

## REAL PINOR BUNDLES AND REAL LIPSCHITZ STRUCTURES\*

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**Abstract.** Let  $(M, g)$  be a pseudo-Riemannian manifold of arbitrary dimension and signature. We prove that there exist mutually quasi-inverse equivalences between the groupoid of weakly faithful real pinor bundles on  $(M, g)$  and the groupoid of weakly faithful real Lipschitz structures on  $(M, g)$ , from which follows that every bundle of weakly faithful real Clifford modules is associated to a real Lipschitz structure. The latter gives a generalization of spin structures based on certain groups which we call real Lipschitz groups. In the irreducible case, we classify real Lipschitz groups in all dimensions and signatures. Using this classification and the previous correspondence we obtain the topological obstruction to existence of a bundle of irreducible real Clifford modules over a pseudo-Riemannian manifold  $(M, g)$  of arbitrary dimension and signature. As a direct application of the previous results, we show that the supersymmetry generator of eleven-dimensional supergravity in “mostly plus” signature can be interpreted as a global section of a bundle of irreducible Clifford modules if and *only if* the underlying eleven-manifold is orientable and spin.

**Key words.** Spinor bundles, pseudo-Riemannian manifolds, Clifford algebras, Lipschitz structures.

**Mathematics Subject Classification.** 53C27, 53C10.

**Introduction.** The classical approach to spin geometry [1, 2, 3, 4] assumes existence of a spin structure  $Q$  on a pseudo-Riemannian manifold  $(M, g)$ . Given a linear representation of the corresponding spin group,  $Q$  induces a spinor bundle  $S$  through the associated vector bundle construction. One then shows that  $S$  carries a globally-defined “internal” Clifford multiplication  $TM \otimes S \rightarrow S$ , which turns  $S$  into a bundle of modules over the Clifford bundle  $\text{Cl}(M, g)$ . This allows one to lift a metric connection on  $(M, g)$  to a connection on  $S$  and to define a corresponding Dirac operator. This construction generalizes to  $\text{Spin}^c$  and  $\text{Spin}^q$  structures [5], which can also be used to construct “spinor bundles” with globally-defined internal Clifford multiplication. Thus, existence of such a spinorial structure implies existence of a bundle of Clifford modules over  $(M, g)$ , but the converse is not always true.<sup>1</sup>

For various applications, it is important to develop spin geometry starting from the assumption that  $(M, g)$  admits a bundle of Clifford modules  $S$ , without first choosing by hand a particular spinor structure to which  $S$  is associated. The condition that  $S$  be a bundle of Clifford modules is equivalent to the requirement that  $S$  be endowed with a globally-defined “internal” Clifford multiplication  $TM \otimes S \rightarrow S$ . In general, this condition is weaker than existence of a spin structure, as illustrated by the theory of  $\text{Spin}^c$  and  $\text{Spin}^q$  structures. However, a systematic study of the necessary and sufficient conditions under which a bundle  $S$  of Clifford modules exists on  $(M, g)$

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<sup>1</sup>A well-known variant [1, 6] starts with a Pin structure, leading to a bundle  $S$  which need not admit an “internal” Clifford multiplication, but rather a map that takes  $TM \otimes S$  into a bundle  $S'$  which (depending on dimension and signature) may be non-isomorphic with  $S$ . In that case, one can sometimes define a “modified” version of the Dirac operator (see, for example, [7, 8, 9]). We stress that in this paper we are interested only in vector bundles  $S$  which admit an *internal* Clifford multiplication  $TM \otimes S \rightarrow S$ . A brief discussion of external and internal Clifford multiplications can be found in Appendix B.

does not appear to have been carried out for the case of real Clifford representations. The purpose of the present paper is to perform such a study.

Our approach relies on an equivalence of categories (which we establish in Section 7) between the groupoid of bundles of real Clifford modules obeying a certain “weak faithfulness” condition and the groupoid of so-called *real Lipschitz structures*, a notion which generalizes that of complex Lipschitz structure which was introduced in [10]. This result allows us to extract the *necessary and sufficient* conditions under which  $(M, g)$  admits a bundle of real Clifford modules with given fiberwise representation type.

A real Lipschitz structure is a generalization of a spin structure where the spin group is replaced by a so-called *real Lipschitz group*. The latter is the group of all implementers of pseudo-orthogonal transformations through operators acting in the representation space of a weakly faithful real Clifford representation  $\gamma$  and arises naturally as the automorphism group of  $\gamma$  in a certain category of real Clifford representations and unbased morphisms. The character of the Lipschitz group depends on  $\gamma$  and on the signature  $(p, q)$  of  $g$ . When  $\gamma$  is irreducible, then it is automatically weakly faithful and the character of its Lipschitz group depends on the  $\pmod 8$  reduction of  $p - q$ . We identify all such *elementary Lipschitz groups* as well as certain homotopy-equivalent reduced forms thereof, the latter being summarized in Table 0.1. The  $\text{Spin}^o(V, h)$  structures arising when  $p - q \equiv_8 3, 7$  (and whose real Lipschitz groups are discussed in Subsection 2.3) appear to be new and are studied in detail in the companion paper [11].

$p - q$ $\pmod 8$	Reduced Lipschitz group
0, 2	$\text{Pin}(V, h)$
3, 7	$\text{Spin}^o(V, h)$
4, 6	$\text{Pin}^q(V, h)$
1	$\text{Spin}(V, h)$
5	$\text{Spin}^q(V, h)$

TABLE 0.1

*The reduced elementary Lipschitz group of a pseudo-Riemannian manifold  $(M, g)$  of dimension  $d = p + q$  and signature  $(p, q)$ .*

We also study certain representations of Lipschitz groups, which turn out to play an important role for further developments of the theory. Using these results, we extract the topological obstructions to existence of bundles of irreducible real Clifford modules on  $(M, g)$  in any dimension and signature. This allow us to identify the minimal requirements for developing a version of real spin geometry based on such bundles.

Our study is motivated, in particular, by physical theories such as supergravity and string theory, where it is important to understand the weakest assumptions under which certain models can be defined globally. As we shall show in later papers, this leads to new questions and problems which do not appear to have been systematically considered before and which are of mathematical and physical interest.

The results of this paper lead to various questions which may be of interest for further study. For example, one can ask what modifications may arise in the index theorem for Dirac operators defined on pinor bundles which are associated to real

Lipschitz structures in various dimensions and signatures. Our results afford a systematic study of Killing spinors and generalized Killing spinors on the most general pseudo-Riemannian manifolds admitting bundles of irreducible real Clifford modules. For example, they could be used to extend Wang's results [12] from spin manifolds to manifolds of arbitrary signature which admit bundles of irreducible real Clifford modules. Killing spinors [13] and generalized Killing spinors [14] were studied in the literature on manifolds admitting Spin, Spin<sup>c</sup> and Spin<sup>q</sup> structures [15, 16]; more general spinorial structures (whose associated vector bundles need not be bundles of irreducible Clifford modules) are studied in references [17, 18]. However, Killing spinors on Spin<sup>o</sup> manifolds do not seem to have been studied systematically. In addition, and especially for applications to supergravity and string theory, it would be very interesting to perform an analysis similar to the one presented in this work for the case of bundles of irreducible as well as faithful Clifford modules over the *even* bundle of Clifford algebras of the underlying manifold. Work in this direction is already in progress.

The paper is organized as follows. Section 1 summarizes some facts on real Clifford algebras and associated groups, with the purpose of fixing terminology and notation; it also proves some results which will be needed later on, some of which are not readily available in the literature. Section 2 summarizes certain enlargements of the Spin group which will turn out to provide models for the reduced Lipschitz group of irreducible real Clifford representations in various dimensions and signatures. The same section considers certain representations of these groups. Section 3 considers a certain category of real Clifford representations and unbased morphisms and discusses certain subspaces associated to such representations as well as the notion of weak faithfulness. Section 4 defines real Lipschitz groups and discusses some properties of their elementary representations in a general setting. Section 5 considers the case of irreducible real Clifford representations, which always turn out to be weakly faithful. In that section, we classify the Lipschitz groups of such representations and establish isomorphisms between their reduced versions and various enlarged spinor groups introduced in Section 2. We also describe the elementary representations of such Lipschitz groups and connect them to those of the enlarged spinor groups. Section 6 discusses bundles of weakly faithful real Clifford modules as well as real Lipschitz structures, establishing a general equivalence between the corresponding groupoids. Section 7 discusses certain enlarged spinorial structures which are relevant later on. Section 8 considers the case of bundles of irreducible real Clifford modules and the corresponding Lipschitz structures, which we call *elementary*. Using the results of the previous sections, we determine the topological obstructions to existence of such bundles in every dimension and signature. Section 9 discusses a direct application of our results to the global formulation of M-theory on an eleven-dimensional Lorentzian manifold. Section 10 outlines the relation of our work with certain results in the literature. The appendices contain technical material.

**0.1. Notations, conventions and terminology.** In this paper, a quadratic vector space means a pair  $(V, h)$ , where  $V$  is a finite-dimensional  $\mathbb{R}$ -vector space and  $h : V \times V \rightarrow \mathbb{R}$  is a non-degenerate symmetric bilinear form. Throughout the paper, we assume  $V \neq 0$ . The Clifford algebra  $\text{Cl}(V, h)$  is considered only over  $\mathbb{R}$ . We use the *plus* convention for Clifford algebras, so  $\text{Cl}(V, h)$  is the unital associative algebra generated by  $V$  over  $\mathbb{R}$  with the relations:

$$v^2 = h(v, v) \quad \forall v \in V \quad .$$

A *Clifford representation* is a finite-dimensional unital representation  $\gamma : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S)$  through endomorphisms of a *real* finite-dimensional vector space  $S$  — we never use the complexification of  $\text{Cl}(V, h)$  or the complexification of  $S$ . An irreducible Clifford representation is always assumed to be realized in a space  $S$  of positive dimension (i.e.  $S \neq 0$ ).

Given two groups  $A, B$  such that  $A \times B$  contains a given central  $\mathbb{Z}_2$  subgroup  $C$ , we use the notation  $A \cdot B$  for the quotient  $A \times B/C$ . Let  $\mathbb{G}_m \simeq \mathbb{Z}_m$  denote the group of complex roots of unity of order  $m \in \mathbb{N}_{>0}$  and  $D_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$  denote the dihedral group of order 4.

For  $\mathbb{S} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$  any of the three finite-dimensional associative division algebras over  $\mathbb{R}$ , let  $|\cdot| : \mathbb{S} \rightarrow \mathbb{R}_+$  denote the canonical norm and  $\text{U}(\mathbb{S})$  denote the group of unit norm elements:

$$\text{U}(\mathbb{S}) = \begin{cases} \mathbb{G}_2 & \mathbb{S} = \mathbb{R} \\ \text{U}(1) & \mathbb{S} = \mathbb{C} \\ \text{Sp}(1) & \mathbb{S} = \mathbb{H} \end{cases} .$$

For the algebras  $\mathbb{R}, \mathbb{C}$  and  $\mathbb{H}$ , we define  $M : \mathbb{S} \rightarrow \mathbb{R}_+$  through  $M(s) \stackrel{\text{def.}}{=} |s|^2$ . For the algebra  $\mathbb{D}$  of hyperbolic (a.k.a. split complex) numbers, let  $M : \mathbb{D} \rightarrow \mathbb{R}$  denote the hyperbolic modulus and  $\text{U}(\mathbb{D})$  denote the group of unit hyperbolic numbers (see Appendix A).

Let  $\text{Alg}$  denote the category of finite-dimensional associative and unital  $\mathbb{R}$ -algebras,  $\text{Gp}$  denote the category of groups and  $\text{Set}$  denote the category of sets. For any ring  $R$ , let  $R^\times$  denote its group of invertible elements. For any category  $\mathcal{C}$ , let  $\mathcal{C}^\times$  denote its unit groupoid (the groupoid obtained from  $\mathcal{C}$  by keeping as morphisms only the isomorphisms of  $\mathcal{C}$ ). The symbol  $\simeq_{\mathcal{C}}$  indicates existence of an isomorphism between two objects of  $\mathcal{C}$ . For notational uniformity, we define:

$$\hat{\text{O}}(V, h) \stackrel{\text{def.}}{=} \begin{cases} \text{O}(V, h) & \text{if } d = \text{ev} \\ \text{SO}(V, h) & \text{if } d = \text{odd} \end{cases}$$

for any finite-dimensional quadratic vector space  $(V, h)$  over  $\mathbb{R}$ .

All manifolds  $M$  considered in the paper are connected, Hausdorff and paracompact (hence also second countable). All fiber bundles considered are smooth. We assume  $\dim M > 0$  throughout.

**1. Real Clifford algebras and extended Clifford groups.** This section summarizes various facts about Clifford algebras and associated spinor groups, with the purpose of fixing notations and terminology; some statements which are well-known or easily established are given without proof. We draw the reader’s attention to our discussion of extended Clifford groups, extended Clifford norms and extended Pin groups as well as to the delicate distinction between twisted and untwisted vector representations. The extended (sometimes called “untwisted”) Clifford group  $\text{G}^e(V, h)$  consists of those elements of the multiplicative subgroup of  $\text{Cl}(V, h)$  whose *untwisted* adjoint action fixes  $V$ . It differs from the ordinary (or “twisted”) Clifford group  $\text{G}(V, h)$  by elements belonging to the center of  $\text{Cl}(V, h)$ . Unlike the ordinary Clifford group, the extended Clifford group need not be  $\mathbb{Z}_2$ -graded, since it may contain inhomogeneous elements from the center of the Clifford algebra. The extended Clifford group and its untwisted vector representation (adjoint representation restricted to  $V$ ) are classical, but have seen decreased use after the work of [1], which promoted the use of the ordinary Clifford group and of its twisted vector representation — none of which

are, however, natural for our purpose. In this paper, the extended Clifford group and its vector representation will play a central role since, as we shall see in later sections, these objects relate most directly to the Lipschitz group of irreducible Clifford representations and to the natural representation of that Lipschitz group on  $V$ . The distinction between the twisted and untwisted vector representations of pin groups will also be essential when discussing topological obstructions to existence of elementary real Lipschitz structures. As is well-known, that distinction also plays an important role in the theory of Pin structures and associated Dirac operators [7, 8, 9, 6].

**1.1. The category of real quadratic spaces.** A real quadratic space is a pair  $(V, h)$ , where  $V \neq 0$  is a finite-dimensional  $\mathbb{R}$ -vector space and  $h : V \times V \rightarrow \mathbb{R}$  is a non-degenerate symmetric bilinear pairing on  $V$ . A morphism of quadratic spaces from  $(V, h)$  to  $(V', h')$  (also known as an isometry) is an  $\mathbb{R}$ -linear map  $\varphi : V \rightarrow V'$  such that  $h'(\varphi(v_1), \varphi(v_2)) = h(v_1, v_2)$  for all  $v_1, v_2 \in V$ . Quadratic spaces over  $\mathbb{R}$  and their morphisms form a category  $\text{Quad}$  whose unit groupoid  $\text{Quad}^\times$  we call the *groupoid of real quadratic spaces*; its objects coincide with those of  $\text{Quad}$  while its morphisms are the invertible isometries. When  $h$  is positive-definite, an isometry  $\varphi : (V, h) \rightarrow (V', h')$  is necessarily injective.

**1.2. Non-degenerate vectors and reflections.** Let  $(V, h)$  be a real quadratic space.

DEFINITION 1.1. *The signature  $\epsilon(v) \in \{-1, 0, 1\}$  of a vector  $v \in V$  is the signature of the real number  $h(v, v)$ . The vector  $v$  is called non-degenerate if  $h(v, v) \neq 0$ . It is called a unit vector if  $|h(v, v)| = 1$ .*

The reflection determined by a non-degenerate vector  $v \in V$  is the linear map  $R_v \in \text{O}(V, h)$  given by:

$$R_v(x) \stackrel{\text{def.}}{=} x - 2 \frac{h(x, v)}{h(v, v)} v = x - 2\epsilon(v) \frac{h(x, v)}{|h(v, v)|} v \quad , \tag{1.1}$$

which describes the  $h$ -orthogonal reflection of  $V$  with respect to the hyperplane  $v^\perp = \{y \in V | h(y, v) = 0\} \subset V$  orthogonal to  $v$ . We have  $R_v = R_{\lambda v}$  for any  $\lambda \in \mathbb{R}^\times$ .

**1.3. The category of real Clifford algebras.** The Clifford algebra construction gives a functor  $\text{Cl} : \text{Quad} \rightarrow \text{Alg}$ , where  $\text{Alg}$  denotes the category of unital associative  $\mathbb{R}$ -algebras and unital algebra morphisms. For each object  $(V, h)$  of  $\text{Quad}$ ,  $\text{Cl}(V, h)$  is the Clifford algebra of the quadratic space  $(V, h)$  while for each isometry  $\varphi : (V, h) \rightarrow (V', h')$ ,  $\text{Cl}(\varphi) : \text{Cl}(V, h) \rightarrow \text{Cl}(V', h')$  denotes the unique unital morphism of algebras which satisfies the condition  $\text{Cl}(\varphi)|_V = \varphi$ . The image of the functor  $\text{Cl}$  is a *non-full* sub-category of  $\text{Alg}$  which we denote by  $\text{Cl}$  and whose unit groupoid we denote by  $\text{Cl}^\times$ . Namely:

DEFINITION 1.2. *A morphism of Clifford algebras is a morphism  $\alpha : \text{Cl}(V, h) \rightarrow \text{Cl}(V', h')$  in the category  $\text{Cl}$ , i.e. a morphism of unital algebras which satisfies  $\alpha(V) \subset V'$  and hence is necessarily of the form  $\alpha = \text{Cl}(\varphi)$  for a (uniquely-determined) isometry  $\varphi : (V, h) \rightarrow (V', h')$ , given by  $\varphi \stackrel{\text{def.}}{=} \alpha|_V$ .*

By definition, isomorphisms of Clifford algebras are the morphisms of  $\text{Cl}^\times$ . These are those isomorphisms of unital algebras  $\alpha : \text{Cl}(V, h) \rightarrow \text{Cl}(V', h')$  which satisfy  $\alpha(V) = V'$  and hence are of the form  $\alpha = \text{Cl}(\varphi)$  for a uniquely-determined invertible isometry  $\varphi : (V, h) \rightarrow (V', h')$  (given by  $\varphi = \alpha|_V$ ). The corestriction of the functor  $\text{Cl}$

to its image gives an isomorphism of categories  $\text{Quad} \simeq \text{Cl}$ , which in turn restricts to an isomorphism  $\text{Quad}^\times \simeq \text{Cl}^\times$ . Note that two Clifford algebras may be isomorphic as unital associative algebras *without* being isomorphic as *Clifford* algebras (i.e. without being isomorphic in the category  $\text{Cl} \simeq \text{Quad}$ ).

The category  $\text{Quad}$  admits a skeleton whose objects are the *standard quadratic spaces*  $\mathbb{R}^{p,q} \stackrel{\text{def.}}{=} (\mathbb{R}^{p+q}, h_{p,q})$ , where  $h_{p,q} : \mathbb{R}^{p+q} \times \mathbb{R}^{p+q} \rightarrow \mathbb{R}$  is the *standard symmetric bilinear form of signature*  $(p, q)$ :

$$h_{p,q}(x, y) \stackrel{\text{def.}}{=} \sum_{i=1}^p x_i y_i - \sum_{j=p+1}^{p+q} x_j y_j \quad \forall x, y \in \mathbb{R}^{p+q} .$$

The objects of this skeleton form a countable set indexed by the pairs  $(p, q) \in \mathbb{N} \times \mathbb{N}$ . Accordingly, the category  $\text{Cl}$  admits a skeleton whose objects are the *standard real Clifford algebras*  $\text{Cl}_{p,q} \stackrel{\text{def.}}{=} \text{Cl}(\mathbb{R}^{p,q})$ .

**1.4. Parity, reversion and twisted reversion.** The Clifford algebra  $\text{Cl}(V, h)$  admits three canonical involutive (anti-)automorphisms:

1. The *parity involution* of  $\text{Cl}(V, h)$  is the unique unital  $\mathbb{R}$ -algebra automorphism  $\pi \in \text{Aut}_{\text{Alg}}(\text{Cl}(V, h))$  such that  $\pi(v) = -v$  for all  $v \in V$ .
2. The *reversion* is the unique unital anti-automorphism  $\tau$  of  $\text{Cl}(V, h)$  such that  $\tau(v) = v$  for all  $v \in V$ .
3. The *twisted reversion* is the unique unital anti-automorphism  $\tilde{\tau}$  of  $\text{Cl}(V, h)$  such that  $\tilde{\tau}(v) = -v$ .

We have:

$$\pi^2 = \tau^2 = \tilde{\tau}^2 = \text{id}_{\text{Cl}(V, h)} \quad , \quad \tilde{\tau} = \tau \circ \pi = \pi \circ \tau$$

and the group  $\{\text{id}_{\text{Cl}(V, h)}, \pi, \tau, \tilde{\tau}\}$  is isomorphic with  $D_4$ .

REMARK 1.1. The underlying vector space of  $\text{Cl}(V, h)$  is canonically isomorphic with  $\wedge V$  through the Chevalley-Riesz-Crumeyrolle isomorphism [19] (which depends on  $h$ ). This gives a decomposition  $\text{Cl}(V, h) = \bigoplus_{k=0}^d \text{Cl}_k(V, h)$ , where  $\text{Cl}_k(V, h)$  is the subspace corresponding to  $\wedge^k V$  (this decomposition does *not* give a  $\mathbb{Z}$ -grading of the associative algebra  $\text{Cl}(V, h)$ ). We have:

$$\pi|_{\text{Cl}_k(V, h)} = (-1)^k \text{id}_{\text{Cl}_k(V, h)} \quad , \quad \tau|_{\text{Cl}_k(V, h)} = (-1)^{\frac{k(k-1)}{2}} \text{id}_{\text{Cl}_k(V, h)} .$$

**1.5. The canonical  $\mathbb{Z}_2$ -grading.** The algebra  $\text{Cl}(V, h)$  admits the *canonical  $\mathbb{Z}_2$ -grading*:

$$\text{Cl}(V, h) = \text{Cl}_+(V, h) \oplus \text{Cl}_-(V, h) \quad , \quad \text{Cl}_\pm(V, h) \stackrel{\text{def.}}{=} \ker(\pi \mp \text{id}_{\text{Cl}(V, h)}) .$$

Namely,  $\text{Cl}_+(V, h)$  is the subalgebra generated by all Clifford monomials  $v_1 \dots v_k$  ( $v_j \in V$ ) with even  $k$  and  $\text{Cl}_-(V, h)$  is the subspace generated by all Clifford monomials with odd  $k$ .

**1.6. The group of Clifford units and its adjoint and twisted adjoint actions.** The group of Clifford units is the group  $\text{Cl}(V, h)^\times$  formed by all invertible elements of the algebra  $\text{Cl}(V, h)$ . Let  $\text{Cl}_\pm^\times(V, h) \stackrel{\text{def.}}{=} \text{Cl}(V, h)^\times \cap \text{Cl}_\pm(V, h)$ . Then:

$$\text{Cl}_{\text{hom}}^\times(V, h) \stackrel{\text{def.}}{=} \text{Cl}_+^\times(V, h) \sqcup \text{Cl}_-^\times(V, h)$$

is a subgroup of  $\text{Cl}(V, h)^\times$  called the *group of homogeneous units*. This group is  $\mathbb{Z}_2$ -graded by the disjoint union decomposition given above.

The *adjoint action* of  $\text{Cl}(V, h)^\times$  is the group morphism  $\text{Ad}^{\text{Cl}} : \text{Cl}(V, h)^\times \rightarrow \text{Aut}_{\text{Alg}}(\text{Cl}(V, h))$  given by:

$$\text{Ad}^{\text{Cl}}(a)(x) \stackrel{\text{def.}}{=} axa^{-1} \quad \forall a \in \text{Cl}(V, h)^\times \quad \forall x \in \text{Cl}(V, h) \quad ,$$

which gives a representation of  $\text{Cl}(V, h)^\times$  through unital  $\mathbb{R}$ -algebra automorphisms of  $\text{Cl}(V, h)$ .

The *twisted adjoint action* of  $\text{Cl}(V, h)^\times$  is the group morphism  $\widetilde{\text{Ad}}^{\text{Cl}} : \text{Cl}(V, h)^\times \rightarrow \text{Aut}_{\mathbb{R}}(\text{Cl}(V, h))$  given by:

$$\widetilde{\text{Ad}}^{\text{Cl}}(a)(x) \stackrel{\text{def.}}{=} \pi(a)xa^{-1} \quad \forall a \in \text{Cl}(V, h)^\times \quad \forall x \in \text{Cl}(V, h) \quad .$$

This gives a representation of the group of units through automorphisms of the underlying vector space of  $\text{Cl}(V, h)$ , which (unlike the adjoint action) need not be  $\mathbb{R}$ -algebra automorphisms. We have:

$$\widetilde{\text{Ad}}^{\text{Cl}}|_{\text{Cl}_\pm^\times(V, h)} = \pm \text{Ad}^{\text{Cl}}|_{\text{Cl}_\pm^\times(V, h)} \quad .$$

Notice that  $\pi(\text{Cl}_\pm^\times(V, h)) = \text{Cl}_\pm^\times(V, h)$  as well as the relations:

$$\text{Ad}^{\text{Cl}} \circ \pi|_{\text{Cl}_{\text{hom}}^\times(V, h)} = \text{Ad}^{\text{Cl}}|_{\text{Cl}_{\text{hom}}^\times(V, h)} \quad , \quad \widetilde{\text{Ad}}^{\text{Cl}} \circ \pi|_{\text{Cl}_{\text{hom}}^\times(V, h)} = \widetilde{\text{Ad}}^{\text{Cl}}|_{\text{Cl}_{\text{hom}}^\times(V, h)} \quad .$$

**1.7. Clifford volume elements.** Any orientation of  $V$  determines a *Clifford volume element*  $\nu = e_1 \dots e_d \in \text{Cl}(V, h)^\times$ , where  $(e_1, \dots, e_d)$  is any oriented orthonormal basis of  $(V, h)$ . This element is independent of the choice of oriented orthonormal basis; it depends only on  $h$  and on the chosen orientation of  $V$ . Moreover, the Clifford volume element determined by the opposite orientation of  $V$  equals  $-\nu$ . The Clifford volume element has the following properties, which will be used intensively later on:

$$\nu^2 = \sigma_{p,q} \stackrel{\text{def.}}{=} (-1)^{q+\lfloor \frac{d}{2} \rfloor} = \begin{cases} (-1)^{\frac{p-q}{2}} & \text{if } d = \text{even} \\ (-1)^{\frac{p-q-1}{2}} & \text{if } d = \text{odd} \end{cases} = \begin{cases} +1 & \text{if } p - q \equiv_4 0, 1 \\ -1 & \text{if } p - q \equiv_4 2, 3 \end{cases} \quad (1.2)$$

$$\tau(\nu) = (-1)^{\frac{d(d-1)}{2}} \nu = (-1)^{\lfloor \frac{d}{2} \rfloor} \nu = \begin{cases} +\nu & \text{if } d \equiv_4 0, 1 \\ -\nu & \text{if } d \equiv_4 2, 3 \end{cases} = \begin{cases} (-1)^{\frac{p+q}{2}} \nu & \text{if } d = \text{even} \\ (-1)^{\frac{p+q-1}{2}} \nu & \text{if } d = \text{odd} \end{cases}$$

$$\tau(\nu) =_{d=\text{odd}} \begin{cases} -(-1)^q \nu & \text{if } p - q \equiv_8 3, 7 \\ +(-1)^q \nu & \text{if } p - q \equiv_8 1, 5 \end{cases} \quad , \quad \nu^2 =_{d=\text{odd}} \begin{cases} -1 & \text{if } p - q \equiv_8 3, 7 \\ +1 & \text{if } p - q \equiv_8 1, 5 \end{cases} \quad .$$

Notice that  $\tau(\nu) = (-1)^q \nu^{-1}$ .

**1.8. The volume grading.** We have:

$$\text{Ad}^{\text{Cl}}(\nu) = \pi^{d-1} = \begin{cases} \text{id}_{\text{Cl}(V, h)} & \text{if } d = \text{odd} \\ \pi & \text{if } d = \text{even} \end{cases} \quad . \quad (1.3)$$

In particular,  $\pi$  is an inner automorphism of  $\text{Cl}(V, h)$  when  $d$  is even. The involutive  $\mathbb{R}$ -algebra automorphism  $\text{Ad}^{\text{Cl}}(\nu) \in \text{Aut}_{\text{Alg}}(\text{Cl}(V, h))$  induces a  $\mathbb{Z}_2$ -grading called the *volume grading* of  $\text{Cl}(V, h)$ :

$$\begin{aligned} \text{Cl}^0(V, h) &\stackrel{\text{def.}}{=} \{x \in \text{Cl}(V, h) | \text{Ad}^{\text{Cl}}(\nu)(x) = +x\} , \\ \text{Cl}^1(V, h) &\stackrel{\text{def.}}{=} \{x \in \text{Cl}(V, h) | \text{Ad}^{\text{Cl}}(\nu)(x) = -x\} . \end{aligned} \quad (1.4)$$

Relation (1.3) implies:



1. For odd  $d$ , we have  $\text{Cl}^0(V, h) = \text{Cl}(V, h)$  and  $\text{Cl}^1(V, h) = 0$  and hence the volume grading is concentrated in degree zero.
2. For even  $d$ , we have  $\text{Cl}^0(V, h) = \text{Cl}_+(V, h)$  and  $\text{Cl}^1(V, h) = \text{Cl}_-(V, h)$  and hence the volume grading coincides with the canonical  $\mathbb{Z}_2$ -grading.

**1.9. The Clifford center and pseudocenter.**

DEFINITION 1.3. *An element  $x \in \text{Cl}(V, h)$  is called:*

1. Central, *if it commutes with all elements of  $\text{Cl}(V, h)$ :*

$$xy = yx \quad \forall y \in \text{Cl}(V, h)$$

2. Twisted central *if it satisfies the condition:*

$$xy = \pi(y)x \quad \forall y \in \text{Cl}(V, h)$$

3. Pseudocentral, *if it is central or twisted central.*

Let  $Z(V, h)$ ,  $A(V, h)$  and  $T(V, h)$  denote the subspaces of  $\text{Cl}(V, h)$  consisting of all central, twisted central and pseudocentral elements, respectively. Then  $Z(V, h)$  is the center of the Clifford algebra, while  $T(V, h)$  is called its *pseudocenter*. Clearly both  $Z(V, h)$  and  $T(V, h)$  are unital subalgebras of  $\text{Cl}(V, h)$ , while  $A(V, h)$  is only a subspace. Since  $\text{Cl}(V, h)$  is generated by  $V$  (over  $\mathbb{R}$ ), we have:

$$\begin{aligned} Z(V, h) &= \{x \in \text{Cl}(V, h) \mid xv = vx \quad \forall v \in V\} \\ A(V, h) &= \{x \in \text{Cl}(V, h) \mid xv = -vx \quad \forall v \in V\} . \end{aligned}$$

Moreover, it is easy to see that  $Z(V, h) \cap A(V, h) = 0$  and that  $T(V, h)$  has the decomposition:

$$T(V, h) = Z(V, h) \oplus A(V, h) ,$$

which gives a  $\mathbb{Z}_2$ -grading of the algebra  $T(V, h)$  with components  $T^0(V, h) \stackrel{\text{def.}}{=} Z(V, h)$  and  $T^1(V, h) \stackrel{\text{def.}}{=} A(V, h)$ .

PROPOSITION 1.1. *We have:*

$$T(V, h) = \mathbb{R} \oplus \mathbb{R}\nu \simeq_{\text{Alg}} \mathbb{R}[\nu]/(\nu^2 = \sigma_{p,q}) \simeq_{\text{Alg}} \begin{cases} \mathbb{D} & \text{if } p - q \equiv_4 0, 1 \\ \mathbb{C} & \text{if } p - q \equiv_4 2, 3 \end{cases} ,$$

where  $\mathbb{D}$  is the  $\mathbb{R}$ -algebra of hyperbolic (a.k.a. split complex, or double) numbers (see Appendix A) and  $\nu$  corresponds to the hyperbolic unit  $j \in \mathbb{D}$  or to the imaginary unit  $i \in \mathbb{C}$ . Moreover:

1. When  $d$  is even, we have  $\nu \in \text{Cl}_+(V, h)$ ,  $Z(V, h) = \mathbb{R}$  and  $A(V, h) = \mathbb{R}\nu$ .
2. When  $d$  is odd, we have  $\nu \in \text{Cl}_-(V, h)$ ,  $A(V, h) = 0$  and:

$$Z(V, h) = T(V, h) \simeq_{\text{Alg}} \begin{cases} \mathbb{D} & \text{if } p - q \equiv_8 1, 5 \\ \mathbb{C} & \text{if } p - q \equiv_8 3, 7 \end{cases} .$$

*Proof.* The statements regarding  $Z(V, h)$  are well-known (for example, see [20]). For the statements regarding  $A(V, h)$ , distinguish the cases:

1.  $d$  is even. Then clearly  $\nu$  belongs to  $A(V, h)$ . For any  $x \in A(V, h)$ , we thus have  $x\nu \in Z(V, h) = \mathbb{R}$ , which implies  $x \in \mathbb{R}\nu$  since  $\nu^2 = \sigma_{p,q}$ . Thus  $A(V, h) = \mathbb{R}\nu$  in this case.



2.  $d$  is odd. Then  $\nu$  is central in  $\text{Cl}(V, h)$  and belongs to  $\text{Cl}_-(V, h)$ . These two facts imply that any  $x \in A(V, h)$  satisfies both  $x\nu = \nu x$  and  $x\nu = -\nu x$ , which implies  $x\nu = 0$  and hence  $x = 0$  since  $\nu$  is invertible in  $\text{Cl}(V, h)$ . Thus  $A(V, h) = 0$ . The statements regarding  $T(V, h)$  now follow immediately.  $\square$

When  $d$  is odd, the above gives isomorphisms of groups:

$$Z(V, h)^\times \simeq_{d=\text{odd}} \begin{cases} \mathbb{D}^\times \simeq \mathbb{R}_{>0} \times \text{U}(\mathbb{D}) & \text{if } p - q \equiv_8 1, 5 \\ \mathbb{C}^\times \simeq \mathbb{R}_{>0} \times \text{U}(1) & \text{if } p - q \equiv_8 3, 7 \end{cases} . \tag{1.5}$$

REMARK 1.2. Notice that  $T(V, h)$  does *not* coincide with the super-center  $Z_{\text{super}}(V, h)$  of  $\text{Cl}(V, h)$ , when the latter is viewed as a  $\mathbb{Z}_2$ -graded associative algebra. By definition of the super-center, we have:

$$Z_{\text{super}}(V, h) = Z(V, h) \cap \text{Cl}_+(V, h) \oplus A(V, h) \cap \text{Cl}_-(V, h) = \mathbb{R} .$$

In particular  $Z_{\text{super}}(V, h)$  is a sub-superalgebra of  $\text{Cl}(V, h)$ , while  $T(V, h)$  is not.

**1.10. Normal, simple and quaternionic cases.** It is useful to distinguish various cases according to whether  $\text{Cl}(V, h)$  is a simple  $\mathbb{R}$ -algebra and according to the isomorphism type of the centralizer (a.k.a. Schur algebra)  $\mathbb{S}$  of the irreps of  $\text{Cl}(V, h)$ . This gives the following classification controlled by the value of  $p - q \pmod 8$  (see Table 1.1):

- A. Simple cases:
  - 1 The normal simple case, when  $p - q \equiv_8 0, 2$
  - 2 The complex case, when  $p - q \equiv_8 3, 7$
  - 3 The quaternionic simple case, when  $p - q \equiv_8 4, 6$
- B. Non-simple cases:
  - 1 The normal non-simple case, when  $p - q \equiv_8 1$
  - 2 The quaternionic non-simple case, when  $p - q \equiv_8 5$

$p - q \pmod 8$	$\mathbb{S}$	type	simplicity	$Z(V, h)$	$A(V, h)$	$T(V, h)$	$\nu^2$
0, 2	$\mathbb{R}$	normal	simple	$\mathbb{R}$	$\mathbb{R}\nu$	$\mathbb{D}, \mathbb{C}$	1, -1
3, 7	$\mathbb{C}$	complex	simple	$\mathbb{C}$	0	$\mathbb{C}$	-1
4, 6	$\mathbb{H}$	quaternionic	simple	$\mathbb{R}$	$\mathbb{R}\nu$	$\mathbb{D}, \mathbb{C}$	1, -1
1	$\mathbb{R}$	normal	non-simple	$\mathbb{D}$	0	$\mathbb{D}$	+1
5	$\mathbb{H}$	quaternionic	non-simple	$\mathbb{D}$	0	$\mathbb{D}$	+1

TABLE 1.1

*Classification of Clifford algebras. Here,  $\nu$  is the Clifford volume element with respect to an orientation of  $V$  while  $Z(V, h)$  and  $T(V, h)$  are the center and pseudocenter of  $\text{Cl}(V, h)$ . Moreover,  $A(V, h)$  denotes the subspace of twisted-central elements. Finally,  $\mathbb{S} \subset \text{End}_{\mathbb{R}}(S)$  the Schur algebra (centralizer) of any irreducible representation of  $\text{Cl}(V, h)$  (see Section 3).*

**1.11. Clifford norm and twisted Clifford norm.** The *Clifford norm* is the map  $N : \text{Cl}(V, h) \rightarrow \text{Cl}(V, h)$  defined through:

$$N(x) \stackrel{\text{def.}}{=} \tau(x)x .$$

The *twisted Clifford norm* is the map  $\tilde{N} : \text{Cl}(V, h) \rightarrow \text{Cl}(V, h)$  defined through:

$$\tilde{N}(x) \stackrel{\text{def.}}{=} \tilde{\tau}(x)x \ .$$

These maps are  $\mathbb{R}$ -quadratic, in particular we have  $N(\lambda x) = \lambda^2 N(x)$  for any  $x \in \text{Cl}(V, h)$  and a similar relation for  $\tilde{N}$ . We also have:

$$\begin{aligned} N(1) &= \tilde{N}(1) = 1 \\ N(v) &= -\tilde{N}(v) = h(v, v) \text{ for } v \in V \ . \end{aligned}$$

Notice the relation:

$$\tilde{N}|_{\text{Cl}_{\pm}(V, h)} = \pm N|_{\text{Cl}_{\pm}(V, h)} \ .$$

**1.12. The ordinary Clifford group.** The *ordinary Clifford group*<sup>2</sup> is the following subgroup of the group of homogeneous units:

$$G(V, h) \stackrel{\text{def.}}{=} \{a \in \text{Cl}_{\text{hom}}^{\times}(V, h) | \text{Ad}^{\text{Cl}}(a)(V) = V\} \subset \text{Cl}_{\text{hom}}^{\times}(V, h) \ .$$

It admits the  $\mathbb{Z}_2$ -grading inherited from  $\text{Cl}_{\text{hom}}^{\times}(V, h)$ , which has components:

$$G_{\pm}(V, h) \stackrel{\text{def.}}{=} \{a \in \text{Cl}_{\pm}^{\times}(V, h) | \text{Ad}^{\text{Cl}}(a)(V) = V\} \subset \text{Cl}_{\pm}^{\times}(V, h) \ .$$

This  $\mathbb{Z}_2$ -grading is induced by the *signature morphism*  $\tilde{\delta} : G(V, h) \rightarrow \mathbb{G}_2$ :

$$\tilde{\delta}(a) = \begin{cases} +1 & \text{if } a \in G_+(V, h) \\ -1 & \text{if } a \in G_-(V, h) \end{cases} \ . \tag{1.6}$$

The subgroup  $G_+(V, h)$  is called the *special Clifford group*. Since  $G(V, h)$  is generated by non-degenerate vectors, the restriction of the Clifford norm to the ordinary Clifford group takes values in  $\mathbb{R}^{\times}$  and hence gives a group morphism  $N_G \stackrel{\text{def.}}{=} N|_{G(V, h)} : G(V, h) \rightarrow \mathbb{R}^{\times}$ . Composing this with the absolute value epimorphism  $\mathbb{R}^{\times} \xrightarrow{|\cdot|} \mathbb{R}_{>0}$  gives a surjective group morphism  $|N_G| \stackrel{\text{def.}}{=} |\cdot| \circ N_G : G(V, h) \rightarrow \mathbb{R}_{>0}$  called *absolute Clifford norm*. We also have  $\tilde{N}(G(V, h)) \subset \mathbb{R}^{\times}$  and the twisted Clifford norm gives a group morphism  $\tilde{N}_G \stackrel{\text{def.}}{=} \tilde{N}|_{G(V, h)} : G(V, h) \rightarrow \mathbb{R}^{\times}$ . We have  $\tilde{N}|_{G_{\pm}(V, h)} = \pm N|_{G_{\pm}(V, h)}$  and hence  $|\tilde{N}_G| = |N_G|$ , where  $|\tilde{N}_G| \stackrel{\text{def.}}{=} |\cdot| \circ \tilde{N}_G$ .

**1.13. Vector representations of the ordinary Clifford group.**

DEFINITION 1.4. *The untwisted vector representation of  $G(V, h)$  is the group morphism  $\text{Ad}_0^{\text{Cl}} : G(V, h) \rightarrow \text{O}(V, h)$  given by:*

$$\text{Ad}_0^{\text{Cl}}(a) \stackrel{\text{def.}}{=} \text{Ad}^{\text{Cl}}(a)|_V \quad \forall a \in G(V, h) \ .$$

*The twisted vector representation of  $G(V, h)$  is the group morphism  $\widetilde{\text{Ad}}_0^{\text{Cl}} : G(V, h) \rightarrow \text{O}(V, h)$  given by:*

$$\widetilde{\text{Ad}}_0^{\text{Cl}}(a) \stackrel{\text{def.}}{=} \widetilde{\text{Ad}}^{\text{Cl}}(a)|_V \quad \forall a \in G(V, h) \ .$$

---

<sup>2</sup>Sometimes called the *twisted Clifford group* (“groupe de Clifford tordu” in reference [20]).

We have  $\text{Ad}_0^{\text{Cl}}(v) = -R_v$  and  $\widetilde{\text{Ad}}_0(v) = +R_v$ , for any non-degenerate vector  $v \in V$ . This implies the following well-known results (see, for example, [20]):

PROPOSITION 1.2. *The ordinary Clifford group  $G(V, h)$  is generated by non-degenerate vectors  $v \in V$  and we have:*

$$\det \circ \widetilde{\text{Ad}}_0^{\text{Cl}} = \tilde{\delta} \quad . \tag{1.7}$$

PROPOSITION 1.3. *The twisted vector representation of the ordinary Clifford group satisfies  $\widetilde{\text{Ad}}_0^{\text{Cl}}(G(V, h)) = O(V, h)$  and gives a short exact sequence:*

$$1 \longrightarrow \mathbb{R}^\times \hookrightarrow G(V, h) \xrightarrow{\widetilde{\text{Ad}}_0^{\text{Cl}}} O(V, h) \longrightarrow 1 \quad , \tag{1.8}$$

which restricts to a short exact sequence:

$$1 \longrightarrow \mathbb{R}^\times \hookrightarrow G_+(V, h) \xrightarrow{\widetilde{\text{Ad}}_0^{\text{Cl}}|_{G_+(V, h)} = \text{Ad}_0^{\text{Cl}}|_{G_+(V, h)}} \text{SO}(V, h) \longrightarrow 1 \quad . \tag{1.9}$$

Let:

$$\hat{O}(V, h) \stackrel{\text{def.}}{=} \begin{cases} O(V, h) & \text{if } d = \text{even} \\ \text{SO}(V, h) & \text{if } d = \text{odd} \end{cases} \quad . \tag{1.10}$$

PROPOSITION 1.4. *The untwisted vector representation of the ordinary Clifford group satisfies  $\text{Ad}_0^{\text{Cl}}(G(V, h)) = \hat{O}(V, h)$  and gives a short exact sequence:*

$$1 \longrightarrow \mathbb{R}^\times \hookrightarrow G(V, h) \xrightarrow{\text{Ad}_0^{\text{Cl}}} \hat{O}(V, h) \longrightarrow 1 \quad , \tag{1.11}$$

which restricts to (1.9).

Consider the morphism of groups  $f : O(V, h) \rightarrow O(V, h)$  given by  $f(R) \stackrel{\text{def.}}{=} (\det R)R$ .

PROPOSITION 1.5. *We have  $\text{Ad}_0^{\text{Cl}} = f \circ \widetilde{\text{Ad}}_0^{\text{Cl}}$  on  $G(V, h)$ . Moreover:*

1. *When  $d$  is even,  $f$  is an automorphism of  $O(V, h)$ .*
2. *When  $d$  is odd,  $f$  induces an isomorphism  $O(V, h)/\{-\text{id}_V, \text{id}_V\} \simeq \text{SO}(V, h)$ . In this case, the map  $\varphi \stackrel{\text{def.}}{=}} f \times \det : O(V, h) \xrightarrow{\sim} \text{SO}(V, h) \times \mathbb{G}_2$  given by  $\varphi(R) = (f(R), \det R)$  is an isomorphism of groups and we have:*

$$\text{Ad}_0^{\text{Cl}} \times \tilde{\delta} = \varphi \circ \widetilde{\text{Ad}}_0^{\text{Cl}} \quad .$$

*Proof.* The relation  $\det f(R) = (\det R)^{d+1}$  gives:

$$f(O(V, h)) = \begin{cases} O(V, h) & \text{if } d = \text{even} \\ \text{SO}(V, h) & \text{if } d = \text{odd} \end{cases} \quad , \quad \ker f = \begin{cases} \{\text{id}_V\} & \text{if } d = \text{even} \\ \{-\text{id}_V, \text{id}_V\} & \text{if } d = \text{odd} \end{cases} \quad .$$

The remaining statements are obvious.  $\square$

REMARK 1.3. In general, the group  $O(p, q)$  has non-trivial outer automorphisms. Determining the full outer automorphism group of  $O(p, q)$  for general  $p, q$  turns out to be a surprisingly subtle problem.

REMARK 1.4. It is well-known that the ordinary Clifford group can also be described as:

$$G(V, h) = \{a \in \text{Cl}(V, h)^\times \mid \widetilde{\text{Ad}}^{\text{Cl}}(a)(V) = V\} \quad .$$

**1.14. The pin and spin groups and their vector representations.**

DEFINITION 1.5. *The pin group  $\text{Pin}(V, h)$  is the subgroup of  $G(V, h)$  generated by the vectors  $v \in V$  satisfying  $|h(v, v)| = 1$ . The spin group is the subgroup  $\text{Spin}(V, h) \stackrel{\text{def.}}{=} \text{Pin}(V, h) \cap \text{Cl}_+(V, h)$ .*

The decomposition  $\text{Pin}(V, h) = \text{Pin}_+(V, h) \sqcup \text{Pin}_-(V, h)$ , where  $\text{Pin}_+(V, h) \stackrel{\text{def.}}{=} \text{Spin}(V, h)$  and  $\text{Pin}_-(V, h) \stackrel{\text{def.}}{=} \text{Pin}(V, h) \cap \text{Cl}_-(V, h)$  gives a  $\mathbb{Z}_2$ -grading of the pin group. We have  $\ker |N_G| = \text{Pin}(V, h)$  and an exact sequence:

$$1 \longrightarrow \text{Pin}(V, h) \hookrightarrow G(V, h) \xrightarrow{|N_G|} \mathbb{R}_{>0} \longrightarrow 1 \quad , \tag{1.12}$$

which restricts to an exact sequence:

$$1 \longrightarrow \text{Spin}(V, h) \hookrightarrow G_+(V, h) \xrightarrow{|N_{G_+}|} \mathbb{R}_{>0} \longrightarrow 1 \quad , \tag{1.13}$$

where  $N_{G_+} \stackrel{\text{def.}}{=} N|_{G_+(V, h)}$ . In particular,  $G(V, h) \simeq \mathbb{R}_{>0} \times \text{Pin}(V, h)$  is homotopy-equivalent with  $\text{Pin}(V, h)$  while  $G_+(V, h) \simeq \mathbb{R}_{>0} \times \text{Spin}(V, h)$  is homotopy-equivalent with  $\text{Spin}(V, h)$ .

DEFINITION 1.6. *The untwisted vector representation of  $\text{Pin}(V, h)$  is the restriction of  $\text{Ad}_0^{\text{Cl}}$  to  $\text{Pin}(V, h)$ . The twisted vector representation of  $\text{Pin}(V, h)$  is the restriction of  $\widetilde{\text{Ad}}_0^{\text{Cl}}$  to  $\text{Pin}(V, h)$ . The vector representation of  $\text{Spin}(V, h)$  is the common restriction of  $\text{Ad}_0^{\text{Cl}}$  or  $\widetilde{\text{Ad}}_0^{\text{Cl}}$  to  $\text{Spin}(V, h)$ .*

The twisted vector representation gives an exact sequence:

$$1 \longrightarrow \mathbb{G}_2 \hookrightarrow \text{Pin}(V, h) \xrightarrow{\widetilde{\text{Ad}}_0^{\text{Cl}}} \text{O}(V, h) \longrightarrow 1 \tag{1.14}$$

which restricts to an exact sequence:

$$1 \longrightarrow \mathbb{G}_2 \hookrightarrow \text{Spin}(V, h) \xrightarrow{\widetilde{\text{Ad}}_0^{\text{Cl}}|_{\text{Spin}(V, h)} = \text{Ad}_0^{\text{Cl}}|_{\text{Spin}(V, h)}} \text{SO}(V, h) \longrightarrow 1 \quad . \tag{1.15}$$

The situation is summarized in the following commutative diagram with exact rows and columns, where  $\text{sq}(x) = x^2$  for  $x \in \mathbb{R}^\times$ :

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & (1.16) \\
 & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & \mathbb{G}_2 & \longrightarrow & \text{Pin}(V, h) & \xrightarrow{\widetilde{\text{Ad}}_0^{\text{Cl}}} & \text{O}(V, h) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \parallel \\
 1 & \longrightarrow & \mathbb{R}^\times & \longrightarrow & G(V, h) & \xrightarrow{\widetilde{\text{Ad}}_0^{\text{Cl}}} & \text{O}(V, h) \longrightarrow 1 \\
 & & \downarrow \text{sq} & & \downarrow |N_G| & & \\
 & & \mathbb{R}_{>0} & \xlongequal{\quad} & \mathbb{R}_{>0} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 
 \end{array}$$

The untwisted vector representation gives an exact sequence:

$$1 \longrightarrow \mathbb{G}_2 \hookrightarrow \text{Pin}(V, h) \xrightarrow{\text{Ad}_0^{\text{Cl}}} \hat{\text{O}}(V, h) \longrightarrow 1 \tag{1.17}$$

which restricts to (1.15) and we have a commutative diagram similar to (1.16).

**1.15. Relation between the twisted and untwisted vector representations of  $\text{G}(V, h)$  and  $\text{Pin}(V, h)$  for even  $d$ .** The following result relates the groups  $\text{G}(V, h)$  and  $\text{G}(V, -\sigma_{p,q}h)$  when  $d$  is even (cf. [8]):

**PROPOSITION 1.6.** *When  $d$  is even, there exists an isomorphism of groups  $\varphi : \text{G}(V, h) \xrightarrow{\sim} \text{G}(V, -\sigma_{p,q}h)$  such that  $\text{Ad}_0^{\text{Cl}} \circ \varphi = \widetilde{\text{Ad}}_0^{\text{Cl}}$ . This restricts to an isomorphism from  $\text{Pin}(V, h)$  to  $\text{Pin}(V, -\sigma_{p,q}h)$  having the same property. Thus:*

1. *When  $p - q \equiv 0, 4$ , there exists an isomorphism of groups  $\varphi : \text{G}(V, h) \xrightarrow{\sim} \text{G}(V, -h)$  such that  $\text{Ad}_0^{\text{Cl}} \circ \varphi = \widetilde{\text{Ad}}_0^{\text{Cl}}$ . This restricts to an isomorphism from  $\text{Pin}(V, h)$  to  $\text{Pin}(V, -h)$  having the same property.*
2. *When  $p - q \equiv 2, 6$ , there exists a group automorphism  $\varphi : \text{G}(V, h) \xrightarrow{\sim} \text{G}(V, h)$  such that  $\text{Ad}_0^{\text{Cl}} \circ \varphi = \widetilde{\text{Ad}}_0^{\text{Cl}}$ . This restricts to an automorphism of  $\text{Pin}(V, h)$  having the same property.*

*Proof.* Since  $d$  is even, we have:

$$\nu^2 = \sigma_{p,q} = (-1)^{q+\frac{d}{2}} = (-1)^{\frac{p-q}{2}} = \begin{cases} +1 & \text{if } p - q \equiv 0, 4 \\ -1 & \text{if } p - q \equiv 2, 6 \end{cases} .$$

Moreover,  $\nu$  anticommutes with  $v$  for all  $v \in V$  and hence  $(\nu v)^2 = -\sigma_{p,q}v^2$ . Therefore, the map  $V \ni v \rightarrow \nu v \in V' \stackrel{\text{def.}}{=} \nu V \subset \text{Cl}(V, h)$  extends (upon identifying the vector space  $V'$  with  $V$ ) to a unital isomorphism of  $\mathbb{R}$ -algebras  $\varphi_0 : \text{Cl}(V, h) \xrightarrow{\sim} \text{Cl}(V, -\sigma_{p,q}h)$ , which restricts to an isomorphism  $\varphi : \text{G}(V, h) \xrightarrow{\sim} \text{G}(V, -\sigma_{p,q}h)$ . For any non-degenerate vectors  $v_1, \dots, v_k \in V$ , we have:

$$\begin{aligned} \text{Ad}_0^{\text{Cl}}(\varphi(v_1 \dots v_k)) &= \text{Ad}_0^{\text{Cl}}(\nu v_1) \circ \dots \circ \text{Ad}_0^{\text{Cl}}(\nu v_k) = \\ \text{Ad}_0^{\text{Cl}}(\nu) \circ \text{Ad}_0^{\text{Cl}}(v_1) \circ \dots \circ \text{Ad}_0^{\text{Cl}}(\nu) \circ \text{Ad}_0^{\text{Cl}}(v_k) &= \widetilde{\text{Ad}}_0^{\text{Cl}}(v_1 \dots v_k) \quad , \end{aligned}$$

where we noticed that  $\text{Ad}_0^{\text{Cl}}(\nu) = -\text{id}_V$  since  $d$  is even. Thus  $\text{Ad}_0^{\text{Cl}}(\varphi(g)) = \widetilde{\text{Ad}}_0^{\text{Cl}}(g)$  for all  $g \in \text{G}(V, h)$ .  $\square$

**1.16. Connected components of the pin and pseudo-orthogonal groups.**

When  $pq \neq 0$ , the group  $\text{Spin}(V, h)$  has two connected components given by:

$$\text{Spin}^\pm(V, h) \stackrel{\text{def.}}{=} \{a \in \text{Spin}(V, h) | N(a) = \pm 1\} \quad ,$$

where  $\text{Spin}^+(V, h)$  is the connected component of the identity. The decomposition  $\text{Spin}(V, h) = \text{Spin}^+(V, h) \sqcup \text{Spin}^-(V, h)$  is the  $\mathbb{Z}_2$ -grading induced by the group morphism  $N|_{\text{Spin}(V, h)} : \text{Spin}(V, h) \rightarrow \mathbb{G}_2$ . Accordingly,  $\text{Pin}(V, h)$  has four connected components given by:

$$\text{Pin}_\epsilon^\eta(V, h) = \{a \in \text{Pin}(V, h) | \tilde{\delta}(a) = \epsilon, N(a) = \eta\} \quad , \quad (\epsilon, \eta \in \{-1, 1\}) \quad ,$$

where  $\text{Pin}_+^\pm(V, h) = \text{Spin}^\pm(V, h)$  is the connected component of the identity. The decomposition  $\text{Pin}(V, h) = \sqcup_{\epsilon, \eta \in \{-1, +1\}} \text{Pin}_\epsilon^\eta(V, h)$  is the  $D_4$ -grading induced by the

group morphism  $(\tilde{\delta}|_{\text{Pin}(V,h)}) \times (N|_{\text{Pin}(V,h)}) \rightarrow \mathbb{G}_2 \times \mathbb{G}_2$ . In particular, the component group  $\pi_0(\text{Pin}(V,h)) = \text{Pin}(V,h)/\text{Spin}(V,h)$  is isomorphic with  $D_4$ . Since  $N(-1) = N(1) = 1$ , the Clifford norm descends through the 2-fold covering map of (1.14) to a group morphism  $N_0 : \text{O}(V,h) \rightarrow \mathbb{G}_2$ . Similarly, we have  $\tilde{\delta}(-1) = \tilde{\delta}(1) = 1$  and  $\tilde{\delta}$  descends to the determinant morphism  $\det : \text{O}(V,h) \rightarrow \mathbb{G}_2$ . These two morphisms give a  $D_4$ -grading  $\text{O}(V,h) = \sqcup_{\epsilon,\eta \in \{-1,+1\}} \text{O}_\epsilon^\eta(V,h)$  which coincides with the decomposition of  $\text{O}(V,h)$  into connected components:

$$\text{O}_\epsilon^\eta(V,h) = \{a \in \text{O}(V,h) \mid \det a = \epsilon, N_0(a) = \eta\} .$$

The group  $\text{O}_+^+(V,h)$  is the connected component of the identity and the component group  $\pi_0(\text{O}(V,h)) = \text{O}(V,h)/\text{SO}(V,h)$  is isomorphic with  $D_4$ . Accordingly, the group  $\text{SO}(V,h) = \{a \in \text{O}(V,h) \mid \det a = +1\}$  has two connected components distinguished by the morphism  $N_0$ , which gives a  $\mathbb{Z}_2$ -grading of  $\text{SO}(V,h)$ :

$$\text{SO}^\pm(V,h) = \{a \in \text{SO}(V,h) \mid N_0(a) = \pm 1\} ,$$

where  $\text{SO}^+(V,h) = \text{O}_+^+(V,h)$  is the connected component of the identity. The groups  $\text{O}(V,h)$ ,  $\text{SO}(V,h)$  and  $\text{SO}^+(V,h)$  are homotopy equivalent with their maximal compact forms  $\text{O}(p) \times \text{O}(q)$ ,  $\text{S}[\text{O}(p) \times \text{O}(q)]$  and  $\text{SO}(p) \times \text{SO}(q)$  respectively. When  $p = 0$  and  $q \neq 0$ , we have  $\text{O}(V,h) \simeq \text{O}(q)$  while for  $p \neq 0$  and  $q = 0$  we have  $\text{O}(V,h) \simeq \text{O}(p)$ , hence in these cases  $\text{SO}(V,h)$  is connected and  $\text{O}(V,h)$  has two connected components distinguished by the determinant; similar remarks apply to  $\text{Spin}(V,h)$  and  $\text{Pin}(V,h)$ .

**1.17. Presentation of the pin group in terms of the spin group.** For any  $d$ , we have  $\nu \in \text{Pin}(V,h)$  and:

$$\text{Ad}_0^{\text{Cl}}(\nu) = (-1)^{d-1} \text{id}_V, \quad \widetilde{\text{Ad}}_0^{\text{Cl}}(\nu) = -\text{id}_V .$$

Thus:

$$\widetilde{\text{Ad}}_0^{\text{Cl}}(\nu a) = \widetilde{\text{Ad}}_0^{\text{Cl}}(\nu) \widetilde{\text{Ad}}_0^{\text{Cl}}(a) = -\widetilde{\text{Ad}}_0^{\text{Cl}}(a) \text{ for } a \in \text{Pin}(V,h) .$$

When  $d$  is even, we have  $\nu \in \text{Spin}(V,h)$  and  $\text{Ad}_0^{\text{Cl}}(\nu) = -\text{id}_V \in \text{SO}(V,h)$ . When  $d$  is odd, we have  $\nu \in \text{Pin}_-(V,h)$ ,  $\text{Ad}_0^{\text{Cl}}(\nu) = \text{id}_V \in \text{SO}(V,h)$ ,  $\widetilde{\text{Ad}}_0^{\text{Cl}}(\nu) = -\text{id}_V \in \text{O}_-(V,h)$  and every element of  $\text{Pin}_-(V,h)$  can be written as  $\nu a$  for a uniquely-determined  $a \in \text{Spin}(V,h)$ . Thus:

$$\text{Pin}_-(V,h) =_{d=\text{odd}} \text{Spin}(V,h)\nu$$

and:

$$\text{Pin}(V,h) =_{d=\text{odd}} \text{Spin}(V,h)\langle \nu \rangle \simeq \begin{cases} \text{Spin}(V,h) \times \mathbb{G}_2 & \text{if } p - q \equiv_8 1, 5 \\ \text{Spin}(V,h) \cdot \mathbb{G}_4 & \text{if } p - q \equiv_8 3, 7 \end{cases} . \quad (1.18)$$

In this presentation, the twisted adjoint representation of  $\text{Pin}(V,h)$  for odd  $d$  is given as follows:

1. If  $p - q \equiv_8 1, 5$ , then:

$$\widetilde{\text{Ad}}_0(a,g) = g \text{Ad}_0(a) \quad \forall a \in \text{Spin}(V,h) \text{ and } g \in \mathbb{G}_2$$

2. If  $p - q \equiv_8 3, 7$ , then:

$$\widetilde{\text{Ad}}_0([a,g]) = g^2 \text{Ad}_0(a) \quad \forall a \in \text{Spin}(V,h) \text{ and } g \in \mathbb{G}_4$$

Notice that  $\text{Ad}_0(-a) = \text{Ad}_0(a)$  and  $\widetilde{\text{Ad}}_0(-a) = \widetilde{\text{Ad}}_0(a)$  for all  $a \in \text{Pin}(V, h)$ .

REMARK 1.5. Notice the equivalences:

$$p - q \equiv_8 3 \iff q - p \equiv_8 5 \quad , \quad p - q \equiv_8 7 \iff q - p \equiv_8 1 \quad , \quad (1.19)$$

which interchange the complex case in signature  $(p, q)$  with the non-simple case in signature  $(q, p)$ . In fact, the interchange  $p \leftrightarrow q$  corresponds to  $h \leftrightarrow -h$ . Since  $d$  (which is assumed odd) is invariant under this transformation, the volume forms  $\nu_h = \nu_{p,q}$  and  $\nu_{-h} = \nu_{q,p}$  (taken with respect to some fixed orientation of  $V$ ) are central in  $\text{Cl}(V, h)$  and  $\text{Cl}(V, -h)$ , respectively and we have:

$$\nu_{-h}^2 = -\nu_h^2 \quad .$$

Since  $\text{Spin}(V, -h) \simeq \text{Spin}(V, h)$ , we have:

$$\text{Pin}(V, -h) \simeq \begin{cases} \text{Spin}(V, h) \cdot \mathbb{G}_4 & \text{if } p - q \equiv_8 1, 5 \\ \text{Spin}(V, h) \times \mathbb{G}_2 & \text{if } p - q \equiv_8 3, 7 \end{cases} \quad .$$

The group  $\text{Pin}(V, -h)$  is sometimes denoted  $\text{Pin}^-(V, h)$  and is used in the theory of  $\text{Pin}^-$  structures [6].

**1.18. The extended Clifford group.**

DEFINITION 1.7. *The extended Clifford group<sup>3</sup>  $G^e(V, h)$  is the subgroup of  $\text{Cl}(V, h)^\times$  defined through:*

$$G^e(V, h) \stackrel{\text{def.}}{=} \{x \in \text{Cl}(V, h)^\times \mid \text{Ad}^{\text{Cl}}(x)(V) = V\} \quad .$$

The ordinary Clifford group  $G(V, h)$  coincides with the subgroup formed by those elements of  $G^e(V, h)$  which are homogeneous with respect to the canonical  $\mathbb{Z}_2$ -grading of  $\text{Cl}(V, h)$ . We have:

$$G^e(V, h) \cap \text{Cl}_\pm(V, h) = G_\pm(V, h) \quad , \quad G^e(V, h) \cap \text{Cl}_{\text{hom}}^\times(V, h) = G(V, h) \quad .$$

PROPOSITION 1.7. *We have:*

$$\begin{aligned} G^e(V, h) &= Z(V, h)^\times G(V, h) \\ Z(G^e(V, h)) &= Z(V, h)^\times \\ Z(G(V, h)) &= G(V, h) \cap Z(V, h)^\times \end{aligned}$$

and  $G^e(V, h) \simeq [G(V, h) \times Z(V, h)^\times] / Z(G(V, h))$ . *Namely:*

A. *When  $d$  is even, we have  $G^e(V, h) = G(V, h)$ .*

B. *When  $d$  is odd, we have:*

$$Z(G(V, h)) = \mathbb{R}^\times \sqcup \mathbb{R}^\times \nu \simeq_{\text{Gp}} \mathbb{R}_{>0} \times \{1, \nu, -1, -\nu\} \simeq_{\text{Gp}} \begin{cases} \mathbb{R}_{>0} \times D_4 & \text{if } p - q \equiv_8 1, 5 \\ \mathbb{R}_{>0} \times \mathbb{G}_4 & \text{if } p - q \equiv_8 3, 7 \end{cases} \quad .$$

and:

$$G^e(V, h) \simeq_{\text{Gp}} \begin{cases} \mathbb{R}_{>0} \times G(V, h) & \text{if } p - q \equiv_8 1, 5 \\ [\text{U}(1) \times G(V, h)] / \mathbb{G}_4 & \text{if } p - q \equiv_8 3, 7 \end{cases} \quad .$$

<sup>3</sup>Sometimes called the *untwisted* Clifford group or simply the ‘‘Clifford group’’ in older literature such as [20].



Furthermore:

$$Z(V, h)^\times / Z(G(V, h)) \simeq_{\mathbb{G}_p} \begin{cases} \mathbb{U}(\mathbb{D})/D_4 \simeq_{\mathbb{G}_p} \mathbb{R}_{>0} & \text{if } p - q \equiv_8 1, 5 \\ \mathbb{U}(1)/\mathbb{G}_4 \simeq_{\mathbb{G}_p} \mathbb{U}(1) & \text{if } p - q \equiv_8 3, 7 \end{cases}$$

and there exists a short exact sequence:

$$1 \longrightarrow G(V, h) \hookrightarrow G^e(V, h) \longrightarrow Z(V, h)^\times / Z(G(V, h)) \longrightarrow 1 \quad (1.20)$$

*Proof.* The extended Clifford group is generated by the Clifford group and by the elements of  $Z(V, h)^\times$  (see [20]), thus  $G^e(V, h) = Z(V, h)^\times G(V, h)$ . Let  $e_1 \dots e_d$  be an orthonormal basis of  $(V, h)$ . Since  $e_i \in G(V, h) \subset G^e(V, h)$ , any  $a \in Z(G^e(V, h))$  satisfies  $ae_i = e_i a$  for all  $i = 1 \dots d$ , which implies  $av = va$  for all  $v \in V$ . Thus  $a \in Z(V, h)^\times$ . This shows that  $Z(G^e(V, h)) \subset Z(V, h)^\times$  and also that  $Z(G(V, h)) \subset Z(V, h)^\times$ . Since the inverse to the first inclusion is obvious, we conclude that  $Z(G^e(V, h)) = Z(V, h)^\times$ . Since  $Z(G(V, h)) \subset Z(V, h)^\times$ , we have  $Z(G(V, h)) \subset G(V, h) \cap Z(V, h)^\times$ . The inverse of this inclusion is obvious, so we have  $Z(G(V, h)) = G(V, h) \cap Z(V, h)^\times$ . The surjective morphism of groups given by  $G(V, h) \times Z(V, h)^\times \ni (a, \alpha) \rightarrow \alpha a \in G^e(V, h)$  has kernel equal to  $\{(\alpha^{-1}, \alpha) | \alpha \in Z(G(V, h))\} \simeq Z(G(V, h))$ , which shows that  $G^e(V, h) \simeq [G(V, h) \times Z(V, h)^\times] / Z(G(V, h))$ . Proposition 1.1 implies statements A. and B., where in the case  $p - q \equiv_8 1, 5$  we used the isomorphisms  $\mathbb{D}^\times \simeq \mathbb{R}_{>0} \times \mathbb{U}(\mathbb{D})$  and  $\mathbb{U}(\mathbb{D}) \simeq \mathbb{U}^{++}(\mathbb{D}) \times D_4 \simeq \mathbb{R}_{>0} \times D_4$  (see Appendix A).  $\square$

PROPOSITION 1.8. *The following statements hold:*

- A. *When  $d$  is even, we have  $G^e(V, h) = G(V, h) \simeq_{\mathbb{G}_p} \mathbb{R}_{>0} \times \text{Pin}(V, h)$ .*
- B. *When  $d$  is odd, we have:*

$$G^e(V, h) \simeq_{\mathbb{G}_p} Z(V, h)^\times \text{Pin}(V, h) \simeq_{\mathbb{G}_p} \begin{cases} [\mathbb{D}^\times \times \text{Pin}(V, h)] / D_4 & \text{if } p - q \equiv_8 1, 5 \\ [\mathbb{C}^\times \times \text{Pin}(V, h)] / \mathbb{G}_4 & \text{if } p - q \equiv_8 3, 7 \end{cases}$$

and:

$$G^e(V, h) \simeq_{\mathbb{G}_p} Z(V, h)^\times \text{Spin}(V, h) \simeq_{\mathbb{G}_p} \begin{cases} \mathbb{D}^\times \cdot \text{Spin}(V, h) & \text{if } p - q \equiv_8 1, 5 \\ \mathbb{C}^\times \cdot \text{Spin}(V, h) & \text{if } p - q \equiv_8 3, 7 \end{cases}$$

*Proof.* Follows from relations (1.18) and Proposition 1.7.  $\square$

**1.19. The vector representation of the extended Clifford group.**

DEFINITION 1.8. *The vector representation of  $G^e(V, h)$  is the group morphism  $\text{Ad}_0^e : G^e(V, h) \rightarrow \hat{\text{O}}(V, h)$  given by:*

$$\text{Ad}_0^e(a) \stackrel{\text{def.}}{=} \text{Ad}^{\text{Cl}}(a)|_V \quad (a \in G^e(V, h)) \quad .$$

*Its restriction to  $G(V, h)$  coincides with the untwisted vector representation of  $G(V, h)$ .*

When  $d$  is even, the vector representation of the extended Clifford group gives the exact sequence (1.11):

$$1 \longrightarrow Z(V, h)^\times = \mathbb{R}^\times \hookrightarrow G^e(V, h) = G(V, h) \xrightarrow{\text{Ad}_0^e = \text{Ad}_0^{\text{Cl}}} \text{O}(V, h) \longrightarrow 1$$

while when  $d$  is odd it gives an exact sequence (cf. [20, Proposition (1.1.8)]):

$$1 \longrightarrow Z(V, h)^\times \hookrightarrow G^e(V, h) \xrightarrow{\text{Ad}_0^e} \text{SO}(V, h) \longrightarrow 1 \quad ,$$

where  $Z(V, h)^\times$  was given in (1.5).

**1.20. Volume grading of the extended Clifford group.** The volume grading of  $\text{Cl}(V, h)$  induces a grading of  $G^e(V, h)$ . From the properties of the volume grading of  $\text{Cl}(V, h)$  we obtain the following proposition.

PROPOSITION 1.9. *The following statements hold:*

- A. *When  $d$  is even, the volume grading of  $G^e(V, h)$  coincides with the canonical  $\mathbb{Z}_2$ -grading of  $G(V, h)$ .*
- B. *When  $d$  is odd, the volume grading of  $G^e(V, h)$  is concentrated in degree zero:*

$$G^e(V, h)^0 = G^e(V, h) \quad , \quad G^e(V, h)^1 = \emptyset \quad .$$

**1.21. The improved reversion and improved Clifford norm.** Since  $\tau(Z(V, h)) \subset Z(V, h)$ , we have  $N(Z(V, h)) \subset Z(V, h)$ , which implies  $N(G^e(V, h)) \subset Z(V, h)^\times$ . Since  $Z(V, h)^\times$  is an Abelian group, it follows that the restriction of  $N$  gives a group morphism from  $G^e(V, h)$  to  $Z(V, h)^\times$ . In general,  $N(G^e(V, h))$  is larger than  $\mathbb{R}^\times$ . Let:

$$\epsilon_d \stackrel{\text{def.}}{=} -(-1)^{\lfloor \frac{d}{2} \rfloor} = \begin{cases} -1 & \text{if } d \equiv_4 0, 1 \\ +1 & \text{if } d \equiv_4 2, 3 \end{cases} \quad . \tag{1.21}$$

DEFINITION 1.9. *The improved reversion is the unique unital anti-automorphism  $\tau_e$  of  $\text{Cl}(V, h)$  which satisfies  $\tau_e(v) = \epsilon_d v$  for all  $v \in V$ , namely:*

$$\tau_e(x) \stackrel{\text{def.}}{=} \tau \circ \pi^{\frac{1-\epsilon_d}{2}} = \begin{cases} \tilde{\tau} = \tau \circ \pi & \text{if } d \equiv_4 0, 1 \\ \tau & \text{if } d \equiv_4 2, 3 \end{cases} \quad .$$

With this definition, we have:

$$\tau_e(\nu) = \begin{cases} -\nu & \text{if } d \equiv_4 1, 2, 3 \\ +\nu & \text{if } d \equiv_4 0 \end{cases} \quad .$$

In particular,  $\tau_e(\nu) = -\nu$  when  $d$  is odd ( $d \equiv_4 1, 3$ ). Thus  $\tau_e$  acts as conjugation of  $Z(V, h) \simeq \mathbb{C}$  in the complex case  $p - q \equiv_8 3, 7$  and as conjugation of  $Z(V, h) \simeq \mathbb{D}$  in the normal non-simple and quaternionic non-simple cases  $p - q \equiv_8 1, 5$ , i.e. in all cases when  $Z(V, h)$  is not reduced to  $\mathbb{R}$ .

DEFINITION 1.10. *The improved Clifford norm is the map  $N_e : \text{Cl}(V, h) \rightarrow \text{Cl}(V, h)$  defined through:*

$$N_e(x) \stackrel{\text{def.}}{=} \tau_e(x)x = \begin{cases} \tilde{N}(x) & \text{if } d \equiv_4 0, 1 \\ N(x) & \text{if } d \equiv_4 2, 3 \end{cases} \quad (x \in \text{Cl}(V, h)) \quad . \tag{1.22}$$

Notice that  $N_e(x) = x^2$  for all  $x \in \mathbb{R}$ .

PROPOSITION 1.10. *The improved Clifford norm satisfies  $N_e(Z(V, h)) \subset \mathbb{R}$  and*

$$N_e|_{Z(V, h)} = \text{M} \quad ,$$

namely:

- 1. *When  $d$  is even,  $N_e|_{Z(V, h)}$  coincides with the squared absolute value (and hence with the squaring function) on  $Z(V, h) = \mathbb{R}$*

- 2. When  $p-q \equiv_8 3, 7$ ,  $N_e|_{Z(V,h)}$  coincides with the squared absolute value on  $Z(V, h) \simeq \mathbb{C}$
- 3. When  $p-q \equiv 1, 5$ ,  $N_e|_{Z(V,h)}$  coincides with the hyperbolic modulus on  $Z(V, h) \simeq \mathbb{D}$ .

*Proof.* The case of even  $d$  is obvious (since  $Z(V, h) = \mathbb{R}$  in that case). For odd  $d$  and  $\alpha, \beta \in \mathbb{R}$ , we have:

$$N_e(\nu) = -\nu^2 = \begin{cases} +1 & \text{if } p - q \equiv_8 3, 7 \\ -1 & \text{if } p - q \equiv_8 1, 5 \end{cases}$$

and:

$$N_e(\alpha + \beta\nu) = \begin{cases} \alpha^2 + \beta^2 = |z|^2 & \text{if } p - q \equiv_8 3, 7 \\ \alpha^2 - \beta^2 = M(z) & \text{if } p - q \equiv_8 1, 5 \end{cases} ,$$

where:

$$z = \begin{cases} \alpha + i\beta \in \mathbb{C} & \text{if } p - q \equiv_8 3, 7 \\ \alpha + j\beta \in \mathbb{D} & \text{if } p - q \equiv_8 1, 5 \end{cases} .$$

□

PROPOSITION 1.11.  $N_e$  induces a group morphism  $N_e|_{G^e(V,h)} : G^e(V, h) \rightarrow \mathbb{R}^\times$ .

*Proof.* Since  $N(G(V, h)) \subset \mathbb{R}^\times$  and  $\tilde{N}(G(V, h)) \subset \mathbb{R}^\times$ , it is clear that  $N_e(G(V, h)) \subset \mathbb{R}^\times$ . The conclusion follows from the previous proposition using the fact that  $G^e(V, h) = Z(V, h) \times G(V, h)$ . □

Composing  $N_e|_{G^e(V,h)}$  with the absolute value morphism  $|| : \mathbb{R}^\times \rightarrow \mathbb{R}_{>0}$  gives a group morphism  $|N_e| : G^e(V, h) \rightarrow \mathbb{R}_{>0}$ :

$$|N_e|(x) = |N_e(x)| \quad \forall x \in G^e(V, h) .$$

For any  $v \in V$ , we have:

$$N_e(v) = \begin{cases} -N(v) & \text{if } d \equiv_4 0, 1 \\ +N(v) & \text{if } d \equiv_4 2, 3 \end{cases}$$

and hence:

$$|N_e|_{G(V,h)} = |N|_{G(V,h)} . \tag{1.23}$$

**1.22. The extended pin group.**

DEFINITION 1.11. *The extended pin group is defined through:*

$$\text{Pin}^e(V, h) \stackrel{\text{def.}}{=} \ker(|N_e| : G^e(V, h) \rightarrow \mathbb{R}_{>0}) \tag{1.24}$$

We have a short exact sequence:

$$1 \longrightarrow \text{Pin}^e(V, h) \hookrightarrow G^e(V, h) \xrightarrow{|N_e|} \mathbb{R}_{>0} \longrightarrow 1 \tag{1.25}$$

and a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & (1.26) \\
 & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & U(Z(V, h)) & \longrightarrow & \text{Pin}^e(V, h) & \xrightarrow{\text{Ad}_0^e} & \hat{O}(V, h) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \parallel \\
 1 & \longrightarrow & Z(V, h)^\times & \longrightarrow & G^e(V, h) & \xrightarrow{\text{Ad}_0^e} & \hat{O}(V, h) \longrightarrow 1 \\
 & & \downarrow |M| & & \downarrow |N_e| & & \\
 & & \mathbb{R}_{>0} & \xlongequal{\quad} & \mathbb{R}_{>0} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 
 \end{array}$$

where  $|M| \stackrel{\text{def.}}{=} |\cdot| \circ M$  and:

$$\begin{aligned}
 U(Z(V, h)) &\stackrel{\text{def.}}{=} \{x \in Z(V, h) \mid |N_e(x)| = 1\} = \{x \in Z(V, h) \mid |M(x)| = 1\} \\
 &= \begin{cases} \mathbb{G}_2 & p - q \equiv_8 0, 2, 4, 6 \\ U(1) & p - q \equiv_8 3, 7 \\ U(\mathbb{D}) & p - q \equiv_8 1, 5 \end{cases} . \end{aligned} \tag{1.27}$$

In all cases, we have  $Z(V, h)^\times = \{z \in Z(V, h) \mid |M(z)| \neq 0\}$  and any element  $z \in Z(V, h)^\times$  can be written as  $z = u\sqrt{|M(z)|}$ , where  $u \in U(Z(V, h))$  is uniquely determined by  $z$ . Thus:

$$G^e(V, h) = \mathbb{R}_{>0}\text{Pin}^e(V, h) \simeq \mathbb{R}_{>0} \times \text{Pin}^e(V, h) . \tag{1.28}$$

In particular,  $G^e(V, h)$  is homotopy equivalent with  $\text{Pin}^e(V, h)$ .

DEFINITION 1.12. *Define:*

$$\begin{aligned}
 \text{Spin}^c(V, h) &\stackrel{\text{def.}}{=} \text{Spin}(V, h) \cdot U(1) \\
 \text{Spin}^h(V, h) &\stackrel{\text{def.}}{=} \text{Spin}(V, h) \cdot U(\mathbb{D}) . \end{aligned} \tag{1.29}$$

PROPOSITION 1.12. *We have  $\text{Pin}^e(V, h) = U(Z(V, h))\text{Pin}(V, h)$ , namely:*

1. *In the simple normal and simple quaternionic cases ( $p - q \equiv_8 0, 2, 4, 6$ ), we have  $\text{Pin}^e(V, h) = \text{Pin}(V, h)$ .*
2. *In the complex case ( $p - q \equiv_8 3, 7$ ), we have  $\text{Pin}^e(V, h) = \text{Spin}(V, h)U(1) \simeq \text{Spin}^c(V, h)$ .*
3. *In the non-simple cases ( $p - q \equiv_8 1, 5$ ), we have  $\text{Pin}^e(V, h) = \text{Spin}(V, h)U(\mathbb{D}) \simeq \text{Spin}^h(V, h)$ .*

*Proof.*

1. Follows from the fact that  $d$  is even and hence  $Z(V, h) = \mathbb{R}$  and  $G^e(V, h) = G(V, h)$  in the simple normal and simple quaternionic cases.

2. and 3. In the complex and non-simple cases,  $d$  is odd and any  $x \in G^e(V, h)$  can be written as  $x = za$  for some  $z \in Z(V, h)^\times$  and  $a \in \text{Spin}(V, h)$ . Since  $|N_e(a)| = |N(a)| = 1$ , we have  $|N_e(x)| = |N_e(z)| = |M(z)|$ , which equals 1 iff  $z \in U(Z(V, h))$ . In both cases we have  $U(Z(V, h)) \cap \text{Spin}(V, h) = \{-1, 1\}$ , which gives the conclusion.  $\square$

Proposition 1.7 implies  $Z(\text{Pin}^e(V, h)) = U(Z(V, h))$ . The situation is summarized in Table 1.2.

$\frac{p-q}{\text{mod } 8}$	type	$Z(V, h)$	$U(Z(V, h))$	$\text{Pin}^e(V, h)$
0, 2	normal simple	$\mathbb{R}$	$\mathbb{G}_2$	$\text{Pin}(V, h)$
3, 7	complex simple	$\mathbb{C}$	$U(1)$	$\text{Spin}^e(V, h)$
4, 6	quat. simple	$\mathbb{R}$	$\mathbb{G}_2$	$\text{Pin}(V, h)$
1	normal non-simple	$\mathbb{D}$	$U(\mathbb{D})$	$\text{Spin}^h(V, h)$
5	quat. non-simple	$\mathbb{D}$	$U(\mathbb{D})$	$\text{Spin}^h(V, h)$

TABLE 1.2  
*Extended pin groups.*

Table 1.3 summarizes the relation of some low-dimensional Clifford algebras with certain classical algebras and our notation for the latter<sup>4</sup>. Table 1.4 describes the corresponding Spin, Pin and Pin<sup>e</sup> groups.

$\text{Cl}(V, h)$	name of numbers	notation	$Z(V, h)$	isomorphic descriptions
$\text{Cl}_{0,1}$	complex	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
$\text{Cl}_{1,0}$	double/split/hyperbolic	$\mathbb{D}$	$\mathbb{D}$	$\mathbb{R} \oplus \mathbb{R}$
$\text{Cl}_{0,2}$	quaternions	$\mathbb{H}$	$\mathbb{R}$	$\mathbb{H}$
$\text{Cl}_{2,0} \simeq \text{Cl}_{1,1}$	para/split/co-quaternions	$\mathbb{P}$	$\mathbb{R}$	$\text{Mat}(2, \mathbb{R})$
$\text{Cl}_{0,3}$	split biquaternions	$\mathbb{D}_{\mathbb{H}}$	$\mathbb{D}$	$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{D} \simeq \mathbb{H} \oplus \mathbb{H}$
$\text{Cl}_{3,0}$	biquaternions	$\mathbb{C}_{\mathbb{H}}$	$\mathbb{C}$	$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = \text{Mat}(2, \mathbb{C})$

TABLE 1.3  
*Some classical Clifford algebras and their isomorphic descriptions.*

**2. Enlarged spinor groups and their elementary representations.** In this section, we discuss certain enlargements of  $\text{Spin}(V, h)$  which, together with the spin and pin groups, will arise later as canonical models of the reduced Lipschitz group  $\mathcal{L}$  of irreducible real representations of  $\text{Cl}(V, h)$  for various dimensions and signatures. Depending on the value of  $p - q \text{ mod } 8$ ,  $\mathcal{L}$  turns out to be isomorphic with one of the groups  $\text{Spin}(V, h)$ ,  $\text{Pin}(V, h)$ ,  $\text{Spin}^q(V, h)$ ,  $\text{Pin}^q(V, h)$  or  $\text{Spin}^{\pm}_q(V, h)$  discussed below, while certain natural representations of  $\mathcal{L}$  are isomorphic with the elementary representations discussed in this section. The groups  $\text{Spin}^q(V, h)$  and their elementary representations were studied in [5] for the case  $p = d, q = 0$ . To our knowledge, the groups  $\text{Spin}^{\pm}_q(V, h)$  and their elementary representations were not considered before in this context; they are studied in detail in [11], to which we refer the reader for more information.

<sup>4</sup>Notice that other references, such as [20], use a convention in which  $p$  and  $q$  are interchanged.

$Cl(V, h)$	$\frac{p-q}{\text{mod } 8}$	type	$Spin(V, h)$	$Pin(V, h)$	$Pin^e(V, h)$
$Cl_{0,1}$	7	complex	$\mathbb{G}_2$	$\mathbb{G}_4$	$Spin_{0,1}^c = U(1)$
$Cl_{1,0}$	1	normal non-simple	$\mathbb{G}_2$	$D_4$	$Spin_{1,0}^h = U(\mathbb{D})$
$Cl_{0,2}$	6	quat. simple	$U(1)$	$Pin_{0,2}$	$Pin_{0,2}$
$Cl_{2,0}$	2	normal simple	$U(1)$	$Pin_{2,0}$	$Pin_{2,0}$
$Cl_{0,3}$	5	quat. non-simple	$Sp(1)$	$Pin_{0,3}$	$Spin_{3,0}^h = Sp(1) \cdot U(\mathbb{D})$
$Cl_{3,0}$	3	complex	$Sp(1)$	$Pin_{3,0}$	$Spin_{3,0}^c = Sp(1) \cdot U(1)$

TABLE 1.4  
 The groups  $Spin(V, h)$ ,  $Pin(V, h)$  and  $Pin^e(V, h)$  for some classical Clifford algebras.

**2.1. The group  $Pin^q(V, h)$  and its elementary representations.**

DEFINITION 2.1. *Define:*

$$Pin^q(V, h) \stackrel{\text{def.}}{=} Pin(V, h) \cdot Sp(1) = [Pin(V, h) \times Sp(1)] / \{-1, 1\} \quad (2.1)$$

Let  $Ad_\bullet : Sp(1) = U(\mathbb{H}) \rightarrow SO(Im\mathbb{H}) = SO(3, \mathbb{R})$  be the adjoint representation<sup>5</sup> of  $U(\mathbb{H}) = Sp(1)$ :

$$Ad_\bullet(q)(u) = quq^{-1} \quad \forall q \in U(\mathbb{H}) \quad , \quad \forall u \in Im\mathbb{H} \quad ,$$

where  $Im\mathbb{H} = \mathbb{R}^3$  is endowed with the canonical scalar product. We have:

$$Ad_\bullet(q) = Ad_{\mathbb{H}}(q)|_{Im\mathbb{H}} \quad \forall q \in U(\mathbb{H}) = Sp(1) \quad ,$$

where  $Ad_{\mathbb{H}} : U(\mathbb{H}) \rightarrow Aut_{Alg}(\mathbb{H})$  is the morphism of groups given by:

$$Ad_{\mathbb{H}}(q)(q') \stackrel{\text{def.}}{=} qq'q^{-1} \quad \forall q \in U(\mathbb{H}) = Sp(1) \quad \forall q' \in \mathbb{H} \quad .$$

The latter gives a four-dimensional representation over  $\mathbb{R}$  which decomposes as a direct sum of the trivial representation (whose underlying subspace is  $\mathbb{R}1_{\mathbb{H}}$ ) and the representation  $Ad_\bullet$  (which is supported on  $Im\mathbb{H}$ ).

DEFINITION 2.2. *The vector representation of  $Pin^q(V, h)$  is the group morphism  $\lambda : Pin^q(V, h) \rightarrow \hat{O}(V, h)$  given by:*

$$\lambda([a, q]) \stackrel{\text{def.}}{=} Ad_0(a) \quad \forall [a, q] \in Pin^q(V, h) \quad .$$

*The twisted vector representation of  $Pin^q(V, h)$  is the group morphism  $\tilde{\lambda} : Pin^q(V, h) \rightarrow O(V, h)$  given by:*

$$\tilde{\lambda}([a, q]) \stackrel{\text{def.}}{=} \widetilde{Ad}_0(a) \quad \forall [a, q] \in Pin^q(V, h) \quad .$$

*The characteristic representation of  $Pin^q(V, h)$  is the group morphism  $\mu : Pin^q(V, h) \rightarrow SO(3, \mathbb{R})$  given by:*

$$\mu([a, q]) \stackrel{\text{def.}}{=} Ad_\bullet(q) \quad \forall [a, q] \in Pin^q(V, h) \quad ,$$

<sup>5</sup>This coincides with the vector representation of  $Spin(3) \simeq Sp(1) \simeq SU(2)$ ,  $Ad_\bullet$  being the double covering morphism  $Spin(3) \rightarrow SO(3, \mathbb{R})$ .

The basic representation of  $\text{Pin}^q(V, h)$  is the group morphism  $\rho \stackrel{\text{def.}}{=} \lambda \times \mu : \text{Pin}^q(V, h) \rightarrow \hat{\text{O}}(V, h) \times \text{SO}(3, \mathbb{R})$ :

$$\rho([a, q]) \stackrel{\text{def.}}{=} (\text{Ad}_0(a), \text{Ad}_\bullet(q)) \quad . \tag{2.2}$$

The twisted basic representation of  $\text{Pin}^q(V, h)$  is the group morphism  $\tilde{\rho} \stackrel{\text{def.}}{=} \tilde{\lambda} \times \mu : \text{Pin}^q(V, h) \rightarrow \text{O}(V, h) \times \text{SO}(3, \mathbb{R})$ :

$$\tilde{\rho}([a, q]) \stackrel{\text{def.}}{=} (\widetilde{\text{Ad}}_0(a), \text{Ad}_\bullet(q)) \quad . \tag{2.3}$$

We have exact sequences:

$$1 \longrightarrow \mathbb{Z}_2 \hookrightarrow \text{Pin}^q(V, h) \xrightarrow{\rho} \hat{\text{O}}(V, h) \times \text{SO}(3, \mathbb{R}) \longrightarrow 1 \quad , \tag{2.4}$$

and:

$$1 \longrightarrow \mathbb{Z}_2 \hookrightarrow \text{Pin}^q(V, h) \xrightarrow{\tilde{\rho}} \text{O}(V, h) \times \text{SO}(3, \mathbb{R}) \longrightarrow 1 \quad , \tag{2.5}$$

where  $\mathbb{Z}_2 = \{[-1, 1] = [1, -1], [1, 1] = [-1, -1]\}$ .

**PROPOSITION 2.1.** *Let  $d$  be even and  $\varphi : \text{Pin}(V, h) \xrightarrow{\sim} \text{Pin}(V, -\sigma_{p,q}h)$  be the isomorphism of Proposition 1.6. Then the isomorphism of groups  $\theta : \text{Pin}^q(V, h) \xrightarrow{\sim} \text{Pin}^q(V, -\sigma_{p,q}h)$  defined through:*

$$\theta([a, q]) \stackrel{\text{def.}}{=} [\varphi(a), q] \quad \forall a \in \text{Pin}(V, h) \quad \text{and} \quad q \in \text{U}(\mathbb{H}) \quad ,$$

satisfies  $\rho \circ \theta = \tilde{\rho}$ .

*Proof.* Follows from Proposition 1.6.  $\square$

**2.2. The group  $\text{Spin}^q(V, h)$  and its elementary representations.** Define:

$$\text{Spin}^q(V, h) \stackrel{\text{def.}}{=} \text{Spin}(V, h) \cdot \text{Sp}(1) = [\text{Spin}(V, h) \times \text{Sp}(1)] / \{-1, 1\} \quad . \tag{2.6}$$

In the case  $p = d, q = 0$ , this group was studied in [5].

**DEFINITION 2.3.** *The vector representation  $\lambda : \text{Spin}^q(V, h) \rightarrow \text{SO}(V, h)$  is the restriction of any of the vector representations of  $\text{Pin}^q(V, h)$ . The characteristic representation  $\mu : \text{Spin}^q(V, h) \rightarrow \text{SO}(3, \mathbb{R})$  is the restriction of the characteristic representation of  $\text{Pin}^q(V, h)$ . The basic representation  $\rho \stackrel{\text{def.}}{=} \lambda \times \mu : \text{Spin}^q(V, h) \rightarrow \text{SO}(V, h) \times \text{SO}(3, \mathbb{R})$  is the restriction of any of the basic representations of  $\text{Pin}^q(V, h)$ .*

The sequences (2.4) and (2.5) restrict to the same exact sequence:

$$1 \longrightarrow \mathbb{Z}_2 \hookrightarrow \text{Spin}^q(V, h) \xrightarrow{\rho} \text{SO}(V, h) \times \text{SO}(3, \mathbb{R}) \longrightarrow 1 \quad . \tag{2.7}$$

**2.3. The group  $\text{Spin}_\alpha^o(V, h)$  and its elementary representations.** In this subsection, we assume that  $d = p + q$  is odd. Let  $\alpha \in \{-1, 1\}$  be a sign factor and define:

$$\text{Pin}_2(\alpha) \stackrel{\text{def.}}{=} \begin{cases} \text{Pin}_{2,0} & \text{if } \alpha = +1 \\ \text{Pin}_{0,2} & \text{if } \alpha = -1 \end{cases} \quad .$$



DEFINITION 2.4. *Define:*

$$\text{Spin}_\alpha^o(V, h) = \text{Spin}(V, h) \cdot \text{Pin}_2(\alpha) \stackrel{\text{def.}}{=} [\text{Spin}(V, h) \times \text{Pin}_2(\alpha)] / \{-1, 1\} .$$

Let  $\text{Ad}_0^{(2)} : \text{Pin}_2(\alpha) \rightarrow \text{O}(2, \mathbb{R})$  and  $\widetilde{\text{Ad}}_0^{(2)} : \text{Pin}_2(\alpha) \rightarrow \text{O}(2, \mathbb{R})$  be the untwisted and twisted vector representations of  $\text{Pin}_2(\alpha)$ . In signatures  $(2, 0)$  and  $(0, 2)$ , Proposition 1.6 gives an automorphism of  $\text{Pin}_2(\alpha)$  which exchanges the two representations  $\text{Ad}_0^{(2)}$  and  $\widetilde{\text{Ad}}_0^{(2)}$ . Notice that  $\text{Spin}(2) \simeq \text{SO}(2, \mathbb{R}) \simeq \text{U}(1)$  and that  $\text{U}(1)/\{-1, 1\} \simeq \text{U}(1)$ .

DEFINITION 2.5.

1. *The vector representation of  $\text{Spin}_\alpha^o(V, h)$  is the group morphism  $\lambda : \text{Spin}_\alpha^o(V, h) \rightarrow \text{O}(V, h)$  defined through:*

$$\lambda([a, u]) \stackrel{\text{def.}}{=} \det(\text{Ad}_0^{(2)}(u))\text{Ad}_0(a) \quad \forall [a, u] \in \text{Spin}_\alpha^o(V, h) .$$

2. *The characteristic representation of  $\text{Spin}_\alpha^o(V, h)$  is the group morphism  $\mu : \text{Spin}_\alpha^o(V, h) \rightarrow \text{O}(2, \mathbb{R})$  defined through:*

$$\mu([a, u]) \stackrel{\text{def.}}{=} \text{Ad}_0^{(2)}(u) \quad \forall [a, u] \in \text{Spin}_\alpha^o(V, h) .$$

3. *The basic representation is the group morphism  $\rho \stackrel{\text{def.}}{=} \lambda \times \mu : \text{Spin}_\alpha^o(V, h) \rightarrow \text{S}[\text{O}(V, h) \times \text{O}(2, \mathbb{R})]$ :*

$$\rho([a, u]) \stackrel{\text{def.}}{=} (\det(\text{Ad}_0^{(2)}(u))\text{Ad}_0(a), \text{Ad}_0^{(2)}(u)) .$$

We have a short exact sequence:

$$1 \longrightarrow \text{Spin}(V, h) \longrightarrow \text{Spin}_\alpha^o(V, h) \xrightarrow{\mu} \text{O}(2, \mathbb{R}) \longrightarrow 1 .$$

Since  $d$  is odd, we also have short exact sequences:

$$1 \longrightarrow \text{U}(1) \longrightarrow \text{Spin}_\alpha^o(V, h) \xrightarrow{\lambda} \text{O}(V, h) \longrightarrow 1$$

(where  $\text{U}(1) = \text{Spin}(2) \subset \text{Pin}_2(\alpha) \subset \text{Spin}_\alpha^o(V, h)$ ) and:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}_\alpha^o(V, h) \xrightarrow{\rho} \text{S}[\text{O}(V, h) \times \text{O}(2, \mathbb{R})] \longrightarrow 1 ,$$

where  $\mathbb{Z}_2 = \{[1, 1] = [-1, -1], [1, -1] = [-1, 1]\} \subset \text{Spin}_\alpha^o(V, h)$ . One can identify  $\text{Pin}_2(\alpha)$  with the abstract group  $\text{O}_2(\alpha)$  defined below.

DEFINITION 2.6. *Let  $\text{O}_2(\alpha)$  be the compact non-Abelian Lie group with underlying set  $\text{U}(1) \times \mathbb{Z}_2$  and composition given by:*

$$\begin{aligned} (z_1, \hat{0})(z_2, \hat{0}) &= (z_1 z_2, \hat{0}), & (z_1, \hat{0})(z_2, \hat{1}) &= (z_1 z_2, \hat{1}), \\ (z_1, \hat{1})(z_2, \hat{0}) &= (z_1 \bar{z}_2, \hat{1}), & (z_1, \hat{1})(z_2, \hat{1}) &= (\alpha z_1 \bar{z}_2, \hat{0}), \end{aligned} \tag{2.8}$$

where  $\mathbb{Z}_2 = \{\hat{0}, \hat{1}\}$ . The unit in  $\text{O}_2(\alpha)$  is given by  $1 \equiv (1, \hat{0})$ .

The group  $\text{U}(1)$  embeds into  $\text{O}_2(\alpha)$  as the non-central subgroup  $\text{U}(1) \times \{\hat{0}\}$ . The element  $\mathbf{c} \stackrel{\text{def.}}{=} (1, \hat{1}) \in \text{O}_2(\alpha)$  satisfies  $\mathbf{c}^2 = (\alpha, \hat{0}) = \alpha 1$  and  $\mathbf{c}^{-1} = (\alpha, \hat{1}) = \alpha \mathbf{c}$ . Thus  $\mathbf{c}$  has order two when  $\alpha = 1$  and order four when  $\alpha = -1$ . This element

generates a subgroup  $\Gamma_c$  of  $O_2(\alpha)$  which is isomorphic with  $\mathbb{Z}_2$  when  $\alpha = +1$  and with  $\mathbb{G}_4$  when  $\alpha = -1$ . Together with  $\mathbf{c}$ , the subgroup  $U(1)$  generates  $O_2(\alpha)$ . In fact,  $O_2(\alpha) = \{z|z \in U(1)\} \sqcup \{zc|z \in U(1)\}$  is the group generated by  $U(1)$  and  $\mathbf{c}$  with the relations  $\mathbf{c}^2 = \alpha 1$  and  $\mathbf{c}z = \bar{z}\mathbf{c}$ . We have:

$$\text{Ad}(\mathbf{c})(x) = K(x), \quad \forall x \in O_2(\alpha),$$

where  $K : O_2(\alpha) \rightarrow O_2(\alpha)$  is the *conjugation automorphism*, given by:

$$K(z, \kappa) = (\bar{z}, \kappa), \quad \forall z \in U(1), \quad \kappa \in \mathbb{Z}_2.$$

Notice that  $K(z) = \bar{z}$  for  $z \in U(1)$ ,  $K(\mathbf{c}) = \mathbf{c}$  and  $K^2 = \text{id}_{O_2(\alpha)}$ . The group  $O_2(\alpha)$  admits a  $\mathbb{Z}_2$ -grading with homogeneous components:

$$O_2(\alpha)^+ = \{(z, \hat{0})|z \in U(1)\} \simeq U(1), \quad O_2(\alpha)^- = \{(z, \hat{1})|z \in U(1)\} = U(1)\mathbf{c}.$$

These coincide with the connected components of  $O_2(\alpha)$ . We have  $Z(O_2(\alpha)) = \{-1, 1\} = \mathbb{G}_2$ .

DEFINITION 2.7. *The abstract determinant is the grading morphism  $\eta_\alpha : O_2(\alpha) \rightarrow \mathbb{G}_2$  of  $O_2(\alpha)$ :*

$$\eta_\alpha(x) \stackrel{\text{def.}}{=} (-1)^{\text{pr}_2(x)},$$

where  $\text{pr}_2(z, \kappa) \stackrel{\text{def.}}{=} \kappa$  for any  $(z, \kappa) \in O_2(\alpha)$ .

We have  $\eta_\alpha(z) = 1$ ,  $\eta_\alpha(\mathbf{c}) = -1$  and a short exact sequence:

$$1 \longrightarrow U(1) \longrightarrow O_2(\alpha) \xrightarrow{\eta_\alpha} \mathbb{G}_2 \longrightarrow 1 \quad . \tag{2.9}$$

Moreover:

1. For  $\alpha = +1$ , the sequence (2.9) splits (a splitting morphism  $\theta : \mathbb{G}_2 \rightarrow O_2(\alpha)$  being given by  $\theta(1) = 1$  and  $\theta(-1) = \mathbf{c}$ ) and we have<sup>6</sup>  $O_2(+)$   $\simeq_{\text{Gp}}$   $O(2, \mathbb{R})$  by an isomorphism which identifies  $U(1)$  with  $SO(2, \mathbb{R})$  and  $\mathbf{c}$  with reflection of  $\mathbb{R}^2$  with respect to some axis.
2. For  $\alpha = -1$ , the sequence (2.9) presents  $O_2(-)$  as a *non-split* extension of  $\mathbb{Z}_2$  by  $U(1)$ . In particular, we have  $O_2(-) \not\simeq O(2, \mathbb{R})$ .

DEFINITION 2.8. *The squaring morphism is the surjective group morphism  $\sigma_\alpha : O_2(\alpha) \rightarrow O_2(+)$  given by:*

$$\sigma_\alpha(z, \kappa) \stackrel{\text{def.}}{=} (z^2, \kappa), \quad \forall (z, \kappa) \in O_2(\alpha) \quad .$$

We have a short exact sequence:

$$1 \longrightarrow \{-1, 1\} \hookrightarrow O_2(\alpha) \xrightarrow{\sigma_\alpha} O_2(+) \longrightarrow 1 \quad .$$

In particular,  $O_2(+)$   $\simeq$   $\text{Pin}_{2,0}$  and  $O_2(-)$   $\simeq$   $\text{Pin}_{0,2}$  are inequivalent central extensions of  $O(2, \mathbb{R})$  by  $\mathbb{Z}_2$ . The reflection:

$$C_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in O_-(2, \mathbb{R}) \tag{2.10}$$

---

<sup>6</sup>This is the well-known isomorphism between  $\text{Pin}_{2,0} \simeq O_2(+)$  and the orthogonal group  $O(2, \mathbb{R})$ .

of  $\mathbb{R}^2$  with respect to the horizontal axis (=real axis of  $\mathbb{C} \equiv \mathbb{R}^2$ ) gives isomorphisms of groups  $\Phi_0^{(\pm)} : O_2(+) \xrightarrow{\sim} O(2, \mathbb{R})$  through the formula:

$$\Phi_0^{(\pm)}(e^{i\theta}, \hat{0}) = R(\pm\theta), \quad \Phi_0^{(\pm)}(e^{i\theta}, \hat{1}) = R(\pm\theta)C_0, \tag{2.11}$$

where  $\theta \in \mathbb{R}$  and:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \in SO(2, \mathbb{R}) . \tag{2.12}$$

Let  $e_1, e_2$  be the canonical basis of  $\mathbb{R}^2$  and  $\nu_2(\alpha) = e_1e_2$  be the Clifford volume element of  $Cl_2(\alpha) \stackrel{\text{def.}}{=} \begin{cases} Cl_{2,0} & \text{if } \alpha = +1 \\ Cl_{0,2} & \text{if } \alpha = -1 \end{cases}$  with respect to the standard orientation of  $\mathbb{R}^2$ . The following result is proved in [11]:

**PROPOSITION 2.2.** *There exists an isomorphism of  $\mathbb{Z}_2$ -graded groups  $\psi_\alpha : O_2(\alpha) \xrightarrow{\sim} Pin_2(\alpha)$  which satisfies:*

$$\psi_\alpha(\mathbf{i}) = \nu_2 = e_1e_2 \quad \text{and} \quad \psi_\alpha(\mathbf{c}) = e_1 .$$

Moreover, the untwisted vector representation of  $Pin_2(\alpha)$  agrees with the squaring morphism  $\sigma_\alpha$  through the isomorphisms  $\psi_\alpha$  and  $\Phi_0^{(-\alpha)}$ :

$$Ad_0^{(2)} \circ \psi_\alpha = \Phi_0^{(-\alpha)} \circ \sigma_\alpha . \tag{2.13}$$

and the abstract determinant agrees with the grading morphism  $\det \circ Ad_0^{(2)}$  of  $Pin_2(\alpha)$ :

$$\det \circ Ad_0^{(2)} \circ \psi_\alpha = \eta_\alpha . \tag{2.14}$$

### 2.4. Adapted $Spin^o$ groups.

**DEFINITION 2.9.** *Let  $(V, h)$  be a quadratic vector space belonging to the complex case, i.e. such that the signature  $(p, q)$  of  $(V, h)$  satisfies  $p - q \equiv_8 3, 7$  (in particular,  $d = p + q$  is odd). Let  $\alpha_{p,q} \stackrel{\text{def.}}{=} (-1)^{\frac{p-q+1}{4}}$ . Then the adapted  $Spin^o$  group of  $(V, h)$  is defined through:*

$$Spin^o(V, h) \stackrel{\text{def.}}{=} Spin_{\alpha_{p,q}}^o(V, h) = \begin{cases} Spin_-^o(V, h) & \text{if } p - q \equiv_8 3 \\ Spin_+^o(V, h) & \text{if } p - q \equiv_8 7 \end{cases} .$$

### 2.5. The canonical spinor group and its elementary representations.

It is convenient to introduce a group depending on  $p - q \bmod 8$  which, as we shall see later on, provides a canonical presentation of the reduced Lipschitz group of real Clifford irreps.

**DEFINITION 2.10.** *The canonical spinor group  $\Lambda(V, h)$  of  $(V, h)$  is defined as follows:*

1. In the normal simple case, we set  $\Lambda(V, h) \stackrel{\text{def.}}{=} Pin(V, h)$ .
2. In the complex case, we set  $\Lambda(V, h) \stackrel{\text{def.}}{=} Spin^o(V, h)$ .
3. In the quaternionic simple case, we set  $\Lambda(V, h) \stackrel{\text{def.}}{=} Pin^q(V, h)$ .
4. In the normal non-simple case, we set  $\Lambda(V, h) \stackrel{\text{def.}}{=} Spin(V, h)$ .

5. In the quaternionic non-simple case, we set  $\Lambda(V, h) \stackrel{\text{def.}}{=} \text{Spin}^q(V, h)$ .

The situation is summarized in Table 2.1.

$p - q$ mod 8	type	$\Lambda(V, h)$
0, 2	normal simple	$\text{Pin}(V, h)$
3, 7	complex	$\text{Spin}^o(V, h)$
4, 6	quaternionic simple	$\text{Pin}^q(V, h)$
1	normal non-simple	$\text{Spin}(V, h)$
5	quaternionic non-simple	$\text{Spin}^q(V, h)$

TABLE 2.1  
Canonical spinor groups

DEFINITION 2.11. The vector representation  $\lambda$  of  $\Lambda(V, h)$  is defined as follows:

1. In the normal simple case,  $\lambda \stackrel{\text{def.}}{=} \text{Ad}_0^{\text{Cl}} : \text{Pin}(V, h) \rightarrow \text{O}(V, h)$  is the untwisted vector representation of  $\text{Pin}(V, h)$ .
2. In the complex case,  $\lambda : \text{Spin}^o(V, h) \rightarrow \text{O}(V, h)$  is the vector representation of  $\text{Spin}_{\alpha_{p,q}}^o(V, h)$ .
3. In the quaternionic simple case,  $\lambda : \text{Pin}^q(V, h) \rightarrow \text{O}(V, h)$  is the untwisted vector representation of  $\text{Pin}^q(V, h)$ .
4. In the normal non-simple case,  $\lambda \stackrel{\text{def.}}{=} \text{Ad}_0^{\text{Cl}} : \text{Spin}(V, h) \rightarrow \text{SO}(V, h)$  is the vector representation of  $\text{Spin}(V, h)$ .
5. In the quaternionic non-simple case,  $\lambda : \text{Spin}^q(V, h) \rightarrow \text{SO}(V, h)$  is the vector representation of  $\text{Spin}^q(V, h)$ .

DEFINITION 2.12. The characteristic representation  $\mu$  of  $\Lambda(V, h)$  is defined as follows:

1. In the normal simple case,  $\mu : \text{Pin}(V, h) \rightarrow 1$  is the trivial one-dimensional representation.
2. In the complex case,  $\mu : \text{Spin}^o(V, h) \rightarrow \text{O}(2, \mathbb{R})$  is the characteristic representation of  $\text{Spin}_{\alpha_{p,q}}^o(V, h)$ .
3. In the quaternionic simple case,  $\mu : \text{Pin}^q(V, h) \rightarrow \text{SO}(3, \mathbb{R})$  is the characteristic representation of  $\text{Pin}^q(V, h)$ .
4. In the normal non-simple case,  $\mu : \text{Spin}(V, h) \rightarrow 1$  is the trivial one-dimensional representation.
5. In the quaternionic non-simple case,  $\mu : \text{Spin}^q(V, h) \rightarrow \text{SO}(3, \mathbb{R})$  is the characteristic representation of  $\text{Spin}^q(V, h)$ .

DEFINITION 2.13. The basic representation of  $\Lambda(V, h)$  is the representation  $\rho = \lambda \times \mu$ , namely:

1. In the normal simple case,  $\rho = \lambda : \text{Pin}(V, h) \rightarrow \text{O}(V, h)$  is the untwisted vector representation of  $\text{Pin}(V, h)$ .
2. In the complex case,  $\rho : \text{Spin}^o(V, h) \rightarrow \text{S}[\text{O}(V, h) \times \text{O}(2, \mathbb{R})]$  is the basic representation of  $\text{Spin}_{\alpha_{p,q}}^o(V, h)$ .
3. In the quaternionic simple case,  $\rho : \text{Pin}^q(V, h) \rightarrow \text{O}(V, h) \times \text{SO}(3, \mathbb{R})$  is the untwisted basic representation of  $\text{Pin}^q(V, h)$ .

- 4. In the normal non-simple case,  $\rho : \text{Spin}(V, h) \rightarrow \text{SO}(V, h)$  is the vector representation of  $\text{Spin}(V, h)$ .
- 5. In the quaternionic non-simple case,  $\rho : \text{Spin}^q(V, h) \rightarrow \text{SO}(V, h) \times \text{SO}(3, \mathbb{R})$  is the basic representation of  $\text{Spin}^q(V, h)$ .

**3. Clifford representations over  $\mathbb{R}$ .** In this section, we discuss finite-dimensional real Clifford representations, the notion of weak faithfulness for such representations as well as certain natural subspaces associated to them. Real Lipschitz groups (to be introduced in the next section) arise most naturally as the automorphism groups of weakly faithful real Clifford representations in a certain category which has more morphisms than the usual category of representations. Therefore, we start by introducing that category.

**3.1. The unbased category of Clifford representations over  $\mathbb{R}$ .** DEFINITION 3.1. A Clifford representation over  $\mathbb{R}$  is a morphism of unital algebras  $\gamma : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S)$ , where  $S$  is a finite-dimensional  $\mathbb{R}$ -vector space.

Let  $\gamma : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S)$  and  $\gamma' : \text{Cl}(V', h') \rightarrow \text{End}_{\mathbb{R}}(S')$  be two Clifford representations.

DEFINITION 3.2. A morphism from  $\gamma$  to  $\gamma'$  is a pair  $(f_0, f)$  such that:

- 1.  $f_0 : V \rightarrow V'$  is an isometry from  $(V, h)$  to  $(V', h')$
- 2.  $f : S \rightarrow S'$  is an  $\mathbb{R}$ -linear map
- 3.  $\gamma'(\text{Cl}(f_0)(x)) \circ f = f \circ \gamma(x)$  for all  $x \in \text{Cl}(V, h)$ .

A based morphism is a morphism  $(f_0, f)$  such that  $f_0 = \text{id}_V$ . A (not necessarily based) isomorphism from  $\gamma$  to itself is called an automorphism.

In our language, a morphism of representations in the traditional sense corresponds to a based morphism. Since  $\text{Cl}(V, h)$  is generated by  $V$  while  $\text{Cl}(V', h')$  is generated by  $V'$ , condition (3) is equivalent with the weaker requirement:

$$\gamma'(f_0(v)) \circ f = f \circ \gamma(v) \quad \forall v \in V \quad ,$$

and can also be written as (see the diagram in Figure 3.1):

$$R_f \circ \gamma' \circ f_0 = L_f \circ \gamma|_V \quad \text{or} \quad R_f \circ \gamma' \circ \text{Cl}(f_0) = L_f \circ \gamma \quad ,$$

where  $L_f : \text{End}_{\mathbb{R}}(S) \rightarrow \text{Hom}_{\mathbb{R}}(S, S')$  and  $R_f : \text{End}_{\mathbb{R}}(S') \rightarrow \text{Hom}_{\mathbb{R}}(S, S')$  are defined through:

$$L_f(\varphi) \stackrel{\text{def.}}{=} f \circ \varphi \quad , \quad R_f(\psi) = \psi \circ f \quad , \quad \forall \varphi \in \text{End}_{\mathbb{R}}(S) \quad \text{and} \quad \psi \in \text{End}_{\mathbb{R}}(S') \quad .$$

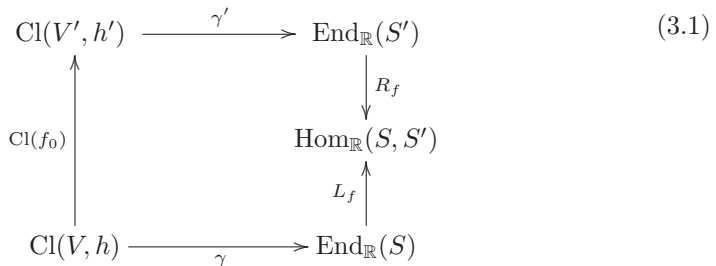


FIG. 3.1. Commutativity of the diagram is the condition that  $(f_0, f)$  gives an unbased morphism of Clifford representations.

With this definition, Clifford representations form a category denoted  $\text{ClRep}$ , were compatible morphisms  $(f_0, f)$  and  $(f'_0, f')$  compose pairwise, i.e.  $(f'_0, f') \circ (f_0, f) \stackrel{\text{def.}}{=} (f'_0 \circ f_0, f' \circ f)$ . The forgetful functor  $\Pi : \text{ClRep} \rightarrow \text{Cl}$  which takes  $\gamma$  into  $\text{Cl}(V, h)$  and  $(f_0, f)$  into  $\text{Cl}(f_0)$  is a fibration whose fiber above  $\text{Cl}(V, h)$  is the usual category  $\text{Rep}(\text{Cl}(V, h))$  of representations of  $\text{Cl}(V, h)$  (whose morphisms are the based morphisms of representations). Isomorphisms in  $\text{Rep}(\text{Cl}(V, h))$  are the usual equivalences of representations. Hence equivalences of representations of Clifford algebras coincide with based isomorphisms of  $\text{ClRep}$ ; in particular, any isomorphism class of Clifford representations in the category  $\text{ClRep}$  decomposes as a disjoint union of equivalence classes. The surjective functor  $F = \text{Cl}^{-1} \circ \Pi : \text{ClRep} \rightarrow \text{Quad}$  sends  $\text{Cl}(V, h)$  to  $(V, h)$  and  $(f_0, f)$  to  $f_0$ . A morphism  $(f_0, f)$  is an isomorphism in  $\text{ClRep}$  iff both  $f_0$  and  $f$  are bijective.

**PROPOSITION 3.1.** *Let  $(f_0, f) : \gamma \xrightarrow{\sim} \gamma'$  be an isomorphism of Clifford representations. Then  $\text{Ad}(f)(\gamma(V)) = \gamma'(V')$  and  $\gamma' \circ f_0 = \text{Ad}(f) \circ \gamma|_V$ , where the unital isomorphism of algebras  $\text{Ad}(f) : \text{End}_{\mathbb{R}}(S) \rightarrow \text{End}_{\mathbb{R}}(S')$  is defined through:*

$$\text{Ad}(f)(\varphi) \stackrel{\text{def.}}{=} f \circ \varphi \circ f^{-1}$$

for all  $\varphi \in \text{End}_{\mathbb{R}}(S)$ .

*Proof.* Follows immediately from the fact that  $\text{Cl}(f_0)|_V = f_0$ .  $\square$

When  $(f_0, f)$  is an isomorphism, condition 3. in Definition 3.2 becomes:

$$\gamma' \circ \text{Cl}(f_0) = \text{Ad}(f) \circ \gamma \quad , \tag{3.2}$$

being equivalent with the condition  $\gamma' \circ f_0 = \text{Ad}(f) \circ \gamma|_V$ , which states that  $f$  implements the isometry  $f_0 : (V, h) \rightarrow (V', h')$  at the level of the representation spaces.

**3.2. Clifford image, vector image and Schur algebra.**

**DEFINITION 3.3.** *The Clifford image of  $\gamma$  is the unital subalgebra  $C(\gamma) \stackrel{\text{def.}}{=} \gamma(\text{Cl}(V, h)) \subset \text{End}_{\mathbb{R}}(S)$ . The vector image is the subspace  $W(\gamma) \stackrel{\text{def.}}{=} \gamma(V) \subset C(\gamma)$ .*

Notice that the Clifford image coincides with the unital subalgebra of  $\text{End}_{\mathbb{R}}(S)$  generated by the vector image and hence  $C(\gamma)$  is uniquely determined by  $W(\gamma)$ . Every element  $w \in W(\gamma)$  can be written as  $w = \gamma(v)$  for some  $v \in V$ , but this presentation need not be unique<sup>7</sup>.

**DEFINITION 3.4.** *The Schur algebra  $\mathbb{S}(\gamma)$  is the centralizer algebra of  $C(\gamma)$  inside the algebra  $(\text{End}_{\mathbb{R}}(S), \circ)$ . It consists of all  $\mathbb{R}$ -linear endomorphisms of  $S$  which commute with every element of  $C(\gamma)$ .*

Since  $C(\gamma)$  is generated by  $W(\gamma)$ , we have:

$$\begin{aligned} \mathbb{S}(\gamma) &= \{a \in \text{End}_{\mathbb{R}}(S) \mid aw = wa \quad \forall w \in W(\gamma)\} \\ &= \{a \in \text{End}_{\mathbb{R}}(S) \mid a\gamma(v) = \gamma(v)a \quad \forall v \in V\} . \end{aligned} \tag{3.3}$$

Since  $\mathbb{S} := \mathbb{S}(\gamma)$  is a unital subalgebra of  $\text{End}_{\mathbb{R}}(S)$ , we can view  $S$  as a left  $\mathbb{S}$ -module upon defining the left multiplication with scalars through:

$$s\xi \stackrel{\text{def.}}{=} s(\xi) \quad \forall s \in \mathbb{S} \quad , \quad \xi \in S \quad ,$$

---

<sup>7</sup>It is unique when  $\gamma$  is weakly faithful, see below.

where  $s(\xi)$  denotes the result of applying the operator  $s$  to  $\xi$ . Also notice that  $\text{End}_{\mathbb{R}}(S)$  is an  $\mathbb{S}$ -bimodule (with external multiplications given by composition from the left and from the right with operators belonging to  $\mathbb{S}$ ). To simplify notation, we will often denote composition of operators by juxtaposition. Let:

$$\begin{aligned} \text{End}_{\mathbb{S}}(S) &\stackrel{\text{def.}}{=} \{a \in \text{End}_{\mathbb{R}}(S) \mid a(s\xi) = sa(\xi) \quad \forall \xi \in S \text{ and } s \in \mathbb{S}\} \\ &= \{a \in \text{End}_{\mathbb{R}}(S) \mid as = sa \quad \forall s \in \mathbb{S}\}, \end{aligned} \tag{3.4}$$

denote the unital algebra of  $\mathbb{S}$ -linear maps (endomorphisms of  $S$  as an  $\mathbb{S}$ -module) and  $\text{Aut}_{\mathbb{S}}(S)$  denote the group of invertible  $\mathbb{S}$ -linear maps ( $\mathbb{S}$ -module automorphisms). By the definition of  $C(\gamma)$ , we have:

$$C(\gamma) \subset \text{End}_{\mathbb{S}}(S) \ .$$

REMARK 3.1. Recall that  $\text{Cl}(V, h)$  is a semisimple  $\mathbb{R}$ -algebra. This implies that any finite-dimensional  $\text{Cl}(V, h)$ -module (i.e. any Clifford representation  $\gamma : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S)$ ) decomposes as a direct sum of inequivalent irreps  $\gamma_i : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S_i)$  ( $i = 1 \dots n$ ) with multiplicities  $m_i$ , namely  $S = \bigoplus_{i=1}^n S_i \otimes_{\mathbb{R}} U_i$  with  $U_i = \mathbb{R}^{m_i}$  and  $\gamma(x) = \bigoplus_{i=1}^n \gamma_i(x) \otimes_{\mathbb{R}} \text{id}_{U_i}$ . Schur’s lemma implies the decomposition:

$$\mathbb{S}(\gamma) = \text{End}_{\text{Cl}(V, h)}(S) = \bigoplus_{i=1}^n \mathbb{S}(\gamma_i) \otimes_{\mathbb{R}} \text{End}_{\mathbb{R}}(U_i) \ , \tag{3.5}$$

where  $\mathbb{S}(\gamma_i)$  are division algebras over  $\mathbb{R}$ , hence each  $\mathbb{S}(\gamma_i)$  is isomorphic with one of the  $\mathbb{R}$ -algebras  $\mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ .

**3.3. Pseudocentralizer, anticommutant subspace and Schur pairing.**

DEFINITION 3.5. *The anticommutant subspace of the Clifford representation  $\gamma : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S)$  is the following subspace of  $\text{End}_{\mathbb{R}}(S)$ :*

$$\begin{aligned} A(\gamma) &\stackrel{\text{def.}}{=} \{a \in \text{End}_{\mathbb{R}}(S) \mid aw = -wa \quad \forall w \in W(\gamma)\} \\ &= \{a \in \text{End}_{\mathbb{R}}(S) \mid a\gamma(v) = -\gamma(v)a \quad \forall v \in V\} \\ &= \{a \in \text{End}_{\mathbb{R}}(S) \mid a\gamma(x) = \gamma(\pi(x))a \quad \forall x \in \text{Cl}(V, h)\}. \end{aligned} \tag{3.6}$$

Let  $A := A(\gamma)$ . We have  $\mathbb{S}A \subset A$  and  $A\mathbb{S} \subset A$ , so  $A$  is an  $\mathbb{S}$ -bimodule (a submodule of the  $\mathbb{S}$ -bimodule  $\text{End}_{\mathbb{R}}(S)$ ). For any  $a_1, a_2 \in A$ , we have  $a_1a_2 \in \mathbb{S}$ , so the composition of  $\text{End}_{\mathbb{R}}(S)$  induces an  $\mathbb{R}$ -bilinear map:

$$m : A \times A \rightarrow \mathbb{S} \ . \tag{3.7}$$

In particular,  $a \in A$  implies  $a^2 \in \mathbb{S}$ , so  $A$  consists of square roots of elements of  $\mathbb{S}$  (taken in  $\text{End}_{\mathbb{R}}(S)$ ). These observations imply that the subspace  $\mathbb{S} + A$  is a unital subalgebra of  $\text{End}_{\mathbb{R}}(S)$ .

DEFINITION 3.6. *The pseudocentralizer of  $\gamma$  is the unital subalgebra  $\mathbb{T}(\gamma) \stackrel{\text{def.}}{=} \mathbb{S}(\gamma) + A(\gamma)$  of  $\text{End}_{\mathbb{R}}(S)$ .*

DEFINITION 3.7. *The Schur pairing of  $\gamma$  is the symmetric  $\mathbb{R}$ -bilinear pairing  $\mathfrak{p}_{\gamma} : A(\gamma) \times A(\gamma) \rightarrow \mathbb{S}(\gamma)$  obtained by symmetrizing the composition map (3.7):*

$$\mathfrak{p}_{\gamma}(a_1, a_2) \stackrel{\text{def.}}{=} \frac{1}{2}(a_1a_2 + a_2a_1) \ , \quad \forall a_1, a_2 \in A(\gamma) \ .$$



Notice that  $\mathbb{S}^\times \cap A = \emptyset$  if  $W(\gamma) \neq 0$ . Hence:

$$\mathbb{S} \cap A \subset \mathbb{S} \setminus \mathbb{S}^\times \text{ if } W(\gamma) \neq 0 \text{ .}$$

In particular, we have  $\mathbb{S} \cap A = \{0\}$  when  $W(\gamma) \neq 0$  and  $\mathbb{S}$  is a division algebra (for example, when  $\gamma$  is an irreducible representation with  $S \neq 0$ ).

REMARK 3.2. Let  $(f_0, f) : \gamma \xrightarrow{\sim} \gamma'$  be an isomorphism of Clifford representations. Then  $\text{Ad}(f)(W(\gamma)) = W(\gamma')$ . Since  $\text{Ad}(f)$  is a unital  $\mathbb{R}$ -algebra isomorphism from  $\text{End}_{\mathbb{R}}(S)$  to  $\text{End}_{\mathbb{R}}(S')$ , this implies  $\text{Ad}(f)(C(\gamma)) = C(\gamma')$  and:

$$\text{Ad}(f)(\mathbb{S}(\gamma)) = \mathbb{S}(\gamma') \text{ , } \text{Ad}(f)(A(\gamma)) = A(\gamma') \text{ , } \text{Ad}(f)(\mathbb{T}(\gamma)) = \mathbb{T}(\gamma') \text{ .}$$

Hence restriction gives bijective maps:

$$\begin{aligned} \text{Ad}_s(f) &\stackrel{\text{def.}}{=} \text{Ad}(f)|_{\mathbb{S}(\gamma)} : \mathbb{S}(\gamma) \rightarrow \mathbb{S}(\gamma') \\ \text{Ad}_A(f) &\stackrel{\text{def.}}{=} \text{Ad}(f)|_{A(\gamma)} : A(\gamma) \rightarrow A(\gamma') \\ \text{Ad}_{\mathbb{T}}(f) &\stackrel{\text{def.}}{=} \text{Ad}(f)|_{\mathbb{T}(\gamma)} : \mathbb{T}(\gamma) \rightarrow \mathbb{T}(\gamma') \text{ .} \end{aligned} \tag{3.8}$$

Notice that  $\text{Ad}_s(f)$  and  $\text{Ad}_{\mathbb{T}}(f)$  are unital isomorphisms of  $\mathbb{R}$ -algebras while  $(\text{Ad}_s(f), \text{Ad}_A(f))$  is a twisted isomorphism of left modules (see Appendix C) from the  $\mathbb{S}(\gamma)$ -module  $A(\gamma)$  to the  $\mathbb{S}(\gamma')$ -module  $A(\gamma')$ . Since these maps behave well under composition, this gives functors  $\mathbb{S} : \text{ClRep}^\times \rightarrow \text{Alg}^\times$ ,  $A : \text{ClRep}^\times \rightarrow \text{TwMod}^\times$  (where  $\text{TwMod}$  is the category of left modules over unital associative  $\mathbb{R}$ -algebras and twisted morphisms between such) and  $\mathbb{T} : \text{ClRep}^\times \rightarrow \text{Alg}^\times$  which respectively associate to  $\gamma$  the objects  $\mathbb{S}(\gamma)$ ,  $A(\gamma)$  and  $\mathbb{T}(\gamma)$  and to  $f$  the morphisms  $\mathbb{S}(f) = \text{Ad}_s(f)$ ,  $A(f) = (\text{Ad}_s(f), \text{Ad}_A(f))$  and  $\mathbb{T}(f) = \text{Ad}_{\mathbb{T}}(f)$ . When  $\gamma' = \gamma$  and  $(f_0, f) \in \text{Aut}(\gamma)$ , we have  $\text{Ad}_s(f) \in \text{Aut}_{\text{Alg}}(\mathbb{S}(\gamma))$ ,  $\text{Ad}_A(f) \in \text{Aut}_{\mathbb{S}(\gamma)}^{\text{tw}}(A(\gamma))$  and  $\text{Ad}_{\mathbb{T}}(f) \in \text{Aut}_{\text{Alg}}(\mathbb{T}(\gamma))$ . The fact that  $\text{Ad}_{\mathbb{T}}(f)$  is an algebra automorphism implies  $\mathfrak{p}_\gamma \circ (\text{Ad}_A(f) \otimes_{\mathbb{R}} \text{Ad}_A(f)) = \text{Ad}_s(f) \circ \mathfrak{p}_\gamma$  and hence  $\text{Ad}_A(f) \in \text{Aut}_{\mathbb{S}(\gamma)}^{\text{tw}}(A(\gamma), \mathfrak{p}_\gamma)$  (see Appendix C for notation).

**3.4. The pinor volume element and the volume grading of  $\text{End}_{\mathbb{R}}(S)$ .** Let us fix an orientation of  $V$  and let  $\nu$  denote the corresponding Clifford volume element.

DEFINITION 3.8. *The element  $\omega_\gamma \stackrel{\text{def.}}{=} \gamma(\nu) \in C(\gamma) \subset \text{End}_{\mathbb{R}}(S)$  is called the pinor volume element of  $\gamma$  determined by the chosen orientation of  $V$ .*

The pinor volume element  $\omega := \omega_\gamma$  satisfies:

$$\omega^2 = (-1)^{q+\frac{d(d-1)}{2}} \text{id}_S \text{ , } \omega w = (-1)^{d-1} w \omega \quad \forall w \in W \text{ .} \tag{3.9}$$

Thus  $\omega \in \mathbb{S}$  for odd  $d$  and  $\omega \in A$  for even  $d$ . The first relation implies that  $\text{Ad}(\omega)$  is an involutive automorphism of the algebra  $\text{End}_{\mathbb{R}}(S)$ :

$$\text{Ad}(\omega)^2 = \text{id}_{\text{End}_{\mathbb{R}}(S)}$$

and hence induces a  $\mathbb{Z}_2$ -grading  $\text{End}_{\mathbb{R}}(S) = \text{End}_{\mathbb{R}}^0(S) \oplus \text{End}_{\mathbb{R}}^1(S)$  of this algebra, where:

$$\begin{aligned} \text{End}_{\mathbb{R}}^0(S) &\stackrel{\text{def.}}{=} \ker(\text{Ad}(\omega) - \text{id}_{\text{End}_{\mathbb{R}}(S)}) = \{a \in \text{End}_{\mathbb{R}}(S) | a\omega = \omega a\} \\ \text{End}_{\mathbb{R}}^1(S) &\stackrel{\text{def.}}{=} \ker(\text{Ad}(\omega) + \text{id}_{\text{End}_{\mathbb{R}}(S)}) = \{a \in \text{End}_{\mathbb{R}}(S) | a\omega = -\omega a\} \text{ .} \end{aligned}$$

In particular,  $\text{End}_{\mathbb{R}}^0(S)$  is a unital subalgebra of  $\text{End}_{\mathbb{R}}(S)$ . Since  $\omega = \gamma(\nu)$ , we have:

$$\text{Ad}(\omega) \circ \gamma = \gamma \circ \text{Ad}(\nu) \quad ,$$

which gives:

**PROPOSITION 3.2.** *The morphism of  $\mathbb{R}$ -algebras  $\gamma : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S)$  is homogeneous of degree zero with respect to the  $\mathbb{Z}_2$ -gradings induced by  $\nu$  on  $\text{Cl}(V, h)$  and by  $\omega$  on  $\text{End}_{\mathbb{R}}(S)$ :*

$$\gamma(\text{Cl}^\kappa(V, h)) \subset \text{End}_{\mathbb{R}}^\kappa(S) \quad \forall \kappa \in \mathbb{Z}_2 \quad .$$

When  $d$  is even, we have  $\text{Cl}^0(V, h) = \text{Cl}_+(V, h)$  and  $\text{Cl}^1(V, h) = \text{Cl}_-(V, h)$  and the proposition gives  $C_+(\gamma) \stackrel{\text{def.}}{=} \gamma(\text{Cl}_+(V, h)) \subset \text{End}_{\mathbb{R}}^0(S)$  and  $C_-(\gamma) \stackrel{\text{def.}}{=} \gamma(\text{Cl}_-(V, h)) \subset \text{End}_{\mathbb{R}}^1(S)$ . In this case, we have  $C(\gamma) = C_+(\gamma) \oplus C_-(\gamma)$  and  $C(\gamma)$  is a homogeneous subalgebra of  $\text{End}_{\mathbb{R}}(S)$  with respect to the  $\mathbb{Z}_2$ -grading introduced above. When  $d$  is odd, we have  $\text{Cl}(V, h) = \text{Cl}^0(V, h)$  and the proposition gives  $C(\gamma) \subset \text{End}_{\mathbb{R}}^0(S)$ .

**3.5. Weakly faithful and rigid Clifford representations.**

**DEFINITION 3.9.** *A Clifford representation  $\gamma : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S)$  is called weakly faithful if the restriction  $\gamma_0 \stackrel{\text{def.}}{=} \gamma|_V : V \rightarrow \text{End}_{\mathbb{R}}(S)$  is an injective map. It is called rigid if  $\gamma_V = \text{id}_V$ .*

When  $\gamma$  is weakly faithful, we can use the injection  $\gamma|_V$  to identify  $V$  with the subspace  $W(\gamma) \subset \text{End}_{\mathbb{R}}(S)$ . Let  $\text{ClRep}_w$  (respectively  $\text{ClRep}_r$ ) denote the full subcategories of  $\text{ClRep}$  whose objects are the weakly faithful (respectively rigid) Clifford representations and  $\text{ClRep}_w^\times$  (respectively  $\text{ClRep}_r^\times$ ) denote the corresponding unit groupoids (which are full sub-groupoids of  $\text{ClRep}^\times$ ). Notice that any faithful or rigid Clifford representation is weakly faithful. When  $\gamma$  and  $\gamma'$  are weakly faithful and  $(f_0, f) : \gamma \rightarrow \gamma'$  is an *isomorphism* of Clifford representations, Proposition 3.1 shows that we have  $\text{Ad}(f)(\gamma(V)) = \gamma'(V')$  and that  $f_0$  is uniquely determined by  $f$  through the relation:

$$f_0 = (\gamma'|_{V'})^{-1} \circ \text{Ad}(f) \circ \gamma|_V \quad . \tag{3.10}$$

It is easy to see that the converse also holds, so we have:

**PROPOSITION 3.3.** *Assume that  $\gamma$  and  $\gamma'$  are weakly faithful. Then any isomorphism  $(f_0, f) : \gamma \rightarrow \gamma'$  is determined by the linear isomorphism  $f : S \rightarrow S'$ . Namely, we have  $\text{Ad}(f)(\gamma(V)) = \gamma'(V')$  and  $f_0$  is given by relation (3.10). Conversely, any linear isomorphism  $f : S \rightarrow S'$  which satisfies  $\text{Ad}(f)(\gamma(V)) = \gamma'(V')$  determines an isomorphism of quadratic spaces  $f_0 : (V, h) \rightarrow (V', h')$  through relation (3.10) and we have  $(f_0, f) \in \text{Hom}_{\text{ClRep}^\times}(\gamma, \gamma')$ .*

In view of this, we denote isomorphisms of weakly faithful Clifford representations only by  $f$  (since  $f$  determines  $f_0$  in this case) and we identify  $\text{Hom}_{\text{ClRep}_w^\times}(\gamma, \gamma')$  with a subset of the set  $\text{Isom}_{\mathbb{R}}(S, S')$  of linear isomorphisms from  $S$  to  $S'$ :

$$\begin{aligned} \text{Hom}_{\text{ClRep}_w^\times}(\gamma, \gamma') &\equiv \{f \in \text{Isom}_{\mathbb{R}}(S, S') \mid \text{Ad}(f)(\gamma(V)) = \gamma'(V')\} \\ &\text{for } \gamma, \gamma' \in \text{Ob}(\text{ClRep}_w) \quad . \end{aligned} \tag{3.11}$$

**REMARK 3.3.** One can show that the full inclusion functor  $\text{ClRep}_r^\times \hookrightarrow \text{ClRep}_w^\times$  is an equivalence of categories. This equivalence of categories can be used to show equivalence of our approach with the real version of the formalism of “spin spaces” which was used in [10] for the case of complex Clifford representations.

**4. Real Lipschitz groups and their elementary representations.** In this section, we introduce the real Lipschitz group of a weakly faithful real Clifford representation, which coincides with the automorphism group of the latter in the category ClRep introduced in the previous section. We also study certain elementary representations of real Lipschitz groups. The results of this section apply to any weakly faithful real Clifford representation  $\gamma$ , which need not be irreducible.

**4.1. The Lipschitz group of a weakly faithful real Clifford representation.** When  $\gamma : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S)$  is a weakly faithful real Clifford representation, restriction gives a linear isomorphism:

$$\gamma|_V : V \xrightarrow{\sim} W(\gamma)$$

which can be used to transport  $h$  to a symmetric and non-degenerate bilinear form  $g_\gamma : W(\gamma) \times W(\gamma) \rightarrow \mathbb{R}$ :

$$g_\gamma(w_1, w_2) \stackrel{\text{def.}}{=} h((\gamma|_V)^{-1}(w_1), (\gamma|_V)^{-1}(w_2)) \quad \forall w_1, w_2 \in W(\gamma) .$$

Thus  $(W(\gamma), g_\gamma)$  is a quadratic space and  $\gamma|_V : (V, h) \xrightarrow{\sim} (W(\gamma), g_\gamma)$  is an invertible isometry.

DEFINITION 4.1. *Let  $\gamma$  be a weakly faithful real Clifford representation. The group  $L(\gamma) \stackrel{\text{def.}}{=} \{a \in \text{Aut}_{\mathbb{R}}(S) | \text{Ad}(a)(W(\gamma)) \subset W(\gamma)\}$  is called the real Lipschitz group of  $\gamma$ . For simplicity, we will often denote  $W(\gamma)$  by  $W$ ,  $g_\gamma$  by  $g$ ,  $L(\gamma)$  by  $L$  etc.*

**4.2. The vector representation of the Lipschitz group.** For any  $a \in L$  and any  $w \in W$ , we have  $\text{Ad}(a)(w) \in W$  and  $\text{Ad}(a)(w)^2 = \text{Ad}(a)(w^2)$ . Since  $w^2 = g(w, w)\text{id}_S$  and  $\text{Ad}(a)(w)^2 = g(\text{Ad}(a)(w), \text{Ad}(a)(w))\text{id}_S$ , this implies  $\text{Ad}(a)|_W \in O(W, g)$ .

DEFINITION 4.2. *The group morphism  $\text{Ad}_0^\gamma : L(\gamma) \rightarrow O(W(\gamma), g_\gamma)$  given by:*

$$\text{Ad}_0^\gamma(a) \stackrel{\text{def.}}{=} \text{Ad}(a)|_W$$

*is called the vector representation of  $L(\gamma)$ .*

It is easy to see that the group  $\mathbb{S}(\gamma)^\times$  of units of  $\mathbb{S}(\gamma)$  coincides with the kernel of  $\text{Ad}_0^\gamma$ :

$$\ker(\text{Ad}_0^\gamma) = \mathbb{S}(\gamma)^\times \subset L . \tag{4.1}$$

PROPOSITION 4.1. *Let  $\gamma$  be a weakly faithful real Clifford representation. Then the Lipschitz group  $L(\gamma)$  is naturally isomorphic with the automorphism group  $\text{Aut}_{\text{ClRep}_w}(\gamma)$  of  $\gamma$  in the category  $\text{ClRep}_w$ . Therefore, the isomorphism class of  $L(\gamma)$  depends only on the isomorphism class of  $\gamma$  in the category  $\text{ClRep}_w$ .*

*Proof.* Any  $a \in L$  induces an invertible isometry  $a_0 \in O(V, h)$  through relation (3.10), namely:

$$a_0 = (\gamma|_V)^{-1} \circ \text{Ad}_0^\gamma(a) \circ (\gamma|_V) \in O(V, h) , \tag{4.2}$$

which implies:

$$\gamma \circ \text{Cl}(a_0) = \text{Ad}(a) \circ \gamma . \tag{4.3}$$

Thus  $(a_0, a)$  is the unique automorphism of  $\gamma$  in the category  $\text{ClRep}$  whose second component equals  $a$ . Conversely, we have  $a \in L(\gamma)$  for any  $(a_0, a) \in \text{Aut}_{\text{ClRep}_w}(\gamma) = \text{Aut}_{\text{ClRep}}(\gamma)$  (see Proposition 3.1) and  $a_0$  is determined by  $a$  through relation (4.2) (see Proposition 3.3). Hence the map  $F_1 : \text{Aut}_{\text{ClRep}}(\gamma) \rightarrow L(\gamma)$  given by  $F_1(a_0, a) = a$  is an isomorphism of groups which allows us to identify  $L(\gamma)$  with  $\text{Aut}_{\text{ClRep}}(\gamma)$ .  $\square$

For what follows, we fix a weakly faithful Clifford representation  $\gamma : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S)$  and set  $L := L(\gamma)$ ,  $C := C(\gamma)$ ,  $W := W(\gamma)$ ,  $g := g_\gamma$ ,  $\mathbb{S} := \mathbb{S}(\gamma)$ ,  $A := A(\gamma)$  and  $\text{Ad}_0 := \text{Ad}_0^\gamma$ . Fixing an orientation on  $V$ , we orient  $W$  such that  $\gamma|_V : V \xrightarrow{\sim} W$  is orientation-preserving and let  $\nu$  and  $\omega = \gamma(\nu)$  denote the corresponding Clifford volume and pinor volume element. Since the quadratic spaces  $(V, h)$  and  $(W, g)$  are isometric, we will sometimes identify them using the isometry  $\gamma|_V$ , in which case we view the vector representation of  $L$  as a group morphism  $\text{Ad}_0 : L \rightarrow \text{O}(V, h)$ .

REMARK 4.1. An element  $w \in W$  is invertible in  $\text{End}_{\mathbb{R}}(S)$  iff it is nondegenerate in the quadratic space  $(W, g)$ . In this case, its inverse  $w^{-1} = \frac{1}{g(w,w)}w$  also belongs to  $W$ . If  $w \in W$  is a unit vector, then so is  $w^{-1}$ .

**4.3. Volume grading of the Lipschitz group.** Let  $\det : \text{O}(W, g) \rightarrow \mathbb{G}_2$  be the determinant morphism and  $L = L^0 \sqcup L^1$  be the  $\mathbb{Z}_2$ -grading of  $L$  induced by the group morphism  $\det \circ \text{Ad}_0 : L \rightarrow \mathbb{G}_2$ :

$$\begin{aligned} L^0 &\stackrel{\text{def.}}{=} \{a \in L \mid \det(\text{Ad}_0(a)) = +1\} = \text{Ad}_0^{-1}(\text{SO}(W, g)) \quad , \\ L^1 &\stackrel{\text{def.}}{=} \{a \in L \mid \det(\text{Ad}_0(a)) = -1\} = \text{Ad}_0^{-1}(\text{O}_-(W, g)) \quad . \end{aligned}$$

DEFINITION 4.3. The subgroup  $L^0 \subset L$  is called the special Lipschitz group.

PROPOSITION 4.2. We have:

$$\text{Ad}(\omega)(a) = \det(\text{Ad}_0(a))a \quad , \quad \forall a \in L \tag{4.4}$$

and:

$$L^0 = L \cap \text{End}_{\mathbb{R}}^0(S) \quad , \quad L^1 = L \cap \text{End}_{\mathbb{R}}^1(S) \quad . \tag{4.5}$$

*Proof.* If  $e_1 \dots e_d$  is an oriented orthonormal basis of  $(W, g)$ , then:

$$\epsilon_1 \stackrel{\text{def.}}{=} (\gamma|_V)^{-1}(e_1), \dots, \epsilon_d \stackrel{\text{def.}}{=} (\gamma|_V)^{-1}(e_d) ,$$

is an oriented orthonormal basis of  $(V, h)$ . Thus  $\nu = \epsilon_1 \dots \epsilon_d$  and hence  $\omega = \gamma(\nu) = \epsilon_1 \dots \epsilon_d$ . For any  $a \in L$ , we have:

$$\omega' \stackrel{\text{def.}}{=} \text{Ad}(a)(\omega) = \text{Ad}_0(a)(e_1) \dots \text{Ad}_0(a)(e_d) = e'_1 \dots e'_d \quad ,$$

where  $e'_k \stackrel{\text{def.}}{=} \text{Ad}_0(a)(e_k) \in W$  form an orthonormal basis of  $(W, g)$  (because  $\text{Ad}_0(a) \in \text{O}(W, g)$ ). Thus  $\epsilon'_1 \stackrel{\text{def.}}{=} (\gamma|_V)^{-1}(e'_1), \dots, \epsilon'_d \stackrel{\text{def.}}{=} (\gamma|_V)^{-1}(e'_d)$  is an orthonormal basis of  $(V, h)$  which satisfies  $\epsilon'_k = a_0(\epsilon_k)$ , where  $a_0 \in \text{O}(V, h)$  is given by (4.2). Hence  $\nu' \stackrel{\text{def.}}{=} \epsilon'_1 \dots \epsilon'_d = \det(a_0)\epsilon_1 \dots \epsilon_d = \det(a_0)\nu$ , which implies  $\omega' = \gamma(\nu') = \det(a_0)\omega$ . Relation (4.2) gives  $\det \text{Ad}_0(a) = \det(a_0) \in \{-1, 1\}$  and hence  $\omega' = \det(\text{Ad}_0(a))\omega$ .

Thus  $\text{Ad}(a)(\omega) = \det(\text{Ad}_0(a))\omega$ , which is equivalent with (4.4) upon using the relation  $\text{Ad}_0(a) = \text{Ad}_0(a^{-1}) \forall a \in L$ . Hence:

$$\text{Ad}(\omega)(a) = (-1)^\alpha a \quad \forall a \in L^\alpha \quad (\alpha \in \mathbb{Z}_2) \quad . \tag{4.6}$$

and hence  $L^\alpha \subset \text{End}_{\mathbb{R}}^\alpha(S)$  for all  $\alpha \in \mathbb{Z}_2$ . Since  $L = L^0 \sqcup L^1$  and  $\text{End}_{\mathbb{R}}^0(S) \cap \text{End}_{\mathbb{R}}^1(S) = \{0\}$  while  $0 \notin L$ , this implies (4.5).  $\square$

REMARK 4.2. Notice that  $\omega \in L$  since  $\text{Ad}(\omega)(w) = (-1)^{d-1}w \in W$  for all  $w \in W$ . Since  $\text{Ad}(\omega)(\omega) = +\omega$ , we have  $\omega \in \text{End}_{\mathbb{R}}^0(S)$  and relations (4.5) imply  $\omega \in L^0$ .

PROPOSITION 4.3. *Let  $w \in W$  be a non-degenerate vector. Then  $w \in L$  and  $\text{Ad}_0(w)$  equals minus the orthogonal reflection of  $(W, g)$  in the hyperplane orthogonal to  $w$ :*

$$\text{Ad}_0(w) = -R_w \tag{4.7}$$

*Proof.* Follows by direct computation using the relations  $w^{-1} = \frac{1}{g(w,w)}w$  and  $wx + xw = 2g(x, w)$  for  $x \in W$  as well as relation (1.1).  $\square$

REMARK 4.3. Let  $w \in W$  be non-degenerate. Since  $\text{Ad}(\omega)(w) = (-1)^{d-1}w$ , relations (4.5) give:

1. When  $d$  is even, then  $w \in L^1$
2. When  $d$  is odd, then  $w \in L^0$ .

#### 4.4. Image subgroups of the Lipschitz group.

PROPOSITION 4.4. *We have  $\gamma(G^e(V, h)) \subset L$ , where  $G^e(V, h)$  is the extended Clifford group.*

*Proof.* Since  $\gamma$  is a unital morphism of  $\mathbb{R}$ -algebras, we have  $\gamma(\text{Cl}(V, h)^\times) \subset C^\times \subset \text{Aut}_{\mathbb{R}}(S)$ . Moreover, we have:

$$\text{Ad}(\gamma(x))(\gamma(y)) = \gamma(\text{Ad}^{\text{Cl}}(x)(y)) \quad \forall x \in \text{Cl}(V, h)^\times \quad \text{and} \quad y \in \text{Cl}(V, h) \quad , \tag{4.8}$$

which also reads:

$$R_\gamma \circ \text{Ad} \circ \gamma = L_\gamma \circ \text{Ad}^{\text{Cl}} \quad .$$

Since  $\gamma|_V : V \xrightarrow{\sim} W$  is a bijection, any element  $w \in W$  can be written as  $w = \gamma(v)$  with  $v = \gamma^{-1}(w) \in V$ . Applying (4.8) to  $y = v$  gives:

$$\text{Ad}(\gamma(x))(w) = \gamma(\text{Ad}^{\text{Cl}}(x)(v)) \quad \forall x \in \text{Cl}(V, h)^\times \quad .$$

When  $x \in G^e(V, h)$ , we have  $\text{Ad}^{\text{Cl}}(x)(v) \in V$  and the relation above gives  $\text{Ad}(\gamma(x))(w) \in \gamma(V) = W$ , which implies  $\gamma(x) \in L$ .  $\square$

DEFINITION 4.4. *The image extended Clifford group of  $\gamma$  is the subgroup  $\gamma(G^e(V, h)) \subset L$ . The image Clifford group of  $\gamma$  is the subgroup  $\gamma(G(V, h)) \subset \gamma(G^e(V, h))$ . The image pin group of  $\gamma$  is the subgroup  $\gamma(\text{Pin}(V, h)) \subset \gamma(G(V, h))$ . The image spin group of  $\gamma$  is the subgroup  $\gamma(\text{Spin}(V, h)) \subset \gamma(\text{Pin}(V, h))$ .*

Recall that  $G(V, h)$  is generated by the non-degenerate vectors of  $(V, h)$  while  $\text{Pin}(V, h)$  is generated by the unit vectors of  $(V, h)$ . Since  $\gamma|_V$  is an invertible isometry from  $(V, h)$  to  $(W, g)$ , we have:

PROPOSITION 4.5. *The image Clifford group  $\gamma(G(V, h))$  coincides with the subgroup of  $\text{Aut}_{\mathbb{R}}(S)$  generated by all non-degenerate vectors of  $(W, g)$ , while the image pin group  $\gamma(\text{Pin}(V, h))$  coincides with the subgroup of  $\text{Aut}_{\mathbb{R}}(S)$  generated by all unit vectors of  $(W, g)$*

The  $\mathbb{Z}_2$ -grading of  $L$  induces  $\mathbb{Z}_2$ -gradings on all image subgroups defined above. For example, we have  $\gamma(G^e(V, h))^\kappa = \gamma(G^e(V, h)) \cap L^\kappa$ ,  $\gamma(\text{Pin}(V, h))^\kappa = \gamma(\text{Pin}(V, h)) \cap L^\kappa$  for  $\kappa \in \mathbb{Z}_2$ . Relation (4.5) implies:

$$\begin{aligned} \gamma(G^e(V, h))^\kappa &= \gamma(G^e(V, h)) \cap \text{End}_{\mathbb{R}}^\kappa(S) \quad , \\ \gamma(G(V, h))^\kappa &= \gamma(G(V, h)) \cap \text{End}_{\mathbb{R}}^\kappa(S) \quad , \\ \gamma(\text{Pin}(V, h))^\kappa &= \gamma(\text{Pin}(V, h)) \cap \text{End}_{\mathbb{R}}^\kappa(S) \quad . \end{aligned}$$

When  $d$  is even, the gradings of  $\gamma(G(V, h))$  and  $\gamma(\text{Pin}(V, h))$  coincide with those induced by the canonical  $\mathbb{Z}_2$ -grading of  $\text{Cl}(V, h)$ . When  $d$  is odd, we have:

$$\gamma(G(V, h)) = \gamma(G(V, h))^0 \subset L^0 \quad , \quad \gamma(\text{Pin}(V, h)) = \gamma(\text{Pin}(V, h))^0 \subset L^0 \quad .$$

REMARK 4.4. In general,  $L^1$  can be non-empty even when  $d$  is odd (since  $L$  need not be generated by invertible elements from  $W$ ). We will see later that this indeed happens for certain irreducible Clifford representations. On the other hand,  $G^e(V, h)$  is generated by  $Z(V, h)^\times$  and  $G(V, h)$  and we have  $\gamma(Z(V, h)^\times) \subset \mathbb{S}^\times \subset L^0$ . This implies  $\gamma(G^e(V, h)) \subset L^0$  when  $d$  is odd. When  $d$  is even, we have  $G^e(V, h) = G(V, h)$  and  $\gamma(G^e(V, h)) = \gamma(G(V, h))$ .

**4.5. Twisting elements and twisted image pin groups.**

DEFINITION 4.5. *We say that  $\gamma$  admits twisting elements if the intersection  $A(\gamma) \cap \text{Aut}_{\mathbb{R}}(S)$  is non-empty. In this case, a twisting element of  $\gamma$  is an invertible element of the anticommutant subspace  $A(\gamma)$ , i.e. an element of the set  $A(\gamma) \cap \text{Aut}_{\mathbb{R}}(S)$ . A twisting element  $\mu \in A(\gamma) \cap \text{Aut}_{\mathbb{R}}(S)$  is called nondegenerate if the endomorphism  $\text{id}_S + \mu \in \text{End}_{\mathbb{R}}(S)$  is invertible and special if  $\mu^2 = -\text{id}_S$ .*

PROPOSITION 4.6. *Any special twisting element is non-degenerate.*

*Proof.* If  $\mu$  is a special twisting element, then  $\frac{1}{2}(\text{id}_S - \mu)(\text{id}_S + \mu) = \text{id}_S$  and hence  $\text{id}_S + \mu$  is invertible with inverse  $\frac{1}{2}(\text{id}_S - \mu)$ .  $\square$

For any twisting element  $\mu$  and any  $w \in W$ , we have  $\text{Ad}(\mu)(w) = -w$ . Thus twisting elements form a subset of  $L$ . This set is invariant under multiplication with elements of  $\mathbb{R}^\times$ . The subset of special twisting elements is invariant under changes of sign ( $\mu \rightarrow -\mu$ ).

DEFINITION 4.6. *Let  $\mu$  be a twisting element. The  $\mu$ -twisted image pin group of  $\gamma$  is the sub-group  $\text{Pin}_\mu(\gamma) \subset \text{Aut}_{\mathbb{R}}(S)$  generated by the elements  $\mu w$ , where  $w$  runs over the unit vectors of  $(W, g)$ .*

Any element of  $\text{Pin}_\mu(\gamma)$  can be brought to the form  $(-1)^{\frac{k(k-1)}{2}} \mu^k w_1 \dots w_k$ , where  $w_1, \dots, w_k$  are unit vectors of  $(W, g)$ , but this presentation need not be unique.

PROPOSITION 4.7. *Let  $\mu$  be a special twisting element. Then the  $\mu$ -twisted image pin group is isomorphic with the image pin group, namely:*

$$\text{Ad}(\text{id}_S + \mu)(\gamma(\text{Pin}(V, h))) = \text{Pin}_\mu(\gamma) \tag{4.9}$$

and we have:

$$\text{Ad}(a)(W) = W \quad \forall \quad a \in \text{Pin}_\mu(\gamma) \quad . \tag{4.10}$$

Moreover, the isomorphism of groups  $\varphi_\mu \stackrel{\text{def.}}{=} \text{Ad}(\text{id}_S + \mu)|_{\gamma(\text{Pin}(V,h))} : \gamma(\text{Pin}(V,h)) \xrightarrow{\sim} \text{Pin}_\mu(\gamma)$  satisfies:

$$\text{Ad}_W \circ \varphi_\mu \circ \gamma|_{\text{Pin}(V,h)} = \text{Ad}(\gamma|_V) \circ \widetilde{\text{Ad}}_0 \quad , \tag{4.11}$$

where the group morphism  $\text{Ad}_W : \text{Pin}_\mu(\gamma) \rightarrow \text{O}(W,g)$  is defined through  $\text{Ad}_W(a) = \text{Ad}(a)|_W$  for all  $a \in \text{Pin}_\mu(\gamma)$  and  $\widetilde{\text{Ad}}_0 : \text{Pin}(V,h) \rightarrow \text{O}(V,h)$  is the twisted vector representation of  $\text{Pin}(V,h)$ . Hence the representation  $\text{Ad}_W \circ \varphi_\mu \circ \gamma|_{\text{Pin}(V,h)}$  of  $\text{Pin}(V,h)$  is equivalent with the twisted vector representation.

REMARK 4.5. In the proposition,  $\gamma|_V$  is viewed as an invertible isometry from  $(V,h)$  to  $(W,g)$ , so  $\text{Ad}(\gamma|_V)$  is a unital isomorphism of algebras from  $\text{End}_{\mathbb{R}}(V)$  to  $\text{End}_{\mathbb{R}}(W)$  which restricts to an isomorphism of groups from  $\text{O}(V,h)$  to  $\text{O}(W,g)$ .

*Proof.* Let  $\mu$  be a special twisting element. Then the map  $V \ni v \rightarrow \mu\gamma(v) \in \text{End}_{\mathbb{R}}(S)$  satisfies  $(\mu\gamma(v))^2 = \gamma(v)^2 = h(v,v)\text{id}_S$  and hence induces a unital morphism of algebras  $\gamma_\mu : \text{Cl}(V,h) \rightarrow \text{End}_{\mathbb{R}}(S)$  such that  $\gamma_\mu(v) = \mu\gamma(v)$  for all  $v \in V$ . The identity  $\mu\gamma(v) = (\text{id}_S + \mu)\gamma(v)(\text{id}_S + \mu)^{-1} = \text{Ad}(\text{id}_S + \mu)(\gamma(v))$  implies that this representation is equivalent with  $\gamma$ , namely:

$$\gamma_\mu = \text{Ad}(\text{id}_S + \mu) \circ \gamma \quad . \tag{4.12}$$

Since  $\gamma_\mu(\text{Pin}(V,h)) = \text{Pin}_\mu(\gamma)$ , relation (4.12) implies (4.9). For every  $w \in W$  and any unit vector  $u$  of  $(W,g)$ , we have:

$$\text{Ad}(\mu u)(w) = \mu u w (\mu u)^{-1} = \mu u w u^{-1} \mu^{-1} = -u w u^{-1} = R_u \quad . \tag{4.13}$$

This implies  $\text{Ad}(\mu u)(W) \subset W$  and shows that (4.10) holds. Taking  $u = \gamma(v)$  and  $w = \gamma(v')$  with  $v = (\gamma|_V)^{-1}(u) \in V$  a unit vector of  $(V,h)$  and  $v' = (\gamma|_V)^{-1}(w) \in V$  in (4.13) gives:

$$(\text{Ad} \circ \varphi_\mu \circ \gamma)(v)(w) = \text{Ad}(\mu\gamma(v))(\gamma(v')) = \gamma|_V(\widetilde{\text{Ad}}_0(v)(v')) \quad ,$$

which gives (4.11).  $\square$

**4.6. Surjectivity conditions for the vector representation.**

PROPOSITION 4.8. *Let  $\mu$  be any twisting element and  $w \in W$  be a non-degenerate vector. Then  $\mu w \in \text{L}$  and  $\text{Ad}_0(\mu w)$  equals the  $g$ -orthogonal reflection of  $W$  in the hyperplane orthogonal to  $w$ :*

$$\text{Ad}_0(\mu w) = +R_w \quad .$$

In particular, we have  $\text{Pin}_\mu(\gamma) \subset \text{L}$ .

*Proof.* It suffices to consider the case when  $w$  is a unit vector, so we can assume  $g(w,w) = \epsilon \in \{-1,1\}$ . Then  $w^2 = g(w,w)\text{id}_S = \epsilon \text{id}_S$ , so  $w$  is invertible and  $w^{-1} = \epsilon w$ . For any  $x \in W$ , we compute:

$$\text{Ad}(\mu w)(x) = \text{Ad}(\mu)(w x w^{-1}) = \epsilon \text{Ad}(\mu)(w x w) = -\epsilon w x w = -\text{Ad}(w)(x) = +R_w \quad ,$$

where we used (4.7).  $\square$

THEOREM 4.9. *The following statements hold:*



1. The image of the vector representation of  $L^0$  equals the special orthogonal group of  $(W, g)$ , so we have a short exact sequence:

$$1 \longrightarrow \mathbb{S}^\times \hookrightarrow L^0 \xrightarrow{\text{Ad}_0} \text{SO}(W, g) \longrightarrow 1 \quad . \quad (4.14)$$

2. Suppose that  $d$  is even or that  $\gamma$  admits a twisting element. Then the vector representation of  $L$  is surjective and we have a short exact sequence:

$$1 \longrightarrow \mathbb{S}^\times \hookrightarrow L \xrightarrow{\text{Ad}_0} \text{O}(W, g) \longrightarrow 1 \quad . \quad (4.15)$$

*Proof.* Since  $L$  contains all non-degenerate vectors of  $(W, g)$ , Proposition 4.3 shows that the group  $\text{Ad}_0(L)$  contains all minus reflections  $\text{Ad}_0(w) = -R_w$  of  $\text{O}(W, g)$ . By the definition of the grading of  $L$ , we have  $\text{Ad}_0(L^0) \subset \text{SO}(W, g)$  and  $\text{Ad}_0(L^1) \subset \text{O}_-(W, g)$ , where  $\text{O}_-(W, g) \subset \text{O}(W, g)$  denotes the set of improper orthogonal transformations. Thus  $\text{Ad}_0(L^0)$  and  $\text{Ad}_0(L^1)$  are disjoint. We distinguish the cases:

1. When  $d$  is even, the minus reflections have determinant  $-1$  and they generate  $\text{O}(W, g)$ , thus  $\text{Ad}_0(L) = \text{O}(W, g)$ , which establishes part of the second statement.
2. When  $d$  is odd, minus reflections have determinant  $+1$  and generate  $\text{SO}(W, g)$ , thus  $\text{SO}(W, g) \subset \text{Ad}_0(L)$ . Since  $\text{Ad}_0(L^0) \subset \text{SO}(W, g)$  and  $\text{Ad}_0(L^1) \subset \text{O}_-(W, g)$  are disjoint, we must have  $\text{SO}(W, g) = \text{Ad}_0(L^0)$ , which establishes the first statement.

Suppose now that  $d$  is arbitrary but that  $\gamma$  admits a twisting element  $\mu$ . Then the previous proposition shows that  $L$  contains  $\text{Pin}_\mu(\gamma)$  and that  $\text{Ad}_0(L)$  contains all reflections  $\text{Ad}(\mu w) = R_w$  of  $(W, g)$ , which generate  $\text{O}(W, g)$  by the Cartan-Dieudonne theorem. This completes the proof of the second statement.  $\square$

#### 4.7. The Schur representation.

PROPOSITION-DEFINITION 4.10. For any  $a \in L$ , we have  $\text{Ad}(a)(\mathbb{S}) = \mathbb{S}$ . Thus  $\text{Ad}$  induces a Schur representation:

$$\text{Ad}_s : L \rightarrow \text{Aut}_{\text{Alg}}(\mathbb{S}) \quad , \quad \text{Ad}_s(a) \stackrel{\text{def.}}{=} \text{Ad}(a)|_{\mathbb{S}} \quad (a \in L)$$

through unital algebra automorphisms of  $\mathbb{S}$ . Furthermore, we have:

$$\ker \text{Ad}_s = L \cap \text{End}_{\mathbb{S}}(S) \quad (4.16)$$

*Proof.* Given  $a \in L$ , any vector  $w \in W$  can be written as  $w = \text{Ad}(a)(w')$  for some  $w' \in W$ , namely  $w' = \text{Ad}(a^{-1})(w)$ . For any  $s \in \mathbb{S}$ , we have:

$$\begin{aligned} \text{Ad}(a)(s)w &= \text{Ad}(a)(s)\text{Ad}(a)(w') = \text{Ad}(a)(sw') = \text{Ad}(a)(w's) \\ &= \text{Ad}(a)(w')\text{Ad}(a)(s) = w\text{Ad}(a)(s) \end{aligned} \quad (4.17)$$

where we used the fact that  $s \in \mathbb{S}$  commutes with  $w' \in W$ . This shows that  $\text{Ad}(a)(s)$  commutes with any  $w \in W$  and hence that  $\text{Ad}(a)(s)$  belongs to  $\mathbb{S}$ . Thus  $\text{Ad}(a)(\mathbb{S}) \subset \mathbb{S}$ . Since  $\text{Ad}(a)$  is  $\mathbb{R}$ -linear and bijective while  $\mathbb{S}$  is finite-dimensional over  $\mathbb{R}$ , we in fact have  $\text{Ad}(a)(\mathbb{S}) = \mathbb{S}$ . The fact that  $\text{Ad}_s(a)$  is a unital morphism of algebras is obvious, as is the remaining statement.  $\square$

**4.8. The anticommutant representation.**

PROPOSITION 4.11. *The anticommutant subspace is invariant under the vector representation of L:*

$$\text{Ad}(a)(A) = A \quad , \quad \forall a \in L$$

and hence Ad induces a linear representation  $\text{Ad}_A : L \rightarrow \text{Aut}_{\mathbb{R}}(A)$ ,  $\text{Ad}_A(a) \stackrel{\text{def.}}{=} \text{Ad}(a)|_A$  in the space A. Moreover,  $\text{Ad}_A$  is a representation of L through  $\text{Ad}_s$ -twisted  $\mathbb{S}$ -module automorphisms which are twisted-orthogonal with respect to the Schur pairing  $\mathfrak{p}$ , so the following relations hold for all  $a \in L$ :

$$\text{Ad}_A(a)(sx) = \text{Ad}_s(a)(s)\text{Ad}_A(a)(x) \quad \forall s \in \mathbb{S} \quad \text{and} \quad x \in A \tag{4.18}$$

and:

$$\mathfrak{p}(\text{Ad}_A(a)(x_1), \text{Ad}_A(a)(x_2)) = \text{Ad}_s(a)(\mathfrak{p}(x_1, x_2)) \quad , \quad \forall x_1, x_2 \in A \quad . \tag{4.19}$$

We refer the reader to Appendix C for the notion of twisted module automorphism and for the notation used in the proposition.

*Proof.* For any  $a \in L$ ,  $x \in A$  and  $w \in W$ , we have:

$$\text{Ad}(a)(x)w = ax\text{Ad}(a^{-1})(w)a^{-1} = -a\text{Ad}(a^{-1})(w)xa^{-1} = -w\text{Ad}(a)(x) \quad , \tag{4.20}$$

where in the middle equality we used the fact that  $a^{-1} \in L$  (since L is a group), which implies that  $\text{Ad}(a^{-1})(w)$  belongs to  $W$  and hence that it anticommutes with  $x \in A$  (by the definition of A). Relation (4.20) shows that  $\text{Ad}(a)(x)$  anticommutes with  $w$  for any  $w \in W$  and hence  $\text{Ad}(a)(x)$  belongs to  $A$  for any  $x \in A$ , thus  $\text{Ad}(a)(A) = A$  for any  $a \in L$ . Relation (4.18) is obvious. For any  $x_1, x_2 \in A$ , we have:

$$\mathfrak{p}(\text{Ad}(a)(x_1), \text{Ad}(a)(x_2)) = \frac{1}{2} [\text{Ad}(a)(x_1x_2) + \text{Ad}(a)(x_2x_1)] = \text{Ad}(a)(\mathfrak{p}(x_1, x_2)) \quad ,$$

which gives (4.19) since  $\mathfrak{p}(x_1, x_2) \in \mathbb{S}$ .  $\square$

DEFINITION 4.7. *The group morphism:*

$$\text{Ad}_A : L \rightarrow \text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p})$$

is called the anticommutant representation of L.

**4.9. Adapted pairings and Lipschitz norms.**

DEFINITION 4.8. *A non-degenerate  $\mathbb{R}$ -bilinear pairing  $\mathcal{B} : S \times S \rightarrow \mathbb{R}$  is called adapted to  $\gamma$  if it has the following properties:*

(a)  $\mathcal{B}$  is symmetric or skew-symmetric, that is:

$$\mathcal{B}(\xi, \xi') = \sigma_{\mathcal{B}}\mathcal{B}(\xi', \xi) \quad \forall \xi, \xi' \in S ,$$

where  $\sigma_{\mathcal{B}} \in \{-1, 1\}$  is called the symmetry of  $\mathcal{B}$ .

(b)  $w^t = \epsilon_{\mathcal{B}}w$  for all  $w \in W$ , where  ${}^t : \text{End}_{\mathbb{R}}(S) \rightarrow \text{End}_{\mathbb{R}}(S)$  denotes the  $\mathcal{B}$ -transpose and  $\epsilon_{\mathcal{B}} \in \{-1, 1\}$  is called the type of  $\mathcal{B}$ .

REMARK 4.6. Since an adapted pairing is either symmetric or skew-symmetric, the  $\mathcal{B}$ -transpose is an involutive unital  $\mathbb{R}$ -algebra anti-automorphism of  $\text{End}_{\mathbb{R}}(S)$ . Notice that an admissible bilinear pairing (in the sense of [21, 22]) of an irreducible Clifford representation is adapted; in fact, an admissible pairing is an adapted pairing which has a definite “isotropy” (see op. cit.).

DEFINITION 4.9. We define  $\text{O}(S, \mathcal{B})$  to be the automorphism group of  $(S, \mathcal{B})$ , namely:

$$\text{O}(S, \mathcal{B}) = \{a \in \text{End}_{\mathbb{R}}(S) \mid \mathcal{B}(ax, ay) = \mathcal{B}(x, y)\} , \tag{4.21}$$

for all  $x, y \in S$ . It consists of all  $\mathcal{B}$ -orthogonal invertible linear operators acting in  $S$ :

$$\text{O}(S, \mathcal{B}) = \{a \in \text{End}_{\mathbb{R}}(S) \mid a^t \circ a = \text{id}_S\} .$$

Let  $\mathcal{B}$  be an adapted pairing on  $S$ .

DEFINITION 4.10. The Lipschitz norm determined by  $\mathcal{B}$  is the map  $\mathcal{N}_{\mathcal{B}} : \text{End}_{\mathbb{R}}(S) \rightarrow \text{End}_{\mathbb{R}}(S)$  defined through:

$$\mathcal{N}_{\mathcal{B}}(a) \stackrel{\text{def.}}{=} a^t \circ a \quad (a \in \text{End}_{\mathbb{R}}(S)) .$$

Notice that  $\mathcal{N}_{\mathcal{B}}(a)^t = \mathcal{N}_{\mathcal{B}}(a)$ , so the image of the Lipschitz norm consists of  $\mathcal{B}$ -symmetric linear operators. We have  $\mathcal{N}_{\mathcal{B}}(w) = \epsilon_{\mathcal{B}} w^2 = \epsilon_{\mathcal{B}} g(w, w)\text{id}_S$  for all  $w \in W$ . Notice that an operator  $a \in \text{End}_{\mathbb{R}}(S)$  is  $\mathcal{B}$ -orthogonal if and only if  $\mathcal{N}_{\mathcal{B}}(a) = \text{id}_S$ ; in particular, the intersection:

$$W \cap \text{O}(S, \mathcal{B}) = \{w \in W \mid g(w, w) = \epsilon_{\mathcal{B}}\}$$

coincides with the set of unit vectors of signature equal to  $\epsilon_{\mathcal{B}}$  and we have:

$$L \cap \text{O}(S, \mathcal{B}) = \{a \in L \mid \mathcal{N}_{\mathcal{B}}(a) = \text{id}_S\} . \tag{4.22}$$

However, the restriction  $\mathcal{N}_{\mathcal{B}}|_L$  does not generally give a group morphism, because the elements of  $\mathcal{N}_{\mathcal{B}}(L)$  need not commute with those of  $L$  or  $L^t$ . As we shall see below, the situation is somewhat better for the image extended Clifford group  $\gamma(\text{G}^e(V, h))$ .

DEFINITION 4.11. The modified reversion determined by  $\mathcal{B}$  is the unital anti-automorphism of  $\text{Cl}(V, h)$  given by:

$$\tau_{\mathcal{B}}(x) = \tau \circ \pi^{\frac{1-\epsilon_{\mathcal{B}}}{2}} = \begin{cases} \tau & \text{if } \epsilon_{\mathcal{B}} = +1 \\ \tilde{\tau} = \tau \circ \pi & \text{if } \epsilon_{\mathcal{B}} = -1 \end{cases} .$$

The modified Clifford norm determined by  $\mathcal{B}$  is the map  $N_{\mathcal{B}} : \text{Cl}(V, h) \rightarrow \mathbb{R}$  defined through:

$$N_{\mathcal{B}}(x) \stackrel{\text{def.}}{=} \tau_{\mathcal{B}}(x)x = \begin{cases} N & \text{if } \epsilon_{\mathcal{B}} = +1 \\ \tilde{N} & \text{if } \epsilon_{\mathcal{B}} = -1 \end{cases} \quad (x \in \text{Cl}(V, h)) .$$

Notice that  $\tau_{\mathcal{B}}(v) = \epsilon_{\mathcal{B}}v$  for all  $v \in V$ . For any  $x \in \text{Cl}(V, h)$ , we have:

$$\gamma(x)^t = \gamma(\tau_{\mathcal{B}}(x)) \quad ,$$

which implies:

$$\mathcal{N}_{\mathcal{B}} \circ \gamma = \gamma \circ N_{\mathcal{B}} \quad . \tag{4.23}$$

In particular, we have  $\mathcal{N}_{\mathcal{B}}(C(\gamma)) \subset C(\gamma)$ .

PROPOSITION 4.12. *We have  $\mathcal{N}_{\mathcal{B}}(\gamma(G(V, h))) \subset \mathbb{R}^{\times} \text{id}_S$  and:*

$$\mathcal{N}_{\mathcal{B}}(\gamma(G^e(V, h))) \subset \gamma(N_{\mathcal{B}}(G^e(V, h))) \subset Z(\mathbb{S}) \cap \mathbb{S}^{\times} = Z(\mathbb{S})^{\times} \quad . \tag{4.24}$$

*In particular, we have  $\mathcal{N}_{\mathcal{B}}(\gamma(G^e(V, h))) \subset \mathbb{R}^{\times} \text{id}_S$  if  $N_{\mathcal{B}}$  coincides with the improved Clifford norm.*

*Proof.* Since  $N(G(V, h)) \subset \mathbb{R}^{\times}$  and  $\tilde{N}(G(V, h)) \subset \mathbb{R}^{\times}$ , we have  $N_{\mathcal{B}}(G(V, h)) \subset \mathbb{R}^{\times}$  and (4.23) gives  $\mathcal{N}_{\mathcal{B}}(\gamma(G(V, h))) \subset \mathbb{R}^{\times} \text{id}_S$ . The first inclusion in (4.24) also follows from (4.23). Since  $N(G^e(V, h)) \subset Z(V, h)^{\times}$  and  $\tilde{N}(G^e(V, h)) \subset Z(V, h)^{\times}$ , we have  $N_{\mathcal{B}}(G^e(V, h)) \subset Z(V, h)^{\times}$ . Since  $\gamma(Z(V, h)) \subset Z(\mathbb{S})$ , we obtain the second inclusion of (4.24).  $\square$

Since  $N$  and  $\tilde{N}$  are equal on  $\text{Spin}(V, h)$ , we have  $N_{\mathcal{B}} = N$  on  $\text{Spin}(V, h)$  and (4.23) gives:

$$\mathcal{N}_{\mathcal{B}}(\gamma(a)) = \begin{cases} +\text{id}_S & \text{if } a \in \text{Spin}^+(V, h) \\ -\text{id}_S & \text{if } a \in \text{Spin}^-(V, h) \end{cases}$$

In particular, we have  $\gamma(\text{Spin}^+(V, h)) \subset \text{O}(S, \mathcal{B})$  and hence any adapted pairing is invariant under the action of the subgroup  $\gamma(\text{Spin}^+(V, h)) \subset \text{L}$ . Notice that an adapted pairing *need not* be invariant under the  $\gamma$ -action of the full spin group  $\text{Spin}(V, h)$ . Also notice that the full subgroup  $\text{L} \cap \text{O}(S, \mathcal{B})$  consisting of those elements of  $\text{L}$  which preserve an adapted pairing  $\mathcal{B}$  can be strictly larger than  $\gamma(\text{Spin}^+(V, h))$ , even when  $\mathcal{B}$  is an admissible pairing of an irreducible Clifford representation.

REMARK 4.7. In general, the set  $\mathcal{N}_{\mathcal{B}}(\text{L})$  is larger than  $Z(\mathbb{S})^{\times}$ . The decomposition (3.5) implies:

$$Z(\mathbb{S}) = \bigoplus_{i=1}^n Z(\mathbb{S}_i) \text{id}_{U_i} \simeq_{\text{Alg}} \bigoplus_{i=1}^n Z(\mathbb{S}_i) \quad , \quad Z(\mathbb{S})^{\times} \simeq_{\text{GP}} \bigoplus_{i=1}^n Z(\mathbb{S}_i)^{\times} \quad ,$$

where  $\mathbb{S}_i \stackrel{\text{def.}}{=} \mathbb{S}(\gamma_i)$  are the Schur algebras of the inequivalent irreducible components of  $\gamma$ . We have:

$$Z(\mathbb{S}_i)^{\times} \simeq \begin{cases} \mathbb{R}^{\times} & , \mathbb{S}_i \simeq_{\text{Alg}} \mathbb{R} \\ \mathbb{C}^{\times} & , \mathbb{S}_i \simeq_{\text{Alg}} \mathbb{C} \\ \mathbb{R}^{\times} & , \mathbb{S}_i \simeq_{\text{Alg}} \mathbb{H} \end{cases} \quad .$$

**5. Lipschitz groups of irreducible real Clifford representations.** In this section, we study the Lipschitz groups of irreducible real Clifford representations (all of which turn out to be weakly faithful and to form a single unbased isomorphism class in every signature) as well as their elementary representations. In particular, we show that the reduced Lipschitz groups of such representations (which we call

*elementary reduced Lipschitz groups*) are isomorphic with the canonical spinor groups introduced in Section 2.5 and that their elementary representations agree with those of the canonical spinor groups. This shows, in particular, that the canonical spinor group of  $(V, h)$  arises naturally as the Lipschitz group of the unique unbased isomorphism class of the irreps of  $(V, h)$ , which is always weakly faithful. Notice that this provides a unifying perspective on various extended spinor groups arising in spin geometry, while also including the groups  $\text{Spin}^o(V, h)$ , new in the literature, and identifying the precise spinor group which is relevant when considering irreducible real Clifford representations in every dimension and signature. One should compare this with the traditional approach, where a spinor group is chosen a priori, without worrying about irreducibility of the corresponding Clifford representation. In our approach, irreducibility of the Clifford representation is the central feature of interest.

**5.1. Basics.**

DEFINITION 5.1. A real pin representation is an irreducible finite-dimensional real Clifford representation  $\gamma : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S)$ , where  $V \neq 0$  and  $S \neq 0$ .

A real pin representation  $\gamma : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S)$  is faithful iff  $\text{Cl}(V, h)$  is simple as an associative  $\mathbb{R}$ -algebra, which happens when  $p - q \not\equiv_8 1, 5$  (the *simple case*). The pinor volume element  $\omega = \gamma(\nu)$  is proportional to  $\text{id}_S$  iff we are in the *non-simple case*  $p - q \equiv_8 1, 5$ . In the simple case, all pin representations of  $\text{Cl}(V, h)$  are equivalent. In the non-simple case,  $\text{Cl}(V, h)$  admits two inequivalent irreducible representations, which can be realized in the same space  $S$ . In each of these irreps, the Clifford volume element  $\nu \in \text{Cl}(V, h)$  defined by a given orientation of  $V$  satisfies:

$$\omega \stackrel{\text{def.}}{=} \gamma(\nu) = \epsilon_{\gamma} \text{id}_S \quad ,$$

where  $\epsilon_{\gamma} \in \{-1, 1\}$  is a sign factor called the *signature* of the irrep  $\gamma$ . The two irreps are distinguished by the value of  $\epsilon_{\gamma}$  and we denote them by  $\gamma_{\pm} : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S)$  (where  $\epsilon_{\gamma_{\pm}} = \pm 1$ ). We have:

$$\gamma_+ = \gamma_- \circ \pi \quad , \tag{5.1}$$

where  $\pi : \text{Cl}(V, h) \rightarrow \text{Cl}(V, h)$  is the parity automorphism, which satisfies  $\pi(\nu) = -\nu$  since  $d = \dim V = p + q$  is odd in the non-simple case. Though inequivalent, these two irreps are isomorphic in the category  $\text{ClRep}$  through the isomorphism  $(f_0, f) = (-\text{id}_V, \text{id}_S)$ , where  $-\text{id}_V \in \text{O}_-(V, h)$ . Indeed, we have  $\pi|_V = -\text{id}_V$  hence  $\text{Cl}(-\text{id}_V) = \pi$  and (5.1) reads  $\text{Ad}_{\text{id}_S} \circ \gamma_+ = \gamma_- \circ \text{Cl}(-\text{id}_V)$  (cf. (3.2)), which shows that  $(f_0, f) : \gamma_+ \rightarrow \gamma_-$  is an isomorphism in  $\text{ClRep}$ . Notice that  $(f_0, f) \circ (f_0, f) = \text{id}_{\gamma_+}$ , so  $(f_0, f)^{-1} = (f_0, f)$ . The kernel of  $\gamma_{\epsilon}$  is given by:

$$\ker \gamma_{\epsilon} = \{x \in \text{Cl}(V, h) | x\nu = -\epsilon x\} \quad . \tag{5.2}$$

and we have<sup>8</sup>  $\dim(\ker \gamma_{\epsilon}) = \dim(\text{im} \gamma_{\epsilon}) = \frac{1}{2} \dim \text{Cl}(V, h) = 2^{d-1}$ .

PROPOSITION 5.1. Let  $(V, h)$  be a quadratic space. Then all real irreducible representations of  $\text{Cl}(V, h)$  are weakly faithful. Moreover, there exists a single isomorphism class of such representations in the category  $\text{ClRep}$ , which is uniquely determined by the isomorphism class of  $(V, h)$  and hence by the signature of  $h$ . In the simple

<sup>8</sup>In the non-simple case, we have  $\nu^2 = +1$  and multiplication with  $\nu$  gives a non-unital involutive algebra endomorphism of  $\text{Cl}(V, h)$ .

cases, this isomorphism class is also an equivalence class of representations. In the non-simple cases, this isomorphism class decomposes into two equivalence classes of representations, each of which is determined by the signature of  $h$ .

*Proof.* Injectivity of  $\gamma|_V$  can fail only when  $\gamma$  is not faithful and  $V \cap \ker \gamma \neq \{0\}$ , which by relation (5.2) can happen only when  $\dim V = 2$  since right multiplication with  $\nu$  maps  $V$  into the subspace of  $\text{Cl}(V, h)$  which corresponds to  $\wedge^{d-1}V$  through the Riesz-Chevalley-Crumeyrole isomorphism. However,  $\gamma$  is always faithful when  $\dim V = 2$ , since in this case  $p - q \in \{-2, 0, 2\}$  and hence  $p - q \equiv_8 0, 2, 6$ , which corresponds to the simple case. The remaining statements follow from the discussion above.  $\square$

**PROPOSITION 5.2.** *In the simple case,  $\gamma$  gives a bijection between  $\text{Cl}(V, h)$  and the Clifford image  $C = \text{im} \gamma$ . In the non-simple case, we have  $C = C_+ = C_-$  and  $\gamma$  restricts to linear bijections between  $\text{Cl}_\pm(V, h)$  and  $C$ . In this case, the restriction of  $\gamma$  to  $\text{Cl}_+(V, h)$  is a unital isomorphism of algebras from  $\text{Cl}_+(V, h)$  to  $C$ .*

*Proof.* In the simple case,  $\gamma$  is faithful and hence induces a bijection between  $\text{Cl}(V, h)$  and  $C \stackrel{\text{def.}}{=} \gamma(\text{Cl}(V, h))$ . In the non-simple case, we have  $\omega = \gamma(\nu) = \text{eid}_S$ , where  $\epsilon \stackrel{\text{def.}}{=} \epsilon_\gamma$ . Recall that  $C_\pm \stackrel{\text{def.}}{=} \gamma(\text{Cl}_\pm(V, h))$  and hence  $C = \gamma(\text{Cl}(V, h)) = C_+ + C_-$ . The dimension of  $V$  is odd and the right multiplication  $R_\nu : \text{Cl}(V, h) \rightarrow \text{Cl}(V, h)$  with the Clifford volume element  $\nu$  maps  $\text{Cl}_\pm(V, h)$  into  $\text{Cl}_\mp(V, h)$ . This implies that the right multiplication  $R_\omega : \text{End}_\mathbb{R}(S) \rightarrow \text{End}_\mathbb{R}(S)$  with  $\omega$  satisfies  $R_\omega(C_\pm) = C_\mp$ . Since  $\omega = \text{eid}_S$ , we have  $R_\omega(C_\pm) = C_\pm$  and we conclude that  $C = C_+ = C_-$  since  $C_\pm$  are subspaces of  $\text{End}_\mathbb{R}(S)$ . Since  $\nu^2 = 1$  in the simple cases, we have  $(R_\nu)^2 = \text{id}_{\text{Cl}(V, h)}$ . Hence the linear map  $R_\nu : \text{Cl}_+(V, h) \rightarrow \text{Cl}_-(V, h)$  is bijective and thus  $\dim_\mathbb{R} \text{Cl}_+(V, h) = \dim_\mathbb{R} \text{Cl}_-(V, h) = \frac{1}{2} \dim_\mathbb{R} \text{Cl}(V, h) = \dim(\text{im} \gamma) = \dim C$ , which implies that  $\gamma$  restricts to bijections between  $\text{Cl}_\pm(V, h)$  and  $C$ . Since  $C_+ = \gamma(\text{Cl}_+(V, h))$  and  $\text{Cl}_+(V, h)$  is a unital subalgebra of  $\text{Cl}(V, h)$ , the unital morphism of algebras  $\gamma$  restricts to a unital isomorphism of algebras from  $\text{Cl}_+(V, h)$  to  $C_+$ .  $\square$

**5.2. The Schur algebra.**

**DEFINITION 5.2.** *Let  $U$  be an oriented three-dimensional Euclidean vector space. The quaternion algebra of  $U$  is the normed unital associative  $\mathbb{R}$ -algebra  $\mathbb{H}_U$  whose underlying set equals  $\mathbb{R} \oplus U$  and whose multiplication is defined through:*

$$(q_0 + \mathbf{q})(q'_0 + \mathbf{q}') \stackrel{\text{def.}}{=} q_0q'_0 - (\mathbf{q}, \mathbf{q}') + q_0\mathbf{q}' + q'_0\mathbf{q} + \mathbf{q} \times \mathbf{q}' \quad \forall q_0, q'_0 \in \mathbb{R}, \mathbf{q}, \mathbf{q}' \in U \quad ,$$

where  $(\ , \ )$  and  $\times$  denote scalar and vector products of  $U$ . The norm of  $q := q_0 + \mathbf{q} \in \mathbb{H}_U$  is defined through:

$$\|q\|_U \stackrel{\text{def.}}{=} \sqrt{c_U(q)q} = \sqrt{q_0^2 + \|\mathbf{q}\|^2} \quad ,$$

where  $\| \cdot \|$  is the norm of  $U$  and  $c_U : \mathbb{H}_U \rightarrow \mathbb{H}_U$  is the conjugation of  $\mathbb{H}_U$ , i.e. the unital anti-automorphism given by:

$$c_U(q_0 + \mathbf{q}) = q_0 - \mathbf{q} \quad .$$

The standard algebra  $\mathbb{H}$  of quaternions is the quaternion algebra of  $\mathbb{R}^3$ , when the latter is endowed with its canonical scalar product and orientation. Any quaternion algebra  $\mathbb{H}_U$  is of course isomorphic with  $\mathbb{H}$  as a unital associative normed algebra through

some (non-unique) isomorphism which takes  $\text{Im}\mathbb{H} = \mathbb{R}^3$  into  $U$ . The following result characterizes the Schur algebra of pin representations (see, for example, [23]):

PROPOSITION 5.3. *The Schur algebra of pin representations is as follows:*

1. *In the normal (simple or non-simple) case, we have  $\mathbb{S} = \mathbb{R}\text{id}$ , which we identify with  $\mathbb{R}$  through the isomorphism  $\mathbb{R} \ni x \xrightarrow{\sim} x\text{id}_{\mathbb{S}} \in \mathbb{S}$ . In this case, we set  $c = \text{id}_{\mathbb{S}}$  and endow  $\mathbb{S}$  with the norm induced from  $\mathbb{R}$ .*
2. *In the complex case, we have  $\mathbb{S} = \mathbb{R}\text{id}_{\mathbb{S}} \oplus \mathbb{R}\omega$ , which we identify with  $\mathbb{C}$  through the isomorphism  $\mathbb{C} \ni z = x + iy \xrightarrow{\sim} x + y\omega \in \mathbb{S}$  (in this case, we have  $\nu^2 = -1$  and hence  $\omega^2 = -\text{id}_{\mathbb{S}}$ ). Accordingly, we let  $J \stackrel{\text{def.}}{=} \omega$  denote the imaginary unit of  $\mathbb{S}$ . In particular,  $\mathbb{S}$  is a normed  $\mathbb{R}$ -algebra whose norm and conjugation  $c$  do not depend on the choice of orientation of  $V$  (and hence are invariant under the change  $\omega \rightarrow -\omega$ ). The subspace  $\text{Im}\mathbb{S} \stackrel{\text{def.}}{=} \mathbb{R}\omega$  is also independent of the choice of orientation of  $V$ .*
3. *In the quaternionic (simple or non-simple) case, we have a direct sum decomposition  $\mathbb{S} = \mathbb{R}\text{id}_{\mathbb{S}} \oplus U$  of the underlying  $\mathbb{R}$ -vector space of  $\mathbb{S}$  (where  $U = U(\gamma)$  is an oriented Euclidean vector space determined by  $\gamma$ ) and  $\mathbb{S}$  is isomorphic with the quaternion algebra of  $U$  through the map  $\mathbb{H}_U \ni (q_0, \mathbf{q}) \rightarrow (q_0\text{id}_{\mathbb{S}}, \mathbf{q}) \in \mathbb{S}$ . In particular,  $\mathbb{S}$  has a natural structure of normed  $\mathbb{R}$ -algebra, hence there exists a (non-unique) unital isomorphism of normed algebras  $f : \mathbb{H} \xrightarrow{\sim} \mathbb{S}$  such that  $f(\text{Im}\mathbb{H}) = U$ . The conjugation  $c = c_U$  of  $\mathbb{S}$  is uniquely determined by  $\gamma$ .*

In the quaternionic case, we set  $\text{Im}\mathbb{S} \stackrel{\text{def.}}{=} U$ . The isomorphisms of the proposition map the groups of unit norm elements  $U(\mathbb{R}), U(\mathbb{C})$  and  $U(\mathbb{H})$  to the corresponding subgroup of  $\mathbb{S}$ , which we denote by  $U(\mathbb{S})$ . In the complex case, the isomorphism  $\mathbb{C} \simeq \mathbb{S}$  of Proposition 5.3 depends on the choice of orientation of  $V$ ; changing that orientation amounts to postcomposing that isomorphism with the conjugation  $c$  of  $\mathbb{S}$  or to precomposing it with the conjugation of  $\mathbb{C}$ . The following proposition clarifies the role of the isomorphism  $f$  in the quaternionic case.

PROPOSITION 5.4. *Let  $\mathbb{S}$  be a unital  $\mathbb{R}$ -algebra such that  $\mathbb{S} \simeq_{\text{Alg}} \mathbb{H}$  and let  $m : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{S}$ ,  $m(q_1, q_2) \stackrel{\text{def.}}{=} q_1 q_2$  denote the multiplication map of  $\mathbb{S}$ . Then there exists a surjective map  $\Phi : \text{Isom}_{\text{Alg}}(\mathbb{H}, \mathbb{S}) \rightarrow B(\mathbb{S})$ , where:*

- (a)  *$\text{Isom}_{\text{Alg}}(\mathbb{H}, \mathbb{S})$  is the set of unital isomorphisms of  $\mathbb{R}$ -algebras  $f : \mathbb{H} \xrightarrow{\sim} \mathbb{S}$*
- (b)  *$B(\mathbb{S})$  is the set of  $\mathbb{R}$ -subspaces  $U \subset \mathbb{S}$  such that  $\mathbb{S} = \mathbb{R}1_{\mathbb{S}} \oplus U$ , endowed with a Euclidean scalar product  $(\cdot, \cdot)$  and orientation such that the following condition is satisfied:*
- (c) *The restriction  $m|_{U \times U} : U \times U \rightarrow \mathbb{S}$  of the multiplication map has the form:*

$$m_U(\mathbf{s}, \mathbf{s}') = -(\mathbf{s}, \mathbf{s}') + \mathbf{s} \times \mathbf{s}' \quad ,$$

where  $\times : U \times U \rightarrow U$  is the vector product of  $U$ .

Namely, we have  $\Phi(f) \stackrel{\text{def.}}{=} f(\text{Im}\mathbb{H})$  and the scalar product and orientation of  $\Phi(f)$  are induced by  $f$  from those of  $\text{Im}\mathbb{H}$ . For any  $U \in B(\mathbb{S})$ , the preimage  $\Phi^{-1}(U)$  is a torsor for the right action of  $\text{Aut}_{\text{Alg}}(\mathbb{H})$ :

$$f \rightarrow f \circ \varphi \quad , \quad \varphi \in \text{Aut}_{\text{Alg}}(\mathbb{H}) \simeq_{\text{Gp}} \text{SO}(3, \mathbb{R}) \quad .$$

and hence we have a bijection  $B(\mathbb{S}) \simeq_{\text{Set}} \text{Isom}_{\text{Alg}}(\mathbb{H}, \mathbb{S})/\text{SO}(3, \mathbb{R})$ . For any  $f \in \Phi^{-1}(U)$  and any  $q \in \mathbb{H}$  we have  $f(\bar{q}) = c_U(f(q))$ , where  $c_U : \mathbb{S} \rightarrow \mathbb{S}$  is the involutive unital anti-automorphism of the algebra given by  $c_U = \text{id}_{\mathbb{R}1_{\mathbb{S}}} \oplus (-\text{id}_U)$ .



*Proof.* Surjectivity of  $\Phi$  follows by picking an oriented orthonormal basis  $(e_1, e_2, e_3)$  of  $U$  and noticing that  $U = f(\text{Im}\mathbb{H})$  for the unique  $\mathbb{R}$  linear map  $f : \mathbb{H} \rightarrow \mathbb{S}$  which satisfies  $f(1_{\mathbb{H}}) = 1_{\mathbb{S}}$  and  $f(\epsilon_i) = e_i$ , where  $\epsilon_1, \epsilon_2, \epsilon_3$  is a canonically-oriented orthonormal basis of  $\mathbb{R}^3$ . It is easy to see that this map is a unital isomorphism of  $\mathbb{R}$ -algebras. That  $\Phi^{-1}(U)$  is an  $\text{SO}(3, \mathbb{R})$ -torsor is obvious.  $\square$

REMARK 5.1. Notice that  $m$  is completely determined by the Euclidean scalar product of  $U$  and by its orientation through the formula:

$$m(s_0 + \mathbf{s}, s'_0 + \mathbf{s}') \stackrel{\text{def.}}{=} s_0 s'_0 - (\mathbf{s}, \mathbf{s}') + s_0 \mathbf{s}' + s'_0 \mathbf{s} + \mathbf{s} \times_U \mathbf{s}' \quad \forall s_0, s'_0 \in \mathbb{R} \quad \forall \mathbf{s}, \mathbf{s}' \in U \quad .$$

Conversely,  $m$  determines both the scalar product and orientation of  $U$  though condition (C).

**5.3. The anticommutant subspace.** For the remainder of this section, we fix a real pin representation  $\gamma : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S)$ . Let  $\alpha_{p,q} \in \{-1, 1\}$  be defined as follows:

1. For the normal simple case:

$$\alpha_{p,q} \stackrel{\text{def.}}{=} \sigma_{p,q} = (-1)^{\frac{p-q}{2}} = \begin{cases} +1 & \text{if } p - q \equiv_8 0 \\ -1 & \text{if } p - q \equiv_8 2 \end{cases}$$

2. For the complex case:

$$\alpha_{p,q} \stackrel{\text{def.}}{=} (-1)^{\frac{p-q+1}{4}} = \begin{cases} -1 & \text{if } p - q \equiv_8 3 \\ +1 & \text{if } p - q \equiv_8 7 \end{cases}$$

3. For the quaternionic simple case:

$$\alpha_{p,q} \stackrel{\text{def.}}{=} \sigma_{p,q} = (-1)^{\frac{p-q}{2}} = \begin{cases} +1 & \text{if } p - q \equiv_8 4 \\ -1 & \text{if } p - q \equiv_8 6 \end{cases} \quad .$$

PROPOSITION 5.5. *The following statements hold:*

- I. *In the non-simple cases, we have  $A = 0$ .*
- II. *In the simple cases,  $A$  is a rank one free  $\mathbb{S}$ -module. Namely, there exists an element  $u \in A$  such that:*
  - (a)  *$u$  is a basis of  $A$  over  $\mathbb{S}$*
  - (b)  *$u$  satisfies  $u^2 = \alpha_{p,q} \text{id}_S$  (in particular,  $u$  is invertible)*
  - (c)  *$u$  satisfies:*

$$\text{Ad}_s(u) = \begin{cases} \text{id}_{\mathbb{S}} & \text{for the normal simple case} \\ c & \text{for the complex case} \\ \text{id}_{\mathbb{S}} & \text{for the quaternionic simple case} \end{cases}$$

*In the normal simple and quaternionic simple cases, there exist only two elements  $u \in A$  with these properties, namely  $u = \pm\omega$ . In the complex case, any two elements  $u \in A$  which have these properties are related by  $u' = su$  where  $s = e^{\theta J} \text{id}_S \in \text{U}(\mathbb{S})$  ( $\theta \in \mathbb{R}$ ) corresponds to a complex number of unit modulus under the isomorphism  $\mathbb{S} \simeq_{\text{Alg}} \mathbb{C}$  of Proposition 5.3.*

*Proof.* Consider first the non-simple cases. Then  $x \in A$  must satisfy  $x\gamma(v) = -\gamma(v)x$  for all  $v \in V$ , which implies  $x\gamma(v) = -\gamma(v)x$  since  $d$  is odd. On the other hand, in these cases we have  $\omega = \gamma(v) = \epsilon \text{id}_S$  (with  $\epsilon \in \{-1, 1\}$ ), so the relation



$x\omega = -\omega x$  becomes  $2\epsilon x = 0$ , which implies  $x = 0$ . We conclude that  $A = 0$  in the non-simple cases.

Consider now the simple cases. It was shown in [23] that the element:

$$u \stackrel{\text{def.}}{=} \begin{cases} \omega & \text{for the normal simple case} \\ D & \text{for the complex case} \\ \omega & \text{for the quaternionic simple case} \end{cases} \tag{5.3}$$

(where  $D$  was defined in [23]) satisfies conditions (a),(b) and (c). Let  $u' \in A$  be another element satisfying these three conditions. Then  $u' = su$  for some  $s \in \mathbb{S}^\times$  and hence  $(u')^2 = (su)^2 = susu = s\text{Ad}_s(u)(s)u^2$ . Since  $u^2 = (u')^2 = \alpha_{p,q}\text{id}_S$ , this gives  $s\text{Ad}_s(u)(s) = \text{id}_S$ . In the normal simple and normal quaternionic cases, we have  $\text{Ad}_s(u) = \text{id}_S$ , so the previous relation gives  $s^2 = \text{id}_S$  and hence  $s \in \{-\text{id}_S, \text{id}_S\}$  since the only square roots of unity in the algebras  $\mathbb{C}$  and  $\mathbb{H}$  are  $-1$  and  $+1$  (because  $\mathbb{R}$  and  $\mathbb{H}$  are associative division algebras). It is clear that  $-u$  satisfies (a), (b) and (c). In the complex case, we have  $\text{Ad}_s(u) = c$ , so the relation above becomes  $sc(s) = \text{id}_S$ . This shows that  $s$  corresponds to a complex number of unit modulus under the isomorphism  $\mathbb{S} \simeq_{\text{Alg}} \mathbb{C}$  of Proposition 5.3. In this case, it is obvious that  $e^{\theta J}u$  satisfies the three conditions for any  $\theta \in \mathbb{R}$ .  $\square$

**COROLLARY 5.6.** *In the simple cases, any element  $u$  satisfying conditions (a), (b), (c) of Proposition 5.5 is a twisting element for  $\gamma$ . Moreover:*

1. *In the normal simple case,  $u = \omega$  is a special twisting element iff  $p - q \equiv_8 2$ .*
2. *In the complex case,  $u = D$  is a special twisted element iff  $p - q \equiv_8 3$ .*
3. *In the quaternionic simple case,  $u = \omega$  is a special twisting element iff  $p - q \equiv_8 6$ .  
When  $p - q \equiv_8 4$ , the element  $\mu = J\omega \in A$  is a special twisting element for any  $J \in \text{Im}\mathbb{S} \cap \text{U}(\mathbb{S})$  (we have  $J^2 = -\text{id}_S$ ).*

*Proof.* Follows immediately from Proposition 5.5.  $\square$

**5.4. The Schur pairing in the simple case.** Assume that  $(V, h)$  belongs to the simple case and let  $u \in A$  be an element having the properties given in Proposition 5.5. Then the Schur pairing  $\mathfrak{p} : A \times A \rightarrow \mathbb{S}$  can be identified with a symmetric  $\mathbb{R}$ -bilinear map  $\mathfrak{p}_u : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{S}$  given by:

$$\mathfrak{p}_u(s_1, s_2) \stackrel{\text{def.}}{=} \mathfrak{p}(s_1u, s_2u) = \frac{1}{2} [(s_1u)(s_2u) + (s_2u)(s_1u)] \quad ,$$

(see Appendix C). Identifying  $\mathbb{S}$  with  $\mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$  as in Proposition 5.3, we find:

**PROPOSITION 5.7.** *In the simple cases, the Schur pairing can be identified with one of the following:*

1. *In the normal simple case  $p - q \equiv_8 0, 2$ , the Schur pairing can be identified with the map  $\mathfrak{p}_u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  given by:*

$$\mathfrak{p}_u(x_1, x_2) = \alpha_{p,q}x_1x_2 \quad , \quad \text{where } \alpha_{p,q} = \sigma_{p,q} = (-1)^{\frac{p-q}{2}} \quad .$$

2. *In the complex case  $p - q \equiv_8 3, 7$ , the Schur pairing can be identified with the map  $\mathfrak{p}_u : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$  given by:*

$$\mathfrak{p}_u(z_1, z_2) = \alpha_{p,q}\text{Re}(\bar{z}_1z_2) \in \mathbb{R} \subset \mathbb{C} \quad , \quad \text{where } \alpha_{p,q} = (-1)^{\frac{p-q+1}{4}} \quad .$$

3. *In the quaternionic simple case  $p - q \equiv_8 4, 6$ , the Schur pairing can be identified with the map  $\mathfrak{p}_u : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$  given by:*

$$\mathfrak{p}_u(q_1, q_2) = \alpha_{p,q}\frac{1}{2}(q_1q_2 + q_2q_1) \quad , \quad \text{where } \alpha_{p,q} = \sigma_{p,q} = (-1)^{\frac{p-q}{2}} \quad .$$

**5.5. Pseudocentralizers of pin representations.**

PROPOSITION 5.8. *For any real pin representation, we have  $\mathbb{S} \cap A = 0$ . Hence the pseudocentralizer  $\mathbb{T}$  of  $\gamma$  is a  $\mathbb{Z}_2$ -graded algebra with homogeneous subspaces  $\mathbb{T}^0 = \mathbb{S}$  and  $\mathbb{T}^1 = A$ . In the simple cases, we have  $A = \text{Su}$  (where  $u$  is as in Proposition 5.5), which gives the following unital isomorphisms of  $\mathbb{R}$ -algebras:*

1. *In the normal simple case:*

$$\mathbb{T} \simeq_{\text{Alg}_{\mathbb{Z}_2}} \begin{cases} \text{Cl}_{1,0} \simeq_{\text{Alg}} \mathbb{D} & \text{if } p - q \equiv_8 0 \\ \text{Cl}_{0,1} \simeq_{\text{Alg}} \mathbb{C} & \text{if } p - q \equiv_8 2 \end{cases}$$

2. *In the complex case:*

$$\mathbb{T} \simeq_{\text{Alg}_{\mathbb{Z}_2}} \begin{cases} \text{Cl}_{0,2} \simeq_{\text{Alg}} \mathbb{H} & \text{if } p - q \equiv_8 3 \\ \text{Cl}_{2,0} \simeq \text{Cl}_{1,1} \simeq_{\text{Alg}} \mathbb{P} & \text{if } p - q \equiv_8 7 \end{cases},$$

3. *In the quaternionic simple case:*

$$\mathbb{T} \simeq_{\text{Alg}_{\mathbb{Z}_2}} \begin{cases} \text{Cl}_{0,3} \simeq_{\text{Alg}} \mathbb{D}_{\mathbb{H}} & \text{if } p - q \equiv_8 4 \\ \text{Cl}_{3,0} \simeq_{\text{Alg}} \mathbb{C}_{\mathbb{H}} & \text{if } p - q \equiv_8 6 \end{cases},$$

In the non-simple cases, we have  $\mathbb{T} = \mathbb{T}^0 = \mathbb{S}$ .

*Proof.* Let  $x \in \mathbb{S} \cap A$ . Then  $xw = -wx = wx$  for all  $w \in W = \gamma(V)$ , which implies  $w x = 0$ . Since  $W \simeq V \neq 0$  by our general assumption, there exists a non-degenerate vector  $w_0$  in  $(W, g)$ . Since  $w_0$  is invertible in the algebra  $\text{End}_{\mathbb{R}}(S)$ , the relation  $w_0 x = 0$  implies  $x = 0$ . Thus  $\mathbb{S} \cap A = 0$ . Since  $\mathbb{T} = \mathbb{S} + A$ , we have  $\mathbb{T} = \mathbb{S} \oplus A$ , which is a  $\mathbb{Z}_2$ -grading of the algebra  $\mathbb{T}$  by the discussion of Subsection 3.3. The remaining statements follow from Proposition 5.5.  $\square$

The situation is summarized in Table 5.1

$p - q \pmod 8$	type	$\mathbb{S}$	$u$	$u^2$	$\text{Ad}_s(u)$	$\mathbb{T}$
0	normal simple	$\mathbb{R}$	$\omega$	+1	$\text{id}_{\mathbb{S}}$	$\text{Cl}_{1,0} \simeq \mathbb{D}$
2	normal simple	$\mathbb{R}$	$\omega$	-1	$\text{id}_{\mathbb{S}}$	$\text{Cl}_{0,1} \simeq \mathbb{C}$
3	complex simple	$\mathbb{C}$	$D$	-1	$c$	$\text{Cl}_{0,2} \simeq \mathbb{H}$
7	complex simple	$\mathbb{C}$	$D$	+1	$c$	$\text{Cl}_{2,0} \simeq \mathbb{P}$
4	quaternionic simple	$\mathbb{H}$	$\omega$	+1	$\text{id}_{\mathbb{S}}$	$\text{Cl}_{0,3} \simeq \mathbb{D}_{\mathbb{H}}$
6	quaternionic simple	$\mathbb{H}$	$\omega$	-1	$\text{id}_{\mathbb{S}}$	$\text{Cl}_{3,0} \simeq \mathbb{C}_{\mathbb{H}}$
1	normal non-simple	$\mathbb{R}$	---	---	---	$\text{Cl}_{0,0} \simeq \mathbb{R}$
5	quaternionic non-simple	$\mathbb{H}$	---	---	---	$\text{Cl}_{0,2} \simeq \mathbb{H}$

TABLE 5.1  
Pseudocentralizers of pin representations

**5.6. The Schur representation.**

DEFINITION 5.3. *Let:*

$$L_{\gamma} \stackrel{\text{def.}}{=} \begin{cases} \gamma(G^e(V, h)) & \text{if } p - q \not\equiv_8 1, 5 \text{ (the simple case)} \\ \gamma(G_+(V, h)) & \text{if } p - q \equiv_8 1, 5 \text{ (the non simple case)} \end{cases}. \quad (5.4)$$

PROPOSITION 5.9. *For any real pin representation, the Clifford image is given by  $C = \text{End}_{\mathbb{S}}(S)$  and the kernel of the Schur representation of  $L$  equals  $L_{\gamma}$ :*

$$\ker \text{Ad}_s = L_{\gamma} = L \cap C = L \cap C^{\times} \quad . \tag{5.5}$$

Furthermore:

1. In the simple case,  $\gamma$  restricts to an isomorphism of groups between  $G^e(V, h)$  and  $L_{\gamma}$ .
2. In the non-simple case, we have  $L = L^0$  and  $\gamma$  restricts to an isomorphism of groups between  $G_+(V, h)$  and  $L_{\gamma}$ .

*Proof.* The fact that  $C = \text{End}_{\mathbb{S}}(S)$  is well-known (see, for example, [23]). Thus (4.16) gives  $\ker \text{Ad}_s = L \cap C$ . Let:

$$C^{\times} \stackrel{\text{def.}}{=} \{a \in C \mid \exists b \in C : ab = ba = \text{id}_S\}$$

denote the group of invertible elements of the subalgebra  $C$  of  $\text{End}_{\mathbb{R}}(S)$ . Since  $C = \text{End}_{\mathbb{S}}(S)$ , we have<sup>9</sup>  $C \cap \text{Aut}_{\mathbb{R}}(S) = C^{\times}$ , i.e. an element of  $C$  is invertible in  $\text{End}_{\mathbb{R}}(S)$  iff it is invertible in  $C$ . Since  $L \subset \text{Aut}_{\mathbb{R}}(S)$ , this implies  $L \cap C \subset C^{\times}$  and hence  $L \cap C = L \cap C^{\times}$ . To show that  $L \cap C^{\times} = L_{\gamma}$ , we distinguish the cases:

1. In the simple case,  $\gamma$  restricts to unital isomorphism of algebras between  $\text{Cl}(V, h)$  and  $C$  and between  $V$  and  $W \stackrel{\text{def.}}{=} \gamma(V) \subset C$ . In particular, we have  $C^{\times} = \gamma(\text{Cl}(V, h)^{\times})$ . Using the definitions of  $L$  and  $G^e(V, h)$ , this implies:

$$\begin{aligned} L \cap C^{\times} &= \{a \in C^{\times} \mid \text{Ad}(a)(W) = W\} = \\ &= \gamma(\{x \in \text{Cl}(V, h)^{\times} \mid \text{Ad}(x)(V) = V\}) = \gamma(G^e(V, h)) \quad , \end{aligned}$$

showing that  $\ker \text{Ad}_s = L \cap C = L \cap C^{\times}$  equals  $L_{\gamma}$ . It is clear that  $\gamma$  restricts to an isomorphism of groups between  $G^e(V, h)$  and  $\ker \text{Ad}_s = L_{\gamma} = \gamma(G^e(V, h))$ .

2. In the non-simple case,  $\gamma$  restricts to a unital isomorphism of algebras from  $\text{Cl}_+(V, h)$  to  $C = \text{End}_{\mathbb{S}}(S)$  (see Proposition 5.2) and to an isomorphism of vector spaces from  $V$  to  $W = \gamma(V)$ . In particular, we have  $C^{\times} = \gamma(\text{Cl}_+(V, h)^{\times})$ . Using the definitions of  $L$  and  $G_+(V, h)$ , this implies:

$$\begin{aligned} L \cap C^{\times} &= \{a \in C^{\times} \mid \text{Ad}(a)(W) = W\} = \\ &= \gamma(\{x \in \text{Cl}_+(V, h)^{\times} \mid \text{Ad}(x)(V) = V\}) = \gamma(G_+(V, h)) \quad , \end{aligned}$$

showing that  $\ker \text{Ad}_s = L_{\gamma}$ . We have  $\omega = \gamma(\nu) = \epsilon_{\gamma} \text{id}_S$ , hence  $\text{Ad}(\omega) = \text{id}_{\text{End}_{\mathbb{R}}(S)}$  and the volume grading of  $\text{End}_{\mathbb{R}}(S)$  (see Subsection 3.4) is given by  $\text{End}_{\mathbb{R}}^0(S) = \text{End}_{\mathbb{R}}(S)$  and  $\text{End}_{\mathbb{R}}^1(S) = 0$ . Thus the volume grading of the Lipschitz group (see Subsection 4.3) is given by  $L^0 = L \cap \text{End}_{\mathbb{R}}^0(S) = L$  and  $L^1 = L \cap \text{End}_{\mathbb{R}}^1(S) = \emptyset$ . Since  $G_+(V, h) \subset \text{Cl}_+(V, h)$ , it follows that  $\gamma$  restricts to an isomorphism of groups between  $G_+(V, h)$  and  $\ker \text{Ad}_s = L_{\gamma} = \gamma(G_+(V, h))$ . □

We have:

$$\text{Aut}_{\text{Alg}}(\mathbb{S}) = \begin{cases} \{\text{id}_{\mathbb{S}}\} & \mathbb{S} \simeq \mathbb{R} \\ \{\text{id}_{\mathbb{S}}, c\} & \mathbb{S} \simeq \mathbb{C} \\ \text{U}(\mathbb{S})/\{-\text{id}_S, \text{id}_S\} & \mathbb{S} \simeq \mathbb{H} \end{cases} \quad ,$$

<sup>9</sup>If  $a \in C \cap \text{Aut}_{\mathbb{R}}(S) = \text{End}_{\mathbb{S}}(S) \cap \text{Aut}_{\mathbb{R}}(S)$ , then the relation  $as = sa$  with  $s \in \mathbb{S}$  implies  $a^{-1}s = sa^{-1}$  and hence  $a^{-1} \in \text{End}_{\mathbb{S}}(S) = C$ , where  $a^{-1}$  is the inverse of  $a$  in  $\text{End}_{\mathbb{R}}(S)$ . Thus  $C \cap \text{Aut}_{\mathbb{R}}(S) \subset C^{\times}$ . The opposite inclusion is obvious.

For the third entry, recall that  $\mathbb{H}$  is central simple as an  $\mathbb{R}$ -algebra, so all of its  $\mathbb{R}$ -algebra automorphisms are inner by the Noether-Skolem theorem. We have an exact sequence:

$$1 \longrightarrow \mathbb{R}^\times \hookrightarrow \mathbb{H}^\times \xrightarrow{\text{Ad}} \text{Aut}_{\text{Alg}}(\mathbb{H}) \longrightarrow 1 \quad ,$$

which restricts to an exact sequence:

$$1 \longrightarrow \{-1, 1\} \hookrightarrow \text{U}(\mathbb{H}) \xrightarrow{\text{Ad}} \text{Aut}_{\text{Alg}}(\mathbb{H}) \longrightarrow 1$$

and hence induces isomorphisms  $\text{Aut}_{\text{Alg}}(\mathbb{H}) \simeq \mathbb{H}^\times/\mathbb{R}^\times \simeq \text{U}(\mathbb{H})/\{-1, 1\}$ . We have  $\text{U}(\mathbb{H}) = \text{Sp}(1) \simeq \text{Spin}(3)$  and hence  $\text{Aut}_{\text{Alg}}(\mathbb{H}) \simeq \text{Spin}(3)/\{-1, 1\} \simeq \text{SO}(3, \mathbb{R})$ . The isomorphism  $\text{Aut}_{\text{Alg}}(\mathbb{H}) \simeq \text{SO}(3, \mathbb{R})$  takes  $\alpha \in \text{Aut}_{\text{Alg}}(\mathbb{H})$  into  $\alpha|_{\text{Im}\mathbb{H}} \in \text{SO}(\text{Im}\mathbb{H})$ .

**THEOREM 5.10.** *For any pin representation, we have a short exact sequence:*

$$1 \longrightarrow \text{L}_\gamma \hookrightarrow \text{L} \xrightarrow{\text{Ad}_s} \text{Aut}_{\text{Alg}}(\mathbb{S}) \longrightarrow 1 \quad . \tag{5.6}$$

Moreover:

1. *In the normal simple case, we have  $\text{Aut}_{\text{Alg}}(\mathbb{S}) = \text{Aut}_{\text{Alg}}(\mathbb{R}) \simeq_{\text{Gp}} 1$  and:*

$$\begin{aligned} \text{L} &= \text{L}_\gamma = \gamma(\text{G}(V, h)) \simeq_{\text{Gp}} \text{G}(V, h) \\ \text{L}^0 &= \gamma(\text{G}_+(V, h)) \simeq_{\text{Gp}} \text{G}_+(V, h) \\ \text{L}^1 &= \gamma(\text{G}_-(V, h)) \simeq_{\text{Set}} \text{G}_-(V, h) \quad . \end{aligned}$$

2. *In the complex case, we have  $\text{Aut}_{\text{Alg}}(\mathbb{S}) \simeq \text{Aut}_{\text{Alg}}(\mathbb{C}) = \{\text{id}_S, c\} \simeq_{\text{Gp}} \mathbb{Z}_2$  and:*

$$\begin{aligned} \text{L}_\gamma &= \text{L}^0 = \gamma(\text{G}^e(V, h)) \simeq_{\text{Gp}} \text{G}^e(V, h) = \text{G}(V, h)\text{U}(1) \\ \text{L}^1 &= u\text{L}^0 = \text{L}^0u \simeq_{\text{Set}} \text{L}^0 \quad . \end{aligned}$$

(where  $u \in A$  is as in Proposition 5.5) and:

$$\text{Ad}_s(a) = \begin{cases} \text{id}_S & \text{if } a \in \text{L}^0 \\ c & \text{if } a \in \text{L}^1 \end{cases} \quad . \tag{5.7}$$

3. *In the quaternionic simple case, we have  $\text{Aut}_{\text{Alg}}(\mathbb{S}) \simeq \text{Aut}_{\text{Alg}}(\mathbb{H}) \simeq_{\text{Gp}} \text{SO}(3, \mathbb{R})$  and:*

$$\begin{aligned} \text{L}_\gamma &= \gamma(\text{G}(V, h)) \simeq_{\text{Gp}} \text{G}(V, h) \\ \text{L} &= \gamma(\text{G}(V, h))\text{U}(\mathbb{S}) \simeq_{\text{Gp}} \gamma(\text{G}(V, h)) \cdot \text{U}(\mathbb{S}) \simeq_{\text{Gp}} \text{G}(V, h) \cdot \text{Sp}(1) \\ \text{L}^0 &= \gamma(\text{G}_+(V, h))\text{U}(\mathbb{S}) \simeq_{\text{Gp}} \gamma(\text{G}_+(V, h)) \cdot \text{U}(\mathbb{S}) \simeq_{\text{Gp}} \text{G}_+(V, h) \cdot \text{Sp}(1) \\ \text{L}^1 &= \gamma(\text{G}_-(V, h))\text{U}(\mathbb{S}) \simeq_{\text{Set}} \gamma(\text{G}_-(V, h)) \times \text{U}(\mathbb{S}) \simeq_{\text{Set}} \text{G}_-(V, h) \times \text{Sp}(1) \quad . \end{aligned}$$

4. *In the normal non-simple case, we have  $\text{Aut}_{\text{Alg}}(\mathbb{S}) = \text{Aut}_{\text{Alg}}(\mathbb{R}) \simeq_{\text{Gp}} 1$  and:*

$$\text{L} = \text{L}_\gamma = \text{L}^0 = \gamma(\text{G}_+(V, h)) \simeq_{\text{Gp}} \text{G}_+(V, h) \quad .$$

5. *In the quaternionic non-simple case, we have  $\text{Aut}_{\text{Alg}}(\mathbb{S}) = \text{Aut}_{\text{Alg}}(\mathbb{H}) \simeq_{\text{Gp}} \text{SO}(3, \mathbb{R})$ :*

$$\begin{aligned} \text{L}_\gamma &= \gamma(\text{G}_+(V, h)) \simeq_{\text{Gp}} \text{G}_+(V, h) \\ \text{L} &= \text{L}^0 = \gamma(\text{G}_+(V, h))\text{U}(\mathbb{S}) \simeq_{\text{Gp}} \gamma(\text{G}_+(V, h)) \cdot \text{U}(\mathbb{S}) \simeq_{\text{Gp}} \text{G}_+(V, h) \cdot \text{Sp}(1) \quad . \end{aligned}$$

*Proof.* The sequence (5.6) follows from (5.5) if we show that  $\text{Ad}_s$  is surjective. For this, we distinguish the cases:

- A. The normal simple case. We have  $\text{Aut}_{\text{Alg}}(\mathbb{R}) = \{\text{id}_{\mathbb{R}}\} \simeq_{\text{Gp}} 1$  so surjectivity is automatic. Proposition 1.7 shows that  $G^e(V, h) = G(V, h)$  (since  $d$  is even in the normal simple case) while Proposition 5.9 gives  $L_\gamma = \gamma(G^e(V, h)) = \gamma(G(V, h)) \simeq_{\text{Gp}} G(V, h)$ . Since  $\text{Aut}_{\text{Alg}}(\mathbb{R}) \simeq_{\text{Gp}} 1$ , we have  $L_\gamma = \ker \text{Ad}_s = L$  and hence  $L = L_\gamma = \gamma(G(V, h))$ . In this case,  $\gamma$  is a unital isomorphism of algebras from  $\text{Cl}(V, h)$  to  $\text{End}_{\mathbb{R}}(S)$ , which implies (since  $\omega = \gamma(\nu)$  and  $d$  is even) that we have  $\text{End}_{\mathbb{R}}^0(S) = \gamma(\text{Cl}_+(V, h))$  and  $\text{End}_{\mathbb{R}}^1(S) = \gamma(\text{Cl}_-(V, h))$  (cf. Subsection 1.8). Thus  $L^0 = L \cap \text{End}_{\mathbb{R}}^0(S) = \gamma(G(V, h) \cap \text{Cl}_+(V, h)) = \gamma(G_+(V, h)) \simeq_{\text{Gp}} G_+(V, h)$  and  $L^1 = L \cap \text{End}_{\mathbb{R}}^1(S) = \gamma(G(V, h) \cap \text{Cl}_-(V, h)) = \gamma(G_-(V, h)) \simeq_{\text{Set}} G_-(V, h)$ .
- B. The complex case. In this case, we have  $\text{Ad}_s(u) = c$  and  $\text{Aut}_{\text{Alg}}(\mathbb{S}) = \{\text{id}_{\mathbb{S}}, c\}$ . Since  $u \in L$ , it follows that  $\text{Ad}_s$  is surjective and (5.6) holds. Recalling that  $J = \omega = \gamma(\nu)$  (see Proposition 5.3), we have  $L^0 = L \cap \ker(\text{Ad}(\omega) - \text{id}_S) = L \cap \ker(\text{Ad}(J) - \text{id}_S) = L \cap \text{End}_{\mathbb{S}}(S) = L \cap C = \ker \text{Ad}_s = L_\gamma = \gamma(G^e(V, h)) \simeq_{\text{Gp}} G^e(V, h)$ , where we used Proposition 5.9. The relation  $\text{Ad}_s(ua) = \text{Ad}_s(u) \circ \text{Ad}_s(a) = c \circ \text{Ad}_s(a)$  ( $a \in L$ ) together with the sequence (5.6) and the fact that  $c^2 = \text{id}_{\mathbb{S}}$  implies  $uL^0 = L^1$  and also gives relation (5.7). Similarly, the relation  $\text{Ad}_s(au) = \text{Ad}_s(a) \circ \text{Ad}_s(u) = \text{Ad}_s(a) \circ c$  for  $a \in L$  implies  $L^0u = L^1$ . Thus  $L^1 = uL^0 = L^0u$  and (5.7) holds.
- C. The normal non-simple case. That  $L = L^0$  and  $L_\gamma = \gamma(G_+(V, h)) \simeq_{\text{Gp}} G_+(V, h)$  follows from Proposition 5.9. In this case, we have  $\text{Aut}_{\text{Alg}}(\mathbb{R}) = \{\text{id}_{\mathbb{R}}\}$  and hence  $L_\gamma = \ker \text{Ad}_s = L$ .
- D. The quaternionic (simple or non-simple) case. In this case, every  $\varphi \in \text{Aut}_{\text{Alg}}(\mathbb{S})$  has the form  $\varphi = \text{Ad}(s)$  for some  $s \in U(\mathbb{S})$  (because all automorphisms of  $\mathbb{S} \simeq \mathbb{H}$  are inner). Since  $U(\mathbb{S}) \subset \mathbb{S}^\times \subset L$ , we have  $\varphi \in \text{Ad}_s(L)$ , which proves that  $\text{Ad}_s$  is surjective and hence (5.6) holds. We have  $\mathbb{S} \cap C = \mathbb{S} \cap \text{End}_{\mathbb{S}}(S) = Z(\mathbb{S}) = \mathbb{R} \text{id}_S$  since  $\mathbb{S} \simeq \mathbb{H}$  and  $Z(\mathbb{H}) = \mathbb{R}$ . This implies  $\mathbb{S}^\times \cap C = \mathbb{R}^\times \text{id}_S$  and  $U(\mathbb{S}) \cap C = \{-\text{id}_S, \text{id}_S\}$ . The map  $f : L_\gamma \times U(\mathbb{S}) \rightarrow L$  given by:

$$f(a_0, s) \stackrel{\text{def.}}{=} sa_0 = a_0s$$

is a morphism of groups since every element of  $\mathbb{S}$  commutes with every element of  $L_\gamma \subset C$ . Given  $a \in L$ , we can write  $\text{Ad}_s(a) \in \text{Aut}_{\text{Alg}}(\mathbb{S})$  as  $\text{Ad}_s(a) = \text{Ad}_s(s)$  for some  $s \in U(\mathbb{S}) \subset L$ . Then  $\text{Ad}_s(s^{-1}a) = \text{id}_{\mathbb{S}}$  and hence  $s^{-1}a \in \ker \text{Ad}_s = L_\gamma$  (see Proposition 5.9), thus  $a = sa_0 = f(a_0, s)$  for some  $a_0 \in L_\gamma$ . This shows that  $f$  is surjective and hence  $L = U(\mathbb{S})L_\gamma = L_\gamma U(\mathbb{S})$ . On the other hand,  $f(s, a_0) = \text{id}_S$  implies  $a_0 = s^{-1} \in U(\mathbb{S}) \cap L_\gamma \subset U(\mathbb{S}) \cap C = \{-\text{id}_S, \text{id}_S\}$ , which gives  $\ker f = \{(-1, -\text{id}_S), (1, \text{id}_S)\}$  upon noticing that  $f(-a_0, -s) = f(a_0, s)$ . Thus  $f$  induces an isomorphism  $L \simeq L_\gamma \cdot U(\mathbb{S})$  and we have  $L = L_\gamma U(\mathbb{S}) = U(\mathbb{S})L_\gamma \simeq L_\gamma \cdot U(\mathbb{S})$ .

In the quaternionic simple case, Propositions 5.9 gives  $L_\gamma = \gamma(G^e(V, h)) \simeq G^e(V, h)$  while Proposition 1.7 gives  $G^e(V, h) = G(V, h)$  since  $d$  is even. In this case, we have  $\omega = \gamma(\nu) \in L_\gamma$ . Since  $\gamma$  restricts to a bijection between  $G(V, h)$  and  $L_\gamma$  while  $U(\mathbb{S}) \subset \text{End}_{\mathbb{R}}^0(S)$ , we have  $L^\kappa = L \cap \text{End}_{\mathbb{R}}^\kappa(S) = (L_\gamma \cap \text{End}_{\mathbb{R}}^\kappa(S))U(\mathbb{S})$  for all  $\kappa \in \mathbb{Z}_2$ , which gives  $L^0 = \gamma(G_+(V, h))U(\mathbb{S}) \simeq_{\text{Gp}} \gamma(G_+(V, h)) \cdot U(\mathbb{S})$  and  $L^1 = \gamma(G_-(V, h))U(\mathbb{S}) \simeq_{\text{Set}} \gamma(G_-(V, h)) \times U(\mathbb{S})$  since  $\{-1, 1\} \subset G_+(V, h)$ . We thus obtain the statements at point 3. In the quaternionic non-simple case, Proposition 5.9 gives  $L_\gamma = \gamma(G_+(V, h)) \simeq G_+(V, h)$  and  $L = L^0$ , so we obtain the statements at point 5.  $\square$

For  $\alpha \in \{-1, 1\}$ , let:

$$G_2(\alpha) = \begin{cases} G_{2,0} \simeq_{G_P} \mathbb{R}_{>0} \times \text{Pin}_2(+) \simeq_{G_P} \mathbb{R}_{>0} \times O_2(+) & \text{if } \alpha = +1 \\ G_{0,2} \simeq_{G_P} \mathbb{R}_{>0} \times \text{Pin}_2(-) \simeq_{G_P} \mathbb{R}_{>0} \times O_2(-) & \text{if } \alpha = -1 \end{cases} ,$$

where  $O_2(\alpha)$  is the isomorphic model of  $\text{Pin}_2(\alpha)$  discussed in Section 2.3 and  $G_{p,q}$  denotes the ordinary Clifford group of  $\mathbb{R}^{p,q}$ .

PROPOSITION 5.11. *In the complex case, we have an isomorphism of groups:*

$$L \simeq_{G_P} [G_+(V, h) \times G_2(\alpha_{p,q})] / \{(\lambda 1, \lambda^{-1} 1) \mid \lambda \in \mathbb{R}^\times\} \simeq_{G_P} \mathbb{R}_{>0} \times \text{Spin}^o(V, h) \quad , \quad (5.8)$$

where  $\alpha_{p,q} = (-1)^{\frac{p-q+1}{4}} = \begin{cases} -1 & \text{if } p - q \equiv_8 3 \\ +1 & \text{if } p - q \equiv_8 7 \end{cases}$  and  $\text{Spin}^o(V, h) \stackrel{\text{def.}}{=} \text{Spin}^o_{\alpha_{p,q}}(V, h)$  is the adapted  $\text{Spin}^o$  group of  $(V, h)$  defined in Subsection 2.4.

*Proof.* In the complex case,  $\gamma$  is faithful and hence induces a unital isomorphism of  $\mathbb{R}$ -algebras from  $\text{Cl}(V, h)$  to  $C$ , which restricts to a unital isomorphism of  $\mathbb{R}$ -algebras between  $Z(V, h) = \mathbb{R} \oplus \mathbb{R}\nu$  and  $\mathbb{S} = \mathbb{R} \oplus \mathbb{R}\omega$  (this isomorphism takes  $\nu$  into  $\omega = J$ ). Let  $s : C \xrightarrow{\sim} \mathbb{S}$  be the unital isomorphism of  $\mathbb{R}$ -algebras which takes 1 into  $\text{id}_{\mathbb{S}}$  and  $i$  into  $J$ . Since  $d$  is odd, we have  $G_-(V, h) = \nu G_+(V, h)$  and hence  $G^e(V, h) = Z(V, h)^\times \times G(V, h) = Z(V, h)^\times \times G_+(V, h)$ . Thus  $\gamma(G^e(V, h)) = \mathbb{S}^\times \gamma(G_+(V, h))$ . We have  $Z(V, h)^\times \cap G_+(V, h) = \mathbb{R}^\times$ . Theorem 5.10 implies that any  $b \in L$  is of the form  $s(z)\gamma(a)$  or  $s(z)\gamma(a)u$  for some  $z \in \mathbb{C}^\times$  and some  $a \in G_+(V, h)$ . Hence the map  $\varphi : G_+(V, h) \times \mathbb{C}^\times \times \mathbb{Z}_2 \rightarrow L$  given by:

$$\varphi(a, z, \hat{0}) = s(z)\gamma(a) \quad , \quad \varphi(a, z, \hat{1}) = s(z)\gamma(a)u$$

is surjective. We have:

$$\begin{aligned} (s(z_1)\gamma(a_1))(s(z_2)\gamma(a_2)) &= s(z_1 z_2)\gamma(a_1 a_2) , \\ (s(z_1)\gamma(a_1))(s(z_2)\gamma(a_2)u) &= s(z_1 z_2)\gamma(a_1 a_2)u , \\ (s(z_1)\gamma(a_1)u)(s(z_2)\gamma(a_2)) &= s(z_1 \bar{z}_2)\gamma(a_1 a_2)u , \\ (s(z_1)\gamma(a_1)u)(s(z_2)\gamma(a_2)u) &= \alpha_{p,q} s(z_1 \bar{z}_2)\gamma(a_1 a_2) , \end{aligned} \quad (5.9)$$

and  $\varphi(1, 1, \hat{0}) = \text{id}_{\mathbb{S}}$ , which shows that  $\varphi$  is a morphism of groups from  $G_+(V, h) \times G_2(\alpha_{p,q})$  to  $L$  (notice that  $G_2(\pm) \simeq_{\text{Set}} \mathbb{C}^\times \times \mathbb{Z}_2$ ). We have  $\ker \varphi = \{(a, z, \hat{0}) \in G_+(V, h) \times \mathbb{C}^\times \times \mathbb{Z}_2 \mid \gamma(a) = s(z^{-1})\} = \{(\lambda, \lambda^{-1}, \hat{0}) \equiv (\lambda 1_{G_+(V, h)}, \lambda^{-1} 1_{G_2(\alpha_{p,q})}) \mid \lambda \in \mathbb{R}^\times\}$ , since  $\gamma(G_+(V, h)) \cap \mathbb{S}^\times = \gamma(G_+(V, h) \cap Z(V, h)^\times) = \mathbb{R}^\times \text{id}_{\mathbb{S}}$  and  $\gamma$  and  $s$  are injective. This gives the first isomorphism in (5.8). The second isomorphism follows from  $G_+(V, h) \simeq_{G_P} \mathbb{R}_{>0} \times \text{Spin}(V, h)$  and  $G_2(\pm) \simeq_{G_P} \mathbb{R}_{>0} \times \text{Pin}_2(\pm)$ .  $\square$

PROPOSITION 5.12. *Assume that we are in the complex case with  $p - q \equiv_8 7$ . Then the sequence (5.6) splits and we have  $L \simeq_{G_P} G^e(V, h) \rtimes_{\varphi} \mathbb{Z}_2 \simeq_{G_P} \mathbb{R}_{>0} \times (\text{Spin}^e(V, h) \rtimes \mathbb{Z}_2)$ , where  $\varphi : \mathbb{Z}_2 \rightarrow \text{Aut}_{G_P}(G^e(V, h))$  is the group morphism given by:*

$$\varphi(\hat{0}) = \text{id}_{G^e(V, h)} \quad , \quad \varphi(\hat{1}) = \pi|_{G^e(V, h)}$$

and  $\pi$  is the parity automorphism of  $\text{Cl}(V, h)$ .

*Proof.* In this case, we have  $u^2 = +\text{id}_{\mathbb{S}}$  and the map  $\psi : \text{Aut}_{\text{Alg}}(\mathbb{S}) = \{\text{id}_{\mathbb{S}}, c\} \rightarrow L$  given by  $\psi(\text{id}_{\mathbb{S}}) = \text{id}_{\mathbb{S}}$  and  $\psi(c) = u$  is a group morphism which splits the sequence (5.6). Thus  $L \simeq \gamma(G^e(V, h)) \rtimes_{\hat{\varphi}} \{\text{id}_{\mathbb{S}}, c\}$ , where  $\hat{\varphi} : \{\text{id}_{\mathbb{S}}, c\} \rightarrow \text{Aut}_{G_P}(\gamma(G^e(V, h)))$

is the morphism of groups given by  $\hat{\varphi}(\text{id}_S) = \text{id}_{G^e(V,h)}$  and  $\hat{\varphi}(c) = \text{Ad}(u)|_{\gamma(G^e(V,h))}$ . Since  $\text{Ad}(u)(w) = -w$  for all  $w \in W = \gamma(V)$  and  $\gamma$  is injective, we have  $\text{Ad}(u) \circ \gamma = \gamma \circ \pi$ , which shows that  $L \simeq G^e(V,h) \rtimes_{\varphi} \mathbb{Z}_2$ . The second isomorphism given in the statement follows from Proposition 1.8.  $\square$

REMARK 5.2. The isomorphism  $L \simeq_{G_P} \mathbb{R}_{>0} \times (\text{Spin}^c(V,h) \rtimes \mathbb{Z}_2)$  also follows from Proposition 5.11 and from the fact that  $\text{Spin}^o(V,h) \simeq_{G_P} \text{Spin}^c(V,h) \rtimes \mathbb{Z}_2$  when  $p - q \equiv_8 7$  (see [11]).

**5.7. The canonical pairing.** Let  $\gamma : \text{Cl}(V,h) \rightarrow \text{End}_{\mathbb{R}}(S)$  be a pin representation. Recall that a  $\gamma$ -admissible pairing on  $S$  is a  $\gamma$ -adapted pairing (in the sense of Subsection 4.9) which has a definite ‘‘isotropy’’. See [21, 22] and [23] for details on admissible pairings. Recall that  $c : \mathbb{S} \rightarrow \mathbb{S}$  is the  $\mathbb{R}$ -linear map which corresponds to conjugation when  $\mathbb{S} \simeq_{\text{Alg}} \mathbb{C}, \mathbb{H}$ , while  $c = \text{id}_{\mathbb{S}}$  when  $\mathbb{S} \simeq_{\text{Alg}} \mathbb{R}$ .

PROPOSITION-DEFINITION 5.13. *Up to rescaling by a non-zero real number, there exists a single  $\gamma$ -admissible pairing  $\mathcal{B}_e$  (called canonical pairing) which has the following properties, where  $\epsilon_e$  is the type of  $\mathcal{B}_e$  and  ${}^{t_e} : \text{End}_{\mathbb{R}}(S) \rightarrow \text{End}_{\mathbb{R}}(S)$  denotes  $\mathcal{B}_e$ -transposition:*

(a) *We have:*

$$\epsilon_e = \begin{cases} +\epsilon_d = -(-1)^{\lfloor \frac{d}{2} \rfloor} & \text{in the simple cases} \\ -\epsilon_d = +(-1)^{\lfloor \frac{d}{2} \rfloor} & \text{in the non - simple cases} \end{cases},$$

where  $\epsilon_d$  was defined in (1.21).

(b)  $s^{t_e} = c(s) \quad \forall s \in \mathbb{S}$  (which amounts to  $U(\mathbb{S}) \subset O(S, \mathcal{B}_e)$ ).

(c) *In the complex case, we have  $u^{t_e} = u^{-1}$ , i.e.  $u \in O(S, \mathcal{B}_e)$ , where  $u$  is as in Proposition 5.5.*

*Proof.* We distinguish the cases:

1. Normal simple case ( $p - q \equiv_8 0, 2$ ). In this case,  $d$  is even and there exist two admissible pairings  $\mathcal{B}_{\pm}$  (each determined up to rescaling by a non-zero real number) which are distinguished by their type  $\epsilon_{\pm} = \pm 1$  (see [21, 22, 23]). Hence we must have  $\mathcal{B}_e = \mathcal{B}_{(-1)^{1+\lfloor \frac{d}{2} \rfloor}}$ . In this case,  $\mathbb{S} = \mathbb{R}\text{id}_S$  and  $c = \text{id}_{\mathbb{S}}$ , thus condition (b) is trivially satisfied.
2. Normal non-simple case ( $p - q \equiv_8 1$ ). In this case,  $d$  is odd and there exists a single (up to scale) admissible pairing  $\mathcal{B}$ , whose type is given by  $\epsilon = (-1)^{\lfloor \frac{d}{2} \rfloor}$  (see [21, 22, 23]). Hence  $\mathcal{B}_e = \mathcal{B}$  satisfies condition (a). It obviously also satisfies condition (b).
3. Complex case ( $p - q \equiv_8 3, 7$ ). Up to scale, there exist four admissible pairings  $\mathcal{B}_{\kappa}$  ( $\kappa = 0 \dots 3$ ) whose types equal  $\epsilon_{\kappa} = (-1)^{1+\lfloor \frac{\kappa}{2} \rfloor}$  (see [21, 22, 23]). Hence we must have  $\mathcal{B}_e = \mathcal{B}_{\kappa_e}$  for some  $\kappa_e \in \{0, 1, 2, 3\}$ . Condition (a) requires:

$$\left[ \frac{\kappa_e}{2} \right] \equiv_2 \left[ \frac{d}{2} \right] = \frac{d-1}{2}, \tag{5.10}$$

where we used the fact that  $d$  is odd. On the other hand, relations [23, (3.16)] give:

$$u^{-t_{\kappa}} = (-1)^{\kappa + \frac{p-q+1}{4}} u, \quad J^{-t_{\kappa}} = (-1)^{\lfloor \frac{\kappa}{2} \rfloor + \lfloor \frac{d}{2} \rfloor} J, \tag{5.11}$$



where  $J = \omega = \gamma(\nu)$  and  ${}^t\kappa$  denotes  $\mathcal{B}_\kappa$ -transpose. Thus condition (c) requires  $\kappa_e \equiv_2 \frac{p-q+1}{4}$ . This is equivalent with  $\kappa_e = 2 \left[ \frac{\kappa_e}{2} \right] + t$ , where  $t \stackrel{\text{def.}}{=} \frac{p-q+1}{4} \pmod 2 = \begin{cases} 1 & \text{if } p-q \equiv_8 3 \\ 0 & \text{if } p-q \equiv_8 7 \end{cases}$ . Thus (5.10) gives (recall that  $d > 0$ ):

$$\kappa_e = d + t - 1 \pmod 4 = \begin{cases} d \pmod 4 & \text{if } p-q \equiv_8 3 \\ d-1 \pmod 4 & \text{if } p-q \equiv_8 7 \end{cases} .$$

Let  $\mathcal{B}_e = \mathcal{B}_{\kappa_e}$ , where  $\kappa_e$  is given by this choice. Then conditions (a) and (c) are satisfied. On the other hand, relation (5.10) and the second relation in (5.11) give  $J^{-t_e} = J$ , which (since  $J^2 = -\text{id}_S$ ) amounts to  $J^{t_e} = -J$ , hence condition (b) is also satisfied.

4. Quaternionic simple case ( $p-q \equiv_8 4, 6$ ). In this case,  $d$  is even and there exist (up to scale) eight admissible pairings  $\mathcal{B}_\kappa^\epsilon$  (where  $\epsilon \in \{-1, 1\}$  and  $\kappa \in \{0, 1, 2, 3\}$ ), of which only the two so-called fundamental pairings (see [23, Subsection 3.4])  $\mathcal{B}_0^+$  and  $\mathcal{B}_0^-$  satisfy condition (b). Indeed, let  $J_1, J_2, J_3 \in \mathbb{S}$  be the elements of  $\mathbb{S}$  which correspond to the quaternion units through the isomorphism of Proposition 5.3. Then it was shown in Subsection 3.4.3 of op. cit. that  $J_k$  ( $k = 1, 2, 3$ ) are  $\mathcal{B}_0^\pm$ -orthogonal and hence satisfy condition (b). Together with relation [23, (3.38)], this implies that the  $\mathcal{B}_\kappa^\epsilon$ -transpose of  $J_l$  equals  $J_k J_l J_k$ , which implies that only the fundamental pairings satisfy condition (b). The fundamental pairings have type

$$\epsilon_{\mathcal{B}_0^\pm} = \pm 1, \text{ so condition (a) requires } \mathcal{B}_e = \mathcal{B}_0^{(-1)^{1+\lfloor \frac{d}{2} \rfloor}} .$$

5. Quaternionic non-simple case ( $p-q \equiv_8 5$ ). In this case  $d$  is odd and there exist four admissible pairings and a similar argument using the results of [23, Subsection 3.4] shows that only the basic pairing  $\mathcal{B}_0$  satisfies condition (b). Hence we must take  $\mathcal{B}_e = \mathcal{B}_0$ . The type of the basic pairing is  $\epsilon_0 = (-1)^{\lfloor \frac{d}{2} \rfloor}$ , so condition (a) is satisfied. □

REMARK 5.3. Since  $\tau(\nu) = (-1)^{\lfloor \frac{d}{2} \rfloor} \nu$ , we have  $\omega^{t_e} = \epsilon_e^d (-1)^{\lfloor \frac{d}{2} \rfloor} \omega$  and condition (a) implies:

$$\omega^{t_e} = \begin{cases} (-1)^{\lfloor \frac{d}{2} \rfloor} \omega & p-q \equiv_8 0, 2, 4, 6 \\ -\omega & p-q \equiv_8 3, 7 \\ +\omega & p-q \equiv_8 1, 5 \end{cases} . \tag{5.12}$$

Since  $\omega = \sigma_{p,q} \omega^{-1}$ , this gives:

$$\omega^{t_e} = \beta_{p,q} \omega^{-1} \quad , \quad \text{where } \beta_{p,q} \stackrel{\text{def.}}{=} \begin{cases} (-1)^p & p-q \equiv_8 0, 2, 4, 6 \\ -\sigma_{p,q} & p-q \equiv_8 3, 7 \\ +\sigma_{p,q} & p-q \equiv_8 1, 5 \end{cases} . \tag{5.13}$$

**5.8. The canonical Lipschitz norm.**

DEFINITION 5.4. *The canonical Lipschitz norm  $\mathcal{N}_e$  is the Lipschitz norm determined by the canonical pairing  $\mathcal{B}_e$ .*

REMARK 5.4. In the simple cases, we have  $\tau_{\mathcal{B}_e} = \tau_e$  and hence  $N_{\mathcal{B}_e} = N_e$ . In the non-simple cases, we have  $\tau_{\mathcal{B}_e} = \tau_e \circ \pi$ . Together with the results of Subsection 1.21 and with the fact that  $U(\mathbb{S}) \subset O(S, \mathcal{B}_e)$ , this implies:



1. In the simple cases, we have  $\mathcal{N}_e \circ \gamma = \gamma \circ N_e$ .
2. In the non-simple cases, we have  $\mathcal{N}_e \circ \gamma|_{\text{Cl}_+(V,h)} = \gamma \circ N|_{\text{Cl}_+(V,h)}$ .
3. In all cases, we have  $\mathcal{N}_e|_{\mathbb{S}} = M$ , where  $M$  is the square of the norm of  $\mathbb{S}$ .

PROPOSITION 5.14. *For any pin representation, we have  $\mathcal{N}_e(L) \subset \mathbb{R}^\times$ , hence the restriction of  $\mathcal{N}_e$  gives a group morphism:*

$$\mathcal{N}_e : L \rightarrow \mathbb{R}^\times \text{id}_S \simeq \mathbb{R}^\times .$$

The restriction of  $\mathcal{N}_e$  to  $L_\gamma$  is determined as follows:

1. In the simple cases, we have  $\mathcal{N}_e \circ \gamma|_{G^e(V,h)} = N_e|_{G^e(V,h)}$ .
  2. In the non-simple cases, we have  $\mathcal{N}_e \circ \gamma|_{G_+(V,h)} = N|_{G_+(V,h)}$ .
- Moreover, in the complex case we have  $\mathcal{N}_e(u) = \text{id}_S$  while in the quaternionic cases we have  $\mathcal{N}_e|_{\mathbb{S}} = M$ , where  $u$  is as in Proposition 5.5.

*Proof.* We consider each case in turn.

1. Normal simple case. In this case,  $L = \gamma(G(V, h))$  (see Theorem 5.10) and the conclusion follows because  $N_e(G(V, h)) \subset \mathbb{R}^\times$  (see Proposition 1.10) using Remark 5.4.
2. Normal non-simple case. In this case,  $L = \gamma(G_+(V, h))$  by Theorem 5.10 and the conclusion follows from  $N(G_+(V, h)) \subset \mathbb{R}^\times$  using Remark 5.4.
3. Complex case. By Remark 5.4 and Proposition 1.10, we have  $\mathcal{N}_e(\gamma(G^e(V, h))) \subset \gamma(N_e(G^e(V, h))) \subset \mathbb{R}^\times \text{id}_S$ . On the other hand, we have  $\mathcal{N}_e(u) = u^{te}u = \text{id}_S$ . For any  $a \in \gamma(G^e(V, h))$ , we thus have  $\mathcal{N}_e(a) \in \mathbb{R}^\times \text{id}_S$  and:

$$\mathcal{N}_e(au) = u^{te}a^{te}au = \pi(a)^{te}\pi(a) = \pi(a^{te}a) = \pi(\mathcal{N}_e(a)) = \mathcal{N}_e(a) \in \mathbb{R}^\times \text{id}_S .$$

This implies  $\mathcal{N}_e(L) \subset \mathbb{R}^\times \text{id}_S$  by Theorem 5.10.

4. Quaternionic (simple or non-simple) case. In this case, we have  $\mathcal{N}_e(s) = s^{te}s = c(s)s = |s|^2$  for all  $s \in \mathbb{S}$ , where  $|\cdot|$  is the norm on  $\mathbb{S}$  which corresponds to the quaternionic norm through the isomorphism of Proposition 5.3. Thus  $\mathcal{N}_e(\mathbb{S}) \subset \mathbb{R}_{\geq 0}$ . By Theorem 5.10, we have  $L = U(\mathbb{S})L_\gamma$ . Consider the subcases:
  - (a) In the quaternionic simple case,  $d$  is even and  $L_\gamma = \gamma(G(V, h))$ . Thus  $\mathcal{N}_e(L) \subset \mathcal{N}_e(\mathbb{S})\gamma(N_e(G(V, h))) \subset \mathbb{R}^\times$ , where we used Remark 5.4.
  - (b) In the quaternionic non-simple case, we have  $L_\gamma = \gamma(G_+(V, h))$ . Since  $N(G_+(V, h)) \subset \mathbb{R}^\times$ , we again conclude  $\mathcal{N}_e(L) \subset \mathbb{R}^\times$  using Remark 5.4.

The remaining statements follow from Remark 5.4 and from relation (5.4).  $\square$

The composition of  $\mathcal{N}_e|_L$  with the absolute value morphism  $|\cdot| : \mathbb{R}^\times \rightarrow \mathbb{R}_{>0}$  gives a morphism of groups  $|\mathcal{N}_e| : L \rightarrow \mathbb{R}_{>0}$ .

### 5.9. The reduced Lipschitz group.

DEFINITION 5.5. *The reduced Lipschitz group  $\mathcal{L}$  is the kernel of the group morphism  $|\mathcal{N}_e| : L \rightarrow \mathbb{R}_{>0}$ .*

We have a short exact sequence:

$$1 \longrightarrow \mathcal{L} \longrightarrow L \xrightarrow{|\mathcal{N}_e|} \mathbb{R}_{>0} \longrightarrow 1 . \tag{5.14}$$

Every  $a \in L$  can be written as  $a = \sqrt{|\mathcal{N}_e(a)|}a_0$  for some uniquely determined  $a_0 \in \mathcal{L}$ . Thus:

$$L = \mathbb{R}_{>0}\mathcal{L} \simeq \mathbb{R}_{>0} \times \mathcal{L} . \tag{5.15}$$

The morphism of groups  $\pi_0 : L \rightarrow \mathcal{L}$  given by projection on the second factor:

$$\pi_0(a) = a_0 = \frac{1}{\sqrt{|\mathcal{N}_e(a)|}} a \tag{5.16}$$

will be called the *normalization morphism*. The adjoint representation  $\text{Ad} : L \rightarrow \text{Aut}_{\mathbb{R}}(\text{End}_{\mathbb{R}}(S))$  of the Lipschitz group factors through this morphism:

$$\text{Ad}(a) = \text{Ad}(\pi_0(a)) \quad \forall a \in L \quad . \tag{5.17}$$

The volume grading of  $L$  induces a  $\mathbb{Z}_2$ -grading  $\mathcal{L} = \mathcal{L}^0 \sqcup \mathcal{L}^1$ , where  $\mathcal{L}^\kappa = \mathcal{L} \cap L^\kappa = \{a \in L^\kappa \mid |\mathcal{N}_e(a)| = 1\}$  for all  $\kappa \in \mathbb{Z}_2$ .

**COROLLARY 5.15.**  *$\mathcal{L}$  is homotopy equivalent with  $L$ .*

*Proof.* Follows from (5.15).  $\square$

**5.10. The canonical presentation of the reduced Lipschitz group.** The following result describes the reduced Lipschitz groups of all real pin representations, and hence their automorphism group in the category  $\text{ClRep}$ .

**THEOREM 5.16.** *The following maps  $\varphi : \Lambda(V, h) \rightarrow \mathcal{L}$  are well-defined and give isomorphisms of groups from the canonical spinor group  $\Lambda(V, h)$  of Section 2 to the reduced Lipschitz group  $\mathcal{L}$  of the pin representation  $\gamma : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S)$ :*

1. *In the normal simple case, we have  $\Lambda(V, h) = \text{Pin}(V, h)$  and:*

$$\varphi(x) = \gamma(x) \quad \forall x \in \Lambda(V, h) = \text{Pin}(V, h) \quad .$$

2. *In the complex case, we have  $\Lambda(V, h) = \text{Spin}^o(V, h)$  and:*

$$\varphi_{\nu, u}([x, \psi_\alpha(e^{i\theta}, \kappa)]) = \gamma(x)e^{i\theta J_\nu} u^\kappa \quad \forall x \in \text{Spin}(V, h) \quad , \quad \forall \theta \in \mathbb{R} \quad , \quad \forall \kappa \in \mathbb{Z}_2 \quad ,$$

where  $J_\nu = \gamma(\nu)$ ,  $\nu$  is the Clifford volume element of  $(V, h)$  determined by an orientation of  $V$  and  $u \in A$  is as in Proposition 5.5. Here, we have  $(e^{i\theta}, \kappa) \in \text{O}_2(\alpha)$  and  $\psi_\alpha : \text{O}_2(\alpha) \xrightarrow{\sim} \text{Pin}_2(\alpha)$  is the isomorphism of Proposition 2.2, where  $\alpha \stackrel{\text{def.}}{=} \alpha_{p, q}$ .

3. *In the quaternionic simple case, we have  $\Lambda(V, h) = \text{Pin}^q(V, h)$  and:*

$$\varphi_f([x, q]) = \gamma(x)f(q) \quad \forall x \in \text{Pin}(V, h) \quad , \quad \forall q \in \text{U}(\mathbb{H}) = \text{Sp}(1) \quad ,$$

where  $f : \mathbb{H} \xrightarrow{\sim} \mathbb{S}$  is a unital isomorphism of normed  $\mathbb{R}$ -algebras (thus  $f(\text{Im}\mathbb{H}) = U(\gamma)$ ) as in Proposition 5.3.

4. *In the normal non-simple case, we have  $\Lambda(V, h) = \text{Spin}(V, h)$  and:*

$$\varphi(x) = \gamma(x) \quad \forall x \in \text{Spin}(V, h) \quad .$$

5. *In the quaternionic non-simple case, we have  $\Lambda(V, h) = \text{Spin}^q(V, h)$  and:*

$$\varphi_f([x, q]) = \gamma(x)f(q) \quad \forall x \in \text{Spin}(V, h) \quad , \quad \forall q \in \text{U}(\mathbb{H}) = \text{Sp}(1) \quad ,$$

where  $f : \mathbb{H} \xrightarrow{\sim} \mathbb{S}$  is a unital isomorphism of normed  $\mathbb{R}$ -algebras (thus  $f(\text{Im}\mathbb{H}) = U(\gamma)$ ) as in Proposition 5.3.

*Proof.* We consider each case in turn.

1. Normal simple case. Theorem 5.10 gives  $L = \gamma(G(V, h))$ . By Proposition 5.14, we have:  $|\mathcal{N}_e|(\gamma(x)) = |\mathcal{N}_e(\gamma(x))| = |\mathcal{N}_e(x)| = |N|(x)$  for all  $x \in G(V, h)$ . Since  $\gamma$  is an isomorphism from  $G(V, h)$  to  $L$ , this gives  $\mathcal{L} = \ker(|\mathcal{N}_e|_L) = \gamma(\ker(|N_G|)) = \gamma(\text{Pin}(V, h))$  and the statement follows.
2. Complex case. Theorem 5.10 gives  $L^0 = \gamma(G^e(V, h))$  and  $L^1 = L^0 u$ . Since  $u^{t_e} u = \text{id}_{\mathbb{S}}$ , Proposition 5.14 gives:  $\mathcal{N}_e(au) = \mathcal{N}_e(a)$  for all  $a = \gamma(x) \in L^0$  (with  $x \in G^e(V, h)$ ) and Proposition 5.14 gives  $|\mathcal{N}_e|(au) = |\mathcal{N}_e|(a)$ , which implies  $\mathcal{L}^0 = \gamma(\text{Spin}^c(V, h))$  and  $\mathcal{L}^1 = \gamma(\text{Spin}^c(V, h))u$  since  $\gamma$  is injective. This gives the desired statement.
3. Quaternionic simple case. Theorem 5.10 gives  $L = \gamma(G(V, h))U(\mathbb{S})$  while Proposition 5.14 gives  $|\mathcal{N}_e|(\gamma(x)) = |N_G|(x)$  for  $x \in G(V, h)$ , where we used the fact that  $U(\mathbb{S}) \subset O(\mathcal{S}, \mathcal{B}_e)$ . Since  $\gamma$  is injective, this gives  $\mathcal{L} = \gamma(\ker |N_G|) = \gamma(\text{Pin}(V, h))$  and the statement follows.
4. Normal non-simple case. Theorem 5.10 gives  $L = \gamma(G_+(V, h))$  while Proposition 5.14 gives  $|\mathcal{N}_e|(\gamma(x)) = |N|(x)$  for  $x \in G_+(V, h)$ . Since  $\gamma|_{G_+(V, h)}$  is injective, this gives  $\mathcal{L} = \gamma(\ker |N|_{G_+(V, h)}) = \gamma(\text{Spin}(V, h))$  and the statement follows.
5. Quaternionic non-simple case. Theorem 5.10 gives  $L = \gamma(G_+(V, h)U(\mathbb{S}))$  while Proposition 5.14 gives  $|\mathcal{N}_e|(\gamma(x)s) = |N|(x)$  for  $x \in G_+(V, h)$  and  $s \in U(\mathbb{S})$ , where we used the fact that  $U(\mathbb{S}) \subset O(\mathcal{S}, \mathcal{B}_e)$ . Since  $\gamma|_{G_+(V, h)}$  is injective, this gives  $\mathcal{L} = \gamma(\ker |N|_{G_+(V, h)})U(\mathbb{S}) = \gamma(\text{Spin}(V, h))U(\mathbb{S})$  and the statement follows.

□

DEFINITION 5.6. Any of the group isomorphisms given in the previous proposition is called an admissible isomorphism from the enlarged spinor group  $\Lambda(V, h)$  to  $\mathcal{L}$ .

REMARK 5.5. In the normal (simple or non-simple case), the unique admissible isomorphism is given by the restriction of  $\gamma$  and hence it is canonically determined by  $\gamma$ . In the complex case, the admissible isomorphisms are determined by  $\gamma$ , by a choice of orientation of  $V$  and by a choice of element  $u \in A$  as in Proposition 5.5. In the quaternionic (simple or non-simple case), there exists an infinite set of admissible isomorphisms, each of which is determined by  $\gamma$  and by a choice of unital isomorphism  $f \in \text{Isom}_{\text{Alg}}(\mathbb{H}, \mathbb{S})$  satisfying  $f(\text{Im}\mathbb{H}) = U(\gamma)$  (recall that the set of such  $f$  forms an  $\text{SO}(3, \mathbb{R})$ -torsor).

**5.11. The pairing induced by  $\mathcal{B}_e$  on  $\text{End}_{\mathbb{R}}(S)$ .** The canonical pairing  $\mathcal{B}_e$  induces an  $\mathbb{R}$ -bilinear, symmetric and non-degenerate pairing  $(\ , \ )_e : \text{End}_{\mathbb{R}}(S) \times \text{End}_{\mathbb{R}}(S) \rightarrow \mathbb{R}$  defined through:

$$(T_1, T_2)_e \stackrel{\text{def.}}{=} \frac{1}{\dim_{\mathbb{R}} S} \text{tr}(T_1^{t_e} T_2) \ , \ \forall T_1, T_2 \in \text{End}_{\mathbb{R}}(S) \ . \tag{5.18}$$

Symmetry of  $(\ , \ )_e$  follows from involutivity of  ${}^{t_e}$  and from the property  $\text{tr}(T^{t_e}) = \text{tr}(T)$  for all  $T \in \text{End}_{\mathbb{R}}(S)$ .

PROPOSITION 5.17. For any  $a \in \mathcal{L}$  and  $T \in \text{End}_{\mathbb{R}}(S)$ , we have:

$$\text{Ad}(a)(T)^{t_e} = \text{Ad}(a)(T^{t_e}) \ .$$

Moreover, the adjoint representation of  $\mathcal{L}$  preserves the pairing (5.18):

$$(\text{Ad}(a)(T_1), \text{Ad}(a)(T_2))_e = (T_1, T_2)_e \ \forall a \in \mathcal{L} \ , \ \forall T_1, T_2 \in \text{End}_{\mathbb{R}}(S) \ .$$

*Proof.* For all  $a \in \mathcal{L}$ , we have  $\mathcal{N}_e(a) = \pm \text{id}_S$ , hence  $a^{t_e} = \pm a^{-1}$  and  $\text{Ad}(a)(T) = aTa^{-1} = \pm aTa^{t_e}$ . Thus  $\text{Ad}(a)(T)^{t_e} = \pm aT^{t_e}a^{t_e} = aT^{t_e}a^{-1} = \text{Ad}(a)(T^{t_e})$ . This gives:

$$\begin{aligned} (\text{Ad}(a)(T_1), \text{Ad}(a)(T_2))_e &= \frac{1}{\dim_{\mathbb{R}} S} \text{tr}(\text{Ad}(a)(T_1^{t_e})\text{Ad}(a)(T_2)) = \\ &= \frac{1}{\dim_{\mathbb{R}} S} \text{tr}(\text{Ad}(a)(T_1^{t_e}T_2)) = \frac{1}{\dim_{\mathbb{R}} S} \text{tr}(T_1^{t_e}T_2) = (T_1, T_2)_e, \end{aligned} \tag{5.19}$$

where we used the cyclic property of the trace.  $\square$

Let  $(\ , \ )_{\mathbb{S}}$  and  $(\ , \ )_A$  denote the restrictions of  $(\ , \ )_e$  to the subspaces  $\mathbb{S}$  and  $A$  of  $\text{End}_{\mathbb{R}}(S)$ . Since  $(\ , \ )_e$  is invariant under the adjoint representation of  $\mathcal{L}$ , these restricted pairings are invariant respectively under the Schur and anticommutant representations of  $\mathcal{L}$ . In the complex and quaternionic cases, let  $(\ , \ )_{\text{Im}\mathbb{S}}$  denote the restriction of  $(\ , \ )$  to  $\text{Im}\mathbb{S}$ .

**PROPOSITION 5.18.** *The pairing  $(\ , \ )_{\mathbb{S}}$  coincides with the canonical Euclidean scalar product on  $\mathbb{S}$  (the scalar product which induces the norm of  $\mathbb{S}$ ). Moreover, the Schur representation of the reduced Lipschitz group  $\mathcal{L}$  preserves  $(\ , \ )_{\mathbb{S}}$ .*

*Proof.* The fact that  $\text{Ad}_{\mathbb{S}}$  preserves  $(\ , \ )_{\mathbb{S}}$  follows from Proposition 5.17. Since  $s^{t_e} = c(s)$  for all  $s \in \mathbb{S}$  (see Proposition 5.13), we have:

$$(s_1, s_2)_{\mathbb{S}} = \frac{1}{\dim_{\mathbb{R}} S} \text{tr}(c(s_1)s_2) \quad \forall s_1, s_2 \in \mathbb{S} . \tag{5.20}$$

Thus:

1. In the normal simple or non-simple case, we have  $s_i = \alpha_i \text{id}_S$  with  $\alpha_i \in \mathbb{R}$  and  $(s_1, s_2)_{\mathbb{S}} = \alpha_1 \alpha_2$ , which is the Euclidean scalar product on  $\mathbb{S} \simeq \mathbb{R}$ . In this case, we have  $\text{Ad}_{\mathbb{S}}(a) = \text{id}_{\mathbb{S}}$  for all  $a \in \mathcal{L}$ .
2. In the complex case, we have  $s_i = \alpha_i \text{id}_S + \beta_i J$  with  $\alpha_i, \beta_i \in \mathbb{R}$ , which gives:

$$c(s_1)s_2 = (\alpha_1\alpha_2 + \beta_1\beta_2)\text{id}_S + (\alpha_1\beta_2 - \alpha_2\beta_1)J .$$

Since  $J^{t_e} = c(J) = -J$ , we have  $\text{tr}(J) = 0$ . Thus  $(s_1, s_2)_{\mathbb{S}} = \alpha_1\alpha_2 + \beta_1\beta_2$ , which is the canonical scalar product on  $\mathbb{S} \simeq \mathbb{C}$  (that scalar product which induces the canonically normalized norm of the normed algebra  $\mathbb{S}$ ). In this case, Theorem 5.10 gives:

$$\text{Ad}_{\mathbb{S}}(a) = \begin{cases} \text{id}_{\mathbb{S}} & \text{if } a \in \mathcal{L}^0 \\ c & \text{if } a \in \mathcal{L}^1 \end{cases} .$$

3. In the quaternionic case, let  $J_i$  be an orthonormal and oriented basis of  $\text{Im}\mathbb{S} = U(\gamma)$ . We have  $s_i = \alpha_i \text{id}_S + \sum_{k=1}^3 \beta_i^k J_k$  with  $\alpha_i, \beta_i^k \in \mathbb{R}$ , which gives:

$$c(s_1)s_2 = (\alpha_1\alpha_2 + \sum_{k=1}^3 \beta_1^k \beta_2^k)\text{id}_S + \sum_{k=1}^3 (\alpha_1\beta_2^k - \alpha_2\beta_1^k)J_k - \sum_{k,l,m=1}^3 \epsilon_{klm} \beta_1^k \beta_2^l J_m .$$

Since  $J_k^{t_e} = c(J_k) = -J_k$ , we have  $\text{tr}(J_k) = 0$ . Thus  $(s_1, s_2)_{\mathbb{S}} = \alpha_1\alpha_2 + \sum_{k=1}^3 \beta_1^k \beta_2^k$ , which is the canonical scalar product of the normed algebra  $\mathbb{S}$ .  $\square$

REMARK 5.6. Relation (5.20) implies that  $(\cdot, \cdot)_{\mathbb{S}}$  satisfies the following identities for all  $s_1, s_2 \in \mathbb{S}$ :

$$(c(s_1), c(s_2))_{\mathbb{S}} = (s_1, s_2)_{\mathbb{S}} = (s_2, s_1)_{\mathbb{S}} \quad , \tag{5.21}$$

which shows that  $c : \mathbb{S} \rightarrow \mathbb{S}$  is  $(\cdot, \cdot)_{\mathbb{S}}$ -orthogonal (and hence also  $(\cdot, \cdot)_{\mathbb{S}}$ -symmetric, since  $c^2 = \text{id}_{\mathbb{S}}$ ).

PROPOSITION 5.19. *In the simple cases, the pairing  $(\cdot, \cdot)_A$  on  $A = \mathbb{S}u$  agrees up to sign with the pairing  $(\cdot, \cdot)_{\mathbb{S}}$ :*

1. *In the normal and quaternionic simple cases, we have:*

$$(s_1\omega, s_2\omega)_A = (-1)^p (s_1, s_2)_{\mathbb{S}} \quad \forall s_1, s_2 \in \mathbb{S} \quad ,$$

where  $\beta_{p,q} = (-1)^p$  (see (5.13)).

2. *In the complex case, we have:*

$$(s_1u, s_2u)_A = (s_1, s_2)_{\mathbb{S}} \quad \forall s_1, s_2 \in \mathbb{S} \simeq \mathbb{C}$$

Moreover, the anticommutant representation of  $\mathcal{L}$  preserves the pairing  $(\cdot, \cdot)_A$ .

*Proof.* The last statement follows from the fact that the adjoint representation of  $\mathcal{L}$  preserves  $(\cdot, \cdot)_e$ . Statement 1. follows immediately from Proposition 5.5 using the fact that  $\omega$  commutes with the elements of  $\mathbb{S}$  while  $\omega^{t_e}\omega = \beta_{p,q}\text{id}_S$  (see relation (5.13)), where  $\beta_{p,q} = (-1)^p$  for  $p - q \equiv 0, 2, 4, 6$ . Statement 2 follows from Proposition 5.5 using the fact that  $\text{Ad}_s(u) = c$  while  $u^{t_e}u = \text{id}_S$ .  $\square$

**5.12. The vector representations of  $L$  and  $\mathcal{L}$ .** The following result shows that the vector representation of  $L$  exhausts the full pseudo-orthogonal group in the simple cases; this is a consequence of the fact that pin representations admit twisting elements in the simple case (see Corollary 5.6). In the non-simple case, the vector representation of  $L = L^0$  exhausts the special pseudo-orthogonal group. We identify  $O(W, g)$  with  $O(V, h)$  using the invertible isometry  $\gamma|_V : V \xrightarrow{\sim} W$ , thus viewing the vector representation of the Lipschitz group as a morphism from  $L$  to  $O(V, h)$ .

THEOREM 5.20. *In the simple case, there exists a short exact sequence:*

$$1 \longrightarrow \mathbb{S}^\times \longrightarrow L \xrightarrow{\text{Ad}_q} O(V, h) \longrightarrow 1 \quad , \tag{5.22}$$

which restricts to a short exact sequence:

$$1 \longrightarrow \mathbb{S}^\times \longrightarrow L^0 \xrightarrow{\text{Ad}_q} \text{SO}(V, h) \longrightarrow 1 \quad . \tag{5.23}$$

*In the non-simple case, we have  $L = L^0$  and there exists a short exact sequence:*

$$1 \longrightarrow \mathbb{S}^\times \hookrightarrow L \xrightarrow{\text{Ad}_q} \text{SO}(V, h) \longrightarrow 1 \quad . \tag{5.24}$$

*Proof.* The simple case follows from Theorem 4.9 and Corollary 5.6. In the non-simple case, we have  $L = L^0$  by Theorem 5.10. Hence the exact sequence (4.14) of Theorem 4.9 becomes (5.24).  $\square$

Since the adjoint representation of  $L$  factors through the normalization morphism (5.16) while the restriction of  $\mathcal{N}_e$  to  $\mathbb{S}$  equals the square norm morphism  $M : \mathbb{S} \rightarrow \mathbb{R}_{>0}$ , we obtain:

COROLLARY 5.21. *In the simple case, there exists a short exact sequence:*

$$1 \longrightarrow U(\mathbb{S}) \longrightarrow \mathcal{L} \xrightarrow{\text{Ad}_0} O(V, h) \longrightarrow 1 \quad , \tag{5.25}$$

*which restricts to a short exact sequence:*

$$1 \longrightarrow U(\mathbb{S}) \longrightarrow \mathcal{L}^0 \xrightarrow{\text{Ad}_0} SO(V, h) \longrightarrow 1 \quad . \tag{5.26}$$

*In the non-simple case, we have  $\mathcal{L} = \mathcal{L}^0$  and there exists a short exact sequence:*

$$1 \longrightarrow U(\mathbb{S}) \hookrightarrow \mathcal{L} \xrightarrow{\text{Ad}_0} SO(V, h) \longrightarrow 1 \quad . \tag{5.27}$$

We have a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \tag{5.28} \\
 & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & U(\mathbb{S}) & \longrightarrow & \mathcal{L} & \xrightarrow{\text{Ad}_0} & O(V, h) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \parallel \\
 1 & \longrightarrow & \mathbb{S}^\times & \longrightarrow & L & \xrightarrow{\text{Ad}_0} & O(V, h) \longrightarrow 1 \\
 & & \downarrow M & & \downarrow \mathcal{N}_e & & \\
 & & \mathbb{R}^\times & \xlongequal{\quad} & \mathbb{R}^\times & & \\
 & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 
 \end{array}$$

REMARK 5.7. Recall the sign factor  $\epsilon_d$  which was defined in (1.21) and notice that  $L \cap O(S, \mathcal{B}_e) = \{a \in L | \mathcal{N}_e(a) = +\text{id}_S\} \subset \mathcal{L}$ . Also recall that  $\mathcal{N}_e(u) = +\text{id}_S$  in the complex case. Relation (1.22) gives:

$$N_e|_{\text{Spin}(V, h)} = N|_{\text{Spin}(V, h)} \quad , \quad N_e|_{\text{Pin}_-(V, h)} = \epsilon_d N|_{\text{Pin}_-(V, h)} \quad ,$$

which implies:

$$\ker(N_e : \text{Pin}(V, h) \rightarrow \mathbb{R}^\times) = \text{Spin}^+(V, h) \sqcup \text{Pin}_-^{\epsilon_d}(V, h) \quad .$$

When  $pq \neq 0$ , these observations together with Proposition 5.14 show that the connected components of the reduced Lipschitz group  $\mathcal{L}$  are as follows:

1. In the normal simple case,  $\mathcal{L} \simeq \text{Pin}(V, h)$  has four connected components, which are distinguished by the  $\mathbb{Z}_2$  grading inherited from  $\text{Cl}(V, h)$  and by the value of the canonical Lipschitz norm  $\mathcal{N}_e$ . We have  $\mathcal{L} \cap O(S, \mathcal{B}_e) = \gamma(\text{Spin}^+(V, h) \sqcup \text{Pin}_-^{\epsilon_d}(V, h)) \simeq_{\text{Gp}} \text{Spin}^+(V, h) \sqcup \text{Pin}_-^{\epsilon_d}(V, h)$ , which has two connected components.
2. In the complex case,  $\mathcal{L} \simeq \text{Spin}_\alpha^o(V, h)$  (where  $\alpha \stackrel{\text{def.}}{=} \alpha_{p, q}$ ) has four connected components, which are distinguished by the volume grading  $\mathcal{L} = \mathcal{L}^0 \sqcup \mathcal{L}^1$  (where  $\mathcal{L}^0 \simeq_{\text{Gp}} \text{Spin}^c(V, h)$  and  $\mathcal{L}^1 = u\mathcal{L}^0$ ) and by the value of  $\mathcal{N}_e$ . We have  $\mathcal{L} \cap O(S, \mathcal{B}_e) = \gamma(\text{Spin}^+(V, h) \cdot U(1)) \sqcup \gamma(\text{Spin}^+(V, h) \cdot U(1))u \simeq_{\text{Gp}} \text{Spin}^+(V, h) \cdot O_2(\alpha)$ , which has two connected components.

3. In the quaternionic simple case,  $\mathcal{L} \simeq \text{Pin}^q(V, h)$  has four connected components, which are distinguished by the  $\mathbb{Z}_2$  grading inherited from  $\text{Cl}(V, h)$  and by the value of  $\mathcal{N}_e$ . We have  $\mathcal{L} \cap \text{O}(S, \mathcal{B}_e) = \gamma(\text{Spin}^+(V, h) \sqcup \text{Pin}_-^{\varepsilon_d}(V, h)) \cdot \text{Sp}(1) \simeq [\text{Spin}^+(V, h) \sqcup \text{Pin}_-^{\varepsilon_d}(V, h)] \cdot \text{Sp}(1)$ , which has two connected components.
4. In the normal non-simple case,  $\mathcal{L} \simeq \text{Spin}(V, h)$  has two connected components, which are distinguished by the value of  $\mathcal{N}_e$ . We have  $\mathcal{L} \cap \text{O}(S, \mathcal{B}_e) = \gamma(\text{Spin}^+(V, h))$ , which is connected.
5. In the quaternionic non-simple case,  $\mathcal{L} \simeq \text{Spin}^q(V, h)$  has two connected components, which are distinguished by the value of  $\mathcal{N}_e$ . We have  $\mathcal{L} \cap \text{O}(S, \mathcal{B}_e) = \gamma(\text{Spin}^+(V, h)) \cdot \text{Sp}(1) \simeq \text{Spin}^+(V, h) \cdot \text{Sp}(1)$ , which is connected.

**5.13. The anticommutant representation of  $L$  in the simple case.** Assume that we are in the simple case. Recall that  $A \subset \text{End}_{\mathbb{R}}(S)$  denotes the anticommutant subspace of  $\gamma : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S)$ , which is a left  $\mathbb{S}$ -module of rank one (see Proposition (5.5)). Also recall the anticommutant representation  $\text{Ad}_A : L \rightarrow \text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p})$  of the Lipschitz group (introduced in Subsection 4.8), where  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p})$  denotes the group of those twisted automorphisms of the  $\mathbb{S}$ -module  $A$  which are twisted-orthogonal with respect to the Schur pairing  $\mathfrak{p}$  (see Appendix C).

PROPOSITION 5.22. *The following statements hold in the simple case:*

1. In the normal simple case, we have  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A) = \text{Aut}_{\mathbb{R}}(A) = \mathbb{R}^{\times} \text{id}_A \simeq \mathbb{R}^{\times}$  and  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p}) = \{-\text{id}_A, \text{id}_A\} \simeq \mathbb{Z}_2$ .
2. In the complex case, we have  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A) \simeq \mathbb{C}^{\times} \rtimes_{\Phi} \mathbb{Z}_2 \simeq \text{GL}(2, \mathbb{R})$  and  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p}) \simeq \text{U}(1) \rtimes_{\Phi} \mathbb{Z}_2 \simeq \text{O}(2, \mathbb{R})$ , where  $\Phi : \mathbb{Z}_2 \rightarrow \text{Aut}_{\text{Gp}}(\mathbb{C}^{\times})$  is the group morphism given by  $\Phi(\hat{0})(z) = z$  and  $\Phi(\hat{1})(z) = \bar{z}$  ( $z \in \mathbb{C}^{\times}$ ).
3. In the quaternionic simple case, we have  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A) \simeq (\mathbb{H}^{\times})^{\text{op}} \rtimes_{\Phi} \text{SO}(3, \mathbb{R})$ , where  $\Phi : \text{SO}(3, \mathbb{R}) \rightarrow \text{Aut}_{\text{Gp}}(\mathbb{H}^{\times})$  is the group morphism given by  $\Phi([q]) = \text{Ad}(q)|_{\mathbb{H}^{\times}}$  for all  $[q] \in \text{U}(\mathbb{H})/\{-1, 1\} = \text{SO}(3, \mathbb{R})$ . We also have  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p}) \simeq \{-\text{id}_A, \text{id}_A\} \times \text{Aut}_{\text{Alg}}(\mathbb{S}) \simeq \mathbb{Z}_2 \times \text{SO}(3, \mathbb{R})$ .

*Proof.*

1. Normal simple case. In this case, we have  $\mathbb{S} \simeq \mathbb{R}$  and  $A = \mathbb{R}\omega$ . Since  $\text{Aut}_{\text{Alg}}(\mathbb{R}) = \{\text{id}_{\mathbb{R}}\}$ , Proposition C.2 of Appendix C gives  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A) = \text{Aut}_{\mathbb{S}}(A) = \mathbb{R}^{\times} \text{id}_A \simeq \mathbb{R}^{\times} \simeq \text{GL}(1, \mathbb{R})$ . Proposition 5.7 shows that the Schur pairing coincides up to sign with the usual scalar product on  $\mathbb{R}$ . Since  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A) = \text{Aut}_{\mathbb{S}}(A)$ , the  $\mathfrak{p}$ -orthogonality condition (C.2) gives  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p}) \simeq \text{O}(1, \mathbb{R}) \simeq \mathbb{Z}_2$ .
2. In the complex case, we have  $\text{Aut}_{\text{Alg}}(\mathbb{S}) = \{\text{id}_{\mathbb{S}}, c\} \simeq \mathbb{Z}_2$  and the element  $u = D$  of Proposition 5.5 is a basis of  $A$  over  $\mathbb{S} \simeq \mathbb{C}$ . Hence the splitting morphism  $G_u : \text{Aut}_{\text{Alg}}(\mathbb{S}) \rightarrow \text{Aut}_{\mathbb{S}}^{\text{tw}}(A)$  in the proof of Proposition C.2 takes  $c$  into the twisted morphism  $\varphi \in \text{Aut}_{\mathbb{S}}^{\text{tw}}(A)$  given by  $\varphi(sD) = c(s)D$  for all  $s \in \mathbb{S}$ . Since  $c$  corresponds to complex conjugation, Proposition C.2 gives  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A) \simeq \mathbb{C}^{\times} \rtimes_{\Phi} \mathbb{Z}_2 \simeq \text{O}(2, \mathbb{R})$  if we identify  $\mathbb{S}$  with  $\mathbb{C}$  as in Proposition 5.3. With this identification, Proposition 5.7 shows that the Schur pairing coincides (up to sign) with the Euclidean scalar product on  $\mathbb{C} \equiv \mathbb{R}^2$ , which is  $\mathbb{R}$ -valued. Hence the Schur pairing satisfies  $c(\mathfrak{p}(x_1, x_2)) = \mathfrak{p}(x_1, x_2)$  and condition (C.2) gives  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p}) \simeq \text{U}(1) \rtimes_{\Phi} \mathbb{Z}_2 \simeq \text{O}(2, \mathbb{R})$ .
3. In the quaternionic simple case, we have  $\text{Aut}_{\text{Alg}}(\mathbb{S}) \simeq \text{U}(\mathbb{S})/\{-\text{id}_{\mathbb{S}}, \text{id}_{\mathbb{S}}\} \simeq \text{SO}(3, \mathbb{R})$  and the element  $u = \omega$  generates  $A$  over  $\mathbb{S}$ . The splitting morphism  $G_u : \text{Aut}_{\text{Alg}}(\mathbb{S}) \rightarrow \text{Aut}_{\mathbb{S}}^{\text{tw}}(A)$  in the proof of Proposition C.2 takes  $[q] \in \text{U}(\mathbb{S})/\{-\text{id}_{\mathbb{S}}, \text{id}_{\mathbb{S}}\}$  into the twisted morphism  $\varphi_q \in \text{Aut}_{\mathbb{S}}^{\text{tw}}(A)$  given by  $\varphi_q(s\omega) = \text{Ad}(q)(s)\omega$  for all  $s \in \mathbb{S}$ . Identifying  $\mathbb{S}$  with  $\mathbb{H}$  as in Proposition 5.3, Proposition C.2 gives

$\text{Aut}_{\mathbb{S}}^{\text{tw}}(A) \simeq (\mathbb{H}^{\times})^{\text{op}} \rtimes_{\text{Res}} \text{Aut}_{\text{Alg}}(\mathbb{H}) \simeq (\mathbb{H}^{\times})^{\text{op}} \rtimes_{\Phi} \text{SO}(3, \mathbb{R})$ . Any twisted automorphism  $(\varphi_0, \varphi)$  of  $A = \mathbb{S}\omega$  satisfies  $\varphi(\omega) = \sigma\omega$  for some  $\sigma \in \mathbb{S}$  and has the form:

$$\begin{aligned} \varphi_0 &= \text{Ad}_s(s_0) \\ \varphi(s\omega) &= \varphi_0(s)\sigma\omega \end{aligned}$$

for some  $s_0 \in U(\mathbb{S})$ . Since  $s$  and  $\omega$  commute, it is easy to see that  $(\varphi_0, \varphi)$  satisfies condition (C.2) with the Schur pairing given in Proposition 5.7 iff:

$$(\varphi_0(s)\sigma)^2 = \varphi_0(s^2) \quad \forall s \in \mathbb{S} . \tag{5.29}$$

For  $s = \text{id}_{\mathbb{S}}$ , this gives  $\sigma^2 = \text{id}_{\mathbb{S}}$ , which amounts to  $\sigma \in \{-\text{id}_{\mathbb{S}}, \text{id}_{\mathbb{S}}\}$  (because  $\mathbb{S}$  is a division algebra); both of these solutions satisfy (5.29). Thus  $\varphi(s\omega) = \pm \text{Ad}(s_0)(s)\omega$  and  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p}) \simeq \{-\text{id}_{\mathbb{S}}, \text{id}_{\mathbb{S}}\} \times \text{Aut}_{\text{Alg}}(\mathbb{S})$ . □

**THEOREM 5.23.** *The following statements hold in the simple case:*

1. *In the normal simple case, the anticommutant representation gives a short exact sequence:*

$$1 \longrightarrow L^0 \hookrightarrow L \xrightarrow{\text{Ad}_A} \{-\text{id}_A, \text{id}_A\} \simeq \mathbb{Z}_2 \longrightarrow 1 , \tag{5.30}$$

where  $L^0 = \gamma(G_+(V, h)) \simeq G_+(V, h)$  and  $L^1 = \gamma(G_-(V, h)) \simeq G_-(V, h)$ . We have:

$$\text{Ad}_A(a) = \begin{cases} +\text{id}_A & \text{if } a \in L^0 = \gamma(G_+(V, h)) \\ -\text{id}_A & \text{if } a \in L^1 = \gamma(G_-(V, h)) \end{cases} . \tag{5.31}$$

2. *In the complex case, the anticommutant representation of  $L$  gives a short exact sequence:*

$$1 \longrightarrow \gamma(G_+(V, h)) \hookrightarrow L \xrightarrow{\text{Ad}_A} \text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p}) \simeq O(2, \mathbb{R}) \longrightarrow 1 \tag{5.32}$$

which restricts to a short exact sequence:

$$1 \longrightarrow \gamma(G_+(V, h)) \hookrightarrow L^0 = \gamma(G^e(V, h)) \xrightarrow{\text{Ad}_A} \text{Aut}_{\mathbb{S}}(A, \mathfrak{p}) \simeq \text{SO}(2, \mathbb{R}) \longrightarrow 1 \tag{5.33}$$

and for all  $a \in G_+(V, h)$  and all  $\theta \in \mathbb{R}$  we have:

$$\text{Ad}_A(\gamma(ae^{\theta\nu}))(u) = \text{Ad}_A(\gamma(ae^{\theta\nu})u)(u) = e^{2\theta J}u . \tag{5.34}$$

3. *In the quaternionic simple case, we have  $L = \gamma(G(V, h))U(\mathbb{S})$  with  $L^0 = U(\mathbb{S})\gamma(G_+(V, h))U(\mathbb{S})$  and  $L^1 = \gamma(G_-(V, h))U(\mathbb{S})$ . The anticommutant representation of  $L$  gives a short exact sequence:*

$$1 \longrightarrow \gamma(G_+(V, h)) \hookrightarrow L \xrightarrow{\text{Ad}_A} \text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p}) \simeq \mathbb{Z}_2 \times \text{SO}(3, \mathbb{R}) \longrightarrow 1 \tag{5.35}$$

which restricts to a short exact sequence:

$$1 \longrightarrow \gamma(G_+(V, h)) \hookrightarrow L^0 \longrightarrow \text{SO}(3, \mathbb{R}) \longrightarrow 1 \tag{5.36}$$

and for all  $s \in \mathbb{S}$  we have:

$$\text{Ad}_A(a)(s\omega) = \begin{cases} +\text{Ad}_s(a)(s)\omega & \text{if } a \in L^0 \\ -\text{Ad}_s(a)(s)\omega & \text{if } a \in L^1 \end{cases} . \tag{5.37}$$



*Proof.*

1. The normal simple case. We have  $A = \mathbb{R}\omega$  and  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p}) = \{\text{id}_A, -\text{id}_A\}$ . Relation (5.31) follows from the fact that  $a \in L$  commutes with  $\omega$  iff  $a \in L^0$  and anticommutes with  $\omega$  iff  $a \in L^1$ . Thus  $\ker \text{Ad}_A = L^0$ . The map  $\gamma$  induces an isomorphism from  $G(V, h)$  to  $L$  and (since  $d$  is even) we have  $L^0 = \gamma(G_+(V, h))$  and  $L^1 = \gamma(G_-(V, h))$ . The set  $L^1 = \gamma(G_-(V, h))$  is non-empty since  $V \neq 0$ ; this shows that  $\text{Ad}_A$  is surjective.
2. The complex case. In this case, we have  $A = \mathbb{S}u$  for  $u$  as in Proposition 5.5 and  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p}) \simeq O(2, \mathbb{R})$ . For  $a \in L^0 = \gamma(G^e(V, h))$ , we have  $\text{Ad}_s(a) = \text{id}_{\mathbb{S}}$ , hence  $\text{Ad}(a)$  is  $\mathbb{S}$ -linear and  $\text{Ad}_A(a)$  corresponds to an element of  $SO(2, \mathbb{R})$ . For  $a \in L^1 = uL^0$ , we have  $\text{Ad}_s(a) = c$  (see Theorem 5.10), hence  $\text{Ad}(a)$  is  $\mathbb{S}$ -antilinear and  $\text{Ad}_A(a)$  corresponds to an element of  $O_-(2, \mathbb{R})$ . Thus  $\text{Ad}_A(a) = \text{id}_A$  iff  $a = \gamma(x) \in L^0$  for some  $x \in G^e(V, h)$  which satisfies  $\text{Ad}(u)(\gamma(x)) = \gamma(x)$ . Since  $\gamma$  is injective and  $\text{Ad}(u) \circ \gamma = \gamma \circ \pi$ , this amounts to  $\pi(x) = x$  i.e.  $x \in G_+(V, h)$ . This shows that  $\ker \text{Ad}_A = \gamma(G_+(V, h))$ . We have  $Z(V, h) = \mathbb{R} \oplus \mathbb{R}\nu \simeq_{\text{Alg}} \mathbb{C}$  and  $\nu \in Z(V, h)$ . For any  $x, y \in \mathbb{R}$ , we have  $e^{x+y\nu} \in Z(V, h)^\times \subset G^e(V, h)$  (recall that  $\nu^2 = -1$  in the complex case). Since  $\gamma(\nu) = \omega = J$ , we have  $\gamma(Z(V, h)^\times) = \mathbb{S} \subset L$  and  $\gamma(e^{\theta\nu}) = e^{\theta J} \in \gamma(G^e(V, h)) = L^0$ . This implies (5.34) upon using the fact that  $u$  and  $J$  anticommute. In particular,  $\text{Ad}_A(U(\mathbb{S}))$  corresponds to  $SO(2, \mathbb{R}) \simeq U(1)$  and  $\text{Ad}(U(\mathbb{S})u)$  corresponds to  $O_-(2, \mathbb{R})$ . Thus  $\text{Ad}_A$  is surjective, the sequence (5.32) holds and it restricts to (5.33).
3. The quaternionic simple case. In this case, we have  $A = \mathbb{S}\omega$ . Clearly  $a \in \ker \text{Ad}_A$  implies  $\text{Ad}_s(a) = \text{id}_{\mathbb{S}}$  and hence  $a \in L_\gamma = \gamma(G(V, h))$  by Proposition 5.9. Since  $d$  is even, we have  $\text{Ad}(a)(\omega) = \epsilon\omega$  for  $a \in \gamma(G_\epsilon(V, h))$  (where  $\epsilon \in \{-1, 1\}$ ) and we conclude that  $\ker \text{Ad}_A = \gamma(G_+(V, h))$ . Since  $L = U(\mathbb{S})\gamma(G(V, h))$ , any  $a \in L$  can be written as  $a = s_a a_0$  for some  $s_a \in U(\mathbb{S})$  and some  $a_0 \in \gamma(G(V, h))$ . Since  $s_a$  commutes with  $a_0$  and  $\omega$ , we have  $\text{Ad}(a)(\omega) = \text{Ad}(a_0)(\omega)$  and hence:

$$\text{Ad}_A(a)(\omega) = \epsilon\omega \quad \text{for } a_0 \in \gamma(G_\epsilon(V, h)) \quad . \tag{5.38}$$

This gives:

$$\text{Ad}_A(a)(s\omega) = \text{Ad}_s(a)(s)\text{Ad}_A(a)(\omega) = \epsilon\text{Ad}_s(a)(s)\omega \quad \text{for } a_0 \in \gamma(G_\epsilon(V, h)) \quad .$$

Thus (5.37) holds. Since  $\text{Ad}_s(a_0) = \text{id}_{\mathbb{S}}$ , we have  $\text{Ad}_s(a) = \text{Ad}_s(s_0)\text{Ad}_s(a_0) = \text{Ad}_s(s_0)$ , which together by (5.37) implies that the sequences (5.35) and (5.36) are exact.

□

**5.14. Relation between the elementary representations of  $\mathcal{L}$  and those of  $\Lambda(V, h)$ .**

DEFINITION 5.7. *The characteristic representation  $\mu_{\mathcal{L}}$  of the reduced Lipschitz group  $\mathcal{L}$  is defined as follows:*

1. *In the normal simple or non-simple case, it is the Schur representation of  $\mathcal{L}$ , i.e. the trivial one-dimensional representation  $\mu_{\mathcal{L}} \stackrel{\text{def.}}{=} \text{Ad}_{\mathbb{S}} : \mathcal{L} \rightarrow 1$ .*
2. *In the complex case, it is the anticommutant representation  $\mu_{\mathcal{L}} \stackrel{\text{def.}}{=} \text{Ad}_A : \mathcal{L} \rightarrow O(A, ( , )_A) \simeq O(2, \mathbb{R})$ .*
3. *In the quaternionic simple or non-simple case,  $\mu_{\mathcal{L}} : \mathcal{L} \rightarrow SO(\text{Im}\mathbb{S}, ( , )_{\mathbb{S}}|_{\text{Im}\mathbb{S}}) \simeq SO(3, \mathbb{R})$  is the Schur representation restricted to  $\text{Im}\mathbb{S}$ :*

$$\mu_{\mathcal{L}}(a) \stackrel{\text{def.}}{=} \text{Ad}_s(a)|_{\text{Im}\mathbb{S}} \quad \forall a \in L \quad .$$

DEFINITION 5.8. *The basic representation of  $\mathcal{L}$  is the representation  $\rho_{\mathcal{L}} = \text{Ad}_0 \times \mu_{\mathcal{L}}$ .*

Notice that  $\rho_{\mathcal{L}} = \text{Ad}_0$  in the normal (simple and non-simple) cases.

To state the next result precisely, we need certain identifications which we discuss in turn.

1. Recall that  $\gamma|_V : V \xrightarrow{\sim} W = \gamma(V)$  is an invertible isometry between the quadratic spaces  $(V, h)$  and  $(W, g)$ . This induces an isomorphism of groups  $\text{Ad}(\gamma|_V) : \text{O}(V, h) \xrightarrow{\sim} \text{O}(W, g)$  given by:

$$\text{Ad}(\gamma|_V)(R) \stackrel{\text{def.}}{=} (\gamma|_V) \circ R \circ (\gamma|_V)^{-1} \in \text{O}(W, g) \quad \forall R \in \text{O}(V, g)$$

and we have:

$$\text{Ad}_0 \circ \gamma|_{\text{G}^e(V, h)} = \text{Ad}(\gamma|_V) \circ \text{Ad}_0^e \quad , \tag{5.39}$$

where  $\text{Ad}_0^e : \text{G}^e(V, h) \rightarrow \text{O}(V, h)$  is the untwisted vector representation of the extended Clifford group while  $\text{Ad}_0 : \text{L} \rightarrow \text{O}(W, g)$  is the vector representation of the Lipschitz group.

2. In the complex case, any choice of orientation of  $V$  and of an element  $u \in A$  as in Proposition 5.5 gives an  $\mathbb{R}$ -linear isomorphism  $g_{\nu, u} : \mathbb{R}^2 \xrightarrow{\sim} A$  given by:

$$g_{\nu, u}(x, y) = (x - \alpha_{p, q} y J_{\nu})u \in A \quad \forall x, y \in \mathbb{R} \quad ,$$

where  $J_{\nu} = \gamma(\nu)$  and  $\nu$  is the Clifford volume element of  $V$  with respect to the given orientation. This isomorphism transports both  $(, )_A$  and the Schur pairing  $\mathfrak{p}$  to scalar products on  $\mathbb{R}^2$  which coincide up to sign with the canonical scalar product. It follows that the unital isomorphism of  $\mathbb{R}$ -algebras  $\text{Ad}(g_{\nu, u}) : \text{End}_{\mathbb{R}}(\mathbb{R}^2) \xrightarrow{\sim} \text{End}_{\mathbb{R}}(A)$  restricts to an isomorphism of groups  $\text{Ad}(g_{\nu, u}) : \text{O}(2, \mathbb{R}) \xrightarrow{\sim} \text{O}(A, (, )_A)$ .

3. In the quaternionic (simple and non-simple) cases, any unital isomorphism of  $\mathbb{R}$ -algebras  $f : \mathbb{H} \xrightarrow{\sim} \mathbb{S}$  induces a unital isomorphism of  $\mathbb{R}$ -algebras  $\text{Ad}(f) : \text{End}_{\mathbb{R}}(\mathbb{H}) \rightarrow \text{End}_{\mathbb{R}}(\mathbb{S})$  given by:

$$\text{Ad}(f)(g) \stackrel{\text{def.}}{=} f \circ g \circ f^{-1} \quad , \quad \forall g \in \text{End}_{\mathbb{R}}(\mathbb{H}) \quad .$$

Consider the morphism of groups  $\text{Ad}_{\mathbb{S}} : \text{U}(\mathbb{S}) \rightarrow \text{Aut}_{\text{Alg}}(\mathbb{S})$  defined through:

$$\text{Ad}_{\mathbb{S}}(s)(s') \stackrel{\text{def.}}{=} s s' s^{-1} \quad \forall s \in \text{U}(\mathbb{S}) \quad \forall s' \in \mathbb{S} \quad .$$

Then:

$$\text{Ad}_{\mathbb{S}} \circ f|_{\text{U}(\mathbb{H})} = \text{Ad}(f) \circ \text{Ad}_{\mathbb{H}} \quad . \tag{5.40}$$

When viewed as a linear representation of  $\text{U}(\mathbb{S})$  over  $\mathbb{R}$ ,  $\text{Ad}_{\mathbb{S}}$  decomposes as a direct sum of the trivial representation supported on the subspace  $\mathbb{R}\text{id}_{\mathbb{S}}$  and the representation  $\text{Ad}_{\mathbb{S}}^{\bullet} : \text{U}(\mathbb{S}) \rightarrow \text{SO}(\text{Im}\mathbb{S}, (, )_{\text{Im}\mathbb{S}})$  given by  $\text{Ad}_{\mathbb{S}}^{\bullet}(s) \stackrel{\text{def.}}{=} \text{Ad}_{\mathbb{S}}(s)|_{\text{Im}\mathbb{S}}$  for all  $s \in \text{U}(\mathbb{S})$ . We have  $f(\text{Im}\mathbb{H}) = \text{Im}\mathbb{S}$  and  $f|_{\text{Im}\mathbb{H}}$  is a linear isomorphism from  $\text{Im}\mathbb{H}$  to  $\text{Im}\mathbb{S}$ , which induces an isomorphism of groups  $\text{Ad}(f|_{\text{Im}\mathbb{H}}) : \text{SO}(\text{Im}\mathbb{H}) \xrightarrow{\sim} \text{SO}(\text{Im}\mathbb{S}, (, )_{\text{Im}\mathbb{S}})$ . Relation (5.40) implies:

$$\text{Ad}_{\mathbb{S}}^{\bullet} \circ (f|_{\text{U}(\mathbb{H})}) = \text{Ad}(f|_{\text{Im}\mathbb{H}}) \circ \text{Ad}_{\bullet} \quad , \tag{5.41}$$

where  $\text{Ad}_{\bullet} : \text{U}(\mathbb{H}) \rightarrow \text{SO}(\text{Im}\mathbb{H}) = \text{SO}(3, \mathbb{R})$  is the adjoint representation of  $\text{U}(\mathbb{H}) = \text{Sp}(1)$  (see Section 2).

Recall that  $\lambda, \mu$  and  $\rho$  are the vector, characteristic and basic representations of the extended spinor group  $\Lambda(V, h)$ . The following result shows that the elementary representations  $\text{Ad}_0, \mu_{\mathcal{L}}$  and  $\rho_{\mathcal{L}}$  of the reduced Lipschitz group  $\mathcal{L}$  of a pin representation  $\gamma$  agree with the elementary representations  $\lambda, \mu$  and  $\rho$  of the enlarged spinorial group  $\Lambda(V, h)$  studied in Section 2.

**THEOREM 5.24.** *Let  $\varphi : \Lambda(V, h) \xrightarrow{\sim} \mathcal{L}$  be any of the admissible isomorphisms given in Theorem 5.16. Then:*

1. *In the normal (simple and non-simple) cases, we have  $\varphi = \gamma|_{\Lambda(V, h)}$  and the following relations hold:*

$$\begin{aligned} \text{Ad}(\gamma|_V) \circ \lambda &= \text{Ad}_0 \circ \varphi \quad , \\ \mu &= \mu_{\mathcal{L}} \circ \varphi \quad , \\ \text{Ad}(\gamma|_V) \circ \rho &= \rho_{\mathcal{L}} \circ \varphi \quad . \end{aligned} \tag{5.42}$$

2. *In the complex case,  $\varphi = \varphi_{\nu, u}$  is determined by an orientation of  $(V, h)$  and by a choice of element  $u \in A$  as in Proposition 5.5 and the following relations hold:*

$$\begin{aligned} \text{Ad}(\gamma|_V) \circ \lambda &= \text{Ad}_0 \circ \varphi_{\nu, u} \quad , \\ \text{Ad}(g_{\nu, u}) \circ \mu &= \mu_{\mathcal{L}} \circ \varphi_{\nu, u} \quad , \\ [\text{Ad}(\gamma|_V) \times \text{Ad}(g_{\nu, u})] \circ \rho &= \rho_{\mathcal{L}} \circ \varphi_{\nu, u} \quad . \end{aligned} \tag{5.43}$$

3. *In the quaternionic (simple and non-simple) cases,  $\varphi = \varphi_f$  is determined by  $\gamma$  and by a unital isomorphism of normed  $\mathbb{R}$ -algebras  $f : \mathbb{H} \xrightarrow{\sim} \mathbb{S}$  and the following relations hold:*

$$\begin{aligned} \text{Ad}(\gamma|_V) \circ \lambda &= \text{Ad}_0 \circ \varphi_f \quad , \\ \text{Ad}(f|_{\text{Im}\mathbb{H}}) \circ \mu &= \mu_{\mathcal{L}} \circ \varphi_f \quad , \\ [\text{Ad}(\gamma|_V) \times \text{Ad}(f|_{\text{Im}\mathbb{H}})] \circ \rho &= \rho_{\mathcal{L}} \circ \varphi_f \quad . \end{aligned} \tag{5.44}$$

This result gives the intrinsic meaning of the elementary representations of the enlarged spinor groups considered in Section 2. It also shows that one can use the model  $\Lambda(V, h)$  of the reduced Lipschitz group  $\mathcal{L}$  (which is homotopy equivalent with the full Lipschitz group  $L$ ) when developing the theory of Lipschitz structures for the Lipschitz groups of pin representations.

*Proof.*

1. **Normal simple case.** In this case, we have  $\varphi = \gamma|_{\text{Pin}(V, h)} : \text{Pin}(V, h) \xrightarrow{\sim} \mathcal{L}$ . Since  $\text{Pin}(V, h)$  is a subgroup of  $G^e(V, h)$ , relation (5.39) gives  $\text{Ad}_0 \circ \varphi = \text{Ad}(\gamma|_V)(\text{Ad}_0^{\text{Cl}}|_{\text{Pin}(V, h)}) = \text{Ad}(\gamma|_V) \circ \lambda$ , which is the first equation in (5.42). The second relation holds since  $\mu$  and  $\mu_{\mathcal{L}}$  are the trivial representations, while the last relation in (5.42) holds because  $\rho = \lambda$  and  $\rho_{\mathcal{L}} = \text{Ad}_0|_{\mathcal{L}}$ .
2. **Normal non-simple case.** In this case, we have  $\varphi = \gamma|_{\text{Spin}(V, h)} : \text{Spin}(V, h) \xrightarrow{\sim} \mathcal{L}$ , with  $\mu_{\mathcal{L}}$  and  $\mu$  trivial and  $\rho = \lambda = \text{Ad}_0^{\text{Cl}}|_{\text{Spin}(V, h)}$ ,  $\rho_{\mathcal{L}} = \text{Ad}_0|_{\mathcal{L}}$ . Relations (5.42) hold by an argument similar to that for the normal simple case.
3. **Complex case.** Let  $\alpha := \alpha_{p, q}$  and  $\psi := \psi_{\alpha}$  be the isomorphism of Proposition 2.2. We have  $\Lambda(V, h) = \text{Spin}^o(V, h)$  and  $\varphi([a, \psi(e^{i\theta}, \kappa)]) = \gamma(a)e^{\theta J_{\nu}}u^{\kappa}$  for all  $a \in \text{Spin}(V, h)$ , with  $(e^{i\theta}, \kappa) \in \text{O}_2(\alpha)$ , where  $\theta \in \mathbb{R}$ ,  $\kappa \in \mathbb{Z}_2$ . Thus:

$$(\text{Ad}_0 \circ \varphi)([a, \psi(e^{i\theta}, \kappa)]) = (\text{Ad}_0 \circ \gamma)(a)\text{Ad}_0(e^{\theta J_{\nu}}) \text{Ad}_0(u^{\kappa}) = (-1)^{\kappa}(\text{Ad}_0 \circ \gamma)(a) \quad ,$$

where we used the fact that  $\text{Ad}_0(e^{\theta J_\nu}) = \text{id}_W$  (since  $e^{\theta J_\nu} \in \mathbb{S}^\times$ ) and  $\text{Ad}_0(u) = -\text{id}_W$ . This can also be written as:

$$(\text{Ad}_0 \circ \varphi)([a, \psi(e^{i\theta}, \kappa)]) = \eta(e^{i\theta}, \kappa)(\text{Ad}_0 \circ \gamma)(a) = \eta(e^{i\theta}, \kappa)\text{Ad}(\gamma|_V) \circ \text{Ad}_0^{\text{Cl}}(a) \quad ,$$

where  $\eta := \eta_\alpha : \text{O}_2(\alpha) \rightarrow \mathbb{G}_2$  is the abstract determinant introduced in Subsection 2.3 and in the last equality we used relations (5.39). Since  $\eta = \det \circ \text{Ad}_0^{(2)} \circ \psi$  (see Proposition 2.2), the relation above becomes:

$$\begin{aligned} (\text{Ad}_0 \circ \varphi)([a, \psi(e^{i\theta}, \kappa)]) &= \det(\text{Ad}_0^{(2)}(\psi(e^{i\theta}, \kappa)))\text{Ad}(\gamma|_V) \circ \text{Ad}_0^{\text{Cl}}(a) = \\ &= \text{Ad}(\gamma|_V) \circ \lambda([a, \psi(e^{i\theta}, \kappa)]) \quad , \end{aligned}$$

where in the last line we used Definition 2.5. This shows that the first relation of (5.43) holds. Since  $\mu_{\mathcal{L}} = \text{Ad}_A$ , Theorem 5.23 gives:

$$(\mu_{\mathcal{L}} \circ \varphi)([a, \psi(e^{i\theta}, \kappa)])(u) = \text{Ad}_A(\gamma(a)e^{\theta J_\nu} u^\kappa)(u) = e^{2\theta J_\nu} u \quad ,$$

Setting  $g := g_{\nu, u}$ , we have  $g(1) = u$  and  $g(e^{-2i\alpha\theta}) = e^{2\theta J} u$  and the relation above gives:

$$\begin{aligned} (\mu_{\mathcal{L}} \circ \varphi)([a, \psi(e^{i\theta}, \kappa)]) &= \text{Ad}(g)(R(-2\alpha\theta)C_0^\kappa) = \\ &= (\text{Ad}(g) \circ \Phi_0^{(-\alpha)} \circ \sigma_\alpha)(e^{i\theta}, \kappa) = (\text{Ad}(g) \circ \text{Ad}_0^{(2)})(\psi(e^{i\theta}, \kappa)) \quad , \end{aligned}$$

where we identified  $\mathbb{R}^2 = \mathbb{C}$  and used Proposition 2.2 in the last line. Here,  $\sigma_\alpha$  is the squaring representation of  $\text{O}_2(\alpha)$  discussed in Subsection 2.3. This shows that the second relation in (5.43) holds. The last relation in (5.43) now follows because  $\rho_{\mathcal{L}} = \lambda_{\mathcal{L}} \times \mu_{\mathcal{L}}$  and  $\rho = \lambda \times \mu$ .

4. Quaternionic simple case. In this case, we have  $\Lambda(V, h) = \text{Pin}(V, h) \cdot \text{Sp}(1)$  and  $\varphi([a, q]) = \gamma(a)f(q) \in \mathcal{L}$  for all  $a \in \text{Pin}(V, h)$  and  $q \in \text{Sp}(1) = \text{U}(\mathbb{H})$ , where  $f : \mathbb{H} \rightarrow \mathbb{S}$  is any unital isomorphism of normed  $\mathbb{R}$ -algebras. Thus:

$$(\text{Ad}_0 \circ \varphi)([a, q]) = \text{Ad}_0(\gamma(a))\text{Ad}_0(f(q)) = \text{Ad}_0(\gamma(a)) \quad ,$$

where we used the fact that  $\text{Ad}_0(s) = \text{id}_W$  for all  $s \in \text{U}(\mathbb{S})$  (since  $W = \gamma(V) \subset C$ ). Using relation (5.39), the equation above gives  $\text{Ad}_0 \circ \varphi = \text{Ad}(\gamma|_V) \circ \lambda$  (where  $\lambda$  is the vector representation of  $\text{Pin}^q(V, h)$  introduced in Section 2), showing that the first relation in (5.44) holds. For the Schur representation of  $\mathcal{L}$ , we have:

$$(\text{Ad}_s \circ \varphi)([a, q]) = \text{Ad}_s(\gamma(a))\text{Ad}(f(q))|_{\mathbb{S}} = \text{Ad}_{\mathbb{S}}(f(q)) \quad ,$$

where we used the fact that  $\text{Ad}_s(\gamma(a)) = \text{id}_{\mathbb{S}}$ . Restricting this to the subspace  $\text{Im}\mathbb{S}$  and using relation (5.41) gives the second relation in (5.44). The third relation also holds, because  $\rho_{\mathcal{L}} = \lambda_{\mathcal{L}} \times \mu_{\mathcal{L}}$  and  $\rho = \lambda \times \mu$ .

5. Quaternionic non-simple case. The argument is almost identical to that for the quaternionic simple case. □

**6. Real pinor bundles and real Lipschitz structures.** In this section, we discuss real Lipschitz structures and bundles of Clifford modules over a pseudo-Riemannian manifold  $(M, g)$ , establishing an equivalence between the corresponding groupoids. This shows that the classification of the former agrees with that of the latter. For the case of bundles of irreducible Clifford modules, the relevant Lipschitz

structures are called *elementary* and can be described using the enlarged spinor groups of Section 2. Combining this with the results of Section 7 will allow us, in the next section, to extract the topological obstructions to existence of elementary Lipschitz structures and hence to existence of bundles of irreducible Clifford modules in each dimension and signature.

Let  $(M, g)$  be a connected second countable smooth pseudo-Riemannian manifold of dimension greater than zero and let  $g^*$  denote the metric induced on  $T^*M$ . For reasons having to do with various applications (to be discussed in other papers), we choose to work with the Clifford bundle of the pseudo-Euclidean vector bundle  $(T^*M, g^*)$ , rather than with the Clifford bundle of  $(TM, g)$  (as is more customary). Since the pseudo-Euclidean vector bundles  $(TM, g)$  and  $(T^*M, g^*)$  are isometric through the musical isomorphism, this is of course equivalent with the more traditional approach.

**6.1. Real pinor bundles.**

DEFINITION 6.1. A real pinor bundle is a smooth bundle  $S \neq 0$  of finite-dimensional modules over the Clifford bundle  $\text{Cl}(T^*M, g^*)$ , i.e. a pair  $(S, \gamma)$  where  $S$  is a real vector bundle over  $M$  and  $\gamma : \text{Cl}(T^*M, g^*) \rightarrow \text{End}(S)$  is a smooth morphism of vector bundles (called the structure morphism) such that the fiber map  $\gamma_m : \text{Cl}(T_m^*M, g_m^*) \rightarrow \text{End}_{\mathbb{R}}(S_m)$  is a unital morphism of associative  $\mathbb{R}$ -algebras for each  $m \in M$ .

Hence any fiber  $S_m$  is a real representation of the Clifford algebra  $\text{Cl}(T_m^*M, g_m^*) \simeq \text{Cl}(T_mM, g_m)$ . For any based smooth morphism  $f \in \text{Hom}(S, S')$  of real vector bundles, we let  $L_f : \text{End}(S) \rightarrow \text{Hom}(S, S')$  and  $R_f : \text{End}(S') \rightarrow \text{Hom}(S, S')$  denote the vector bundle morphisms defined through:

$$L_f(\varphi) \stackrel{\text{def.}}{=} f \circ \varphi, \quad R_f(\varphi') \stackrel{\text{def.}}{=} \varphi' \circ f, \quad \varphi \in \text{End}_{\mathbb{R}}(S), \quad \varphi' \in \text{End}_{\mathbb{R}}(S').$$

DEFINITION 6.2. A based morphism of real pinor bundles  $f : (S, \gamma) \rightarrow (S', \gamma')$  is a smooth based morphism  $f : S \rightarrow S'$  of real vector bundles such that:

$$L_f \circ \gamma = R_f \circ \gamma' \quad ,$$

i.e. such that the fiber  $f_m : S_m \rightarrow S'_m$  at any point  $m \in M$  is a based morphism of Clifford representations from  $\gamma_m : \text{Cl}(T_m^*M, g_m^*) \rightarrow \text{End}_{\mathbb{R}}(S_m)$  to  $\gamma'_m : \text{Cl}(T_m^*M, g_m^*) \rightarrow \text{End}_{\mathbb{R}}(S'_m)$ .

Since  $M$  is connected, all quadratic spaces  $(T_m^*, g_m^*)$  are mutually isomorphic and hence isomorphic to some model quadratic space  $(V, h)$ . Similarly, all fibers of  $S$  are isomorphic as  $\mathbb{R}$ -vector spaces and hence isomorphic with some model vector space  $S_0$ . Using a common trivializing cover of  $TM$  and  $S$ , this implies that the real Clifford representations  $\gamma_m : T_mM \rightarrow \text{End}_{\mathbb{R}}(S_m)$  ( $m \in M$ ) are mutually isomorphic in the category  $\text{ClRep}$  and hence isomorphic with some model representation  $\gamma_0 : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S_0)$ . The isomorphism class of  $\gamma_0$  in  $\text{ClRep}$  is invariant under isomorphism of real pinor bundles.

DEFINITION 6.3. The type  $\eta$  of a real pinor bundle  $(S, \gamma)$  is the isomorphism class of its fiberwise Clifford representation  $\gamma_m : (T_m^*M, g_m^*) \rightarrow \text{Aut}_{\mathbb{R}}(S_m)$  in the category  $\text{ClRep}$ .

DEFINITION 6.4. A real pinor bundle  $(S, \gamma)$  is called weakly faithful if  $\gamma|_{T^*M}$  is a monomorphism of vector bundles from  $T^*M$  to  $\text{End}(S)$ , i.e. if the map  $\gamma_m|_{T_m^*M} :$

$T_m^*M \rightarrow \text{End}_{\mathbb{R}}(S_m)$  is injective (and thus a weakly faithful Clifford representation) for each  $m \in M$ .

Let  $\text{ClB}(M, g)$  denote the category of real pinor bundles over  $(M, g)$  and based pinor bundle morphisms and  $\text{ClB}_w(M, g)$  denote the full sub-category whose objects are the weakly faithful real pinor bundles. Clearly  $(S, \gamma)$  is weakly faithful iff its type is. If  $\eta : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S)$  is a weakly faithful Clifford representation, we let  $\text{ClB}_w^\eta(M, g)$  denote the full sub-category of  $\text{ClB}_w(M, g)$  consisting of all real pinor bundles of type equal to the isomorphism class of  $\eta$  and  $\text{ClB}_w^\eta(M, g)^\times$  denote the corresponding unit groupoid.

**6.2. The pseudo-orthogonal coframe bundle.** Let  $(p, q)$  denote the signature of  $(M, g)$  and  $d = p+q$  denote the dimension of  $M$ . Let  $(V, h)$  be a quadratic space isomorphic with any (and hence all) fibers of the pseudo-Euclidean bundle  $(T^*M, g^*)$ .

DEFINITION 6.5. *The pseudo-orthogonal coframe bundle  $P_{\text{O}(V, h)}(M, g)$  of  $(M, g)$  relative to  $(V, h)$  is the principal bundle with structure group  $\text{O}(V, h) = \text{Aut}_{\text{Quad}^\times}(V, h)$ , total space:*

$$P_{\text{O}(V, h)}(M, g) \stackrel{\text{def.}}{=} \sqcup_{m \in M} \text{Hom}_{\text{Quad}^\times}((V, h), (T_m^*M, g_m^*))$$

and right  $\text{O}(V, h)$ -action given by right composition of  $r \in P_{\text{O}(V, h)}(M, g)_m$  with elements  $R \in \text{O}(V, h)$ :

$$rR \stackrel{\text{def.}}{=} r \circ R \quad .$$

Notice that the group  $\text{O}(T_m^*M, g_m^*)$  acts from the left on each fiber  $P_{\text{O}(V, h)}(M, g)_m$  by left composition:

$$R_m r \stackrel{\text{def.}}{=} R_m \circ r \quad \forall r \in \text{Hom}_{\text{Quad}^\times}((V, h), (T_m^*M, g_m^*)) \text{ and } R_m \in \text{O}(T_m^*M, g_m^*) .$$

The pseudo-orthogonal coframe bundle of  $(M, g)$  relative to  $\mathbb{R}^{p, q}$  is denoted  $P_{p, q}(M, g)$  and is called the *canonical pseudo-orthogonal coframe bundle* of  $(M, g)$ . Its fiber at  $m \in M$  is the set of all invertible isometries  $f : \mathbb{R}^{p, q} \rightarrow (T_m^*M, g_m^*)$ , which can be identified with the set of all pseudo-orthogonal frames of  $(T_m^*M, g_m^*)$  through the map:

$$f \rightarrow (f(\epsilon_1), \dots, f(\epsilon_d)) \quad ,$$

where  $(\epsilon_1, \dots, \epsilon_d)$  is the canonical basis of  $\mathbb{R}^{p+q}$ . The pseudo-orthogonal coframe bundle of  $(M, g)$  relative to any model  $(V, h)$  of the fiber of  $(T^*M, g^*)$  is isomorphic with the canonical pseudo-orthogonal coframe bundle.

REMARK 6.1. Let  $(V_0, h_0)$  be an isometric model of the tangent spaces  $(T_m M, g_m)$  of  $(M, g)$ . The bundle  $P_{\text{O}(V_0, h_0)}(M, g)$  of pseudo-orthogonal frames of  $(M, g)$  relative to  $(V_0, h_0)$  is the principal  $\text{O}(V_0, h_0)$ -bundle with fibers  $P_{\text{O}(V_0, h_0)}(M, g)_m \stackrel{\text{def.}}{=} \text{Hom}_{\text{Quad}^\times}((V_0, h_0), (T_m M, g_m))$ . Taking  $V_0 = V^* = \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$  and canonically identifying  $V_0^* = (V^*)^*$  with  $V$ , the musical isomorphism gives an invertible isometry  $\zeta : (V, h) \rightarrow (V_0, h_0)$  defined through:

$$\zeta(x)(y) \stackrel{\text{def.}}{=} h(x, y) \quad \forall x, y \in V_0 \quad .$$

This induces an isomorphism of groups  $\text{Ad}(\zeta) : \text{O}(V, h) \xrightarrow{\sim} \text{O}(V_0, h_0)$  which identifies the tautological representation of  $\text{O}(V_0, h_0)$  with the dual (a.k.a. contragradient) representation of  $\text{O}(V, h)$ :

$$\text{Ad}(\zeta)(R)(\eta) = \eta \circ R^{-1} \quad \forall \eta \in V_0 = V^* \quad \forall R \in \text{O}(V, h) \quad .$$

This implies that  $P_{\text{O}(V_0, h_0)}(M, g)$  is naturally isomorphic with  $P_{\text{O}(V, h)}(M, g)$  and that  $TM$  is associated to  $P_{\text{O}(V, h)}(M, h)$  through the contragradient representation of  $\text{O}(V, h)$ .

**6.3. Real Lipschitz structures.** Let  $\varphi : H \rightarrow G$  be a morphism of groups and  $P$  be a principal  $G$ -bundle over  $M$ . We say that  $P$  admits a  $\varphi$ -reduction to  $H$  if there exists a principal  $H$ -bundle  $Q$  over  $M$  and a  $\varphi$ -equivariant bundle map  $\tau : Q \rightarrow P$ , where  $\varphi$ -equivariance means that we have<sup>10</sup>  $\tau(qh) = \tau(q)\varphi(h)$  for all  $q \in Q$  and  $h \in H$ . In this case, the pair  $(Q, \tau)$  is called a  $\varphi$ -reduction of  $P$  to  $H$ . Notice that we do not require that  $\varphi$  be injective or surjective. Given two  $\varphi$ -reductions  $(Q, \tau)$  and  $(Q', \tau')$  of  $P$  to  $H$ , an *isomorphism of  $\varphi$ -reductions* from  $(Q, \tau)$  to  $(Q', \tau')$  is a based isomorphism of principal  $H$ -bundles  $f : Q \rightarrow Q'$  such that  $\tau' \circ f = \tau$ . Notice that  $\varphi$ -reductions of  $P$  to  $H$  form a groupoid whose morphisms are given by isomorphisms of reductions.

REMARK 6.2. Suppose that  $\varphi$  is not surjective and let  $I(\varphi) \stackrel{\text{def.}}{=} \varphi(H)$ , which is a normal subgroup of  $G$ . Let  $j : I(\varphi) \rightarrow G$  be the inclusion morphism and  $\varphi_0 : H \rightarrow I(\varphi)$  be the corestriction of  $\varphi$ . Then  $P$  admits a  $\varphi$ -reduction to  $H$  iff  $P$  admits a  $j$ -reduction  $(P', \tau')$  to  $I(\varphi)$  and  $(P', \tau')$  admits a  $\varphi_0$ -reduction to  $H$ . Of course, a  $j$ -reduction of  $P$  to  $I(\varphi)$  is the same as an ordinary reduction of structure group.

The principal bundle  $P$  admits a  $\varphi$ -reduction to  $H$  iff  $[P] \in H^1(M, G)$  lies in the image of the induced map  $\varphi_* : H^1(M, H) \rightarrow H^1(M, G)$ . In this case, we have  $\varphi_*([Q]) = [P]$  for any  $\varphi$ -reduction  $(Q, \tau)$ . Let  $\eta : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S_0)$  be a weakly faithful Clifford representation,  $L(\eta) \stackrel{\text{def.}}{=} \text{Aut}_{\text{ClRep}}(\eta)$  be its Lipschitz group and  $\text{Ad}_0 : L(\eta) \rightarrow \text{O}(V, h)$  be the vector representation of  $L(\eta)$ .

DEFINITION 6.6. Let  $P$  be a principal  $\text{O}(V, h)$ -bundle over  $M$ . A real Lipschitz structure on  $P$  relative to  $\eta \in \text{ClRep}$  is an  $\text{Ad}_0$ -reduction  $(Q, \tau)$  of  $P$  to  $L(\eta)$ . A real Lipschitz structure on  $(M, g)$  relative to  $\eta$  is a real Lipschitz structure on  $P_{\text{O}(V, h)}(M, g)$  relative to  $\eta$ .

$\text{Ad}_0$ -equivariance for a real Lipschitz structure  $(Q, \tau)$  on  $(M, g)$  means that the following relation is satisfied by the map  $\tau_m : Q_m \rightarrow P_{\text{O}(V, h)}(M, g)_m = \text{Hom}_{\text{Quad}^\times}((V, h), (T_m^*M, g_m^*))$  for all  $m \in M, q \in Q_m$  and  $a \in L(\eta)$ :

$$\tau_m(qa) = \tau_m(q) \circ \text{Ad}_0(a) \quad . \tag{6.1}$$

DEFINITION 6.7. Let  $(Q, \tau)$  and  $(Q', \tau')$  be two real Lipschitz structures on  $(M, g)$  relative to  $\eta$ . An isomorphism of Lipschitz structures from  $(Q, \tau)$  to  $(Q', \tau')$  is an isomorphism of  $\text{Ad}_0$ -reductions of  $P_{\text{O}(V, h)}(M, g)$  to  $L(\eta)$ .

Let  $L_\eta(M, g)$  be the groupoid of real Lipschitz structures of  $(M, g)$  relative to  $\eta$ .

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<sup>10</sup>The element  $q \in Q$  should not be confused with the number appearing in the signature  $(p, q)$  of  $g$ .



**6.4. Relation between weakly faithful real pinor bundles and real Lipschitz structures.** Let  $\eta : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S_0)$  be a weakly faithful real Clifford representation, where  $(V, h)$  is a model of the fiber of  $(T^*M, g^*)$  and let  $L \stackrel{\text{def.}}{=} L(\eta)$  be the Lipschitz group of  $\eta$ . Let  $[\eta]$  be the isomorphism class of  $\eta$  in the category  $\text{ClRep}$ . Recall the surjective functor  $F : \text{ClRep} \rightarrow \text{Quad}$  of Subsection 3.1. This sends every object  $\gamma' : \text{Cl}(V', h') \rightarrow \text{End}_{\mathbb{R}}(S'_0)$  of  $\text{ClRep}$  to the quadratic space  $(V', h')$ . Given two objects  $\gamma_i : \text{Cl}(V_i, h_i) \rightarrow \text{End}_{\mathbb{R}}(S_{0i})$  ( $i = 1, 2$ ) of  $\text{ClRep}$ , the functor  $F$  sends any morphism  $(f_0, f) \in \text{Hom}_{\text{ClRep}}(\gamma_1, \gamma_2)$  (where  $f \in \text{Hom}_{\mathbb{R}}(S_{01}, S_{02})$  and  $f_0 \in \text{Hom}_{\text{Quad}}((V_1, h_1), (V_2, h_2))$ ) to the morphism of quadratic spaces  $F(f_0, f) = f_0$ . When  $\gamma_1$  and  $\gamma_2$  are weakly faithful and  $f$  is bijective, Proposition 3.3 shows that  $f_0$  is also bijective and that it is uniquely determined by  $f$  (namely, we have  $\text{Ad}(f)(\gamma_1(V_1)) = \gamma_2(V_2)$  and  $f_0 = (\gamma_2|_{V_2})^{-1} \circ \text{Ad}(f)|_{\gamma_1(V_1)} \circ (\gamma_1|_{V_1})$ ). Moreover, morphisms  $(f_0, f) \in \text{Hom}_{\text{ClRep}^\times_w}(\gamma_1, \gamma_2)$  of the groupoid  $\text{ClRep}^\times_w$  can be identified with those linear isomorphisms  $f : S_{01} \rightarrow S_{02}$  which satisfy  $\text{Ad}(f)(\gamma_1(V_1)) = \gamma_2(V_2)$  (see (3.11)). We will use this identification throughout this subsection.

**PROPOSITION-DEFINITION 6.1.** *There exists a correspondence between real pinor bundles and real Lipschitz structures given as follows.*

A. Let  $(S, \gamma)$  be a weakly faithful real pinor bundle of type  $\eta$  on  $(M, g)$ . Let  $Q \stackrel{\text{def.}}{=} Q_\eta(S, \gamma)$  be the principal bundle with structure group  $L = L(\eta) = \text{Aut}_{\text{ClRep}_w}(\eta)$ , total space:

$$Q \stackrel{\text{def.}}{=} \sqcup_{m \in M} \text{Hom}_{\text{ClRep}^\times}(\eta, \gamma_m),$$

projection given by  $\pi(q) = m$  for  $q \in Q_m = \text{Hom}_{\text{ClRep}^\times}(\eta, \gamma_m)$  and right  $L$ -action given by  $qa \stackrel{\text{def.}}{=} q \circ a$  for  $a \in L$ . Let  $\tau \stackrel{\text{def.}}{=} \tau_\eta(S, \gamma) : Q_\eta(S, \gamma) \rightarrow P_{O(V, h)}(M, g)$  be the map defined through:

$$\tau_m(q) = F(q) = q_0 \in \text{Hom}_{\text{Quad}^\times}((V, h), (T_m^*M, g_m^*)) = P_{O(V, h)}(M, g)_m \quad . \quad (6.2)$$

Then  $(Q, \tau)$  is a Lipschitz structure on  $(M, g)$  relative to  $\eta$ , called the Lipschitz structure defined by  $(S, \gamma)$ . A based isomorphism of weakly faithful real pinor bundles  $f : (S, \gamma) \rightarrow (S', \gamma')$  of type  $\eta$  induces an isomorphism  $Q_\eta(f) : (Q_\eta(S, \gamma), \tau_\eta(S, \gamma)) \rightarrow (Q_\eta(S', \gamma'), \tau_\eta(S', \gamma'))$  of Lipschitz structures relative to  $\eta$ , which is defined through (notice that  $f_m \in \text{Hom}_{\text{ClRep}^\times}(\gamma_m, \gamma'_m)$  and  $(f_m)_0 = F(f_m) = \text{id}_{T_m^*M}$ ):

$$Q_\eta(f)(q) \stackrel{\text{def.}}{=} (\text{id}_{T_m^*M}, f_m) \circ q \quad , \quad \forall q \in Q_\eta(S, \gamma)_m = \text{Hom}_{\text{ClRep}^\times}(\eta, \gamma_m) \quad . \quad (6.3)$$

B. Let  $(Q, \tau)$  be a Lipschitz structure on  $(M, g)$  relative to  $\eta$ . Then the vector bundle  $S := S_\eta(Q, \tau) \stackrel{\text{def.}}{=} Q \times_{\rho_\eta} S_0$  associated to  $Q$  through the tautological representation  $\rho_\eta : L \rightarrow \text{Aut}_{\mathbb{R}}(S_0)$  of  $L$  becomes a weakly faithful real pinor bundle of type  $\eta$  when endowed with the structure morphism  $\gamma := \gamma(Q, \tau) : \text{Cl}(T^*M, g^*) \rightarrow \text{End}_{\mathbb{R}}(S)$  given by:

$$\gamma_m(x)([q, s]) = [q, \eta(\text{Cl}(\tau_m(q)^{-1})(x))(s)] \quad , \quad \forall x \in \text{Cl}(T_m^*M, g_m^*), \quad (6.4)$$

for all  $q \in Q_m$  and  $s \in S_0$ . The pair  $S_\eta(Q, \tau) = (S, \gamma)$  thus constructed is called the real pinor bundle defined by the Lipschitz structure  $(Q, \tau)$  (in particular, it is weakly faithful). An isomorphism of Lipschitz structures  $f :$



$(Q, \tau) \rightarrow (Q', \tau')$  relative to  $\eta$  induces a based isomorphism of real pinor bundles  $S_\eta(f) = (S_\eta(Q, \tau), \gamma_\eta(Q, \tau)) \rightarrow (S_\eta(Q', \tau'), \gamma_\eta(Q', \tau'))$  given by:

$$S_\eta(f)_m([q, s]) = [f_m(q), s] \quad , \quad \forall q \in Q_m \quad \forall s \in S_0 \quad . \quad (6.5)$$

Furthermore, the correspondences defined above give functors  $Q_\eta : \text{ClB}_w^\eta(M, g)^\times \rightarrow L_\eta(M, g)$  and  $S_\eta : L_\eta(M, g) \rightarrow \text{ClB}_w^\eta(M, g)^\times$ .

At point B. of the proposition, notice that  $\tau_m(q) \in \text{Hom}_{\text{Quad}^\times}((V, h), (T_m^*, g_m^*))$ , is an invertible isometry which induces the unital isomorphism of  $\mathbb{R}$ -algebras  $\text{Cl}(\tau_m(q)) : \text{Cl}(V, h) \xrightarrow{\sim} \text{Cl}(T_m^*, g_m^*)$ , whose inverse appears in relation (6.4). Thus  $\eta \circ [\text{Cl}(\tau_m(q))]^{-1} = \eta \circ \text{Cl}(\tau_m(q)^{-1}) : \text{Cl}(T_m^*, g_m^*) \rightarrow \text{End}_{\mathbb{R}}(S_0)$  is a representation of the Clifford algebra  $\text{Cl}(T_m^*, g_m^*)$  in the vector space  $S_0$ . This representation is isomorphic with  $\eta$  in the category  $\text{ClRep}$ .

REMARK 6.3. Notice that the definition of  $Q_\eta(S, \gamma)$  is similar to that of the pseudo-orthogonal coframe bundle  $P_{O(V, h)}(M, g)$ , where the groupoid of quadratic spaces is replaced by the groupoid  $\text{ClRep}_w^\times$  while the quadratic spaces  $(V, h)$  and  $(T_m^*M, g_m^*)$  are replaced by the Clifford representations  $\eta : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S_0)$  and  $\gamma_m : \text{Cl}(T_m^*M, g_m^*) \rightarrow \text{End}_{\mathbb{R}}(S_m)$  (the former being a model of the latter in the category  $\text{ClRep}_w$ ). In particular, the Lipschitz group  $L = L(\eta)$  is a model for the groups  $\text{Aut}_{\text{ClRep}_w}(\gamma_m)$  of automorphisms of the fibers of  $S$ , when the latter is considered as a bundle of Clifford representations. It is crucial that these automorphisms are considered in the category  $\text{ClRep}$  (thus they need not be based) and not in the ordinary category of representations of unital algebras. This is because  $\eta$  and  $\gamma_m$  can only be identified if one picks an invertible isometry between  $(V, h)$  and  $(T_m^*M, g_m^*)$ ; this forces one to use the structure group  $L = \text{Hom}_{\text{ClRep}_w^\times}(\eta)$  in order to make  $Q$  into a principal bundle.

*Proof.* The fact that (6.2) is  $\text{Ad}_0$ -equivariant follows from the relation  $(q \circ a)_0 = q_0 a_0 = q_0 \circ \text{Ad}_0(a)$  (which holds because  $F$  is a functor which restricts to  $\text{Ad}_0$  on  $L = \text{Hom}_{\text{ClRep}_w^\times}(\eta)$ , in particular we have  $a_0 = \text{Ad}_0(a)$ ). This implies that (6.1) holds and hence  $(Q, \tau)$  is a Lipschitz structure of type  $\eta$ . To show that (6.4) is well-defined, notice that relation (3.2) gives  $\text{Ad}(a) \circ \eta = \eta \circ \text{Cl}(\text{Ad}_0(a))$  for any  $a \in L$ , which implies (using  $[\text{Cl}(\tau_m(q))]^{-1} = \text{Cl}(\tau_m(q)^{-1})$  and  $\text{Ad}_0(a^{-1}) = \text{Ad}_0(a)^{-1}$ ):

$$\text{Ad}(a^{-1}) \circ \eta \circ \text{Cl}(\tau_m(q)^{-1}) = \eta \circ \text{Cl}(\text{Ad}_0(a^{-1}) \circ \tau_m(q)^{-1}) = \eta \circ [\text{Cl}(\tau_m(q) \circ \text{Ad}_0(a))]^{-1} \quad .$$

Using relation (6.1), this gives:

$$\text{Ad}(a^{-1}) \circ \eta \circ \text{Cl}(\tau_m(q)^{-1}) = \eta \circ \text{Cl}(\tau_m(qa)^{-1}) \quad \forall a \in L \quad . \quad (6.6)$$

Thus:

$$\begin{aligned} [qa^{-1}, \eta(\text{Cl}(\tau_m(qa^{-1})^{-1})(x))(as)] &= [q, a^{-1}\eta(\text{Cl}(\tau_m(qa^{-1})^{-1})(x))(as)] = \\ [q, (\text{Ad}(a^{-1}) \circ \eta \circ \text{Cl}(\tau_m(qa^{-1})^{-1}))(x)(s)] &= [q, \eta(\text{Cl}(\tau_m(q)^{-1})(x))(s)] \quad , \end{aligned} \quad (6.7)$$

where in the last equality we used (6.6). This shows that (6.4) is well-defined. The fact that  $\gamma_m$  defined in (6.4) is a Clifford representation is obvious, as are the remaining statements.  $\square$

THEOREM 6.2. *There exist invertible natural transformations:*

$$\mathcal{F}_\eta : S_\eta \circ Q_\eta \xrightarrow{\sim} \text{id}_{\text{ClB}_w^\eta(M, g)^\times} \quad \text{and} \quad \mathcal{G}_\eta : \text{id}_{L_\eta(M, g)} \xrightarrow{\sim} Q_\eta \circ S_\eta \quad .$$

Hence the functors  $Q_\eta$  and  $S_\eta$  are mutually quasi-inverse equivalences between the groupoids  $\text{ClB}_w^\eta(M, g)^\times$  and  $L_\eta(M, g)$ .

REMARK 6.4. Theorem 6.2 shows, in particular, that the classification of weakly faithful real pinor bundles of type  $[\eta]$  over  $(M, g)$  (up to based isomorphism of real pinor bundles) is equivalent with that of real Lipschitz structures relative to  $[\eta]$  (up to isomorphism of Lipschitz structures). The first classification problem asks one to determine the set of isomorphism classes of objects in the category  $\text{ClB}_w^\eta(M, g)$ , while the latter asks for the set of isomorphism classes of objects in the category  $L_\eta(M, g)$ . The theorem implies that there exists a canonically-defined bijection between these two sets.

*Proof.* Fixing  $\eta$ , we denote  $\mathcal{F}_\eta$  by  $\mathcal{F}$  and  $\mathcal{G}_\eta$  by  $\mathcal{G}$  for ease of notation.

**I. Construction of  $\mathcal{F}$ .** Let  $(S, \gamma)$  be an object of  $\text{ClB}_w^\eta(M, g)$ ,  $(Q, \tau) \stackrel{\text{def.}}{=} Q_\eta(S, \gamma)$  and  $(S', \gamma') \stackrel{\text{def.}}{=} S_\eta(Q, \tau)$ . For any  $m \in M$ , we have  $Q_m = \text{Hom}_{\text{ClRep}^\times}(\eta, \gamma_m)$  and  $S'_m = Q_m \times_{\rho_\eta} S_0 = Q_m \times S_0 / \sim$ , where  $\sim$  is the following equivalence relation on pairs  $(q, s) \in Q_m \times S_0$ :

$$(q, s) \simeq (q', s') \text{ if } \exists a \in L = \text{Aut}_{\text{ClRep}}(\eta) : q' = q \circ a \text{ and } s' = a^{-1}(s) .$$

The smooth map  $\bar{\mathcal{F}} : Q \times S_0 \rightarrow S$  given by:

$$\bar{\mathcal{F}}_m(q, s) = q(s) \in S_m \quad \forall (q, s) \in Q_m \times S_0$$

satisfies  $\bar{\mathcal{F}}(qa, a^{-1}(s)) = (q \circ a)(a^{-1}(s)) = q(s)$  and hence descends to a based morphism of vector bundles  $\mathcal{F} := \mathcal{F}_{S, \gamma} : S' \rightarrow S$ . Since  $q$  is an  $\mathbb{R}$ -linear bijection from  $S_0$  to  $S_m$ , the condition  $\bar{\mathcal{F}}(q, s) = 0$  is equivalent with  $s = 0$ , which implies that  $\mathcal{F}_m$  is injective for all  $m \in M$ . Since  $S$  has type  $\eta$ , we have  $\text{rk} S' = \dim_{\mathbb{R}} S_0 = \text{rk} S$  and hence  $\dim_{\mathbb{R}} S'_m = \dim_{\mathbb{R}} S_m$ . Thus  $\mathcal{F}_m$  is bijective for all  $m$  and hence  $\mathcal{F}$  is a based isomorphism of vector bundles. The bundle  $S'$  is endowed with the Clifford module structure given by:

$$\gamma'_m(x)([q, s]) = [q, \eta(\text{Cl}(\tau(q)^{-1})(x))(s)] \quad , \quad \forall x \in \text{Cl}(T_m^* M, g_m^*) \quad , \quad (q \in Q_m, s \in S_0) \quad ,$$

where  $\tau(q) = q_0 \in \text{Hom}_{\text{Quad}^\times}((V, h), (T_m^* M, g_m^*))$  for any  $q \in Q_m$ . Since  $q^{-1} \in \text{Hom}_{\text{ClRep}^\times}(\gamma_m, \eta)$  is an isomorphism of Clifford representations, we have  $\eta \circ \text{Cl}(\tau(q)^{-1}) = \eta \circ \text{Cl}(q_0^{-1}) = \text{Ad}(q^{-1}) \circ \gamma_m$  (see relation (3.2)). Thus  $\eta(\text{Cl}(\tau(q)^{-1})(x)) = q^{-1} \circ \gamma_m(x) \circ q \in \text{End}_{\mathbb{R}}(S_0)$  and:

$$(\mathcal{F}_m \circ \gamma'_m(x))([q, s]) = \mathcal{F}_m([q, q^{-1}(\gamma_m(x)(q(s)))])) = \gamma_m(x)(q(s)) = (\gamma_m(x) \circ \mathcal{F}_m)([q, s]),$$

which shows that  $\mathcal{F}_m \circ \gamma'_m(x) = \gamma_m(x) \circ \mathcal{F}_m$  for all  $m \in M$ . Hence  $\mathcal{F} = \mathcal{F}_{S, \gamma}$  is a based isomorphism of pinor bundles, i.e.  $\mathcal{F}_{S, \gamma} \in \text{Hom}_{\text{ClB}(M, g)^\times}((S', \gamma'), (S, \gamma)) = \text{Hom}_{\text{ClB}(M, g)^\times}((S_\eta \circ Q_\eta)(S, \gamma), (S, \gamma))$ . To show that  $\mathcal{F}_{S, \gamma}$  is a natural transformation, let  $(S_1, \gamma_1)$  and  $(S_2, \gamma_2)$  be two isomorphic weakly faithful pinor bundles of type  $\eta$  and  $f \in \text{Hom}_{\text{ClB}(M, g)^\times}((S_1, \gamma_1), (S_2, \gamma_2))$  be a based isomorphism of pinor bundles. Then  $f_m \in \text{Hom}_{\text{ClRep}^\times}(\gamma_{1, m}, \gamma_{2, m})$  is an isomorphism of Clifford representations for every  $m \in M$ . Let  $(Q_i, \tau_i) \stackrel{\text{def.}}{=} Q_\eta(S_i, \gamma_i)$  and  $(S'_i, \gamma'_i) \stackrel{\text{def.}}{=} S_\eta(Q_i, \tau_i)$ . Let  $\mathcal{F}_i \stackrel{\text{def.}}{=} \mathcal{F}_{S_i, \gamma_i} : (S'_i, \gamma'_i) \xrightarrow{\sim} (S_i, \gamma_i)$  be the based isomorphisms of pinor bundles constructed as above. Finally, let  $g \stackrel{\text{def.}}{=} Q_\eta(f) : (Q_1, \tau_1) \xrightarrow{\sim} (Q_2, \tau_2)$  be the isomorphism of Lipschitz

structures induced by  $f$  and  $f' \stackrel{\text{def.}}{=} S_\eta(g) = S_\eta(Q_\eta(f)) : (S'_1, \gamma'_1) \xrightarrow{\sim} (S'_2, \gamma'_2)$  be the isomorphism of pinor bundles induced by  $g$ . For any  $[q_1, s] \in S'_{1,m} = Q_{1,m} \times_{\rho_\eta} S_0 = \text{Hom}_{\text{ClRep}^\times}(\eta, \gamma_{1,m}) \times_{\rho_\eta} S_0$ , we have  $f'_m([q_1, s]) = [g(q_1), s] = [f_m \circ q_1, s]$  (see Proposition 6.1). Thus:

$$(\mathcal{F}_2 \circ f')_m([q_1, s]) = (f_m \circ q_1)(s) = f_m(q_1(s)) = (f \circ \mathcal{F}_1)_m([q_1, s]) \quad ,$$

which shows that  $\mathcal{F}_2 \circ f' = f \circ \mathcal{F}_1$ , i.e.  $\mathcal{F}_2 \circ (S_\eta \circ Q_\eta)(f) = f \circ \mathcal{F}_1$ . Hence the diagram in Figure 6.8 commutes. Thus  $\mathcal{F} : S_\eta \circ Q_\eta \xrightarrow{\sim} \text{id}_{\text{ClB}_w^0(M,g)^\times}$  is an invertible natural transformation.

$$\begin{array}{ccc} (S'_1, \gamma'_1) & \xrightarrow{\mathcal{F}_1} & (S_1, \gamma_1) \\ (S_\eta \circ Q_\eta)(f) \downarrow & & \downarrow f \\ (S'_2, \gamma'_2) & \xrightarrow{\mathcal{F}_2} & (S_2, \gamma_2) \end{array} \quad (6.8)$$

**II. Construction of  $\mathcal{G}$ .** Let  $(Q, \tau)$  be an object of  $L_\eta(M, g)$  and  $(S, \gamma) = S_\eta(Q, \tau)$ . Let  $(Q', \tau') \stackrel{\text{def.}}{=} Q_\eta(S, \gamma)$ . We have  $Q'_m = \text{Hom}_{\text{ClRep}^\times}(\eta, \gamma_m)$  and  $\tau'_m(q') = q'_0$  for any  $q' \in Q'_m$ . On the other hand, we have  $S_m = Q_m \times_{\rho_\eta} S_0$  and  $\gamma_m$  is given by (6.4). For any  $q \in Q_m$ , let  $\mathcal{G}_m(q) : S_0 \rightarrow S_m$  be the linear map defined through:

$$\mathcal{G}_m(q)(s) = [q, s] \quad (s \in S_0) \quad .$$

Then  $\mathcal{G}_m(q)$  is a linear isomorphism from  $S_0$  to  $S_m$  and for all  $a \in L$  and  $s \in S_0$  we have  $\mathcal{G}_m(qa)(s) = [qa, s] = [q, a(s)] = \mathcal{G}_m(q)(a(s))$ , i.e.:

$$\mathcal{G}_m(qa) = \mathcal{G}_m(q) \circ \rho_\eta(a) = \mathcal{G}_m(q) \circ a \quad (a \in L) \quad . \quad (6.9)$$

For any  $x \in \text{Cl}(T_m^*M, g_m^*)$  and any  $s \in S_0$ , relation (6.4) gives:

$$\begin{aligned} (\gamma_m(x) \circ \mathcal{G}_m(q))(s) &= \gamma_m(x)([q, s]) = [q, \eta(\text{Cl}(\tau(q)^{-1})(x))(s)] \\ &= (\mathcal{G}_m(q) \circ \eta(\text{Cl}(\tau(q)^{-1})(x)))(s) \end{aligned} \quad (6.10)$$

i.e.  $\gamma_m(x) \circ \mathcal{G}_m(q) = \mathcal{G}_m(q) \circ \eta(\text{Cl}(\tau(q)^{-1})(x))$ . Recall that  $\tau(q) \in P_{\text{O}(V,h)}(M, g)_m = \text{Hom}_{\text{Quad}^\times}((V, h), (T_m^*M, g_m^*))$ , so  $\text{Cl}(\tau(q))$  is a unital isomorphism of  $\mathbb{R}$ -algebras from  $\text{Cl}(V, h)$  to  $\text{Cl}(T_m^*M, g_m^*)$ . Replacing  $x$  with  $\text{Cl}(\tau(q))(y)$  where  $y \in \text{Cl}(V, h)$ , the relation above gives:

$$\text{Ad}(\mathcal{G}_m(q)) \circ \eta = \gamma_m \circ \text{Cl}(\tau(q)) \quad .$$

Using relation (3.2), this shows that  $\mathcal{G}_m(q) \in Q'_m = \text{Hom}_{\text{ClRep}^\times}(\eta, \gamma_m)$  and that we have:

$$\tau'(\mathcal{G}_m(q)) = (\mathcal{G}_m(q))_0 = \tau(q) \in \text{Hom}_{\text{Quad}^\times}((V, h), (T_m^*M, g_m^*)) \quad . \quad (6.11)$$

Relation (6.9) reads:

$$\mathcal{G}_m(qa) = \mathcal{G}_m(q)a \quad ,$$

where in the right hand side we use the right L-action on the bundle  $Q'$ . Thus  $(\mathcal{G}_m)_{m \in M}$  define a morphism of principal L-bundles  $\mathcal{G} = \mathcal{G}_{Q,\tau} : Q \rightarrow Q'$  (which is

automatically an isomorphism). Relation (6.11) gives  $\tau' \circ \mathcal{G} = \tau$ , showing that we have  $\mathcal{G} \in \text{Hom}_{L_\eta(M,g)}(Q, Q')$ .

To show that  $\mathcal{G}$  gives a natural transformation from  $\text{id}_{L_\eta(M,g)}$  to  $Q_\eta \circ S_\eta$ , consider two isomorphic objects  $(Q_i, \tau_i)$  ( $i = 1, 2$ ) of  $L_\eta(M, g)$  and let  $(S_i, \gamma_i) \stackrel{\text{def.}}{=} S_\eta(Q_i, \tau_i)$  and  $(Q'_i, \tau'_i) \stackrel{\text{def.}}{=} Q_\eta(S_i, \gamma_i)$ . Let  $\mathcal{G}_i \stackrel{\text{def.}}{=} \mathcal{G}_{Q_i, \tau_i} : Q_i \xrightarrow{\sim} Q'_i$  be the isomorphisms of Lipschitz structures defined as above. For any isomorphism  $f \in \text{Hom}_{L_\eta(M,g)}((Q_1, \tau_1), (Q_2, \tau_2))$  and any elements  $q_1 \in Q_{1,m}$  and  $s \in S_0$ , we have:

$$\begin{aligned} (\mathcal{G}_2 \circ f)(q_1)(s) &= \mathcal{G}_2(f(q_1))(s) = [f(q_1), s] = S_\eta(f)([q_1, s]) = S_\eta(f)(\mathcal{G}_1(q_1)(s)) \\ &= (S_\eta(f) \circ \mathcal{G}_1(q_1))(s) = (Q_\eta \circ S_\eta)(f)(\mathcal{G}_1(q_1))(s), \end{aligned} \tag{6.12}$$

which implies  $\mathcal{G}_2 \circ f = (Q_\eta \circ S_\eta)(f) \circ \mathcal{G}_1$  and hence the diagram in Figure 6.13 commutes. Thus  $\mathcal{G} : \text{id}_{L_\eta(M,g)} \xrightarrow{\sim} Q_\eta \circ S_\eta$  is an isomorphism of functors.

$$\begin{array}{ccc} (Q_1, \tau_1) & \xrightarrow{\mathcal{G}_1} & (Q'_1, \tau'_1) \\ f \downarrow & & \downarrow (Q_\eta \circ S_\eta)(f) \\ (Q_2, \tau_2) & \xrightarrow{\mathcal{G}_2} & (Q'_2, \tau'_2) \end{array} \tag{6.13}$$

□

**7. Some enlarged spinorial structures in general signature.** In this section, we discuss certain enlarged spinorial structures associated to the groups appearing in the list of canonical spinor groups of Section 2. In particular, we discuss the topological obstructions to existence of such structures (some of which are known or are extensions of known results to arbitrary signature and some of which are new) as well as the behavior of  $\text{Pin}$  and  $\text{Pin}^q$  structures under sign reversal of the metric in even dimensions.

**7.1. Modified Stiefel-Whitney classes of a pseudo-Riemannian manifold.** Let  $M$  be a connected pseudo-Riemannian manifold (which need not be orientable) and  $(V, h)$  be a quadratic vector space of signature  $(p, q)$  and dimension  $d = p + q$ . Let  $P$  be a principal  $\text{O}(V, h)$ -bundle. Recall that  $\text{O}(p) \times \text{O}(q)$  is a maximal compact form of  $\text{O}(p, q)$  and the inclusion morphism  $j : \text{O}(p) \times \text{O}(q) \rightarrow \text{O}(p, q)$  is a deformation retract. On the other hand, any pseudo-orthonormal basis of  $(V, h)$  determines an isomorphism of groups  $\xi : \text{O}(p, q) \xrightarrow{\sim} \text{O}(V, h)$  and the isomorphisms determined by two such bases differ by conjugation. The morphism of groups  $j_\xi \stackrel{\text{def.}}{=} \xi \circ j : \text{O}(p) \times \text{O}(q) \rightarrow \text{O}(V, h)$  is a deformation retract which induces a bijection  $(j_\xi)_* : H^1(M, \text{O}(p)) \times H^1(M, \text{O}(q)) \xrightarrow{\sim} H^1(M, \text{O}(V, h))$ . It follows that every principal  $\text{O}(V, h)$ -bundle  $P$  over  $M$  is isomorphic with a fiber product  $P_+ \times_M P_-$ , where  $P_+$  is a principal  $\text{O}(p)$ -bundle and  $P_-$  is a principal  $\text{O}(q)$ -bundle and where  $P_+$  and  $P_-$  are determined by  $P$  up to isomorphism. The same conclusion also follows from the homotopy equivalence of classifying spaces  $\text{BO}(V, h) \simeq \text{BO}(p) \times \text{BO}(q)$ . The *modified Stiefel-Whitney classes* of  $P$  are defined [20] as the Stiefel-Whitney classes of  $P_\pm$ :

$$w_k^\pm(P) \stackrel{\text{def.}}{=} w_k(P_\pm) \in H^k(M, \mathbb{Z}_2) \ .$$

Notice that  $w_1(P) = w_1^+(P) + w_1^-(P)$ .

Assume now that  $P = P_{O(V_0, h_0)}(M, g) \simeq P_{O(V, h)}(M, g)$  is the pseudo-orthogonal frame bundle of a connected pseudo-Riemannian manifold  $(M, g)$ , where  $(V_0, h_0)$  is an isometric model of the tangent spaces  $(T_m M, g_m)$  of  $M$ . Then:

$$w_k^\pm(M, g) \stackrel{\text{def.}}{=} w_k^\pm(P_{O(V_0, h_0)}(M, g)) = w_k^\pm(P_{O(V, h)}(M, g)) \ .$$

are called the *modified Stiefel-Whitney classes of  $(M, g)$* . Since  $TM$  is associated to  $P_{O(V_0, h_0)}(M, g)$  through the tautological representation of  $O(V_0, h_0)$ , we have a corresponding Whitney sum decomposition  $TM = T_+M \oplus T_-M$ , where  $T_\pm M$  are vector sub-bundles of  $TM$  which are associated to  $O(p)$  and  $O(q)$  through the tautological representations of those groups; these sub-bundles are determined only up to isomorphism <sup>11</sup>. We have  $w_k^\pm(M, g) = w_k(T^\pm M)$  and  $w_1(M) = w_1^+(M, g) + w_1^-(M, g)$ , where  $w_k(M)$  are the usual Stiefel-Whitney classes of  $TM$ .

**7.2. Spin structures in general signature.** Recall that a spin structure on  $(M, g)$  is an  $\text{Ad}_0^{\text{Cl}}$ -reduction of  $P_{O(V, h)}(M, g) \simeq P_{O(V_0, h_0)}(M, g)$  to  $\text{Spin}(V, h)$ . Since  $\text{Ad}_0(\text{Spin}(V, h)) = \text{SO}(V, h)$ , Remark 6.2 implies that a spin structure can exist on  $(M, g)$  only when the structure group of  $P_{O(V, h)}(M, g)$  reduces to  $\text{SO}(V, h)$ , i.e. only when  $M$  is orientable. In that case, a spin structure is the same as an  $\text{Ad}_0^{\text{Cl}}$ -reduction of  $P_{\text{SO}(V, h)}(M, g)$  to  $\text{Spin}(V, h)$ , where  $P_{\text{SO}(V, h)}(M, g)$  is the bundle of positively-oriented pseudo-orthogonal coframes (which is naturally isomorphic with the bundle of positively-oriented pseudo-orthogonal frames) with respect to an orientation of  $(M, g)$  determined by the spin structure. Notice that changing the orientation of  $(M, g)$  changes  $P_{\text{SO}(V, h)}(M, g)$  into an isomorphic bundle. The following result was proved in [20]:

PROPOSITION 7.1 ([20]). *The following statements are equivalent:*

- (a)  $(M, g)$  admits a Spin structure
- (b) The following conditions are satisfied:

$$w_1(M) = 0 \quad \text{and} \quad w_2^+(M, g) + w_2^-(M, g) = 0 \ .$$

**7.3. Twisted and untwisted Pin structures in even dimension.** Assume that  $d$  is even. In this case, both the twisted and untwisted vector representations of  $\text{Pin}(V, h)$  are surjective onto  $O(V, h)$ , hence one can define *two* kinds of pin structures on  $(M, g)$ .

DEFINITION 7.1. *A twisted pin structure on  $(M, g)$  is a  $\widetilde{\text{Ad}}_0^{\text{Cl}}$ -reduction of  $P_{O(V, h)}(M, g)$  to  $\text{Pin}(V, h)$ . An untwisted pin structure on  $(M, g)$  is an  $\text{Ad}_0^{\text{Cl}}$ -reduction of  $P_{O(V, h)}(M, g)$  to  $\text{Pin}(V, h)$ .*

Both twisted and untwisted pin structures on  $(M, g)$  form groupoids if one takes morphisms to be isomorphisms of  $\widetilde{\text{Ad}}_0^-$ , respectively  $\text{Ad}_0^-$ -reductions. Notice the equality  $P_{O(V, h)}(M, g) = P_{O(V, h)}(M, -g)$ . Proposition 1.6 implies the following relation between the two types of pin structure:

PROPOSITION 7.2. *When  $d$  is even, there exists an equivalence of categories between the groupoid of twisted pin structures of  $(M, g)$  and the groupoid of untwisted pin structures of  $(M, -\sigma_{p, q}g)$ . Namely:*

<sup>11</sup>They can be constructed as the sums of positive and negative eigenbundles of the  $g_0$ -symmetric endomorphism  $A$  of  $TM$  which represents  $g$  with respect to any Riemannian metric  $g_0$  on  $M$ ; changing  $g_0$  leads to isomorphic bundles.

1. When  $p - q \equiv_8 0, 4$ , the groupoid of twisted pin structures of  $(M, g)$  is equivalent with the groupoid of untwisted pin structures of  $(M, -g)$ .
2. When  $p - q \equiv_8 2, 6$ , the groupoids of twisted and untwisted pin structures of  $(M, g)$  are equivalent to each other.

In particular, twisted and untwisted pin structures are equivalent notions when  $p - q \equiv_8 2, 6$ .

*Proof.* Let  $\sigma \stackrel{\text{def.}}{=} \sigma_{p,q}$ . If  $(Q, \tau)$  is an untwisted pin structure on  $(M, -\sigma g)$ , define  $Q'$  to be the principal  $\text{Pin}(V, h)$ -bundle over  $M$  having the same total space as  $Q$  and right group action given by:

$$q * a \stackrel{\text{def.}}{=} q\varphi(a) \quad \forall q \in Q \quad \forall a \in \text{Pin}(V, h) \quad ,$$

where  $\varphi : \text{Pin}(V, h) \xrightarrow{\sim} \text{Pin}(V, -\sigma h)$  is the isomorphism of groups given in Proposition 1.6. Let  $\tau' \stackrel{\text{def.}}{=} \tau : Q \rightarrow P_{O(V, -\sigma h)}(M, -\sigma g) = P_{O(V, h)}(M, g)$ . Since  $\text{Ad}_0^{\text{Cl}} \circ \varphi = \widetilde{\text{Ad}}_0^{\text{Cl}}$ , we have:

$$\tau'(q * a) = \tau(q\varphi(a)) = \tau(q)\text{Ad}_0^{\text{Cl}}(\varphi(a)) = \tau'(q)\widetilde{\text{Ad}}_0^{\text{Cl}} \quad \forall q \in Q \quad \forall a \in \text{Pin}(V, h) \quad ,$$

where we used the fact that  $\tau$  is  $\text{Ad}_0$ -equivariant. This shows that  $\tau'$  is  $\widetilde{\text{Ad}}_0$ -equivariant and hence  $(Q', \tau')$  is a twisted pin structure on  $(M, g)$ . Let  $(Q_1, \tau_1), (Q_2, \tau_2)$  be untwisted pin structures on  $(M, -\sigma g)$  and  $f : (Q_1, \tau_1) \rightarrow (Q_2, \tau_2)$  be an isomorphism of untwisted pin structures. Let  $(Q'_1, \tau'_1)$  and  $(Q'_2, \tau'_2)$  be the twisted pin structures on  $(M, g)$  defined by  $(Q_1, \tau_1)$  and  $(Q_2, \tau_2)$  as above. For all  $q_1 \in Q_1$  and  $a \in \text{Pin}(V, h)$ , we have:

$$f(q_1 * a) = f(q_1\varphi(a)) = f(q_1)\varphi(a) = f(q_1) * a \quad ,$$

and:

$$\tau'_2 \circ f = \tau_2 \circ f = \tau_1 = \tau'_1 \quad ,$$

which shows that  $f' \stackrel{\text{def.}}{=} f$  is an isomorphism of twisted pin structures from  $(Q'_1, \tau'_1)$  to  $(Q'_2, \tau'_2)$ . It is easy to check that the correspondence defined above is an equivalence of categories.  $\square$

REMARK 7.1. In most of the literature, the name ‘‘Pin structure’’ is reserved for what we call a *twisted* pin structure. One sometimes also encounters the notion of ‘‘Pin<sup>-</sup> structure’’ of  $(M, g)$ , which is defined as a *twisted* pin structure of  $(M, -g)$ . The proposition implies the following:

1. When  $p - q \equiv_8 0, 4$ , the groupoid of  $\text{Pin}^-$  structures of  $(M, g)$  is equivalent with the groupoid of untwisted  $\text{Pin}$  structures of  $(M, g)$ .
2. When  $p - q \equiv_8 2, 6$ , the groupoid of  $\text{Pin}^-$  structures of  $(M, g)$  is equivalent with the groupoid of untwisted  $\text{Pin}$  structures of  $(M, -g)$ .

PROPOSITION 7.3. *Let  $d$  be even and  $\sigma \stackrel{\text{def.}}{=} \sigma_{p,q}$ . Then the following statements are equivalent:*

- (a)  $(M, g)$  admits an untwisted  $\text{Pin}$  structure
- (b)  $(M, -\sigma g)$  admits a twisted  $\text{Pin}$  structure
- (c) The following condition is satisfied:

$$w_2^+(M, g) + w_2^-(M, g) + w_1^\sigma(M, g)^2 + w_1^-(M, g)w_1^+(M, g) = 0 \quad .$$

*Proof.* The equivalence of (a) and (b) follows from Proposition 7.2. The equivalence of (b) and (c) follows from [20, Proposition (1.1.26)] upon noticing the relation:

$$w_1^\pm(M, -g) = w_1^\mp(M, g) \quad .$$

□

**7.4. Twisted and untwisted  $\text{Pin}^q(V, h)$  structures in even dimension.** In even dimension, both the twisted and untwisted basic representation of  $\text{Pin}^q(V, h)$  have image equal to  $\text{O}(V, h) \times \text{SO}(3, \mathbb{R})$  (see Subsection 2.1). This allows us to define two kinds of  $\text{Pin}^q$  structures on  $(M, g)$ .

DEFINITION 7.2. *Let  $d$  be even. Then:*

1. A twisted  $\text{Pin}^q$  structure on  $(M, g)$  is a triplet  $(E, Q, \tau)$ , where  $E$  is a principal  $\text{SO}(3, \mathbb{R})$ -bundle over  $M$  and  $(Q, \tau)$  is a  $\tilde{\rho}$ -reduction of  $P_{\text{O}(V, h)}(M, g) \times_M E$  to  $\text{Pin}^q(V, h)$ .
2. An untwisted  $\text{Pin}^q$  structure on  $(M, g)$  is a triplet  $(E, Q, \tau)$ , where  $E$  is a principal  $\text{SO}(3, \mathbb{R})$ -bundle over  $M$  and  $(Q, \tau)$  is a  $\rho$ -reduction of  $P_{\text{O}(V, h)}(M, g) \times_M E$  to  $\text{Pin}^q(V, h)$ .

Proposition 2.1 implies the following result, whose proof is similar to that of Proposition 7.2:

PROPOSITION 7.4. *When  $d$  is even, there exists an equivalence of categories between the groupoid of twisted  $\text{Pin}^q(V, h)$  structures of  $(M, g)$  and that of untwisted  $\text{Pin}^q(V, -\sigma_{p, q}h)$  structures of  $(M, g)$ . Namely:*

1. When  $p - q \equiv_8 0, 4$ , the groupoid of twisted  $\text{Pin}^q$  structures of  $(M, g)$  is equivalent with the groupoid of untwisted  $\text{Pin}^q$  structures of  $(M, -g)$ .
2. When  $p - q \equiv_8 2, 6$ , the groupoids of twisted and untwisted  $\text{Pin}^q$  structures of  $(M, g)$  are equivalent to each other.

PROPOSITION 7.5. *Let  $d$  be even and  $\sigma \stackrel{\text{def.}}{=} \sigma_{p, q}$ . Then the following statements are equivalent:*

- (a)  $(M, g)$  admits an untwisted  $\text{Pin}^q$  structure
- (b)  $(M, -\sigma g)$  admits a twisted  $\text{Pin}^q$  structure
- (c) There exists a principal  $\text{SO}(3, \mathbb{R})$ -bundle  $E$  over  $M$  such that the following condition is satisfied:

$$w_2^+(M, g) + w_2^-(M, g) + w_1^\sigma(M, g)^2 + w_1^-(M, g)w_1^+(M, g) = w_2(E) \quad .$$

*Proof.* Equivalence of (a) and (b) follows from Proposition 7.4. To show equivalence of (b) and (c), notice that the short exact sequence (2.5) induces the exact sequence of pointed sets:

$$H^1(M, \text{Pin}^q(V, h)) \xrightarrow{\tilde{\rho}_*} H^1(M, \text{O}(V, h) \times \text{SO}(3, \mathbb{R})) \xrightarrow{\partial} H^2(M, \mathbb{Z}_2) \quad , \quad (7.1)$$

where  $H^1(M, \text{O}(V, h) \times \text{SO}(3, \mathbb{R})) = H^1(M, \text{O}(V, h)) \oplus H^1(M, \text{SO}(3, \mathbb{R}))$ . When  $P$  is an  $\text{O}(V, h)$ -bundle on  $M$ , the connecting map is given by  $\partial([P \times_M E]) = \partial'([P]) + w_2([E]) = w_2^+(P) + w_2^-(P) + w_1^-(P)^2 + w_1^-(P)w_1^+(P) + w_2(E)$ , where we used the fact that  $\text{Sp}(1) \simeq \text{Spin}(3) = \text{Spin}_{3,0}$  and the connecting map in the sequence:

$$H^1(M, \text{Sp}(1)) \xrightarrow{(\text{Ad}_0^{(2)})^*} H^1(M, \text{SO}(3, \mathbb{R})) \longrightarrow H^2(M, \mathbb{Z}_2)$$



induced by the short exact sequence  $1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}(3) \rightarrow \text{SO}(3, \mathbb{R}) \rightarrow 1$  is given by  $[E] \rightarrow w_2(E)$ , while the connecting map  $\partial'$  in the sequence:

$$H^1(M, \text{Pin}(V, h)) \xrightarrow{(\widetilde{\text{Ad}}_0^{\text{Cl}})^*} H^1(M, \text{O}(V, h)) \xrightarrow{\partial'} H^2(M, \mathbb{Z}_2)$$

induced by the exact sequence (1.14) is given by  $\partial'([P]) = w_2^+(P) + w_2^-(P) + w_1^-(P)^2 + w_1^-(P)w_1^+(P)$  (see [20]). Thus  $(M, g)$  admits a twisted  $\text{Pin}^q(V, h)$  structure iff  $w_2^+(M, g) + w_2^-(M, g) + w_1^-(M, g)^2 + w_1^-(M, g)w_1^+(M, g) = w_2(E)$ . Applying this to  $(M, -\sigma g)$  shows that (b) is equivalent with (c).  $\square$

REMARK 7.2. The connecting map  $\partial$  in the sequence (7.1) can also be determined directly as follows. First, notice that we must have  $\partial([P] \oplus [E]) = a_1w_2^+(P) + a_2w_2^-(P) + a_3w_1^-(P)^2 + a_4w_1^-(P)w_1^+(P) + a_5w_1^+(P)^2 + a_6w_2(E)$  (since  $w_1(E) = 0$ ). When  $E$  is trivial, we have  $w_2(E) = 0$  and it is easy to see that the bundle  $P \times_M E$  admits a twisted  $\text{Pin}^q$  structure iff  $P$  admits a twisted pin structure<sup>12</sup>. This implies that  $\partial([P])$  must be given by the expression computed in [20], i.e. we must have  $a_1 = a_2 = a_3 = a_4 = 1$  and  $a_5 = 0$ . On the other hand, when  $p = d, q = 0$  and  $w_1(P) = 0$ , a  $\text{Pin}^q$  structure on  $P$  reduces to a  $\text{Spin}^q$  structure on  $P$ , hence in this case we have  $\partial([P] \oplus [E]) = w_2(P) + w_2(E)$  by the results of [5]; this shows that  $a_6 = 1$ .

**7.5. Spin<sup>q</sup> structures in general signature.** Let  $\rho : \text{Spin}^q(V, h) \rightarrow \text{SO}(V, h) \times \text{SO}(3, \mathbb{R})$  denote the basic representation of  $\text{Spin}^q(V, h)$  (see Subsection 2.2).

DEFINITION 7.3. A  $\text{Spin}^q$  structure on  $(M, g)$  is a triplet  $(E, Q, \tau)$ , where  $E$  is a principal  $\text{SO}(3, \mathbb{R})$ -bundle over  $M$  and  $(Q, \tau)$  is a  $\rho$ -reduction of  $P_{\text{O}(V, h)}(M, g) \times_M E$  to  $\text{Spin}^q(V, h)$ .

Since the image of  $\rho$  equals  $\text{SO}(V, h) \times \text{SO}(3, \mathbb{R})$ , it follows that  $(M, g)$  can admit a  $\text{Spin}^q$  structure only when  $M$  is orientable; hence we can assume that  $w_1(M) = 0$ . When this condition is satisfied, a  $\text{Spin}^q$  structure is the same as a  $\rho$ -reduction of  $P_{\text{SO}(V, h)}(M, g) \times_M E$ , where  $P_{\text{SO}(V, h)}(M, g)$  is the principal bundle of positively-oriented pseudo-orthogonal frames with respect to an orientation of  $M$  determined by the  $\text{Spin}^q$  structure. In fact, we have:

PROPOSITION 7.6. *The following statements are equivalent:*

- (a)  $(M, g)$  admits a  $\text{Spin}^q$  structure
- (b) There exists a principal  $\text{SO}(3, \mathbb{R})$ -bundle  $E$  over  $M$  such that the following conditions are satisfied:

$$w_1(M) = 0 \quad \text{and} \quad w_2^+(M, g) + w_2^-(M, g) = w_2(E) \quad .$$

*Proof.* The short exact sequence (2.7) induces the exact sequence of pointed sets:

$$H^1(M, \text{Spin}^q(V, h)) \xrightarrow{\rho^*} H^1(M, \text{SO}(V, h) \times \text{SO}(3, \mathbb{R})) \xrightarrow{\partial} H^2(M, \mathbb{Z}_2) \quad ,$$

whose connecting map satisfies  $\partial([P \times_M E]) = \partial'([P]) + w_2(E) = w_2^+(P) + w_2^-(P) + w_2(E)$ , where we used the fact that the connecting map  $\partial'$  in the sequence:

$$H^1(M, \text{Spin}(V, h)) \xrightarrow{(\text{Ad}_0^{\text{Cl}})^*} H^1(M, \text{SO}(V, h)) \xrightarrow{\partial'} H^2(M, \mathbb{Z}_2)$$

<sup>12</sup>The inverse implication is obvious. For the direct implication, notice that the natural inclusion  $\text{Pin}(V, h) \simeq \text{Pin}(V, h) \cdot \mathbb{Z}_2 \subset \text{Pin}^q(V, h)$  allows one to reduce the structure group of a pin structure  $Q$  to  $\text{Pin}(V, h)$  when  $E$  is trivial; indeed, the characteristic representation  $\mu$  of  $\text{Pin}^q(V, h)$  induces an isomorphism  $Q/\text{Pin}(V, h) \simeq E$  and  $E$  has a section when it is trivial.



induced by (1.15) is given by  $\partial'([P]) = w_2^+(P) + w_2^-(P)$  (see [20]). Thus  $(M, g)$  admits a  $\text{Spin}^q(V, h)$  structure iff  $w_1(M) = 0$  and  $w_2^+(M, g) + w_2^-(M, g) = w_2(E)$ .  $\square$

**7.6.  $\text{Spin}^o$  structures in signature  $p - q \equiv_8 3, 7$ .** Assume that the signature  $(p, q)$  of  $(M, g)$  belongs to the complex case, i.e. it satisfies  $p - q \equiv_8 3, 7$  (in particular,  $d = p + q$  is odd). Recall that  $\alpha_{p,q} \stackrel{\text{def.}}{=} (-1)^{\frac{p-q+1}{4}} = \begin{cases} -1 & \text{if } p - q \equiv_8 3 \\ +1 & \text{if } p - q \equiv_8 7 \end{cases}$ . Let  $\rho : \text{Spin}^o(V, h) \rightarrow \text{S}[\text{O}(V, h) \times \text{O}(2, \mathbb{R})] \subset \text{O}(V, h) \times \text{O}(2, \mathbb{R})$  denote the basic representation of the adapted  $\text{Spin}^o$  group  $\text{Spin}^o(V, h) = \text{Spin}^o_{\alpha_{p,q}}(V, h)$  (see Subsection 2.3).

**DEFINITION 7.4.** *An adapted  $\text{Spin}^o$  structure on  $(M, g)$  is a triplet  $(E, Q, \tau)$ , where  $E$  is a principal  $\text{O}(2, \mathbb{R})$ -bundle over  $M$  and  $(Q, \tau)$  is a  $\rho$ -reduction of  $P_{\text{O}(V,h)}(M, g) \times E$  to  $\text{Spin}^o(V, h)$ .*

The following result is proved in [11]:

**PROPOSITION 7.7** ([11]). *Let  $d$  be odd and  $\alpha \stackrel{\text{def.}}{=} \alpha_{p,q}$ . Then the following statements are equivalent:*

- (a)  $(M, g)$  admits an adapted  $\text{Spin}^o$  structure
- (b) There exists a principal  $\text{O}(2, \mathbb{R})$ -bundle  $E$  over  $M$  such that the following two conditions are satisfied:

$$w_1(M) = w_1(E)$$

and:

$$w_2^+(M, g) + w_2^-(M, g) = w_2(E) + w_1(E)(pw_1^+(M, g) + qw_1^-(M, g)) + \left[ \delta_{\alpha, -1} + \frac{p(p+1)}{2} + \frac{q(q+1)}{2} \right] w_1(E)^2 \quad (7.2)$$

**8. Elementary real pinor bundles and elementary real Lipschitz structures.** In this section, we consider the classification of bundles of irreducible Clifford modules over  $(M, g)$  and extract the topological obstruction to existence of such bundles in every dimension and signature. Let  $(M, g)$  be a connected and second countable pseudo-Riemannian manifold of signature  $(p, q)$  and dimension  $d = p + q$ , where we assume that  $d > 0$ . Let  $(V, h)$  be a model for the fibers of the pseudo-Euclidean vector bundle  $(T^*M, g^*)$ .

**DEFINITION 8.1.** *An elementary real pinor bundle over  $(M, g)$  is a real pinor bundle  $(S, \gamma)$  such that the Clifford representation  $\gamma_m : \text{Cl}(T_m^*M, g_m^*) \rightarrow \text{End}_{\mathbb{R}}(S_m)$  is a pin representation for every point  $m \in M$ .*

Notice that an elementary real pinor bundle is weakly faithful. If  $\eta : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S_0)$  is a real Clifford representation such that  $(S, \gamma)$  has type  $\eta$ , then  $(S, \gamma)$  is elementary iff  $\eta$  is a pin representation.

**DEFINITION 8.2.** *A elementary real Lipschitz structure of  $(M, g)$  is a real Lipschitz structure relative to a pin representation  $\eta : \text{Cl}(V, g) \rightarrow \text{End}_{\mathbb{R}}(S_0)$ . A reduced elementary real Lipschitz structure of  $(M, g)$  is a reduced real Lipschitz structure relative to a pin representation.*

**DEFINITION 8.3.** *The characteristic group of  $(V, h)$  is the compact Lie group  $\mathfrak{G}(V, h)$  defined as follows:*

1.  $\mathfrak{G} \stackrel{\text{def.}}{=} 1$  (the trivial one-element group), in the normal simple or non-simple case (i.e. if  $p - q \equiv_8 0, 1, 2$ )
2.  $\mathfrak{G}(V, h) \stackrel{\text{def.}}{=} \text{O}(2, \mathbb{R})$ , in the complex case (i.e. if  $p - q \equiv_8 3, 7$ )
3.  $\mathfrak{G}(V, h) \stackrel{\text{def.}}{=} \text{SO}(3, \mathbb{R})$ , in the quaternionic simple or non-simple case (i.e. if  $p - q \equiv_8 4, 5, 6$ ).

DEFINITION 8.4. A characteristic bundle for  $(M, g)$  is a principal bundle  $E$  over  $M$  with structure group  $\mathfrak{G}(V, h)$ .

In the normal (simple or non-simple) case,  $E$  is the trivial  $1 : 1$  cover bundle. Recall the canonical spinor group  $\Lambda(V, h)$  and its vector, characteristic and basic representations  $\lambda : \Lambda(V, h) \rightarrow \text{O}(V, h)$ ,  $\mu : \Lambda(V, h) \rightarrow \mathfrak{G}(V, h)$  and  $\rho : \Lambda(V, h) \rightarrow \text{O}(V, h) \times \mathfrak{G}(V, h)$  discussed in Subsection 2.5.

DEFINITION 8.5. A canonical spinor structure on  $(M, g)$  is a  $\lambda$ -reduction  $(Q, \tau_1)$  of  $P_{\text{O}(V, h)}$  to the canonical spinor group  $\Lambda(V, h)$ . A modified canonical spinor structure on  $(M, g)$  is a triplet  $(E, Q, \tau)$ , where  $E$  is a characteristic bundle for  $(M, g)$  and  $(Q, \tau)$  is a  $\rho$ -reduction of  $P_{\text{O}(V, h)} \times_M E$  to  $\Lambda(V, h)$ .

Canonical and modified canonical spinor structures on  $(M, g)$  form groupoids whose morphisms are the isomorphisms of  $\lambda$ -reductions and  $\rho$ -reductions respectively.

PROPOSITION 8.1. The groupoids of canonical and modified canonical spinor structures on  $(M, g)$  are equivalent.

*Proof.* Given a canonical spinor structure  $(Q, \tau_1)$  on  $(M, g)$ , the characteristic morphism  $\mu : \Lambda(V, h) \rightarrow \mathfrak{G}(V, h)$  induces a principal  $\mathfrak{G}(V, h)$ -bundle  $E \stackrel{\text{def.}}{=} Q \times_{\mu} \mathfrak{G}(V, h)$  on  $M$  and a  $\mu$ -equivariant bundle map  $\tau_2 : Q \rightarrow E$  given by  $\tau_2(q) \stackrel{\text{def.}}{=} [q, 1]$ . Since  $\rho = \lambda \times \mu$ , the bundle map  $\tau \stackrel{\text{def.}}{=} \tau_1 \times \tau_2 : Q \rightarrow P_{\text{O}(V, h)}(M, g) \times_M E$  is  $\rho$ -equivariant, hence  $(E, Q, \tau)$  is a modified canonical spinor structure on  $(M, g)$ . Conversely, let  $(E, Q, \tau)$  be a modified canonical spinor structure on  $P_{\text{O}(V, h)}(M, g)$  and set  $\tau_1 = \pi_1 \circ \tau$ , where  $\pi_1 : P_{\text{O}(V, h)}(M, g) \times_M E \rightarrow P_{\text{O}(V, h)}(M, g)$  is the bundle map given by fiberwise projection on the first factor. Then  $(Q, \tau_1)$  is a canonical spinor structure on  $(M, g)$ . It is easy to see that the correspondences defined above extend to mutually quasi-inverse functors between the groupoids of canonical and modified canonical spinor structures on  $(M, g)$ .  $\square$

THEOREM 8.2. The following groupoids are equivalent for any pseudo-Riemannian manifold  $(M, g)$ :

- (a) The groupoid of elementary real pinor bundles of  $(M, g)$ .
- (b) The groupoid of elementary real Lipschitz structures.
- (c) The groupoid of elementary reduced real Lipschitz structures.
- (d) The groupoid of canonical spinor structures.
- (e) The groupoid of modified canonical spinor structures.

Depending on the dimension and signature, this groupoid equals:

1. When  $p - q \equiv_8 0, 2$  (normal simple case): the groupoid of untwisted Pin structures.
2. When  $p - q \equiv_8 3, 7$  (complex case): the groupoid of  $\text{Spin}^o$  structures
3. When  $p - q \equiv_8 4, 6$  (quaternionic simple case): the groupoid of untwisted  $\text{Pin}^q$  structures.
4. When  $p - q \equiv_8 1$  (normal non-simple case): the groupoid of Spin structures.
5. When  $p - q \equiv_8 5$  (quaternionic non-simple case): the groupoid of  $\text{Spin}^q$  structures.

*Proof.* The equivalence between the groupoids at (a) and (b) follows from Theorem 6.2. For any pin representation  $\eta$ , the vector representation of  $L(\eta)$  factors through the normalization morphism  $\pi_0 : L(\eta) \simeq \mathbb{R}_{>0} \times \mathcal{L}(\eta) \rightarrow \mathcal{L}(\eta)$  of Subsection 5.9. The correspondence which takes a Lipschitz structure  $(Q, \tau)$  into the reduced Lipschitz structure  $(Q \times_{\pi_0} \mathcal{L}(\eta), \tau_0)$  (where  $\tau_0([q, a_0]) \stackrel{\text{def.}}{=} \tau(q)\text{Ad}_0(a_0)$  for all  $q \in Q$  and  $a_0 \in \mathcal{L}(\eta)$ ) induces an equivalence of categories between the groupoid of real Lipschitz structures relative to  $\eta$  and the groupoid of reduced real Lipschitz structures relative to  $\eta$ . This establishes the equivalence between the groupoids at points (b) and (c). The equivalence between the groupoids at (c) and (d) follows from Theorem 5.16 and Theorem 5.24. The equivalence between the groupoids at (d) and (e) follows from Proposition 8.1. The remaining statements follow from the definition of the canonical spinor group  $\Lambda(V, h)$  and of its basic representation.  $\square$

Theorem 8.2 and the results of Section 7 immediately imply:

THEOREM 8.3. *Let  $\sigma \stackrel{\text{def.}}{=} \sigma_{p,q}$ .*

1. *In the normal simple case ( $p - q \equiv_8 0, 2$ ), the following statements are equivalent:*

- (a) *There exists an elementary real pinor bundle on  $(M, g)$*
- (b)  *$(M, g)$  admits an elementary real Lipschitz structure*
- (c)  *$(M, g)$  admits an untwisted  $\text{Pin}(V, h)$  structure*
- (d)  *$(M, g)$  admits a twisted  $\text{Pin}(V, -\sigma h)$  structure*
- (e) *The following condition is satisfied:*

$$w_2^+(M, g) + w_2^-(M, g) + w_1^\sigma(M, g)^2 + w_1^-(M, g)w_1^+(M, g) = 0 \quad .$$

2. *In the complex case, the following statements are equivalent:*

- (a) *There exists an elementary real pinor bundle on  $(M, g)$*
- (b)  *$(M, g)$  admits an elementary real Lipschitz structure*
- (c)  *$(M, g)$  admits a  $\text{Spin}^\circ$  structure*
- (d) *There exists a principal  $\text{O}(2, \mathbb{R})$ -bundle  $E$  over  $M$  such that the following two conditions are satisfied:*

$$w_1(M) = w_1(E)$$

and:

$$w_2^+(M, g) + w_2^-(M, g) = w_2(E) + w_1(E)(pw_1^+(M, g) + qw_1^-(M, g)) + \left[ \delta_{\alpha, -1} + \frac{p(p+1)}{2} + \frac{q(q+1)}{2} \right] w_1(E)^2 \quad (8.1)$$

3. *In the quaternionic simple case ( $p - q \equiv_8 4, 6$ ), the following statements are equivalent:*

- (a) *There exists an elementary real pinor bundle on  $(M, g)$*
- (b)  *$(M, g)$  admits an elementary real Lipschitz structure*
- (c)  *$(M, g)$  admits an untwisted  $\text{Pin}^q$  structure*
- (d)  *$(M, -\sigma g)$  admits a twisted  $\text{Pin}^q$  structure*
- (e) *There exists a principal  $\text{SO}(3, \mathbb{R})$ -bundle  $E$  over  $M$  such that the following condition is satisfied:*

$$w_2^+(M, g) + w_2^-(M, g) + w_1^\sigma(M, g)^2 + w_1^-(M, g)w_1^+(M, g) = w_2(E) \quad .$$

4. *In the normal non-simple case ( $p - q \equiv_8 1$ ), the following statements are equivalent:*

- (a) *There exists an elementary real pinor bundle on  $(M, g)$*
- (b)  *$(M, g)$  admits an elementary real Lipschitz structure*
- (c)  *$(M, g)$  admits a Spin structure*
- (d) *The following two conditions are satisfied:*

$$w_1(M) = 0 \quad \text{and} \quad w_2^+(M, g) + w_2^-(M, g) = 0 \quad .$$

5. *In the quaternionic non-simple case  $(p - q \equiv_8 5)$ , the following statements are equivalent:*

- (a) *There exists an elementary real pinor bundle on  $(M, g)$*
- (b)  *$(M, g)$  admits an elementary real Lipschitz structure*
- (c)  *$(M, g)$  admits a  $\text{Spin}^q$  structure*
- (d) *There exists a principal  $\text{SO}(3, \mathbb{R})$ -bundle  $E$  over  $M$  such that the following two conditions are satisfied:*

$$w_1(M) = 0 \quad \text{and} \quad w_2^+(M, g) + w_2^-(M, g) = w_2(E) \quad .$$

REMARK 8.1. Since  $TM = T_+M \oplus T_-M$ , we have  $w_2(M) = w_2^+(M, g) + w_2^-(M, g) + w_1^+(M, g)w_1^-(M, g)$ . This allows one to express the conditions in the Theorem in various equivalent forms. For example, the conditions for existence of a spin structure can also be written as  $w_1(M) = 0$  and  $w_2(M) = w_1^-(M, g)^2$ .

**9. Some remarks on the spin geometry of M-theory.** In this section, we apply our results to a theory of physical interest, deriving a no-go result regarding the interpretation of its spinorial fields. Eleven-dimensional supergravity is a physical theory formulated on a connected and paracompact smooth 11-manifold  $M$ , which involves a metric  $g$  of Lorentzian signature, a four-form field strength  $F$  and a spin 3/2 fermion called the gravitino. In the standard local formulation, the gravitino is a real local field  $\psi_\mu^\alpha$  carrying a covector index  $\mu$  and a spinorial index  $\alpha$ , the latter running from 1 to 32. The theory admits supersymmetry transformations parameterized by a real fermionic supersymmetry generator  $\chi^\alpha$ . It is natural to ask how the local formulas appearing in the construction of this theory found in the Physics literature should be interpreted globally and what are the minimal conditions on  $(M, g)$  under which a consistent global interpretation is possible. When approaching this question, one has to consider the two possible choices of Lorentzian signature:

1. “Mostly plus” signature, i.e.  $(p, q) = (10, 1)$ , which belongs to the normal non-simple case  $p - q \equiv_8 1$ . In this case, the smallest real representations of  $\text{Cl}_{10,1}$  are the two irreducible representations, which have dimension 32 and are distinguished by the choice of signature  $\epsilon \in \{-1, 1\}$ .
2. “Mostly minus” signature, i.e.  $(p, q) = (1, 10)$ , which belongs to the complex case with  $p - q \equiv_8 7$ . In this case, the smallest real representation of  $\text{Cl}_{1,10}$  is the irreducible real representation, which has dimension 64. However, the smallest real representations of the even subalgebra  $\text{Cl}_{1,10}^{\text{ev}}$  are the two real chiral (Majorana-Weyl) representations, both of which have dimension 32 and are distinguished by the condition that the Clifford volume element maps to  $\epsilon \text{id}$  in the representation space, where  $\epsilon \in \{-1, 1\}$ .

When  $M$  is non-compact, Lorentzian metrics on  $M$  always exist. When  $M$  is compact, it is well-known that they exist iff the Euler characteristic of  $M$  vanishes. The results of this paper imply the following.

PROPOSITION 9.1. *Let  $(M, g)$  be a Lorentzian manifold of dimension  $d = 11$  and “mostly plus” signature  $(p, q) = (10, 1)$ . Then  $(M, g)$  admits an elementary real pinor bundle if and only if it admits a spin structure, i.e. if and only if  $w_1(M) = 0$  and  $w_2^+(M, g) = 0$ , which is equivalent with the conditions  $w_1(M) = 0$  and  $w_2(M) + w_1^-(M, g)^2 = 0$ . In that case,  $S$  has rank 32.*

*Proof.* We have  $p - q \equiv_8 1$ , which corresponds to the normal non-simple case. Hence a reduced elementary Lipschitz structure on  $(M, g)$  is a spin structure and the topological condition for existence of such is  $w_1(M) = w_2^+(M, g) + w_2^-(M, g) = 0$ . We have  $w_2^-(M, g) = w_2(T_-M) = 0$  since  $q = 1$  and  $T_-M$  is a real line bundle, hence the second condition reduces to  $w_2^+(M, g) = 0$ . We also have  $w_2(M) = w_2(T_+M) + w_2(T_-M) + w_1(T_+M)w_1(T_-M) = w_2(T_+M) + w_1(T_-M)^2$ , where we used the condition  $w_1(T_+M) + w_1(T_-M) = w_1(TM) = 0$ . Hence the second topological condition is equivalent modulo the first with the condition  $w_2(M) + w_1^-(M, g)^2 = 0$ .  $\square$

REMARK 9.1. Suppose that  $(M, g)$  is time-orientable, i.e. it admits a globally-defined timelike vector field  $X$ . Then we can take  $T_-M$  to be the real line bundle generated by  $X$ . Hence  $T_-M$  is topologically trivial and we have  $w_1^-(M, g) = 0$ . In this case, the structure group of  $P_{O(V, h)}(M, g)$  reduces to  $O(10, \mathbb{R})$  and the topological conditions for existence of a spin structure reduce to  $w_1(M) = 0$  and  $w_2(M) = 0$ . Notice that  $M$  admits a time-orientable Lorentzian metric iff it admits an arbitrary Lorentzian metric.

The result above implies:

*Assume that eleven-dimensional supergravity is formulated on a smooth Lorentzian eleven-manifold  $(M, g)$  of mostly plus signature. Then the supersymmetry generator  $\chi$  of the theory can be interpreted as a smooth global section of a bundle  $S$  of irreducible real Clifford modules if and only if  $M$  is oriented and spin. In that case, the gravitino  $\psi$  can be interpreted as a global section of the bundle  $T^*M \otimes S$ . Up to isomorphism, there are in fact two real pinor bundles  $S$  which can be considered in that case (assuming that the spin structure is fixed), which are distinguished by the signature  $\epsilon$ .*

Since physics should be invariant under changing  $g$  into  $-g$ , one expects a similarly simple interpretation in mostly minus signature when  $(M, g)$  admits a spin structure. In order to provide an argument in favor of the equivalence between the spinor bundles in both signatures, when  $M$  admits a spin structure, we need to further elaborate on the theory of  $\text{Spin}^o$  structures and hence we defer this analysis to Reference [11].

The global interpretation of the local formulas of supergravity is affected by cover ambiguities. This implies that one is not forced apriori to interpret  $\chi$  as a global section of a bundle  $S$  of irreducible Clifford modules. In fact, eleven-dimensional supergravity can be defined on unoriented eleven-manifolds, as explained in references [26] and [27]. In the approach of op. cit., one assumes a Pin structure on  $(M, g)$  and constructs the theory using the modified Dirac operator [9], even though a bundle of irreducible real Clifford modules does not exist on  $(M, g)$ . In view of this, the results above tell us *precisely* when it is possible to globally construct the theory using a vector bundle  $S$  endowed with *internal* Clifford multiplication. We show that this is possible exactly when  $(M, g)$  admits a spin structure.

We mention that the situation is considerably more involved when considering supergravity theories in lower dimensions (coupled to matter). As we show in forthcoming work, the results of this paper can be used to construct certain such theories without assuming that the corresponding space-time admits a Spin or Pin structure.

**10. Relation to other work.** Lipschitz groups for *complex* Clifford representations were considered in [10, 24, 25]. As apparent from the present work, the corresponding theory for real Clifford representations is considerably more involved.  $\text{Spin}^q(V, h)$  structures in positive signature  $p = d, q = 0$  were introduced in [5]. However, reference [5] considers so-called “quaternionic spinor bundles”, i.e. vector bundles associated to a  $\text{Spin}^q(V, h)$  structure through a *quaternionic* representation  $\gamma_{\mathbb{H}} : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{H}}(S)$  which is irreducible *over*  $\mathbb{H}$ . Here,  $S$  is a right  $\mathbb{H}$ -module and the representation is through  $\mathbb{H}$ -module endomorphisms. Any such representation is also a real representation upon viewing  $S$  as an  $\mathbb{R}$ -vector space by restriction of scalars, but that real representation need not be irreducible as a representation over  $\mathbb{R}$  (since  $S$  may admit invariant  $\mathbb{R}$ -subspaces which are not  $\mathbb{H}$ -submodules). In fact, a brief look at Table 1 on page 98 of [5] shows that the  $\mathbb{H}$ -irreducible quaternionic Clifford representations listed there are reducible over  $\mathbb{R}$  except for  $d \equiv_8 4$ , which in our terminology corresponds to a sub-case of the quaternionic simple case. We stress that “quaternionic spinor bundles” based on  $\mathbb{H}$ -irreducible quaternionic Clifford representations (as in [5]) are not directly relevant for most physical theories, where one is interested instead in elementary pinor bundles in the sense of this paper (namely, vector bundles whose fibers are  $\mathbb{R}$ -irreducible real Clifford representations). Similar remarks apply to the work of [17, 18], which extend the constructions of [5] by replacing  $\text{Sp}(1) = \text{Spin}(3)$  with a higher spin group. We study  $\text{Spin}^o$  structures in detail in reference [11].

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**Appendix A. Hyperbolic numbers.** Let  $\mathbb{D}$  be the commutative algebra of hyperbolic (a.k.a. split complex/double) numbers. Any element  $z \in \mathbb{D}$  can be written uniquely in the form  $z = x + jy$ , where  $x, y \in \mathbb{R}$  and  $j^2 = +1$ . The map  $\varphi : \mathbb{D} \rightarrow \mathbb{R} \times \mathbb{R}$  given by:

$$\varphi(x + jy) = (x + y, x - y)$$

is a unital isomorphism of  $\mathbb{R}$ -algebras which satisfies  $\varphi(-1) = (-1, -1)$ ,  $\varphi(j) = (1, -1)$  and  $\varphi(-j) = (-1, 1)$ . In particular,  $\varphi$  induces an isomorphism of groups  $\mathbb{D}^\times \simeq \mathbb{R}^\times \times \mathbb{R}^\times$  and the component maps  $\varphi_\pm : \mathbb{D} \rightarrow \mathbb{R}$  given by  $\varphi_\pm(x + jy) = x \pm y$  are unital morphisms of  $\mathbb{R}$ -algebras which satisfy  $\varphi_\pm(j) = \pm 1$  and  $\varphi_\pm(\mathbb{D}^\times) = \mathbb{R}^\times$ . The group  $\mathbb{D}^\times$  has four connected components:

$$\mathbb{D}^{\epsilon_1, \epsilon_2} \stackrel{\text{def.}}{=} \{z \in \mathbb{D}^\times \mid \text{sign}(\varphi_+(z)) = \epsilon_1 \quad , \quad \text{sign}(\varphi_-(z)) = \epsilon_2\} \quad ,$$

where  $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ . This gives a  $D_4$ -grading of  $\mathbb{D}^\times$  which corresponds to the grading morphism  $z \rightarrow (\text{sign}(\varphi_+(z)), \text{sign}(\varphi_-(z))) \in \mathbb{G}_2 \times \mathbb{G}_2 \simeq D_4$ . We have

$$(-1)\mathbb{D}^{\epsilon_1, \epsilon_2} = \mathbb{D}^{-\epsilon_1, -\epsilon_2} \quad , \quad j\mathbb{D}^{\epsilon_1, \epsilon_2} = \mathbb{D}^{\epsilon_1, -\epsilon_2} \quad , \quad (-j)\mathbb{D}^{\epsilon_1, \epsilon_2} = \mathbb{D}^{-\epsilon_1, \epsilon_2}$$

and hence  $\mathbb{D}^\times \simeq \mathbb{D}^{++} \times D_4$ . Moreover, we have  $\mathbb{D}^{++} \simeq \mathbb{R}_{>0} \times \mathbb{R}_{>0}$  and  $1 \in \mathbb{D}^{++}$ ,  $-1 \in \mathbb{D}^{--}$ .

Recall that *hyperbolic conjugation* is the unital  $\mathbb{R}$ -algebra automorphism of  $\mathbb{D}$  defined through:

$$(x + jy)^* = x - jy \quad \forall x, y \in \mathbb{R}$$

and that the *hyperbolic modulus* is the surjective map  $M : \mathbb{D} \rightarrow \mathbb{R}$  given by:

$$M(x) = z^*z = x^2 - y^2 = \varphi_+(z)\varphi_-(z) \quad \forall z = x + jy \in \mathbb{D} \quad (x, y \in \mathbb{R}) .$$

We have  $M(z_1z_2) = M(z_1)M(z_2)$ ,  $M(1) = 1$ ,  $M(j) = -1$  and  $M(jz) = -M(z)$ . A hyperbolic number  $z = x + jy \in \mathbb{D}$  is invertible iff  $M(z) \neq 0$ . The zero divisors of  $\mathbb{D}$  are characterized by  $M(z) = 0$  and correspond to the union of the lines  $y = \pm x$ . The hyperbolic modulus induces a group morphism  $M : \mathbb{D}^\times \rightarrow \mathbb{R}^\times$ .

The *group of unit hyperbolic numbers* is the subgroup of  $\mathbb{D}^\times$  given by:

$$\begin{aligned} U(\mathbb{D}) &\stackrel{\text{def.}}{=} \{z \in \mathbb{D}^\times \mid |M(z)| = 1\} = \{z \in \mathbb{D} \mid \varphi_+(z)\varphi_-(z) \in \{-1, 1\}\} = \\ &= \{x + jy \in \mathbb{D} \mid x^2 - y^2 \in \{-1, 1\}\} , \end{aligned}$$

and fits into the exact sequence:

$$1 \longrightarrow U(\mathbb{D}) \hookrightarrow \mathbb{D}^\times \xrightarrow{|M|} \mathbb{R}_{>0} \longrightarrow 1 .$$

This group has four connected components given by  $U^{\epsilon_1, \epsilon_2}(\mathbb{D}) \stackrel{\text{def.}}{=} U(\mathbb{D}) \cap \mathbb{D}^{\epsilon_1, \epsilon_2}$  and we have  $U(\mathbb{D}) \simeq U^{++}(\mathbb{D}) \times D_4$ . The map:

$$\mathbb{R} \ni \theta \rightarrow \cosh \theta + j \sinh \theta \in U^{++}(\mathbb{D})$$

gives a group isomorphism  $U^{++}(\mathbb{D}) \simeq (\mathbb{R}, +) \simeq \mathbb{R}_{>0}$ . Moreover, the map:

$$\mathbb{D}^\times \ni z \rightarrow \left( |M(z)|, \frac{z}{\sqrt{|M(z)|}} \right) \in \mathbb{R}_{>0} \times U(\mathbb{D})$$

is an isomorphism of groups and  $U(\mathbb{D})$  is homotopy-equivalent with  $\mathbb{D}^\times$ .

**Appendix B. On internal and external Clifford multiplication.** In this appendix, we give a general construction of Clifford multiplication for weakly faithful real Clifford representations and certain types of associated vector bundles, explaining the difference between the “internal” and “external” versions of the former.

Let  $\gamma : Cl(V, h) \rightarrow \text{End}(S_0)$  be a weakly faithful real Clifford representation,  $H$  be a Lie group and  $\theta, \theta' : H \rightarrow \text{Aut}_{\mathbb{R}}(S_0)$  be linear representations of  $H$  in  $S_0$ . Let  $\lambda : H \rightarrow O(V, h)$  be a representation of  $H$  through isometries of  $(V, h)$  and  $\rho : H \rightarrow GL(V \otimes S_0) \simeq \text{Aut}_{\mathbb{R}}(V) \otimes \text{Aut}_{\mathbb{R}}(S_0)$  denote the inner tensor product of  $\lambda$  and  $\theta$ :

$$\rho(a) \stackrel{\text{def.}}{=} \lambda(a) \otimes \theta(a) \quad \forall a \in H .$$

Consider the linear map  $\mu : V \otimes S_0 \rightarrow S_0$  given by  $\mu(v, \xi) \stackrel{\text{def.}}{=} \gamma(v)\xi$ .

**PROPOSITION B.1.** *The map  $\mu$  is a based morphism of representations from  $\rho$  to  $\theta'$  iff the following relation holds for all  $a \in H$  and all  $v \in V$ :*

$$\gamma(\lambda(a)v) = \theta'(a) \circ \gamma(v) \circ \theta(a)^{-1} . \tag{B.1}$$

*In particular,  $\theta$  and  $\theta'$  determine  $\lambda$  (since  $\gamma$  is weakly faithful).*

*Proof.* We have:

$$\begin{aligned} (L_\mu \circ \rho)(a)(v \otimes \xi) &= (\mu \circ \rho(a))(v \otimes \xi) = (\gamma(\lambda(a)(v)) \circ \theta(a))\xi \\ (\theta' \circ R_\mu)(a)(v \otimes \xi) &= (\theta'(a) \circ \mu)(v \otimes \xi) = (\theta'(a) \circ \gamma(v))\xi . \end{aligned}$$



Recall that  $\mu$  is a based morphism of representations iff  $\mu \circ \rho(a) = \theta'(a) \circ \mu$  for all  $a \in H$ , i.e. iff  $L_\mu \circ \rho = \theta' \circ R_\mu$ . Using the relations above, this amounts to the condition:

$$\gamma(\lambda(a)v) \circ \theta(a) = \theta'(a) \circ \gamma(v) \quad \forall v \in V \quad , \tag{B.2}$$

which amounts to (B.1).  $\square$

Relation (B.1) says that the pair of representations  $(\theta, \theta')$  of  $H$  in  $S_0$  “implements” the pseudo-orthogonal representation  $\lambda$  of  $H$  in  $V$ . In particular, this relation requires  $\theta'(a) \circ \gamma(V) \circ \theta(a)^{-1} \subset \gamma(V)$  for all  $a \in H$ , which is a non-trivial condition on  $\theta$  and  $\theta'$ .

Let  $P_{O(V,h)}$  be a principal  $O(V,h)$ -bundle over a manifold  $M$  and  $P_H$  be a  $\lambda$ -reduction of  $P$ . Let  $T$  be the vector bundle associated to  $P_{O(V,h)}$  through the tautological representation of  $O(V,g)$  on  $V$  and  $S = P_H \times_\theta S_0, S' = P_H \times_{\theta'} S_0$  be the vector bundles associated to  $P_H$  through the representations  $\theta$  and  $\theta'$  on  $S_0$ . Notice that  $T = P_H \times_\lambda V$ , i.e.  $T$  is associated to  $P_H$  through the representation  $\lambda$ . It follows that  $T \otimes S$  is associated to  $P_H$  through the representation  $\rho = \lambda \otimes \theta$ . Hence when condition (B.2) holds, the morphism of representations  $\mu$  induces a morphism of vector bundles  $\mu_* : T \otimes S \rightarrow S'$  which is called (*generalized*) *Clifford multiplication*. The Clifford multiplication is called *internal* if  $\theta$  and  $\theta'$  are equivalent representations of  $H$ . In this case, the vector bundles  $S$  and  $S'$  are isomorphic and we can identify them, thus obtaining a map  $T \otimes S \rightarrow S$  which induces a fiberwise  $\text{Cl}(T)$ -module structure on  $S$ . The Clifford multiplication is called *external* if  $\theta$  and  $\theta'$  are inequivalent representations of  $H$ . When  $P_{O(V,h)}$  is the orthogonal coframe bundle of a pseudo-Riemannian manifold  $(M,g)$ , we can take  $T = TM$  and have  $\mu_* : TM \otimes S \rightarrow S'$ , which in the internal case makes  $S$  into a bundle of Clifford modules over  $(M,g)$ . Consider the following applications of this construction, in those dimensions and signature where (B.1) can be satisfied with the choices listed below:

1.  $\gamma$  is irreducible over  $\mathbb{R}$ ,  $H = \text{Spin}(V,h)$ ,  $\theta = \theta' = \gamma|_{\text{Spin}(V,h)}$  and  $\lambda = \text{Ad}_0$ , the vector representation of  $\text{Spin}(V,h)$ . In this case,  $S$  is a spinor bundle associated to the spin structure  $P_H$  and  $\mu_* : TM \otimes S \rightarrow S$  is the ordinary Clifford multiplication of  $S$ .
2.  $\gamma$  is irreducible over  $\mathbb{R}$ ,  $H = \text{Pin}(V,h)$ ,  $\theta = \theta' = \gamma|_{\text{Pin}(V,h)}$  and  $\lambda = \text{Ad}_0$ , the *untwisted* vector representation of  $\text{Pin}(V,h)$ . Then  $P_H$  is an untwisted pin structure. We have  $\mu_* : TM \otimes S \rightarrow S$ , i.e. Clifford multiplication with a vector maps  $S$  into itself.
3.  $\gamma$  is irreducible over  $\mathbb{R}$ ,  $H = \text{Pin}(V,h)$ ,  $\theta = \gamma|_{\text{Pin}(V,h)}$ ,  $\theta' = \theta \circ \pi|_{\text{Pin}(V,h)}$  and  $\lambda = \widetilde{\text{Ad}}_0$ , where  $\pi$  is the parity involution of  $\text{Cl}(V,h)$  (recall that  $\pi(\text{Pin}(V,h)) = \text{Pin}(V,h)$ ). In this case,  $P_H$  is a twisted pin structure and the bundles  $S$  and  $S'$  are generally non-isomorphic. In this case, we obtain a generally external Clifford multiplication  $\mu_* : T^*M \otimes S \rightarrow S'$ .
4.  $H = \mathcal{L}$  (the reduced Lipschitz group) and  $\theta = \theta'$  coincide with the tautological representation of  $\mathcal{L}$  on  $S$ . Then (B.1) can always be satisfied with  $\lambda = \text{Ad}_0$ , the vector representation of  $\mathcal{L}$ . In this case,  $P_H$  is a Lipschitz structure while  $S$  is the pinor bundle associated to  $P_H$ . We have  $\mu_* : TM \otimes S \rightarrow S$ , i.e. the Clifford multiplication is internal.

In the case of twisted pin structures, the usual definition of the Dirac operator gives an operator which maps  $S$  into  $S'$ , an inconvenient feature which (in the case of complex pinor bundles) was noticed and discussed in [7, 8, 9]. Notice that Lipschitz structures



always lead to well-defined internal Clifford multiplication on  $S$ . In fact, Lipschitz structures are *designed* to make this happen.

**Appendix C. Twisted automorphisms of  $\mathbb{S}$ -modules and  $\mathbb{S}$ -valued pairings.**

**C.1. Twisted morphisms of modules.** Let  $\mathbb{S}, \mathbb{S}'$  be unital associative  $\mathbb{R}$ -algebras.

DEFINITION C.1. A twisted morphism from a left  $\mathbb{S}$ -module  $A$  to a left  $\mathbb{S}'$ -module  $A'$  is a pair  $(\varphi_0, \varphi)$ , where  $\varphi_0 \in \text{Hom}_{\text{Alg}}(\mathbb{S}, \mathbb{S}')$  is a unital morphism of  $\mathbb{R}$ -algebras and  $\varphi \in \text{Hom}_{\mathbb{R}}(A, A')$  is an  $\mathbb{R}$ -linear map, such that the following condition is satisfied:

$$\varphi(sa) = \varphi_0(s)\varphi(a) \quad \forall s \in \mathbb{S} \quad \text{and} \quad a \in A \quad .$$

Left modules over unital associative  $\mathbb{R}$ -algebras and twisted morphisms form a category denoted  $\text{TwMod}$ . This fibers over the category  $\text{Alg}$  of unital associative  $\mathbb{R}$ -algebras through the forgetful functor which takes a left  $\mathbb{S}$ -module  $A$  to  $\mathbb{S}$  and a twisted morphism  $(\varphi_0, \varphi)$  to  $\varphi_0$ . The fiber over  $\mathbb{S}$  is the usual category  $\text{Mod}_{\mathbb{S}}$  of left  $\mathbb{S}$ -modules and ordinary  $\mathbb{S}$ -module morphisms (those twisted module morphisms  $(\varphi_0, \varphi)$  for which  $\varphi_0 = \text{id}_{\mathbb{S}}$ ).

**C.2. Twisted automorphisms.** Let  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A)$  denote the group of twisted automorphisms of the left  $\mathbb{S}$ -module  $A$  and  $\text{Aut}_{\mathbb{S}}(A)$  denote the group of usual  $\mathbb{S}$ -module automorphisms. We have the following obvious result:

PROPOSITION C.1. There exists an exact sequence of groups:

$$1 \longrightarrow \text{Aut}_{\mathbb{S}}(A) \hookrightarrow \text{Aut}_{\mathbb{S}}^{\text{tw}}(A) \xrightarrow{F} \text{Aut}_{\text{Alg}}(\mathbb{S}) \quad , \tag{C.1}$$

where  $F(\varphi_0, \varphi) = \varphi_0$ .

**C.3.  $\mathbb{S}$ -valued symmetric pairings.**

DEFINITION C.2. An  $\mathbb{S}$ -valued symmetric pairing on the left  $\mathbb{S}$ -module  $A$  is an  $\mathbb{R}$ -bilinear symmetric map  $\mathfrak{p} : A \times A \rightarrow \mathbb{S}$ . The image algebra  $I(\mathfrak{p})$  determined by  $\mathfrak{p}$  is the subalgebra of  $\mathbb{S}$  generated by the set  $\mathfrak{p}(A \times A)$  over  $\mathbb{R}$ .

Notice that  $\mathfrak{p}$  is uniquely determined by its diagonal quadratic form  $\mathfrak{p}_d : A \rightarrow \mathbb{R}$ , which is defined through  $\mathfrak{p}_d(a) \stackrel{\text{def.}}{=} \mathfrak{p}(a, a)$ . Indeed, we have the polarization identity:

$$\mathfrak{p}(a_1, a_2) = \frac{1}{2}(\mathfrak{p}_d(a_1 + a_2) - \mathfrak{p}_d(a_1) - \mathfrak{p}_d(a_2)) \quad \forall a_1, a_2 \in A \quad .$$

We have  $\mathfrak{p}_d(\lambda a) = \lambda^2 \mathfrak{p}_d(a)$  for all  $\lambda \in \mathbb{R}$  and  $a \in A$  as well as the parallelogram identity:

$$\mathfrak{p}_d(a_1 + a_2) + \mathfrak{p}_d(a_1 - a_2) = 2[\mathfrak{p}_d(a_1) + \mathfrak{p}_d(a_2)] \quad (a_1, a_2 \in A) \quad .$$

DEFINITION C.3. Let  $\mathfrak{p}$  be an  $\mathbb{S}$ -valued symmetric pairing on the left  $\mathbb{S}$ -module  $A$ . A twisted automorphism  $(\varphi_0, \varphi) \in \text{Aut}_{\mathbb{S}}^{\text{tw}}(A)$  is called  $\mathfrak{p}$ -orthogonal if the following condition is satisfied:

$$\mathfrak{p}(\varphi(a_1), \varphi(a_2)) = \varphi_0(\mathfrak{p}(a_1, a_2)) \quad \forall a_1, a_2 \in A \quad . \tag{C.2}$$

Notice that the restriction  $\varphi_0|_{I(\mathfrak{p})}$  is uniquely determined by  $\varphi$ . Using the polarization identity, condition (C.2) is equivalent with:

$$\mathfrak{p}_a(\varphi(a)) = \varphi_0(\mathfrak{p}_a(a)) \quad \forall a \in A \quad . \tag{C.3}$$

Let  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p})$  denote the group of  $\mathfrak{p}$ -orthogonal twisted automorphisms of  $A$  and  $\text{Aut}_{\mathbb{S}}(A, \mathfrak{p})$  denote the subgroup of  $\mathfrak{p}$ -orthogonal module automorphisms (those  $\mathfrak{p}$ -orthogonal twisted automorphism with  $\varphi_0 = \text{id}_{\mathbb{S}}$ ).

DEFINITION C.4. *The twist group of  $\mathfrak{p}$  is the following subgroup of  $\text{Aut}_{\text{Alg}}(\mathbb{S})$ :*

$$G_{\mathfrak{p}} \stackrel{\text{def.}}{=} F(\text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p})) \subset \text{Aut}_{\text{Alg}}(\mathbb{S}) \quad .$$

The sequence (C.1) induces a short exact sequence:

$$1 \longrightarrow \text{Aut}_{\mathbb{S}}(A, \mathfrak{p}) \hookrightarrow \text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p}) \xrightarrow{F} G_{\mathfrak{p}} \longrightarrow 1 \quad . \tag{C.4}$$

Any  $\mathbb{R}$ -algebra automorphism of  $\mathbb{S}$  restricts to a group automorphism of  $\mathbb{S}^{\times}$ . This gives a morphism of groups  $\text{Res} : \text{Aut}_{\text{Alg}}(\mathbb{S}) \rightarrow \text{Aut}_{\text{Gp}}(\mathbb{S}^{\times}) = \text{Aut}_{\text{Gp}}((\mathbb{S}^{\times})^{\text{op}})$ . Let  $F_0 \stackrel{\text{def.}}{=} \text{Res} \circ F : \text{Aut}_{\mathbb{S}}^{\text{tw}}(A) \rightarrow \text{Aut}_{\text{Gp}}(\mathbb{S}^{\times})$  denote the morphism of groups induced by the map  $F$  of (C.1).

**C.4. The case of rank one free  $\mathbb{S}$ -modules.**

PROPOSITION C.2. *Let  $A$  be a free left  $\mathbb{S}$ -module of rank one and  $u$  be a basis of  $A$  over  $\mathbb{S}$ . Then  $\text{Aut}_{\mathbb{S}}(A) \simeq (\mathbb{S}^{\times})^{\text{op}}$ , the sequence (C.1) gives a split short exact sequence:*

$$1 \longrightarrow (\mathbb{S}^{\times})^{\text{op}} \rightarrow \text{Aut}_{\mathbb{S}}^{\text{tw}}(A) \xrightarrow{F} \text{Aut}_{\text{Alg}}(\mathbb{S}) \longrightarrow 1 \tag{C.5}$$

and there exists an isomorphism of groups:

$$\text{Aut}_{\mathbb{S}}^{\text{tw}}(A) \simeq (\mathbb{S}^{\times})^{\text{op}} \rtimes_{\text{Res}} \text{Aut}_{\text{Alg}}(\mathbb{S}) \quad . \tag{C.6}$$

*Proof.* Since  $A$  is a free rank one left  $\mathbb{S}$ -module with basis  $u$ , we have  $A = \mathbb{S}u$  and any  $x \in A$  can be written as  $x = su$  for some uniquely-determined  $s$ . Thus any  $\mathbb{R}$ -linear map  $\varphi \in \text{End}_{\mathbb{R}}(A)$  defines an element  $\sigma_u(\varphi) \in \mathbb{S}$  through the relation  $\varphi(u) = \sigma_u(\varphi)u$ . Conversely, any  $s \in \mathbb{S}$  defines an  $\mathbb{R}$ -linear operator  $x \rightarrow sx$  acting in  $A$ , which takes  $u$  into  $su$ . This gives a surjective  $\mathbb{R}$ -linear map:

$$\sigma_u : \text{End}_{\mathbb{R}}(A) \longrightarrow \mathbb{S} \tag{C.7}$$

which satisfies  $\sigma_u(\text{id}_A) = 1$ . For  $(\varphi_0, \varphi), (\varphi'_0, \varphi') \in \text{Aut}_{\mathbb{S}}^{\text{tw}}(A)$ , we have:

$$\sigma_u(\varphi \circ \varphi') = \varphi_0(\sigma_u(\varphi'))\sigma_u(\varphi) = \sigma_u(\varphi) \cdot^{\text{op}} \varphi_0(\sigma_u(\varphi')) \quad , \tag{C.8}$$

where  $\cdot^{\text{op}}$  denotes multiplication in  $\mathbb{S}^{\text{op}}$ . It is easy to see that (C.8) implies the inclusion  $\sigma_u(\text{Aut}_{\mathbb{R}}^{\text{tw}}(A)) \subset \mathbb{S}^{\times}$  as well as the relation:

$$\sigma_u(\varphi)^{-1} = \varphi_0(\sigma_u(\varphi^{-1})) \quad \text{for } (\varphi_0, \varphi) \in \text{Aut}_{\mathbb{S}}^{\text{tw}}(A) \quad .$$

When  $\varphi \in \text{Aut}_{\mathbb{S}}(A)$  is an untwisted automorphism of  $A$  (thus  $\varphi_0 = \text{id}_{\mathbb{S}}$ ), we have  $\varphi(su) = s\varphi(u) = s\sigma_u(\varphi)u$  for all  $s \in \mathbb{S}$ , which shows that  $\varphi$  is uniquely-determined

by  $\sigma_u(\varphi) \in \mathbb{S}^\times$ . This implies that the map  $\sigma_u$  of (C.7) restricts to a bijection between  $\text{Aut}_{\mathbb{S}}(A)$  and  $\mathbb{S}^\times$  while (C.8) with  $\varphi_0 = \text{id}_{\mathbb{S}}$  shows that this bijection is an isomorphism of groups between  $\text{Aut}_{\mathbb{S}}(A)$  and  $(\mathbb{S}^{\text{op}})^\times = (\mathbb{S}^\times)^{\text{op}}$ .

Relation (C.8) also implies that the map  $T_u : \text{Aut}_{\mathbb{S}}^{\text{tw}}(A) \rightarrow (\mathbb{S}^\times)^{\text{op}} \rtimes_{\text{Res}} \text{Aut}_{\text{Alg}}(\mathbb{S})$  given by:

$$T_u(\varphi_0, \varphi) \stackrel{\text{def.}}{=} (\sigma_u(\varphi), \varphi_0)$$

is a morphism of groups. This map is bijective since, for any pair  $(\sigma, \varphi_0) \in \mathbb{S}^\times \times \text{Aut}_{\text{Alg}}(\mathbb{S})$ , there exists a unique  $\varphi \in \text{Aut}_{\mathbb{R}}(A)$  such that  $(\varphi_0, \varphi) \in \text{Aut}_{\mathbb{S}}^{\text{tw}}(A)$  and  $\sigma_u(\varphi) = \sigma$ , namely:

$$\varphi(su) = \varphi_0(s)\sigma u \quad (s \in \mathbb{S}) \ ;$$

we have  $T_u^{-1}(\sigma, \varphi_0) = (\varphi_0, \varphi)$ . Taking  $\sigma = 1$  shown that the map  $F$  of (C.1) is surjective. In view of (C.1), its kernel coincides with  $\text{Aut}_{\mathbb{S}}(A)$ , which (as show above) is isomorphic with  $(\mathbb{S}^\times)^{\text{op}}$ .

In fact, the morphism  $G_u : \text{Aut}_{\text{Alg}}(\mathbb{S}) \rightarrow \text{Aut}_{\mathbb{S}}^{\text{tw}}(A)$  given by  $G_u(\varphi_0) \stackrel{\text{def.}}{=} T_u^{-1}(1, \varphi_0)$  is a section of  $F$ :

$$F \circ G_u = \text{id}_{\text{Aut}_{\text{Alg}}(\mathbb{S})}$$

and hence splits the sequence (C.5). The semidirect product presentation (C.6) is the one induced by the splitting morphism  $G_u$ .  $\square$

For the following, we fix a basis  $u \in A$ . Identifying  $A$  with  $\mathbb{S}$  through the isomorphism  $\mathbb{S} \ni s \rightarrow su \in A$ , an  $\mathbb{S}$ -valued  $\mathbb{R}$ -bilinear symmetric form  $\mathfrak{p}$  on  $A$  corresponds to an  $\mathbb{S}$ -valued  $\mathbb{R}$ -bilinear symmetric form:

$$\mathfrak{p}_u : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{S} \tag{C.9}$$

on  $\mathbb{S}$ , namely:

$$\mathfrak{p}_u(s_1, s_2) \stackrel{\text{def.}}{=} \mathfrak{p}(s_1u, s_2u) \ .$$

Using relation (C.6), a twisted  $\mathfrak{p}$ -orthogonal automorphism  $(\varphi_0, \varphi) \in \text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p})$  corresponds to a pair  $(\sigma_u, \varphi_0) \stackrel{\text{def.}}{=} (\sigma_u(\varphi), \varphi_0) \in \mathbb{S}^\times \times \text{Aut}_{\text{Alg}}(\mathbb{S})$  which satisfies:

$$\mathfrak{p}_u(\varphi_0(s_1)\sigma_u, \varphi_0(s_2)\sigma_u) = \varphi_0(\mathfrak{p}_u(s_1, s_2)) \quad \forall s_1, s_2 \in \mathbb{S} \ . \tag{C.10}$$

Accordingly, the group  $\text{Aut}_{\mathbb{S}}^{\text{tw}}(A, \mathfrak{p})$  identifies with the subgroup of  $(\mathbb{S}^\times)^{\text{op}} \rtimes \text{Aut}_{\text{Alg}}(\mathbb{S})$  consisting of all such pairs. Obviously, this is a subgroup of  $(\mathbb{S}^\times)^{\text{op}} \times G_{\mathfrak{p}}$ .

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