

ON STATIONARY SOLUTIONS TO THE VACUUM EINSTEIN FIELD EQUATIONS*

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Abstract. We prove that any 4-dimensional geodesically complete spacetime with a timelike Killing field satisfying the vacuum Einstein field equation $Ric(g_M) = \lambda g_M$ with nonnegative cosmological constant $\lambda \geq 0$ is flat. When $\dim \geq 5$, if the spacetime is assumed to be static additionally, we prove that its universal cover splits isometrically as a product of a Ricci flat Riemannian manifold and a real line \mathbb{R} .

Key words. Stationary solutions, vacuum Einstein field equations.

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1. Introduction. A Lorentzian manifold (M, g_M) or a spacetime is a differentiable manifold M equipped with a Lorentzian metric g_M of signature $(-1, +1, \dots, +1)$. In general relativity, the gravity is described by a spacetime 4-manifold (M, g_M) , the Lorentzian metric g_M satisfies the Einstein field equation:

$$Ric(g_M) - \frac{1}{2}Rg_M + \Lambda g_M = \kappa T \tag{1.1}$$

where T is the energy-momentum tensor due to the presence of matter or fields, κ and Λ are constants.

In this paper, we are interested in the solutions to (1.1) with timelike Killing fields. These solutions are called stationary solutions. Stationary solutions are used to model the possible time-independent limit states of a cosmological system. For instance, Kerr metrics are stationary and vacuum solutions ($T = 0, \Lambda = 0$) to (1.1), while Schwarzschild metrics are static and vacuum solutions ($T = 0, \Lambda = 0$). Here, static means that the spacetime has a timelike Killing field whose orthogonal complement is an integrable distribution, i.e., the timelike Killing field is locally orthogonal to spacelike hypersurfaces. These stationary solutions, including Schwarzschild, Kerr, Reissner-Nordstrom (electrovac static), Kerr-Newmann metrics (electrovac stationary), have been central to the study of the black hole spacetimes, see [7] [16].

If the spacetime (M, g_M) admits an isometric \mathbb{R} -action such that the \mathbb{R} -orbits are timelike curves, the infinitesimal generator of the \mathbb{R} -action is a timelike Killing field. In many literatures, the terminology "stationary" was also used to referring to the existence of such global \mathbb{R} -action. Here, our usage of "stationary" is in a broader sense, it only refers to the existence of a timelike Killing field.

Recall that a Lorentzian manifold (M, g_M) is said to be chronological if it contains no closed timelike curves. In [1], M. T. Anderson proved that if the spacetime (M^4, g_M) is geodesically complete, chronological and admits an isometric timelike \mathbb{R} -action such that g_M satisfies the vacuum Einstein field equation $Ric(g_M) \equiv 0$, then (M^4, g_M) must be flat. Here, (M^4, g_M) is said to be geodesically complete if the affine parameters of any g_M -geodesic on M can be extended to the whole real line \mathbb{R} . When the \mathbb{R} -orbit space M/\mathbb{R} is an asymptotically flat 3-manifold, the result was due to A. Lichnerowicz [14] in 1955. See [9] for the previous work of A. Einstein and W. Pauli.

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The chronological condition in [1] and [14] was used to ensure that the \mathbb{R} -orbit space M/\mathbb{R} (denoted by N) is a paracompact Hausdorff and smooth manifold, see [11]. Actually, in this case, the manifold M is diffeomorphic to $\mathbb{R} \times N$, and the metric g_M has the following global form (see [1],[11], [13])

$$g_M = -u^2(dt + \pi^*\theta)^2 + \pi^*g_N, \quad (1.2)$$

on $M \approx \mathbb{R} \times N$, where u, θ are some function and 1-form on N , g_N is a Riemannian metric on N , $\pi : M \rightarrow N$ is the projection map from M to the space of \mathbb{R} -orbits N . The argument in [1] used the collapsing theory (cf.[3]) for a sequence of 3-Riemannian manifolds which are the orbit spaces of the isometric \mathbb{R} -actions. When the orbit spaces N are noncompact and have dimension equal to 3, M. T. Anderson [1] argued that the collapsing can be unwrapped by considering their universal covers. Recently, J. Cortier and V. Minerbe [6] gave a new proof of M. T. Anderson's theorem [1] under an extra assumption on the norm of the timelike Killing field $\frac{\partial}{\partial t}$.

In [1] (§1 second paragraph), M.T. Anderson asked whether the chronological condition can be removed in his theorem. We remark that without chronological condition, the orbit space could be very "bad". A simple compact example is Minkowski flat torus T^2 (see [11]), here we take the constant vector field with irrational slope as a timelike Killing field. In this case, any Killing orbit is dense in T^2 , so the quotient topology just consists of two elements: the empty set and the whole space. For noncompact examples, one can take a product of such a torus with a real line.

One of the main results of this paper gives an affirmative answer to M. T. Anderson's question:

THEOREM 1.1. *Let (M, g_M) be a geodesically complete spacetime of dimension 4 with a timelike Killing field X such that g_M satisfies the Einstein equation $\text{Ric}(g_M) = \lambda g_M$, where $\lambda \geq 0$. Then (M, g_M) is flat.*

The Einstein equation satisfied by the spacetime in Theorem 1.1 is equivalent to $T = 0$ and $\Lambda \geq 0$ in (1.1). We remark that when $\lambda < 0$, the result of Theorem 1.1 is not true. The simplest counterexamples are anti-De Sitter spacetimes, which are static, geodesically complete, and satisfy $\text{Ric}(g_M) = \lambda g_M$ for $\lambda < 0$.

Direct generalization of Theorem 1.1 to higher dimensions is not true, because we have to allow non-flat examples which are product of a Ricci flat Riemannian manifold with a real line. So when dimension ≥ 5 , the best we can hope is a splitting result. Actually, if the spacetimes are assumed to be static, we can prove that it is really the case:

THEOREM 1.2. *Let (M, g_M) be a geodesically complete spacetime of dimension $n+1$ with a timelike Killing field whose orthogonal complement is integrable. Suppose the metric g_M satisfies the Einstein equation $\text{Ric}(g_M) = \lambda g_M$, where $\lambda \geq 0$. Then $\lambda = 0$ and the universal cover of (M, g_M) is isometric to $\mathbb{R} \times N$ equipped with a product metric $-dt^2 + g_N$, where (N, g_N) is a complete Ricci flat Riemannian manifold of dimension n .*

As we mentioned before, the result in Theorem 1.2 is not true for $\lambda < 0$.

It should be noted that recently M. Reiris [15] has shown Theorem 1.2 under the chronological condition. More precisely, M. Reiris [15] has obtained the same result for static solutions to Einstein-scalar equation under the assumption that the spacetime splits topologically as $M \approx \mathbb{R} \times N$ and the metric has global form (1.2) with $\theta = 0$.

Theorems 1.1 and 1.2 are derived by proving a local curvature estimate or a local gradient estimate of the norm of the Killing field X . To state the result, we need to introduce a Riemannian metric which is naturally associated to the stationary spacetime (M, g_M, X) .

Let X^* be the 1-form on M obtained from X by lowering indices. We define

$$\hat{g} = -\frac{2}{g_M(X, X)}X^* \otimes X^* + g_M, \tag{1.3}$$

which is a Riemannian metric on M . It can be shown that the vector field X is still a Killing field for the metric \hat{g} . In other words, one can associate a stationary Riemannian metric \hat{g} to a stationary Lorentzian metric g_M with the same Killing field. See [4] and [5] for similar ideas in treating the injectivity radius estimate and local optimal regularity of Einstein spacetimes.

Our local curvature or gradient estimates are the followings:

THEOREM 1.3. *Let (M, g_M) be a spacetime of dimension 4 with a timelike Killing field X and g_M satisfies the Einstein equation $Ric(g_M) = \lambda g_M$. Let $\hat{B}(x_0, a)$ be a \hat{g} -metric ball centered at x_0 of radius $a > 0$ with compact closure in M . Then there is a universal constant $C > 0$ such that*

$$\sup_{x \in \hat{B}(x_0, \frac{a}{2})} |Rm(g_M)|_{\hat{g}} \leq C(a^{-2} + \max\{-\lambda, 0\}). \tag{1.4}$$

THEOREM 1.4. *Let (M, g_M) be a spacetime of dimension $n + 1$ with a timelike Killing field X whose orthogonal complement is integrable, and g_M satisfies Einstein equation $Ric(g_M) = \lambda g_M$. Let $\hat{B}(x_0, a)$ be a \hat{g} -metric ball centered at x_0 of radius $a > 0$ with compact closure in M . Then there is a universal constant $C > 0$ such that*

$$\sup_{x \in \hat{B}(x_0, \frac{a}{2})} |\hat{\nabla} \log(-g_M(X, X))|_{\hat{g}} \leq C(\sqrt{na}^{-1} + \sqrt{\max\{-\lambda, 0\}}). \tag{1.5}$$

Note that $\max\{-\lambda, 0\} = 0$ if $\lambda \geq 0$ in Theorems 1.3 and 1.4. The central idea of the proof of Theorem 1.3 is a harmonic map estimate as in [1] in a slightly different setting. The map involved in the proof is called Ernst potential, which was favored previously to be treated as a 3-d object as a reduction of the Einstein field equation. In our paper, treating the Ernst potential map as defined locally on 4-d Riemannian manifold (M, \hat{g}) is crucial in the proof of Theorem 1.3.

When dimension equals to 4, we can actually show that a local curvature estimate holds on more general spacetimes which are not necessarily vacuum (see Theorems 5.3, 5.4). The result roughly says that if the energy momentum tensor is controlled, then the full curvature tensor of the spacetime can also be controlled in a quantitative manner.

The paper is organized as follows. In section 2, we prepare some preliminary formulas that will be used throughout the paper. Sections 2.1, 2.2 and 2.3 involve many straight forward computations on the connections and curvatures, and the formulas work for stationary Lorentzian and Riemannian manifolds. In section 3, we prove that the g_M -geodesic completeness implies the \hat{g} -geodesic completeness. In section 4, we prove Theorems 1.4 and 1.2. In section 5, we prove Theorems 1.3 and 1.1.

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2. Stationary spacetime and its associated Riemannian metric. Suppose (M, g_M, X) is a stationary spacetime of dimension $n + 1$, where g_M is a smooth Lorentzian metric on M and X is a timelike Killing field. Denote the set of integral curves of X by N , $\pi : M \rightarrow N$ the projection map.

2.1. A local coordinate system. Fix a point $P \in M$, we will construct a natural coordinate system $\{x^\alpha\}$ around P in the followings so that the metric has the form (1.2) locally.

Let Ψ_τ be the (local) diffeomorphisms generated by X such that $\Psi_0 = id$, and $\Psi_{\tau_1}\Psi_{\tau_2} = \Psi_{\tau_1+\tau_2}$ wherever they are defined. Now we fix a codimensional one spacelike submanifold $\Sigma \subset M$ passing through P such that $\bar{\Sigma}$ is compact. Considering the affine parameters of all integral curves of X starting from Σ , we obtain a function t defined on an open neighborhood of Σ in M such that $t = 0$ on Σ . Given a local coordinate system (x^1, \dots, x^n) on Σ around P . We can construct a local coordinate system (x^0, x^1, \dots, x^n) on M around P , where $x^0 = t$. Actually, for any point $Q \in \Sigma$ lying in the coordinate chart in Σ , we require $x^i(\Psi_t(Q)) = x^i(Q)$ for $i = 1, 2, \dots, n$, and $x^0(\Psi_t(Q)) = t$. Throughout the paper, we use Greek letters α, β, \dots to indicate the indices varying from 0 to n and Latin letters i, j, k, \dots varying from 1 to n . The coordinate system $\{x^\alpha\}$ depends on the choice of spacelike submanifold Σ and the coordinate system $\{x^i\}$ on Σ .

Let X^* be the 1-form obtained by lowering indices of X . The induced Riemannian metric on the horizontal distribution $\mathcal{H} = X^\perp$ is given by $g_{\mathcal{H}} = g_M - \frac{1}{g_M(X, X)}X^* \otimes X^*$. It is clear that the horizontal metric $g_{\mathcal{H}}$ and 1-form $\theta = -u^{-2}(X^* + u^2 dt)$ satisfy $\mathcal{L}_X g_{\mathcal{H}} = 0$, $\mathcal{L}_X \theta = 0$, where $u^2 = -g_M(X, X)$, \mathcal{L}_X is the Lie derivative of the vector field X .

It we choose a different spacelike submanifold, say Σ' , and denote the corresponding time function and one-form by t' and θ' , we have $\theta - \theta' = d\psi$, where $\psi = t' - t$ is a locally defined smooth function. Note that the integrability of the horizontal distribution \mathcal{H} is equivalent to $dX^* = 0 \text{ mod } X^*$ (Frobenius condition), or $d\theta = 0$.

The metric g_M now has the following form

$$g_M = -u^2(dt + \theta)^2 + g_{\mathcal{H}}, \quad (2.1)$$

on a neighborhood of P . In the above local coordinate system $\{x^\alpha\}$, we have

$$\begin{aligned} g_{\mathcal{H}}\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^\alpha}\right) &= \theta\left(\frac{\partial}{\partial x^0}\right) = 0, \\ \frac{\partial}{\partial t}g_{ij} &= \frac{\partial}{\partial t}\theta_i = \frac{\partial}{\partial t}u^2 = 0, \end{aligned} \quad (2.2)$$

where $g_{ij} \triangleq g_{\mathcal{H}}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$, $\theta_i = \theta\left(\frac{\partial}{\partial x^i}\right)$. Roughly speaking, the equations in (2.2) say that u , θ , g_{ij} are essentially quantities on the space N of X -integral curves. Actually, if we identify Σ with $\pi(\Sigma)$ using the projection map $\pi : M \rightarrow N$ and equip $\pi(\Sigma) \subset N$ the Riemannian structure from $(\Sigma, g_{\mathcal{H}})$. The equations in (2.2) are equivalent to $u = \pi^*u$, $\theta = \pi^*(\theta)$, $\pi^*g = g$.

2.2. Connection and curvature matrices. Now we will do some straightforward calculations for metrics of the form $\bar{g} = w(dt + \theta)^2 + g$, where w, θ, g are t -independent. The metric is Lorentzian if $w < 0$, and Riemannian if $w > 0$.

Since $\bar{g}_{00} = w$, $\bar{g}_{0i} = w\theta_i$, $\bar{g}_{ij} = g_{ij} + w\theta_i\theta_j$, one can calculate the inverse matrix $(\bar{g}^{\alpha\beta})$ of $(\bar{g}_{\alpha\beta})$:

$$\bar{g}^{00} = w^{-1} + |\theta|^2, \bar{g}^{0i} = -\theta^i, \bar{g}^{ij} = g^{ij}, \tag{2.3}$$

where $\theta^i = g^{ij}\theta_j$ and $|\theta|^2 = g^{ij}\theta_i\theta_j$. It is useful to choose a good frame to calculate the connection coefficients. Let $e_0 = \frac{\partial}{\partial t}$, $e_i = \frac{\partial}{\partial x^i} - \theta_i \frac{\partial}{\partial t}$. It can be shown that $[e_0, e_j] = 0$, $[e_i, e_j] = -\Lambda_{ij}e_0$ and $\langle e_0, e_i \rangle = 0$, $\langle e_i, e_j \rangle = g_{ij}$, where $\Lambda_{ij} = \nabla_i\theta_j - \nabla_j\theta_i$, $\nabla_i\theta_j$ is the covariate derivative of the tensor θ w.r.t. the horizontal metric g .

The dual frame of $\{e_\alpha\}$ is $\{\omega^\alpha\}$, $\omega^0 = dt + \theta$, $\omega^i = dx^i$, $i = 1, 2, \dots, n$. They satisfy $\langle \omega^0, \omega^0 \rangle = w^{-1}$, $\langle \omega^0, \omega^i \rangle = 0$, $\langle \omega^i, \omega^j \rangle = g^{ij}$. Denote the Levi-Civita connection matrix w.r.t. the basis e_α by $\bar{\omega}_\alpha^\beta$, $De_\alpha = \bar{\omega}_\alpha^\beta \otimes e_\beta$. Recall Cartan's equations,

$$\begin{aligned} d\omega^\alpha &= \omega^\beta \wedge \bar{\omega}_\beta^\alpha \\ d\langle \omega^\alpha, \omega^\beta \rangle &= -\bar{\omega}_\gamma^\alpha \langle \omega^\gamma, \omega^\beta \rangle - \bar{\omega}_\gamma^\beta \langle \omega^\gamma, \omega^\alpha \rangle \\ \bar{\Omega}_\alpha^\beta &= d\bar{\omega}_\alpha^\beta - \bar{\omega}_\alpha^\gamma \wedge \bar{\omega}_\gamma^\beta. \end{aligned} \tag{2.4}$$

The 1st equation in (2.4) says that the connection is torsion free, the 2nd says it is compatible with the metric \bar{g} . The 3rd equation in (2.4) is the definition of the curvature matrix $\{\bar{\Omega}_\alpha^\beta\}$. The connection matrix $\{\bar{\omega}_\alpha^\beta\}$ is completely determined by the first two equations in (2.4). Actually, the 2nd equation (2.4) takes the following form

$$\begin{aligned} dg^{ij} &= -\bar{\omega}_k^i g^{kj} - \bar{\omega}_k^j g^{ki} \\ 0 &= -\bar{\omega}_k^0 g^{ki} - \bar{\omega}_0^i w^{-1} \\ dw^{-1} &= -2\bar{\omega}_0^0 w^{-1}. \end{aligned} \tag{2.5}$$

The 1st equation in (2.4) is

$$\begin{aligned} d\theta &= d\omega^0 = \omega^k \wedge \bar{\omega}_k^0 + \omega^0 \wedge \bar{\omega}_0^0 \\ 0 &= d\omega^i = \omega^k \wedge \bar{\omega}_k^i + \omega^0 \wedge \bar{\omega}_0^i. \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we have

$$\begin{aligned} \bar{\omega}_j^i &= \omega_j^i + \frac{1}{2}wg^{il}\Lambda_{jl}\omega^0 \\ \bar{\omega}_0^i &= -\frac{1}{2}wg^{il}\Lambda_{lk}\omega^k - \frac{1}{2}g^{il}\nabla_l w\omega^0 \\ \bar{\omega}_i^0 &= \frac{1}{2}\Lambda_{il}\omega^l + \frac{1}{2}w^{-1}\nabla_i w\omega^0 \\ \bar{\omega}_0^0 &= \frac{1}{2}w^{-1}dw, \end{aligned} \tag{2.7}$$

where $\{\omega_j^i\}$ is the connection matrix of the horizontal metric $g = g_{\mathcal{H}}$ w.r.t. the natural frame $\{\frac{\partial}{\partial x^i}\}$. Now one can calculate the curvature matrix by using (2.7) and the 3rd

equation in (2.4):

$$\begin{aligned}
 \bar{\Omega}_j^i &= \Omega_j^i + \frac{w}{4}g^{il}(\Lambda_{jp}\Lambda_{lq} + \Lambda_{jl}\Lambda_{pq})\omega^p \wedge \omega^q \\
 &\quad + \left[\frac{1}{2}g^{il}\Lambda_{jl}dw + \frac{1}{2}wg^{il}\nabla_p\Lambda_{jl}\omega^p + \frac{1}{4}(\Lambda_{jl}\omega^l\nabla^i w - \nabla_j w\Lambda_{lp}g^{il}\omega^p)\right] \wedge \omega^0 \\
 \bar{\Omega}_0^i &= g^{il}\left[-\frac{w}{4}\nabla_l\Lambda_{pq} - \frac{1}{8}(w_p\Lambda_{lq} - w_q\Lambda_{lp}) - \frac{1}{4}\nabla_l w\Lambda_{pq}\right]\omega^p \wedge \omega^q \\
 &\quad + g^{il}\left(-\frac{1}{2}\nabla_{pl}w + \frac{1}{4}w^{-1}\nabla_p w\nabla_l w + \frac{w^2}{4}\Lambda_{pq}\Lambda_{lm}g^{qm}\right)\omega^p \wedge \omega^0.
 \end{aligned}
 \tag{2.8}$$

Using the formula $D_{e_\alpha} = \bar{\omega}_\alpha^\beta \otimes e_\beta$, (2.7) may be paraphrased as follows:

$$\begin{aligned}
 D_{e_i}e_j &= \Gamma_{ij}^k e_k - \frac{1}{2}\Lambda_{ij}e_0 \\
 D_{e_0}e_i &= D_{e_i}e_0 = \frac{1}{2}w\Lambda_{ik}g^{kl}e_l + \frac{1}{2}\nabla_i \log |w|e_0 \\
 D_{e_0}e_0 &= -\frac{1}{2}g^{ij}\nabla_i w e_j.
 \end{aligned}
 \tag{2.9}$$

Using $\Omega_\beta^\alpha = \frac{1}{2}\bar{R}_{\beta\gamma\delta}^\alpha \omega^\gamma \wedge \omega^\delta$ and $\bar{R}(e_\alpha, e_\beta, e_\gamma, e_\delta) = \langle e_\alpha, e_\epsilon \rangle \bar{R}_{\beta\gamma\delta}^\epsilon$, (2.8) can be rewritten as

$$\begin{aligned}
 \bar{R}(e_i, e_j, e_k, e_l) &= R_{ijkl} + \frac{w}{4}(\Lambda_{il}\Lambda_{jk} - \Lambda_{ik}\Lambda_{jl}) - \frac{w}{2}\Lambda_{ij}\Lambda_{kl} \\
 \bar{R}(e_i, e_j, e_k, e_0) &= -\frac{1}{2}(w\nabla_k\Lambda_{ij} + \nabla_k w\Lambda_{ij}) + \frac{1}{4}(\nabla_i w\Lambda_{jk} - \nabla_j w\Lambda_{ik}) \\
 \bar{R}(e_i, e_0, e_j, e_0) &= -\frac{1}{2}\nabla_{ij}w + \frac{1}{4}w^{-1}\nabla_i w\nabla_j w + \frac{w^2}{4}\Lambda_{ik}\Lambda_{jl}g^{kl}.
 \end{aligned}
 \tag{2.10}$$

Here, our convention for the sign of the curvature tensor is that we require $R_{ijij} > 0$ on spheres. (2.10) can also be obtained alternatively by using (2.9) and the following formula:

$$\bar{R}(e_\alpha, e_\beta, e_\gamma, e_\delta) = -\langle (D_{e_\alpha}D_{e_\beta} - D_{e_\beta}D_{e_\alpha} - D_{[e_\alpha, e_\beta]})e_\gamma, e_\delta \rangle.$$

Since we have computed all connection coefficients (see (2.9)), it is not difficult to compute the Hessian and the Laplacian of any time-independent function f :

$$\begin{aligned}
 \bar{\nabla}^2 f(e_0, e_0) &= \frac{1}{2}\langle \nabla w, \nabla f \rangle \\
 \bar{\nabla}^2 f(e_0, e_j) &= -\frac{1}{2}wg^{kl}\Lambda_{jk}f_l \\
 \bar{\nabla}^2 f(e_i, e_j) &= \nabla_{ij}f
 \end{aligned}
 \tag{2.11}$$

and

$$\bar{\Delta} f = \Delta f + \frac{1}{2}\langle \nabla \log |w|, \nabla f \rangle.
 \tag{2.12}$$

The formulas (2.11) and (2.12) are important in the calculations of sections 4 (see (4.4)) and 5.

2.3. Ricci curvature. By taking traces on (2.10), we get the Ricci curvature formula:

$$\begin{aligned} \bar{Ric}(e_0, e_0) &= -\frac{\Delta w}{2} + \frac{|\nabla w|^2}{4w} + \frac{w^2}{4}|\Lambda|^2 \\ \bar{Ric}(e_0, e_j) &= \frac{w}{2}g^{kl}(\nabla_k \Lambda_{jl} + \frac{3}{2}\Lambda_{jk}\nabla_l \log w) \\ \bar{Ric}(e_i, e_j) &= R_{ij} - \frac{\nabla_i \nabla_j w}{2w} + \frac{\nabla_i w \nabla_j w}{4w^2} - \frac{w}{2}g^{kl}\Lambda_{ik}\Lambda_{jl}, \end{aligned} \tag{2.13}$$

where R_{ij} is the Ricci curvature of the horizontal metric g_{ij} .

Let $w = -u^2 < 0$ in (2.1) and (2.13), we have

$$\begin{aligned} \Delta u &= -\frac{u^3}{4}|\Lambda|^2 + u^{-1}\bar{Ric}(e_0, e_0) \\ g^{kl}(\nabla_k \Lambda_{jl} + 3\Lambda_{jk}\nabla_l \log u) &= -2u^{-2}\bar{Ric}(e_0, e_j) \\ R_{ij} &= u^{-1}\nabla_i \nabla_j u - \frac{u^2}{2}g^{kl}\Lambda_{ik}\Lambda_{jl} + \bar{Ric}(e_i, e_j). \end{aligned} \tag{2.14}$$

Let

$$\hat{g} = -\frac{2}{g_M(X, X)}X^* \otimes X^* + g_M \tag{2.15}$$

be the Riemannian metric defined in (1.3) on M . Combining (2.14) and (2.13), we get

$$\begin{aligned} \hat{Ric}(e_0, e_0) &= \frac{u^4}{2}|\Lambda|^2 - \bar{Ric}(e_0, e_0) \\ \hat{Ric}(e_0, e_j) &= -\bar{Ric}(e_0, e_j) \\ \hat{Ric}(e_i, e_j) &= -u^2g^{kl}\Lambda_{ik}\Lambda_{jl} + \bar{Ric}(e_i, e_j), \end{aligned} \tag{2.16}$$

where \hat{Ric} and \bar{Ric} are Ricci curvatures of metrics \hat{g} and \bar{g} .

It is helpful to introduce a new metric \tilde{g} conformal to g on horizontal distribution. This metric will play an important role in a priori estimates (see Section 5.3). Let $\tilde{g} = u^{\frac{2}{n-2}}g$ be a conformal change of the horizontal metric g . The Christoffel symbols of \tilde{g} can be given by (see Chapter 5 in [17])

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \frac{1}{n-2}(\nabla_i \log u \delta_j^k + \nabla_j \log u \delta_i^k - g^{kl}\nabla_l \log u g_{ij}). \tag{2.17}$$

This implies

$$u^{\frac{2}{n-2}}\tilde{\Delta}L = \Delta L + \langle \nabla \log u, \nabla L \rangle = \hat{\Delta}L \tag{2.18}$$

for any t -independent smooth function L on M .

The Ricci curvature of \tilde{g} can be computed by the following formula (see Chapter 5 in [17]):

$$\begin{aligned} \tilde{R}_{ij} &= R_{ij} - \nabla_{ij}^2 \log u + \frac{1}{n-2}(\log u)_i(\log u)_j - \frac{1}{n-2}\frac{\Delta u}{u}g_{ij} \\ &= \frac{u^2}{4(n-2)}|\Lambda|^2 g_{ij} - \frac{u^2}{2}g^{kl}\Lambda_{ik}\Lambda_{jl} + \frac{n-1}{n-2}\frac{u_i u_j}{u^2} \\ &\quad + \bar{Ric}(e_i, e_j) - \frac{u^{-2}\bar{Ric}(X, X)}{n-2}g_{ij}, \end{aligned} \tag{2.19}$$

where we have used (2.14). By direct computations, we also have

$$\begin{aligned}
 u^{\frac{2}{n-2}} \tilde{\Delta} \log u &= -\frac{u^2}{4} |\Lambda|^2 + u^{-2} \bar{Ric}(X, X) \\
 g^{kl} (\tilde{\nabla}_k \Lambda_{jl} + (3 + \frac{4-n}{n-2}) \Lambda_{jk} \nabla_l \log u) &= -2u^{-2} \bar{Ric}(e_0, e_j).
 \end{aligned}
 \tag{2.20}$$

COROLLARY 2.1. *Let (M, g_M, X) be a static Einstein spacetime of dimension $n + 1$ satisfying $Ric(g_M) = \lambda g_M$. Then (M, \hat{g}) is a Riemannian Einstein manifold with the same cosmological constant as g_M , i.e., $Ric(\hat{g}) = \lambda \hat{g}$.*

Proof. This follows from (2.16) by noting that $d\theta = 0$ on static spacetimes. \square

COROLLARY 2.2. *Let (M, g_M, X) be a geodesically complete stationary and chronological spacetime of dimension $n + 1$ with a timelike Killing field X such that the X -orbit space N is a **compact** smooth manifold. Then the followings hold*

i) If $Ric_M(X, X) \leq 0$ holds everywhere, then (M, g_M, X) is static, $|X|_{g_M}^2 \equiv const.$ and there is a closed 1-form θ on N such that (M, g_M) is isometric to a metric $-(d\tau + \theta)^2 + g_N$ on $\mathbb{R} \times N$.

ii) If g_M is Einstein and static, $Ric(g_M) = \lambda g_M$, we have $\lambda = 0$, the conclusion of i) holds, and (N, g_N) is Ricci flat.

Proof. First of all, by [11] (c.f.[1],[13]), M is diffeomorphic to $\mathbb{R} \times N$, and the metric g_M now has the global form (1.2). From the first equation of (2.14), we know $Ric_M(X, X) = u\Delta u + \frac{u^4}{4} |\Lambda|^2$. For i), since $\Delta u \leq 0$, by strong maximum principle, we have $Ric_M(X, X) = 0$ and $d\theta = \Lambda = 0$ and $u \equiv const.$. This shows i). If we assume $Ric(g_M) = \lambda g_M$, then $Ric_M(X, X) = -\lambda u^2$. By i), we know $\lambda \leq 0$. The first equation of (2.14) implies $\Delta u = -\lambda u$. Since $u > 0$, we know $\lambda = 0$ and $u = const.$ by strong maximum principle. The conclusion of ii) holds. \square

REMARK 2.1. *Under the assumptions of Corollary 2.2 and i), (M, g_M) is isometric to a product $-dt^2 + g_N$ on $\mathbb{R} \times N$ if $H^1(M, \mathbb{R}) = 0$.*

3. Completeness. Before the discussion of completeness, we need to do some preliminary work on projecting curves to horizontal ones. Here, we say a curve is horizontal if its tangent vectors are horizontal.

LEMMA 3.1. *Let (M, g_M) be a spacetime with a timelike Killing field X . Let $\bar{\gamma} : I \rightarrow M$ be a smooth curve on M , where $I \subset \mathbb{R}$ is an interval. Fix $s_0 \in I$, $p_0 = \Psi_{\tau_0}(\bar{\gamma}(s_0))$, there is a unique maximal smooth horizontal curve $\sigma : I' \rightarrow M$ such that $I' \subset I$, $\tau(s_0) = \tau_0$, $\Psi_{\tau(s)}(\bar{\gamma}(s)) = \sigma(s)$ for any $s \in I'$, where $\tau : I' \rightarrow \mathbb{R}$ is a smooth function, Ψ_τ is the local flow generated by X . Moreover, $I' = I$ provided that the vector field X is complete.*

Here σ is maximal means that any such horizontal projection curve of $\bar{\gamma}$ passing p_0 is only a part of σ . The vector field X is said to be complete if any integral curves of X can be defined, for their affine parameters, on the whole real line \mathbb{R} .

Proof. Consider the curve $\Psi_{\tau_0}(\bar{\gamma})$. Note that the map Ψ_{τ_0} might not be defined on whole $\bar{\gamma}$, so the parameters of the curve $\Psi_{\tau_0}(\bar{\gamma})$ lie in a connected subinterval of I . Let $\Psi_{\tau_0}(\bar{\gamma}(s))$ be parameterized by $(x^0(s), x^1(s), \dots, x^n(s))$ on a chronological chart $\{x^\alpha\}$ around p_0 used in §2.1. Let $T = \sum_\alpha T^\alpha e_\alpha$ be the tangent vector of $\Psi_{\tau_0}(\bar{\gamma}(s))$, where $T^i = \frac{dx^i}{ds}$ and $T^0 = \frac{dx^0}{ds} + \sum T^i \theta_i$.

Consider another curve

$$\sigma(s) = (y(s), x^1(s), \dots, x^n(s)) \tag{3.1}$$

on the coordinate system $\{x^\alpha\}$, where we require $\frac{d}{ds}y(s) + \sum T^i\theta_i = 0$, $\sigma(s_0) = \Psi_{\tau_0}(\bar{\gamma}(s_0))$. It is clear $\frac{d\sigma(s)}{ds} = \sum_{i=1}^n T^i e_i$, i.e., $\sigma(s)$ is horizontal. Note that the function $y(s)$ is determined uniquely by these requirements. So $\bar{\gamma}$ has a unique horizontal projection $\Psi_{\tau(s)}(\bar{\gamma}(s))$ on this chart, where $\tau(s) = \tau_0 + y(s) - x^0(s)$ is clearly a smooth function. Because the manifold may be covered by chronological coordinate charts used in §2.1, one can extend the horizontal projection curve of $\bar{\gamma}$ to a maximal one. \square

REMARK 3.1. *It is clear that in Lemma 3.1, we have*

$$|\dot{\sigma}|_{\hat{g}}(s) \leq |\dot{\bar{\gamma}}|_{\hat{g}}(s), \quad s \in I', \tag{3.2}$$

which implies that if $\bar{\gamma}$ is not horizontal, then the \hat{g} -length of the horizontal projection curve σ will become strictly smaller.

LEMMA 3.2. *Let (M, g_M) be a time-geodesically complete spacetime with a time-like Killing field X . Then X is complete.*

Proof. We only need to show that any integral curve $\zeta : [a, b) \rightarrow M$ of X can be extended over b . When $c > a$ is close to a , there is a timelike geodesic $\gamma : [0, d] \rightarrow M$ such that $\gamma(0) = \zeta(a)$ and $\gamma(d) = \zeta(c)$. By time-completeness assumption, γ can be extended to be defined on all affine parameters. Ψ_{b-c} is clearly defined near $\zeta(a)$ and $\Psi_{b-c}(\zeta(a)) = \zeta(b - (c - a))$, where Ψ_t are the local diffeomorphisms generated by X . To prove the lemma, it suffices to show that the maps Ψ_τ , for $\tau \in [0, b - c]$ can be defined on whole γ .

Suppose this is not true, there will be a smooth family of X -integral curves connecting $\gamma(s)$ and $\sigma(s) = \Psi_{b-c}(\gamma(s))$ for s lying in a maximal interval $I = [0, b') \subset [0, d]$. The integral curve η of X starting at $\gamma(b')$ can only be defined on a maximal interval $[0, c')$ where $c' < b - c$. Considering the timelike geodesic $\xi(s) = \Psi_{c'}(\gamma(s))$ defined on $[0, b')$, it can be extended to all affine parameters by time-geodesic completeness assumption. Near the point $\xi(b')$, by considering the integral curves of $-X$ starting at $\xi(s)$ for $s \in (b' - \epsilon, b']$, we find that η can actually be extended over c' , and $\eta(c') = \xi(b')$. This is a contradiction. \square

In general, the horizontal projection curve of a geodesic is no longer a geodesic, but it still satisfies a ‘good’ ODE. Indeed, in a coordinate system $\{x^\alpha\}$ in §2.1, by using (2.9) and direct computations, we have

$$\begin{aligned} \bar{\nabla}_T T &= w^{-1} \frac{d(T^0 w)}{ds} e_0 \\ &+ \left[\frac{dT^i}{ds} + \Gamma_{kl}^i T^k T^l + \frac{1}{2} (T^0 w)^2 \nabla_j w^{-1} g^{jj} + T^0 w \Lambda_{lm} T^l g^{mi} \right] e_i, \end{aligned} \tag{3.3}$$

where $\bar{\gamma}(s) = (x^0(s), x^1(s), \dots, x^n(s))$, $T = \dot{\bar{\gamma}} = \sum_\alpha T^\alpha e_\alpha$.

This implies that $\bar{\gamma}$ is a geodesic on (M, \bar{g}) if and only if the curve $\gamma = \pi(\bar{\gamma})$, $\gamma(s) = (x^1(s), \dots, x^n(s))$, satisfies

$$\begin{aligned} T^0 w &= \langle T, X \rangle = \text{const.} \triangleq c \\ \nabla_{\dot{\gamma}} \dot{\gamma} &= -\frac{c^2}{2} \nabla w^{-1} - c(i_{\dot{\gamma}} d\theta)^\sharp, \end{aligned} \tag{3.4}$$

where $i_{\dot{\gamma}}d\theta$ is the 1-form obtained from the contraction of 2-form $\Lambda = d\theta$ with the tangent vector $\dot{\gamma} = \sum T^i \frac{\partial}{\partial x^i}$.

The first equation of (3.4) can be derived alternatively by

$$T\langle T, X \rangle = \langle T, \bar{\nabla}_T X \rangle = 0$$

since X is Killing and $\bar{\nabla}_T T = 0$.

THEOREM 3.3. *Let (M, g_M) be a geodesically complete spacetime with a timelike Killing field X . Then (M, \hat{g}) is a complete Riemannian manifold, where \hat{g} is defined in (1.3).*

Proof. Fix $p \in M$, we will show that the exponential map $\exp_{\hat{g}}$ of \hat{g} can be defined on the whole tangent space $T_p M$ at p . Let \bar{r} be the supremum of all $r > 0$ such that the exponential map $\exp_{\hat{g}}$ can be defined on a ball of radius r centered at 0 in $T_p M$. We have to show $\bar{r} = \infty$. We argue by contradiction. Suppose $\bar{r} < \infty$. Let $\bar{\gamma} : [0, \bar{r}) \rightarrow M$ be a normal \hat{g} -geodesic parameterized by arclength, $\bar{\gamma}(0) = p$, $|\dot{\bar{\gamma}}(0)|_{\hat{g}} = 1$, and $\bar{\gamma}$ can not be extended over time \bar{r} (as a \hat{g} -geodesic). From the definition of \bar{r} , for any $r < \bar{r}$, we know $B(p, r) = \exp_{\hat{g}}(B(0, r))$ and $\bar{B}(p, r) = \exp_{\hat{g}}(\bar{B}(0, r))$, where $B(p, r) = \{q \in M : d_{\hat{g}}(q, p) < r\}$, $\bar{B}(p, r) = \{q \in M : d_{\hat{g}}(q, p) \leq r\}$, $B(0, r) = \{v \in T_p M : |v| < r\}$, and $\bar{B}(0, r) = \{v \in T_p M : |v| \leq r\}$. Therefore, $\bar{B}(p, r) = \{q \in M : d_{\hat{g}}(q, p) \leq r\}$ is compact provided $r < \bar{r}$.

We assume that $\bar{\gamma}$ is not horizontal, otherwise $\bar{\gamma}$ can be extended to whole real line \mathbb{R} , because a horizontal \hat{g} -geodesic is also a horizontal g_M -geodesic (see (3.4)). By Lemmas 3.1, 3.2, one can construct a smooth horizontal projection curve $\sigma : [0, \bar{r}) \rightarrow M$ such that $\sigma(0) = \gamma(0) = p$, $\tau(0) = 0$, $\sigma(s) = \Psi_{\tau(s)}\gamma(s)$, where $\tau : [0, \bar{r}) \rightarrow \mathbb{R}$ is a smooth function. Since $\bar{\gamma}$ is not horizontal, the \hat{g} -length of σ is less than that of γ (see (3.2)), i.e., $L \triangleq L(\sigma) < L(\gamma) = \bar{r}$. By (3.2), for any $0 < a < b < \bar{r}$, we have $d_{\hat{g}}(\sigma(a), \sigma(b)) \leq b - a$. For any sequence $r_k < \bar{r}$, $r_k \rightarrow \bar{r}$, $\{\sigma(r_k)\}$ is a Cauchy sequence in a **compact** subset $\bar{B}(p, L)$. So $\lim_{s \rightarrow \bar{r}} \sigma(s)$ must exist. Denote the limit by q .

Choose a local coordinate system $\{x^\alpha\}$ as in §2.1 around q . Since $\pi(\sigma(s)) \triangleq \gamma(s)$, $s < \bar{r}$, satisfies the 2nd equation of the ODE (3.4) near q , we know $\sigma(s)$ can be extended smoothly over \bar{r} , i.e., σ is now defined on $[0, \bar{r} + \epsilon]$ for some $\epsilon > 0$, and $\pi(\sigma) |_{[\bar{r}-\epsilon, \bar{r}+\epsilon]}$ satisfies (3.4) on the coordinate system $\{x^\alpha\}$. By solving the x^0 -coordinate function from the 1st equation of ODE (3.4) for $s \in [\bar{r} - \epsilon, \bar{r} + \epsilon]$, we get a \hat{g} -geodesic $\tilde{\gamma} : [\bar{r} - \epsilon, \bar{r} + \epsilon]$ lying in the coordinate system whose horizontal projection curve is $\sigma |_{[\bar{r}-\epsilon, \bar{r}+\epsilon]}$. Since X is complete (see Lemma 3.2), one can choose a suitable $t_0 \in \mathbb{R}$, such that $\Psi_{t_0}(\tilde{\gamma})$ coincides with γ on $[\bar{r} - \epsilon, \bar{r})$. Now $\gamma \cup \Psi_{t_0}(\tilde{\gamma} |_{[\bar{r}, \bar{r}+\epsilon]})$ will be a smooth \hat{g} -geodesic which is an extension of $\bar{\gamma}$. This is a contradiction with the definition of \bar{r} . The proof is complete. \square

THEOREM 3.4. *Let (M, g_M) be a geodesically complete static Einstein spacetime of dimension $n + 1$, i.e. $Ric(g_M) = \lambda g_M$. Then $\lambda \leq 0$.*

Proof. Suppose $\lambda > 0$. We know $Ric(\hat{g}) = \lambda \hat{g}$ by (2.16). On the other hand, (M, \hat{g}) is complete by Theorem 3.3. This implies that M is compact by Bonnet-Myers theorem. From (2.12) and (2.14), we have $\hat{\Delta} \log u = -\lambda$ on M . At the minimum point of $\log u$, we find $-\lambda \geq 0$, which is a contradiction with $\lambda > 0$. \square

4. Static solutions. In this section, we will handle static spacetimes. We will first derive a local gradient estimate on the norm of the Killing field. The idea comes from Yau’s gradient estimate of harmonic functions on Riemannian manifolds (see [18] or §1.3 in [17]).

4.1. Static vacuum solutions. In the following Theorem 4.1, we assume (M, g_M, X) is a static Einstein spacetime of dimension $n + 1$ with a timelike Killing field X whose orthogonal complement is integrable, and $Ric(g_M) = \lambda g_M$.

Let \hat{g} be the Riemannian metric defined by (1.3). Note that $u = [-g_M(X, X)]^{\frac{1}{2}}$ is a time-independent function, we have $|\hat{\nabla} \log u|_{\hat{g}}^2 = |\nabla \log u|_g^2$ by (2.3).

THEOREM 4.1. *Let $\hat{B}(x_0, a)$ be a \hat{g} -geodesic ball centered at x_0 of radius $a > 0$ with compact closure in M . Then there is a universal constant C such that*

$$\sup_{x \in \hat{B}(x_0, \frac{a}{2})} |\hat{\nabla} \log u|_{\hat{g}} \leq C(\sqrt{na}^{-1} + \sqrt{\max\{-\lambda, 0\}}). \tag{4.1}$$

Proof. By (2.11), we know

$$\begin{aligned} \hat{\nabla}^2 \log u(e_0, e_0) &= |\nabla u|^2 \\ \hat{\nabla}^2 \log u(e_0, e_i) &= 0 \\ \hat{\nabla}^2 \log u(e_i, e_j) &= \nabla_i \nabla_j \log u, \end{aligned} \tag{4.2}$$

on a local coordinate system $\{x^\alpha\}$ in §2.1, where $e_0 = \frac{\partial}{\partial t}$, $e_i = \frac{\partial}{\partial x^i} - \theta_i \frac{\partial}{\partial t}$, $i = 1, 2, \dots, n$. This implies $\hat{\Delta} \log u = -\lambda$.

According to the Corollary 2.1, we know \hat{g} is also an Einstein metric and $\hat{R}ic = \lambda \hat{g}$. By Bochner formula, we have

$$\hat{\Delta} |\hat{\nabla} \log u|^2 = 2|\hat{\nabla}^2 \log u|_{\hat{g}}^2 + 2\lambda |\hat{\nabla} \log u|^2. \tag{4.3}$$

From (4.2), we get

$$|\hat{\nabla}^2 \log u|_{\hat{g}}^2 = |\nabla_{ij}^2 \log u|_g^2 + |\nabla \log u|_g^4, \tag{4.4}$$

and

$$\hat{\Delta} |\nabla \log u|^2 = 2|\nabla^2 \log u|_g^2 + 2|\nabla \log u|_g^4 + 2\lambda |\nabla \log u|^2. \tag{4.5}$$

Let ρ be the \hat{g} -distance function centered at x_0 . Let $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a smooth nonnegative decreasing cutoff function such that $\psi = 1$ on $[0, \frac{1}{2}]$, $\psi = 0$ outside $[0, 1]$, and $|\psi''| + \frac{(\psi')^2}{\psi} \leq C\sqrt{\psi}$.

We consider the nonnegative function $f = \psi(\frac{\rho}{a})|\nabla \log u|^2$ on $\hat{B}(x_0, a)$. Suppose f achieves its maximum at some smooth point $x_1 \in \hat{B}(x_0, a)$ of ρ . Then we have $\hat{\Delta} f(x_1) \leq 0$ and $\hat{\nabla} f(x_1) = 0$. Hence

$$\begin{aligned} 0 &\geq \hat{\Delta} f(x_1) \geq 2\psi |\nabla^2 \log u|_g^2 + 2\psi |\nabla \log u|_g^4 + 2\lambda \psi |\nabla \log u|^2 \\ &\quad - \frac{1}{a^2} (|\psi''| + 2\frac{(\psi')^2}{\psi}) |\nabla \log u|^2 + a^{-1} \hat{\Delta} \rho \psi' |\hat{\nabla} \log u|^2. \end{aligned} \tag{4.6}$$

We first consider $\lambda \geq 0$ case. In this case, we have the Laplacian comparison theorem $\hat{\Delta} \rho \leq \frac{n}{a}$. Hence $a^{-1} \hat{\Delta} \rho \psi' \geq 2na^{-2} \psi'$. Multiplying both sides of (4.6) by ψ , we get $2f(x_1)^2 - Cna^{-2} f(x_1) \leq 0$, which implies $f(x_1) \leq Cna^{-2}$. In particular, we have

$$\sup_{x \in \hat{B}(x_0, \frac{a}{2})} |\hat{\nabla} \log u|_{\hat{g}} \leq C\sqrt{na}^{-1}. \tag{4.7}$$

If x_1 lies in the cut locus of x_0 , by applying a standard support function technique (see [18], or Theorem 3.1 in [17]), (4.6) and (4.7) still hold.

Now we assume $\lambda < 0$. In this case, Laplacian comparison theorem tells us $\hat{\Delta}\rho \leq \frac{n}{\rho}(1 + \sqrt{\frac{|\lambda|}{n}}\rho)$ (see Corollary 1.2 in [17]). This gives $a^{-1}\hat{\Delta}\rho\psi' \geq (2na^{-2} + a^{-1}\sqrt{|\lambda|n})\psi'$. Multiplying both sides of (4.6) by ψ , we get $f(x_1) \leq C(|\lambda| + na^{-2})$, where C is a universal constant (independent of n). The proof is complete. \square

Proof of Theorem 1.2. If (M, g_M) is geodesically complete, we know (M, \hat{g}) is complete from Lemma 3.3. If $\lambda \geq 0$, letting $a \rightarrow \infty$ in Theorem 4.1, one can prove $u = const.$. Now the 1-form X^* dual to X becomes closed. On the universal cover, $X^* = df$ must hold for some function f , the level set $\{f = const.\}$ is a global integrable submanifold of the horizontal distribution. It is easy to see that $\{f = const.\}$ is complete and Ricci flat. The universal cover of (M, g_M) will be isometric to $\mathbb{R} \times \{f = const.\}$.

The argument of Theorem 4.1 essentially provides a proof of the following:

COROLLARY 4.2. *Let (M^n, g) be a Einstein Riemannian manifold with a nowhere vanishing Killing field X such that the orthogonal complement of X is integrable and $Ric = \lambda g$. Then for any metric ball $B(x_0, a)$ with compact closure in M^n , we have*

$$\sup_{B(x_0, \frac{a}{2})} |\nabla \log |X|| \leq C(\sqrt{na}^{-1} + \sqrt{\max\{-\lambda, 0\}}), \tag{4.8}$$

where C is a universal constant. Moreover, if $\lambda \geq 0$ and (M^n, g) is complete, then $|X| = const.$ and the universal cover of (M^n, g) is isometric to $\mathbb{R} \times N$, where N is a complete Ricci flat Riemannian manifold.

COROLLARY 4.3. *Let (N^n, g) be a Riemannian manifold, u a smooth positive function on N satisfying $R_{ij} = u^{-1}\nabla_{ij}u + \lambda g_{ij}$, $\Delta u = -\lambda u$. Then for any metric ball $B(x_0, a)$ with compact closure in N , we have*

$$\sup_{B(x_0, \frac{a}{2})} |\nabla \log u| \leq C(\sqrt{na}^{-1} + \sqrt{\max\{-\lambda, 0\}}), \tag{4.9}$$

where C is a universal constant. Moreover, $u = const.$ and (N^n, g) is Ricci flat if $\lambda \geq 0$ and (N^n, g) is complete.

Proof. Let $M = \mathbb{R} \times N^n$ be a manifold equipped with a static Riemannian metric $\hat{g} = u^2 dt^2 + g$. Now $X \triangleq \frac{\partial}{\partial t}$ is a Killing field. One can show that $Ric(\hat{g}) = \lambda \hat{g}$ (see (2.13)). The same argument as in Theorem 4.1 will give (4.9). Because the projection from M to N^n of any \hat{g} -Cauchy sequence on M is also a g -Cauchy sequence on N^n , the completeness of N^n will imply the completeness of M . The last assertion of the corollary holds. \square

5. 4-d stationary vacuum spacetimes.

5.1. Preliminaries. In this section, we focus on the usual dimension of space-time, i.e., $\dim M = 4$. Now fix a point $x_0 \in M$, let $\{x^\alpha\}$ be a coordinate system used in §2.1, which covers some open neighborhood $M' \subset M$ of x_0 . Let $\pi : M \rightarrow N$ be the projection from M to the X -orbit space N . Equip $N' = \pi(M')$ the horizontal Riemannian metric g_{ij} . Since $X^* = -u^2(dt + \theta)$, the Hodge dual of $X^* \wedge dX^*$ (on (M, \bar{g})) is $\pm u^3 *d\theta$, where $*d\theta$ is the Hodge dual of $d\theta$ (on (N', g)). Denote $\omega = u^3 *d\theta$. Note that $|\omega|^2 = u^6 |d\theta|^2 = \frac{u^6}{2} |\Lambda|^2$, where the norm for a 2-form is taken by requiring $|e^1 \wedge e^2|^2 = 1$ if e^1, e^2 are orthonormal. Now $d^2\theta = 0$ is equivalent to $d(u^{-3} * \omega) = 0$, or $g^{ij} \nabla_i \omega_j = 3(d \log u, \omega)$.

It should be noted that $\omega \otimes \omega$ is globally defined on M no matter whether M is orientable or not.

Now we can rewrite equations (2.14) and (2.16) as follows:

$$\begin{aligned} R_{ij} &= u^{-1} \nabla_i \nabla_j u + \frac{1}{2} u^{-4} (\omega_i \omega_j - |\omega|^2 g_{ij}) + \bar{Ric}(e_i, e_j) \\ \Delta u &= -\frac{1}{2} u^{-3} |\omega|^2 + u^{-1} \bar{Ric}(X, X) \\ g^{kl} \nabla_k \omega_l &= 3g^{kl} \omega_k \nabla_l \log u \\ (*d\omega)_j &= \pm 2u \bar{Ric}(X, e_j), \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \hat{Ric}(X, X) &= u^{-2} |\omega|^2 - \bar{Ric}(X, X) \\ \hat{Ric}(X, e_j) &= -\bar{Ric}(X, e_j) \\ \hat{Ric}(e_i, e_j) &= -u^{-4} (|\omega|^2 g_{ij} - \omega_i \omega_j) + \bar{Ric}(e_i, e_j). \end{aligned} \tag{5.2}$$

Recall that $\tilde{g} \triangleq u^2 g$ in Section 2.3, and we have (see (2.19) (2.20) (5.1)):

$$\begin{aligned} \tilde{R}_{ij} &= \frac{1}{2} u^{-4} \omega_i \omega_j + 2 \frac{u_i u_j}{u^2} + \bar{Ric}(e_i, e_j) - u^{-2} \bar{Ric}(X, X) g_{ij} \\ u^2 \tilde{\Delta} \log u &= -\frac{1}{2} u^{-4} |\omega|^2 + u^{-2} \bar{Ric}(X, X) \\ \tilde{g}^{kl} \tilde{\nabla}_k \omega_l &= 4 \tilde{g}^{kl} \omega_k \nabla_l \log u \\ (*_{\tilde{g}} d\omega) \left(\frac{\partial}{\partial x^j} \right) &= \pm 2 \bar{Ric}(X, e_j). \end{aligned} \tag{5.3}$$

By taking trace on the first equation of (5.3), we find that the scalar curvature \tilde{R} of \tilde{g} satisfies

$$u^2 \tilde{R} = \frac{1}{2} u^{-4} |\omega|^2 + 2u^{-2} |\nabla u|^2 + \bar{R} - 2u^{-2} \bar{Ric}(X, X), \tag{5.4}$$

where \bar{R} is the scalar curvature of (M, \bar{g}) .

5.2. Ernst potential map Φ . Throughout this subsection, we assume the following condition holds:

$$\bar{Ric}(X, Y) = 0 \quad \text{whenever} \quad \bar{g}(X, Y) = 0, \tag{5.5}$$

where X is the timelike Killing field. In general, condition (5.5) does not hold, while it holds when (M, \bar{g}) is Einstein, i.e., $Ric(\bar{g}) = c\bar{g}$.

When condition (5.5) holds, from the last equation of (5.1), we know $\omega = d\psi$ holds locally for some function ψ by Poincare lemma. In this case, we have

$$\begin{aligned} \hat{\Delta}\psi &= 4\langle \nabla\psi, \nabla \log u \rangle_{\hat{g}}, \\ \tilde{\Delta}\psi &= 4\langle \nabla\psi, \nabla \log u \rangle_{\tilde{g}}. \end{aligned} \tag{5.6}$$

Let $g_{-1} = y^{-2}(dx^2 + dy^2)$ be the hyperbolic metric (sectional curvature $\equiv -1$) on Poincare upper half plane $H = \{(x, y) : x \in \mathbb{R}, y > 0\}$. We define the Ernst potential map $\Phi : M' \rightarrow H$ by $\Phi = (\psi, u^2) = (x, y)$, see [1] [10]. Because ψ and u are time-independent, Φ is also a map from $N' = \pi(M')$ to H .

LEMMA 5.1. *The map Φ satisfies*

- i) $\Phi^*g_{-1} = u^{-4}\omega \otimes \omega + 4d \log u \otimes d \log u$;
- ii) $\hat{\Delta}\Phi = u^2\tilde{\Delta}\Phi = 2\bar{Ric}(X, X)\frac{\partial}{\partial y}$, where $\hat{\Delta}\Phi$ (or $\tilde{\Delta}\Phi$) is the harmonic map Laplacian between two Riemannian manifolds (M', \hat{g}) (or (N', \tilde{g})) and (H, g_{-1}) .

Proof. We can calculate the Christoffel symbols Γ_{bc}^a of g_{-1} as follows:

$$\begin{aligned} \Gamma_{11}^1 &= \Gamma_{22}^1 = \Gamma_{12}^2 = 0 \\ -\Gamma_{12}^1 &= -\Gamma_{22}^2 = \Gamma_{11}^2 = y^{-1}. \end{aligned} \tag{5.7}$$

From the definition of the Hessian of a map from (M, \hat{g}) to (H, g_{-1}) , we have

$$(\hat{\nabla}_{\alpha\beta}\Phi)^a = \hat{\nabla}_{\alpha\beta}\Phi^a + \Gamma_{bc}^a \nabla_{\alpha}\Phi^b \nabla_{\beta}\Phi^c. \tag{5.8}$$

Combining with (5.7), it follows

$$\begin{aligned} (\hat{\nabla}_{\alpha\beta}\Phi)^1 &= \hat{\nabla}_{\alpha\beta}\psi - \psi_{\alpha}(\log u^2)_{\beta} - \psi_{\beta}(\log u^2)_{\alpha} \\ (\hat{\nabla}_{\alpha\beta}\Phi)^2 &= 2u^2\hat{\nabla}_{\alpha\beta} \log u + u^{-2}\psi_{\alpha}\psi_{\beta}. \end{aligned} \tag{5.9}$$

Taking traces on (5.9) with respect to \hat{g} , we get

$$\begin{aligned} (\hat{\Delta}\Phi)^1 &= \hat{\Delta}\psi - 4\langle \nabla\psi, \nabla \log u \rangle_g = 0 \\ (\hat{\Delta}\Phi)^2 &= 2u^2\hat{\Delta} \log u + u^{-2}|\omega|_g^2 = 2\bar{Ric}(X, X), \end{aligned} \tag{5.10}$$

where we have used (5.6) (5.1) and (2.12)). \square

COROLLARY 5.2. *When (M, \bar{g}) is Ricci flat, the map $\Phi = (x, y) = (\psi, u^2)$ is a harmonic map from (M', \hat{g}) (or (N, \tilde{g})) to (H, g_{-1}) .*

Now we can apply the standard Bochner formula (see [8]):

$$\begin{aligned} \hat{\Delta}e(\Phi) &= 2\langle \hat{\nabla}\Phi, \hat{\nabla}\hat{\Delta}\Phi \rangle + 2|\hat{\nabla}_{\alpha\beta}\Phi|^2 + 2\langle \hat{Ric}, \Phi^*g_{-1} \rangle_{\hat{g}} \\ &\quad - 2R_{abcd}\Phi_{\alpha}^a \Phi_{\beta}^b \Phi_{\gamma}^c \Phi_{\delta}^d \hat{g}^{\beta\delta} \hat{g}^{\alpha\gamma} \end{aligned} \tag{5.11}$$

where

$$\begin{aligned} e(\Phi) &= \hat{g}^{\alpha\beta}(\Phi^*g_{-1})_{\alpha\beta} = u^{-4}|\omega|^2 + 4|\nabla \log u|^2 \\ &= 2u^2\bar{R} - 2(\bar{R} - 2u^{-2}\bar{Ric}(X, X)). \end{aligned} \tag{5.12}$$

Here we have used (5.4).

Now we compute the term $I_1 = \langle \hat{\nabla}\Phi, \hat{\nabla}\hat{\Delta}\Phi \rangle$ first.

By Lemma 5.1 and (5.7), we have

$$\begin{aligned} \hat{\nabla}_\alpha \hat{\Delta} \Phi^a &= \frac{\partial}{\partial x^\alpha} (\hat{\Delta} \Phi^a) + \Gamma_{bc}^a \hat{\Delta} \Phi^b \frac{\partial \Phi^c}{\partial x^\alpha} \\ &= \frac{\partial}{\partial x^\alpha} (\hat{\Delta} \Phi^a) - 2u^{-2} \bar{Ric}(X, X) \frac{\partial \Phi^a}{\partial x^\alpha}, \end{aligned}$$

which implies

$$\begin{aligned} \hat{\nabla}_\alpha \hat{\Delta} \Phi^1 &= -2u^{-2} \bar{Ric}(X, X) \frac{\partial \psi}{\partial x^\alpha}, \\ \hat{\nabla}_\alpha \hat{\Delta} \Phi^2 &= \frac{\partial}{\partial x^\alpha} (2\bar{Ric}(X, X)) - 2u^{-2} \bar{Ric}(X, X) \frac{\partial u^2}{\partial x^\alpha}. \end{aligned}$$

Hence,

$$\begin{aligned} I_1 = \langle \hat{\nabla} \Phi, \hat{\nabla} \hat{\Delta} \Phi \rangle &= -2u^{-2} \bar{Ric}(X, X) (u^{-4} |\omega|^2 + 4|\nabla \log u|^2) \\ &\quad + u^{-4} \langle \nabla (2\bar{Ric}(X, X)), \nabla u^2 \rangle. \end{aligned} \tag{5.13}$$

The 3rd term on the right hand side of (5.11) can be computed as follows:

$$\begin{aligned} \langle \hat{Ric}, \Phi^* g_{-1} \rangle_{\hat{g}} &= [u^{-4} (\omega^i \omega^j - |\omega|^2 g^{ij}) + \bar{Ric}(e_k, e_l) g^{ik} g^{jl}] [u^{-4} \omega_i \omega_j + 4 \frac{u_i u_j}{u^2}] \\ &= -4u^{-6} (|\omega|^2 |\nabla u|^2 - \langle \omega, \nabla u \rangle^2) \\ &\quad + g^{ik} g^{jl} (u^{-4} \omega_i \omega_j + 4 \frac{u_i u_j}{u^2}) \bar{Ric}(e_k, e_l). \end{aligned} \tag{5.14}$$

Since g_{-1} has constant sectional curvature $K \equiv -1$, we have

$$\begin{aligned} &- R_{abcd} \Phi_\alpha^a \Phi_\beta^b \Phi_\gamma^c \Phi_\delta^d \hat{g}^{\alpha\gamma} \hat{g}^{\beta\delta} \\ &= [(\Phi^* g_{-1})_{\alpha\gamma} (\Phi^* g_{-1})_{\beta\delta} - (\Phi^* g_{-1})_{\alpha\delta} (\Phi^* g_{-1})_{\beta\gamma}] \hat{g}^{\alpha\gamma} \hat{g}^{\beta\delta} \\ &= 8u^{-6} (|\omega|^2 |\nabla u|^2 - \langle \omega, \nabla u \rangle^2). \end{aligned} \tag{5.15}$$

Now we compute the term $|\hat{\nabla}_{\alpha\beta} \Phi|^2$ at a given point (\bar{x}, t_0) . By definition, $|\hat{\nabla}_{\alpha\beta} \Phi|^2 = u^{-4} (|(\hat{\nabla}_{\alpha\beta} \Phi)^1|_{\hat{g}}^2 + |(\hat{\nabla}_{\alpha\beta} \Phi)^2|_{\hat{g}}^2)$.

Let $\{x^i\}$ be a normal coordinate system around the fixed point $\bar{x} \in N$. Let $E_0 = u^{-1} \frac{\partial}{\partial t}$, $E_i = \frac{\partial}{\partial x^i} - \theta_i \frac{\partial}{\partial t}$, then $\{E_\alpha\}$ is an orthonormal basis of \hat{g} at (\bar{x}, t_0) , hence

$$|(\hat{\nabla}_{\alpha\beta} \Phi)^a|_{\hat{g}}^2 = \sum_{\alpha, \beta} [(\hat{\nabla}_{\alpha\beta} \Phi)^a]^2. \tag{5.16}$$

From (2.11), we have

$$\begin{aligned} (\hat{\nabla}^2 \log u)(E_0, E_0) &= u^{-2} |\nabla u|_g^2 \\ (\hat{\nabla}^2 \log u)(E_0, E_i) &= -\frac{1}{2} \Lambda_{il} u_k g^{kl} = \pm \frac{u^{-3}}{2} * (\omega \wedge du)_i \\ (\hat{\nabla}^2 \log u)(E_i, E_j) &= (\nabla^2 \log u) \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right). \end{aligned} \tag{5.17}$$

On the other hand, we have

$$\begin{aligned} (d\psi \otimes d\psi)(E_0, E_0) &= (d\psi \otimes d\psi)(E_0, E_i) = 0 \\ (d\psi \otimes d\psi)(E_i, E_j) &= \psi_i \psi_j. \end{aligned} \tag{5.18}$$

From (5.9), we get

$$u^{-4} |(\hat{\nabla}_{\alpha\beta}\Phi)^2|_{\hat{g}}^2 = 4|\nabla \log u|^4 + 2|u^{-2}\omega \wedge d \log u|^2 + |2\nabla_{ij} \log u + u^{-4}\omega_i\omega_j|^2. \tag{5.19}$$

To compute the term

$$\begin{aligned} |(\hat{\nabla}_{\alpha\beta}\Phi)^1|_{\hat{g}}^2 &= (\hat{\nabla}^2\psi(E_0, E_0))^2 + 2 \sum_i (\hat{\nabla}^2\psi(E_0, E_i))^2 \\ &\quad + \sum_{i,j} [\hat{\nabla}^2\psi(E_i, E_j) - 2\psi_i(\log u)_j - 2\psi_j(\log u)_i]^2, \end{aligned}$$

we need (see (2.11))

$$\begin{aligned} \hat{\nabla}^2\psi(E_0, E_0) &= \langle \nabla \log u, \nabla \psi \rangle \\ \hat{\nabla}^2\psi(E_0, E_i) &= -\frac{1}{2}u\Lambda_{il}g^{lk}\nabla_k\psi = \pm \frac{u^{-2}}{2} * (\omega \wedge \omega)_i = 0 \\ \hat{\nabla}^2\psi(E_i, E_j) &= (\nabla^2\psi)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \end{aligned} \tag{5.20}$$

hence

$$u^{-4} |(\hat{\nabla}_{\alpha\beta}\Phi)^1|_{\hat{g}}^2 = \langle d \log u, u^{-2}\omega \rangle^2 + u^{-4} |\nabla_i\omega_j - \omega_i(\log u^2)_j - \omega_j(\log u^2)_i|^2. \tag{5.21}$$

Combining (5.21) (5.19) (5.15)(5.14)(5.13), we have

$$\begin{aligned} \hat{\Delta}\left(\frac{1}{2}e(\Phi)\right) &= \Delta\left(\frac{1}{2}e(\Phi)\right) + \langle \nabla \log u, \nabla\left(\frac{1}{2}e(\Phi)\right) \rangle \\ &= 4|\nabla \log u|^4 + \langle d \log u, u^{-2}\omega \rangle^2 + |2\nabla_{ij} \log u + u^{-4}\omega_i\omega_j|^2 \\ &\quad + u^{-4} |\nabla_i\omega_j - 2\omega_i(\log u)_j - 2\omega_j(\log u)_i|^2 + 6|u^{-2}\omega \wedge d \log u|^2 \\ &\quad + I_2 + I_3, \end{aligned} \tag{5.22}$$

where

$$\begin{aligned} \frac{1}{2}e(\Phi) &= \frac{1}{2}u^{-4}|\omega|^2 + 2|\nabla \log u|^2 \\ I_2 &= [\bar{Ric}(e_k, e_l) - 2u^{-2}\bar{Ric}(X, X)g_{kl}]g^{ik}g^{jl}[u^{-4}\omega_i\omega_j + 4\frac{u_iu_j}{u^2}] \\ I_3 &= 4u^{-2}\langle \nabla(\bar{Ric}(X, X)), \nabla \log u \rangle. \end{aligned} \tag{5.23}$$

5.3. A priori estimates. The main result of this section is the following:

THEOREM 5.3. *Let (M, g_M) be a spacetime of dimension 4 with a timelike Killing field X . Denote $\hat{g} = -\frac{2}{g_M(X, X)}X^* \otimes X^* + g_M$ the Riemannian metric associated to X . Let $\hat{B}(x_0, a)$ be a \hat{g} -metric ball centered at x_0 of radius $a > 0$ with compact closure in M , and assume*

$$\sup_{\hat{B}(x_0, a)} |Ric(g_M)|_{\hat{g}} \leq a^{-2}. \tag{5.24}$$

Then there is a universal constant C such that

$$\sup_{x \in \hat{B}(x_0, \frac{a}{2})} |\nabla \log u|^2 + u^{-4}|\omega|^2 \leq \frac{C}{a^2}, \tag{5.25}$$

where $u^2 = -g_M(X, X)$. Moreover, for any $p > 1$, there is a constant $C_p > 0$ depending only on p such that

$$\left(\frac{1}{\text{vol}_{\hat{g}}(\hat{B}(x_0, \frac{a}{2}))}\int_{\hat{B}(x_0, \frac{a}{2})}|Rm(g_M)|_{\hat{g}}^p d\text{vol}_{\hat{g}}\right)^{\frac{1}{p}} \leq \frac{C_p}{a^2}. \tag{5.26}$$

Proof. By scaling invariance of the estimates, we may assume $a = 1$. The argument is divided into two cases:

Case 1: $\partial\hat{B}(x_0, a) \neq \phi$;

Case 2: $\partial\hat{B}(x_0, a) = \phi$.

We treat Case 1 first. Let $h(x) = 2|\nabla \log u|^2(x) + \frac{1}{2}u^{-4}|\omega|^2(x)$, and $f(x) = h(x)d_{\hat{g}}^2(x, \partial\hat{B}(x_0, 1))$. Since f is a nonnegative function on $\hat{B}(x_0, 1)$, vanishes on $\partial\hat{B}(x_0, 1)$, there is a point $\bar{x} \in \hat{B}(x_0, 1)$ such that $f(\bar{x}) = \sup_{x \in \hat{B}(x_0, 1)} f(x)$. To prove the theorem, it suffices to prove that there is a universal constant $C > 0$ such that $f(\bar{x}) < C$. We will argue by contradiction. Suppose there are a sequence of spacetimes (M_l, g_{M_l}) and \hat{g}_l -balls $\hat{B}(x_l, 1) \subset M_l$ with compact closure satisfying (5.24) with $a = 1$, but $f(\bar{x}_l) \rightarrow \infty$ as $l \rightarrow \infty$, where

$$f(\bar{x}_l) = \sup_{x \in \hat{B}(x_l, 1)} h_l(x)d_{\hat{g}_l}^2(x, \partial\hat{B}(x_l, 1)), \quad h_l(x) = 2|\nabla \log u_l|^2 + \frac{1}{2}u_l^{-4}|\omega_l|^2.$$

Now we will work on a fixed space (M_l, \hat{g}_l) . For simplicity, we drop the subscript l .

For any fixed $0 < \epsilon < 1$, and any $x \in \hat{B}(x_0, 1)$ with

$$d_{\hat{g}}(x, \bar{x}) \leq \epsilon f(\bar{x})^{\frac{1}{2}} h^{-\frac{1}{2}}(\bar{x}) = \epsilon d_{\hat{g}}(\bar{x}, \partial\hat{B}(x_0, 1)),$$

we have

$$h(x) \leq \frac{1}{(1 - \epsilon)^2} h(\bar{x}). \tag{5.27}$$

Note that the function $f(x)$ is invariant under the scaling of the metric. The metric $\bar{g} = -u^2(dt + \theta)^2 + g$ is invariant under normalizations $u \rightarrow u(\bar{x})^{-1}u$, $t \rightarrow u(\bar{x})t$, $\theta \rightarrow u(\bar{x})\theta$, $\omega \rightarrow u(\bar{x})^{-2}\omega$. Therefore, the equation (5.1) remains invariant under such normalizations. So without loss of generality, by scaling u and the metric \bar{g} by suitable positive constants, we may assume $u(\bar{x}) = 1$ and $h(\bar{x}) = 1$. Now (5.27) becomes

$$h(x) \leq \frac{1}{(1 - \epsilon)^2} \quad \text{on } B(\bar{x}, \epsilon f(\bar{x})^{\frac{1}{2}}). \tag{5.28}$$

From (5.28), we know $|\nabla \log u| \leq \frac{1}{\sqrt{2}}(1 - \epsilon)^{-1}$ on $B(\bar{x}, \epsilon f(\bar{x})^{\frac{1}{2}})$.

Take $\epsilon = \frac{D}{\sqrt{f(\bar{x})}}$, where $D \geq 1$ is a fixed constant independent of l , we have

$$e^{-\frac{D}{\sqrt{2}}(1 - \frac{D}{\sqrt{f(\bar{x})}})^{-1}} \leq u(x) \leq e^{\frac{D}{\sqrt{2}}(1 - \frac{D}{\sqrt{f(\bar{x})}})^{-1}} \quad \text{on } \hat{B}(\bar{x}, D). \tag{5.29}$$

By (5.3) and (5.29), it can be shown that the sectional curvature of \tilde{g} on $\hat{B}(\bar{x}, D)$ satisfies

$$-\tilde{K}_{max} \leq \tilde{K}(x) \leq e^{10D(1 - \frac{D}{\sqrt{f(\bar{x})}})^{-1}} (1 - Df(\bar{x})^{-\frac{1}{2}})^{-2} \triangleq \tilde{K}_{max}. \tag{5.30}$$

Let $E = \mathcal{H}_{\bar{x}} \subset T_{\bar{x}}M$ be the orthogonal complement of X at \bar{x} equipped with the Euclidean metric induced from \tilde{g} or \hat{g} . Let $\exp_E : E \rightarrow M$ be the restriction of the exponential map of the conformal metric $u^2(\tilde{g})$ (or $u^2(\hat{g})$, see (3.4)). Clearly, \exp_E is a smooth map.

Claim 1: There is a universal constant $\delta_1 > 0$ such the exponential map \exp_E is an immersion from $B(0, \delta_1) \subset E$ to $\hat{B}(\bar{x}, 1)$. Moreover, the pull back $(0, 2)$ -tensor field $\exp_E^* \tilde{g}$ is strictly positive definite everywhere on $B(0, \delta_1)$, where $\tilde{g} = u^2[\tilde{g} - \frac{1}{g_M(X, X)} X^* \otimes X^*]$.

Let $v_1 \in E, 0 \neq v_2 \in T_{v_1}E = E$, we will show $(\exp_E)_{*v_1}(v_2) \neq 0$ when $|v_1|$ is small. Let $\gamma(s) = \exp_E(sv_1), s \in [0, 1]$ be a horizontal geodesic w.r.t. $u^2(\tilde{g})$ such that $\gamma(0) = \bar{x}, \gamma(1) = \exp_E(v_1)$. The variation $\epsilon \rightarrow \gamma_\epsilon(s) = \exp_E(s(v_1 + \epsilon v_2))$ of horizontal geodesics gives a nontrivial Jacobi field $U = s(\exp_E)_{*sv_1}(v_2)$ on γ such that $U(0) = 0, U(1) = (\exp_E)_{*v_1}(v_2)$.

Note that one can always construct an open contractible 3-dimensional smooth spacelike immersed submanifold $\sigma : \Sigma \rightarrow M$ so that there is a smooth map $\tilde{\gamma} : [0, 1] \rightarrow \Sigma$ satisfying $\gamma = \sigma \circ \tilde{\gamma}$. Actually, extend γ a little so that it is defined on $[-\epsilon', 1 + \epsilon']$ for some small $\epsilon' > 0$, let $\{E_1(s), E_2(s), E_3(s)\}$, where $E_3(s) = \dot{\gamma}(s)$, be a parallel and horizontal orthogonal $u^2\tilde{g}$ -frame along γ , and define $\Sigma = (\epsilon', 1 + \epsilon') \times \{|w_1|^2 + |w_2|^2 < \epsilon'^2\} \subset \mathbb{R}^3$, then the map $\sigma : \Sigma \rightarrow M$, where $\sigma(s, w_1, w_2) \triangleq \exp_{\gamma(s)}^{u^2\tilde{g}}(w_1 E_1(s) + w_2 E_2(s))$, will be an immersion when ϵ' is sufficiently small. Here $\exp_{\gamma(s)}^{u^2\tilde{g}}$ is the exponential map w.r.t. the metric $u^2\tilde{g}$ at $\gamma(s)$. By considering the integral curves of X passing through $\sigma(\Sigma)$ and setting the affine parameters $t = 0$ on Σ , there is a small positive number $\delta > 0$ such that the map $F : \Sigma \times (-\delta, \delta) \rightarrow M$, where $F(x, t) = \Psi_t(\sigma(x))$, is also an immersion. Let $\pi : \Sigma \times (-\delta, \delta) \rightarrow \Sigma, \pi(x, t) = x$ be the natural projection map. Now we pull back the metric \tilde{g} by F to $\Sigma \times (-\delta, \delta)$, and equip Σ the horizontal metric(still denoted by u^2g) induced from the metric $F^*(u^2\tilde{g})$.

Note that we can lift the family of horizontal geodesics $\epsilon \rightarrow \gamma_\epsilon$ on M to a family of horizontal geodesics $\epsilon \rightarrow \tilde{\gamma}_\epsilon$ on $\Sigma \times (-\delta, \delta)$ such that $F(\tilde{\gamma}_\epsilon) = \gamma_\epsilon$. Denote the variational vector field on $\tilde{\gamma}$ by \tilde{U} . Now $\pi(\tilde{\gamma}_\epsilon)$ is a variation of geodesics on (Σ, u^2g) (see (3.4)), $\pi_*\tilde{U}$ is the variational Jacobi field such that $\pi_*\tilde{U}(0) = 0$. $\pi_*\tilde{U}$ is nontrivial since its derivative at $\tilde{\gamma}(0)$ with respect to $\dot{\tilde{\gamma}}(0)$ is $\pi_*(v_2) \neq 0$. By (5.30), the conjugate radius of the exponential map of (Σ, u^2g) at $\tilde{\gamma}(0)$ is greater than $\pi\tilde{K}_{max}^{-\frac{1}{2}}$. So if $|v_1| < \pi\tilde{K}_{max}^{-\frac{1}{2}}$, we must have $\pi_*\tilde{U}(1) \neq 0$ and hence $(\exp_E^* \tilde{g})_{v_1}(v_2) \neq 0$. This finishes the proof of *Claim 1*.

Claim 2: The identity map from E to itself is the exponential map at point 0 of $(B(0, \delta_1), \exp_E^* \tilde{g})$.

The *Claim 2* is clear by our construction. We denote the metric $\exp_E^* \tilde{g}$ still by \tilde{g} . The point is that the injectivity radius of \tilde{g} at 0 is at least δ_1 . One can also pull back the functions u , the 1-form ω by the exponential map \exp_E to $B(0, \delta_1)$. Since $B(0, \delta_1)$ is contractible, the function ψ satisfying $d\psi = \omega$ can be globally defined on $B(0, \delta_1)$. We denote these pulled back quantities still by the same notations u, ψ, ω . From the first equation of (5.3) and our assumption, it is important to know that the curvature of \tilde{g} is bounded on $B(0, \delta_1)$. By [12], one can construct a harmonic coordinate system $\{z^i\}$ of radius $2\delta_2 > 0$ around 0 such that for any $0 < \alpha < 1, p = \alpha^{-1}$, there is a constant $C_\alpha > 0$ so that the estimate

$$\frac{1}{2}\delta_{ij} \leq \tilde{g}_{ij} \leq 2\delta_{ij}, |\tilde{g}_{ij}|_{C^{1,\alpha}} + |\tilde{g}_{ij}|_{W^{2,p}} \leq C_\alpha \tag{5.31}$$

holds on $\{|z| < 2\delta_2\}$. By the second and third equations of (5.3), we know $\tilde{\Delta}u$ and $d\omega$ and $\delta\omega$ are uniformly bounded by our assumption. By elliptic regularity, $|u|_{C^{1,\alpha}}$ and $|\omega|_{C^\alpha}$ are uniformly bounded on a smaller ball for any fixed $0 < \alpha < 1$. By Arzela-Ascoli theorem, one can take a $C^{1,\alpha}$ and $W^{2,p}$ convergent subsequence for \tilde{g} and u , and C^α , $W^{1,p}$ convergent subsequence of ω . We denote the limit by \tilde{g}^∞ , u^∞ , ω^∞ . Since h only involves ∇u , ω and g , we know that the C^α norm of h is uniformly bounded on $\{|z| < \frac{3}{2}\delta_2\}$ (independent of l). The limit h^∞ is C^α and must satisfy

$$h^\infty(x) \leq h^\infty(0) = 1 \quad (5.32)$$

on $\{|z| < \frac{3}{2}\delta_2\}$, see (5.28). Now we attempt to show that the limit $(\tilde{g}^\infty, u^\infty, \omega^\infty)$ is actually smooth and satisfies the vacuum Einstein equation.

Recall that in the above harmonic coordinate system $\{z^i\}$, Ricci curvature $2\tilde{R}_{ij} = -\tilde{g}^{kl}\frac{\partial^2}{\partial z^k\partial z^l}\tilde{g}_{ij} + Q_{ij}(\partial\tilde{g}, \tilde{g})$, where Q is quadratic in $\partial\tilde{g}$, with polynomial coefficients in \tilde{g} , \tilde{g}^{-1} .

For each scaled solution g_l, u_l, ω_l , in the above harmonic coordinate system $\{z^i\}$, multiplying the first equation of (5.3) by a function $\xi \in W_0^{1,p}(B(0, \delta_2))$, $p > 1$, and integrating by parts, we get:

$$\begin{aligned} & \int_{B(0, \delta_2)} \tilde{g}^{kl} \frac{\partial \tilde{g}_{ij}}{\partial z^k} \frac{\partial \xi}{\partial z^l} + [\partial_l \tilde{g}^{kl} \partial_k \tilde{g}_{ij} + Q(\partial \tilde{g}, \tilde{g})_{ij}] \xi dz^1 dz^2 dz^3 \\ & = 2 \int_{B(0, \delta_2)} \left(\frac{1}{2} u^{-4} \omega_i \omega_j + 2 \frac{u_i u_j}{u^2} \right) \xi dz^1 dz^2 dz^3 + 2I_4 \end{aligned} \quad (5.33)$$

where

$$I_4 = \int_{B(0, \delta_2)} (\bar{R}ic(e_i, e_j) - u^{-2} \bar{R}ic(X, X) g_{ij}) \xi dz^1 dz^2 dz^3.$$

Since the norm of $\bar{R}ic(e_i, e_j) - u^{-2} \bar{R}ic(X, X) g_{ij}$ is bounded by $Ch(\bar{x}_l)^{-1} \rightarrow 0$ by our scaling, we know $I_4 \rightarrow 0$ as $l \rightarrow \infty$. Note that the C^α -norms of $\partial\tilde{g}$, ω , ∂u are uniformly bounded, (5.33) must converge to

$$\begin{aligned} & \int_{B(0, \delta_2)} (\tilde{g}^\infty)^{kl} \frac{\partial}{\partial z^k} g_{ij}^\infty \frac{\partial}{\partial z^l} \xi \\ & = \int_{B(0, \delta_2)} \left[-\partial_l (\tilde{g}^\infty)^{kl} \partial_k \tilde{g}_{ij}^\infty - Q(\partial \tilde{g}^\infty, \tilde{g}^\infty)_{ij} + (u^\infty)^{-4} \omega_i^\infty \omega_j^\infty + 4 \frac{u_i^\infty u_j^\infty}{(u^\infty)^2} \right] \xi. \end{aligned} \quad (5.34)$$

Because $A_{ij} = -\partial_l (\tilde{g}^\infty)^{kl} \partial_k \tilde{g}_{ij}^\infty - Q(\partial \tilde{g}^\infty, \tilde{g}^\infty)_{ij} + (u^\infty)^{-4} \omega_i^\infty \omega_j^\infty + 4 \frac{u_i^\infty u_j^\infty}{(u^\infty)^2} \in W^{1,p}$, and coefficients $(\tilde{g}^\infty)^{kl} \in W^{2,p}$, we know $\tilde{g}^\infty \in W^{3,p}$ by standard L^p -estimate for elliptic equations of divergence form.

Now we can apply the same technique to the rest equations of (5.3) to obtain

$$\begin{aligned} (u^\infty)^2 \tilde{\Delta} \log u^\infty & = -\frac{1}{2} (u^\infty)^{-4} |\omega^\infty|^2 \\ (\tilde{g}^\infty)^{kl} \tilde{\nabla}_k \omega_l^\infty & = 4 (\tilde{g}^\infty)^{kl} \omega_k^\infty \nabla_l \log u^\infty \\ d\omega^\infty & = 0 \end{aligned} \quad (5.35)$$

where the above equations hold in the sense of integration by parts as in (5.34). Since $\omega^\infty \in W^{1,p}$, $u^\infty \in W^{2,p}$, by applying L^p estimates to the first equation of (5.35), we

know $u^\infty \in W^{3,p}$. This implies $\omega^\infty \in W^{3,p}$ by the 2nd and 3rd equations in (5.35). Hence $A_{ij} \in W^{2,p}$, this gives $\tilde{g}_{ij}^\infty \in W^{4,p}$. Repeating this arguments, we find that $\tilde{g}^\infty, u^\infty, \omega^\infty$ are actually smooth and satisfy the vacuum Einstein equations on $\{|z| < \delta_2\}$. Since $d\omega^\infty = 0$, from the calculations in Section 5.2, we know equation (5.22) must hold for the limit $(g^\infty, u^\infty, \omega^\infty)$. Moreover, $I_2 = 0$ and $I_3 = 0$ hold in (5.22) and (5.23). That is to say, we have

$$\begin{aligned} &\Delta_{g^\infty}(h^\infty) + \langle \nabla \log u^\infty, \nabla(h^\infty) \rangle \\ &= 4|\nabla \log u^\infty|^4 + |\omega^\infty|^2 |\nabla \log u^\infty|^2 + |2\nabla_{ij} \log u^\infty + (u^\infty)^{-4} \omega_i^\infty \omega_j^\infty|^2 \\ &\quad + (u^\infty)^{-4} |\nabla_i \omega_j^\infty - 2\omega_i^\infty (\log u^\infty)_j - 2\omega_j^\infty (\log u^\infty)_i|^2 \\ &\quad + 5|(u^\infty)^{-2} \omega^\infty \wedge d \log u^\infty|^2 \\ &\geq 0 \end{aligned} \tag{5.36}$$

on $\{|z| < \delta_3\}$.

Now we can apply the strong maximum principle on equation (5.36) since (5.32) holds. It follows that $h^\infty \equiv \text{const.}$, and the right hand side of (5.36) vanishes everywhere on $\{|z| < \delta_3\}$. In particular, this implies $|\nabla \log u^\infty|^4 \equiv 0$ and $|2\nabla_{ij} \log u^\infty + (u^\infty)^{-4} \omega_i^\infty \omega_j^\infty|^2 = 0$ on $\{|z| < \delta_3\}$, which give us $u^\infty \equiv 1, \omega^\infty \equiv 0$. Hence $h^\infty \equiv 0$, which is a contradiction with $h^\infty(0) = 1$. This proves Case 1.

For Case 2, the maximum of $h(x)$ can be achieved at some point \bar{x} by the compactness of M . The result can be proved by following the same argument of Case 1. The proof of the theorem is complete. \square

From the proof of Theorem 5.3, if we integrate the vector field X for a short time along the image of the horizontal exponential map, one can obtain a local covering map which provides a ‘good’ local ‘coordinate system’ (c.f. (5.31)).

THEOREM 5.4. *Under the assumptions of Theorem 5.3, there is a smooth non-degenerate map $\Psi : \{|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 < C_0^{-2} a^2\} \rightarrow \hat{B}(x_0, a)$, $\Psi(0) = x_0$ such that $\Psi^* g_M = \tilde{g}_{\alpha\beta} dz^\alpha dz^\beta = -u^2 (dz^0 + \sum \theta_i dz^i)^2 + g_{ij} dz^i dz^j$ satisfies*

$$\begin{aligned} &\frac{1}{1 + C_0} < u < 1 + C_0, \quad |\theta| < C_0 \\ &(1 + C_0)^{-1} \delta_{ij} < g_{ij} < (1 + C_0) \delta_{ij} \\ &\frac{1}{a^4} \int_{\{|z| < ca\}} (a|\partial\bar{g}| + a^2|\partial^2\bar{g}|)^p dz < C_p \end{aligned} \tag{5.37}$$

where u, θ and g_{ij} are z^0 - independent, (z^1, z^2, z^3) are harmonic coordinates for $g_{ij} dz^i dz^j$, $C_p, p = 0, 1, \dots$, are a family of universal constants.

THEOREM 5.5. *Let (M, g_M) be a Einstein spacetime of dimension 4 with a time-like Killing field X , $\text{Ric}(g_M) = \lambda g_M$, where $\lambda \geq 0$, and let $\hat{B}(x_0, a)$ be a \hat{g} -metric ball in M with compact closure. Then we have*

$$\sup_{\hat{B}(x_0, \frac{a}{2})} |\nabla \log u|^2 + u^{-4} |\omega|^2 \leq Ca^{-2}, \tag{5.38}$$

for some universal constant C .

Proof. When $\lambda = 0$, one can apply Theorem 5.3 to derive (5.38) since condition (5.24) holds trivially in this case. We only need to handle $\lambda > 0$ case. By scaling

invariance of the estimate, one can assume $a = 1$. We mimic the proof of Theorem 5.3.

We treat the case $\partial\hat{B}(x_0, 1) \neq \phi$ first.

Let $h(x) = 2|\nabla \log u|^2(x) + \frac{1}{2}u^{-4}|\omega|^2(x) + 6\lambda$, $f(x) = h(x)d_{\hat{g}}^2(x, \partial\hat{B}(x_0, 1))$, and $\bar{x} \in \hat{B}(x_0, 1)$ such that $f(\bar{x}) = \sup_{x \in \hat{B}(x_0, 1)} f(x)$. To prove $f(\bar{x}) < C$ for some universal constant C , we will argue by contradiction. Suppose there are a sequence of 4-Lorentzian manifolds (M_l, \bar{g}_l) satisfying $Ric(\bar{g}_l) = \lambda_l \bar{g}_l$ ($\lambda_l \geq 0$) and a sequence of \hat{g}_l -balls $\hat{B}(x_l, 1) \subset M_l$ with compact closure such that $f(\bar{x}_l) \rightarrow \infty$ as $l \rightarrow \infty$, where

$$f(\bar{x}_l) = \sup_{x \in \hat{B}(x_l, 1)} h_l(x)d_{\hat{g}_l}^2(x, \partial\hat{B}(x_l, 1))$$

$$h_l(x) = 2|\nabla \log u_l|^2 + \frac{1}{2}u_l^{-4}|\omega_l|^2 + 6\lambda_l. \tag{5.39}$$

Scaling u_l and \bar{g}_l by $u_l(\bar{x}_l)^{-1}$ and $h_l(\bar{x}_l)$ respectively, one can assume $u_l(\bar{x}_l) = 1$, $h_l(\bar{x}_l) = 1$. We still use the same notations u_l, ω_l, \bar{g}_l , etc., to denote the corresponding scaled quantities.

Note that the boundedness of h_l implies that the sectional curvature of \bar{g} is uniformly bounded on $B_{\bar{g}_l}(\bar{x}_l, 1)$. As in Theorem 5.3, one can use the horizontal exponential map (w.r.t. metric $u_l^2\bar{g}_l$) to pull back \bar{g}_l, ω_l and u_l to horizontal tangent space. Using the harmonic coordinates $\{z^i\}$ on the horizontal tangent space and a boot strap argument as in Theorem 5.3, one can show that $\{u_l, \omega_l, \bar{g}_l\}$ has a subsequence converging to a smooth limit $(u^\infty, \omega^\infty, \bar{g}^\infty)$. Note that on (5.22), $I_2 + I_3 = 4\lambda|\nabla \log u|^2 + 3\lambda u^{-4}|\omega|^2 \geq 0$ for each $(u_l, \omega_l, \bar{g}_l)$. So $I_2 + I_3 \geq 0$ still holds for the limit $\{u^\infty, \omega^\infty, \bar{g}^\infty\}$. By applying the strong maximum principle to equation (5.22) for the limit as in Theorem 5.3, we find $u^\infty \equiv 1, \omega^\infty = 0, h^\infty \equiv 1$, and $\lambda^\infty = \frac{1}{6}$. From the second equation of (5.1), we have $\Delta u^\infty = -\frac{1}{6}u^\infty$, which is a contradiction.

If $\partial\hat{B}(x_0, a) = \phi$, M will be compact. The maximum point of $h(x)$ can be achieved. One can apply the strong maximum principle directly on (5.22) to find a contradiction with $\lambda > 0$ as in the preceding argument. \square

THEOREM 5.6. *Assume (M, g_M, X) to be a spacetime of dimension 4 with a timelike Killing field X such that $Ric(g_M) = \lambda g_M$. Let $\hat{B}(x_0, a)$ be a \hat{g} -metric ball with compact closure in M , where $0 < a < \frac{1}{\sqrt{\max\{-\lambda, 0\}}}$. Then we have*

$$\sup_{x \in \hat{B}(x_0, \frac{a}{2})} |Rm(g)|(x) + |Rm(g_M)|_{\hat{g}}(x) + |Rm(\hat{g})|_{\hat{g}}(x) \leq \frac{C_0}{a^2}, \tag{5.40}$$

and

$$\sup_{x \in \hat{B}(x_0, \frac{a}{2})} |\nabla_g^k Rm(g)|_g(x) + |\nabla_{g_M}^k Rm(g_M)|_{\hat{g}}(x) + |\nabla_{\hat{g}}^k Rm(\hat{g})|_{\hat{g}}(x) \leq \frac{C_k}{a^{k+2}}, \tag{5.41}$$

where $Rm(g)$ is the Riemann curvature tensor of the horizontal metric g , $C_k, k = 0, 1, 2, \dots$, are constants.

Proof. By scaling invariance of the estimates, we can assume $a = 1, \lambda \geq 0$ or $\lambda = -1$. From Theorems 5.3 and 5.5, we know $|\nabla \log u|^2 + u^{-4}|\omega|^2 \leq C$ on $\hat{B}(x_0, \frac{15}{16})$. Hence, we know the curvature $Rm(\tilde{g})$ of the conformal horizontal metric $\tilde{g} = u^2g$ is bounded. Next we may apply the regularity argument as in the proof of Theorem 5.3

to prove

$$\sup_{x \in \tilde{B}(x_0, \frac{1}{2} + \frac{1}{4^{k+1}})} |\tilde{\nabla}^k u|_{\tilde{g}}(x) + |\tilde{\nabla}^k \text{Rm}(\tilde{g})|_{\tilde{g}}(x) + |\tilde{\nabla}^k \omega|_{\tilde{g}}(x) \leq C_k. \quad (5.42)$$

Now (5.41) can be easily deduced from (5.42) and (2.10). \square

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