Asian J. Math.
Vol. 23, No. 3, pp. 455–478, June 2019

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ORDINARY AND ALMOST ORDINARY PRYM VARIETIES*

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Abstract. We study the p-rank stratification of the moduli space of Prym varieties in characteristic $p > 0$. For arbitrary primes p and ℓ with $\ell \neq p$ and integers $g \geq 3$ and $0 \leq f \leq g$, the first theorem generalizes a result of Nakajima by proving that the Prym varieties of all the unramified $\mathbb{Z}/\ell$-covers of a generic curve X of genus g and p-rank f are ordinary. Furthermore, when $p \geq 5$ and $\ell = 2$, the second theorem implies that there exists a curve of genus g and p-rank f having an unramified double cover whose Prym has p-rank f' for each $\frac{g}{2} - 1 \leq f' \leq g - 2$; (these Pryms are not ordinary). Using work of Raynaud, we use these two theorems to prove results about the (non)-intersection of the ℓ-torsion group scheme with the theta divisor of the Jacobian of a generic curve X of genus g and p-rank f.

Key words. Prym, curve, abelian variety, Jacobian, p-rank, theta divisor, torsion point, moduli space.

Mathematics Subject Classification. Primary 11G10, 14H10, 14H30, 14H40, 14K25; Secondary 11G20, 11M38, 14H42, 14K10, 14K15.

1. Introduction. Suppose X is a smooth projective connected curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$. Suppose $\pi : Y \to X$ is an unramified cyclic cover of degree ℓ for some prime $\ell \neq p$. Then Y has genus $g_Y = \ell(g-1)+1$ by the Riemann-Hurwitz formula. For each of the $\ell^{2g} - 1$ unramified $\mathbb{Z}/\ell$-covers $\pi : Y \to X$, the Jacobian J_Y is isogenous to $J_X \oplus P_\pi$ for an abelian variety P_π of dimension $(\ell-1)(g-1)$, called the *Prym variety* of π. In particular, when $\ell = 2$ and $\pi : Y \to X$ is an unramified double cover, then Y has genus $2g-1$ and P_π is a principally polarized abelian variety of dimension $g-1$.

In this paper, we study the relationship between the p-ranks of J_X and P_π. The p-rank f_A of an abelian variety A/k of dimension g_A is the integer $0 \leq f_A \leq g_A$ such that the number of p-torsion points in $A(k)$ is p^{f_A}. One says that A is *ordinary* if its p-rank is as large as possible ($f_A = g_A$) and is *almost ordinary* if its p-rank equals $g_A - 1$.

Consider the moduli space $\mathcal{M}_g$ whose points represent smooth curves X of genus g and the moduli space $\mathcal{R}_{g,\ell}$ whose points represent unramified $\mathbb{Z}/\ell$-covers $\pi : Y \to X$. There is a finite flat morphism of degree $\ell^{2g} - 1$, denoted

$$\Pi_\ell : \mathcal{R}_{g,\ell} \to \mathcal{M}_g,$$

which takes the point representing a cover $\pi : Y \to X$ to the point representing the curve X [9, Page 6]. For $0 \leq f \leq g$, let $\mathcal{M}_g^f$ denote the p-rank f stratum of $\mathcal{M}_g$. For most g and f, it is not known whether $\mathcal{M}_g^f$ is irreducible; however, every component of $\mathcal{M}_g^f$ has dimension $2g - 3 + f$ [10, Theorem 2.3].

By a result of Nakajima, the Prym varieties of the unramified $\mathbb{Z}/\ell$-covers of the generic curve X/k of genus $g \geq 2$ are ordinary [16, Theorem 2]. In other words, the cover represented by the generic point of $\mathcal{R}_{g,\ell}$ has an ordinary Prym.

*Received December 1, 2016; accepted for publication February 9, 2018.

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The first theorem in the paper generalizes Nakajima's result by adding a condition on the p-rank of X. Specifically, if X is a generic k-curve of genus g and p-rank f, then Theorem 1.1 implies that the Prym varieties of all of the unramified $\mathbb{Z}/\ell$-covers of X are ordinary.

Raynaud used the theta divisor Θ_X in J_X to study the p-rank of Prym varieties [19, 20, 21]. (See Section 8 for the definition of Θ_X.) Using Raynaud's work, our theorems yield new results about the (non)-existence of points of order ℓ contained in Θ_X.

THEOREM 1.1. *Let $\ell \neq p$ be prime. Let $g \geq 2$ and $0 \leq f \leq g$ with $f \neq 0$ if $g = 2$. Let S be an irreducible component of $\mathcal{M}_g^f$.*

(1) *(See Theorem 4.5) Then $\Pi_\ell^{-1}(S)$ is irreducible (of dimension $2g-3+f$) and the cover represented by the generic point of $\Pi_\ell^{-1}(S)$ has an ordinary Prym.*
(2) *(See Theorem 8.3) If X is the curve represented by the generic point of S, then the theta divisor Θ_X of the Jacobian of X does not contain any point of order ℓ.*

The second theorem demonstrates the existence of unramified double covers $\pi : Y \to X$ such that the Prym P_π is almost ordinary (with p-rank $f' = g-2$).

THEOREM 1.2. *Let $\ell = 2$. Let $g \geq 2$ and $0 \leq f \leq g$ (with $f \geq 2$ when $p = 3$). Let S be an irreducible component of $\mathcal{M}_g^f$.*

(1) *(See Theorem 7.1) The locus of points of $\Pi_2^{-1}(S)$ representing covers whose Prym P_π is almost ordinary is non-empty with codimension 1 in $\Pi_2^{-1}(S)$.*
(2) *(See Theorem 8.4) The locus of points of S representing curves X for which Θ_X contains a point of order 2 is non-empty with codimension 1 in S.*

As an application of Theorem 1.2, we prove:

APPLICATION 1.3 (See Corollary 7.3). *Let $\ell = 2$ and $p \geq 5$. Let $g \geq 2$ and $0 \leq f \leq g$. Then there exists a smooth curve $X/\bar{\mathbb{F}}_p$ of genus g and p-rank f having an unramified double cover $\pi : Y \to X$ for which the Prym has p-rank f' for each $\frac{g}{2} - 1 \leq f' \leq g-1$.*

Here is an outline of the paper. Section 2 contains background about Prym varieties and the p-rank stratification of $\mathcal{M}_g$. In Section 3, we analyze the p-ranks of Pryms of covers of singular curves and the p-rank stratification of the boundary of $\mathcal{R}_{g,\ell}$.

Section 4 contains the proof of Theorem 1.1(1). The proof mirrors Nakajima's technique of degeneration to the boundary of $\mathcal{R}_{g,\ell}$; the argument is more complicated, however, because $\mathcal{M}_g^f$ may not be irreducible. To avoid this difficulty, for each irreducible component S of $\mathcal{M}_g^f$, we consider the $\mathbb{Z}/\ell$-monodromy of the tautological curve $X \to S$, namely the image of the fundamental group $\pi_1(S, s_0)$ in $\mathrm{Aut}(\mathrm{Pic}^0(X)[\ell]_{s_0})$. A key point is that the $\mathbb{Z}/\ell$-monodromy of $X \to S$ is as large as possible, namely $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ [1, Theorem 4.5]. We use this to prove that $\Pi_\ell^{-1}(S)$ is irreducible and that it degenerates to a particular boundary component $\Delta_{i:g-i}$.

In Sections 5-7, we restrict to the case $\ell = 2$. In Section 5, we stratify $\mathcal{R}_{g,2}$ by (f, f') where f (resp. f') is the p-rank of X (resp. P_π). Using purity, we prove that the dimension of each component of the (f, f') stratum of $\mathcal{R}_{g,2}$ is at least $g-2+f+f'$ Proposition 5.2. Section 6 contains results about non-ordinary Pryms in the low genus cases $g = 2, 3$, generalizing [10, Theorem 6.1].

Section 7 contains the proof of Theorem 1.2(1). The proof is again an inductive argument which uses the boundary component $\Delta_{i:g-i}$, but it relies on more refined information from each of Sections 3, 4, 5, and 6.

Application 1.3 (Corollary 7.3) follows from Theorems 1.1 and 1.2 using a 'straight-forward' deformation argument for $f' \leq g-3$: Suppose π_s is an unramified double cover of a *singular* curve of genus g and p-rank f for which the Prym has p-rank f'. Applying [1, Section 3], one can deform π_s to an unramified double cover of a *smooth* curve of genus g whose p-rank is still f. However, it is possible that the p-rank f' of the Prym increases in this deformation. In fact, there are situations where this is guaranteed to happen, see Remark 6.3. Under the hypotheses of Corollary 7.3, we construct a deformation of π_s for which the p-rank of the Prym remains constant. We emphasize that the technique used in Theorems 1.1 - 1.2 is stronger than the straight-forward approach and gives more information about the p-rank stratification of $\mathcal{R}_{g,\ell}$.

Section 8 contains the definition of the theta divisor Θ_X and the proofs of Theorems 1.1(2) and 1.2(2). We then compare our results with those of Raynaud [19, 20] and Pop/Saidi [18]. Briefly, Raynaud's results are stronger in that they apply to an arbitrary base curve X but are weaker in other ways: his result about ordinary Pryms applies only when $\ell > (p-1)3^{g-1}g!$, in which case he shows that at least one of the Pryms is ordinary; and, in his result for non-ordinary Pryms, the Galois group of the cover is solvable but not cyclic and the p-rank of the Prym is not determined. In [18, Proposition 2.3], the result is stronger in that it applies to an arbitrary curve which is either non-ordinary or whose Jacobian is absolutely simple, but is weaker in that the degree of the cyclic unramified cover and the p-rank of the Prym are not determined.

Section 9 contains some open questions. For example, in Section 9.1, we illustrate the difficulty in proving these results computationally, even for $g = 2$ and a fixed small prime p. See [4] for results about p-ranks of *ramified* cyclic covers of curves.

2. Prym varieties and p-rank stratifications.

Suppose X is a smooth projective curve of genus g defined over k. The Jacobian J_X of X is a principally polarized abelian variety of dimension g. A $\mathbb{Z}/\ell$-cover $\pi : Y \to X$ is a Galois cover together with an isomorphism $\iota : \mathrm{Gal}(Y/X) \to \mathbb{Z}/\ell$. For a prime $\ell \neq p$, there is a bijection between points of order ℓ on J_X and unramified connected $\mathbb{Z}/\ell$-covers $\pi : Y \to X$.

2.1. Prym varieties.

Suppose $\pi : Y \to X$ is an unramified $\mathbb{Z}/\ell$-cover. The *Prym variety* P_π is the connected component containing 0 of the norm map on Jacobians. More precisely, if σ is the endomorphism of J_Y induced by a generator σ of $\mathrm{Gal}(Y/X)$, then

$$P_\pi = \mathrm{Im}(1-\sigma) = \ker(1+\sigma+\ldots+\sigma^{\ell-1})^0.$$

The canonical principal polarization of J_Y induces a polarization on P_π [15, Page 6]. This polarization is principal when $\ell = 2$.

2.2. Moduli spaces of unramified cyclic covers.

Let $\mathcal{R}_{g,\ell}$ denote the moduli space whose points represent unramified $\mathbb{Z}/\ell$-covers of smooth projective curves of genus g; it is a smooth Deligne-Mumford stack [8, Page 5].

The points of $\mathcal{R}_{g,\ell}$ can also represent triples (X, η, ϕ) where X is a smooth genus g curve equipped with a line bundle $\eta \in \mathrm{Pic}(X)$ and an isomorphism $\phi : \eta^{\otimes \ell} \xrightarrow{\sim} \mathcal{O}_X$. This is because the data of the $\mathbb{Z}/\ell$-cover $\pi : Y \to X$ is equivalent to the data (X, η, ϕ).

The morphism $\Pi_\ell : \mathcal{R}_{g,\ell} \to \mathcal{M}_g$, which sends the point representing $\pi : Y \to X$ to the point representing X, is surjective, étale, and finite of degree $\ell^{2g} - 1$. Thus

$\dim(\mathcal{R}_{g,\ell}) = 3g - 3$.

2.3. Marked covers. A point of $\mathcal{M}_{g;1}$ represents a smooth curve X of genus g together with a marking, namely the choice of a point $x \in X$. A point of $\mathcal{R}_{g,\ell;1}$ represents an unramified $\mathbb{Z}/\ell$-cover $\pi : Y \to X$, where X is a smooth curve of genus g, together with a marking $x' \mapsto x$, namely the choice of a point $x' \in \pi^{-1}(x)$. The marking $x' \mapsto x$ determines a labeling of the ℓ points of the fiber $\pi^{-1}(x)$ because of the $\mathbb{Z}/\ell$-action. There are forgetful maps $\psi_M : \mathcal{M}_{g;1} \to \mathcal{M}_g$ and $\psi_R : \mathcal{R}_{g,\ell;1} \to \mathcal{R}_{g,\ell}$.

LEMMA 2.1. *If $S \subset \mathcal{M}_g$ is irreducible, then $\psi_M^{-1}(S)$ is irreducible in $\mathcal{M}_{g;1}$. If $Q \subset \mathcal{R}_{g,\ell}$ is irreducible, then $\psi_R^{-1}(Q)$ is irreducible in $\mathcal{R}_{g,\ell;1}$.*

Proof. The fiber of ψ_M above the point of $\mathcal{M}_g$ representing X is isomorphic to X and is thus irreducible. The fiber of ψ_R above the point of $\mathcal{R}_{g,\ell}$ representing $\pi : Y \to X$ is isomorphic to Y and is thus irreducible. The result follows from Zariski's theorem. □

2.4. The p-rank. Let μ_p be the kernel of Frobenius on $\mathbb{G}_m$. The p-rank of a semi-abelian variety A' is $f_{A'} = \dim_{\mathbb{F}_p} \mathrm{Hom}(\mu_p, A')$. If A' is an extension of an abelian variety A by a torus T, then $f_{A'} = f_A + \mathrm{rank}(T)$.

For an abelian variety A, the p-rank can also be defined as the integer f_A such that the number of p-torsion points in $A(k)$ is p^{f_A}. If A has dimension g_A then $0 \le f_A \le g_A$. The p-rank is invariant under isogeny $\sim$ of abelian varieties.

The p-rank of a stable curve X is that of $\mathrm{Pic}^0(X)$. There are three p-ranks associated with an unramified cover $\pi : Y \to X$, namely the p-rank f of X, the p-rank f' of P_π, and the p-rank of Y which equals $f + f'$.

2.5. The p-rank stratification. Suppose A/S is a semi-abelian scheme over a Deligne-Mumford stack in characteristic p. There is a stratification $S = \cup S^f$ by locally closed reduced substacks such that $s \in S^f(k)$ if and only if $f(A_s) = f$ ([14, Theorem 2.3.1], see also [1, Lemma 2.1]). Similarly, $\mathcal{M}_g^f$ denotes the locally closed reduced substack of $\mathcal{M}_g$ whose points represent smooth curves of genus g with p-rank f.

In studying the p-rank stratification, one can work over a scheme S after pulling back to an étale atlas. The p-rank is a lower semicontinuous function, which means that it can only decrease under specialization. The following purity result for the p-rank states that if the p-rank does change, then it does so in codimension one.

LEMMA 2.2 (Purity for p-rank) [17, Lemma 1.6] (see also [10, Lemma 2.1]). *Let A/S be a semi-abelian variety over an integral scheme in characteristic p, and suppose that A has generic p-rank f. Let $S^{<f} \subset S$ be the locus parametrizing the $s \in S$ such that A_s has p-rank strictly less than f. Then $S^{<f}$ is either empty or is pure of codimension one in S.*

2.6. Compactification of $\mathcal{M}_g$. Suppose X is a stable curve with irreducible components C_i, for $1 \le i \le s$. Let $\tilde{C}_i$ be the normalization of C_i. By [3, Example 8, Page 246], J_X is canonically an extension of an abelian variety by a torus T. There is a short exact sequence:

$$1 \to T \to J_X \to \oplus_{i=1}^{s} J_{\tilde{C}_i} \to 1. \tag{1}$$

The rank r_T of T is the rank of the cohomology group $H^1(\Gamma_X, \mathbb{Z})$, where Γ_X denotes the dual graph of X. One says that X has *compact type* if T is trivial.

Let $\bar{\mathcal{M}}_g$ be the Deligne-Mumford compactification of $\mathcal{M}_g$; it is a smooth proper Deligne-Mumford stack. The boundary $\partial\mathcal{M}_g = \bar{\mathcal{M}}_g - \mathcal{M}_g$ is the union of the components $\Delta_0[\bar{\mathcal{M}}_g]$ and $\Delta_i[\bar{\mathcal{M}}_g]$ for $1 \le i \le \lfloor g/2 \rfloor$ defined as in [1, Section 2.3].

For $1 \le i \le \lfloor g/2 \rfloor$, the p-rank f stratum of the boundary component $\Delta_i[\bar{\mathcal{M}}_g]$ is the union of the images of the clutching morphisms:

$$\kappa_i : \bar{\mathcal{M}}_{i;1}^{f_1} \times \bar{\mathcal{M}}_{g-i;1}^{f_2} \to \Delta_i[\bar{\mathcal{M}}_g^{f_1+f_2}], \tag{2}$$

for all pairs $\{f_1, f_2\}$ of non-negative integers such that $f_1 + f_2 = f$. The p-rank f stratum of the boundary component $\Delta_0[\bar{\mathcal{M}}_g]$ is the image of the clutching morphism:

$$\kappa_0 : \bar{\mathcal{M}}_{g-1;2}^{f-1} \to \Delta_0[\bar{\mathcal{M}}_g^f]. \tag{3}$$

3. The p-rank stratification of the boundary of $\mathcal{R}_{g,\ell}$. In this section, we study Pryms of unramified covers of singular curves. Then we analyze the p-rank stratification of the boundary of $\mathcal{R}_{g,\ell}$, whose points typically represent unramified $\mathbb{Z}/\ell$-covers of singular curves. Lemma 3.3 states that every component of the boundary of $\Pi_\ell^{-1}(\mathcal{M}_g^f)$ has dimension $2g - 4 + f$. This is used for Propositions 4.4 and 6.4. We compute the p-rank of the Prym of the cover represented by the generic point of each boundary strata; in particular, Lemma 3.7 is used in Sections 4, 6, and 7.

This section relies on structural results from [7] and [8]. It is necessary to include some material from these references. The following lemma will also be useful.

LEMMA 3.1 ([24, page 614]). *If A and B are substacks of a smooth proper stack S then*

$$\operatorname{codim}(A \cap B, S) \le \operatorname{codim}(A, S) + \operatorname{codim}(B, S).$$

3.1. Compactification of $\mathcal{R}_{g,\ell}$. By [8, Definition 1.2], a twisted curve $\mathtt{C}$ is a one dimensional stack such that the corresponding coarse moduli space C is a stable curve whose smooth locus is represented by a scheme and whose singularities are nodes with local picture $[\{xy = 0\}/\mu_r]$ with $\zeta \in \mu_r$ acting as $\zeta(x, y) = (\zeta x, \zeta^{-1} y)$. The definition of a faithful line bundle $\eta \in \mathrm{Pic}(\mathtt{C})$ is in [8, Definition 1.3]. By [8, Definition 1.5], a level-ℓ twisted curve of genus g is a triple $[\mathtt{X}, \eta, \phi]$ where $\mathtt{X}$ is a twisted curve of genus g, $\eta \in \mathrm{Pic}(\mathtt{X})$ is a faithful line bundle, and $\phi : \eta^{\otimes \ell} \to \mathcal{O}_{\mathtt{X}}$ is an isomorphism.

By [8, page 6], the moduli space $\mathcal{R}_{g,\ell}$ admits a compactification $\bar{\mathcal{R}}_{g,\ell}$ whose points represent level-ℓ twisted curves of genus g. It is a smooth Deligne-Mumford stack and there is a finite forgetful morphism $\Pi_\ell : \bar{\mathcal{R}}_{g,\ell} \to \bar{\mathcal{M}}_g$.

Let $\partial\mathcal{R}_{g,\ell} = \bar{\mathcal{R}}_{g,\ell} - \mathcal{R}_{g,\ell}$. Some points of $\partial\mathcal{R}_{g,\ell}$ cannot be interpreted in terms of ℓ-torsion line bundles or $\mathbb{Z}/\ell$-covers of a scheme-theoretic curve. For the sake of intuition, whenever possible, we describe the generic point of a boundary component of $\bar{\mathcal{R}}_{g,\ell}$ in terms of the cover $\pi : Y \to X$ it represents.

3.2. Definition of W_g^f.

DEFINITION 3.2. For $0 \le f \le g$, define $W_g^f = \Pi_\ell^{-1}(\mathcal{M}_g^f)$ and $\bar{W}_g^f = \Pi_\ell^{-1}(\bar{\mathcal{M}}_g^f)$.

The points of W_g^f represent unramified $\mathbb{Z}/\ell$-covers $\pi : Y \to X$ of a smooth curve X of genus g and p-rank f.

LEMMA 3.3. *Let $g \ge 1$ and $0 \le f \le g$. Then W_g^f is non-empty. For $g \ge 2$, let Q be an irreducible component of $\bar{W}_g^f$. Then*

(1) *Q has dimension $2g-3+f$;*
(2) *$W_g^f \cap Q$ is open and dense in Q (the generic point of Q represents a smooth curve);*
(3) *the dimension of every component of $Q \cap \partial\mathcal{R}_{g,\ell}$ is $2g-4+f$.*

Proof. Since $\Pi_\ell : \bar{\mathcal{R}}_{g,\ell} \to \bar{\mathcal{M}}_g$ is finite, flat, and surjective, these facts follow from the analogous facts for $\bar{\mathcal{M}}_g^f$; see [10, Theorem 2.3] for part (1) and [1, Lemmas 3.1, 3.2(a)] for parts (2) - (3). □

3.3. Boundary components of $\bar{\mathcal{R}}_{g,\ell}$. Let $\partial\mathcal{R}_{g,\ell} = \bar{\mathcal{R}}_{g,\ell} - \mathcal{R}_{g,\ell}$ denote the boundary of $\mathcal{R}_{g,\ell}$. Informally, the points of $\partial\mathcal{R}_{g,\ell}$ represent unramified $\mathbb{Z}/\ell$-covers of singular curves, although we make this more precise below. For covers of singular curves of compact type, the boundary components lie above $\Delta_i[\bar{\mathcal{M}}_g]$ for some $1 \le i \le \lfloor g/2 \rfloor$ and are denoted $\Delta_{i:g-i}$, Δ_i, and Δ_{g-i}. For covers of singular curves of non-compact type, the boundary components lie above $\Delta_0[\bar{\mathcal{M}}_g]$ and are denoted $\Delta_{0,I}$, $\Delta_{0,II}$ and $\Delta_{0,III}^{(a)}$.

In Sections 3.4 and 3.5, we recall the definition of these boundary components and investigate them in terms of the p-rank. Before doing this, recall the following results.

Proposition 3.4 ([7, Equation (16)]). *For $1 \le i < \lfloor g/2 \rfloor$, there is an equality of divisors*

$$\Pi_\ell^*(\Delta_i[\bar{\mathcal{M}}_g]) = \Delta_i[\bar{\mathcal{R}}_{g,\ell}] + \Delta_{g-i}[\bar{\mathcal{R}}_{g,\ell}] + \Delta_{i:g-i}[\bar{\mathcal{R}}_{g,\ell}].$$

If g is even, there is an equality of divisors

$$\Pi_\ell^*(\Delta_{g/2}[\bar{\mathcal{M}}_g]) = \Delta_{g/2}[\bar{\mathcal{R}}_{g,\ell}] + \Delta_{g/2:g/2}[\bar{\mathcal{R}}_{g,\ell}].$$

Proposition 3.5 ([7, page 14] or [8, Equation (17)]). *There is an equality of divisors*

$$\Pi_\ell^*(\Delta_0[\bar{\mathcal{M}}_g]) = \Delta_{0,I}[\bar{\mathcal{R}}_{g,\ell}] + \Delta_{0,II}[\bar{\mathcal{R}}_{g,\ell}] + \ell \sum_{a=0}^{\lfloor \ell/2 \rfloor} \Delta_{0,III}^{(a)}[\bar{\mathcal{R}}_{g,\ell}].$$

3.4. Pryms of covers of singular curves of compact type. Let X be a singular curve formed by intersecting two curves C_1 and C_2 (at points $x_1 \in C_1$ and $x_2 \in C_2$) in an ordinary double point. By (1), $J_X \backsimeq J_{C_1} \oplus J_{C_2}$. Let i be the genus of C_1 and $g-i$ be the genus of C_2. Let ξ be the point of $\Delta_i[\bar{\mathcal{M}}_g]$ representing X. Then an unramified cyclic degree ℓ cover $\pi : Y \to X$ is determined by two line bundles $\eta_{C_1} \in \mathrm{Pic}^0(C_1)[\ell]$ and $\eta_{C_2} \in \mathrm{Pic}^0(C_2)[\ell]$, which are not both trivial. The points of $\Delta_{i:g-i}[\bar{\mathcal{R}}_{g,\ell}]$ above ξ represent covers π for which both η_{C_1} and η_{C_2} are nontrivial; the points of $\Delta_i[\bar{\mathcal{R}}_{g,\ell}]$ (resp. $\Delta_{g-i}[\bar{\mathcal{R}}_{g,\ell}]$) above ξ represent covers π for which η_{C_1} (resp. η_{C_2}) is trivial.

3.4.1. The boundary component $\Delta_{i:g-i}$. The boundary divisor $\Delta_{i:g-i}[\bar{\mathcal{R}}_{g,\ell}]$ is the image of the clutching map

$$\kappa_{i:g-i} : \bar{\mathcal{R}}_{i,\ell;1} \times \bar{\mathcal{R}}_{g-i,\ell;1} \to \bar{\mathcal{R}}_{g,\ell},$$

defined on a generic point as follows. Let τ_1 be a point of $\bar{\mathcal{R}}_{i,\ell;1}$ representing $(\pi_1 : C_1' \to C_1, x_1' \mapsto x_1)$ and let τ_2 be a point of $\bar{\mathcal{R}}_{g-i,\ell;1}$ representing $(\pi_2 : C_2' \to C_2, x_2' \mapsto x_2)$.

Let Y be the curve with components C_1' and C_2', formed by identifying $\sigma^k(x_1')$ and $\sigma^k(x_2')$ in an ordinary double point for $0 \le k \le \ell - 1$. Then $\kappa_{i:g-i}(\tau_1, \tau_2)$ is the point representing the unramified $\mathbb{Z}/\ell$-cover $Y \to X$. This is illustrated in Figure 1 for $\ell = 2$.

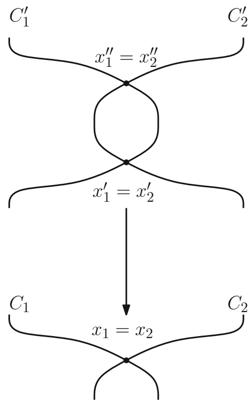

FIG. 1. $\Delta_{i:g-i} : \eta_{C_1} \not\cong \mathcal{O}_{C_1},\ \eta_{C_2} \not\cong \mathcal{O}_{C_2}$

LEMMA 3.6. *The clutching map $\kappa_{i:g-i}$ restricts to a map*

$$\kappa_{i:g-i} : \bar{W}^{f_1}_{i;1} \times \bar{W}^{f_2}_{g-i;1} \to \Delta_{i:g-i}[\bar{W}^{f_1+f_2}_{g}].$$

Proof. This follows from (2). □

LEMMA 3.7. *Suppose $\pi : Y \to X$ is an unramified $\mathbb{Z}/\ell$-cover represented by a point of $\Delta_{i:g-i}[\bar{\mathcal{R}}_{g,\ell}]$. Then P_π is an extension of a semi-abelian variety $P_{\pi_1} \oplus P_{\pi_2}$ by a torus T of rank $r_T = \ell - 1$. If f_i' is the p-rank of P_{π_i}, then the p-rank of P_π is $f_1' + f_2' + (\ell - 1)$.*

Proof. By (1), J_Y is an extension of $J_{C_1'} \oplus J_{C_2'}$ by a torus T whose rank r_T is the rank of $H^1(\Gamma_Y, \mathbb{Z})$. Then $r_T = \ell - 1$ since Γ_Y consists of two vertices, for the two irreducible components C_1', C_2', which are connected with ℓ edges, corresponding to the ℓ intersection points. There is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & 1 & \longrightarrow & J_X & \longrightarrow & J_{C_1} \oplus J_{C_2} & \longrightarrow & 1 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 1 & \longrightarrow & T & \longrightarrow & J_Y & \longrightarrow & J_{C_1'} \oplus J_{C_2'} & \longrightarrow & 1. \end{array}$$

By the snake lemma, there is an exact sequence

$$1 \to \mathrm{Coker}(a) \to \mathrm{Coker}(b) \to \mathrm{Coker}(c) \to 1,$$

and thus an exact sequence

$$1 \to T \to P_\pi \to P_{\pi_1} \oplus P_{\pi_2} \to 1. \tag{4}$$

From (4), $f_{P_\pi} = f_{P_{\pi_1} \oplus P_{\pi_2}} + \mathrm{rank}(T) = f_1' + f_2' + (\ell - 1)$. □

3.4.2. The boundary component $\Delta_i, i > 0$:. The boundary divisor $\Delta_i[\bar{\mathcal{R}}_g]$ is the image of the clutching map

$$\kappa_i : \bar{\mathcal{R}}_{i,\ell;1} \times \bar{\mathcal{M}}_{g-i;1} \to \bar{\mathcal{R}}_{g,\ell},$$

defined on a generic point as follows. Let τ be a point of $\mathcal{R}_{i,\ell;1}$ representing $(\pi'_1 : C'_1 \to C_1, x' \mapsto x)$ and let ω be a point of $\mathcal{M}_{g-i;1}$ representing (C_2, x_2). Let Y be the curve with components C'_1 and ℓ copies of C_2, formed by identifying the point x_2 on each copy of C_2 with a point of $\pi_1^{-1}(x_1)$. Then $\kappa_i(\tau, \omega)$ represents the unramified $\mathbb{Z}/\ell$-cover $\pi : Y \to X$. See Figure 2 for $\ell = 2$.

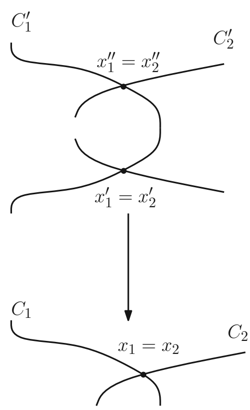

FIG. 2. $\Delta_i : \eta_{C_1} \not\simeq \mathcal{O}_{C_1}, \eta_{C_2} \simeq \mathcal{O}_{C_2}$

LEMMA 3.8. *The clutching map κ_i restricts to a map:*

$$\kappa_i : \bar{W}^{f_1}_{i;1} \times \bar{\mathcal{M}}^{f_2}_{g-i;1} \to \Delta_i[\bar{W}^{f_1+f_2}_g].$$

Proof. This follows from (2). □

LEMMA 3.9. *Suppose $\pi : Y \to X$ is an unramified $\mathbb{Z}/\ell$-cover represented by a point of $\Delta_i[\bar{\mathcal{R}}_{g,\ell}]$. Then $P_\pi \simeq P_{\pi_1} \oplus J^{\ell-1}_{C_2}$. If f'_1 is the p-rank of P_{π_1}, then the p-rank of P_π is $f'_1 + (\ell - 1)f_2$.*

Proof. By construction, $J_Y \backsimeq J_{C'_1} \oplus J^{\ell}_{C_2}$. The image of $(1-\sigma)$ on J_Y is P_π, while on $J_{C'_1} \oplus J^{\ell}_{C_2}$ it is $P_{\pi_1} \oplus J^{\ell-1}_{C_2}$. Then the p-rank is the sum of the p-ranks of P_{π_1} and $J^{\ell-1}_{C_2}$. □

3.5. Pryms of covers of singular curves of non-compact type. This material is needed only for future work. The main reference is [11, Example 6.5] when $\ell = 2$ and [7, Section 1.4] and [8, Section 1.5.2] for general ℓ. Let (X', x, y) be a curve of genus $g - 1$ with 2 marked points. Let X be a curve of genus g of non-compact type formed by identifying two points x, y on X'. By (1), if X' has p-rank f_1, then X has p-rank $f = f_1 + 1$.

3.5.1. The Boundary Component $\Delta_{0,I}$. The boundary divisor $\Delta_{0,I}[\bar{\mathcal{R}}_{g,\ell}]$ is the image of the clutching map

$$\kappa_{0,I} : \bar{\mathcal{R}}_{g-1,\ell;2} \to \bar{\mathcal{R}}_{g,\ell},$$

defined on a generic point as follows. Let τ be a point of $\mathcal{R}_{g-1,\ell;2}$ representing $(\pi' : Y' \to X', x' \mapsto x, y' \mapsto y)$ (two markings). Let Y be the nodal curve of non-compact type with normalization Y', formed by identifying $\sigma^k(x')$ and $\sigma^k(y')$, for $0 \leq k \leq \ell - 1$, in an ordinary double point. Then $\kappa_{0,I}(\tau)$ is the point representing the unramified $\mathbb{Z}/\ell$-cover $\pi : Y \to X$. See Figure 3 for the case $\ell = 2$.

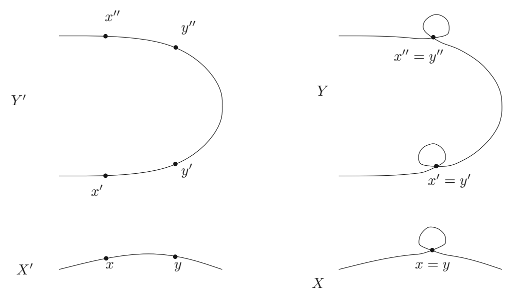

FIG. 3. $\Delta_{0,I}, \ell = 2$

LEMMA 3.10. *The clutching map $\kappa_{0,I}$ restricts to a map: $\kappa_{0,I} : \bar{W}^{f_1}_{g-1;2} \to \Delta_{0,I}[\bar{W}^{f_1+1}_g]$.*

Proof. This follows from (3). □

LEMMA 3.11. *Suppose $\pi : Y \to X$ is an unramified $\mathbb{Z}/\ell$-cover represented by a point of $\Delta_{0,I}[\bar{\mathcal{R}}_{g,\ell}]$. Then P_π is an extension of a semi-abelian variety $P_{\pi'}$ by a torus T with $r_T = \ell - 1$. If f'_1 is the p-rank of $P_{\pi'}$, then the p-rank of P_π is $f' = f'_1 + (\ell - 1)$.*

Proof. There is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T_X & \longrightarrow & J_X & \longrightarrow & J_{X'} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T_Y & \longrightarrow & J_Y & \longrightarrow & J_{Y'} & \longrightarrow & 1, \end{array}$$

where T_X is a torus of rank 1 and T_Y is a torus of rank ℓ. Then T_Y/T_X is a torus T of rank $\ell - 1$. By the snake lemma, there is an exact sequence $1 \to T \to P_\pi \to P_{\pi'} \to 1$. □

3.5.2. The Boundary Component $\Delta_{0,II}$. The boundary divisor $\Delta_{0,II}[\bar{\mathcal{R}}_{g,\ell}]$ is the image of the clutching map

$$\kappa_{0,II} : \bar{\mathcal{M}}_{g-1;2} \to \bar{\mathcal{R}}_{g,\ell},$$

defined on a generic point as follows. Let ω be a point of $\bar{\mathcal{M}}_{g-1;2}$ representing (X', x, y) (with 2 markings). Consider a disconnected curve with components (X'_i, x_i, y_i) indexed by $i \in \mathbb{Z}/\ell$ such that each component is isomorphic to (X', x, y).

Let Y be the nodal curve of non-compact type formed by identifying $\sigma^k(x_1)$ and $\sigma^{k+1}(y_1)$, for $0 \leq k \leq \ell - 1$, in an ordinary double point. Then $\kappa_{0,II}(\omega)$ is the point representing the $\mathbb{Z}/\ell$-cover $\pi : Y \to C$. This is illustrated in Figure 4 for $\ell = 2$.

LEMMA 3.12. *The clutching map $\kappa_{0,II}$ restricts to a map $\kappa_{0,II} : \bar{\mathcal{M}}^{f_1}_{g-1;2} \to \Delta_{0,II}[\bar{W}^{f_1+1}_g]$.*

Proof. This follows from (3). □

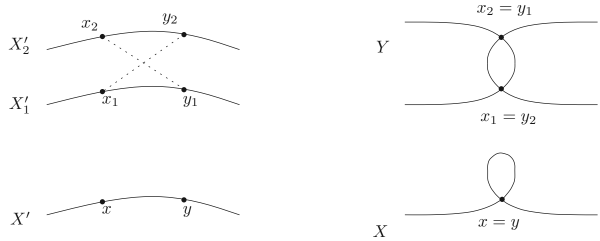

FIG. 4. $\Delta_{0,II,\ell=2}$

LEMMA 3.13. *Suppose $\pi : Y \to X$ is an unramified $\mathbb{Z}/\ell$-cover represented by a point of $\Delta_{0,II}[\bar{\mathcal{R}}_{g,\ell}]$. Then $P_\pi \simeq J_{X'}^{\ell-1}$. If f_1 is the p-rank of X', then the p-rank of P_π is $(\ell-1)f_1$.*

Proof. Like Lemma 3.11, but T_X and T_Y have rank 1. □

3.5.3. The Boundary Component $\Delta_{0,III}$. The points of the last boundary component(s) represent level-ℓ twisted curves $[X, \tilde{\eta}, \tilde{\phi}]$ which have the following structure. Let (X', x, y) be a point of $\bar{\mathcal{M}}_{g-1;2}$ and let E be a projective line. The curve $X = X' \cup_{x,y} E$ of genus g has components X' and E, with two ordinary double points formed by identifying x with 0 and y with ∞. Since E is an exceptional component, the restriction of $\tilde{\eta}$ to E is $\eta_E = \mathcal{O}_E(1)$. This implies that the restriction $\eta_{X'}$ of $\tilde{\eta}$ to X' has degree -1 and that $\eta_{X'}^{\otimes -\ell} = \mathcal{O}_{X'}(ax + (\ell - a)y)$, for some $1 \le a \le \ell - 1$.

The boundary divisor $\Delta_{0,III}^{(a)}$ is the closure in $\bar{\mathcal{R}}_{g-1,\ell}$ of points representing such level-ℓ twisted curves $[X, \tilde{\eta}, \tilde{\phi}]$. For complete details, see [7, Section 1.4]. There is a clutching map

$$\kappa_{0,III}^{(a)} : \bar{M}_{g-1;2} \to \Delta_{0,III}^{(a)}[\bar{R}_{g,\ell}].$$

LEMMA 3.14. *The clutching map $\kappa_{0,III}^{(a)}$ restricts to a map*

$$\kappa_{0,III}^{(a)} : \bar{W}_{g-1;2}^{f_1} \to \Delta_{0,III}^{(a)}[\bar{W}_g^{f_1+1}].$$

Proof. This follows from (3). □

4. Ordinary Pryms of a generic curve of given p-rank. The main result of this section is that the Prym variety of an unramified $\mathbb{Z}/\ell$-cover of a generic curve of genus g and p-rank f is ordinary, Theorem 4.5. We prove this using degeneration to $\partial\mathcal{R}_{g,\ell}$ and information about the $\mathbb{Z}/\ell$-monodromy of components of $\mathcal{M}_g^f$. The monodromy results are needed since all curves represented by a point of $\Delta_i[\bar{\mathcal{M}}_g^f]$ have an unramified $\mathbb{Z}/\ell$-cover with non-ordinary Prym, when $f < g$ and $0 \le i < g$, as seen in Sections 3.4.2 and 3.5.

4.1. Earlier work on the p-rank stratification of $\mathcal{M}_g$.

PROPOSITION 4.1 ([1, Proposition 3.4]). *Let $g \ge 2$ and $0 \le f \le g$. Suppose $1 \le i \le g-1$ and (f_1, f_2) is a pair such that $f_1 + f_2 = f$ and $0 \le f_1 \le i$ and $0 \le f_2 \le g - i$. Let S be an irreducible component of $\mathcal{M}_g^f$ and let $\bar{S}$ be its closure in $\bar{\mathcal{M}}_g$.*

(1) *Then $\bar{S}$ intersects $\kappa_{i:g-i}(\bar{\mathcal{M}}_{i;1}^{f_1} \times \bar{\mathcal{M}}_{g-i;1}^{f_2})$.*

(2) *Each irreducible component of the intersection contains the image of a component of* $\bar{\mathcal{M}}^{f_1}_{i;1} \times \bar{\mathcal{M}}^{f_2}_{g-i;1}$.

Let $\mathcal{C} \to S$ be a relative proper semi-stable curve of compact type of genus g over S. Then $\mathrm{Pic}^0(\mathcal{C})[\ell]$ is an étale cover of S with geometric fiber isomorphic to $(\mathbb{Z}/\ell)^{2g}$. For each $n \in \mathbb{N}$, the fundamental group $\pi_1(S, s)$ acts linearly on the fiber $\mathrm{Pic}^0(\mathcal{C})[\ell^n]_s$, and the monodromy group $\mathsf{M}_{\ell^n}(\mathcal{C} \to S, s)$ is the image of $\pi_1(S, s)$ in $\mathrm{Aut}(\mathrm{Pic}^0(\mathcal{C})[\ell^n]_s)$. Also $\mathsf{M}_{\mathbb{Z}_\ell}(S, s) := \varprojlim_n \mathsf{M}_{\ell^n}(S, s)$ is the ℓ-adic monodromy group. When S is an irreducible component of $\mathcal{M}^f_g$ and $\mathcal{C} \to S$ is the tautological curve, the next result states that $\mathsf{M}_\ell(S) := \mathsf{M}_\ell(\mathcal{C} \to S, s)$ and $\mathsf{M}_{\mathbb{Z}_\ell}(S) := \mathsf{M}_{\mathbb{Z}_\ell}(\mathcal{C} \to S, s)$ are as large as possible.

Theorem 4.2 ([1, Theorem 4.5]). *Let ℓ be a prime distinct from p; let $g \geq 2$ and $0 \leq f \leq g$ with $f \neq 0$ if $g = 2$. Let S be an irreducible component of $\mathcal{M}^f_g$, the p-rank f stratum in $\mathcal{M}_g$. Then $\mathsf{M}_\ell(S) \simeq \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ and $\mathsf{M}_{\mathbb{Z}_\ell}(S) \simeq \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$.*

4.2. Irreducibility of fibers of Π_ℓ over $\mathcal{M}^f_g$. Recall that the morphism $\Pi_\ell : \mathcal{R}_{g,\ell} \to \mathcal{M}_g$, which sends the point representing the cover $\pi : Y \to X$ to the point representing the curve X, is finite and flat with degree $\ell^{2g} - 1$.

Proposition 4.3. *Under the hypotheses of Theorem 4.2, if S is an irreducible component of $\mathcal{M}^f_g$, then $\Pi_\ell^{-1}(S)$ is irreducible.*

Proof. Equip $(\mathbb{Z}/\ell)^{2g}$ with the standard symplectic pairing $\langle \cdot, \cdot \rangle_{\mathrm{std}}$. The principal polarization λ on $\mathrm{Pic}^0(\mathcal{C}/S)$ induces a symplectic pairing $\langle \cdot, \cdot \rangle_\lambda$ on the ℓ-torsion $\mathrm{Pic}^0(\mathcal{C}/S)[\ell]$. Let

$$S_{[\ell]} := \mathrm{Isom}((\mathrm{Pic}^0(\mathcal{C}/S)[\ell], \langle \cdot, \cdot \rangle_\lambda), ((\mathbb{Z}/\ell)^{2g}_S, \langle \cdot, \cdot \rangle_{\mathrm{std}})).$$

There is an ℓth root of unity on S, so $S_{[\ell]} \to S$ is an étale Galois cover, possibly disconnected, with covering group $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. By Theorem 4.2, $\mathsf{M}_\ell(S) \simeq \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. The geometric interpretation of this is that $S_{[\ell]}$ is irreducible.

Suppose $\tilde{\xi}$ is a point of $S_{[\ell]}$. Then $\tilde{\xi}$ represents a curve X, together with an isomorphism between $(\mathbb{Z}/\ell)^{2g}$ and $\mathrm{Pic}^0(X)[\ell]$. The isomorphism identifies $(1, 0, \ldots, 0)$ with a point of order ℓ on J_X. It follows that $\tilde{\xi}$ determines an unramified $\mathbb{Z}/\ell$-cover $\pi : Y \to X$. Thus there is a forgetful morphism $F : S_{[\ell]} \to \Pi_\ell^{-1}(S)$. Then $\Pi_\ell^{-1}(S)$ is irreducible because $S_{[\ell]}$ is. □

4.3. Key degeneration result.

Proposition 4.4. *Let $g \geq 3$ and $0 \leq f \leq g$. Let Q be an irreducible component of $\bar{W}^f_g$.*

(1) *Then Q intersects $\Delta_{i:g-i}$ for each $1 \leq i \leq \lfloor g/2 \rfloor$.*
(2) *If (f_1, f_2) is a pair such that $f_1 + f_2 = f$ and $0 \leq f_1 \leq i$ and $0 \leq f_2 \leq g - i$, then Q contains the image of a component of $\kappa_{i;g-i}(\bar{W}^{f_1}_{i;1} \times \bar{W}^{f_2}_{g-i;1})$.*

Proof. By Proposition 4.3, $Q = \Pi_\ell^{-1}(S)$ for some irreducible component S of $\bar{\mathcal{M}}^f_g$. By Proposition 4.1, S contains the image of a component of $\bar{\mathcal{M}}^{f_1}_{i;1} \times \bar{\mathcal{M}}^{f_2}_{g-i;1}$. Consider a point ξ of Q lying above this image. Then ξ represents an unramified $\mathbb{Z}/\ell$-cover $\pi : Y \to X$ as in Section 3.4.1. By definition, X is a stable curve having components C_1 and C_2 of genera i and $g - i$ and p-ranks f_1 and f_2.

The $\mathbb{Z}/\ell$-cover π is determined by a point of order ℓ on J_X. Now $J_X \simeq J_{C_1} \oplus J_{C_2}$, so $J_X[\ell] \simeq J_{C_1}[\ell] \oplus J_{C_2}[\ell]$. The point ξ is in Δ_i or Δ_{g-i} if and only if the point of order ℓ is in either $J_{C_1}[\ell] \oplus \{0\}$ or $\{0\} \oplus J_{C_2}[\ell]$. There are $\ell^{2g} - \ell^{2i} - \ell^{2(g-i)} + 1$ points of order ℓ which do not have this property. Since $Q = \Pi_\ell^{-1}(S)$, without loss of generality, one can suppose that the point of order ℓ is one of these or, equivalently, that ξ is in $\Delta_{i:g-i}$, completing part (1).

Every component of $Q \cap \Delta_{i:g-i}$ has dimension $2g-4+f$. By Lemma 3.3(3), this equals the dimension of the components of $\kappa_{i:g-i}(\bar{W}_{i;1}^{f_1} \times \bar{W}_{g-i;1}^{f_2})$, finishing part (2). □

4.4. Ordinary Pryms. The first theorem is that the Prym of an unramified $\mathbb{Z}/\ell$-cover of a generic curve of genus g and p-rank f is ordinary, for any $0 \leq f \leq g$.

THEOREM 4.5. *Let ℓ be a prime distinct from p; let $g \geq 2$ and $0 \leq f \leq g$ with $f \neq 0$ if $g = 2$. If Q is an irreducible component of $W_g^f = \Pi_\ell^{-1}(\mathcal{M}_g^f)$, then the Prym of the cover represented by the generic point of Q is ordinary (with p-rank $f'_Q = (\ell-1)(g-1)$).*

Proof. The proof is by induction on g, with the base case $g = 1$ being vacuous. Suppose the result is true for all $1 \leq g' < g$. Let Q be an irreducible component of W_g^f. Let $\bar{Q}$ be its closure in $\bar{\mathcal{R}}_{g,\ell}$. Choose i such that $1 \leq i \leq g-1$ and a pair (f_1, f_2) such that $f_1 + f_2 = f$ and $0 \leq f_1 \leq i$ and $0 \leq f_2 \leq g-i$. Note that one can avoid the choice $i = 2$ and $f_1 = 0$. By Proposition 4.4, $\bar{Q}$ contains a component of $\kappa_{i:g-i}(\bar{W}_{i,1}^{f_1} \times \bar{W}_{g-i,1}^{f_2})$.

Let f'_Q, $f'_{\partial Q}$, f'_1, and f'_2 respectively denote the p-rank of the Prym of the cover represented by the generic point of a component of Q, ∂Q, $\bar{W}_{i;1}^{f_1}$ and $\bar{W}_{g-i;1}^{f_2}$. By semi-continuity $f'_Q \geq f'_{\partial Q}$. By Lemma 3.7, $f'_{\partial Q} = f'_1 + f'_2 + (\ell-1)$. By the inductive hypothesis, $f'_1 = (\ell-1)(i-1)$ and $f'_2 = (\ell-1)(g-i-1)$. Thus $f'_Q \geq (\ell-1)(g-1)$ which equals $\dim(P_\pi)$. □

Theorem 1.1(1) follows from Proposition 4.3 and Theorem 4.5.

5. Purity results.

5.1. A stratification of $\bar{\mathcal{R}}_g$ by the p-ranks of X and P_π. When $\ell = 2$ and p is odd, we consider the stratification of $\mathcal{R}_{g,2}$ by p-rank. Proposition 5.2 gives a lower bound for the dimension of the p-rank strata. Since this section is only about double covers, the subscript $\ell = 2$ is dropped from the notation for simplicity.

If $\pi : Y \to X$ is an unramified double cover, let f' denote the p-rank of P_π. Let $\tilde{\mathcal{A}}_{g-1}^{f'}$ denote the p-rank f' stratum of the toroidal compactification $\tilde{\mathcal{A}}_{g-1}$ of the moduli space $\mathcal{A}_{g-1}$ of principally polarized abelian varieties of dimension $g-1$. The Prym map $Pr_g : \bar{\mathcal{R}}_g \to \tilde{\mathcal{A}}_{g-1}$ sends the point representing $\pi : Y \to X$ to the point representing the principally polarized abelian variety P_π. The image and fibers of Pr_g are well understood only for $2 \leq g \leq 6$.

DEFINITION 5.1. Let $0 \leq f \leq g$ and $0 \leq f' \leq g-1$. Define $\bar{V}_g^{f'} = Pr_g^{-1}(\tilde{\mathcal{A}}_{g-1}^{f'})$ and $V_g^{f'} = \bar{V}_g^{f'} \cap \mathcal{R}_g$. Define $\bar{\mathcal{R}}_g^{(f,f')} = \bar{W}_g^f \cap \bar{V}_g^{f'}$ and $\mathcal{R}_g^{(f,f')} = \bar{\mathcal{R}}_g^{(f,f')} \cap \mathcal{R}_g$.

Hence, the points of $V_g^{f'}$ (resp. $\mathcal{R}_g^{(f,f')}$) represent unramified double covers $\pi : Y \to X$ of a smooth curve X of genus g (resp. and p-rank f) such that P_π has p-rank f'.

By Theorem 4.5, if $p \geq 3$, then $\mathcal{R}_g^{f,g-1}$ is non-empty of dimension $2g-3+f$ for all $g \geq 2$ and $0 \leq f \leq g$ with $f \neq 0$ if $g = 2$. The following purity result follows indirectly from Lemma 2.2, which states that the p-rank can only change in codimension one.

PROPOSITION 5.2. *Let $\ell = 2$, $p \geq 3$, $g \geq 2$ and $0 \leq f \leq g$. For $0 \leq f' \leq g-2$, if $\mathcal{R}_g^{(f,f')}$ (resp. $\bar{\mathcal{R}}_g^{(f,f')}$) is non-empty, then each of its components has dimension at least $g-2+(f+f')$.*

Proof. Consider the forgetful morphism $\tau_g : \mathcal{R}_g \to \mathcal{M}_{2g-1}$ which sends the point representing $\pi : Y \to X$ to the point representing Y. If $g \geq 2$, then the genus of Y is at least 3 and $\mathrm{Aut}_k(Y)$ is finite. So τ_g is finite-to-1 and its image has dimension $3g-3$.

Let U be a component of $\mathcal{R}_g^{(f,f')}$ and let Z be its image under τ_g. If $\pi : Y \to X$ is represented by a point of U, then the p-rank of Y is $f+f'$. Thus Z is contained in $\mathcal{M}_{2g-1}^{f+f'}$. Note that Z is a component of $\mathrm{Im}(\tau_g) \cap \mathcal{M}_{2g-1}^{f+f'}$. By Lemma 3.1,

$$\mathrm{codim}(\mathrm{Im}(\tau_g) \cap \mathcal{M}_{2g-1}^{f+f'}, \mathrm{Im}(\tau_g)) \leq \mathrm{codim}(\mathcal{M}_{2g-1}^{f+f'}, \mathcal{M}_{2g-1}).$$

By [10, Theorem 2.3], $\mathrm{codim}(\mathcal{M}_{2g-1}^{f+f'}, \mathcal{M}_{2g-1}) = 2g-1-(f+f')$. Now $\dim(U) = \dim(Z)$ and so

$$\dim(U) \geq 3g-3-(2g-1-(f+f')) = g-2+f+f'.$$

The statement is also true for $\bar{\mathcal{R}}_g^{(f,f')}$ since $\mathcal{R}_g^{(f,f')}$ is open and dense in it. □

REMARK 5.3. The hypothesis in Proposition 5.2 that $\mathcal{R}_g^{(f,f')} \neq \emptyset$ is not superfluous. When $p = 3$, then $\mathcal{R}_2^{(0,0)} = \emptyset$ [10, Theorem 6.1].

REMARK 5.4. The strategy of the proof of Proposition 5.2 does not give much information for covers of degree $\ell \geq 3$ because g_Y is too big relative to $3g-3$.

5.2. Increasing the p-rank of the Prym variety. We show that geometric information about $\mathcal{R}_g^{(f,f')}$ can be used to deduce geometric information about $\mathcal{R}_g^{(f,F')}$ when $f' \leq F' \leq g-1$.

PROPOSITION 5.5. *Let $g \geq 2$. If $\mathcal{R}_g^{(f,f')}$ is non-empty and has a component of dimension $g-2+f+f'$ in characteristic p, then $\mathcal{R}_g^{(f,F')}$ is non-empty and has a component of dimension $g-2+f+F'$ in characteristic p for each F' such that $f' \leq F' \leq g-1$.*

Proof. Let $S_{f'}$ be a component of $\mathcal{R}_g^{(f,f')}$ having dimension $g-2+f+f'$. Then $S_{f'}$ is contained in $W_g^f := \Pi^{-1}(\mathcal{M}_g^f)$. Each component of the latter has dimension $2g-3+f$ since $\Pi : \mathcal{R}_g \to \mathcal{M}_g$ is finite and flat and $\mathcal{M}_g^f$ is pure of dimension $2g-3+f$ by [10, Theorem 2.3]. Thus $S_{f'}$ has codimension $g-1-f'$ in W_g^f. Also, the generic geometric point of W_g^f represents a cover π such that the Prym P_π has p-rank $g-1$ by Theorem 4.5.

Consider the forgetful morphism $\tau_g : \mathcal{R}_g \to \mathcal{M}_{2g-1}$ which sends the point representing $\pi : Y \to X$ to the point representing Y. Since $g \geq 3$, the map τ_g is finite-to-1. Now $\tau_g(S_{f'}) \subset \mathcal{M}_{2g-1}^{f+f'}$ and $\tau_g(W_g^f)$ is contained in the closure of $\mathcal{M}_{2g-1}^{f+g-1}$. Thus the p-ranks and the dimensions for $\tau_g(S_{f'})$ and $\tau_g(W_g^f)$ both differ by exactly $g-1-f'$.

By Lemma 2.2, the p-rank can only change in codimension 1. It follows that there is a nested sequence $T_{f'} \subset \cdots \subset T_i \cdots \subset T_{g-1}$, indexed by i from f' to $g-1$, with $T_{f'} = \tau_g(S_{f'})$ and $T_{g-1} = \tau_g(W_g^f)$, such that $\dim(T_i) = g-2+f+i$ and the generic geometric point of T_i represents a curve Y_i with p-rank $f+i$.

Then T_i is in the image of τ_g, so there is a sequence $S_{f'} \subset \cdots \subset S_i \subset \cdots \subset W_g^f$ such that $\tau_g(S_i) = T_i$. Thus $\dim(S_i) = g - 2 + f + i$. Also, the generic geometric point of S_i represents an unramified double cover $\pi : Y_i \to X_i$ such that Y_i has p-rank $f + i$ and X_i has genus g and p-rank f; it follows that P_{π_i} has p-rank i. Thus $R_g^{(f,f')}$ contains an open dense subset of S_i, which we denote again by S_i at the risk of causing confusion.

The next claim is that S_i is open and dense in a component of $R_g^{(f,i)}$ for $f' \leq i \leq g - 1$. This is true for $i = f'$ by hypothesis. If it is not true for all i, let j be the minimal index for which it is false. Then S_{j-1} has codimension at least 2 inside a component Σ of $R_g^{(f,j)}$, and $\tau_g(S_{j-1})$ has codimension at least 2 in $\tau_g(\Sigma)$. This contradicts Lemma 2.2, since the p-rank drops by 1 on a subset of codimension 2. This completes the proof. □

6. Results for low genus when $\ell = 2$. This section contains results about non-ordinary Pryms of unramified double covers of curves of low genus $g = 2$ and $g = 3$. These results provide the base cases for the results in Section 7. Since this section is only about double covers, the subscript $\ell = 2$ is dropped from the notation for simplicity.

Recall that $\mathcal{A}_g^f$ is irreducible for all $g \geq 2$ and $0 \leq f \leq g$ except $(g, f) = (2, 0)$ [6, Theorem A]. When either $g = 2$, $f = 1, 2$ or $g = 3$, $0 \leq f \leq 3$, the image of $\mathcal{M}_g^f$ under the Torelli map is open and dense in $\mathcal{A}_g^f$ and thus $\mathcal{M}_g^f$ is irreducible as well.

6.1. Base Case: Genus 2. This section contains a proof that $\mathcal{R}_2^{(f,f')}$ is non-empty with the expected dimension for all six choices of (f, f') when $p \geq 5$.

PROPOSITION 6.1. *Let $g = 2$, $0 \leq f \leq 2$, and $0 \leq f' \leq 1$. Then $\mathcal{R}_2^{(f,f')}$ is non-empty (except when $p = 3$, $f = 0, 1$, and $f' = 0$) and each of its components has dimension $f + f'$.*

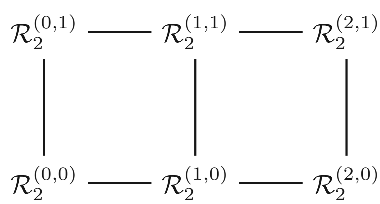

FIG. 5. *The p-rank stratification of $\mathcal{R}_2$*

Proof. By Lemma 3.3, W_2^f is non-empty with dimension $1 + f$ for $0 \leq f \leq 2$. If $f = 1, 2$, then $\mathcal{M}_2^f$ is irreducible and so W_2^f is irreducible by Proposition 4.3.

By [10, Section 7.1], $\mathcal{R}_2^{(1,0)}$ and $\mathcal{R}_2^{(0,0)}$ are empty when $p = 3$.

- (0,0). By [10, Theorem 6.1], if $p \geq 5$, then $\mathcal{R}_2^{(0,0)}$ is nonempty with dimension 0.
- (0,1). When $f = 0$, then $W_2^0 = \mathcal{R}_2^{(0,1)} \cup \mathcal{R}_2^{(0,0)}$. So $\mathcal{R}_2^{(0,1)}$ is open and dense in W_2^0 and thus $\dim(\mathcal{R}_2^{(0,1)}) = 1$.
- (2,1). Since W_2^2 is open and dense in $\mathcal{R}_2$, which contains $\mathcal{R}_2^{(0,1)}$, the generic point of W_2^2 has $f' = 1$. This implies that $\mathcal{R}_2^{(2,1)}$ is open and dense in W_2^2 and $\dim(\mathcal{R}_2^{(2,1)}) = 3$.

- (1,1). Lemma 2.2, applied to $\mathcal{R}_2^{(0,1)} \subset \mathcal{R}_2$, shows that $\mathcal{R}_2^{(1,1)}$ is non-empty with dimension 2. Thus $\mathcal{R}_2^{(1,1)}$ is open and dense in W_2^1.
- (2,0). The fiber product construction in Section 9.1 shows that $\dim(V_2^0) = 2$. Namely, a point of $\mathcal{A}_1^0$ is represented by a supersingular elliptic curve E_λ. By (5) in Section 9.1, there is a 2-dimensional family of curves X with an unramified double cover $\pi : Y \to X$ with $P_\pi \sim E_\lambda$.
  Note that $\mathcal{R}_2^{(2,0)}$ is non-empty and open and dense in V_2^0: if $p = 3$, then $\mathcal{R}_2^{(2,0)} = V_2^0$ since $\mathcal{R}_2^{(0,0)}$ and $\mathcal{R}_2^{(1,0)}$ are empty; if $p \geq 5$, then W_2^1 is irreducible with dimension 2 and generic $f' = 1$ and so no component of V_2^0 is contained in W_2^1.
- (1,0). If $p \geq 5$, applying Lemma 2.2 to $\mathcal{R}_2^{(0,0)} \subset V_2^0$ shows that $\mathcal{R}_2^{(1,0)}$ is non-empty. Also $\mathcal{R}_2^{(1,0)} \subset W_2^1$ which is irreducible of dimension 2 and generic $f' = 1$. Thus $\dim(\mathcal{R}_2^{(1,0)}) \leq 1$. By Proposition 5.2, every component of $\mathcal{R}_2^{(1,0)}$ has dimension 1.

□

Remark 6.2. When $g = 2$, then each of the 15 connected unramified $\mathbb{Z}/2$-covers $\pi : Y \to X$ arises via a fiber product construction. For a fixed (small) prime p, it is thus computationally feasible to find equations for curves represented by points of $\mathcal{R}_2^{(f,f')}$. However, if $f + f'$ is small, then it is not feasible to prove that $\mathcal{R}_2^{(f,f')}$ is non-empty for all primes p using a computational perspective; this is explained in more detail in Section 9.1.

Remark 6.3. Consider $\mathcal{M}_{1;2}^0$, whose points represent supersingular elliptic curves with 2 marked points. Then $\kappa_{0,II}(\mathcal{M}_{1;2}^0)$ has dimension 1 and is fully contained in $\partial\mathcal{R}_2^{(1,0)}$.

6.2. Base case: $g = 3$.

Proposition 6.4. *Let $\ell = 2$ and $p \geq 3$. Let $g = 3$ and $0 \leq f \leq 3$ (with $f \neq 0, 1$ when $p = 3$). Then $\Pi^{-1}(\mathcal{M}_3^f)$ is irreducible and $\mathcal{R}_3^{(f,1)} = \Pi^{-1}(\mathcal{M}_3^f) \cap V_3^1$ is non-empty with dimension $2 + f$.*

Proof. For $0 \leq f \leq 3$, $\mathcal{M}_3^f$ is irreducible and so $\Pi^{-1}(\mathcal{M}_3^f)$ is irreducible by Proposition 4.3. Also $\dim(\Pi^{-1}(\mathcal{M}_3^f)) = 3 + f$. By Theorem 4.5, the Prym of the unramified $\mathbb{Z}/2$-cover represented by the generic point of $\Pi^{-1}(\mathcal{M}_3^f)$ has p-rank $f' = 2$. By Proposition 5.2, if $\mathcal{R}_3^{(f,1)} = \Pi^{-1}(\mathcal{M}_3^f) \cap V_3^1$ is non-empty, then its components have dimension $2 + f$.

Recall that $\mathcal{R}_2^{(f,0)}$ is non-empty and has dimension f by Proposition 6.1 when $f = 1, 2$ and by [10, Theorem 6.1] (except when $p = 3$ and $f = 0, 1$).

If $0 \leq f \leq 2$, consider $I = \kappa_{2:1}(\mathcal{R}_{2;1}^{(f,0)} \times \mathcal{R}_{1;1}^{(0,0)}) \subset \bar{\mathcal{R}}_3$. Then I is non-empty. The choice of base point increases the dimension by 1 so $\dim(I) = \dim(\bar{\mathcal{R}}_2^{(f,0)}) + 1 = f + 1$. If $f = 3$, consider $I = \kappa_{2:1}(\mathcal{R}_{2;1}^{(2,0)} \times \mathcal{R}_{1;1}^{(1,0)}) \subset \bar{\mathcal{R}}_3$. Then $I \neq \emptyset$ and $\dim(I) = 4$.

By Lemmas 3.6 and 3.7, I is contained in a component T of $\bar{\mathcal{R}}_3^{(f,1)}$. Now $\dim(T) \geq f + 2$ by Proposition 5.2. The generic point of T is not contained in $\Delta_{2:1}[\bar{\mathcal{R}}_3^{(f,1)}]$ since the dimension of the latter is bounded by $f + 1$. Moreover, from the construction of I, the generic point of T is not contained in any other boundary component and is thus in $\mathcal{R}_3^{(f,1)}$. □

7. Non-ordinary Pryms of unramified double covers. In this section, we demonstrate the existence of smooth curves of given genus and p-rank having an unramified double cover whose Prym is not ordinary. Since this section is only about double covers, the subscript $\ell = 2$ is dropped from the notation for simplicity.

7.1. Almost ordinary. Theorem 1.2(1) follows from Theorem 7.1, which states that there is a codimension one condition on a generic curve X of genus g and p-rank f for which the Prym of an unramified double cover $\pi : Y \to X$ is almost ordinary. The almost ordinary condition means that the p-rank of P_π is $g-2 = \dim(P_\pi) - 1$.

Consider the stratum $\mathcal{R}_g^{(f,g-2)} = W_g^f \cap V_g^{g-2}$ whose points represent unramified double covers $\pi : Y \to X$ such that X is a smooth curve of genus g and p-rank f and such that P_π has p-rank $f' = g-2$ (or, equivalently, such that P_π is almost ordinary).

Theorem 7.1. *Let $\ell = 2$ and $p \geq 3$. Let $g \geq 2$ and $0 \leq f \leq g$ (with $f \geq 2$ when $p = 3$). Then $\mathcal{R}_g^{(f,g-2)}$ is non-empty and each of its components has dimension $2g-4+f$.*

More generally, let S be a component of $\mathcal{M}_g^f$. Then the locus of points of $\Pi^{-1}(S)$ representing unramified double covers for which the Prym P_π is almost ordinary is non-empty and codimension 1 in $\Pi^{-1}(S)$ (dimension $2g-4+f$).

Proof. The first statement follows from the second since every component of $\mathcal{R}_g^{(f,g-2)}$ is contained in $\Pi^{-1}(S)$ for some component S of $\mathcal{M}_g^f$. It thus suffices to prove that $\Pi^{-1}(S) \cap V_g^{g-2}$ is non-empty and its components have dimension $2g-4+f$.

Dimension: Let T be a component of $\Pi^{-1}(S) \cap V_g^{g-2}$. Then Proposition 5.2 implies that $\dim(T) \geq 2g-4+f$. The generic point of $\Pi^{-1}(S)$ represents a cover whose Prym has p-rank $f' = g-1$ by Theorem 4.5 (or Proposition 6.1 if $g = 2$ and $f = 0$). Thus $\dim(T) = \dim(\Pi^{-1}(S)) - 1 = 2g-4+f$. It thus suffices to show $\Pi^{-1}(S) \cap V_g^{g-2}$ is non-empty.

Base cases: When $g = 2$ and $0 \leq f \leq 2$, then $\Pi^{-1}(S) \cap V_2^0$ is non-empty with dimension f by Proposition 6.1 (unless $p = 3$ and $f = 0, 1$). When $g = 3$ and $0 \leq f \leq 3$, then $\Pi^{-1}(\mathcal{M}_3^f)$ is irreducible and $\Pi^{-1}(\mathcal{M}_3^f) \cap V_g^1$ is non-empty with dimension $2+f$ by Proposition 6.4.

Strategy: Suppose $g \geq 4$. Let $\bar{S}$ be the closure of S in $\bar{\mathcal{M}}_g^f$. The plan is to show that $\Pi^{-1}(\bar{S}) \cap \bar{V}_g^{g-2}$ is non-empty and that one of its components is not contained in $\partial\mathcal{R}_g$.

Non-empty: Let $g_1 = 3$ and $g_2 = g-3$. Choose f_1, f_2 such that $f_1 + f_2 = f$ with $0 \leq f_i \leq g_i$. By Proposition 4.1, there are components $S_{i;1}$ of $\bar{\mathcal{M}}_{g_i;1}^{f_i}$ such that

$$\kappa_{g_1:g_2}(S_{1;1} \times S_{2;1}) \subset \bar{S}.$$

Recall the forgetful map $\psi_M : \mathcal{M}_{g;1} \to \mathcal{M}_g$ from Section 2.3. Let $S_i = \psi_M(S_{i;1})$ which is an irreducible component of $\bar{\mathcal{M}}_{g_i}^{f_i}$.

The Prym of the cover represented by the generic point of $\Pi^{-1}(S_2)$ has p-rank $f_2' = g_2 - 1$ by Theorem 4.5. By Proposition 4.3, $\Pi^{-1}(S_1)$ is irreducible. By Proposition 6.4, there exists a point of $\Pi^{-1}(S_1)$ representing a cover whose Prym has p-rank $f_1' = g_1 - 2 = 1$. Since $\bar{\mathcal{M}}_3^{f_1}$ is irreducible, $\Pi^{-1}(S_1) = \bar{W}_3^{f_1}$. Consider

$$N := \kappa_{g_1:g_2}(\Pi^{-1}(S_{1;1}) \times \Pi^{-1}(S_{2;1})) \subset \Pi^{-1}(\bar{S}).$$

By Lemma 3.7, N contains a point representing a cover whose Prym has p-rank $f' = f_1' + f_2' + 1 = g - 2$, i.e., whose Prym is almost ordinary. Thus $\Pi^{-1}(\bar{S}) \cap V_g^{g-2}$ is non-empty.

Generically smooth: Let T be a component of $\Pi^{-1}(\bar{S}) \cap \bar{V}_g^{g-2}$ containing N. By the remarks above, T intersects the image of

$$\kappa_{g_1:g_2} : \bar{\mathcal{R}}_{g_1;1}^{(f_1,1)} \times \bar{W}_{g_2;1}^{f_2} \to \Delta_{3:g-3}[\bar{\mathcal{R}}_g^{(f,g-2)}].$$

This image has dimension

$$(2 + f_1) + 1 + (2(g - 3) - 3 + f_2) + 1 = 2g - 5 + f.$$

By Proposition 5.2, $\dim(T) \geq 2g - 4 + f$. Thus the generic point τ of T is not contained in $\Delta_{3:g-3}[\bar{\mathcal{R}}_g]$. Furthermore, τ is not contained in any other component of $\partial\mathcal{R}_g$ because the generic points of S_1 and S_2 represent smooth curves. Thus $\tau \subset \Pi^{-1}(S) \cap V_g^{g-2}$. □

7.2. Pryms with low p-rank. As an application, we demonstrate the existence of smooth curves of given genus and p-rank having an unramified double cover whose Prym has any p-rank between $\frac{g}{2} - 1$ and $g - 3$.

THEOREM 7.2. *Let $p \geq 5$. Let $g \geq 2$ and write $g = 3r + 2s$ for integers $r, s \geq 0$. Let $0 \leq f \leq g$. Let $2r + s - 1 \leq f' \leq g - 1$. Then $\mathcal{R}_g^{(f,f')}$ is non-empty and has a component of dimension $g - 2 + f + f'$ in characteristic p.*

Proof. In light of Proposition 5.5, it suffices to prove the result when $f' = 2r+s-1$. The proof is by induction on $r+s$. In the base case $(r, s) = (0, 1)$, then $g = 2$ and the result is true by Proposition 6.1. In the base case $(r, s) = (1, 0)$, then $g = 3$ and the result is true by Proposition 6.4. As an inductive hypothesis, suppose that the result is true for all pairs (r', s') such that $1 \leq r' + s' < r + s$.

Case 1: suppose $r \geq 1$. Let $g_1 = 3$ and $g_2 = g - 3$. There exist f_1, f_2 such that $f_1 + f_2 = f$ and $0 \leq f_1 \leq g_1$ and $0 \leq f_2 \leq g_2$. Let $f_1' = 1$ and $f_2' = 2r + s - 3$. By Proposition 6.4, $\mathcal{R}_3^{(f_1,1)}$ is non-empty and has a component S_1 of dimension $d_1 = 2+f_1$. (The points of S_1 represent unramified double covers $\pi_1 : Y_1 \to X_1$ of a smooth curve of genus g_1 and p-rank f_1, such that P_{π_1} has p-rank 1.)

By the inductive hypothesis applied to $(r - 1, s)$, it follows that $\mathcal{R}_{g_2}^{(f_2,2r+s-3)}$ is non-empty and has a component S_2 of dimension $d_2 = g_2 - 2 + f_2 + f_2'$. (The points of S_2 represent unramified double covers $\pi_2 : Y_2 \to X_2$ of a smooth curve of genus g_2 and p-rank f_2, such that P_{π_2} has p-rank f_2'.) Adding a marking increases the dimension by 1, so $U_1 := \psi_R^*(S_1) = S_1 \times_{\mathcal{R}_3} \mathcal{R}_{3;1}$ has dimension $d_1 + 1 = 3 + f_1$ and $U_2 := \psi_R^*(S_2) = S_2 \times_{\mathcal{R}_{g_2}} \mathcal{R}_{g_2;1}$ has dimension $d_2 + 1 = g_2 - 1 + f_2 + f_2'$.

Let $\mathcal{K}$ be a component of $\kappa_{3,g_2}(U_1 \times U_2)$; then $\mathcal{K}$ has dimension $d_1 + d_2 + 2$. By Lemmas 3.6 and 3.7, $\mathcal{K}$ is contained in a component $\mathcal{Z}$ of $\bar{\mathcal{R}}_{g_1+g_2}^{(f_1+f_2,f_1'+f_2'+1)} = \bar{\mathcal{R}}_g^{(f,2r+s-1)}$. In other words, the points of $\mathcal{K}$ represent unramified double covers of curves (of compact type) having genus g and p-rank f whose Prym varieties have p-rank f'. By Lemma 3.1, the dimension of $\mathcal{Z}$ is at most

$$d_1 + d_2 + 3 = (g_2 + 3) - 2 + (f_1 + f_2) + (2r + s - 1) = g - 2 + f + f'.$$

Also $\dim(\mathcal{Z}) \geq g - 2 + f + f'$ by Proposition 5.2. Thus $\dim(\mathcal{Z}) = g - 2 + f + f'$ and the generic point of $\mathcal{Z}$ is not contained in $\mathcal{K}$. The generic geometric points of S_1

and S_2 represent unramified double covers of smooth curves by hypothesis. Thus the generic geometric point of $\mathcal{Z}$ is not contained in any other boundary component of $\bar{\mathcal{R}}_g$ and so it represents an unramified double cover of a smooth curve.

Case 2: suppose $s \geq 1$. Let $g_1 = 2$ and $g_2 = g - 2$. There exist f_1, f_2 such that $f_1 + f_2 = f$ and $0 \leq f_1 \leq g_1$ and $0 \leq f_2 \leq g_2$. Let $f_1' = 0$ and $f_2' = 2r + s - 2$. By Proposition 6.1, $\mathcal{R}_2^{(f_1,0)}$ is non-empty and has a component S_1 of dimension $d_1 = f_1$. By the inductive hypothesis applied to $(r, s-1)$, it follows that $\mathcal{R}_{g_2}^{(f_2,2r+s-2)}$ is non-empty and has a component S_2 of dimension $d_2 = g_2 - 2 + f_2 + (2r + s - 2)$. The rest of the proof follows the same reasoning as in Case (1). □

Corollary 7.3. *Let $\ell = 2$ and $p \geq 5$. Let $g \geq 4$ and $0 \leq f \leq g$. Suppose $\frac{g}{2} - 1 \leq f' \leq g - 3$. Then $\mathcal{R}_g^{(f,f')}$ is non-empty and has a component of dimension $g - 2 + f + f'$. In particular, there exists a smooth curve $X/\bar{\mathbb{F}}_p$ of genus g and p-rank f having an unramified double cover $\pi : Y \to X$ for which the Prym P_π has p-rank f'.*

Proof. If g is even, let $r = 0$ and $s = g/2$. If g is odd, let $r = 1$ and $s = (g-3)/2$. In either case, the condition $\frac{g}{2} - 1 \leq f' \leq g - 3$ implies that the hypothesis $2r + s - 1 \leq f' \leq g - 1$ in Theorem 7.2 is satisfied and the result follows from Theorem 7.2. □

8. Applications to Theta divisors.

8.1. Background. Recall the definition of the theta divisor from [21, Section 1.1]. Given a relative curve X/S, let X^1 be the curve induced by base change by the absolute Frobenius of S. Consider the relative Frobenius morphism $F : X \to X^1$. The *sheaf of locally exact differentials* B is the image of $F_*d : F_*(\mathcal{O}_X) \to F_*(\Omega^1_X)$. There is an exact sequence of $\mathcal{O}_{X^1}$-modules:

$$0 \to \mathcal{O}_{X^1} \to F_*(\mathcal{O}_X) \stackrel{F_*d}{\to} B \to 0.$$

Also, B is the kernel of the Cartier operator $C : F_*(\Omega^1_X) \to \Omega^1_{X^1}$, and there is an exact sequence of $\mathcal{O}_{X^1}$-modules:

$$0 \to B \to F_*(\Omega^1_X) \stackrel{C}{\to} \Omega^1_{X^1} \to 0.$$

Now B is a vector bundle on X^1 of rank $p - 1$ and slope $g - 1$, where the slope is the quotient of the degree by the rank. More precisely, if X^1 is not smooth, then B is a torsion-free sheaf, which is locally free of rank $p - 1$ outside the singularities of X^1.

By [19, Theorem 4.1.1], B admits a theta divisor Θ_X. This is a positive Cartier divisor on the Jacobian J^1 of X^1 (the determinant of the universal cohomology). A point $a \in J^1(k)$ is in the support of Θ_X if and only if $H^0(X^1, B \otimes L_a) \neq 0$ where $L_a \in \mathrm{Pic}^0(X^1)$ is the invertible sheaf identified with a.

8.2. The theta divisor. By work of Raynaud, the theta divisor θ_X determines whether unramified covers of the curve X are ordinary. By [20, Proposition 1], X is ordinary if and only if θ_X does not contain the identity of J^1.

To generalize this, consider a non-trivial point $a \in J^1[\ell]$ with $p \nmid \ell$. The point a determines an unramified $\mathbb{Z}/\ell$-cover $\pi_a : Y_a \to X$, and an invertible sheaf $L_a \in \mathrm{Pic}^0(X^1)$ of order ℓ. Denote the orbit of a under $(\mathbb{Z}/\ell)^*$ as

$$\mathrm{Sat}(a) = \{ia \mid \gcd(i, \ell) = 1\}.$$

Proposition 8.1 ([21, Proposition 2.1.4]). *Let $a \in J^1[\ell]$ be non-trivial with $p \nmid \ell$. The new part of $\pi_a : Y_a \to X$ is ordinary if and only if* $\mathrm{Sat}(a)$ *does not intersect the theta divisor* Θ_X.

Using the geometry of Θ_X, Raynaud and Pop/Saidi prove:

Theorem 8.2. *Let X be a smooth projective k-curve of genus $g \geq 2$.*
(1) *[19, Theorem 4.3.1] For sufficiently large ℓ, there is an unramified $\mathbb{Z}/\ell$-cover $\pi : Y \to X$ such that P_π is ordinary. It suffices to take $\ell > (p-1)3^{g-1}g!$ by [22, Remark 3.1.1].*
(2) *[20, Theorem 2] There is an unramified Galois cover $Z \to X$, with solvable prime-to-p Galois group, with a non-ordinary representation (so Z is not ordinary).*
(3) *[18, Proposition 2.3] If X is non-ordinary or if J_X is simple then there is an unramified $\mathbb{Z}/\ell$-cover $\pi_\ell : Y_\ell \to X$ such that P_{π_ℓ} is not ordinary for infinitely many primes ℓ.*

8.3. Comparison with previous work. The results in this paper strengthen the results in Theorem 8.2 for a generic curve X of genus g and p-rank f for all $g \geq 2$ and $0 \leq f \leq g$. Specifically, Theorem 1.1 removes the condition on ℓ in Theorem 8.2(1) and shows that all (not just one) of the Pryms of the $\mathbb{Z}/\ell$-covers of X are ordinary, for a generic curve X of genus g and p-rank f. Theorem 1.2 and Corollary 7.3 are about double covers, rather than covers of unknown degree, and they determine the value of the p-rank of the Prym which gives more information than saying that the new part of the Prym is not ordinary.

8.4. New results on theta divisors. We apply Proposition 8.1 in the opposite direction from Raynaud and Pop/Saidi to complete the proofs of Theorems 1.1(2) and 1.2(2).

Theorem 8.3. *Let $\ell \neq p$ be prime. Let $g \geq 2$ and $0 \leq f \leq g$ with $f \neq 0$ if $g = 2$. Let S be an irreducible component of $\mathcal{M}_g^f$. If X is the curve represented by the generic point of S, then the theta divisor Θ_X of the Jacobian of X does not contain any point of order ℓ.*

Proof. By Proposition 8.1, this statement is equivalent to Theorem 1.1(1) since the Prym is the new part of $\pi : Y \to X$. □

Theorem 8.4. *Let $\ell = 2$. Let $g \geq 2$ and $0 \leq f \leq g$ (with $f \geq 2$ when $p = 3$). Let S be an irreducible component of $\mathcal{M}_g^f$. The locus of points of S representing curves X for which Θ_X contains a point of order 2 is non-empty with codimension 1 in S.*

Proof. This follows from Theorem 1.2(1) by Proposition 8.1. □

Remark 8.5. Let g be odd and $d = (g-1)/2$. If X has genus g, then J_X contains the top difference variety $V_d = X_d - X_d$, which consists of divisors of the form $\sum_{i=1}^d P_i - \sum_{i=1}^d Q_i$. By [7, Corollary 0.4], V_d contains no points of order ℓ when X is generic.

9. Examples and open questions. This section contains examples for $g = 2$, a question about Pryms of hyperelliptic curves, and questions about non-ordinary Pryms whose answers would generalize Theorem 1.2(1).

9.1. The fiber product construction when $g = 2$. We explain why the fiber product construction of unramified double covers is not useful for proving Proposition 6.1.

Suppose X is a genus 2 curve and $f_1 : X \to \mathbb{P}^1$ is a hyperelliptic cover branched above a set B_X of cardinality 6. For a set $B_E \subset B_X$ of cardinality 4, let $f_2 : E \to \mathbb{P}^1$ be the hyperelliptic cover branched above B_E. The fiber product $f : Y \to \mathbb{P}^1$ of f_1 and f_2 is a Klein four cover of $\mathbb{P}^1$. By Abhyankar's Lemma, the degree two subcover $\pi : Y \to X$ is unramified since $B_E \subset B_X$. Then $J_Y \sim J_X \oplus E$ by [12, Theorem B]. Thus $P_\pi \sim E$.

Furthermore, each of the 15 connected unramified double covers $\pi : Y \to X$ arises via the fiber product construction (from one of the 15 choices of $B_E \subset B_X$). This is because the hyperelliptic involution ι on X fixes each point of order 2 on J_X and thus extends to Y.

For $\lambda \in k - \{0, 1\}$, let $E_\lambda : y_2^2 = x(x-1)(x-\lambda)$. For distinct $t_1, t_2 \in k - \{0, 1, \lambda\}$, consider the genus two curve

$$X : y_1^2 = f_\lambda(t_1, t_2) := x(x-1)(x-\lambda)(x-t_1)(x-t_2). \quad (5)$$

As above, $E_\lambda \sim P_\pi$ for an unramified double cover $\pi : Y \to X$. One says that λ is supersingular when E_λ is supersingular.

Let $M_\lambda(t_1, t_2)$ be the matrix of the Cartier operator on $H^0(X, \Omega^1)$ with respect to the basis $\{dx/y, xdx/y\}$. Let c_i be the coefficient of x^i in $f_\lambda(t_1, t_2)^{(p-1)/2}$. By [25, page 381],

$$M_\lambda(t_1, t_2) = \begin{pmatrix} c_{p-1} & c_{p-2} \\ c_{2p-1} & c_{2p-2} \end{pmatrix}.$$

Let $D_\lambda = \det(M_\lambda(t_1, t_2))$ and let $S_\lambda \subset \mathbb{A}^2$ be the vanishing locus of D_λ. By [25, Theorem 2.2], X is ordinary if and only if $D_\lambda \neq 0$; the p-rank of X is the rank of $N_\lambda(t_1, t_2) = M_\lambda(t_1, t_2)^{(p)} M_\lambda(t_1, t_2)$ (where (p) means to raise each entry of the matrix to the pth power).

(1) The case $(f, f') = (2, 0)$. For each supersingular λ, to show X is generically ordinary, one needs to check that $D_\lambda \in k[t_1, t_2]$ is non-zero.
(2) The case $(f, f') = (\leq 1, 0)$. To show $\mathcal{R}_2^{(1,0)} \cup \mathcal{R}_2^{(0,0)} \neq \emptyset$, one needs to find λ supersingular such that $D_\lambda \in k[t_1, t_2]$ is non-constant and S_λ is not contained in the union L of the lines $t_i = 0$, $t_i = 1$, $t_i = \lambda$, and $t_1 = t_2$.
(3) The case $(f, f') = (0, 0)$ for $p \geq 5$. To show $\dim(\mathcal{R}_2^{(0,0)}) = 0$, one needs to show that $N_\lambda(t_1, t_2)$ has rank 1 (not 0) for every supersingular λ and for each generic point of S_λ not in L. To show $\mathcal{R}_2^{(0,0)} \neq \emptyset$, one needs to find λ supersingular and distinct $t_1, t_2 \in k - \{0, 1, \lambda\}$ such that $N_\lambda(t_1, t_2)$ has rank 0.

Example 9.1 (Example of Proposition 6.1). Let $p = 5$. Let $\lambda = a^4$ for a root a of $x^2 + 4x + 2$. Then E_λ is supersingular and

$$D_\lambda = (t_1 + 4t_2)^2(t_1^2 t_2 + t_1 t_2^2 + a^{17} t_1^2 + a^{17} t_2^2 + a^5 t_1 t_2 + a^4 t_1 + a^4 t_2).$$

(1) $(f, f') = (2, 0)$. Since $D_\lambda \neq 0$, X is generically ordinary.
(2) $(f, f') = (1, 0)$. When $(t_1, t_2) = (a^{16}, a)$, then $f_X = 1$.
(3) $(f, f') = (0, 0)$. By [10, Section 7.2], there is exactly one unramified double cover $\pi : Y \to X$ up to isomorphism such that X has genus 2 and Y has p-rank 0. An equation for X is $y^2 = x(x^4 + x^3 + 2x + 3)$.

Example 9.1 illustrates Remark 6.2: it is not feasible to prove that $\mathcal{R}_2^{(f,0)}$ is non-empty for all primes p computationally.

9.2. The hyperelliptic case. We expect there is an analogue of Theorem 4.5 for the hyperelliptic locus $\mathcal{H}_g$. One can ask if the Prym of the cover represented by the generic point of each irreducible component of $\Pi_\ell^{-1}(\mathcal{H}_g^f)$ is ordinary for $1 \leq f \leq g$. Propositions 4.1 and 4.4 are true (for $i = 1$) for $\mathcal{H}_g^f$ [2, Corollary 3.13] and Theorem 4.2 is true for $\mathcal{H}_g^f$ when $f > 0$ (or for $f = 0$ and $\ell >> 0$) [2, Theorems 5.2, 5.7]. However, there may be complications with Propositions 3.4, 3.5 for $\mathcal{H}_g^f$, especially when $\ell = 2$.

9.3. A question for $g = 3$ about Pryms of p-rank 0.

Question 9.2. For a prime p, is $\mathcal{R}_3^{(0,0)}$ non-empty? Does there exist an unramified double cover $\pi : Y \to X$ of a smooth curve $X/\bar{\mathbb{F}}_p$ of genus 3 such that Y has p-rank 0?

The answer to Question 9.2 is yes when $p = 3$ by [10, Example 5.5] but is unknown for $p \geq 5$. By [10, Proposition 4.2], if it is non-empty, then $\dim(\mathcal{R}_3^{(0,0)}) = 1$; however, there are components of $\partial\bar{R}_3^{(0,0)}$ which have dimension 1 or 2.

9.4. Non-ordinary Pryms for odd degree cyclic covers. It is unknown whether Theorem 7.1 can be generalized to the case $\ell \geq 3$, for a given prime p.

Question 9.3. Suppose $\ell \neq p$ is an odd prime. For which (g, f) does there exist a curve X of genus g and p-rank f with an unramified $\mathbb{Z}/\ell$-cover $\pi : Y \to X$ such that P_π is non-ordinary?

Example 9.4 ([16, Section 6]). Let $p = 2$ and $g = 2$ and $\ell = 3$. If X is a curve of genus 2 which is not ordinary ($f < 2$), then the Prym of every unramified $\mathbb{Z}/3$-cover of X is ordinary.

9.5. A question about purity. Let $\ell = 2$ and $p \geq 3$. The points of $\mathcal{R}_g^{(f,f')} = W_g^f \cap V_g^{f'}$ represent unramified double covers $\pi : Y \to X$ such that X is a smooth curve of genus g and p-rank f and P_π has p-rank f'. By Proposition 5.2, $\dim(\mathcal{R}_{g;2}^{(f,f')}) \geq g - 2 + f + f'$.

Question 9.5. Let $g \geq 2$ and $0 \leq f \leq g$ and $0 \leq f' \leq g - 1$. If $\mathcal{R}_{g;2}^{(f,f')}$ is non-empty, do all its components have dimension exactly $g - 2 + f + f'$?

The answer to Question 9.5 is yes for any $0 \leq f \leq g$ when:
(1) $f' = g - 1$ by Theorem 4.5 (or Proposition 6.1);
(2) and $f' = g - 2$ (with $f \geq 2$ when $p = 3$) by Theorem 7.1.
One complication in answering Question 9.5 for $f' < g-2$ is that there are families of *singular* curves in $\bar{\mathcal{R}}_{g;2}^{(f,f')}$ whose dimension exceeds $g-2+f+f'$ as in Remark 6.3.

9.6. Pryms with p-rank zero. Let $\ell = 2$ and $p \geq 5$ and $g = 3$. In [5], the authors study genus 3 curves having Pryms with p-rank 0. Here is an open question about this case. Consider $\bar{V}_3^1 = Pr_3^{-1}(\tilde{\mathcal{A}}_2^1)$ whose points represent unramified double covers $\pi : Y \to X$, where X has genus 3 and P_π has p-rank at most 1. By [23, Theorem 4.2, equations 3.14-3.16], $\bar{V}_3^1$ has one component of dimension 5, and three exceptional components of lower dimension.

In $\bar{V}_3^1$ is the locus V_3^0 (additional constraint that $f' = 0$). By [23, Theorem 4.2], $\dim(V_3^0) = 3 + \dim(\mathcal{A}_2^0) = 4$. If $p > 11$, then $\mathcal{A}_2^0$ is not irreducible by [13, Theorem 5.8], and so V_3^0 is not irreducible. Also in $\bar{V}_3^1$ is the locus $W_3^2 \cap \bar{V}_3^1$, (additional constraint that $f = 2$). By Proposition 6.4, $\dim(W_3^2 \cap \bar{V}_3^1) = 4$; it is not known whether it is irreducible.

QUESTION 9.6. Is $\dim(W_3^2 \cap V_3^0) = 3$?

Acknowledgement. We would like to thank Achter, Bruin, Casalaina-Martin, Farkas, and Grushevsky for good conversations, and the referees for insightful comments.

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