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LOCAL DIMENSIONS OF MEASURES OF FINITE TYPE ON THE TORUS*

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Abstract. The structure of the set of local dimensions of a self-similar measure has been studied by numerous mathematicians, initially for measures that satisfy the open set condition and, more recently, for measures on $\mathbb{R}$ that are of finite type.

In this paper, our focus is on finite type measures defined on the torus, the quotient space $\mathbb{R}/\mathbb{Z}$. We give criteria which ensures that the set of local dimensions of the measure taken over points in special classes generates an interval. We construct a non-trivial example of a measure on the torus that admits an isolated point in its set of local dimensions. We prove that the set of local dimensions for a finite type measure that is the quotient of a self-similar measure satisfying the strict separation condition is an interval. We show that sufficiently many convolutions of Cantor-like measures on the torus never admit an isolated point in their set of local dimensions, in stark contrast to such measures on $\mathbb{R}$. Further, we give a family of Cantor-like measures on the torus where the set of local dimensions is a strict subset of the set of local dimensions, excluding the isolated point, of the corresponding measures on $\mathbb{R}$.

Key words. Multi-fractal analysis, local dimension, IFS, finite type, quotient space.

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1. Introduction. The local dimension of a probability measure μ defined on a metric space, at a point x in the support of μ, is the number

$$\dim_{loc} \mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

It is of interest to determine the local dimensions of a given measure as these numbers quantify the local concentration of the measure. For self-similar measures that satisfy the open set condition (OSC), it is well known that the set of local dimensions is a closed interval whose endpoints are given by a simple formula.

When the OSC condition fails, the situation is more complicated, less understood and can be quite different. Indeed, in [12], Hu and Lau discovered that when μ is the three-fold convolution of the classic middle-third Cantor measure on $\mathbb{R}$, then there is an isolated point in the set of local dimensions, namely at $x = 0, 3$. This fact was later established for other 'overlapping' Cantor-like measures in [1, 6, 7, 15].

These Cantor-like measures are special examples of equicontractive self-similar measures of finite type on $\mathbb{R}$, a notion introduced by Ngai and Wang [13] that is weaker than the OSC, but stronger than the weak separation condition [14]. Such measures have a very structured geometry, which makes them more tractable than arbitrary self-similar measures. In [3, 4, 5], Feng conducted a detailed study of equicontractive, self-similar measures of finite type on $\mathbb{R}$, focussing primarily on Bernoulli convolutions with contraction factors the inverse of a simple Pisot number. In [9, 11], the authors, together with Ng, showed that if μ is a measure of finite type on $\mathbb{R}$ that arises from regular probabilities and has full interval support, then the set of local dimensions of μ at the points in any positive loop class is a closed interval and the local dimensions

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at periodic points in the loop class are dense within this interval. See Section 2.1 for a definition of regular probabilities. Moreover, there is always a distinguished positive loop class, known as the essential class, which has full measure and is often all but the endpoints of the support of the measure. This is the situation for the Bernoulli convolutions with contraction a simple Pisot inverse and the 3-fold convolution of the middle-third Cantor measure, for instance. If the self-similar measure does not arise from regular probabilities, it is still true that the set of local dimensions at points in the interior of the essential class is a closed interval.

In this paper, we introduce the notion of finite type for measures on the torus $\mathbb{T}$ (the quotient space $\mathbb{R}/\mathbb{Z}$) that are quotients of equicontractive, self-similar measures on $\mathbb{R}$. Examples of such measures include convolutions on the torus of Cantor measures or Bernoulli convolutions with contraction factor the inverse of a Pisot number. These measures are the quotients of the convolutions on $\mathbb{R}$ of the (initial) Cantor measures or Bernoulli convolutions.

We develop a general method for calculating the local dimensions of finite type measures on $\mathbb{T}$ and obtain a simple formula for the local dimensions at the periodic points. With these tools, the same techniques as used for finite type measures on $\mathbb{R}$ show that if the self-similar measure is associated with regular probabilities and the quotient measure has support $\mathbb{T}$, then the set of local dimensions of the quotient measure at the points in any positive loop class is an interval. If we do not assume regular probabilities, under a mild technical assumption it is still true that the set of local dimensions at points in the interior of any positive essential class is an interval. As with finite type measures on $\mathbb{R}$, in either case this interval is the closure of the local dimensions at the periodic points in the class. However, in contrast to the case for finite type measures on $\mathbb{R}$, the essential class for measures of finite type on $\mathbb{T}$ need not be unique or of positive type.

We use these results to prove that if a self-similar measure of finite type on the torus is associated with an IFS that satisfies the strong separation condition, then the set of local dimensions of the quotient measure is not only an interval, but coincides with the set of local dimensions of the original measure. We also give the first (as far as we are aware) non-trivial example of a quotient measure on $\mathbb{T}$ whose set of local dimensions admits an isolated point.

In [1], it was shown that the sets of local dimensions for quotients of k-fold convolutions of Cantor measures with contraction factor $1/d$ are intervals whenever $k \geq d$. Although these quotient measures do not have an essential class of positive type, we are able to modify our general approach to give a new proof of this fact. Moreover, we extend this result to what we call complete quotient Cantor-like measures and also prove that the set of local dimensions is the closure of the set of local dimensions of periodic points.

In [7] it was explicitly shown that set of local dimensions of the 3-fold convolution of the Cantor measure with contraction factor $1/3$ on the torus is a strict subset of the set of local dimensions at points in the essential class, $(0, 3)$, of the corresponding measure on $\mathbb{R}$. The authors also comment that a similar proof can be used for all d-fold convolutions of the Cantor measure with contraction factor $1/d$ for $d \geq 3$. In the last section we extend this result to show that for all $k \geq 0$ and all d sufficiently large the $(k + d)$-fold convolution of the Cantor measure with contraction factor $1/d$ shares this property.

2. Finite type quotient measures on $\mathbb{T}$.

2.1. Basic definitions and notation. Assume

$$\{S_j(x) = \varrho x + d_j : j = 0, \dots, k\}$$

is an iterated function system (IFS) of equicontractive similarities on $\mathbb{R}$ and let p_j for $j \in \mathcal{A} = \{0, ..., k\}$ denote probabilities, meaning $p_j > 0$ and $\sum p_j = 1$. By the associated self-similar measure μ, we mean the unique measure satisfying the identity

$$\mu = \sum_{j=0}^{k} p_j \mu \circ S_j^{-1}. \tag{1}$$

Its support is the associated self-similar set $K \subseteq \mathbb{R}$, the unique non-empty compact set satisfying $K = \bigcup_{j=0}^{k} S_j(K)$.

There is no loss of generality in assuming $d_0 < d_1 < \cdots < d_k$. If $p_0 = p_k = \min p_j$, then the probabilities are referred to as *regular*.

We first recall what it means to say such an equicontractive IFS or self-similar measure on $\mathbb{R}$ is of finite type.

Definition 2.1. The iterated function system, $\{S_j(x) = \varrho x + d_j : j = 0, \dots, k\}$, is said to be of **finite type** if there is a finite set $F \subseteq \mathbb{R}$ such that for each positive integer n and any two sets of indices $\sigma, \tau \in \mathcal{A}^n$, either

$$\varrho^{-n} |S_\sigma(0) - S_\tau(0)| > \delta \text{ or } \varrho^{-n}(S_\sigma(0) - S_\tau(0)) \in F,$$

where $\delta = (1 - \varrho)^{-1}(\max d_j - \min d_j)$ is the diameter of K. If $\{S_j\}$ is of finite type and μ is an associated self-similar measure, we also say that μ is of finite type.

Hereafter, we will refer to this notion as 'finite type on $\mathbb{R}$' to distinguish it from the notion of 'finite type on the torus', which will be the focus of this paper and will be defined shortly.

Measures which satisfy the open set condition are of finite type on $\mathbb{R}$ and measures of finite type on $\mathbb{R}$ satisfy the weak open set condition [14]. The structure of finite type measures and aspects of their multi-fractal analysis is explained in detail in [3, 4, 5, 9, 11].

Example 2.2. The self-similar measures associated with the IFS $\{S_0(x) = \varrho x, S_1(x) = \varrho x + (1 - \varrho)\}$ and probabilities $p_0 = p_1 = 1/2$ are known as (uniform) Bernoulli convolutions when $\varrho > 1/2$ and Cantor measures when $\varrho < 1/2$. Cantor measures satisfy the OSC. Bernoulli convolutions do not, but when ϱ is the inverse of a Pisot number[1] the measures are of finite type.

Given two measures μ, ν on $\mathbb{R}$, by their convolution, $\mu * \nu$, we mean the measure on $\mathbb{R}$ defined by

$$\mu * \nu(E) = \int_{\mathbb{R}} \mu(E - x)\, d\nu(x) \text{ for all Borel sets } E \subseteq \mathbb{R}.$$

The k-fold convolutions of the Bernoulli convolutions or Cantor measures are (also) the self-similar measures associated with the IFS $\{S_j(x) = \varrho x + j(1 - \varrho) : j = 0, 1, ..., k\}$

[1] A Pisot number is an algebraic integer greater than one, all of whose Galois conjugates are strictly less than one in modulus.

and probabilities $p_j = \binom{k}{j}2^{-k}$. These measures do not satisfy the OSC when $k \geq 1/\varrho$, but are of finite type on $\mathbb{R}$ whenever ϱ is the inverse of a Pisot number, such as a positive integer [13].

By the torus, $\mathbb{T}$, we mean the quotient group, $\mathbb{R}/\mathbb{Z}$. We let π denote the canonical quotient map and denote the usual metric on $\mathbb{T}$ by $d_{\mathbb{T}}$. Sometimes it is convenient to identify the torus as the group $[0,1)$ under addition mod 1.

Given a measure μ on $\mathbb{R}$, we let μ_π be the quotient measure defined by $\mu_\pi(E) = \mu(\pi^{-1}(E))$ for any Borel set $E \subseteq \mathbb{T}$. Of course, if μ has support K, then μ_π has support $\pi(K)$. If μ has support contained in $[0,1]$, the only difference between the local dimensions of μ and μ_π is that (identifying $\mathbb{T}$ with $[0,1)$) $\dim_{loc}\mu_\pi(0) = \min(\dim_{loc}\mu(0), \dim_{loc}\mu(1))$, so this situation is trivial.

Definition 2.3. The iterated function system, $\{S_j(x) = \varrho x + d_j : j = 0, \ldots, k\}$, defined on $\mathbb{R}$, is said to be of **finite type on the torus** $\mathbb{T}$ if there is a finite set $\Lambda \subseteq \mathbb{R}$ such that for each positive integer n and any two sets of indices $\sigma, \tau \in \mathcal{A}^n$, either

$$d_{\mathbb{T}}(\pi(S_\sigma(0)), \pi(S_\tau(0))) > \varrho^n\delta \text{ or } d_{\mathbb{T}}(\pi(S_\sigma(0)), \pi(S_\tau(0))) \in \varrho^n\Lambda,$$

where $\delta = (1-\varrho)^{-1}(\max d_j - \min d_j)$ is the diameter of the self-similar set $K \subseteq \mathbb{R}$. If the IFS is of finite type on the torus and μ_π is the quotient measure of a self-similar measure μ associated with the IFS, we also say that μ_π is of finite type on $\mathbb{T}$.

Later in this section we will show that quotients of the k-fold convolutions of Bernoulli convolutions or Cantor measures with contraction factors the inverse of a Pisot number are of finite type on $\mathbb{T}$. We will also give an example of a measure which is of finite type on $\mathbb{R}$, but whose quotient is not of finite type on $\mathbb{T}$. It is worth observing that the converse cannot happen. If the measure is of finite type on the torus, then it will be of finite type on $\mathbb{R}$. To see this we note that for all σ and τ there exists an integer k such that $|S_\sigma(0) - S_\tau(0)| = k \pm d_{\mathbb{T}}(\pi(S_\sigma(0)), \pi(S_\tau(0)))$. If $k = 0$ then $|S_\sigma(0) - S_\tau(0)| = d_{\mathbb{T}}(\pi(S_\sigma(0)), \pi(S_\tau(0))) \in \varrho^n\Lambda$. If $k \neq 0$ and n is sufficiently large then $|S_\sigma(0) - S_\tau(0)| > \varrho^n\delta$. There are only a finite number of characteristic vectors arising from when n is not sufficently large, and hence the measure is of finite type.

A translation argument shows there is no loss of generality in assuming $d_0 = 0$. We will also suppose that the diameter of the self-similar set K is an integer δ. We will leave it to the reader to consider what modifications to our arguments need to be made when the diameter is not an integer.

First, we introduce the important notions of quotient net intervals, neighbours and characteristic vectors. These are motivated by the analogous ideas for measures of finite type on $\mathbb{R}$.

For each integer $n \geq 1$, let $0 = h_1 < h_2 < \cdots < h_{s_n - 1} < 1$ be the collection of elements $\{\pi(S_\sigma(0)), \pi(S_\sigma(\delta)) : \sigma \in \mathcal{A}^n\}$ in the torus which here it is convenient to identify with $[0,1)$. Put $h_{s_n} = 1$ and let

$$\mathcal{F}_n^{(\pi)} = \{[h_j, h_{j+1}] : 1 \leq j \leq s_n - 1, \pi^{-1}(h_j, h_{j+1}) \bigcap K \neq \emptyset\}.$$

Put $\mathcal{F}_0^{(\pi)} = \{[0,1]\}$. Elements in $\mathcal{F}_n^{(\pi)}$ are called the *quotient net intervals of level* n. In what follows, we will often omit the adjective 'quotient' if this is clear from the context. We note that these are intervals in $\mathbb{R}$ that are contained in $[0,1]$. Each quotient net interval Δ, of level $n \geq 1$, is contained in a unique quotient net interval $\widehat{\Delta}$ of level $n-1$, called its *parent*.

There is a similar notion of net intervals for measures on $\mathbb{R}$. It is worth observing that an interval in $\mathcal{F}_n^{(\pi)}$ may not correspond directly to a net interval of level n in $\mathbb{R}$ because $\pi(S_\sigma(\epsilon_1))$ and $\pi(S_\tau(\epsilon_2))$ may be adjacent in $\mathbb{T}$ without being adjacent in $\mathbb{R}$.

Let $\Delta = [a,b] \in \mathcal{F}_n^{(\pi)}$, $n \geq 1$ and for $l = 0, ..., \delta-1$ let $\Delta^{(l)} = \Delta + l$. As $\pi^{-1}(a,b) \cap K$ is not empty there must be some l such that $int(\Delta^{(l)}) \bigcap K = (a+l, b+l) \bigcap K$ is not empty. Put

$$\{a_1, \ldots, a_m\} = \{\varrho^{-n}(a - S_\sigma(0) + l) : \sigma \in \mathcal{A}^n \ , l \in \mathbb{N}, \, int(\Delta^{(l)}) \bigcap S_\sigma(K) \neq \emptyset\},$$

where we assume the real numbers $\{a_i\}$ satisfy $a_1 < a_2 < \cdots < a_m$. By the *quotient neighbour set* of Δ we mean the tuple

$$V_n^{(\pi)}(\Delta) = (a_1, \ldots, a_m).$$

The normalized length of $\Delta = [a,b] \in \mathcal{F}_n^{(\pi)}$ is denoted

$$\ell_n(\Delta) = \varrho^{-n}|b-a|.$$

As $|S_\sigma(0) - S_\sigma(\delta)| = \rho^n \delta$ when $\sigma \in \mathcal{A}^n$, it follows that for large enough n, $|b-a| = d_{\mathbb{T}}(a,b)$.

Suppose $\widehat{\Delta}$ is the parent of $\Delta \in \mathcal{F}_n^{(\pi)}$ and $\Delta_1, ..., \Delta_j$ are the quotient net intervals of level n that are also children of $\widehat{\Delta}$, listed from left to right, with the same normalized length and quotient neighbour set as Δ. Define $r_n(\Delta)$ to be the integer r such that $\Delta = \Delta_r$. By the *quotient characteristic vector* of $\Delta \in \mathcal{F}_n^{(\pi)}$ for $n \geq 1$ we mean the triple

$$\mathcal{C}_n^{(\pi)}(\Delta) = (\ell_n(\Delta), V_n^{(\pi)}(\Delta), r_n(\Delta)).$$

We call $(\ell_n(\Delta), V_n^{(\pi)}(\Delta))$ the *reduced quotient characteristic vector* of Δ. For $n = 0$, we define

$$\mathcal{C}_0^{(\pi)}([0,1]) = (1, (0, 1, 2, \ldots, \delta - 1), 1).$$

The reason for this choice will be clear later.

We remark that this structure depends only on the similarities $\{S_j\}_{j=0}^k$ and not on the measure itself.

Here is a very important fact about measures of finite type on $\mathbb{T}$. The same statement (with the appropriate definitions) is known to be true for measures of finite type on $\mathbb{R}$ [3].

Lemma 2.4. *If the IFS is of finite type on $\mathbb{T}$, then there are only finitely many quotient characteristic vectors.*

Proof. First, we will check there are only finitely many normalized lengths of quotient net intervals. It is certainly enough to verify this for net intervals of level n for large enough n, thus we can assume that if $\Delta = [a,b] \in \mathcal{F}_n^{(\pi)}$, then $\rho^n \delta < 1/2$. Since $|S_\sigma(0) - S_\sigma(\delta)| \leq \rho^n \delta$ when $\sigma \in \mathcal{A}^n$, it follows that $|b-a| = d_{\mathbb{T}}(\pi(a), \pi(b)) \leq \rho^n \delta$.

Suppose $a = \pi(S_\sigma(0))$ and $b = \pi(S_\tau(0))$ for some $\sigma, \tau \in \mathcal{A}^n$ (viewing $\mathbb{T}$ as $[0,1)$.) Then

$$\rho^{-n} |b-a| = \rho^{-n} d_{\mathbb{T}}(\pi(S_\sigma(0)), \pi(S_\tau(0))) \leq \rho^{-n} \delta$$

and thus by definition $\rho^{-n}|b-a|$ belongs to the finite set Λ. The argument is similar if a, b are both images of δ.

Now suppose $a = \pi(S_\sigma(\delta))$ and $b = \pi(S_\tau(0))$. Since $\pi^{-1}(a,b) \cap K$ is not empty, the definition of a net interval ensures there is some $\alpha \in \mathcal{A}^n$ such that $\pi^{-1}[a,b] \subseteq [S_\alpha(0), S_\alpha(\delta)]$. Of course, $\rho^{-n} d_{\mathbb{T}}(\pi(S_\alpha(0)), \pi(S_\alpha(\delta))) = \delta$. Since

$$\begin{aligned} d_{\mathbb{T}}(\pi(S_\sigma(\delta)), \pi(S_\tau(0))) = {} & d_{\mathbb{T}}(\pi(S_\sigma(\delta)), \pi(S_\alpha(\delta))) + d_{\mathbb{T}}(\pi(S_\tau(0)), \pi(S_\alpha(0))) \\ & - d_{\mathbb{T}}(\pi(S_\alpha(\delta)), \pi(S_\alpha(0))), \end{aligned}$$

we see that $\rho^{-n}|b-a| = \rho^{-n} d_{\mathbb{T}}(\pi(S_\sigma(\delta)), \pi(S_\tau(0)))$ belongs to the finite set $\Lambda \pm \Lambda \pm \Lambda$ (where we assume, without loss of generality, that $\delta \in \Lambda$).

Similar, but easier, arguments apply if $a = \pi(S_\sigma(0))$, $b = \pi(S_\tau(\delta))$. Consequently, there are only finitely many normalized lengths.

This fact guarantees that each quotient net interval has a bounded number of children. In particular, there can only be finitely many choices for the 3rd component of the characteristic vectors.

Lastly, we need to show there are only finitely many neighbours. So suppose $a_i = \rho^{-n}(a - S_\sigma(0) + l)$ where $\Delta = [a,b]$ and $(\Delta + l) \cap S_\sigma(0,\delta)$ is not empty for $\sigma \in \mathcal{A}^n$ and integer l. This guarantees that $|S_\sigma(0) - (a+l)| \le \rho^n \delta$ and therefore

$$\rho^{-n}|S_\sigma(0) - (a+l)| = \rho^{-n} d_{\mathbb{T}}(\pi(S_\sigma(0)), \pi(a+l)) \in \Lambda \pm \Lambda \pm \Lambda,$$

completing the proof. □

2.2. Examples and Counterexamples. We begin this subsection by exhibiting a family of examples of quotient measures of finite type on the torus.

PROPOSITION 2.5. *Let β be a Pisot number and $S_j(x) = \beta^{-1}x + d_j$ for $d_j \in \mathbb{Q}[\beta]$ and $j = 0, 1, 2, ..., k$. Assume the self-similar set has convex hull the interval $[0,\delta]$ with δ an integer. Then the quotient of any associated self-similar measure is of finite type on the torus.*

LEMMA 2.6. *Let $S \subset \mathbb{Q}[\beta]$ be a finite set and $\Lambda^S(\beta) = \left\{\sum_{i=0}^n a_i \beta^I \colon a_i \in S, n \in \mathbb{N}\right\}$. Then there exists a constant $c > 0$ such that if $y, z \in \Lambda^S(\beta)$, then either $y = z$ or $|y - z| > c$.*

Proof. This is essentially done in [8], but we include it here for completeness.

We first observe that we can assume $S \subset \mathbb{Z}[\beta]$. To see this we multiply by the least common multiple of the denominators of the a_i. This scales the constant c, but does not alter its existence.

Thus we have that $\Lambda^S(\beta) \in \mathbb{Z}[\beta]$. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the Galois automorphisms on $\mathbb{Q}[\beta]$. Either $y = z$ or $y - z \neq 0$. In the latter case, we have $y - z \in \mathbb{Z}[\beta]$, and hence the algebraic norm, $N(y-z)$, is a non-zero integer. This implies that

$$|N(y-z)| = \left|\prod_i \sigma_i(y-z)\right| \ge 1$$

and hence if $y = \sum a_j \beta^j$ and $z = \sum a'_j \beta^j$, with $a_j, a'_j \in S$, then

$$|y - z| \ge \frac{1}{\left|\prod_{\sigma_i \neq \mathrm{id}} \sigma_i(y-z)\right|} \ge \frac{1}{\prod_{\sigma_i \neq \mathrm{id}} \left|\sum_j \sigma_i(a_j - a'_j)\sigma_i(\beta^j)\right|}$$

where id is the identity automorphism. Let $c_\sigma = \max_{a,b\in S} |\sigma(a-b)| > 0$. We have that

$$|y-z| \geq \frac{1}{\prod_{\sigma_i \neq \mathrm{id}} \sum_{j=0}^{\infty} c_{\sigma_i} |\sigma_i(\beta)|^j}.$$

As β is Pisot, $|\sigma_i(\beta)| < 1$ for all Galois actions, and hence the right hand side is bounded below, giving the result. □

Proof of Proposition 2.5. Let

$$F = \{\delta, d_j - \ell : j = 0, ..., k; \ell = 0, ..., \delta\} \subseteq \mathbb{Q}[\beta]$$

and consider the choices of $\sigma, \tau \in \mathcal{A}^n$ such that $\rho^{-n} d_{\mathbb{T}}(\pi(S_\sigma(0)), \pi(S_\tau(0))) \leq \delta$. These normalized distances are equal to $\beta^n |S_\sigma(0) - S_\tau(0) + \ell|$ for suitable integers $\ell \in \{-\delta, ..., \delta\}$ and hence $\beta^n d_{\mathbb{T}}(\pi(S_\sigma(0)), \pi(S_\tau(0))) \in \Lambda^{F-F}(\beta)$. According to Lemma 2.6, the absolute values of the non-zero elements of $\Lambda^{F-F}(\beta)$ are bounded away from zero, say $\geq \varepsilon$. Thus there can be at most $(2\delta+1)/\varepsilon$ elements in $\Lambda^{F-F}(\beta) \cap [-\delta, \delta]$. This proves the finite type property. □

Remark 2.7. If μ_π, ν_π are two measures on $\mathbb{T}$, then their convolution (on $\mathbb{T}$) is defined by

$$\mu_\pi * \nu_\pi(E) = \int_{\mathbb{T}} \mu_\pi(E-x)\ d\nu_\pi(x)$$

for all Borel sets $E \subseteq \mathbb{T}$. (Here the group operation is understood on the torus.) If μ_π and ν_π are the quotients of measures μ and ν on $\mathbb{R}$ respectively, then the convolution $\mu_\pi * \nu_\pi$ is equal to the quotient of the convolution (on $\mathbb{R}$) $\mu * \nu$. In other words, $(\mu * \nu)_\pi = \mu_\pi * \nu_\pi$.

From Proposition 2.5 and Remark 2.7 we immediately deduce the following:

Corollary 2.8. *Any k-fold convolution (taken on $\mathbb{T}$) of a Cantor measure or Bernoulli convolution, where the contraction factor is the inverse of a Pisot number, is of finite type on $\mathbb{T}$.*

However, there are also measures of finite type on $\mathbb{R}$ whose quotients are not of finite type on $\mathbb{T}$. Here is an example.

Example 2.9. Let ϱ be a solution to

$$2(1-\varrho)\sum_{i=1}^{\infty} \varrho^{i^2} = 1/2$$

(approximately 0.384) and consider the IFS with similarities $S_0(x) = \varrho x$ and $S_1(x) = \varrho x + 2(1-\varrho)$. The convex hull of the self-similar set is $[0,2]$. Let μ be the associated self-similar measure. As $\varrho < 1/2$, the IFS satisfies the OSC and even the strong separation condition, and thus μ is of finite type on $\mathbb{R}$. Let $\sigma^{(n)} = (\sigma_0, \sigma_1, ..., \sigma_{n^2})$ and $\tau^{(n)} = (\tau_0, \tau_1, ..., \tau_{n^2}) \in \mathcal{A}^{n^2+1}$ be defined by $\sigma_{i^2} = 1$ if $i \neq 0$, $\sigma_i = 0$ otherwise, and $\tau_i = 1 - \sigma_i$. The points $a_n = S_{\sigma^{(n)}}(0)$ and $b_n = S_{\tau^{(n)}}(0)$ are easily seen to be symmetric about 1 with $a_n < 1/2$ and $b_n > 3/2$. Consequently,

$$d_{\mathbb{T}}(\pi(a_n), \pi(b_n)) = 2|a - 1/2| = 4(1-\varrho)\sum_{i=n+1}^{\infty} \varrho^{i^2} \approx 4(1-\varrho)\varrho^{(n+1)^2}.$$

Hence there exists a net interval in $\mathcal{F}_{n^2+1}^{(\pi)}$ with normalized length at most

$$\approx 4\varrho^{-(n^2+1)}\varrho^{(n+1)^2}(1-\varrho) = 4(1-\varrho)\varrho^{2n}.$$

These normalized lengths are not bounded below and hence there cannot be a finite number of characteristic vectors. Therefore μ_π is not of finite type.

2.3. Symbolic representations and the essential class(es). By an *admissible path* we mean a finite tuple $\eta = (\gamma_j)$ where each γ_j is the quotient characteristic vector of Δ_j and Δ_j is the parent of Δ_{j+1}. Each $\Delta \in \mathcal{F}_n^{(\pi)}$ can be identified with a unique admissible path $\eta = \eta(\Delta) = (\gamma_j)_{j=0}^n$ where $\gamma_0 = \mathcal{C}_0[0,1]$ and $\gamma_n = \mathcal{C}_n(\Delta)$. We call this the *symbolic representation* of Δ.

We will often write $\Delta_n(x)$ for a net interval of level n containing x. By the *quotient symbolic representation* of $x \in K$ we mean the sequence

$$[x] = (\mathcal{C}_0(\Delta_0(x)), \dots, \mathcal{C}_n(\Delta_n(x)), ...)$$

where $\Delta_n(x) \subseteq \Delta_{n-1}(x)$. If x is an endpoint of $\Delta_n(x)$ for some n, then x is called a *boundary point* and it can have two symbolic representations. Otherwise, the symbolic representation of x is unique.

We can also define the notion of quotient loop classes and essential classes in the same manner as was done for measures of finite type on $\mathbb{R}$. A non-empty subset Ω' of quotient characteristic vectors will be called a *quotient loop class* if whenever $\alpha, \beta \in \Omega'$, then there are quotient characteristic vectors γ_j, $j = 1, \dots, n$, such that $\gamma_1 = \alpha$, $\gamma_n = \beta$ and $(\gamma_1, \dots, \gamma_n)$ is an admissible path with all $\gamma_j \in \Omega'$. A loop class Ω' is called a *quotient essential class* if, in addition, whenever $\alpha \in \Omega'$ and $\beta \in \Omega$ is a child of α, then $\beta \in \Omega'$. We say that an element x with symbolic representation $(\gamma_0, \gamma_1, ., , ,)$ is *in a loop class* (such as *in the essential class*) if there is some J such that $\gamma_j \in$ loop class for all $j \geq J$.

The finite type property ensures that every element in $\pi(K)$ is contained in a quotient loop class if the associated IFS is of finite type on $\mathbb{T}$.

Feng, in [5, Lemma 6.4], proved that if μ is of finite type on $\mathbb{R}$, then there is exactly one essential class. Surprisingly, this is not true for self-similar measures of finite type on the torus, as the example below demonstrates.

Example 2.10. Consider the IFS with maps $S_j(x) = x/4 + d_j/5$ for $j = 0, ..., 4$, $d_j = 3j$ for $j = 0, ..., 3$ and $d_4 = 15$ and any probabilities p_j. According to Proposition 2.5 this IFS is of finite type on $\mathbb{T}$. The convex hull of K is $[0, 4]$. Using the computer, we determined that any corresponding quotient measure has 10 reduced quotient characteristic vectors. From the reduced transition diagram, Figure 1, one can see that there are two different essential classes, the first from the reduced characteristic vector labelled 7 and the second consisting of the three reduced characteristic vectors $5, 9, 10$.

Remark 2.11. In [11, Proposition 3.6] it was shown that, for self-similar measures of finite type on $\mathbb{R}$, the set of points in the essential class has full μ-measure, and full H^s-measure. Here H^s is the Hausdorff measure associated to the support of μ. Despite not necessarily having a unique essential class, a similar proof will show that for μ_π a self-similar measure of finite type on the torus, the set of points in the union of all essential classes of μ_π will have full μ_π-measure and full H^s-measure where H_s is the Hausdorf measure associated with the support of μ_π on the torus.

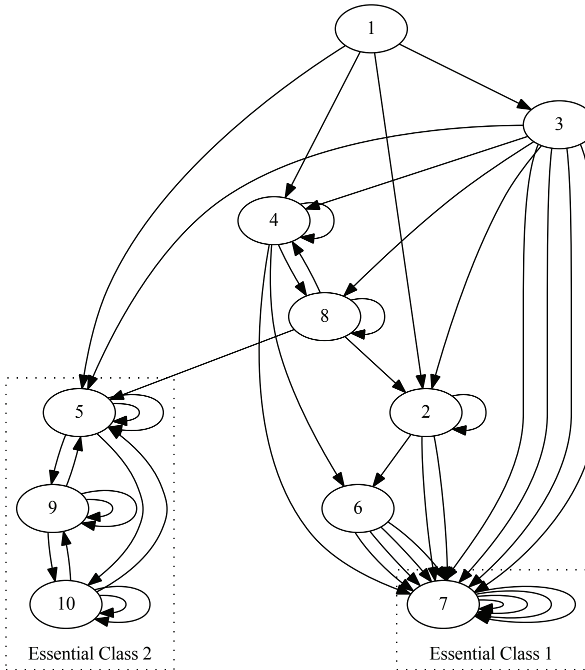

FIG. 1. *Transition diagram for Example 2.10*

3. Computing local dimensions. The *upper* and *lower local dimensions* at a point x in the support of a probability measure μ defined on a metric space are defined as

$$\overline{\dim}_{loc}\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$
$$\underline{\dim}_{loc}\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

When the two coincide the number is known as the *local dimension* of μ at x.

As was the case for measures of finite type on $\mathbb{R}$, there is an iterative strategy for computing local dimensions for measures of finite type on $\mathbb{T}$, based upon suitable transition matrices.

NOTATION 3.1. *Suppose $\Delta = [a,b] \in \mathcal{F}_n^{(\pi)}$ has parent $\widehat{\Delta} = [c,d] \in \mathcal{F}_{n-1}^{(\pi)}$ and assume their quotient neighbour sets are $V_n^{(\pi)}(\Delta) = (a_1,\ldots,a_J)$ and $V_{n-1}^{(\pi)}(\widehat{\Delta}) = (c_1,\ldots,c_I)$, respectively. The* ***quotient primitive transition matrix,***

$$T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta)) = (T_{ji})$$

is the $J \times I$ matrix defined by the rule that $T_{ji} = p_s$ if there is some $\sigma \in \mathcal{A}^{n-1}$ and integer l with $S_\sigma(0) = c - \varrho^{n-1}c_j + l$ and $S_{\sigma s}(0) = a - \varrho^n a_i + l$. We set $T_{ji} = 0$ otherwise.

Given the net interval Δ with symbolic representation $\eta = (\gamma_j)_{j=0}^n$, we write

$$T(\eta) = T(\gamma_0,\gamma_1)\cdots T(\gamma_{n-1},\gamma_n).$$

Any such product of primitive transition matrices will be called a *quotient transition matrix.* By an essential primitive transition matrix, we mean a transition matrix $T(\gamma_{j-1},\gamma_j)$ where γ_{j-1},γ_j belongs to the essential class.

The definition ensures that each column of a primitive transition matrix contains a non-zero entry. If $\pi(K)=\mathbb{T}$, then the same statement holds for each row.

We note that as $S_{\sigma s}(0)=\varrho^{n-1}d_s+S_\sigma(0)$, if $S_\sigma(0)=c-\varrho^{n-1}c_j+l$ and $S_{\sigma s}(0)=a-\varrho^n a_i+l$ for integer l, then

$$a-\varrho^n a_i+l=\varrho^{n-1}d_s+c-\varrho^{n-1}c_j+l,$$

so s is uniquely determined (even if σ and l are not).

Proposition 3.2. *Let μ be a self-similar measure satisfying (1). Assume the convex hull of the self-similar set is $[0,\delta]$ and that the quotient measure μ_π is of finite type on $\mathbb{T}$. Let $\Delta=[a,b]$ be a quotient net interval in $\mathcal{F}_n^{(\pi)}$ with reduced quotient characteristic vector $(\ell_n(\Delta),\,(a_1,\ldots,a_m))$. Then*

$$\mu_\pi(\Delta)=\sum_{i=1}^{m}\mu\left([a_i,a_i+\ell_n(\Delta)]\right)\sum_{l=0}^{\delta-1}\sum_{\substack{\sigma\in\mathcal{A}_n\\ a_i=(S_\sigma(0)-a-l)\varrho^{-n}}}p_\sigma.$$

Proof. Assume $\pi^{-1}(\Delta)\cap[0,\delta]=\bigcup_{l=0}^{\delta-1}\Delta^{(l)}$ where $\Delta^{(l)}=\Delta+l=[a+l,b+l]$. Then

$$\mu_\pi(\Delta)=\mu(\pi^{-1}(\Delta))=\sum_{l=0}^{\delta-1}\mu(\Delta^{(l)})=\sum_{l}\sum_{\sigma\in\mathcal{A}_n}p_\sigma\mu(S_\sigma^{-1}(\Delta^{(l)})).$$

As μ is non-atomic this equals

$$\mu_\pi(\Delta)=\sum_{l=0}^{\delta-1}\sum_{\substack{\sigma\in\mathcal{A}_n\\ S_\sigma(K)\cap int\Delta^{(l)}\neq\emptyset}}p_\sigma\mu(S_\sigma^{-1}(\Delta^{(l)})).$$

Note that if $S_\sigma(K)\cap int\Delta^{(l)}\neq\emptyset$, then by definition $(S_\sigma(0)-a-l)\varrho^{-n}=a_i$ belongs to the quotient neighbour set of Δ. Hence

$$\mu_\pi(\Delta)=\sum_{l=0}^{\delta-1}\sum_{i=1}^{m}\sum_{\substack{\sigma\in\mathcal{A}_n\\ a_i=(S_\sigma(0)-a-l)\varrho^{-n}}}p_\sigma\mu(S_\sigma^{-1}(\Delta^{(l)})).$$

As $S_\sigma(x)=a-a_i\varrho^n+l+\varrho^n x$, we see that $S_\sigma([a_i,a_i+\ell(\Delta)])=[a+l,b+l]$. Hence

$$\begin{aligned}\mu_\pi(\Delta)&=\sum_{l=0}^{\delta-1}\sum_{i=1}^{m}\sum_{\substack{\sigma\in\mathcal{A}_n\\ a_i=(S_\sigma(0)-a-l)\varrho^{-n}}}p_\sigma\mu([a_i,a_i+\ell(\Delta)])\\&=\sum_{i=1}^{m}\mu([a_i,a_i+\ell_n(\Delta)])\sum_{l=0}^{\delta-1}\sum_{\substack{\sigma\in\mathcal{A}_n\\ a_i=(S_\sigma(0)-a-l)\varrho^{-n}}}p_\sigma\end{aligned}$$

as claimed. □

Notation 3.3. *For $n \geq 1$, let*

$$P_n^{(i)} = P_n^{(i)}(\Delta) = \sum_{l=0}^{\delta-1} \sum_{\substack{\sigma \in \mathcal{A}_n \\ a_i = (S_\sigma(0) - a - l)\varrho^{-n}}} p_\sigma,$$

$$Q_n(\Delta) = (P_n^{(1)}, ..., P_n^{(m)}),$$

and

$$Q_0([0,1]) = (1, ..., 1) \in \mathbb{R}^\delta.$$

We remark that as $[a_i, a_i + \ell_n(\Delta)] = S_\sigma^{-1}([a+l, b+l])$ and $K \bigcap S_\sigma^{-1}(a+l, b+l)$ is not empty, we have $\mu([a_i, a_i + \ell_n(\Delta)]) > 0$. Consequently,

$$\mu_\pi(\Delta) \sim \sum_i P_n^{(i)}(\Delta) = \|Q_n(\Delta)\|,$$

with the constants of comparability independent of Δ or n. (Here the norm of the vector is the sum of the absolute values of the entries.)

Proposition 3.4. *For all $n \geq 1$, we have*

$$Q_n(\Delta) = Q_{n-1}(\widehat{\Delta}) T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta)).$$

Proof. First, assume $n \geq 2$. We want to show that for each i,

$$P_n^{(i)} = \sum_j P_{n-1}^{(j)} t_{ji} \tag{2}$$

where $T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta)) = (t_{ji})$

Consider a typical summand in the formula for $P_n^{(i)}$, say p_σ where $\sigma \in \mathcal{A}^n$ and $\varrho^{-n}(a - S_\sigma(0) + l) = a_i$ for suitable integer l. Put $\sigma = \widehat{\sigma} s$ where $\widehat{\sigma} \in \mathcal{A}^{n-1}$ and $s \in \mathcal{A}$. As $S_{\widehat{\sigma}s}(0) = a + l -\ \varrho^n a_i$, by construction there is some index j such that $S_{\widehat{\sigma}}(0) = c + l -\ \varrho^{n-1} c_j$. But then $p_\sigma = p_{\widehat{\sigma}} p_s = p_{\widehat{\sigma}} t_{ji}$ and $p_{\widehat{\sigma}}$ is a summand of $P_{n-1}^{(j)}$.

On the other hand, assume $t_{ji} \neq 0$ and $p_{\widehat{\sigma}}$ is a summand of $P_{n-1}^{(j)}$. Then there must be some l such that $S_{\widehat{\sigma}}(0) = c + l -\ \varrho^{n-1} c_j$. Furthermore, there is some integer k and $\tau \in \mathcal{A}^n$ such that $S_\tau(0) = a + k -\ \varrho^n a_i$ and $S_{\widehat{\tau}}(0) = c + k -\ \varrho^{n-1} c_j$. Assume $\tau = \widehat{\tau} s$. Then $S_{\widehat{\tau}s}(0) = S_{\widehat{\tau}}(0) + \varrho^{n-1} d_s = S_{\widehat{\sigma}}(0) + k - l + \varrho^{n-1} d_s = S_{\widehat{\sigma}s}(0) + k - l$, hence $S_{\widehat{\sigma}s}(0) = a + l - \varrho^n a_i$. This observation shows that $p_\sigma = p_{\widehat{\sigma}} p_s$ is a summand of $P_n^{(i)}$ and $t_{ji} = p_s$. Together, these observations prove (2), as required.

Now suppose $n = 1$. In this case, we need to show that $P_1^{(i)} = \sum_j t_{ji}$ where $(t_{ji}) = T(\mathcal{C}_0[0,1]), \mathcal{C}_1[a,b])$. By definition $t_{ji} = p_s$ if $S_s(0) = a + l - \varrho a_i$ and $0 = l - c_j$. Thus the definition of the neighbour set of $[0,1]$ as $\{0, 1, ..., \delta - 1\}$ ensures we have $P_1^{(i)} = \sum_j t_{ji}$ for each i. □

By the matrix norm of matrix $M = (M_{jk})$ we mean $\|M\| = \sum_{jk} |M_{jk}|$. In terms of this notation the previous results combine to yield

Corollary 3.5. *There are positive constants c_1, c_2 such that*

$$c_1 \|T(\eta)\| \leq \mu_\pi(\Delta) \leq c_2 \|T(\eta)\|$$

whenever $\Delta \in \mathcal{F}_n^{(\pi)}$ *has symbolic representation* η.

Given $\Delta_n \in \mathcal{F}_n^{(\pi)}$, we let $\Delta_n{}^+$ and Δ_n^- be the quotient net intervals of level n sharing endpoints with Δ_n, (in the torus sense). If x belongs to the interior of $\Delta_n(x)$ we put

$$M_n(x) = \mu_\pi(\Delta_n(x)) + \mu_\pi(\Delta_n^+(x)) + \mu_\pi(\Delta_n^-(x)),$$

while if x is a boundary point of $\Delta_n(x)$ we put $M_n(x) = \mu_\pi(\Delta_n(x)) + \mu_\pi(\Delta_n'(x))$ where $\Delta_n'(x)$ is the other net interval of level n with x as endpoint. Since all quotient net intervals of level n have lengths comparable to ρ^n one can see in the same manner as in [11, Thm. 2.6] that

$$\dim_{loc} \mu_\pi(x) = \lim_{n\to\infty} \frac{\log M_n(x)}{n \log \rho},$$

with a similar formula for upper and lower local dimensions. In the special case that the probabilities defining the self-similar measure are regular and $\pi(K) = \mathbb{T}$, the same arguments as given in [9, Lemma 3.5 - Cor. 3.7] show that if $[x] = (\gamma_0, \gamma_1, ...)$, then we have the simpler formula,

$$\dim_{loc} \mu_\pi(x) = \lim_{n\to\infty} \frac{\log \mu_\pi(\Delta_n(x))}{n \log \rho} = \lim_{n\to\infty} \frac{\log \|T(\gamma_0, .., \gamma_n)\|}{n \log \rho}. \tag{3}$$

In the study of finite type measures on $\mathbb{R}$, periodic points played an important role. In a similar fashion, we will say that a point $x \in \pi(K)$ is *periodic* if it has quotient symbolic representation $[x] = (\gamma_0, .., \gamma_J, \theta^-, \theta^-, ...)$ where θ is an admissible cycle (a path whose first and last letters coincide) and θ^- is the path with the last letter of θ deleted. Any point which is a boundary point of some quotient net interval is a periodic point.

As was the case for finite type measures in $\mathbb{R}$, we have the following formula for local dimensions of periodic points. We write $L(\theta^-)$ for the length of the path θ^-.

Proposition 3.6. *If* μ_π *is a measure of finite type on* $\mathbb{T}$ *and* x *is a periodic point with period* θ, *then the local dimension exists and is given by*

$$\dim_{loc} \mu_\pi(x) = \frac{\log sp(T(\theta))}{L(\theta^-) \log \varrho},$$

where if x *is a boundary point of a quotient net interval with two different symbolic representations given by periods* θ *and* ϕ *of the same length, then* θ *is chosen with* $sp(T(\theta)) \geq sp(T(\phi))$.

The proof is the same as in [11, Proposition 2.7] as that argument only required the formula we have developed for $\mu_\pi(\Delta)$ and the fact that the primitive transition matrices have a non-zero entry in each column.

We will call a quotient loop class *positive* if there is some path η in the loop class for which $T(\eta)$ has strictly positive entries. With the preliminary results we have established, the same arguments as in [9, Section 5] prove the following important result. The details are left to the reader.

Theorem 3.7. *Suppose* μ_π *is a quotient measure of finite type on* $\mathbb{T}$ *associated with regular probabilities and with supp*$\mu_\pi = \mathbb{T}$. *Then the set of local dimensions at* *points in any positive quotient loop class is a closed interval and the local dimensions at the periodic points in the loop class are dense in that interval.*

REMARK 3.8. It was shown in [9] that for measures of finite type on $\mathbb{R}$ the essential class is always a positive loop class. In the next section we will see that the quotient essential class (even if unique) need not be positive.

This theorem can be used to see that, as with measures of finite type on $\mathbb{R}$, the sets of local dimensions of measures of finite type on $\mathbb{T}$ may or may not admit isolated points. Here are some examples. The details of these examples can be found in [10].

EXAMPLE 3.9. Consider ν, the convolution square (on $\mathbb{R}$) of the uniform Bernoulli convolution with contraction factor ρ the inverse of the golden mean. This is the self-similar measure associated with the IFS with similarities ρx, $\rho x+1-\rho$ and $\rho x+2(1-\rho)$ and probabilities $1/4, 1/2$ and $1/4$ respectively. In [9, Sect. 8.2] it was shown that the set of local dimensions of ν admits an isolated point. The quotient measure, ν_π, has one essential class which is positive and hence generates an interval of local dimensions. The quotient essential primitive transition matrices and admissible paths precisely coincide with those of ν. It follows that the set of local dimensions of ν_π at points in the quotient essential class coincides with the interval of local dimensions of ν at points in its essential class.

There are also two additional maximal quotient loop classes, both of which are simple loops. These generate the same three (periodic boundary) points, namely, $0, \rho, 1-\rho$. The three points have the same local dimension and it can be shown that this value is contained in the interval of local dimensions generated by the quotient essential class. It follows that the set of local dimensions of the quotient measure has no isolated point.

EXAMPLE 3.10. Consider the IFS $\{S_j(x) = x/4+d_j/8 : j = 0, ..., 7\}$ where $d_j = j$ for $j = 0, ..., 4$, $d_5 = 7, d_6 = 9$ and $d_7 = 12$. This IFS generates the self-similar set $[0, 2]$ and is of finite type on $\mathbb{T}$. There are 10 reduced quotient characteristic vectors and one quotient essential class which consists of one reduced quotient characteristic vector (specifically, 4 non-reduced quotient characteristic vectors). Let μ be the self-similar measure arising from the regular probabilities $p_0 = p_7 = 1/2402$, $p_1 = p_2 = 1000/2402$, $p_3 = p_4 = p_5 = p_6 = 100/2402$. We have verified that the quotient essential class is of positive type when these probabilities are used. Computational work also shows that

$$[.6283, 1.885] \subseteq \{\dim \mu_\pi(x) : x \text{ in essential class}\} \subseteq [.614, 2.053].$$

There are three other maximal loop classes, all of which are simple loops. Two of these loops correspond to the single point $7/8$ whose local dimension is $\frac{\log(100/2402)}{\log 4} \sim 2.293$. One of these two loops corresponds to the path approaching $7/8$ from the left, and the other to the path approaching $7/8$ from the right. There are countably many points associated with the other maximal loop class, all having local dimension ~ 2.286 (the spectral radius of the normalized transition matrix of the length one cycle is the root of the polynomial $x^2 - 100x - 100$, approximately 100.99). Consequently, the set of local dimensions of μ_π consists of a closed interval and two isolated points. It is again the case that the interval in the set of local dimensions of μ_π coincides with the interval component of the set of local dimensions of μ. The two isolated points are also both isolated points in the set of local dimensions of μ.

A weaker result can be proven if we do not assume that the probabilities are regular or that the support of the quotient measure is the torus. Note that by an essential quotient transition matrix, we mean a transition matrix $T(\eta)$ where the path η belongs to the essential class. Recall that if $\text{supp}\mu_\pi = \mathbb{T}$, then there is a non-zero entry on each row of each primitive transition matrix.

THEOREM 3.11. *Suppose μ_π is a quotient measure of finite type on $\mathbb{T}$. Assume the essential class E is positive and that each essential primitive transition matrix has a non-zero entry in each row. Then the set of local dimensions of μ_π at points in the relative interior of E is a closed interval and the local dimensions at periodic points from E are dense in this interval.*

This follows by similar arguments to those given in [11, Sect 3.3]. (The proofs given in Section 4.2 are also similar.)

Next, we will apply this theorem to show that finite-type quotients of self-similar measures on $\mathbb{R}$ satisfying the strong separation condition have the same local dimension as the original measure. Recall that an IFS $\{S_j : j = 1, .., m\}$ with self-similar set K is said to satisfy the strong separation condition if the sets $S_j(K)$ are pairwise disjoint.

THEOREM 3.12. *Assume the equicontractive IFS $\{S_j = \varrho x + d_j : j = 0, ..., k\}$ with $k \geq 1$ satisfies the strong separation condition and is of finite type on $\mathbb{T}$. Let $\{p_j\}_{j=0}^k$ be probabilities and assume μ is the associated self-similar measure and μ_π is its quotient measure. Then*

$$\begin{aligned}\{\dim_{loc} \mu(x) \,:\, x \in K\} &= \left[\frac{\log(\max p_i)}{\log \varrho}, \frac{\log(\min p_i)}{\log \varrho}\right] := I \\ &= \{\dim_{loc} \mu_\pi(x) : x \in \pi(K)\}.\end{aligned}$$

Proof. The first equality is well known (c.f. [2, ch. 11]), so we only need to prove the second.

As usual, denote by $[0, \delta]$ the convex hull of the self-similar set K. Since the IFS satisfies the strong separation condition, the Lebesgue measure of K is zero. As $\pi(K)$ can be identified with $\bigcup_{n\in\mathbb{Z}}(K_n - n)$ where $K_n = K \cap [n, n+1)$, it follows that $\pi(K)$ also has Lebesgue measure (on $\mathbb{T}$) equal to zero. Consequently, for n sufficiently large, the Lebesgue measure of $\pi\left(\bigcup_{|\sigma|=n} S_\sigma([0, \delta])\right) < 1$ and hence there must be intervals $(a_i, a_{i+1}) \subseteq (0, 1)$ such that both a_i and a_{i+1} are in $\pi\left(\bigcup_{|\sigma|=n} S_\sigma([0, \delta])\right)$, while $(a_i, a_{i+1}) \cap \pi\left(\bigcup_{|\sigma|=n} S_\sigma([0, \delta])\right)$ is empty.

As $\pi(K)$ is a perfect set, a_{i+1} is a limit point and thus there is a quotient net interval of level n, say $\Delta_n = [a_{i+1}, a_{i+2}]$, adjacent (in the torus sense) to the empty interval. Moreover, since $(a_i, a_{i+1}) \cap \pi(S_\sigma([0, \delta])) = \emptyset$ for all $|\sigma| = n$, we see that if $[a_{i+1}, a_{i+2}] \subset \pi(S_\sigma([0, \delta]))$, then $\pi(S_\sigma(0)) = a_{i+1}$. This implies that the neighbour of Δ_n is simply the singleton (0).

Let Δ_{n+1} be the left-most descendent of Δ_n. The same reasoning shows the only neighbour of Δ_{n+1} is the singleton (0). The same holds more generally for the left-most descendent of each generation.

Choose ℓ such that $\delta\varrho^{n+\ell} < |\Delta_n|$ and let $\Delta_{n+\ell}$ be the left-most descendent of Δ_n at level $n + \ell$. As $\Delta_{n+\ell}$ is the intersection of a set of size $\delta\varrho^{n+\ell}$ with Δ_n, we see that $\Delta_{n+\ell}$ will have normalized length δ. This proves that $\gamma = (\delta, (0))$ is a reduced quotient characteristic vector.

Assume $\pi(S_\sigma([0,\delta])) = \Delta_{n+\ell}$. As the IFS satisfies the strong separation condition, we see that $\pi(S_{\sigma i}([0,\delta]))$ are all disjoint scaled copies of $\Delta_{n+\ell}$, with precisely the same geometry. This shows that the $k+1$ children of γ have again the same reduced characteristic vector γ. Moreover the probabilities associated to these children, in order from left to right, are $p_0, p_1, \ldots, p_k$, thus the corresponding transition matrices are these 1×1 positive matrices.

To this point, we have proven that $\{\gamma\}$ is *one* (reduced) essential class of μ_π. Next, we will show that it is *the* essential class. We prove this by showing γ is a descendent of any quotient characteristic vector.

Let Δ be a net interval in $\mathcal{F}_n^{(\pi)}$. By an argument similar to above we see that there exists a j such that Δ will contain a quotient net interval $\Delta_0 = [a_{i+1}, a_{i+2}] \subseteq \mathcal{F}_{n+j}^{(\pi)}$, that is adjacent to a set (a_i, a_{i+1}) which is disjoint from all $\pi\left(S_\sigma([0,\delta])\right)$ for $|\sigma| = n+j$. We then proceed as before, taking the left-most descendents of Δ_0 to find a quotient net subinterval that has characteristic vector γ. Hence $\{\gamma\}$ is *the* (reduced) essential class and the essential class consists of the $k+1$ vectors $\gamma_j = (\delta, (0), j)$, $j = 0, ..., k$.

From the previous theorem it follows that the set of local dimensions at points in the relative interior of the essential class is a closed interval. Assume $p_\alpha = \min p_i$ and $p_\beta = \max p_i$. It is easy to see that $[x] = (\chi_0, \chi_1, ..., \chi_J, \gamma_0, \gamma_k, \gamma_\alpha, \gamma_\alpha, ...)$ and $[y] = (\chi_0, \chi_1', ..., \chi_J', \gamma_0, \gamma_k, \gamma_\beta, \gamma_\beta, ...)$ belong to the interior of the essential class for suitable admissible paths $\chi_0, \chi_1, ..., \chi_J$ and $\chi_0, \chi_1', ..., \chi_j'$ and that

$$\dim_{loc} \mu_\pi(x) = \frac{\log(\min p_i)}{\log \varrho}, \dim_{loc} \mu_\pi(y) = \frac{\log(\max p_i)}{\log \varrho}.$$

Consequently, the set of local dimensions of μ_π contains the interval I.

The definition of the quotient measure implies that

$$\min_\ell \left(\underline{\dim}_{loc}\mu(x+\ell)\right) \le \dim_{loc} \mu_\pi(x) \le \min_\ell \left(\overline{\dim}_{loc}\mu(x+\ell)\right)$$

where the minimum is taken over all integers ℓ such that $x + \ell \in$suppμ. Since the upper and lower local dimensions of μ at any point in its support lie in the interval I, it follows that the same is true for $\dim_{loc} \mu_\pi(x)$ at any $x \in \pi(K)$. This completes the proof. □

Remark 3.13. We remark that Example 2.9 demonstrates that satisfying the strong separation condition and having the convex hull of the support being $[0,2]$ is not enough to guarantee that the quotient measure is of finite type.

4. Quotients of Cantor-like measures of finite type.

4.1. Finite-type structure of quotients of Cantor-like measures. In this section, we will investigate the finite type structure and local dimensions of quotients of the Cantor-like measures associated with the IFS

$$\left\{S_j(x) = \frac{1}{d}x + \frac{j(d-1)}{d} \text{ for } j \in \Lambda\right\} \text{ and probabilities } p_j > 0, \tag{4}$$

where $\Lambda \subseteq \{0, 1, ..., k\}$ and $d \geq 3$. We will assume the convex hull of K is the interval $[0, k]$, equivalently, $0, k \in \Lambda$. The strong separation condition is satisfied if $k < d - 1$, so this case is handled by Theorem 3.12 giving the following.

COROLLARY 4.1. *Let μ_π be the quotient of the self-similar measure μ associated with the IFS (4) with $k < d-1$ and $\{0,k\} \subseteq \Lambda$. Then*

$$\{\dim_{loc} \mu_\pi(x) \;:\; x \in supp\mu_\pi\} = \left[\frac{-\log(\max p_i)}{\log d}, \frac{-\log(\min p_i)}{\log d}\right]$$
$$= \{\dim_{loc} \mu(x) : x \in supp\mu\}.$$

Thus we will assume $k \geq d-1$. If $\Lambda = \{0,1,..,k\}$, the associated self-similar measures have support equal to the (full) interval $[0,k]$, while the OSC fails if $k > d-1$.

In the special case that $\Lambda = \{0,1,..,k\}$ and the probabilities satisfy $p_j = \binom{k}{j} 2^{-k}$, then the associated self-similar measure on $\mathbb{R}$ is the k-fold convolution of the uniform Cantor measure with contraction ratio $1/d$. In this case, the corresponding quotient measure is also equal to the k-fold convolution (taken on the torus) of the uniform Cantor measure.

Cantor-like measures are of finite type on $\mathbb{R}$ (c.f. [9, 11]). In [11] this fact was used to study their local dimensions, extending earlier work of [1, 12, 15]. For instance, it was shown that if $\Lambda = \{0,1,..,k\}$, $k \geq d$ and $p_0 < p_j$ for all $j \neq 0,k$, then there is an isolated point in the set of local dimensions.

Of course, the corresponding quotient measures are also of finite type on $\mathbb{T}$ according to Proposition 2.5. In this section we will prove that for many of these quotient measures the set of local dimensions is a closed interval. Typically, the essential class is the full torus (and hence is unique), however it is not in general of positive type, so we cannot appeal to either Theorem 3.7 or 3.11. Rather, we will modify the previous approach.

We begin our study of these Cantor-like measures by investigating their finite-type structures. Observe that for all these IFS the endpoints of quotient net intervals of level n belong to the set $\{jd^{-n} : j = 0, ..., k\}$ and their lengths are at most kd^{-n}. Neighbours will always belong to $\{0, ..., k\}$.

When $\Lambda = \{0, 1, ..., k\}$ more can be said.

LEMMA 4.2. *Consider the IFS (4) where $\Lambda = \{0,1,...,k\}$ and $k \geq d-1$.*
1. *The quotient net intervals of level n are the sets $[jd^{-n}, (j+1)d^{-n}]$ for $j = 0, ..., d^n - 1$.*
2. *There is only one reduced quotient characteristic vector, namely $(1,(0,1,...,k-1))$. It has d (identical) children.*

Proof. The iterates of 0 at level one are the points $j(d-1)/d$ for $j = 0, ..., k$. Taking these mod 1 gives the points j/d for $j = 0, ..., d-1$. The iterates of k are also in $\mathbb{Z}/d$ and so taken mod 1 give no additional terms.

At step n the iterates of 0 are the points $\sum_{i=1}^{n} j_i d^{-i}(d-1)$, where $0 \leq j_i \leq k$ and as $k \geq d-1$, taking these mod 1 again this gives all jd^{-n}. The iterates of k add no new terms. Thus the net intervals at each level are as claimed.

To determine the neighbours, consider the net interval $[j/d,\ (j+1)/d]$ of level one. The neighbours are the integers of the form $d\,(j/d + l - S_J(0))$ where l is an integer, $J \in \{0,1,...,k\}$ and

$$\left[\frac{j}{d} + l, \frac{j+1}{d} + l\right] \subseteq [S_J(0), S_J(k)] = \left[\frac{J(d-1)}{d}, \frac{J(d-1)+k}{d}\right]. \tag{5}$$

The inequalities implied by (5) ensure that such integers are contained in $\{0,1,...,k-1\}$. Easy computations show that if $i\in\{0,...,j\}$, $J=d-(j-i)$ and $l=d-1-(j-i)$, then $i=d\,(j/d+l-S_J(0))$ and all the requirements are satisfied. Similarly, for $i\in\{j+1,...,k-1\}$, we see that $d\,(j/d+l-S_J(0))=i$ when $J=i-j=l$. Further, all the additional requirements are met. This proves the neighbours are precisely the set $\{0,1,...,k-1\}$. In particular, there is are d children at level one of the parent interval $[0,1]$, but only one reduced characteristic vector, $(1,(0,1,...,k-1))$.

The same statement holds for the higher levels because of self-similarity. □

REMARK 4.3. We remark that even when Λ is a proper subset of $\{0,1,...,k\}$, the conclusions of this lemma are often true. For instance, one can check that this is the case if $d=4$, $k=7$ and Λ consists of all but one integer j chosen from $\{1,...,6\}$.

From here on we will restrict our attention to IFS and quotients of their self-similar measures for which the conclusion of the lemma holds. We will refer to these as *complete quotient Cantor-like measures.* Note that the support of such quotient measures is the full torus and they have d primitive transition matrices that we label as $T(\ell)$ for $\ell=0,...,d-1$, where ℓ denotes the ℓ'th child from left to right. These matrices are computed in the next lemma.

LEMMA 4.4. *The primitive transition matrices $T(\ell)$ for $\ell\in\{0,\dots,d-1\}$ for a complete quotient Cantor-like measure satisfying (4) are the $k\times k$ matrices with j,i entry equal to*

$$T(\ell)_{ji}=p_s \text{ if } \frac{\ell-(i-1)+(j-1)d}{d-1}=s \text{ with } s\in\Lambda$$

and 0 *otherwise.*

EXAMPLE 4.5. Consider the case when $d=4$, $k=7$ and $\Lambda=\{0,1,...,7\}$. The unique reduced characteristic vector is $(1,(0,1,...,6))$ and there are four transition matrices

$$T(0)=\begin{bmatrix} p_0&0&0&0&0&0&0\\ 0&p_1&0&0&p_0&0&0\\ 0&0&p_2&0&0&p_1&0\\ p_4&0&0&p_3&0&0&p_2\\ 0&p_5&0&0&p_4&0&0\\ 0&0&p_6&0&0&p_5&0\\ 0&0&0&p_7&0&0&p_6 \end{bmatrix}\quad T(1)=\begin{bmatrix} 0&p_0&0&0&0&0&0\\ 0&0&p_1&0&0&p_0&0\\ p_3&0&0&p_2&0&0&p_1\\ 0&p_4&0&0&p_3&0&0\\ 0&0&p_5&0&0&p_4&0\\ p_7&0&0&p_6&0&0&p_5\\ 0&0&0&0&p_7&0&0 \end{bmatrix}$$

$$T(2)=\begin{bmatrix} 0&0&p_0&0&0&0&0\\ p_2&0&0&p_1&0&0&p_0\\ 0&p_3&0&0&p_2&0&0\\ 0&0&p_4&0&0&p_3&0\\ p_6&0&0&p_5&0&0&p_4\\ 0&p_7&0&0&p_6&0&0\\ 0&0&0&0&0&p_7&0 \end{bmatrix}\quad T(3)=\begin{bmatrix} p_1&0&0&p_0&0&0&0\\ 0&p_2&0&0&p_1&0&0\\ 0&0&p_3&0&0&p_2&0\\ p_5&0&0&p_4&0&0&p_3\\ 0&p_6&0&0&p_5&0&0\\ 0&0&p_7&0&0&p_6&0\\ 0&0&0&0&0&0&p_7 \end{bmatrix}.$$

We remark that these are also the primitive transition matrices in the case that Λ omits only one integer j other than 0 or 7, with the understanding that the entries denoted p_j have value zero.

Proof of Lemma 4.4. The ℓ'th child of parent $[0,1]$ is the net interval $\left[\frac{\ell}{d}, \frac{\ell+1}{d}\right]$. The j'th neighbour of $[0,1]$ is $j-1$ and the i'th neighbour of $\left[\frac{\ell}{d}, \frac{\ell+1}{d}\right]$ is $i-1$. The entry $T(\ell)_{ji}$ will be p_s where $s \in \{0, ..., k\}$ is such that

$$0 - (j-1) + \frac{s(d-1)}{d} = \frac{l}{d} - \frac{(i-1)}{d}.$$

Solving for s yields the desired result. □

The transition matrices of complete quotient Cantor-like measures have not only a non-zero entry in each column, but also a non-zero entry in each row as the support of the quotient measure is full.

We next permute the rows and columns of these transition matrices to produce matrices $\tilde{T}(\ell)$ with a natural block structure, where block (i,j) for $i,j \in \{1, ..., d-1\}$ has size $\left(\left\lfloor \frac{k-i}{d-1} \right\rfloor + 1\right) \times \left(\left\lfloor \frac{k-j}{d-1} \right\rfloor + 1\right)$. The (i',j') entry of block (i,j) of $\tilde{T}(\ell)$ will have as its entry the $(i + i'(d-1), j + j'(d-1))$ entry of $T(\ell)$, i' and j' being indexed beginning at 0. Doing this permutation is not necessary, but it makes the algebraic manipulations simpler in what follows.

Given such a block matrix B, we will write $B(i,j)$ for the (i,j) block. We will say that block matrix B (with this structure) is of *type* r if $B(i,j) \neq 0$ only if $j - i \equiv r \mod (d-1)$.

It is easy to see that the permuted transition matrix $\widetilde{T}(\ell)$ is type $\ell \pmod{(d-1)}$. Furthermore, $\widetilde{T}(\ell)$ has the special property that if $j - i \equiv \ell \mod (d-1)$, then the matrix $\widetilde{T}(\ell)(i,j)$ has at least one non-zero entry in each row and column.

EXAMPLE 4.6. Consider a complete quotient Cantor-like measure with $d = 4$ and $k = 7$. Permuting the rows and columns yields:

$$\tilde{T}(0) = \left[\begin{array}{ccc|cc|cc} p_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_4 & p_3 & p_2 & 0 & 0 & 0 & 0 \\ 0 & p_7 & p_6 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & p_1 & p_0 & 0 & 0 \\ 0 & 0 & 0 & p_5 & p_4 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & p_2 & p_1 \\ 0 & 0 & 0 & 0 & 0 & p_6 & p_5 \end{array}\right] \quad \tilde{T}(1) = \left[\begin{array}{ccc|cc|cc} 0 & 0 & 0 & p_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_4 & p_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_7 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & p_1 & p_0 \\ 0 & 0 & 0 & 0 & 0 & p_5 & p_4 \\ \hline p_3 & p_2 & p_1 & 0 & 0 & 0 & 0 \\ p_7 & p_6 & p_5 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\tilde{T}(2) = \left[\begin{array}{ccc|cc|cc} 0 & 0 & 0 & 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_4 & p_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_7 \\ \hline p_2 & p_1 & p_0 & 0 & 0 & 0 & 0 \\ p_6 & p_5 & p_4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & p_3 & p_2 & 0 & 0 \\ 0 & 0 & 0 & p_7 & p_6 & 0 & 0 \end{array}\right] \quad \tilde{T}(3) = \left[\begin{array}{ccc|cc|cc} p_1 & p_0 & 0 & 0 & 0 & 0 & 0 \\ p_5 & p_4 & p_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_7 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & p_2 & p_1 & 0 & 0 \\ 0 & 0 & 0 & p_6 & p_5 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & p_3 & p_2 \\ 0 & 0 & 0 & 0 & 0 & p_7 & p_6 \end{array}\right].$$

Hereafter, when we speak of a *block matrix* we will mean a matrix B with this block structure of type ℓ for some ℓ and having the property that $B(i,j)$ has a non-zero entry in each row and column if $j - i \equiv \ell \mod (d-1)$. The block matrices of type 0 will be called *block diagonal*. By a *block positive* matrix we will mean a block matrix of type ℓ with all entries of $B(i,j)$ strictly positive for $j - i \equiv \ell \mod (d-1)$. It is worth noting that a block matrix of type ℓ is also of type ℓ' for all $\ell' \equiv \ell \mod (d-1)$.

We will say that a periodic point with period θ is a *block diagonal (positive) periodic point* if the block matrix $\widetilde{T}(\theta)$ is block diagonal (and block positive).

We record here some routine facts about block matrices (with this definition) that follow from block multiplication and the fact that each non-zero block has a non-zero entry in each row and column.

LEMMA 4.7. *Suppose A, B, C are block matrices.*

(i) *If A, B are types a, b respectively, then AB is a block matrix of type $(a+b)$ mod$(d-1)$.*

(ii) *If B is block positive, then AB and BA are block positive.*

(iii) *There are positive constants c_1, c_1', depending on B, such that*

$$c_1 \|A\| \leq \min(\|AB\|, \|BA\|) \leq \max(\|AB\|, \|BA\|) \leq c_1' \|A\|.$$

(iv) *If B is block positive, then there is a positive constant c_0, depending on B, such that*

$$\|ABC\| \geq c_0 \|A\| \|C\|.$$

Being a product of block matrices, every permuted transition matrix of complete quotient Cantor-like measures *is* a block matrix.

Block positive matrices have further good properties.

LEMMA 4.8. *Suppose B is a block positive matrix.*

(i) *If AB is block diagonal, then there is a constant $c = c(B)$ such that*

$$sp(AB) \leq \|AB\| \leq c \cdot sp(AB).$$

(ii) *If B is block diagonal, then there is a constant $c_2 = c_2(B)$ such that*

$$\|B^n\| \leq c_2 sp(B^n) \text{ for all } n.$$

Further, there exists an index j such that $sp(B^n) = sp(B(j,j))^n$ for all n.

Proof. (i) Assume A is type a, so B is type $-a$. Then $B(a+j, j)$ is a positive matrix for each j (here $a+j$ should be understood mod$(d-1)$) and hence by [9, Lemma 3.15(2)] there are constants C_j such that

$$\begin{aligned} \|A(j, a+j)B(a+j, j)\| &\leq C_j sp(A(j, a+j)B(a+j, j)) \\ &\leq C_j sp(AB(j,j)). \end{aligned}$$

Since AB is diagonal, this is dominated by $C_j sp(AB)$. But

$$\|AB\| = \sum_{j=1}^{d-1} \|AB(j,j)\| = \sum_j \|A(j, a+j)B(a+j, j)\|,$$

so we can take $c = \max_j C_j (d-1)$.

(ii) As each square matrix $B(j,j)$ is positive, [9, Lemma 3.15(3)] implies that for each j there is a constant C_j' such that $\|B(j,j)^n\| \leq C_j' sp(B(j,j))^n \leq C_j' sp(B^n)$ for each n. Hence

$$\|B^n\| = \sum_j \|B(j,j)^n\| \leq \max_j C_j'(d-1) sp(B^n) \leq c_2 sp(B^n).$$

The final comment follows because B is block diagonal. □

Lemma 4.9. *Suppose* $\Lambda = \{0, ..., k\}$. *The permuted transition matrix* $(\widetilde{T}(0)\widetilde{T}(d-1))^k$ *is block diagonal and positive.*

Proof. It is easiest to see this if we look at $(T(0)T(d-1))^k$. We must show that if $j \equiv i \mod (d-1)$, then $\left((T(0)T(d-1))^k\right)_{ij} > 0$. Let $\ell_0 = i$ and $\ell_{2k} = j$. Note that

$$\begin{aligned}
&\left((T(0)T(d-1))^k\right)_{ij} \\
&= \sum_{\ell_1=1}^{k} \cdots \sum_{\ell_{2k-1}=1}^{k} T(0)_{i,\ell_1} T(d-1)_{\ell_1,\ell_2} \cdots T(0)_{\ell_{2k-2},\ell_{2k-1}} T(d-1)_{\ell_{2k-1},j} \\
&= \sum_{\ell_1=1}^{k} \cdots \sum_{\ell_{2k-1}=1}^{k} \prod_{r=1}^{2k} p_{\ell_{r-1}-1-\frac{\ell_r-\ell_{r-1}}{d-1}+\tau(r)}
\end{aligned}$$

where $\tau(r) = 0$ if r is odd and $\tau(r) = 1$ if r is even. It suffices to show that there exists $\ell_1, \ell_2, \ldots, \ell_{2k-1}$ such that

$$\prod_{r=1}^{2k} p_{\ell_{r-1}-1-\frac{\ell_r-\ell_{r-1}}{d-1}+\tau(r)} > 0. \tag{6}$$

We will define the ℓ_r inductively. We set $\ell_0 = i$. We define

$$\ell_r = \begin{cases} \ell_{r-1} + (d-1) & \text{if } \ell_{r-1} + (d-1) \leq j \text{ and } \ell_{r-1} + \tau(r) \geq 2 \\ \ell_{r-1} - (d-1) & \text{if } \ell_{r-1} - (d-1) \geq j \text{ and } \ell_{r-1} + \tau(r) \leq k \\ \ell_{r-1} & \text{otherwise} \end{cases}.$$

Recall that $1 \leq i, j \leq k$.

- If $\ell_r = \ell_{r-1} + (d-1)$, then $\ell_{r-1} + \tau(r) \geq 2$ and $\ell_{r-1} < j$ so $\ell_{r-1} + \tau(r) \leq j \leq k$. Thus $\ell_{r-1} - 1 - \frac{\ell_r - \ell_{r-1}}{d-1} + \tau(r) = \ell_{r-1} - 1 - 1 + \tau(r) \in \{0, 1, \cdots, k-1\}$, hence the associated probability is non-zero.
- If $\ell_r = \ell_{r-1} - (d-1)$, then $\ell_{r-1} + \tau(r) \leq k$ and $\ell_{r-1} > j$, so $\ell_{r-1} - 1 - \frac{\ell_r - \ell_{r-1}}{d-1} + \tau(r) = \ell_{r-1} - 1 + 1 + \tau(r) \in \{1, \cdots, k\}$ and hence the associated probability is non-zero.
- If $\ell_r = \ell_{r-1}$ then $\ell_{r-1} - 1 - \frac{\ell_r - \ell_{r-1}}{d-1} + \tau(r) = \ell_{r-1} - 1 + \tau(r) \in \{0, 1, \cdots, k\}$ and again the associated probability is non-zero.

We note that for r even we always have $\ell_{r-1} + \tau(r) \geq 2$ and for r odd we always have $\ell_{r-1} + \tau(r) \leq k$. If $i = \ell_0 < j$, say $j = i + m(d-1)$ for $m \geq 1$, then $\ell_t = j$ for $t \geq 2m$ and therefore $\ell_{2k} = j$. A similar statement holds if $i = \ell_0 > j$ and, of course, if $i = j$, then $\ell_r = i = j$ for all r.

Hence equation (6) is satisfied which proves the result. □

When Λ is a proper subset of $\{0, ..., k\}$, it is still possible for this transition matrix to be positive. For example, this can easily be seen to be true in the case that $d = 4, k = 7$, $p_j = 0$ for one of $j = 1, ..., 6$.

Another simple, but useful, fact is the following. By an *interior periodic point*, we mean a periodic point that is not a boundary point.

Lemma 4.10. *Assume the complete quotient Cantor-like measure admits a permuted transition matrix that is block positive. Then there is a finite set* $\mathcal{F}$ *of block (permuted) transition matrices such that for each block matrix* A *there is some* $B \in \mathcal{F}$

so that AB and BA are block positive and diagonal and, furthermore, any periodic point with period θ satisfying $\widetilde{T}(\theta) = AB$ or BA is an interior periodic point.

Proof. Let F denote any block diagonal and positive permuted transition matrix. This is guaranteed to exist since the product of a block positive matrix with any block matrix is still block positive. For each $r = 1, ..., d-1$, let B_r be a block matrix of type r. Put $\mathcal{F} = \{FB_1B_{r-1} : r = 0, 1, ..., d-2\}$. The matrices FB_1B_{r-1} are block positive and type r. If A is any block matrix, it has type $-r$ for some $r = 1, ..., d-1$ and hence $FB_1B_{r-1}A$ and AFB_1B_{r-1} are block diagonal, positive matrices. The choice of B_1B_{r-1} ensures that they give rise to interior periodic points. ☐

4.2. Local dimensions for complete quotient Cantor-like measures. In this section we prove that the set of local dimensions for all complete quotient Cantor-like measures which admit a block positive transition matrix is an interval. We will also prove that the local dimensions at the interior, positive periodic points are dense.

We begin by proving that the local dimensions at periodic points are dense in the set of all local dimensions.

THEOREM 4.11. *Assume μ_π is a complete quotient Cantor-like measure which admits a block positive transition matrix. Then the set of local dimensions at block diagonal, positive, interior periodic points is dense in the set of all local (upper or lower) dimensions of μ_π.*

Proof. Fix $x \in \mathbb{T}$. We will see how to approximate the lower local dimension of μ_π at x by the local dimensions of block diagonal, positive, interior periodic points. The other cases are similar.

Fix a subsequence (n_ℓ) such that

$$\underline{\dim}_{loc}\mu_\pi(x) = \lim_\ell \frac{\log M_{n_\ell}(x)}{n_\ell \log(1/d)}.$$

For each n_ℓ, let Δ'_{n_ℓ} denote a choice of $\Delta_{n_\ell}(x)$, $\Delta^+_{n_\ell}(x)$, $\Delta^-_{n_\ell}(x)$ such that $\mu_\pi(\Delta'_{n_\ell}) \leq M_{n_\ell}(x) \leq 3\mu_\pi(\Delta'_{n_\ell})$ and let $(\gamma_0^{(\ell)}, ..., \gamma_{n_\ell}^{(\ell)})$ be the symbolic representation of Δ'_{n_ℓ}. Then

$$\underline{\dim}_{loc}\mu(x) = \lim_\ell \frac{\log \left\| \widetilde{T}(\gamma_0^{(\ell)}, ..., \gamma_{n_\ell}^{(\ell)}) \right\|}{n_\ell \log(1/d)}.$$

For notational ease, let $A_\ell = \widetilde{T}(\gamma_0^{(\ell)}, ..., \gamma_{n_\ell}^{(\ell)})$ and choose a block positive matrix B_ℓ from the finite set $\mathcal{F}$ of Lemma 4.10 so that $A_\ell B_\ell$ is block diagonal and if the periodic point y_ℓ with period θ_ℓ satisfies $\widetilde{T}(\theta_\ell) = A_\ell B_\ell$, then y_ℓ is a block diagonal, positive, interior periodic point.

By Lemma 4.8 there is a constant c, which depends only on the finite set $\mathcal{F}$, such that

$$sp(A_\ell B_\ell) \leq \|A_\ell B_\ell\| \leq c \cdot sp(A_\ell B_\ell).$$

Thus for each ℓ there is a constant c_ℓ so that $c_\ell \|A_\ell B_\ell\| = sp(A_\ell B_\ell)$, where c_ℓ is bounded above and below from zero. Similarly, by Lemma 4.7 there is a constant c'_ℓ, bounded above and below, such that $\|A_\ell B_\ell\| = c'_\ell \|A_\ell\|$. Moreover,

$$\dim \mu_\pi(y_\ell) = \frac{\log sp(A_\ell B_\ell)}{L(\theta_\ell^-) \log(1/d)} = \frac{\log(c_\ell c'_\ell \|A_\ell\|)}{L(\theta_\ell^-) \log(1/d)}.$$

But $L(\theta_\ell^-) = n_\ell + L(B_\ell)$ (where by $L(B_\ell)$ we mean the number of permuted primitive transition matrices whose product is B_ℓ) and $L(B_\ell)$ is bounded because B_ℓ is chosen from a finite set. Thus an easy calculation shows that

$$\left| \dim \mu_\pi(y_\ell) - \frac{\log \|A_\ell\|)}{n_\ell \log(1/d)} \right| \to 0 \text{ as } \ell \to \infty$$

and therefore $\dim \mu_\pi(y_\ell) \to \underline{\dim}_{loc}\mu_\pi(x)$. □

THEOREM 4.12. *Assume μ_π is a complete quotient Cantor-like measure which admits a block positive transition matrix. Suppose y, z are block diagonal, positive, interior periodic points. Given any real number R, with* $\dim_{loc} \mu_\pi(y) < R < \dim_{loc} \mu_\pi(z)$, *and $\varepsilon > 0$ there is a periodic point x with*

$$|R - \dim_{loc} \mu_\pi(x)| < \varepsilon.$$

Proof. Assume y has period ϕ and z period θ where $A = \widetilde{T}(\phi)$ and $B = \widetilde{T}(\theta)$ are block diagonal and positive. Given R as above, choose $0 < t < 1$ such that $R = t \dim_{loc} \mu_\pi(y) + (1-t) \dim_{loc} \mu_\pi(z)$. Choose integers $n_\ell, m_\ell \to \infty$ such that

$$\frac{L(\phi^-)n_\ell}{L(\phi^-)n_\ell + L(\theta^-)m_\ell} \to t.$$

Let $\alpha = sp(A)$ and $\beta = sp(B)$ and assume $\alpha = sp(A(j,j))$ and $\beta = sp(B(i,i))$. With this notation,

$$R = \frac{t \log \alpha}{L(\phi^-) \log(1/d)} + \frac{(1-t) \log \beta}{L(\theta^-) \log(1/d)}.$$

Choose block positive matrices M, N from the finite set $\mathcal{F}$ of Lemma 4.10 so that M is type $j - i$ and N is type $i - j$.

By Lemmas 4.7 and 4.8 and the block structure, there are positive constants c, c', independent of ℓ, which may vary from one occurrence to another, such that

$$\begin{aligned} sp(B^{m_\ell} M A^{n_\ell} N) &\geq c \|B^{m_\ell} M A^{n_\ell}\| \geq c \|B^{m_\ell} M A^{n_\ell}(i,j)\| \\ &= c \|B^{m_\ell}(i,i) M(i,j) A^{n_\ell}(j,j)\| \\ &\geq c \|B^{m_\ell}(i,i)\| \, \|A^{n_\ell}(j,j)\| \geq c\beta^{m_\ell}\alpha^{n_\ell}. \end{aligned}$$

On the other hand, as A, B are both block diagonal and positive, we also have

$$sp(B^{m_\ell} M A^{n_\ell} N) \leq c' \|B^{m_\ell}\| \, \|A^{n_\ell}\| \leq c' \beta^{m_\ell} \alpha^{n_\ell}.$$

It follows that if we let x_ℓ be the interior periodic point with period $B^{m_\ell} M A^{n_\ell} N$, then

$$\begin{aligned} \lim_\ell \dim_{loc} \mu_\pi(x_\ell) &= \lim_\ell \frac{\log sp(B^{m_\ell} M A^{n_\ell} N)}{(L(\theta^-)m_\ell + L(\phi^-)n_\ell + L(M) + L(N)) \log(1/d)} \\ &= \lim_\ell \frac{m_\ell \log \beta + n_\ell \log \alpha}{(L(\theta^-)m_\ell + L(\phi^-)n_\ell) \log(1/d)} = R. \end{aligned}$$

□

THEOREM 4.13. *Assume μ_π is a complete quotient Cantor-like measure which admits a block positive transition matrix. Suppose x_n are block diagonal, positive,* *interior periodic points. Then there is some $x \in \mathbb{T}$ such that*

$$\overline{\dim}_{loc}\mu_\pi(x) = \limsup_n \dim_{loc} \mu_\pi(x_n)$$
$$\underline{\dim}_{loc}\mu_\pi(x) = \liminf_n \dim_{loc} \mu_\pi(x_n).$$

Proof. Assume x_n has period θ_n with $\widetilde{T}(\theta_n)$ block diagonal and positive. Let

$$S = \overline{\lim} \frac{\log(sp\widetilde{T}(\theta_n))}{L(\theta_n^-)\log(1/d)} \text{ and } I = \underline{\lim} \frac{\log(sp\widetilde{T}(\theta_n))}{L(\theta_n^-)\log(1/d)}.$$

By passing to a subsequence, we can assume

$$\dim_{loc} \mu_\pi(x_{2n}) = \frac{\log(sp\widetilde{T}(\theta_{2n}))}{L(\theta_{2n}^-)\log(1/d)} \to S$$

and

$$\dim_{loc} \mu_\pi(x_{2n+1}) = \frac{\log(sp\widetilde{T}(\theta_{2n+1}))}{L(\theta_{2n+1}^-)\log(1/d)} \to I.$$

We will define a rapidly increasing sequence (i_n) and then put x to be

$$x = (\gamma_0, \underbrace{\theta_1, ..., \theta_1}_{i_1}, \underbrace{\theta_2, ..., \theta_2}_{i_2}, \underbrace{\theta_3, ..., \theta_3}_{i_3}, ...).$$

The sequence (i_n) will be inductively defined, with i_n depending on i_j, θ_j for $j = 1, ..., n-1$, θ_n and θ_{n+1}. The choice will be clear from the arguments that follow.

The proof that x has the required properties is similar to that of [9, Theorem 5.5] and [11, Theorem 3.13]. We only sketch the main ideas here.

Put $N_n = 1 + \sum_{j=1}^n i_j L(\theta_j^-)$ so that

$$\Delta_{N_n}(x) = (\gamma_0, \underbrace{\theta_1, ..., \theta_1}_{i_1}, \underbrace{\theta_2, ..., \theta_2}_{i_2}, ..., \underbrace{\theta_n, ..., \theta_n}_{i_n}).$$

Since x_n is an interior point, $\Delta_{N_n}(x)$, $\Delta_{N_n}^+(x)$, $\Delta_{N_n}^-(x)$ have a common ancestor $(\gamma_0, \underbrace{\theta_1, ..., \theta_1}_{i_1}, \underbrace{\theta_2, ..., \theta_2}_{i_2}, ..., \underbrace{\theta_n, ..., \theta_n}_{i_n-1}) = \Delta_{N_n - L(\theta_n^-)}(x)$. Applying Corollary 3.5 and Lemma 4.7 there are constants a_n, b_n, c_n depending on i_j, θ_j, $j = 1, ..., n-1$ and θ_n, so that

$$\begin{aligned} M_{N_n}(x) &= a_n \mu_\pi(\Delta_{N_n - L(\theta_n^-)}(x)) \\ &= b_n \left\| \widetilde{T}(\gamma_0, \underbrace{\theta_1, ..., \theta_1}_{i_1}, ..., \underbrace{\theta_n, ..., \theta_n}_{i_n-1}) \right\| = c_n \left\| \widetilde{T}(\theta_n)^{i_n-1} \right\|. \end{aligned}$$

By choosing i_n large enough we can ensure that

$$\left| \frac{\log \left\| \widetilde{T}(\theta_n)^{i_n-1} \right\|}{L(\theta_n^-)(i_n-1)} - \frac{\log(sp\widetilde{T}(\theta_n))}{L(\theta_n^-)} \right| \to 0,$$

$$\frac{|a_n|+|b_n|+|c_n|}{i_n} \to 0$$

and

$$\frac{L(\theta_n^-)(i_n-1)}{N_n} \to 1.$$

Consequently,

$$\left|\frac{\log M_{N_n}(x)}{N_n \log(1/d)} - \frac{\log(sp\widetilde{T}(\theta_n))}{L(\theta_n^-)\log(1/d)}\right| \to 0$$

as $n \to \infty$. It follows that $\frac{\log M_{N_n}(x)}{N_n \log(1/d)}$ tends to S and I along the even and odd subsequences. This shows that $\overline{\dim}_{loc}\mu_\pi(x) \geq S$ and $\underline{\dim}_{loc}\mu_\pi(x) \leq I$.

Now consider arbitrary

$$N = 1 + \sum_{j=1}^{n} i_j L(\theta_j^-) + q_{n+1}L(\theta_{n+1}^-) + r$$

where $0 \leq q_{n+1} < i_{n+1}$, $0 \leq r < L(\theta_{n+1}^-)$ and $r > 0$ if $q_{n+1} = 0$. Then $\Delta_N(x)$, $\Delta_N^+(x)$, $\Delta_N^-(x)$ have common ancestor $\Delta_{J_N}(x)$ where

$$J_N = 1 + \sum_{j=1}^{n} i_j L(\theta_j^-) + (q_{n+1}-1)L(\theta_{n+1}^-)$$

if $q_{n+1} > 1$ and $J_N = 1+\sum_{j=1}^{n-1} i_j L(\theta_j^-)+(i_n-1)L(\theta_n^-)$ otherwise. Similar arguments to above show that we should study the limiting behaviour of

$$\frac{\log \mu_\pi(\Delta_{J_N}(x))}{N\log(1/d)},$$

hence it suffices to study the limiting behaviour of

$$E_n = \frac{\log\left\|\widetilde{T}(\gamma_0, \underbrace{\theta_1,...,\theta_1}_{i_1},..., \underbrace{\theta_n,...,\theta_n}_{i_n}, \underbrace{\theta_{n+1},...,\theta_{n+1}}_{q_{n+1}-1})\right\|}{N\log(1/d)},$$

(with suitable modifications if $q_{n+1} = 0, 1$). Applying Lemma 4.7(iv), with the positive matrix $\widetilde{T}(\theta_n)$ as the matrix B, there is a constant $c_0(n)$, depending on i_j, θ_j, $j = 1, ..., n-1$ and θ_n such that E_n dominates

$$\begin{aligned}
&\frac{\log c_0(n) + \log\left\|\widetilde{T}(\theta_n)^{i_n-1}\right\| + \log\left\|\widetilde{T}(\theta_{n+1})^{q_{n+1}-1}\right\|}{N\log(1/d)}\\
&\geq \frac{\log c_0(n)}{N\log(1/d)} + \frac{\log sp(\widetilde{T}(\theta_n))^{i_n-1}}{N\log(1/d)} + \frac{\log sp(\widetilde{T}(\theta_{n+1}))^{q_{n+1}-1}}{N\log(1/d)}.
\end{aligned}$$

Choose a sequence (ε_n) tending to 0. For i_n sufficiently large, $\left|\frac{\log c_0(n)}{N\log(1/d)}\right| < \varepsilon_n$. Furthermore, with possibly larger i_n we have

$$\begin{aligned}\frac{\log sp(\widetilde{T}(\theta_{n+1}))^{q_{n+1}-1}}{N\log(1/d)} &= \frac{\log sp(\widetilde{T}(\theta_{n+1}))^{q_{n+1}-1}}{(q_{n+1}-1)L(\theta_{n+1}^-)\log(1/d)}\frac{(q_{n+1}-1)L(\theta_{n+1}^-)}{N}\\ &\geq \frac{\log sp(\widetilde{T}(\theta_{n+1}))}{L(\theta_{n+1}^-)\log(1/d)}t_n - \varepsilon_n\end{aligned}$$

for

$$t_n = \frac{q_{n+1}L(\theta_{n+1}^-)+r}{N}.$$

As $1-t_n = 1+\sum_{j=1}^n i_j L(\theta_j^-)$, we similarly have

$$\frac{\log sp(\widetilde{T}(\theta_n))^{i_n-1}}{N\log(1/d)} \geq \frac{\log sp(\widetilde{T}(\theta_n))}{L(\theta_n^-)\log(1/d)}(1-t_n) - \varepsilon_n.$$

Thus

$$E_n \geq \frac{\log sp(\widetilde{T}(\theta_n))}{L(\theta_n^-)\log(1/d)}(1-t_n) + \frac{\log sp(\widetilde{T}(\theta_{n+1}))}{L(\theta_{n+1}^-)\log(1/d)}t_n - 3\varepsilon_n.$$

Appealing to Lemma 4.8(ii), we similarly deduce that for large enough i_n

$$\begin{aligned}E_n &\leq \frac{\log\left\|\widetilde{T}(\theta_n)^{i_n}\right\| + \log\left\|\widetilde{T}(\theta_{n+1})^{q_{n+1}-1}\right\|}{N\log(1/d)} + \varepsilon_n\\ &\leq \frac{\log sp(\widetilde{T}(\theta_n))^{i_n}}{N\log(1/d)} + \frac{\log sp(\widetilde{T}(\theta_{n+1}))^{q_{n+1}-1}}{N\log(1/d)} + 2\varepsilon_n\\ &\leq \frac{\log sp(\widetilde{T}(\theta_n))}{L(\theta_n^-)\log(1/d)}(1-t_n) + \frac{\log sp(\widetilde{T}(\theta_{n+1}))}{L(\theta_{n+1}^-)\log(1/d)}t_n + 3\varepsilon_n.\end{aligned}$$

Together these estimates show that E_n lies within $3\varepsilon_n$ of the same convex combinations of $\frac{\log sp(\widetilde{T}(\theta_n))}{L(\theta_n^-)\log(1/d)}$ and $\frac{\log sp(\widetilde{T}(\theta_{n+1}))}{L(\theta_{n+1}^-)\log(1/d)}$. This proves that $\liminf E_n$ and $\limsup E_n$ lie in the interval $[I,S]$ and hence the same is true for $\underline{\dim}_{loc}\mu_\pi(x)$ and $\overline{\dim}_{loc}\mu_\pi(x)$. □

Combining these results we immediately deduce the following.

Corollary 4.14. *The set of (upper, lower) local dimensions of any complete quotient Cantor-like measure that admits a block positive transition matrix is the closed interval which is the closure of the set of local dimensions at block diagonal, positive, interior periodic points. In particular, if μ_π is the quotient of the self-similar measure μ associated with the IFS (4) with $k \geq d-1$ and $\Lambda = \{0,1,...,k\}$, then the set of local dimensions is a closed interval.*

4.3. Bounds on the local dimensions of Cantor-like measures. In [7] it was shown that the set of local dimensions of the quotient of the 3-fold convolution of the middle third Cantor measure ν, is a proper subset of the interval component of the set of local dimensions of ν. In this subsection, we will see that this is true, more generally, for quotients of $(d+k)$-fold convolutions of uniform Cantor measures with contraction factor $1/d$, provided d is sufficiently large.

For matrix M, let $\|M\|_{\min} = \min_j \sum_i |M_{ij}|$ denote the minimum column sum. Of course, $\|M\| \geq \|M\|_{\min}$ and in [11] it is observed that $\|M_1 M_2\|_{\min} \geq \|M_1\|_{\min} \|M_2\|_{\min}$.

There is a refinement of this for block matrices. Let T be a block matrix with D non-zero blocks and recall that by $T(i,j)$ we mean the block (i,j) submatrix of T. The arithmetic/geometric mean inequality implies that

$$\begin{aligned}
\|T\| &\geq \sum_{\substack{(i,j) \\ \text{non-zero blocks}}} \|T(i,j)\|_{\min} \\
&\geq D \left(\prod_{\substack{(i,j) \\ \text{non-zero blocks}}} \|T(i,j)\|_{\min} \right)^{1/D} \\
&\geq \left(\prod_{\substack{(i,j) \\ \text{non-zero blocks}}} \|T(i,j)\|_{\min} \right)^{1/D}.
\end{aligned}$$

More generally, for block matrices T_k,

$$\begin{aligned}
\|T_1 T_2 \cdots T_t\| &\geq \left(\prod_{\substack{(i,j) \\ \text{non-zero blocks of } T_1 T_2 \ldots T_t}} \|T_1 T_2 \cdots T_t(i,j)\|_{\min} \right)^{1/D} \\
&= \left(\prod_{\substack{(i,j) \\ \text{non-zero blocks of } T_1 T_2 \ldots T_t}} \prod_{k=1}^{t} \|T_k(i_k, j_k)\|_{\min} \right)^{1/D} \\
&= \left(\prod_{k=1}^{t} \prod_{\substack{(i,j) \\ \text{non-zero blocks of } T_k}} \|T_k(i,j)\|_{\min} \right)^{1/D}
\end{aligned}$$

where $T_1 T_2 \cdots T_t(i,j) = \prod_{k=1}^{t} T_k(i_k, j_k)$.

Combined with (3), this observation directly yields the following upper bound on local dimensions.

Proposition 4.15. *Suppose μ_π is a complete quotient Cantor-like measure with contraction factor $1/d$ and regular probabilities. If*

$$\left(\prod_{(i,j) \text{ non-zero blocks}} \left\| \widetilde{T}(\ell)(i,j) \right\|_{\min} \right)^{1/(d-1)} \geq \theta$$

for each primitve transition matrix $\widetilde{T}(\ell)$, *then*

$$\sup_x \dim_{loc} \mu_\pi(x) \leq \frac{\log \theta}{\log 1/d}.$$

Proposition 4.16. *Let* d, k *be non-negative integers with* $d \geq 3$ *and let* $\nu = \nu(d, k)$ *be the* $(d+k)$*-fold convolution of the uniform Cantor measure with contraction factor* $1/d$ *and* ν_π *the associated measure on the torus. For any fixed* k,

$$\{\dim_{loc} \nu_\pi(x) : x \in supp\ \nu_\pi\} \subsetneq \{\dim_{loc} \nu(x) : x \in supp\ \nu, x \neq 0, d+k\}$$

provided d *is sufficiently large.*

Proof. It is easy to see that

$$\{\dim_{loc} \nu_\pi(x) : x \in \text{supp } \nu_\pi\} \subseteq \{\dim_{loc} \nu(x) : x \in \text{supp } \nu, x \neq 0, d+k\}$$

by noting that

$$\min_\ell \left(\underline{\dim}_{loc}\mu(x+\ell)\right) \leq \dim_{loc} \mu_\pi(x) \leq \min_\ell \left(\overline{\dim}_{loc}\mu(x+\ell)\right).$$

Hence we need only show the strict inclusion. In fact, we will show that $\sup_x \dim_{loc} \nu_\pi(x) < \sup_{x\neq 0,d+k} \dim_{loc} \nu(x)$.

There is no loss in assuming $k \leq d-2$. In this case, [1, Thm. 6.1] gives the formula

$$\sup_{x\neq 0,d+k} \dim_{loc} \nu(x) = \frac{\log \beta}{\log 1/d}$$

for

$$\beta = \frac{p_{r+d+1} + p_r + \sqrt{(p_{r+d+1} - p_r)^2 + 4p_{r+1}p_{r+d}}}{2},$$

where $r = \left[\frac{k}{2}\right]$ and

$$p_j = \binom{d+k}{j} 2^{-(d+k)}.$$

Using the bound $\binom{d+k}{j} \leq (d+k)^j$ for $j \leq (d+k)/2$ one can easily verify that $\beta \leq C_0(d+k)^{k/2+1}2^{-(d+k)}$ for a constant C_0 independent of d. Thus

$$\sup_{x\neq 0,d+k} \dim_{loc} \nu(x) \geq \frac{\log 2^{d+k} - \log C_0(d+k)^{k/2+1}}{\log d}. \tag{7}$$

Next, we will apply Proposition 4.15 to find an upper bound on $\dim_{loc} \nu_\pi(x)$. For this, we will need to obtain lower bounds on $\left\|\widetilde{T}(\ell)(i,j)\right\|_{\min}$ for the primitive transition matrices $\widetilde{T}(\ell)$, $\ell = 0, ..., d-1$. Each of these block matrices has $d-1$ non-zero blocks (i,j) where $j - i \equiv \ell \mod (d-1)$. We recall that the (i,j) block is of size $\left(\left[\frac{d+k-i}{d-1}\right] + 1\right) \times \left(\left[\frac{d+k-j}{d-1}\right] + 1\right)$ and so has either one or two rows, the latter if and only if $i \leq k+1$, and similarly for the columns. A calculation shows that the (i', j') entry of block (i, j) is equal to $p_{i-1-j'+i'd}$ if $j \geq i$ and $p_{i-j'+i'd}$ if $j < i$.

First, suppose block (i, j) has one row (hence $i > k+1$). Then the j' - column sum is either $p_{i-1-j'}$ (when $j \geq i$) or $p_{i-j'}$ (when $j < i$). If there is also only one column (equivalently, $j > k+1$), then

$$\left\|\widetilde{T}(\ell)(i,j)\right\|_{\min} \geq \begin{cases} p_{i-1} & \text{if } k+1 < i \leq (d+k)/2 \\ p_i & \text{if } i > (d+k)/2 \end{cases}.$$

If, instead, the block has two columns (when $j \leq k+1$ and therefore $j < i$), we obtain the same conclusion for the minimum column sum.

Otherwise, block (i, j) has two rows, (i.e., $i \leq k+1 \leq (d+k)/2$). Similar reasoning shows that then $\left\|\widetilde{T}(\ell)(i,j)\right\|_{\min} \geq \min(p_{i-2}, 2^{-(d+k)})$. Thus

$$\begin{aligned}\prod_{(i,j) \text{ non-zero blocks}} \left\|\widetilde{T}(\ell)(i,j)\right\|_{\min} &\geq \prod_{i=2}^{k+1} p_{i-2} \prod_{i=k+2}^{\left[\frac{d+k}{2}\right]} p_{i-1} \prod_{i>\left[\frac{d+k}{2}\right]}^{d-1} p_i \\ &\geq \frac{1}{p_k} \prod_{i=0}^{\left[\frac{d+k}{2}\right]-1} p_i \prod_{i>\left[\frac{d+k}{2}\right]}^{d-1} p_i.\end{aligned}$$

Let $t = k+2$. For large enough d, $p_k \leq p_{k+1}$ and $\binom{d+k}{t} \geq (d+k)^{t-1}$. Of course, $\binom{d+k}{s} \geq \binom{d+k}{t}$ for $t \leq s \leq \left[\frac{d+k}{2}\right]$ and for $s > \left[\frac{d+k}{2}\right]$, $p_s = p_{d+k-s}$. Applying these facts, it follows that

$$\begin{aligned}\prod_{(i,j)} \left\|\widetilde{T}(\ell)(i,j)\right\|_{\min} &\geq 2^{-(d+k)(d-1)} \prod_{i=k+3}^{\left[\frac{d+k}{2}\right]-1} \binom{d+k}{i} \prod_{i>\left[\frac{d+k}{2}\right]}^{d-3} \binom{d+k}{i} \\ &\geq 2^{-(d+k)(d-1)} (d+k)^{t(d-6-k)}.\end{aligned}$$

Thus for sufficiently large d,

$$\left(\prod_{(i,j)} \left\|\widetilde{T}(\ell)(i,j)\right\|_{\min}\right)^{1/(d-1)} \geq 2^{-(d+k)} (d+k)^{3(k+2)/4}.$$

It follows from Proposition 4.15 and (7) that for large enough d,

$$\begin{aligned}\sup_x \dim_{loc} \nu_\pi(x) &\leq \frac{\log 2^{-(d+k)} (d+k)^{3(k+2)/4}}{\log 1/d} \\ &= \frac{\log 2^{d+k} - \log(d+k)^{3(k+2)/4}}{\log d} \\ &< \sup_{x \neq 0, d+k} \dim_{loc} \nu(x).\end{aligned}$$

$\square$

REMARK 4.17. We conjecture that when $3 \leq d \leq m$, the set of local dimensions of the quotient of the m-fold convolution of a uniform Cantor measure with contraction factor $1/d$ is a proper subset of the set of local dimensions at essential points of the pre-quotient measure. We have checked this numerically for all $3 \leq d \leq m \leq 10$. See [10, Table S1]. Numerical evidence suggests this is also true for $m = d - 1$ for $d \geq 4$. It is known by Theorem 3.12 that the two sets of local dimensions will be equal for $m < d - 1$.

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