ASIAN J. MATH.
Vol. 23, No. 3, pp. 527–538, June 2019

008

DETERMINATION OF BAUM-BOTT RESIDUES OF HIGHER CODIMENSIONAL FOLIATIONS*

MAURÍCIO CORRÊA† AND FERNANDO LOURENÇO‡

Abstract. Let $\mathscr{F}$ be a singular holomorphic foliation, of codimension k, on a complex compact manifold such that its singular set has codimension $\geq k+1$. In this work we determinate Baum-Bott residues for $\mathscr{F}$ with respect to homogeneous symmetric polynomials of degree $k+1$. We drop the Baum-Bott's generic hypothesis and we show that the residues can be expressed in terms of the Grothendieck residue of an one-dimensional foliation on a $(k+1)$-dimensional disc transversal to a $(k+1)$-codimensional component of the singular set of $\mathscr{F}$. Also, we show that Cenkl's algorithm for non-expected dimensional singularities holds dropping the Cenkl's regularity assumption.

Key words. Baum-Bott residues, localization, holomorphic foliations, characteristic classes.

Mathematics Subject Classification. 57R30, 32S65, 32A27.

1. Introduction. In [2] P. Baum and R. Bott developed a general residue theory for singular holomorphic foliations on complex manifolds. More precisely, they proved the following result:

THEOREM 1.1 (Baum-Bott). *Let $\mathscr{F}$ be a holomorphic foliation of codimension k on a complex manifold M and φ be a homogeneous symmetric polynomials of degree d satisfying $k < d \leq n$. Let Z be a compact connected component of the singular set* $\mathrm{Sing}(\mathscr{F})$. *Then, there exists a homology class $\mathrm{Res}_\varphi(\mathscr{F}, Z) \in \mathrm{H}_{2(n-d)}(Z;\mathbb{C})$ such that:*

i) $\mathrm{Res}_\varphi(\mathscr{F}, Z)$ *depends only on φ and on the local behavior of the leaves of $\mathscr{F}$ near Z,*

ii) Suppose that M is compact and denote by $\mathrm{Res}(\varphi, \mathscr{F}, Z) := \alpha_* \mathrm{Res}_\varphi(\mathscr{F}, Z)$, where α_* is the composition of the maps

$$\mathrm{H}_{2(n-d)}(Z;\mathbb{C}) \xrightarrow{i^*} \mathrm{H}_{2(n-d)}(M;\mathbb{C})$$

and

$$\mathrm{H}_{2(n-d)}(M;\mathbb{C}) \xrightarrow{P} \mathrm{H}^{2d}(M;\mathbb{C})$$

with i^* is the induced map of inclusion $i : Z \longrightarrow M$ and P is the Poincaré duality. Then

$$\varphi(\mathcal{N}_{\mathscr{F}}) = \sum_Z \mathrm{Res}(\varphi, \mathscr{F}, Z).$$

The computation and determination of the residues is difficult in general. If the foliation $\mathscr{F}$ has dimension one with isolated singularities , Baum and Bott in [1] show that residues can be expressed in terms of a Grothendieck residue, i.e, for each $p \in \mathrm{Sing}(\mathscr{F})$ we have

$$\mathrm{Res}_\varphi(\mathscr{F}, Z) = \mathrm{Res}_p \left[\varphi(JX) \frac{dz_1 \wedge \cdots \wedge dz_n}{X_1 \cdots X_n} \right],$$

*Received October 26, 2017; accepted for publication March 2, 2018.
†ICEx - UFMG, Departamento de Matemática, Av. Antônio Carlos 6627, 30123-970 Belo Horizonte MG, Brazil (mauriciojr@ufmg.br).
‡DEX - UFLA, Av. Doutor Sylvio Menicucci, 1001, Kennedy, 37200000, Lavras, Brazil (fernando.lourenco@dex.ufla.br).

where X is a germ of holomorphic vector field at p tangent to $\mathscr{F}$ and JX is the jacobian of X.

The subset of $\mathrm{Sing}(\mathscr{F})$ composed by analytic subsets of codimension $k+1$ will be denoted by $\mathrm{Sing}_{k+1}(\mathscr{F})$ and it is called *the singular set of* $\mathscr{F}$ *with expected codimension*. Baum and Bott in [2] exibes the residues for generic componentes of $\mathrm{Sing}_{k+1}(\mathscr{F})$. Let us recall this result:

An irreducible component Z of $\mathrm{Sing}_{k+1}(\mathscr{F})$ comes endowed with a filtration. For given point $p \in Z$ choose holomorphic vector fields $v_1, \ldots, v_s$ defined on an open neighborhood U_p of $p \in M$ and such that for all $x \in U_p$, the germs at x of the holomorphic vector fields $v_1, \ldots, v_s$ are in $\mathscr{F}_x$ and span $\mathscr{F}_x$ as a $\mathcal{O}_x$-module. Define a subspace $V_p(\mathscr{F}) \subset T_pM$ by letting $V_p(\mathscr{F})$ be the subspace of T_pM spanned by $v_1(p), \ldots v_s(p)$. We have

$$Z^{(i)} = \{p \in Z;\ \ \dim(V_p(\mathscr{F})) \leq n-k-i\} \ \text{ for } \ i = 1, \ldots, n-k.$$

Then,

$$Z \supseteq Z^{(1)} \supseteq Z^{(2)} \supseteq \cdots \supseteq Z^{(n-k)}$$

is a filtration of Z. Now, consider a symmetric homogeneous polynomial φ of degree $k+1$. Let $Z \subset \mathrm{Sing}_{k+1}(\mathscr{F})$ be an irreducible component. Take a generic point $p \in Z$ such that p is a point where Z is smooth and disjoint from the other singular components. Now, consider B_p a ball centered at p, of dimension $k+1$ sufficiently small and transversal to Z in p. In [2, Theorem 3, pg 285] Baum and Bott proved under the following generic assumption

$$\mathrm{cod}(Z) = k+1 \quad \text{and} \quad \mathrm{cod}(Z^{(2)}) < k+1$$

that we have

$$\mathrm{Res}(\mathscr{F}, \varphi; Z) = \mathrm{Res}_\varphi(\mathscr{F}|_{B_p}; p)[Z],$$

where $\mathrm{Res}_\varphi(\mathscr{F}|_{B_p}; p)$ represents the Grothendieck residue at p of the one dimensional foliation $\mathscr{F}|_{B_p}$ on B_p and $[Z]$ denotes the integration current associated to Z.

In [5] and [8] the authors determine the residue $\mathrm{Res}(\mathscr{F}, c_1^{k+1}; Z)$, but even in this case they do not show that we can calculate these residues in terms of the Grothendieck residue of a foliation on a transversal disc. In [15] Vishik proved the same result under the Baum-Bott's generic hypotheses but supposing that the foliation has locally free tangent sheaf. In [3] F. Bracci and T. Suwa study the behavior of the Baum-Bott residues under smooth deformations, providing an effective way of computing residues.

In this work we drop the Baum-Bott's generic hypotheses and we prove the following:

Theorem 1.2. *Let* $\mathscr{F}$ *be a singular holomorphic foliation of codimension* k *on a compact complex manifold* M *such that* $\mathrm{cod}(\mathrm{Sing}(\mathscr{F})) \geq k+1$. *Then,*

$$\mathrm{Res}(\mathscr{F}, \varphi; Z) = \mathrm{Res}_\varphi(\mathscr{F}|_{B_p}; p)[Z],$$

where $\mathrm{Res}_\varphi(\mathscr{F}|_{B_p}; p)$ *represents the Grothendieck residue at* p *of the one dimensional foliation* $\mathscr{F}|_{B_p}$ *on a* $(k+1)$*-dimensional transversal ball* B_p.

Finally, in the last section we apply Cenkl's algorithm for non-expected dimensional singularities [7]. Moreover, we drop Cenkl's regularity hypothesis and we conclude that it is possible to calculate the residues for foliations whenever $\mathrm{cod}(\mathrm{Sing}(\mathscr{F})) \geq k+s$, with $s \geq 1$.

Acknowledgments. We are grateful to Jean-Paul Brasselet, Tatsuo Suwa and Marcio G. Soares for interesting conversations. This work was partially supported by CNPq, CAPES, FAPEMIG and FAPESP-2015/20841-5. We are grateful to Institut de Mathématiques de Luminy- Marseille and Imecc–Unicamp for hospitality. Finally, we would like to thank the referee by the suggestions, comments and improvements to the exposition.

2. Holomorphic foliations. Denote by Θ_M the tangent sheaf of M. A foliation $\mathscr{F}$ of codimension k on an n-dimensional complex manifold M is given by a exact sequence of coherent sheaves

$$0 \longrightarrow T\mathscr{F} \longrightarrow \Theta_M \to N_{\mathscr{F}} \longrightarrow 0,$$

such that $[T\mathscr{F}, T\mathscr{F}] \subset T\mathscr{F}$ and the normal sheaf $N_{\mathscr{F}}$ of $\mathscr{F}$ is a torsion free sheaf of rank $k \leq n-1$. The sheaf $T\mathscr{F}$ is called the tangent sheaf of $\mathscr{F}$. The singular set of $\mathscr{F}$ is defined by $\mathrm{Sing}(\mathscr{F}) := \mathrm{Sing}(N_{\mathscr{F}})$. The dimension of $\mathscr{F}$ is $\dim(\mathscr{F}) = n-k$.

Also, a foliation $\mathscr{F}$, of codimension k, can be induced by a exact sequence

$$0 \longrightarrow N_{\mathscr{F}}^{\vee} \longrightarrow \Omega_M^1 \to \mathcal{Q}_{\mathscr{F}} \longrightarrow 0,$$

where $\mathcal{Q}_{\mathscr{F}}$ is a torsion free sheaf of rank $n-k$. Moreover, the singular set of $\mathscr{F}$ is $\mathrm{Sing}(\mathcal{Q}_{\mathscr{F}})$. Now, by taking the wedge product of the map $N_{\mathscr{F}}^{\vee} \longrightarrow \Omega_M^1$ we get a morphism

$$\bigwedge^k N_{\mathscr{F}}^{\vee} \longrightarrow \Omega_M^k$$

and twisting by $(\bigwedge^k N_{\mathscr{F}}^{\vee})^{\vee} = \det(N_{\mathscr{F}})$ we obtain a morphism

$$\omega : \mathcal{O}_M \longrightarrow \Omega_M^k \otimes \det(N_{\mathscr{F}}).$$

Therefore, a foliation is induced by a twisted holomorphic k-form

$$\omega \in \mathrm{H}^0(X, \Omega_M^k \otimes \det(N_{\mathscr{F}}))$$

which is locally decomposable outside the singular set of $\mathscr{F}$. That is, by the classical Frobenius Theorem for each point $p \in X \setminus \mathrm{Sing}(\mathscr{F})$ there exists a neighbourhood U and holomorphics 1-forms $\omega_1, \ldots, \omega_k \in \mathrm{H}^0(U, \Omega_U^1)$ such that

$$\omega|_U = \omega_1 \wedge \cdots \wedge \omega_k$$

and

$$d\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0$$

for all $i = 1, \ldots, k$.

3. Proof of the Theorem. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_j \geq 0$ for $j = 1, \ldots, k$, consider the homogeneous symmetric polynomial of degree $k+1$, $\varphi = c_1^{\alpha_1} c_2^{\alpha_2} \cdots c_k^{\alpha_k}$ such that $1\alpha_1 + 2\alpha_2 + \cdots + k\alpha_k = k+1$.

Let us consider the twisted k-form $\omega \in \mathrm{H}^0(M, \Omega_M^k \otimes \det(N_{\mathscr{F}}))$ induced by $\mathscr{F}$. Denote by $\mathrm{Sing}_{k+1}(\mathscr{F})$ the union of the irreducible components of $\mathrm{Sing}(\mathscr{F})$ of pure codimension $k+1$. Consider an open subset $U \subset M \setminus \mathrm{Sing}(\mathscr{F})$. Thus, the form

$\omega|_U$ is decomposable and integrable. That is, $\omega|_U$ is given by a product of k 1-forms $\omega_1 \wedge \cdots \wedge \omega_k$. Then, it is possible to find a matrix of $(1,0)$-forms (θ^*_{ls}) such that

$$\partial\omega_l = \sum_{s=1}^{k} \theta^*_{ls} \wedge \omega_s, \quad \overline{\partial}\omega_l = 0, \quad \forall \ \ l = 1, \ldots, k.$$

We have that $\omega_1, \ldots, \omega_k$ is a local frame for $N^*_{\mathscr{F}}|_U$ and the identity above induces on U the *Bott partial connection*

$$\nabla : C^\infty(N^*_{\mathscr{F}}|_U) \to C^\infty((T\mathscr{F}^* \oplus \overline{TM}) \otimes N^*_{\mathscr{F}}|_U)$$

defined by

$$\nabla_v(\omega_l) = i_v(\partial\omega_l), \qquad \nabla_u(\omega_l) = i_u(\overline{\partial}\omega_l) = 0,$$

where $v \in C^\infty(T\mathscr{F}|_U)$ and $u \in C^\infty(\overline{TM}|_U)$ which can be extended to a connection $D^* : C^\infty(N^*_{\mathscr{F}}|_U) \to C^\infty((TM^* \oplus \overline{TM}) \otimes N^*_{\mathscr{F}}|_U)$ in the following way

$$D^*_v(\omega_l) = \sum_{s=1}^{k} i_v(\pi(\theta^*_{ls}))\omega_s, \qquad D^*_u(\omega_l) = i_u(\overline{\partial}\omega_l) = 0$$

where $v \in C^\infty(TM|_U)$ and $u \in C^\infty(\overline{TM}|_U)$ and $\pi : TM^*|_U \to N^*_{\mathscr{F}}|_U$ is the natural projection. Let θ^* be the matrix of the connection D^*, then $\theta := [-\theta^*]^t$ is the matrix of the induced connection D with respect to the frame $\{\omega_1, \ldots, \omega_k\}$.

Let K be the curvature of the connection D of $N_{\mathscr{F}}$ on $M \setminus \mathrm{Sing}(\mathscr{F})$. It follows from Bott's vanishing Theorem [13, Theorem 9.11, pg 76] that $\varphi(K) = 0$. Let V be a small neighborhood of $\mathrm{Sing}_{k+1}(\mathscr{F})$. We regularize θ and K on V , i.e. we choose a matrix of smooth forms $\widehat{\theta}$ and $\widehat{K}$ coinciding with θ and K outside of V, respectively. By hypothesis $\dim(\mathrm{Sing}(\mathscr{F})) \leq n-k-1$ we conclude by a dimensional reason that, for $\deg(\varphi) = k+1$, only the components of dimension $n-k-1$ of $\mathrm{Sing}(\mathscr{F})$ play a role. In fact, since $\mathrm{Res}_\varphi(\mathscr{F}, Z) \in \mathrm{H}_{2(n-k-1)}(Z, \mathbb{C})$, components of dimension smaller than $n-k-1$ contribute nothing. This means that $\varphi(\widehat{K})$ localizes on $\mathrm{Sing}_{k+1}(\mathscr{F})$. Then, $\varphi(\widehat{K})$ has compact support on V, where V is a small neighborhood of $\mathrm{Sing}_{k+1}(\mathscr{F})$. That is,

$$\mathrm{Supp}(\varphi(\widehat{K})) \subset \overline{V}.$$

Then

$$\varphi(\widehat{K}) = \sum_{Z_i} \widehat{\lambda}_i(\varphi)[Z_i],$$

where Z_i is an irreducible component of Sing $_{k+1}(\mathscr{F})$ and $\widehat{\lambda}_i(\varphi) \in \mathbb{C}$. On the other hand, we have that

$$\varphi(N_{\mathscr{F}}) = \sum_{Z_i} \mathrm{Res}(\varphi, \mathscr{F}, Z_i) = \sum_{Z_i} \lambda_i(\varphi)[Z_i].$$

We will show that $\lambda_i(\varphi) = \widehat{\lambda}_i(\varphi)$, for all i. In particular, this implies that $\varphi(\widehat{K}) = \varphi(N_{\mathscr{F}})$. Thereafter, we will determinate the numbers $\widehat{\lambda}_i(\varphi)$.

Consider the unique complete polarization of the polynomial φ, denoted by $\widetilde{\varphi}$. That is, $\widetilde{\varphi}$ is a symmetric k-linear function such that

$$\left(\frac{1}{2\pi i}\right)^{k+1}\widetilde{\varphi}(\widehat{K},\ldots,\widehat{K})=\left(\frac{1}{2\pi i}\right)^{k+1}\varphi(\widehat{K}).$$

Take a generic point $p\in Z_i$, that is, p is a point where Z_i is smooth and disjoint from the other components. Let us consider $L\subset M$ a $(k+1)$-ball intersecting transversally $\mathrm{Sing}_{k+1}(\mathscr{F})$ at a single point $p\in Z_i$ and non intersecting other component. Define

$$BB(\mathscr{F},\varphi;Z_i):=\left(\frac{1}{2\pi i}\right)^{k+1}\int_L\varphi(\widehat{K}). \tag{1}$$

Then $\widehat{\lambda}_i(\varphi)=BB(\mathscr{F},\varphi;Z_i)$. In fact

$$BB(\mathscr{F},\varphi;Z_i)=\left(\frac{1}{2\pi i}\right)^{k+1}\int_L\varphi(\widehat{K})=[L]\cap[\varphi(\widehat{K})]=\widehat{\lambda}_i(\varphi)[L]\cap[Z_i]=\widehat{\lambda}_i(\varphi)$$

since $[L]\cap[Z_i]=1$ and $[L]\cap[Z_i]=0$ for all $i\neq j$. For each $j=1,\ldots,k$, define the polynomial

$$\varphi_j(\widehat{\theta},\widehat{K}):=\widetilde{\varphi}(\widehat{\theta},\underbrace{-2\widehat{\theta}\wedge\widehat{\theta},\ldots,-2\widehat{\theta}\wedge\widehat{\theta}}_{j-1},\underbrace{\widehat{K},\ldots,\widehat{K}}_{k-j}).$$

Now, we consider the $(2k+1)$- form

$$\varphi_\alpha(\widehat{\theta},\widehat{K})=\sum_{j=0}^{k-1}(-1)^j\frac{(k-1)!}{2^j(k-j-1)!(k+j)!}\varphi_{j+1}(\widehat{\theta},\widehat{K}).$$

It follows from [15, Lemma 2.3, pg 5] that on $X\setminus Sing_{k+1}(\mathscr{F})$ we have

$$d(\varphi_\alpha(\widehat{\theta},\widehat{K}))=\varphi(\widehat{K}).$$

Consider $i:B\to M$ an embedding transversal to Z_i on p as above, i.e, $i(B)=L$. We have then an one-dimensional foliation $\mathscr{F}|_L=i^*\mathscr{F}$ on B singular only on $i^{-1}(p)=0$. We have that

$$\widehat{\lambda}_i(\varphi)=BB(\mathscr{F},\varphi;Z_i)=\left(\frac{1}{2\pi i}\right)^{k+1}\int_L\varphi(\widehat{K})=\left(\frac{1}{2\pi i}\right)^{k+1}\int_B\varphi(i^*\widehat{K}).$$

Now, by Stokes's theorem we obtain

$$\begin{aligned}\widehat{\lambda}_i(\varphi)&=\left(\frac{1}{2\pi i}\right)^{k+1}\int_B\varphi(i^*\widehat{K})\\&=\left(\frac{1}{2\pi i}\right)^{k+1}\int_B d(\varphi_\alpha(i^*\widehat{\theta},i^*\widehat{K}))\\&=\left(\frac{1}{2\pi i}\right)^{k+1}\int_{\partial B}\varphi_\alpha(i^*\widehat{\theta},i^*\widehat{K}).\end{aligned}$$

Firstly, it follows from [15, Lemma 4.6] that

$$\widehat{\lambda}_i(\varphi) = \left(\frac{1}{2\pi i}\right)^{k+1} \int_{\partial B} \varphi_\alpha(i^*\widehat{\theta}, i^*\widehat{K}) = \operatorname{Res}_\varphi(i^*\mathscr{F}; 0) \tag{2}$$

Now, we will adopt the Baum and Bott construction [2]. Denote by $\mathcal{A}_M$ the sheaf of germs of real-analytic functions on M. Consider on M a locally free resolution of $N_{\mathscr{F}}$

$$0 \to \mathcal{E}_r \to \mathcal{E}_{r-1} \to \cdots \to \mathcal{E}_0 \to N_{\mathscr{F}} \otimes \mathcal{A}_M \to 0.$$

Let $D_q, D_{q-1}, \ldots, D_0$ be connections for $\mathcal{E}_q, \mathcal{E}_{q-1}, \ldots, \mathcal{E}_0$, respectively. Set the curvature of D_i by $K_i = K(D_i)$. By using Baum-Bott notation [2, pg 297] we have that

$$\varphi(K_q|K_{q-1}|\cdots|K_0) = \varphi(N_{\mathscr{F}}).$$

Consider on V a locally free resolution of the tangent sheaf of $\mathscr{F}$:

$$0 \to \mathcal{E}_q \to \mathcal{E}_{q-1} \to \cdots \to \mathcal{E}_1 \to T\mathscr{F} \otimes \mathcal{A}_V \to 0. \tag{3}$$

Combining this sequence with the sequence

$$0 \to T\mathscr{F} \otimes \mathcal{A}_V \to TV \to N_{\mathscr{F}} \otimes \mathcal{A}_V \to 0.$$

We get

$$0 \to \mathcal{E}_q \to \mathcal{E}_{q-1} \to \cdots \to \mathcal{E}_1 \to TV \to N_{\mathscr{F}} \otimes \mathcal{A}_V \to 0. \tag{4}$$

Pulling back the sequence (3) by $i : B \to V$ we obtain an exact sequence on B:

$$0 \to i^*\mathcal{E}_q \to i^*\mathcal{E}_{q-1} \to \cdots \to i^*\mathcal{E}_1 \to i^*(T\mathscr{F} \otimes \mathcal{A}_V) \to 0. \tag{5}$$

Since B is a small ball we have the splitting $i^*TV = TB \oplus N_{B|V}$, where $N_{B|V}$ denotes its normal bundle. We consider the projection $\xi : i^*TV \to TB$ and we map i^*TV to $N_{i^*\mathscr{F}}$ via

$$i^*TV \stackrel{\xi}{\to} TB \to N_{i^*\mathscr{F}}$$

which give us an exact sequence

$$0 \to i^*(T\mathscr{F} \otimes \mathcal{A}_V) \to i^*TV \to N_{i^*\mathscr{F}} \otimes \mathcal{A}_B \to 0. \tag{6}$$

Now, combining the exact sequences (5) and (6) we obtain an exact sequence

$$0 \to i^*\mathcal{E}_q \to i^*\mathcal{E}_{q-1} \to \cdots \to i^*\mathcal{E}_1 \to i^*TV \to N_{i^*\mathscr{F}} \otimes \mathcal{A}_B \to 0.$$

Let $D_q, D_{q-1}, \ldots, D_0$ be connections for $\mathcal{E}_q, \mathcal{E}_{q-1}, \ldots, \mathcal{E}_1, TV$, respectively. Observe that

$$i^*\varphi(K_q|K_{q-1}|\cdots|K_0) = \varphi(i^*K_q|i^*K_{q-1}|\cdots|i^*K_0) = \varphi(N_{i^*\mathscr{F}}).$$

Finally, it follows from [2, Lemma 7.16]

$$\begin{aligned} \operatorname{Res}_\varphi(i^*\mathscr{F}; 0) &= \left(\frac{1}{2\pi i}\right)^{k+1} \int_B \varphi(i^*K_q|i^*K_{q-1}|\cdots|i^*K_0) \\ &= \left(\frac{1}{2\pi i}\right)^{k+1} \int_B i^*\varphi(K_q|K_{q-1}|\cdots|K_0) \end{aligned}$$

and [2, 9.12, pg 326] that

$$\mathrm{Res}_{\varphi}(i^{*}\mathscr{F};0)=\left(\frac{1}{2\pi i}\right)^{k+1}\int_{B} i^{*}\varphi(K_q|K_{q-1}|\cdots|K_0)=\lambda_i(\varphi). \tag{7}$$

Thus, we conclude from (2) and (7) that $\lambda_i(\varphi)=\widehat{\lambda_i}(\varphi)$, for all i. This implies that $\varphi(\widehat{K})=\varphi(N_{\mathscr{F}})$.

Now, we will determinate the numbers $\widehat{\lambda_i}(\varphi)$. Let $X=\sum_{r=1}^{k+1}X_i\partial/\partial z_i$ be a vector field inducing $i^{*}\mathscr{F}$ on B and $J(X)$ denotes the Jacobian of X. Let ω be the 1-form on $B\setminus\{0\}$ such that $i_X(\omega)=1$. It follows from [15, Corollary 4.7] that

$$\widehat{\lambda_i}(\varphi)=\left(\frac{1}{2\pi i}\right)^{k+1}\int_{\partial B}\varphi_{\alpha}(i^{*}\widehat{\theta},i^{*}\widehat{K})=\left(\frac{1}{2\pi i}\right)^{k+1}\int_{\partial B}\omega\wedge(\overline{\partial}\,\omega)^{k}\varphi(-J(X)).$$

Thus,

$$\widehat{\lambda_i}(\varphi)=\left(\frac{1}{2\pi i}\right)^{k+1}\int_{\partial B}(-1)^{k+1}\omega\wedge(\overline{\partial}\,\omega)^{k}\varphi(J(X)).$$

By using Martinelli's formula [9, pg. 655] we have

$$\widehat{\lambda_i}(\varphi)=\left(\frac{1}{2\pi i}\right)^{k+1}\int_{\partial B}(-1)^{k+1}\omega\wedge(\overline{\partial}\,\omega)^{k}\varphi(J(X))=\mathrm{Res}_0\left[\varphi(JX)\frac{dz_1\wedge\cdots\wedge dz_{k+1}}{X_1\cdots X_{k+1}}\right].$$

Therefore,

$$\widehat{\lambda_i}(\varphi)=\ \mathrm{Res}_{\varphi}(i^{*}\mathscr{F};0)=\mathrm{Res}_{\varphi}(\mathscr{F}|_L;p),$$

where $\mathrm{Res}_{\varphi}(\mathscr{F}|_L;p)$ represents the Grothendieck residue at p of the one dimensional foliation $\mathscr{F}|_L$ on a $(k+1)$-dimensional transversal ball L.

4. Examples. In the next examples, with a slight abuse of notation, we write $\mathrm{Res}(\mathscr{F},\varphi;Z_i)=\lambda_i(\varphi)$.

Example 4.1. Let $\mathscr{F}$ be the logarithmic foliation on $\mathbb{P}^3$ induced, locally in $(\mathbb{C}^3,(x,y,z))$ by the polynomial 1-form

$$\omega=yzdx+xzdy+xydz.$$

In this chart, the singular set of ω is the union of the lines $Z_1=\{x=y=0\}$; $Z_2=\{x=z=0\}$ and $Z_3=\{y=z=0\}$. We have $\mathrm{Res}(\mathscr{F},c_1^2;Z_i)=\mathrm{Res}_{c_1^2}(\mathscr{G};p_i)$, where $\mathscr{G}$ is a foliation on D_i with D_i a 2-disc cutting transversally Z_i. Consider $D_1=\{||(x,y)||\leq 1,\quad z=1\}$ then, we have

$$\omega|_{D_1}=:\omega_1=ydx+xdy\quad\text{with dual vector field}\quad X_1=x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}.$$

Then, $D_1\cap Z_1=\{p_1=(0,0,1)\}$. Now, a straightforward calculation shows that

$$JX_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Thus,

$$\mathrm{Res}_{c_1^2}(\mathscr{G};p_1) = \frac{c_1^2(JX_1(p_1))}{\det(JX_1(p_1))} = 0.$$

The same holds for Z_2 and Z_3. The foliation $\mathscr{F}$ is induced, in homogeneous coordinates $[X,Y,Z,T]$, by the form

$$\widetilde{\omega} = YZTdX + XZTdY + XYTdZ - 3XYZdT.$$

The singular set of $\mathscr{F}$ is the union of the lines Z_1, Z_2, Z_3, and

$$Z_4 = \{T = X = 0\}, \quad Z_5 = \{T = Y = 0\} \text{ and } Z_6 = \{T = X = 0\}.$$

For $Z_4 = \{X = T = 0\}$ we can consider the local chart $U_y = \{Y = 1\}$. Then, we have,

$$\omega_y := \widetilde{\omega}|_{U_y} = ztdx + xtdz - 3xzdt.$$

Take a 2-disc transversal $D_2 = \{||(x,t)|| \leq 1, \ z = 1\}$.

$$\omega_2 := \omega_y|_{D_2} = tdx - 3xdt \quad \text{with dual vector field} \quad X_2 = -3x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t}.$$

Thus, $Z_4 \cap D_2 = \{(0,1,0) =: p_4\}$ and

$$JX_2(p_4) = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore, $\mathrm{Res}_{c_1^2}(\mathscr{G};p_4) = \dfrac{c_1^2(JX_2)(p_4)}{\det(JX_2)(p_4)} = \dfrac{16}{3}$. An analogous calculation shows that

$$\mathrm{Res}_{c_1^2}(\mathscr{G};p_5) = \mathrm{Res}_{c_1^2}(\mathscr{G};p_6) = \frac{16}{3}.$$

Now, we will verify the formula

$$c_1^2(N_{\mathscr{F}}) = \sum_{i=1}^{6} \mathrm{Res}(\mathscr{F}, c_1^2; Z_i)[Z_i].$$

On the one hand, Since $\det(N_{\mathscr{F}}) = \mathscr{O}_{\mathbb{P}^3}(4)$, then

$$c_1^2(N_{\mathscr{F}}) = c_1^2(\det(N_{\mathscr{F}})) = 16h^2,$$

where h represents the hyperplane class. On the other hand, by the above calculations and since $[Z_i] = h^2$, for all i, we have

$$\sum_{i=1}^{6} \mathrm{Res}(\mathscr{F}, c_1^2; Z_i)[Z_i] = 0[Z_1] + 0[Z_2] + 0[Z_3] + \frac{16}{3}[Z_4] + \frac{16}{3}[Z_5] + \frac{16}{3}[Z_6] = 16h^2.$$

The following example is due to D. Cerveau and A. Lins Neto, see [6]. It originates from the so-called exceptional component of the space of codimension one holomorphic foliations of degree 2 of $\mathbb{P}^n$. We can simplify the computation as done by M. Soares in [12].

Example 4.2. Consider $\mathscr{F}$ be a holomorphic foliation of codimension one on $\mathbb{P}^3$, given locally by the 1-form

$$\omega = z(2y^2 - 3x)dx + z(3z - xy)dy - (xy^2 - 2x^2 + yz)dz.$$

The singular set of this foliation has one connect component, denoted by Z, with 3 irreducible components, given by:

1) the twisted cubic $\Gamma: \quad y \longmapsto (2/3y^2, y, 2/9y^3)$,
2) the quadric $Q: \quad y \longmapsto (y^2/2, y, 0)$,
3) the line $L: \quad y \longmapsto (0, y, 0)$.

We consider a transversal 2-disc $D \subset \{y = 1\}$ and we take the restriction of $\mathscr{F}$ on the afine open $\{y = 1\}$. We have an one-dimensional holomorphic foliation, denoted by $\mathscr{G}$, given by the 1-form on H

$$\widetilde{\omega} = (2z - 3xz)dx + (2x^2 - x - z)dz$$

with dual vector field

$$X = (2x^2 - x - z)\frac{\partial}{\partial x} + (-2z + 3xz)\frac{\partial}{\partial z}.$$

The singular set of $\mathscr{G}$ is given by

$$\mathrm{Sing}(X) = \Big\{p_1 = (2/3, 1, 2/9); p_2 = (1/2, 1, 0); p_3 = (0, 1, 0)\Big\}.$$

We know how to calculate the Grothendieck residue of the foliation $\mathscr{G}$:

$$\mathrm{Res}_{c_1^2}(\mathscr{G}; p_1) = \frac{c_1^2(JX(p_1))}{\det(JX(p_1))} = \frac{25}{6},$$

$$\mathrm{Res}_{c_1^2}(\mathscr{G}; p_2) = \frac{c_1^2(JX(p_2))}{\det(JX(p_2))} = -\frac{1}{2},$$

$$\mathrm{Res}_{c_1^2}(\mathscr{G}; p_3) = \frac{c_1^2(JX(p_3))}{\det(JX(p_3))} = \frac{9}{2}.$$

Now, we will verify the formula

$$c_1^2(N_{\mathscr{F}}) = \mathrm{Res}(\mathscr{F}, c_1^2; \Gamma)[\Gamma] + \mathrm{Res}(\mathscr{F}, c_1^2; Q)[Q] + \mathrm{Res}(\mathscr{F}, c_1^2; L)[L].$$

On the one hand, Since $\det(N_{\mathscr{F}}) = \mathcal{O}_{\mathbb{P}^3}(4)$, then

$$c_1^2(N_{\mathscr{F}}) = c_1^2(\det(N_{\mathscr{F}})) = 16h^2,$$

where h represents the hyperplane class. On the other hand, by the above calculations and using that $[\Gamma] = 3h^2$, $[Q] = 2h^2$ and $[L] = h$ we have

$$\sum_{i=1}^{3} \mathrm{Res}(\mathscr{F}, c_1^2; Z_i)[Z_i]. = \frac{25}{6}[\Gamma] - \frac{1}{2}[Q] + \frac{9}{2}[L] = \frac{25}{6}[3h^2] - \frac{1}{2}[2h^2] + \frac{9}{2}[L] = 16h^2.$$

Example 4.3. Let $f : M \dashrightarrow N$ be a dominant meromorphic map such that $\dim(N) = k + 1$ and $\mathscr{G}$ is an one-dimensional foliation on N with isolated singular set $\mathrm{Sing}(\mathscr{G})$. Suppose that $f : M \dashrightarrow N$ is a submersion outside its indeterminacy locus $Ind(f)$. Then, the induced foliation $\mathscr{F} = f^*\mathscr{G}$ on M has codimension k and $\mathrm{Sing}(\mathscr{F}) = f^{-1}(\mathrm{Sing}(\mathscr{G})) \cup Ind(f)$. If $Ind(f)$ has codimension $\geq k + 1$, we conclude that $\mathrm{cod}(\mathrm{Sing}(\mathscr{F})) \geq k+1$. If $q \in f^{-1}(p) \subset M$ is a regular point of the map $f : M \dashrightarrow N$, then

$$\mathrm{Res}(f^*\mathscr{G}, \varphi; f^{-1}(p)) = \mathrm{Res}_\varphi(\mathscr{G}; p)[f^{-1}(p)],$$

where $\mathrm{Res}_\varphi(\mathscr{G}; p)$ represents the Grothendieck residue at $p \in \mathrm{Sing}(\mathscr{G})$. In fact, there exist open sets $U \subset M$ and $V \subset N$, with $q \in f^{-1}(p) \subset U$ and $p \in V$, such that $U \simeq f^{-1}(p) \times V$. Now, if we take a $(k+1)$-ball B in V then by theorem 1.2 we have

$$\mathrm{Res}(f^*\mathscr{G}, \varphi; f^{-1}(p)) = \mathrm{Res}_\varphi(\mathscr{G}|_B; p)[f^{-1}(p)] = \mathrm{Res}_\varphi(\mathscr{G}|; p)[f^{-1}(p)].$$

For instance, if $f : \mathbb{P}^n \dashrightarrow (\mathbb{P}^{k+1}, \mathscr{G})$ is a rational linear projection and $\mathscr{G}$ is an one-dimensional foliation with isolated singularities. Since $Ind(f) = \mathbb{P}^{k+1}$, then $\mathrm{cod}(\mathrm{Sing}(f * \mathscr{G})) = k+1$. Therefore

$$\mathrm{Res}(f^*\mathscr{G}, \varphi; f^{-1}(p)) = \mathrm{Res}(f^*\mathscr{G}, \varphi; \mathbb{P}^{k+1}) = \mathrm{Res}_\varphi(\mathscr{G}; p)[\mathbb{P}^{k+1}].$$

5. Cenkl algorithm for singularities with non-expected dimension . In [7] Cenkl provided an algorithm to determinate residues for non-expected dimensional singularities, under a certain regularity condition on the singular set of the foliation. We observe that this condition is not necessary. In fact, Cenkl 's conditions are the following:

Suppose that the singular set $S := \mathrm{Sing}(\mathscr{F})$ of $\mathscr{F}$ has pure codimension $k+s$, with $s \geq 1$, and

(i) $\mathrm{cod}(S) \geq 4$.

(ii) there exists a closed subset $W \subset M$ such that $S \subset W$ with the property

$$\mathrm{H}^j(W, \mathbb{Z}) \simeq \mathrm{H}^j(W \setminus S, \mathbb{Z}), \quad j = 1, 2.$$

Denote by $M' = M \setminus S$, Cenkl show that under the above condition the line bundle $\wedge^k(N_{\mathscr{F}}|_{M'}^\vee)$ on M' can be extended a line bundle on M. We observe that there always exists a line bundle $\det(N_{\mathscr{F}})^\vee = [\wedge^k(N_{\mathscr{F}})^\vee]^{\vee\vee}$ on M which extends $\wedge^k(N_{\mathscr{F}}|_{M'}^\vee)$, since $N_{\mathscr{F}}$ is a torsion free sheaf and $S = \mathrm{Sing}(N_{\mathscr{F}})$. See, for example [11, Proposition 5.6.10 and Proposition 5.6.12]. Now, consider the vector bundle

$$E_{\mathscr{F}} = \det(N_{\mathscr{F}})^\vee \oplus \det(N_{\mathscr{F}})^\vee.$$

Observe that $E_{\mathscr{F}}|_{M'} = \wedge^k(N_{\mathscr{F}}|_{M'}^\vee) \oplus \wedge^k(N_{\mathscr{F}}|_{M'}^\vee)$. Thus, we conclude that Lemma 1 in [7] holds in general:

LEMMA 5.1. *Consider the projective bundle* $\pi : \mathbb{P}(E_{\mathscr{F}}) \to M$. *Then there exist a holomorphic foliation* $\mathscr{F}_\pi$ *on* $\mathbb{P}(E_{\mathscr{F}})$ *with singular set* $\mathrm{Sing}(\mathscr{F}_\pi) = \pi^{-1}(S)$ *such that*

$$\dim(\mathscr{F}_\pi) = \dim(\mathscr{F}) \quad \textit{and} \quad \dim(\mathrm{Sing}(\mathscr{F}_\pi)) = \dim(S) + 1.$$

We succeeded in replacing the compact manifold M with a foliation $\mathscr{F}$ and the singular set S such that $\dim(\mathscr{F}) - \dim(\mathrm{Sing}(\mathscr{F})) = n-s$ by another compact manifold $\mathbb{P}(E_{\mathscr{F}})$ and a foliation $\mathscr{F}_\pi$ with singular set $\dim(\mathscr{F}_\pi) - \dim(\mathrm{Sing}(\mathscr{F}_\pi)) = n-s-1$. If this procedure is repeated $(n-s-1)$-times we end up with a compact complex analytic manifold with a holomorphic foliation whose singular set is a subvariety of complex dimension one less than the leaf dimension of the foliation. That is, we have a tower of foliated manifolds

$$(P_{n-s-1}, \mathscr{F}^{n-s-1}) \xrightarrow{\pi_{n-s-1}} (P_{n-s-2}, \mathscr{F}^{n-s-2}) \longrightarrow \cdots \xrightarrow{\pi_2} (P_1, \mathscr{F}^1) \xrightarrow{\pi_1 := \pi} (M, \mathscr{F})$$

where $(P_i, \mathscr{F}^i)$ is such that $P_i = \mathbb{P}(E_{\mathscr{F}^{i-1}})$ and $(P_1, \mathscr{F}^1) = (\mathbb{P}(E_{\mathscr{F}}), \mathscr{F}_\pi)$. Thus, by Lemma 5.1 we conclude that on P_{n-s-1} we have a foliations $\mathscr{F}^{n-s-1}$ such that $\mathrm{Sing}(\mathscr{F}^{n-s-1}) = (\pi_{n-s-1} \circ \cdots \circ \pi_2 \circ \pi_1)^{-1}(S)$ and

$$\dim(\mathrm{Sing}(\mathscr{F}^{n-s-1})) = \dim(\mathscr{F}^{n-s-1}) - 1.$$

That is, $\mathrm{cod}(\mathrm{Sing}(\mathscr{F}^{n-s-1})) = \mathrm{cod}(\mathscr{F}^{n-s-1}) + 1$.

On the one hand, we can apply the Theorem 1.2 to determinate the residues of $\mathscr{F}^{n-s-1}$. On the other hand, Cenkl show that we can calculate the residue $\mathrm{Res}_\varphi(\mathscr{F}^1, Z_1)$ in terms of the residue $\mathrm{Res}_\varphi(\mathscr{F}, Z)$ for symmetric polynomial φ of degree $k+1$.

Let us recall the Cenkl's construction:

Let $\sigma_1, \ldots, \sigma_\ell$ be the elementary symmetric functions in the n variables $x_1, \ldots, x_n$ and let $\rho_1, \ldots, \rho_\ell$ be the elementary symmetric functions in the $n+1$ variables $x_1, \ldots, x_n, y$. It follows from [7, Corollary, pg 21] that for any polynomial ϕ, of degree ℓ, can be associated a polynomial ψ of degree $\ell + 1$ such that

$$\psi(\rho_1, \ldots, \rho_\ell) = \phi(\sigma_1, \ldots, \sigma_\ell) y + \phi^0(\sigma_1, \ldots, \sigma_\ell) + \sum_{j \geq 2} \phi^j(\sigma_1, \ldots, \sigma_\ell) \cdot y^j,$$

where ϕ^0 has degree $\ell + 1$ and ϕ^j has degree $\ell - j + 1$.

Let $T_{P/M}$ be the tangent bundle associated the one-dimensional foliation induced by the $\mathbb{P}^1$-fibration $(P, \mathscr{F}_\pi) \to (M, \mathscr{F})$.

Therefore, it follows from Lemma 5.1, Cenkl's construction [7, Theorem 1] and Theorem 1.2 the following :

THEOREM 5.2. *Suppose that* $\mathrm{cod}(\mathrm{Sing}(\mathscr{F})) \geq \mathrm{cod}(\mathscr{F}) + 2$. *If* φ *is a homogeneous symmetric polynomials of degree* $\mathrm{cod}(\mathscr{F}) + 1$, *then*

$$\begin{aligned} &\mathrm{Res}_\psi(\mathscr{F}^1|_{B_p}; p)[Z_1] \\ &= \pi^* \mathrm{Res}_\varphi(\mathscr{F}, Z) \cap c_1(T_{P/M}) + \pi^*(\phi^0(N_{\mathscr{F}})) + \sum_{j \geq 2} \pi^*(\phi^j(N_{\mathscr{F}})) \cap c_1(T_{P/M})^j, \end{aligned}$$

where $\mathrm{Res}_\psi(\mathscr{F}^1|_{B_p}; p)$ *represents the Grothendieck residue at* p *of the one dimensional foliation* $\mathscr{F}^1|_{B_p}$ *on a* $(k+1)$*-dimensional transversal ball* B_p.

We believe that this algorithm can be adapted to the context of residues for flags of foliations [4].

REFERENCES

[1] P. BAUM AND R. BOTT, *On the zeros of meromorphic vector fields, Essay on Topology and Related Topics*, Spring-Verlag, New York, 1970, pp. 29–47.

[2] P. BAUM AND R. BOTT, *Singularities of holomorphic foliations*, J. Differential Geom., 7 (1972), pp. 279–342.

[3] F. BRACCI AND T. SUWA, *Perturbation of Baum-Bott residues*, Asian J. Math., 19:5 (2015), pp. 871–886

[4] J-P. Brasselet, M. Corrêa and F. Lourenço, *Residues for flags of holomorphic foliations*, Advances in Mathematics, 320:7 (2017), pp. 1158–1184.
[5] M. Brunella and C. Perrone, *Exceptional singularities of codimension one holomorphic foliations*, Publicacions Matemàtiques, 55 (2011), pp. 295–312.
[6] D. Cerveau and A. Lins Neto, *Irreducible components of the space of holomorphic foliations of degree two in CP(n)* , Ann. Math., 143 (1996), pp. 577–612.
[7] B. Cenkl, *Residues of singularities of holomorphic foliations*, J. of Differential Geometry, 13 (1978), pp. 11–23.
[8] M. Corrêa and A. Fernandéz-Pérez, *Absolutely k-convex domains and holomorphic foliations on homogeneous manifolds*, Journal of the Mathematical Society of Japan, 69:3 (2017), pp. 1235–1246.
[9] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, 1978.
[10] J-P. Jouanolou, *Equations de Pfaff algébriques*, 1979 Lecture Notes in Mathematics, 708. Springer-Verlag, Berlin.
[11] S. Kobayashi, *Differential geometry of complex vector bundles*, Princeton Univ. Press, 1987.
[12] M. Soares, *Holomorphic foliations and characteristic numbers*, Comm. Contemporary Maths., 7:5 (2005), pp. 583–596.
[13] T. Suwa, *Indices of Vector Fields and Residues of Singular Holomorphic Foliations*, Actualités Mathématiques, Hermann, Paris 1998.
[14] T. Suwa, *Residues of Complex analytic Foliations Singularities*, J. Math. Soc. Japan., 36 (1984), pp. 37–45.
[15] M. S. Vishik, *Singularities of analytic foliations and characteristic classes*, Functional Anal. Appl., 7 (1973), pp. 1–15.