neq \emptyset$, then $U_{\frac{r}{4},p} \subset U_{r,q}$. Thus in this case, for any $x \in B_{\frac{r}{4}}$, there is a unique $y \in B_r$ so that $f_p^{-1}(x) = f_q^{-1}(y) \in U_{\frac{r}{4},p}$ and*

$$A_p(x, u_p(x)) = A_q(y, u_q(y)).$$

Next we prove a technical covering lemma.

LEMMA 21 (Covering Lemma). *Suppose that the immersion $f : \Sigma^2 \to \mathbb{R}^3$ is $(r, \alpha)$-immersed in $D(p, R) \Subset \Sigma\backslash\partial\Sigma$ with $0 < \alpha \leq \sqrt{3}$. Let $L \geq 1, \in \mathbb{N}$ and $0 < s \leq \frac{r}{2}$ with $(L+3)s \leq R$. Let $0 < \theta \leq \frac{1}{2}$. There is an integer $K = K(\theta^{-1}) \geq 2$ and a consequence $\{q_i\}_{i=1}^{K^L}$ of points in $D(p, R)$ with $q_1 = p$ so that for each integer $1 \leq l \leq L$, the following holds:*

(1) $D(p, ls) \subset \bigcup_{i=1}^{K^l} U_{\theta s, q_i} \subset D(p, (l+2)s)$.

(2) $\{q_i\}_{i=K^{l-1}+1}^{K^l} \subset D(p, (l+1)s)\backslash D(p, (l-2)s)$.

*Here $D(p, r')$ is assumed to be $\emptyset$ if $r' \leq 0$.*

*Proof.* Observe first that the second inclusion in (1) follows from (2). Indeed, if $q_i \in D(p, (l+1)s)$, then $f$ is $(r, \alpha)$-immersed at $q_i$. Thus it follows by Lemma 19 that

$$U_{\theta s, q_i} \subset D(q_i, (1+\alpha^2)^{\frac{1}{2}}\theta s) \subset D(q_i, s) \subset D(p, (l+2)s).$$

Hence it suffices to construct a sequence satisfying (2) and the first inclusion of (1).

It is easy to find an integer $K = K(\theta^{-1}) \geq 2$, and two sequences $\{x_i\}_{i=1}^K \subset B_s$ and $\{y_i\}_{i=1}^K \subset B_{\frac{3}{2}s}$ with $x_1 = y_1 = 0$, satisfying

$$B_s \subset \bigcup_{i=1}^K B(x_i, \frac{\theta s}{(1+\alpha^2)^{\frac{1}{2}}}) \subset B_{2s} \subset B_r, \tag{4.1}$$
$$B_{\frac{3}{2}s} \subset \bigcup_{i=1}^K B(y_i, \frac{\theta s}{(1+\alpha^2)^{\frac{1}{2}}}) \subset B_{2s} \subset B_r.$$

We shall construct $\{q_i\}_{i=1}^{K^L}$ by induction on $1 \leq l \leq L$. So we first assume $l = 1$. By Lemma 19 and (4.1), it holds

$$D(p, s) \subset U_{s,p} = f_p^{-1}(B_s) \subset \bigcup_{i=1}^K f_p^{-1}(B(x_i, \frac{\theta s}{(1+\alpha^2)^{\frac{1}{2}}})).$$

Set $q_i = f_p^{-1}(x_i)$. Then

$$q_i \in f_p^{-1}(B_s) = U_{s,\,p} \subset D(p,\,(1+\alpha^2)^{\frac{1}{2}} s) \subset D(p,\,2s).$$

Since $f$ is $(r,\,\alpha)$-immersed at $q_i$, it follows from the argument of the proof of Lemma 19 that

$$f_p^{-1}(B(x_i,\,\frac{\theta s}{(1+\alpha^2)^{\frac{1}{2}}})) \subset D(q_i,\,\theta s) \subset U_{\theta s,\,q_i}.$$

This implies

$$D(p,\,s) \subset \bigcup_{i=1}^{K} U_{\theta s,\,q_i}. \tag{4.2}$$

This completes the construction for $l = 1$.

Next suppose $L \geq 2$ and there is a desired sequence $\{q_i\}_{i=1}^{K^{L-1}}$. Note that $f$ is $(r,\,\alpha)$-immersed at each point $q_i$. Thus we have

$$\begin{aligned} D(p,\,Ls) &= \{q \in \Sigma :\, d(q,\,D(p,\,(L-1)s)) < s\} \\ &\subset \bigcup_{i=1}^{K^{L-1}} \{q \in \Sigma :\, d(q,\,U_{\theta s,\,q_i}) < s\} \\ &\subset \bigcup_{i=1}^{K^{L-1}} U_{\theta s+s,\,q_i} \subset \bigcup_{i=1}^{K^{L-1}} U_{\frac{3}{2}s,\,q_i} = \bigcup_{i=1}^{K^{L-1}} f_{q_i}^{-1}(B_{\frac{3}{2}s}) \\ &\subset \bigcup_{i=1}^{K^{L-1}} \bigcup_{j=1}^{K} f_{q_i}^{-1}(B(y_j,\,\frac{\theta s}{(1+\alpha^2)^{\frac{1}{2}}})). \end{aligned} \tag{4.3}$$

Set $q_{i,\,j} = f_{q_i}^{-1}(y_j)$. Clearly $q_{i,\,1} = q_i$. It also holds that

$$q_{i,\,j} \in f_{q_i}^{-1}(B_{\frac{3}{2}s}) = U_{\frac{3}{2}s,\,q_i} \subset D(q_i,\,3s) \subset D(p,\,R),$$

which implies that $f$ is $(r,\,\alpha)$-immersed at $q_{i,\,j}$. Thus we have

$$f_{q_i}^{-1}(B(y_j,\,\frac{\theta s}{(1+\alpha^2)^{\frac{1}{2}}})) \subset D(q_{i,\,j},\,\theta s) \subset U_{\theta s,\,q_{i,\,j}}. \tag{4.4}$$

Since $q_{i,\,1} = q_i$ for $1 \leq i \leq K^{L-1}$, we could re-index $q_{i,\,j}$ to obtain $\{q_i\}_{i=1}^{K^L}$ which extends $\{q_i\}_{i=1}^{K^{L-1}}$. Substituting (4.4) into (4.3), we then obtain

$$D(p,\,Ls) \subset \bigcup_{i=1}^{K^L} U_{\theta s,\,q_i}. \tag{4.5}$$

Moreover we could assume

$$q_i \in D(p,\,(L+1)s)\backslash D(p,\,(L-2)s) \tag{4.6}$$

provided $K^{L-1} < i \leq K^L$. Otherwise it holds either $D(q_i,\,s) \cap D(p,\,Ls) = \emptyset$ or $D(q_i,\,s) \subset D(p,\,(L-1)s)$. In each of these two cases, $U_{\theta s,\,q_i}$ does not contribute to

(4.5), and hence this $q_i$ could be replaced by any point in the right hand side of (4.6) without changing (4.5). Thus $\{q_i\}_{i=1}^{K^L}$ is a desired sequence. □

Although the conclusion of Lemma 21 is far from optima, it is enough for our latter application.

With Lemma 21, we assign each $1 \le i \le K^L$ the subset

$$Z(i) := \{j \in \mathbb{N} :\ 1 \le j \le K^L,\ U_{s,\,q_j} \cap U_{s,\,q_i} \ne \emptyset\}.$$

Clearly, $Z(i)$ is symmetric in the sense that for any $1 \le i,\, j \le K^L$, $j \in Z(i)$ if and only if $i \in Z(j)$. Moreover, we have

LEMMA 22. *If* $1 \le i \le K^L$, *then* $\max Z(i) \le K^8 i$. *In particular,* $|Z(i)| \le K^8 i$.

*Proof.* Let $l_0$ be the integer so that $K^{l_0-1} < i \le K^{l_0}$ if $i > 1$, and let $l_0 = 1$ if $i = 1$. By Lemma 21 it holds $q_i \in D(p,\,(l_0+1)s)\backslash D(p,\,(l_0-2)s)$. Now the proof breaks down into two cases depending upon whether or not $l_0+7 > L$. If $l_0+7 > L$, we have $\max Z(i) \le K^L < K^{l_0+7} \le K^8 i$. If $l_0+7 \le L$, then for any $j$ with $K^{l_0+6} < j$ there holds that $q_j \notin D(p,\,(l_0+5)s)$ by virtue of Lemma 21. Thus $d_h(q_j,\,q_i) > 4s$. Observe that since $U_{s,\,q_i} \subset D(q_i,\,2s)$ and $U_{s,\,q_j} \subset D(q_j,\,2s)$ according to Lemma 19, we obtain $j \notin Z(i)$ and hence $\max Z(i) \le K^{l_0+6} \le K^7 i$. □

Now let us begin the proof proper of Theorem 16. In view of (2.17) and Lemma 18, we could assume

$$\alpha < \frac{1}{2} \tag{4.7}$$

by uniformly shrinking $r$ if necessary. We also fix the numbers

$$s = \frac{r}{16}, \quad \theta = \frac{1}{10}. \tag{4.8}$$

In particular, Lemma 20, 21 and 22 are available in the sequel.

Since $\{f^n(p_n)\}_{n=1}^{\infty}$ is uniformly bounded and $\lim_{n\to\infty} R_n = \infty$, we could assume without loss of generality that $f^n(p_n) = 0$ for all $n \in \mathbb{N}$ and that $R_n \ge (n+3)s$ for all $n \in \mathbb{N}$. The latter assumption allows us to collect by Lemma 21 a sequence $\{q_i^n\}_{i=1}^{K^n}$ of points in $D_{h_{\delta,\,n}}(p_n,\,R_n)$ for each $n$. Here $K \ge 2$ is an integer independent of $n$. Moreover we could define the relative subset $Z^n(i)$, with $1 \le i \le K^n$, in exactly the same way as is done above. By Lemma 22, $|Z^n(i)|$ is uniformly bounded with respect to $n$ for any fixed $i$. For each $q_i^n$ we assign a Langer chart $(U_{r,\,i}^n,\,f_i^n)$ centered at $q_i^n$ with defining Euclidean isometry $A_i^n$ given by $A_i^n v = R_i^n v + T_i^n$, where $R_i^n \in SO(3)$ and $T_i^n \in \mathbb{R}^3$, and with $C^\infty$ defining function $u_i^n :\, B_r \to \mathbb{R}$. Here the superscript $n$ is used to indicate the quantities relevant to the immersion $f^n$. Note that by definition $u_i^n(0) = 0$, $\partial u_i^n(0) = 0$ and $\|\partial u_i^n\|_{C^0(B_r)} \le \alpha$. By (2.17), $\|u_i^n\|_{C^{2,\,\beta}(B_r)}$ is uniformly bounded with respect to $n$ and $i$. Next we claim that for any fixed $i$, $\{T_i^n\}$ is uniformly bounded with respect to $n$. To see this, let $l$ to be the least integer not less than $\log_K i$. Then $i \le K^l$ and $q_i^n \in D_{h_{\delta,\,n}}(p_n,\,(l+1)s)$. Observe that $f^n(p_n) = 0$ and $T_i^n = f^n(q_i^n)$, therefore it holds $|T_i^n| \le (\log_K i + 2)s$.

We infer from the above uniform bounds that there are sequences $\{T_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^3$, $\{R_i\}_{i\in\mathbb{N}} \subset SO(3)$, $\{u_i\}_{i\in\mathbb{N}} \subset C^{2,\,\beta'}(B_r)$ with $0 < \beta' < \beta$, and $\{Z(i)\} \subset 2^{\mathbb{N}}$, and there is a subsequence

$$\{\nu_n\} \subset \{n\} \tag{4.9}$$

so that the following holds

$$Z^{\nu_n}(i) = Z(i), \text{ for all } n,\, i \text{ with } 1 \le i \le n; \tag{4.10}$$

$$\varepsilon_i^{\nu_n} \to 0 \text{ as } n \to \infty, \text{ for each fixed } i \le n \tag{4.11}$$

where

$$\varepsilon_i^{\nu_n} := |T_i^{\nu_n} - T_i| + ||R_i^{\nu_n} - R_i|| + \|u_i^{\nu_n} - u_i\|_{C^{2,\beta'}(B_r)}.$$

Lemma 21 and properties (4.10) and (4.11) are crucial for applying the argument in [Bre14]. We will make implicit use of them throughout the following proof.

For each $i \in \mathbb{N}$, we set $B_s^i = B_s \times \{i\}$, and then consider the disjoint union

$$\bigcup_{i=1}^{\infty} B_s^i.$$

Let "$\sim$" be a relation defined on it by $(x,\, i) \sim (y,\, j)$ if and only if $j \in Z(i)$ and $A_i(x,\, u_i(x)) = A_j(y,\, u_j(y))$. By the argument of [Bre14, Lemma 4.1], we infer:

Lemma 23. *The relation $\sim$ is an equivalence relation.*

Define

$$\Sigma_\infty = \left( \bigcup_{i=1}^{\infty} B_s^i \right) / \sim$$

with the quotient topology, and let

$$P : \bigcup_{i=1}^{\infty} B_s^i \to \Sigma_\infty$$

be the standard quotient projection. Given any open subset $V$ of $B_s$, we set $V^i := V \times \{i\} \subset B_s^i$ and define

$$\begin{aligned} \varphi_V^i :\, & P(V^i) \to V. \\ & [(x,\, i)] \mapsto x \end{aligned}$$

Clearly, $\varphi_V^i$ is well defined and bijective. Set $\mathcal{U} = \{\varphi_V^i : i \in \mathbb{N},\, V \overset{open}{\subset} B_s\}$.

Lemma 24. *The following holds:*
(1) *The quotient projection $P$ is open.*
(2) *$\Sigma_\infty$ is a second countable Hausdorff space.*
(3) *The set $\mathcal{U}$ is a differential atlas on $\Sigma_\infty$, making $\Sigma_\infty$ a $C^2$ boundaryless surface.*
(4) *$\Sigma_\infty = \cup_{i=1}^{\infty} P(B^i_{\frac{1}{6}s})$.*

We omit the proof, but refer the reader to the argument in [Bre14, Section 4], which could be almost repeatedly used to prove Lemma 24. We only point out that the regularity of $\Sigma_\infty$ is easily inferred by taking note that the transition functions of $\mathcal{U}$ are

$$\begin{aligned} \varphi_V^i \circ (\varphi_W^j)^{-1} :\, \varphi_W^j(P(V^i) \cap P(W^j)) &\to \varphi_V^i(P(V^i) \cap P(W^j)). \\ x &\mapsto \pi \circ A_i^{-1} \circ A_j(x,\, u_j(x)) \end{aligned} \tag{4.12}$$

Thus $\Sigma_\infty$ is indeed $C^{2,\beta'}$.

Now we construct the limiting mapping $f = f_\infty$ by

$$\begin{aligned} f : \Sigma_\infty &\to \mathbb{R}^3. \\ [(x, i)] &\mapsto A_i(x, u_i(x)) \end{aligned}$$

It immediately follows from the definition of the relation $\sim$ that $f$ is well defined. Moreover $f$ is a $C^2$ immersion since $\{u_i\}$ are $C^2$. Set

$$p_\infty = [(0, 1)] \in P(B_s^1).$$

Since $T_1^{\nu_n} = f^{\nu_n}(p_n) = 0$, it holds $T_1 = 0$. Hence $f(p_\infty) = A_1(0, u_1(0)) = 0 = f^{\nu_n}(p_{\nu_n})$.

LEMMA 25. *The immersion $f : \Sigma_\infty \to \mathbb{R}^3$ has trivial normal bundle. Equivalently, $\Sigma_\infty$ is orientable.*

*Proof.* It is easy to see that

$$R_i\left(\frac{(-\partial_{x^1}u_i, -\partial_{x^2}u_i, 1)}{\sqrt{1+|\partial u_i|^2}}\right) \tag{4.13}$$

is a normal vector field defined on $P(B_s^i) \subset \Sigma$ for any $i$. Let $[(x, i)] = [(y, j)]$ be any point in $\Sigma_\infty$. To detect the proof of [Bre14, Lemma 4.1], we see that there is a sequence $\{x^{\nu_n}\}$ of points of $B_s$ so that $x^{\nu_n} \to x$ and that $(f_i^{\nu_n})^{-1}(x^{\nu_n}) = (f_j^{\nu_n})^{-1}(y)$ for large $n$. By (2.12), the designated global unit normal vector field of $\Sigma_{\nu_n}$ at $(f_i^{\nu_n})^{-1}(x^{\nu_n}) = (f_j^{\nu_n})^{-1}(y)$ is

$$R_i^{\nu_n}\left(\frac{(-\partial_{x^1}u_i^{\nu_n}, -\partial_{x^2}u_i^{\nu_n}, 1)}{\sqrt{1+|\partial u_i^{\nu_n}|^2}}(x^{\nu_n})\right) = R_j^{\nu_n}\left(\frac{(-\partial_{x^1}u_j^{\nu_n}, -\partial_{x^2}u_j^{\nu_n}, 1)}{\sqrt{1+|\partial u_j^{\nu_n}|^2}}(y)\right).$$

Thus taking $n \to \infty$ in the above equation yields that

$$R_i\left(\frac{(-\partial_{x^1}u_i, -\partial_{x^2}u_i, 1)}{\sqrt{1+|\partial u_i|^2}}(x)\right) = R_j\left(\frac{(-\partial_{x^1}u_j, -\partial_{x^2}u_j, 1)}{\sqrt{1+|\partial u_j|^2}}(y)\right).$$

This implies that the vector fields (4.13) define a global unit normal vector field of the immersion $f$. □

Given any index $i$, we would show that for $n$ sufficiently large (depending on $i$), $f(P(B_s^i))$ is projected into $f^{\nu_n}(U_{r,i}^{\nu_n})$ along the normal direction of $\Sigma_\infty$. Moreover, this projection is diffeomorphic into. In the sequel, we assume without loss of generality that $A_i = Id$, and for simplicity we identify $P(B_s^i)$ with $B_s^i$ as well as $B_s$. For any $x \in B_s$, we set

$$N_x = -\partial u_i(x) = -(\partial_{x^1}u_i(x), \partial_{x^2}u_i(x)).$$

Then the normal bundle of $f : \Sigma_\infty \to \mathbb{R}^3$ at $x$ is given by

$$\nu_f(x) = \{(N_x, 1)t : t \in \mathbb{R}\}.$$

Note that by our notation here, $f(x) = (x, u_i(x))$. We shall show that $f(x) + \nu_f(x)$ intersects $f^{\nu_n}(U^{\nu_n}_{r,i})$ at exactly one point. This is equivalent to show that the equation:

$$f(x) - (N_x, 1)t = A^{\nu_n}_i(y, u^{\nu_n}_i(y)), \tag{4.14}$$

has unique solution $(y, t)$ in $B_r \times \mathbb{R}$. This equation could be rewritten as

$$H_x(y, t) = (y, t), \tag{4.15}$$

where $H_x : B_r \times \mathbb{R} \to \mathbb{R}^3$ is the mapping defined by

$$H_x(y, t) = -(N_x t, u^{\nu_n}_i(y)) - (R^{\nu_n}_i - Id)(y, u^{\nu_n}_i(y)) - T^{\nu_n}_i + f(x).$$

Note that $H_x$ depends on $i$ and $\nu_n$.

LEMMA 26. *For $n$ sufficiently large ($\alpha + \varepsilon^{\nu_n}_i(1+\alpha^2)^{\frac{1}{2}} < \frac{1}{2}$), $H_x$ is a contraction on $B_r \times \mathbb{R}$ and more precisely*

$$\sup_{B_r\times\mathbb{R}} \|dH_x\| \le \frac{1}{2}. \tag{4.16}$$

*Proof.* Given any $(y, t)$, $(y', t') \in B_r \times \mathbb{R}$, we have

$$\begin{aligned}
&|H_x(y, t) - H_x(y', t')| \\
=&|-(N_x(t-t'), u^{\nu_n}_i(y) - u^{\nu_n}_i(y')) - (R^{\nu_n}_i - Id)(y - y', u^{\nu_n}_i(y) - u^{\nu_n}_i(y'))| \\
\le&\Big(|N_x|^2(t-t')^2 + \|\partial u^{\nu_n}_i\|^2_{C^0(B_r)}|y-y'|^2\Big)^{\frac{1}{2}} \\
&\quad + \|R^{\nu_n}_i - Id\|(1 + \|\partial u^{\nu_n}_i\|^2_{C^0(B_r)})^{\frac{1}{2}}|y-y'| \\
\le&\alpha|(y,t) - (y',t')| + \varepsilon^{\nu_n}_i(1+\alpha^2)^{\frac{1}{2}}|y-y'| \\
\le&\frac{1}{2}|(y,t) - (y',t')|.
\end{aligned}$$

Thus the lemma is proved. □

LEMMA 27. *For $n$ sufficiently large ($\varepsilon^{\nu_n}_i < (1+\frac{1}{r})^{-1}$), there is exactly one point $(Y^{\nu_n}_i(x), S^{\nu_n}_i(x))$ in $B_r \times \mathbb{R}$ so that*

$$H_x(Y^{\nu_n}_i(x), S^{\nu_n}_i(x)) = (Y^{\nu_n}_i(x), S^{\nu_n}_i(x)).$$

*Namely, $f(x) + \nu_f(x)$ intersects $f^{\nu_n}(U^{\nu_n}_{r,i})$ exactly at the point $A^{\nu_n}_i(Y^{\nu_n}_i(x), u^{\nu_n}_i(Y^{\nu_n}_i(x)))$. Moreover it holds $|(Y^{\nu_n}_i(x), S^{\nu_n}_i(x))| \le 3s$.*

*Proof.* Pick any $(y, t) \in \bar{B}^3_{3s}$, where $\bar{B}^3_{3s}$ is the closed Euclidean 3-ball in $\mathbb{R}^2 \times \mathbb{R}$ centered at the origin and with radius $3s$. Then we have for any $x \in B_s$ that

$$\begin{aligned}
|H_x(y,t)| &= |-(N_x t, u^{\nu_n}_i(y)) - (R^{\nu_n}_i - Id)(y, u^{\nu_n}_i(y)) - T^{\nu_n}_i + f(x)| \\
&\le \alpha|(y,t)| + \varepsilon^{\nu_n}_i(1+\alpha^2)^{\frac{1}{2}}|y| + \varepsilon^{\nu_n}_i + (1+\alpha^2)^{\frac{1}{2}}|x| \\
&\le 3\alpha s + \varepsilon^{\nu_n}_i(1+\alpha^2)^{\frac{1}{2}}3s + \varepsilon^{\nu_n}_i + (1+\alpha^2)^{\frac{1}{2}}s \\
&\le (\frac{3}{2} + \frac{3\sqrt{5}\varepsilon^{\nu_n}_i}{2} + \frac{16\varepsilon^{\nu_n}_i}{r} + \frac{\sqrt{5}}{2})s \\
&\le 3s,
\end{aligned}$$

where the last two lines are on account to (4.8) and the assumption. This shows that $H_x(\bar{B}^3_{3s}) \subset \bar{B}^3_{3s}$. Furthermore, by Lemma 27, the restriction $H_x|\bar{B}^3_{3s} : \bar{B}^3_{3s} \to \bar{B}^3_{3s}$ is a contraction. Thus, by the usual Fixed Point Theorem, there is a unique point $(Y_i^{\nu_n}(x), S_i^{\nu_n}(x)) \in \bar{B}^3_{3s}$ solving $H_x(Y_i^{\nu_n}(x), S_i^{\nu_n}(x)) = (Y_i^{\nu_n}(x), S_i^{\nu_n}(x))$, which is equivalent to

$$f(x) - (N_x, 1)S_i^{\nu_n}(x) = A_i^{\nu_n}(Y_i^{\nu_n}(x), u_i^{\nu_n}(Y_i^{\nu_n}(x)). \tag{4.17}$$

Since $H_x^{\nu_n}$ is contractive in $B_r \times \mathbb{R}$, this fixed point $(Y_i^{\nu_n}(x), S_i^{\nu_n}(x))$ is also unique in $B_r \times \mathbb{R}$. □

LEMMA 28. *For each $x \in B_s$, it holds that $Y_i^{\nu_n}(x) \to x$ and $S_i^{\nu_n}(x) \to 0$ as $n \to \infty$.*

*Proof.* Since the norm of $(Y_i^{\nu_n}(x), S_i^{\nu_n}(x))$ is uniformly bounded by $3s$, it suffices to show that the lemma is true for each convergent subsequence of $\{(Y_i^{\nu_n}(x), S_i^{\nu_n}(x))\}_n$. By simplicity, we assume that $(Y_i^{\nu_n}(x), S_i^{\nu_n}(x))$ converges to some $(y', t') \in \bar{B}^3_{3s}$ as $n \to \infty$. Note that we have by (4.17) that

$$A_i^{\nu_n}(Y_i^{\nu_n}(x), u_i^{\nu_n}(Y_i^{\nu_n}(x))) = (x, u_i(x)) - (N_x S_i^{\nu_n}(x), S_i^{\nu_n}(x)).$$

Taking $n \to \infty$, we then obtain

$$(y', u_i(y')) = (x, u_i(x)) - (N_x t', t').$$

Hence

$$|y' - x| = |N_x t'| = |N_x(u_i(x) - u_i(y'))| \le \alpha^2 |x - y'|.$$

This together with $\alpha < 1$ implies that $y' = x$ and $t' = 0$. The lemma is proved. □

For the sake of notation simplicity below, subsequence extracted from $\{\nu_n\}$ is still denoted by $\{\nu_n\}$.

LEMMA 29. *Passing to a subsequence of $\{\nu_n\}$ if necessary, there holds*
*(1) $Y_i^{\nu_n}$ and $S_i^{\nu_n}$ are $C^1(B_s)$ for each $n$.*
*(2) $Y_i^{\nu_n} \to Id$ and $S_i^{\nu_n} \to 0$ on $B_s$ in $C^1$ topology as $n \to \infty$.*

*Proof.* Define

$$\begin{aligned} G : B_s \times B_r \times \mathbb{R} &\to \mathbb{R}^2 \times \mathbb{R}. \\ (x, y, t) &\mapsto (y, t) - H_x(y, t) \end{aligned}$$

It is easy to see from the expression $G(x, y, t) = -(N_x, 1)t - A_i^{\nu_n}(y, u_i^{\nu_n}(y)) + (x, u_i(x))$ that $G$ is $C^1$. Observe that $G(x, \cdot, \cdot) : B_r \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}$ is nonsingular for each $x$ by (4.16), and that $(Y_i^{\nu_n}(x), S_i^{\nu_n}(x))$ is the unique solution of $G(x, y, t) = 0$ by Lemma 27. Thus it follows by the Implicit Function Theorem that $Y_i^{\nu_n}$ and $S_i^{\nu_n}$ are $C^1(B_s)$. This proves (1). Differentiating the equation

$$G(x, Y_i^{\nu_n}(x), S_i^{\nu_n}(x)) = 0. \tag{4.18}$$

with respect to $x$, we could solve the first derivatives of $Y_i^{\nu_n}$ and $S_i^{\nu_n}$ in term of the inverse matrix of $Id - dH_x$ and the first derivative of $G$ evaluated at $(x, Y_i^{\nu_n}(x), S_i^{\nu_n}(x))$. Note that $\|u_i^{\nu_n}, u_i\|_{C^{2,\beta'}(B_s)}$ and $\|Y_i^{\nu_n}, S_i^{\nu_n}\|_{C^0(B_s)}$ are uniformly bounded. So we could use a bootstrap argument to assert firstly that $\|Y_i^{\nu_n}, S_i^{\nu_n}\|_{C^1(B_s)}$, and secondly that $\|Y_i^{\nu_n}, S_i^{\nu_n}\|_{C^{1,\beta'}(B_s)}$, are uniformly bounded with respect to $n$. Applying the Arzela-Ascoli's Lemma and Lemma 28, we then conclude (2). □

Recall that $Y_i^{\nu_n}(B_s) \subset \bar{B}_{3s} \subset B_r$, so we could define

$$\phi_i^{\nu_n} : P(B_s^i) \to U_{r,i}^{\nu_n}$$

by assigning each $x \in B_s^i = B_s$ with the unique point $\phi_i^{\nu_n}(x)$ in $U_{r,i}^{\nu_n}$ so that

$$f^{\nu_n}(\phi_i^{\nu_n}(x)) = A_i^{\nu_n}(Y_i^{\nu_n}(x), u_i^{\nu_n}(Y_i^{\nu_n}(x))).$$

Clearly $\phi_i^{\nu_n}$ is $C^1$. Geometrically, in view of (4.17), $f^{\nu_n}\phi_i^{\nu_n}(x)$ is the intersection point of $f(x) + \nu_f(x)$ and $f^{\nu_n}(U_{r,i}^{\nu_n})$.

Lemma 30. *For $n$ sufficiently large, $\phi_i^{\nu_n} : P(B_s^i) \to U_{r,i}^{\nu_n}$ is diffeomorphic into.*

*Proof.* It suffices to show that for $n$ sufficiently large, $Y_i^{\nu_n} : B_s \to B_r$ is diffeomorphic into. However this is just an easy consequence of Lemma 29 (2). □

Passing to a subsequence $\{\nu_n\}$ if necessary, we could assume that Lemma 26-30 hold for all $n$ and $i$ with $i \le n$.

For each $l \in \mathbb{N}$, we set

$$W_l := \bigcup_{i=1}^{l} P(B^i_{(2^{-1}-4^{-l})s}). \tag{4.19}$$

Then it is clear that

$$\begin{aligned} &p_\infty \in W_1, \\ &W_l \Subset W_{l+1}, \quad \forall\, l \in \mathbb{N}, \quad \text{and} \\ &\bigcup_{l=1}^{\infty} W_l = \Sigma_\infty \end{aligned}$$

where the last line is on account to Lemma 24 (4).

For $n \ge l$ large, we then define the map

$$\Phi_l^{\nu_n} : W_l \to \Sigma_{\nu_n}$$

by $\Phi_l^{\nu_n}(x) = \phi_i^{\nu_n}(x)$ if $x \in P(B^i_{(2^{-1}-4^{-l})s})$ and $i \le l$. The reader should take note that we are here essentially repeating constructions of the immersions $\phi^i$'s in [Bre14, Section 5]. Again by the argument therein, we obtain:

Lemma 31. *After passing to a subsequence of $\{\nu_n\}$, the following holds for all $n$ and $l$ with $n \ge l$:*

(1) *$\Phi_l^{\nu_n}$ is well defined and $\Phi_l^{\nu_n}(p_\infty) = p_{\nu_n}$.*
(2) *$\Phi_l^{\nu_n}$ is an embedding.*
(3) *$\Phi_l^{\nu_n}(W_l) \subset D_{h_{\delta,\nu_n}}(p_{\nu_n}, (\nu_n + 3)s)$.*

Lemma 32. *Let $h_\infty := f^*\delta$ be the pulled back metric on $\Sigma_\infty$. Then the following holds*

(1) *For each $l \in \mathbb{N}$, $f^{\nu_n} \circ \Phi_l^{\nu_n}$ converges to $f$ on $W_l$ in $C^1$ topology.*
(2) *$h$ is a complete Riemannian metric on $\Sigma_\infty$.*

*Proof.* Let $1 \le i \le l$, and $x \in P(B^i_{\frac{1}{2}s})$. Since $\varepsilon_i^{\nu_n} \to 0$ and $Y_i^{\nu_n} \to Id$ in $C^1(B_s)$, it follows that

$$f^{\nu_n} \circ \Phi_l^{\nu_n}(x) = A_i^{\nu_n}(Y_i^{\nu_n}(x),\, u_i^{\nu_n}(Y_i^{\nu_n}(x))) \to A_i(x,\, u_i(x)) = f$$

on $P(B^i_{\frac{1}{2}s})$ in $C^1$ topology. By the definition of $W_l$, this implies (1) .

To prove the completeness of $h_\infty$, let $\gamma:\ [0,\, t_0) \to \Sigma_\infty$ be any maximal normal geodesic starting at $p_\infty$. Then it suffices to show that $t_0 = \infty$. We divide the proof into two cases:

*Case 1. $\gamma \subset W_l$ for some $l$.*

Since $\bar{W}_l$ is compact, it follows that $t_0$ must be $\infty$.

*Case 2. $\gamma \nsubseteq W_l$ for any $l \in \mathbb{N}$.*

Together with Lemma 22, it is then easy to construct a subsequence $\{\kappa_i\} \subset \{i\}$ so that

$$\begin{aligned}&\gamma \cap P(B^{\kappa_i}_{\frac{1}{2}s}) \ne \emptyset,\ \forall\, i \ge 1,\\ &P(B^{\kappa_i}_s) \cap P(B^{\kappa_j}_s) = \emptyset,\ \forall\, i \ne j.\end{aligned}$$

These imply that

$$\begin{aligned} t_0 = \text{length }(\gamma) &\ge \sum_m \text{length }(\gamma \cap P(B^{i_m}_s))\\ &\ge \sum_m \text{length }(\pi \circ A^{-1}_{i_m} \circ f(\gamma \cap P(B^{i_m}_s)))\\ &\ge \sum_m \frac{1}{2}s = \infty.\end{aligned}$$

Thus the lemma is proved. □

LEMMA 33. *For each $l \in \mathbb{N}$, it holds that $|\Pi_{\delta,\,\nu_n}|^2_{h_{\delta,\,\nu_n}} \circ \Phi_l^{\nu_n} \to |\Pi_\infty|_{h_\infty}$ on $W_l$ in $C^1$ topology.*

*Proof.* Give any $x \in P(B^i_{\frac{1}{2}s})$ with $i \le l$, observe that the local expression for the second fundamental form is

$$\begin{aligned}&|\Pi_{\delta,\,\nu_n}|^2_{h_{\delta,\,\nu_n}} \circ \Phi_l^{\nu_n}(x)\\ &= \sum_{j,\,k,\,j',\,k'} \left(\delta^{jk} - \frac{\partial_j u_i^{\nu_n}\partial_k u_i^{\nu_n}}{1+|\partial u_i^{\nu_n}|^2}\right)\left(\delta^{j'k'} - \frac{\partial_{j'} u_i^{\nu_n}\partial_{k'} u_i^{\nu_n}}{1+|\partial u_i^{\nu_n}|^2}\right)\frac{\partial_j\partial_{j'}u_i^{\nu_n}\partial_k\partial_{k'}u_i^{\nu_n}}{(1+|\partial u_i^{\nu_n}|^2)}(Y_i^{\nu_n}(x)).\end{aligned}$$

Since $u_i^{\nu_n} \to u_i$ in $C^2(B_r)$ and $Y_i^{\nu_n} \to Id$ in $C^1(B_s)$, it follows that $|\Pi_{\delta,\,\nu_n}|^2_{h_{\delta,\,\nu_n}} \circ \Phi_l^{\nu_n} \to |\Pi_\infty|_{h_\infty}$ in $C^0(B^i_{\frac{1}{2}s})$, and hence in $C^0(W_l)$. □

Finally, we set

$$\Phi_l := \Phi_l^{\nu_l}.$$

Then Theorem 16 follows from combining Lemma 24, 25, 31, 32 and 33.

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