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THE Q_α-RESTRICTION PROBLEM*

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Abstract. Let $\alpha \in [0, 1)$ and Ω be an open connected subset of $\mathbb{R}^{n\geq 2}$. This paper shows that the Q_α-restriction problem $Q_\alpha|_\Omega = \mathscr{Q}_\alpha(\Omega)$ is solvable if and only if Ω is an Ahlfors n-regular domain; i.e., $\mathrm{vol}(B(x,r)\cap\Omega) \gtrsim r^n$ for any Euclidean ball $B(x,r)$ with center $x \in \Omega$ and radius $r \in \big(0, \mathrm{diam}(\Omega)\big)$, thereby not only yielding an exponential Q_α-integrability as a proper adjustment of the John-Nirenberg type inequality for Q_α conjectured in [3, Problem 8.1, (8.2)] but also resolving the quasiconformal extension problem for Q_α posed in [3, Problem 8.5].

Key words. Q space, restriction, Ahlfors regular domain, Uniform domain, Minkowski type dimension.

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1. Introduction.

The conformal Poincaré inequality says that

$$\fint_{B(x_0,r_0)} |u(x) - \bar{u}_{x_0,r_0}|\,dx \lesssim \left(\int_{B(x_0,r_0)} |\nabla u(x)|^n\,dx\right)^{\frac{1}{n}}$$

holds for any ball $B(x_0, r_0)$ (with centre x_0 and radius r_0) of the Euclidean space $\mathbb{R}^{n\geq 2}$ and a given u in $\dot{W}^{1,n} \equiv \dot{W}^{1,n}(\mathbb{R}^n)$ which is the conformal Sobolev space of all functions with their weak derivatives being n-integrable over $\mathbb{R}^n$; i.e.,

$$\|\nabla u\|_{L^n} \equiv \left(\int_{\mathbb{R}^n} |\nabla u(x)|^n\,dx\right)^{\frac{1}{n}} < \infty.$$

Here and henceforth,

$$\bar{v}_{x_0,r_0} = \fint_{B(x_0,r_0)} v(x)\,dx = \mathrm{vol}\big(B(x_0,r_0)\big)^{-1}\int_{B(x_0,r_0)} v(x)\,dx$$

is the integral mean of a function v over $B(x_0, r_0)$; $\mathrm{vol}(\cdot)$ expresses the n-dimensional Lebesgue measure; $A \lesssim B$ or $B \gtrsim A$ means that $A \leq cB$ for a constant $c > 0$; moreover $A \approx B$ is equivalent to both $A \lesssim B$ and $B \lesssim A$. Upon writing $BMO \equiv BMO(\mathbb{R}^n)$ as the John-Nirenberg class of functions u with bounded mean oscillation:

$$[u]_{BMO} \equiv \sup_{B(x_0,r_0)} \fint_{B(x_0,r_0)} |u(x) - \bar{u}_{x_0,r_0}|\,dx < \infty,$$

we get that $\dot{W}^{1,n}$ embeds into BMO; i.e. (cf. [17, p.34]),

$$\dot{W}^{1,n} \hookrightarrow BMO \quad \text{with} \quad [u]_{BMO} \lesssim \|\nabla u\|_{L^n}.$$

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Surprisingly but naturally, this embedding can be split into (cf. [20, Theorem 4.1]):

$$\dot{W}^{1,n} \hookrightarrow Q_{0\le\alpha<1} \hookrightarrow BMO \quad \text{with} \quad [u]_{BMO} \lesssim [u]_{Q_{0\le\alpha<1}} \lesssim \|\nabla u\|_{L^n},$$

where for $\alpha \in (-\infty,\infty)$, as a conformally invariant space, $Q_\alpha \equiv Q_\alpha(\mathbb{R}^n)$, comprises all functions u on $\mathbb{R}^n$ obeying

$$[u]_{Q_\alpha} \equiv \sup_{B(x_0,r_0)} \left(\text{vol}\big(B(x_0,r_0)\big)^{\frac{2\alpha-n}{n}} \fint_{B(x_0,r_0)} \fint_{B(x_0,r_0)} |u(x)-u(y)|^2|x-y|^{-(n+2\alpha)}\,dxdy\right)^{\frac{1}{2}} < \infty,$$

but as a Banach space, $Q_{-\infty<\alpha<0} = BMO$ and $Q_{1\le\alpha<\infty} = \{0\}$ (cf. [3, 19, 1, 2, 21, 22]).

Upon restricting $Q_{\alpha\in[0,1)}$ to an open connected subset (a subdomain) of $\mathbb{R}^n$, we discover

Theorem 1.1. *Let $\alpha \in [0,1)$ and $\Omega \subset \mathbb{R}^n$. Then the following three statements are equivalent:*

(i) *Ω is an Ahlfors n-regular domain; i.e., there exists a constant $C>0$ such that for any cube I centred in Ω with its edge length $\ell(I) < 2\operatorname{diam}\Omega$ one has $vol(I\cap\Omega) \ge C vol(I)$.*

(ii) *Ω is a $\mathscr{Q}_\alpha$-extension domain; i.e., $Q_\alpha|_\Omega \equiv \{u = v|_\Omega : \ v \in Q_\alpha\} = \mathscr{Q}_\alpha(\Omega)$ with equivalence:*

$$\|u\|_{Q_\alpha|_\Omega} \equiv \inf\{[v]_{Q_\alpha} : \ v|_\Omega = u \ \&\ v \in Q_\alpha\} \approx \|u\|_{\mathscr{Q}_\alpha(\Omega)}.$$

(iii) *Ω is a $\mathscr{Q}_\alpha$-embedding domain; i.e., there are constants C_1 and C_2 depending on α and n such that if $0 < \|u\|_{\mathscr{Q}_\alpha(\Omega)} < \infty$ then*

$$\inf_{c\in\mathbb{R}} \sup_{I\subset\mathbb{R}^n} vol(I)^{-1} \int_{I\cap\Omega} \exp\left(C_1 \frac{|u(x)-c|}{\|u\|_{\mathscr{Q}_\alpha(\Omega)}}\right) dx \le C_2.$$

In the above and below, $\mathscr{Q}_\alpha(\Omega)$ is the collection of all functions u on Ω with

$$\|u\|_{\mathscr{Q}_\alpha(\Omega)} \equiv \sup_{I\subset\mathbb{R}^n} \left(\text{vol}(I)^{\frac{2\alpha-n}{n}} \int_{I\cap\Omega}\int_{I\cap\Omega} |u(x)-u(y)|^2|x-y|^{-(n+2\alpha)}\,dx\,dy\right)^{\frac{1}{2}} < \infty,$$

and $\sup_{I\subset\mathbb{R}^n}$ is taken over all cubes I with edges parallel to the coordinate axes. Meanwhile, if $I\subset\mathbb{R}^n$ and $I\cap\Omega$ above are replaced by $I\subset\Omega$ (with its edges parallel to the coordinate axes), then the corresponding space and its semi-norm are written as $Q_\alpha(\Omega)$ and $\|\cdot\|_{Q_\alpha(\Omega)}$ respectively, and hence it is easy to check the following two facts:

- $Q_\alpha = \mathscr{Q}_\alpha(\mathbb{R}^n) \subset \mathscr{Q}_\alpha(\Omega) \subset Q_\alpha(\Omega)$;
- $Q_\alpha(\Omega) = Q_\alpha|_\Omega \Longleftrightarrow Q_\alpha(\Omega) = \mathscr{Q}_\alpha(\Omega) \ \&\ \mathscr{Q}_\alpha(\Omega) = Q_\alpha|_\Omega$.

As the direct consequence of Theorem 1.1, we get

Theorem 1.2. *Let $0\le\alpha<1$. There exists a constant $C>0$ depending on α and n such that if $0 < [u]_{Q_\alpha} < \infty$ then*

$$\langle u\rangle_{Q_\alpha} \equiv \sup_{B(x_0,r_0)} \fint_{B(x_0,r_0)} \exp\left(C\frac{|u(x)-\bar{u}_{x_0,r_0}|}{[u]_{Q_\alpha}}\right) dx < \infty$$

and hence

$$\frac{vol\big(\{x \in B(x_0,r_0):\ |u(x)-\bar{u}_{x_0,r_0}|>t\}\big)}{vol\big(B(x_0,r_0)\big)} \le \langle u\rangle_{Q_\alpha} \exp\left(-\frac{Ct}{[u]_{Q_\alpha}}\right) \quad \forall \quad t>0.$$

This embedding property may be viewed as not only a proper adjustment of the John-Nirenberg type inequality for Q_α conjectured in (8.2) (cf. [24, Theorems 1 & 2]) of

[3, Problem 8.1] – *Let* $\alpha \in (0,1)$. *Give a John-Nirenberg type inequality for* $Q_\alpha(\mathbb{R}^n)$.

but also a connection between the Moser-Trudinger inequality and the John-Nirenberg inequality below:

- Moser-Trudinger's inequality (cf. [15, 18]) - if u is a $C^1(\mathbb{R}^n)$ function supported in Ω with $\mathrm{vol}(\Omega)<\infty$ and $\bar{u}_\Omega = \fint_\Omega u(x)\,dx$ then there is a constant $C>0$ such that

$$\fint_\Omega \exp\left(C\frac{|u(x)-\bar{u}_\Omega|}{\|\nabla u\|_{L^n}}\right) dx < \infty.$$

- John-Nirenberg's inequality (cf. [10]) - if $u \in BMO$ enjoys $[u]_{BMO}>0$ then there is a constant $C>0$ such that

$$\sup_{B(x_0,r_0)} \fint_{B(x_0,r_0)} \exp\left(C\frac{|u(x)-\bar{u}_{x_0,r_0}|}{[u]_{BMO}}\right) dx < \infty.$$

As the indirect consequence of Theorem 1.1, we obtain

Theorem 1.3. *Let* $\alpha \in [0,1)$ *and* $\Omega \subset \mathbb{R}^n$.

(i) *If* Ω *is a uniform domain; i.e., there exists a constant* C *such that for every pair of points* $x, y \in \Omega$ *one can find a curve* $\gamma \subset \Omega$ *joining* x, y *and obeying*

$$l(\gamma) \le C|x-y| \quad \& \quad d(z,\Omega^{\complement}) \ge C^{-1}\frac{|x-z||y-z|}{|x-y|} \quad \forall \ z \in \gamma$$

with $l(\gamma)$ *being the Euclidean length of* γ, *and if*

$$n > \begin{cases} 2\alpha + \overline{\dim}_{\mathscr{L}}\partial\Omega & \text{when } \partial\Omega \text{ is bounded;} \\ 2\alpha + \overline{\dim}_{\mathscr{G}}\partial\Omega & \text{when } \partial\Omega \text{ is unbounded,} \end{cases} \tag{1.1}$$

then $Q_\alpha|_\Omega = Q_\alpha(\Omega)$. *Moreover,* (1.1) *is sharp under* $\alpha \in (0,1)$ *in the sense that there exists a uniform domain* Ω_α *such that*

$$n = \begin{cases} 2\alpha + \overline{\dim}_{\mathscr{L}}\partial\Omega_\alpha & \text{when } \partial\Omega_\alpha \text{ is bounded;} \\ 2\alpha + \overline{\dim}_{\mathscr{G}}\partial\Omega_\alpha & \text{when } \partial\Omega_\alpha \text{ is unbounded,} \end{cases} \tag{1.2}$$

but $Q_\beta|_{\Omega_\alpha} \ne Q_\beta(\Omega_\alpha)\ \forall\ \beta \in [\alpha,1)$, *and especially, if* $\alpha \in (0,\frac{1}{2}]$ *and* $n=2$ *then the domain* Ω_α *can be chosen as a simply connected domain in* $\mathbb{R}^2$.

(ii) *If* $Q_\alpha|_\Omega = Q_\alpha(\Omega)$ *under the hypothesis that* Ω *is a simply-connected domain in* $\mathbb{R}^2$ *or quasiconformally equivalent to a uniform domain in* $\mathbb{R}^{n\ge3}$, *then* Ω *is a uniform domain.*

As a matter of fact, Theorem 1.3 may be regarded as a resolution to Essén-Janson-Peng-Xiao's quasiconformal extension problem posed in:

[3, Problem 8.5] – *Let* $\alpha \in (0, 1)$*. Find a geometric property of* $\partial\Omega$ *such that* $Q_\alpha(\Omega) = Q_\alpha(\mathbb{R}^2)|_\Omega$.

Here it is perhaps appropriate to point out that $\triangle \equiv \{(x, y) \in \mathbb{R}^2 : \ 0 < x < 1,\ 0 < y < 1-x\}$ is a uniform domain with $Q_\beta(\mathbb{R}^2)|_\triangle \neq Q_\beta(\triangle)\ \forall\ \beta \in [\frac{1}{2}, 1)$, and thus Theorem 1.3 is completely different from the known settings for $\dot{W}^{1,n}$ and BMO as seen below:

- The quasiconformal extension problem for $\dot{W}^{1,n}$ - if $W^{1,n}(\Omega)$ is the conformal Sobolev space of all locally-integrable functions u with
$$\|u\|_{W^{1,n}(\Omega)} \equiv \left(\int_\Omega |\nabla u(x)|^n \, dx\right)^{\frac{1}{n}} < \infty,$$
then it is proved in Jones [12] and Gol'dšteĭn-Latfullink-Vodop'janov [6] that $W^{1,2}(\Omega) = W^{1,2}(\mathbb{R}^2)|_\Omega$ when and only when $\Omega \subset \mathbb{R}^2$ is a uniform domain (i.e., $\partial\Omega$ is a quasicircle). But, this is no longer valid for $\mathbb{R}^{n\geq 3}$ - in other words - each uniform domain $\Omega \subset \mathbb{R}^{n\geq 3}$ is a conformal Sobolev extension domain ($\dot{W}^{1,n}(\Omega) = \dot{W}^{1,n}|_\Omega$; cf. [12]) but not conversely - an easy example is provided by a ball from which a countable number of suitably chosen points are excluded; and an uneasy one under some strongest possible topological conditions is presented in [23]. Moreover, a higher dimension analogy can be found in [4].
- The quasiconformal extension problem for BMO - if $BMO(\Omega)$ stands for the class of all functions $u \in L^1_{\mathrm{loc}}(\Omega)$ with
$$\|u\|_{BMO(\Omega)} \equiv \sup_{B(x_0,r_0)\subseteq\Omega} \fint_{B(x_0,r_0)} |u(x) - \bar{u}_{x_0,r_0}| \, dx < \infty,$$
then according to Jones' [11, Theorem 1], every $u \in BMO(\Omega)$ is the restriction to Ω of some function $\tilde{u} \in BMO$; i.e., $BMO(\Omega) = BMO(\mathbb{R}^2)|_\Omega$, if and only if there two distance functions d_1, d_2 on the set Whitney cubes $\mathsf{E} = \{\mathcal{Q}_j\}$ such that $d_1(\mathcal{Q}_j, \mathcal{Q}_k) \lesssim d_2(\mathcal{Q}_j, \mathcal{Q}_k)$ holds for all $\mathcal{Q}_j, \mathcal{Q}_k \in \mathsf{E}$; in particular, under Ω being the interior or exterior domain complementary to a Jordan curve $\Gamma \subset \mathbb{R}^2$, $BMO(\Omega) = BMO(\mathbb{R}^2)|_\Omega$ if and only if $\Omega \subset \mathbb{R}^2$ is a uniform domain; i.e., Γ is a quasicircle - however - this does not hold for $\mathbb{R}^{n\geq 3}$ - more precisely - it is basically proved in Riemann [16] that if $\partial\Omega$ is a quasisphere (i.e., the image of the unit sphere $\mathbb{S}^{n-1}$ under a globally quasiconformal self-homeomorphism of $\mathbb{R}^{n\geq 3}$) then $BMO(\Omega) = BMO|_\Omega$, but not conversely as shown in [11, Theorem 3].

In §2 we prove Theorem 1.1 via utilizing the extension constructed in [12, 9, 26] and the fractional Sobolev inequality established in [26], and then verify Theorem 1.2 through specializing Theorem 1.1. In §3 we turn to determine some natural relationships between $\mathscr{Q}$-spaces and Q-spaces associated with (1.1) and (1.2) in Theorem 1.3 (cf. Theorems 3.1-3.2-3.3). In §4 we validate Theorem 1.3(i) via employing Theorem 1.1 plus Theorems 3.1-3.2-3.3, and then check Theorem 1.3(ii) through Theorem 4.1 (saying that if $Q_{0\leq\alpha<1}|_\Omega = Q_{0\leq\alpha<1}(\Omega)$ then Ω is l.l.c.) and the so-called Q_α-capacity.

Remark 1.1. *Below are several more fundamental conventions.*

- *For a set* $E \subset \mathbb{R}^n$ *and every* $r > 0$, *denote by* $N_{\mathrm{cov}}(r, E)$ *the minimal number of cubes with edge length* r *required to cover* E. *Then (cf. [14])*
  - *The local self-similar Minkowski dimension of* E *is defined as*

$$\overline{\dim}_{\mathscr{L}} E = \liminf_{N\to\infty} \limsup_{r\to 0} \sup_{\substack{B\subset\mathbb{R}^n \\ Nr\le r_B\le 1}} \frac{\log_2 N_{\mathrm{cov}}(r,\, E\cap B)}{\log_2(r_B/r)},$$

    where the supremum is taken over all balls $B = B(x_B,\, r_B) \subset \mathbb{R}^n$ *with centre* $x_B \in \mathbb{R}^n$ *and radius* $r_B \in [Nr, 1]$.
  - *The global self-similar Minkowski dimension of* E *is defined as*

$$\overline{\dim}_{\mathscr{G}} E = \liminf_{N\to\infty} \sup_{r>0} \sup_{\substack{B\subset\mathbb{R}^n \\ r_B\ge Nr}} \frac{\log_2 N_{\mathrm{cov}}(r,\, E\cap B)}{\log_2(r_B/r)},$$

    where the first supremum is taken over all $r \in (0,\, \infty)$ *and the second is over all balls* $B = B(x_B,\, r_B) \subset \mathbb{R}^n$ *with centre* $x_B \in \mathbb{R}^n$ *and radius* $r_B \in [Nr,\, \infty)$.
- $\ell(I)$ *and* I° *stand for the edge length and the interior of a cube* $I \subset \mathbb{R}^n$, *respectively.*
- *denote by* $I(x,\ell)$ *the cube centered at* x *with side length* ℓ; *for* $\delta > 0$ *let* $\delta I(x,\ell)$ *be the cube* $I(x,\delta\ell)$; *set* $I = I(x_I,\ell(I))$ *and* $\delta I = I(x_I,\delta\ell(I))$ *whenever convenient.*
- *For* $\delta > 0$ *and* $B(x,r)$ *- the ball centered at* x *with radius* r, *let* $\delta B(x,r) = B(x,\delta r)$.
- C *represents a positive constant which is independent of the main parameters, but may vary from line to line.*
- *For any locally integrable function* f *and measurable set* E, *denote by* $⨍_E f$ *the average of* f *on* E *with respect to the volume element* dx, *namely,* $⨍_E f = \big(vol(E)\big)^{-1}\int_E f\,dx$.
- *For a subset* E *of* $\mathbb{R}^n$ *and a point* $x \in \mathbb{R}^n$, *define* $d(x,\, E) = \inf_{z\in E}|x-z|$ *and* $\operatorname{diam} E = \sup_{y,z\in E}|y-z|$.

2. Demonstration of Theorems 1.1-1.2.

2.1. Proof of Theorem 1.1. This consists of three implications: (i)$\Longrightarrow$(ii); (ii)$\Longrightarrow$(iii); (iii)$\Longrightarrow$(i).

Proof of (i)$\Longrightarrow$(ii). Suppose Ω is Ahlfors n-regular. Then $\mathrm{vol}(\overline{\Omega}\setminus\Omega) = 0$ (cf. [9, 26]). Let $\overline{\Omega}^{\complement} \equiv \mathbb{R}^n\setminus\overline{\Omega}$. Then $\overline{\Omega}^{\complement}$ is an open set and admits a Whitney decomposition.

Lemma 2.1. *There exists a collection* $W = \{I_i\}_{i\in\mathbb{N}}$ *of countably many (closed) cubes with* x_i *being the center of* I_i *and* $\ell_i = \ell(I_i)$ *being the side length of* I_i *such that:*
- $\overline{\Omega}^{\complement} = \bigcup_{i\in\mathbb{N}} I_i$ *and* $(I_i)^\circ \cap (I_j)^\circ = \emptyset$ *for all* $i,j \in I$ *with* $i \ne j$;
- $4\sqrt{n}\ell_i \le \operatorname{dist}(I_i,\Omega) \equiv d(I_i,\Omega) \le 16\sqrt{n}\ell_i$;
- $4^{-1}\ell I_i \le \ell(I_j) \le 4\ell_i$ *whenever* $I_i \cap I_j \ne \emptyset$;
- *for each* $i \in \mathbb{N}$, *there exists* $x_i{}^* \in \Omega$ *such that* $d(x_i{}^*, I_i) \le 17\sqrt{n}\ell_i$;
- *for a given* I_i, *the cardinality* $\#\{j :\ I_j \cap I_i \ne \emptyset\}$ *is at most* 12^n.

For the decomposition in Lemma 2.1, there exists a family of smooth functions $\{\varphi_i\}$ such that

- supp $\varphi_i \subset 2I(x_i, \ell_i) \quad \forall \quad i \in \mathbb{N}$;
- $|\nabla\varphi_i| \lesssim \ell_i^{-1} \quad \forall \quad i \in \mathbb{N}$;
- $\sum_i \varphi_i = \chi_{\overline{\Omega}^{\complement}}$.

For each $x \in \overline{\Omega}^c$, let Λ_x be the collection of all $i \in \mathbb{N}$ such that $x \in 2I_i$. Applying Lemma 2.1, we have

$$\sharp\Lambda_x \lesssim 1 \quad \forall \quad x \in \overline{\Omega}^c. \tag{2.1}$$

For each $u \in \mathscr{Q}_\alpha(\Omega)$, define

$$Eu(x) \equiv \begin{cases} u(x) & \text{as } \ x \in \Omega; \\ 0 & \text{as } \ x \in \overline{\Omega} \setminus \Omega; \\ \sum_{i\in\Lambda_x} \varphi_i(x) \fint_{I(x_i{}^*, \ell_i)\cap\Omega} u(z)\, dz & \text{as } \ x \in \overline{\Omega}^c, \end{cases} \tag{2.2}$$

where x_i^* and ℓ_i are as in Lemma 2.1.

Obviously, $Eu = u$ on Ω. We are about to show

$$Eu \in \mathscr{Q}_\alpha(\mathbb{R}^n) \;\;\&\;\; \|Eu\|_{\mathscr{Q}_\alpha(\mathbb{R}^n)} \lesssim \|u\|_{\mathscr{Q}_\alpha(\Omega)}. \tag{2.3}$$

By the definition of $\|Eu\|_{\mathscr{Q}_\alpha(\mathbb{R}^n)}$, this is reduced to showing that for every cube $I \subset \mathbb{R}^n$ one has

$$\ell(I)^{2\alpha-n} \int_I \int_I \frac{|Eu(x) - Eu(y)|^2}{|x-y|^{n+2\alpha}} dxdy \lesssim \|u\|^2_{\mathscr{Q}_\alpha(\Omega)}. \tag{2.4}$$

So, we are required to handle the two cases: $\operatorname{diam}\Omega = \infty$ and $\operatorname{diam}\Omega < \infty$.

<u>*Case:* $\operatorname{diam}\Omega = \infty$.</u>

We consider three subcases: $I \cap \Omega \neq \emptyset$, $I \cap \Omega = \emptyset$ but $d(I, \Omega) \le 100\sqrt{n}\ell(I)$ and $I \cap \Omega = \emptyset$ but $d(I, \Omega) > 100\sqrt{n}\ell(I)$.

Subcase 1: $I \cap \Omega \neq \emptyset$. Write

$$\int_I \int_I \frac{|Eu(x) - Eu(y)|^2}{|x-y|^{n+2\alpha}}\, dx\, dy = H_1 + 2H_2 + H_3,$$

where

$$\begin{cases} H_1 \equiv \int_{I\cap\Omega} \int_{I\cap\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\, dx\, dy, \\ H_2 \equiv \int_{I\setminus\Omega} \int_{I\cap\Omega} \frac{|Eu(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\, dy\, dx, \\ H_3 \equiv \int_{I\setminus\Omega} \int_{I\setminus\Omega} \frac{|Eu(x)-Eu(y)|^2}{|x-y|^{n+2\alpha}}\, dx\, dy. \end{cases}$$

It suffices to prove

$$H_i \lesssim \operatorname{vol}(I)^{(n-2\alpha)/n} \|u\|^2_{\mathscr{Q}_\alpha(\Omega)} \quad \forall \quad i = 1, 2, 3.$$

Obviously, H_1 satisfies the above inequality.

To estimate H_2, for $x \in I \setminus \overline{\Omega}$ and $y \in I \cap \Omega$, by $\sum_{i\in\Lambda_x} \varphi_i(x) = 1$ we write

$$\begin{aligned} Eu(x) - u(y) &\le \left[\sum_{i\in\Lambda_x} \varphi_i(x) \fint_{I(x_i^*,\, \ell_i)\cap\Omega} u(z)\, dz\right] - u(y) \\ &= \sum_{i\in\Lambda_x} \varphi_i(x) \left[\fint_{I(x_i^*,\, \ell_i)\cap\Omega} u(z)\, dz \, - u(y)\right] \\ &= \sum_{i\in\Lambda_x} \varphi_i(x) \fint_{I(x_i^*,\, \ell_i)\cap\Omega} [u(z) - u(y)]\, dz, \end{aligned}$$

thereby getting

$$|Eu(x) - u(y)| \le \sum_{i\in\Lambda_x} \varphi_i(x) \fint_{I(x_i^*,\,\ell_i)\cap\Omega} |u(z) - u(y)|\,dz. \tag{2.5}$$

Moreover, we claim

$$|Eu(x) - u(y)| \lesssim \fint_{I(x,\,20\sqrt{n}d(x,\,\Omega))\cap\Omega} |u(z) - u(y)|\,dz. \tag{2.6}$$

To see this, for $x \in I \cap \overline{\Omega}^{\complement}$, since $d(x,\,\Omega) \le \sqrt{n}\ell(I)$, we obtain

$$I(x, 20\sqrt{n}d(x,\Omega)) \subset I(x_I,\,21n\ell(I)). \tag{2.7}$$

For each $i \in \Lambda_x$, by Lemma 2.1 we have

$$d(I_i,\Omega) + 2\sqrt{n}\ell_i \le 18\sqrt{n}\ell_i \ge d(x,\Omega) \ge d(I_i,\Omega) - 2\sqrt{n}\ell_i \ge 2\sqrt{n}\ell_i,$$

thereby getting

$$\ell_i \approx d(I_i,\,\Omega) \approx d(x,\ \Omega). \tag{2.8}$$

Moreover, we have

$$|x_i^* - x| \le d(x_i^*, I_i) + 2\sqrt{n}\ell_i \le 19\sqrt{n}\ell_i,$$

whence reaching

$$I(x_i^*, \ell_i) \subset I(x, 20\sqrt{n}d(x,\Omega)). \tag{2.9}$$

On the other hand, for $x \in \Omega^{\complement}$ and $I(x,\ \lambda d(x,\ \Omega))$ with $\lambda \ge 8$, by the Ahlfors n-regular condition, we have

$$\mathrm{vol}(I(x,\ \lambda d(x,\ \Omega))\cap\Omega) \ge \mathrm{vol}(I(\widehat{x},\ (2^{-1}\lambda - 2)d(x,\ \Omega))) \gtrsim \mathrm{vol}(I(x,\ \lambda d(x,\ \Omega))), \tag{2.10}$$

where $\widehat{x} \in \Omega$ and $d(x,\ \widehat{x}) = 2d(x,\ \Omega)$.

Combining (2.5), (2.8), (2.9) and (2.10), we arrive at (2.6).

For $y \in I \cap \Omega$ and $z \in I(x,\ 20\sqrt{n}d(x,\ \Omega)) \cap \Omega$, we have $|x - y| \ge d(x,\ \Omega)$ and then

$$|z - y| \le |z - x| + |x - y| \le 20\sqrt{n}d(x,\ \Omega) + |x - y| \lesssim |x - y|$$

For $0 \le \alpha < 1$, by (2.7), we obtain

$$\begin{aligned}
\frac{|Eu(x) - u(y)|}{|x - y|^{n/2+\alpha}} &\lesssim \fint_{I(x,\,20\sqrt{n}d(x,\,\Omega))\cap\Omega} \frac{|u(z) - u(y)|}{|z - y|^{n/2+\alpha}}\,dz \\
&= \fint_{I\left(x,\,20\sqrt{n}d(x,\,\Omega)\right)\cap\Omega} \frac{|u(z) - u(y)|}{|z - y|^{n/2+\alpha}}\chi_{I(x_I,\,21n\ell(I))\cap\Omega}(z)\,dz \\
&\lesssim \fint_{B(x,\,20nd(x,\,\Omega))} \frac{|u(z) - u(y)|}{|z - y|^{n/2+\alpha}}\chi_{I(x_I,\,21n\ell(I))\cap\Omega}(z)\,dz \\
&\lesssim \mathcal{M}\left(\frac{|u(\cdot) - u(y)|}{|\cdot - y|^{n/2+\alpha}}\chi_{I(x_I,\,21n\ell(I))\cap\Omega}(\cdot)\right)(x).
\end{aligned}$$

In the above and below, $\mathcal{M}$ denotes the Hardy-Littlewood maximal operator:

$$\mathcal{M}(g)(x) \equiv \sup_{r>0} \left(\fint_{B(x,\,r)} |g(z)|\, dz \right) = \mathcal{M}(|g|)(x).$$

Applying the L^2-boundedness of $\mathcal{M}$, we get

$$\begin{aligned} H_2 &\lesssim \int_{I\setminus\Omega} \int_{I\cap\Omega} \left[\mathcal{M}\left(\frac{|u(\cdot)-u(y)|}{|\cdot - y|^{n/2+\alpha}} \chi_{I(x_I,\,21n\ell(I))\cap\Omega}(\cdot) \right)(x) \right]^2 dy\, dx \\ &\le \int_{I\cap\Omega} \int_{\mathbb{R}^n} \left[\mathcal{M}\left(\frac{|u(\cdot)-u(y)|}{|\cdot - y|^{n/2+\alpha}} \chi_{I(x_I,\,21n\ell(I))\cap\Omega}(\cdot) \right)(x) \right]^2 dx\, dy \\ &\lesssim \int_{I\cap\Omega} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}} \chi_{I(x_I,\,21n\ell(I))\cap\Omega}(x)\, dx\, dy \\ &\lesssim \int_{I\cap\Omega} \int_{I(x_I,\,21n\ell(I))\cap\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\, dx\, dy \\ &\lesssim \int_{I(x_I,\,21n\ell(I))\cap\Omega} \int_{I(x_I,\,21n\ell(I))\cap\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\, dx\, dy, \end{aligned}$$

whence

$$H_2 \le \mathrm{vol}(I)^{1-2\alpha/n} \|u\|^2_{\mathcal{Q}_\alpha(\Omega)}.$$

For each $x \in I \setminus \Omega$, we split $I \setminus \Omega$ into two parts:

$$\begin{cases} X_1(x) \equiv \left\{ y \in I \setminus \Omega : |x-y| \ge 2^{-1} \max\{d(x,\,\Omega),\, d(y,\,\Omega)\} \right\}; \\ X_2(x) \equiv \left\{ y \in I \setminus \Omega :\; |x-y| < 2^{-1} \max\{d(x,\,\Omega),\, d(y,\,\Omega)\} \right\}, \end{cases}$$

and write

$$\begin{cases} H_3 = H_{3,1} + H_{3,2}; \\ H_{3,1} \equiv \int_{I\setminus\Omega} \int_{X_1(x)} \frac{|Eu(x)-Eu(y)|^2}{|x-y|^{n+2\alpha}}\, dy\, dx; \\ H_{3,2} \equiv \int_{I\setminus\Omega} \int_{X_2(x)} \frac{|Eu(x)-Eu(y)|^2}{|x-y|^{n+2\alpha}}\, dy\, dx. \end{cases}$$

For $x \in I \setminus \Omega$ and $y \in X_1(x)$, we have

$$\sum_{i\in\Lambda_x} \varphi_i(x) = 1 = \sum_{i\in\Lambda_y} \varphi_i(y),$$

whence

$$\begin{aligned} |Eu(x)-Eu(y)| &= \left| \sum_{i\in\Lambda_x} \sum_{j\in\Lambda_y} \varphi_i(x)\varphi_j(y) \left[\fint_{I(x_i^*,\,\ell_i)\cap\Omega} u(z)\, dz - \fint_{I(x_j^*,\,\ell_j)\cap\Omega} u(w)\, dw \right] \right| \\ &\le \sum_{i\in\Lambda_x} \sum_{j\in\Lambda_y} \varphi_i(x)\varphi_j(y) \fint_{I(x_i^*,\,\ell_i)\cap\Omega} \fint_{I(x_j^*,\,\ell_j)\cap\Omega} |u(z)-u(w)|\, dw\, dz \end{aligned}$$

Applying (2.8), (2.9) and (2.10) to both x and y, we have

$$\begin{aligned} &|Eu(x)-Eu(y)| \\ &\lesssim \sum_{i\in\Lambda_x} \sum_{j\in\Lambda_y} \varphi_i(x)\varphi_j(y) \fint_{I(x,\,20\sqrt{n}d(x,\,\Omega))\cap\Omega} \fint_{I(y,\,20\sqrt{n}d(y,\,\Omega))\cap\Omega} |u(z)-u(w)|\, dw\, dz \\ &\lesssim \fint_{I(x,\,20\sqrt{n}d(x,\,\Omega))\cap\Omega} \fint_{I(y,\,20\sqrt{n}d(y,\,\Omega))\cap\Omega} |u(z)-u(w)|\, dw\, dz. \end{aligned}$$

Note that $x \in I \setminus \Omega$ and $y \in X_1(x)$ imply

$$|x-y| \geq 2^{-1} \max\{d(x,\,\Omega),\, d(x,\,\Omega)\}.$$

So, for

$$z \in I(x,\, 20\sqrt{n}d(x,\,\Omega)) \cap \Omega \quad \text{and} \quad w \in I(y,\, 20\sqrt{n}d(y,\,\Omega)) \cap \Omega,$$

we have

$$|z-w| \leq |x-y| + 20\sqrt{n}d(x,\,\Omega) + 20\sqrt{n}d(y,\,\Omega) \lesssim |x-y|.$$

This, together with $0 \leq \alpha < 1$ and (2.10), leads to

$$\begin{aligned} &\frac{|Eu(x)-Eu(y)|}{|x-y|^{n/2+\alpha}} \\ &\lesssim \fint_{I(x,\,20\sqrt{n}d(x,\Omega))\cap\Omega} \fint_{I(y,\,20\sqrt{n}d(y,\Omega))\cap\Omega} \frac{|u(z)-u(w)|}{|z-w|^{n/2+\alpha}}\,dw\,dz \\ &\lesssim \fint_{I(x,\,20\sqrt{n}d(x,\Omega))} \fint_{I(x,\,20\sqrt{n}d(x,\Omega))} \frac{|u(z)-u(w)|}{|z-w|^{n/2+\alpha}} \chi_\Omega(z)\chi_\Omega(w)\,dw\,dz. \end{aligned}$$

Set

$$G(w,\,z) \equiv \frac{|u(z)-u(w)|}{|w-z|^{n/2+\alpha}} \chi_{I(x_I,21\sqrt{n}\ell(I))\cap\Omega}(w)\chi_{I(x_I,21\sqrt{n}\ell(I))\cap\Omega}(z).$$

Applying (2.7), we further have

$$\begin{aligned} &\frac{|Eu(x)-Eu(y)|}{|x-y|^{n/2+\alpha}} \\ &\lesssim \fint_{I(x,\,20\sqrt{n}d(x,\Omega))} \fint_{I(x,\,20\sqrt{n}d(x,\Omega))} G(z,w)\,dw\,dz \lesssim (\mathcal{M}\times\mathcal{M})(G)(x,\,y), \end{aligned}$$

where $(\mathcal{M}\times\mathcal{M})(G)(x,\,y)$ denotes the iterated Hardy-Littlewood maximal function of G - for given z, taking maximal function of $G(w,\,z)$ with respect to the variable w and evaluating at x, we have $\mathcal{M}(G(\cdot,\,z))(x)$; for given x, taking maximal function of $\mathcal{M}(G(\cdot,\,z))(x)$ with respect to z and evaluating at y, we obtain $(\mathcal{M}\times\mathcal{M})(G)(x,\,y)$.

By the L^2-boundedness of the Hardy-Littlewood operator, we obtain

$$\begin{aligned} H_{3,1} &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [(\mathcal{M}\times\mathcal{M})(G)(x,\,y)]^2\,dx\,dy \\ &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G(x,\,y)^2\,dx\,dy \\ &\lesssim \int_{I(x_I,\,21n\ell(I))\cap\Omega}\int_{I(x_I,\,21n\ell(I))\cap\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy \\ &\lesssim \mathrm{vol}(I)^{1-2\alpha/n}\|u\|^2_{\mathscr{Q}_\alpha(\Omega)}. \end{aligned}$$

To estimate $H_{3,\,2}$, note that $x \in I \setminus \Omega$ and $y \in X_2(x)$ ensure

$$\sum_{i\in\Lambda_x\cup\Lambda_y} [\varphi_i(x)-\varphi_i(y)] = \sum_{i\in\Lambda_x}\varphi_i(x) - \sum_{j\in\Lambda_y}\varphi_j(y) = 0.$$

So, by Lemma 2.1 and (2.1) we arrive at

$$\begin{aligned}
&|Eu(x) - Eu(y)| \qquad (2.11)\\
&= \left| \sum_{i\in\Lambda_x\cup\Lambda_y} [\varphi_i(x)-\varphi_i(y)] \fint_{I(x_i^*,\,\ell_i)\cap\Omega} u(z)\,dz \right| \\
&= \left| \sum_{i\in\Lambda_x\cup\Lambda_y} [\varphi_i(x)-\varphi_i(y)] \left[\fint_{I(x_i^*,\,\ell_i)\cap\Omega} u(z)\,dz - \fint_{I(x,\,20\sqrt{n}d(x,\Omega))\cap\Omega} u(w)\,dw \right] \right| \\
&= \left| \sum_{i\in\Lambda_x\cup\Lambda_y} [\varphi_i(x)-\varphi_i(y)] \fint_{I(x_i^*,\,\ell_i)\cap\Omega} \fint_{I(x,\,20\sqrt{n}d(x,\Omega))\cap\Omega} [u(z)-u(w)]\;dw\,dz \right| \\
&\lesssim \sum_{i\in\Lambda_x\cup\Lambda_y} \frac{|x-y|}{\ell_i} \fint_{I(x_i^*,\,\ell_i)\cap\Omega} \fint_{I(x,\,20\sqrt{n}d(x,\Omega))\cap\Omega} |u(z)-u(w)|\,dw\,dz.
\end{aligned}$$

We choose $\overline{y}\in\overline{\Omega}$ so that $|y-\overline{y}| = d(y,\,\Omega)$. Since $y\in X_2(x)$, we have

$$|x-y| < 2^{-1}\max\{d(x,\,\Omega),\, d(y,\Omega)\}. \tag{2.12}$$

Consequently,

$$\begin{aligned}
d(x,\Omega) &\le |x-\overline{y}| + |y-\overline{y}| \\
&\le 2^{-1}d(x,\,\Omega) + 2^{-1}d(y,\,\Omega) + d(y,\,\Omega) \\
&= 2^{-1}d(x,\,\Omega) + 2^{-1}3d(y,\,\Omega),
\end{aligned}$$

which implies $d(x,\,\Omega)\le 3d(y,\,\Omega)$. Similarly, we have $d(y,\,\Omega)\le 3d(x,\,\Omega)$. whence

$$3^{-1}d(y,\Omega) \le d(x,\Omega) \le 3d(y,\Omega). \tag{2.13}$$

This in turns derives

$$I(y,\,20\sqrt{n}d(y,\Omega)) \subset I(x,\,80\sqrt{n}d(x,\Omega)),$$

and by (2.10),

$$\mathrm{vol}\big(I(y,\,20\sqrt{n}d(y,\Omega))\cap\Omega\big) \approx \mathrm{vol}\big(I(x,\,80\sqrt{n}d(x,\Omega))\cap\Omega\big).$$

For each $y\in X_2(x)$ and all $i\in\Lambda_x\cup\Lambda_y$, we have $\ell_i\approx d(x,\Omega)$. For such i, we use

$$\ell_i \approx d(I_i,\,\Omega) \approx d(x,\Omega)$$

to achieve

$$\mathrm{vol}\big(I(x_i^*,\,\ell_i)\cap\Omega\big) \approx \mathrm{vol}\big(I(y,\,20\sqrt{n}d(y,\Omega))\cap\Omega\big) \approx \mathrm{vol}\big(I(x,\,80\sqrt{n}d(x,\Omega))\cap\Omega\big).$$

Applying (2.1)), we know $\sharp\Lambda_x + \sharp\Lambda_y \lesssim 1$. Then by the Hölder inequality, we get

$$\begin{aligned}
&|Eu(x) - Eu(y)| \\
&\lesssim \frac{|x-y|}{d(x,\,\Omega)} \fint_{I(x,\,80\sqrt{n}d(x,\Omega))\cap\Omega} \fint_{I(x,\,20\sqrt{n}d(x,\,\Omega))\cap\Omega} |u(z)-u(w)|\,dw\,dz \\
&\lesssim \frac{|x-y|}{d(x,\Omega)} \fint_{I(x,\,80\sqrt{n}d(x,\Omega))\cap\Omega} \left[\fint_{I(x,\,20\sqrt{n}d(x,\Omega))\cap\Omega} |u(z)-u(w)|^2\,dw \right]^{\frac{1}{2}} dz \\
&\lesssim \frac{|x-y|}{d(x,\Omega)^{1-\alpha}} \fint_{I(x,\,80\sqrt{n}d(x,\Omega))\cap\Omega} \left[\int_{I(x,\,20\sqrt{n}d(x,\Omega))\cap\Omega} \frac{|u(z)-u(w)|^2}{|z-w|^{n+2\alpha}}\,dw \right]^{\frac{1}{2}} dz.
\end{aligned}$$

Since

$$I(x,\,80\sqrt{n}d(x,\Omega))\subset I(x_I,\,81n\ell(I)),$$

we have

$$\begin{aligned}
&|Eu(x)-\ Eu(y)|\\
&\lesssim \frac{|x-y|}{d(x,\Omega)^{1-\alpha}}\fint_{I(x,\,80\sqrt{n}d(x,\Omega))}\\
&\quad\times\left[\int_{I(x_I,\,81n\ell(I))\cap\Omega}\frac{|u(z)-u(w)|^2}{|z-w|^{n+2\alpha}}\,dw\right]^{\frac12}\chi_{I(x_I,\,81n\ell(I))\cap\Omega}(z)\,dz\\
&\lesssim \frac{|x-y|}{d(x,\Omega)^{1-\alpha}}\mathcal{M}\left(\left[\int_{I(x_I,\,81n\ell(I))\cap\Omega}\frac{|u(\cdot)-u(w)|^2}{|\cdot-w|^{n+2\alpha}}\,dw\right]^{\frac12}\chi_{I(x_I,\,81n\ell(I))\cap\Omega}(\cdot)\right)(x).
\end{aligned}$$

This, together with (2.12) and (2.13), implies

$$\begin{aligned}
H_{3,\,2}&=\int_{I\setminus\Omega}\int_{X_2(x)}\frac{|Eu(x)-Eu(y)|^2}{|x-y|^{n+2\alpha}}\,dy\,dx\\
&\lesssim\int_{I\setminus\Omega}\left(\int_{X_2(x)}\frac{|x-y|^{2-n-2\alpha}}{d(x,\,\Omega)^{2-2\alpha}}\,dy\right)\\
&\quad\times\left[\mathcal{M}\left(\left[\int_{I(x_I,\,81n\ell(I))\cap\Omega}\frac{|u(\cdot)-u(w)|^2}{|\cdot-w|^{n+2\alpha}}\,dw\right]^{\frac12}\chi_{I(x_I,\,81n\ell(I))\cap\Omega}(\cdot)\right)(x)\right]^2\,dx.
\end{aligned}$$

Since (2.13) implies $X_2(x)\subset B(x,15d(x,\,\Omega))$, we have

$$\int_{X_2(x)}\frac{|x-y|^{2-n-2\alpha}}{d(x,\,\Omega)^{2-2\alpha}}\,dy\lesssim\int_{B(x,15d(x,\,\Omega))}\frac{|x-y|^{2-n-2\alpha}}{d(x,\,\Omega)^{2-2\alpha}}\,dy\lesssim 1.$$

This, together with the $L^2(\mathbb{R}^n)$-boundedness of $\mathcal{M}$, yields

$$\begin{aligned}
H_{3,\,2}&\lesssim\int_{\mathbb{R}^n}\left[\mathcal{M}\left(\left[\int_{I(x_I,\,81n\ell(I))\cap\Omega}\frac{|u(\cdot)-u(w)|^2}{|\cdot-w|^{n+2\alpha}}\,dw\right]^{\frac12}\chi_{I(x_I,\,81n\ell(I))\cap\Omega}(\cdot)\right)(x)\right]^2\,dx\\
&\lesssim\int_{I(x_I,\,81n\ell(I))\cap\Omega}\int_{I(x_I,\,81n\ell(I))\cap\Omega}\frac{|u(x)-u(w)|^2}{|x-w|^{n+2\alpha}}\,dw\,dx\\
&\lesssim \mathrm{vol}(I)^{(2\alpha-n)/n}\|u\|^2_{\mathscr{Q}_\alpha(\Omega)}.
\end{aligned}$$

Combining the estimates for $H_{3,1}$ and $H_{3,\,2}$, we arrive at

$$H_3=H_{3,1}+H_{3,\,2}\lesssim \mathrm{vol}(I)^{(2\alpha-n)/n}\|u\|^2_{\mathscr{Q}_\alpha(\Omega)}.$$

Subcase 2: $I\cap\Omega=\emptyset$ but $d(I,\Omega)\le 100\sqrt{n}\ell(I)$. This subcase reduces to *Subcase 1* if we consider a large cube $\widetilde{I}=200I$, for which $\widetilde{I}\cap\Omega\neq\emptyset$ and

$$\ell(I)^{2\alpha-n}\int_I\int_I\frac{|Eu(x)-Eu(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy\lesssim\ell(\widetilde{I})^{2\alpha-n}\int_{\widetilde{I}}\int_{\widetilde{I}}\frac{|Eu(x)-Eu(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy.$$

Subcase 3: $I \cap \Omega = \emptyset$ but $d(I, \Omega) \geq 100\sqrt{n}\ell(I)$. For $x \in I$ and $y \in I$, we know

$$\sum_{i \in \Lambda_x \cup \Lambda_y} [\varphi_i(x) - \varphi_i(y)] = 0.$$

Applying Lemma 2.1 and the argument for (2.11), we obtain

$$|Eu(x) - Eu(y)| \lesssim \sum_{i \in \Lambda_x \cup \Lambda_y} \frac{|x-y|}{\ell_i} \fint_{I(x_i^*,\, \ell_i) \cap \Omega} \fint_{I(x,\, 20\sqrt{n} d(x,\, \Omega)) \cap \Omega} |u(z) - u(w)|\, dw\, dz.$$

Observe that

$$d(I,\, \Omega) \leq d(x,\, \Omega), d(y,\ \Omega) \leq d(I,\, \Omega) + \sqrt{n}\ell(I) \leq \frac{11}{10}\sqrt{n} d(I,\, \Omega)$$

ensures

$$\begin{cases} d(x,\, \Omega) \approx d(y,\, \Omega) \approx d(I,\, \Omega); \\ I(x, 20\sqrt{n} d(x,\, \Omega)) \subset I(x_I,\, 30nd(I,\, \Omega)); \\ I(y, 20\sqrt{n} d(y,\, \Omega)) \subset I(x_I,\, 30nd(I,\, \Omega)). \end{cases}$$

So, for $i \in \Lambda_x$ we have $x \in I(x_i, 2\ell_i)$, whence $d(x,\, \Omega) \approx d(I_i,\, \Omega)$. Applying Lemma 2.1, we obtain

$$\ell_i \approx d(x,\, \Omega) \approx d(I,\, \Omega).$$

Using (2.10), we arrive at

$$\mathrm{vol}\big(I(x_i^*,\, \ell_i) \cap \Omega\big) \approx \mathrm{vol}\big(I(x, 20\sqrt{n} d(x,\, \Omega)) \cap \Omega\big) \approx \mathrm{vol}\big(I(x_I, 30nd(I,\, \Omega)) \cap \Omega\big).$$

Therefore,

$$\begin{aligned} &|Eu(x) - Eu(y)| \\ &\lesssim \frac{|x-y|}{d(I,\, \Omega)} \fint_{I(x_I,\, 30nd(I,\, \Omega)) \cap \Omega} \fint_{I(x_I,\, 30nd(I,\, \Omega)) \cap \Omega} |u(z) - u(w)|\, dw\, dz \\ &\lesssim \frac{|x-y|}{d(I,\, \Omega)} \left[\fint_{I(x_I,\, 30nd(I,\, \Omega)) \cap \Omega} \fint_{I(x_I,\, 30nd(I,\, \Omega)) \cap \Omega} |u(z) - u(w)|^2\, dw\, dz \right]^{\frac{1}{2}} \\ &\lesssim \frac{|x-y|}{d(I,\, \Omega)^{1-\alpha+n/2}} \left[\int_{I(x_I,\, 30nd(I,\, \Omega)) \cap \Omega} \int_{I(x_I,\, 30nd(I,\, \Omega)) \cap \Omega} \frac{|u(z) - u(w)|^2}{|z-w|^{n+2\alpha}}\, dw\, dz \right]^{\frac{1}{2}} \\ &\lesssim \frac{|x-y|}{d(I,\, \Omega)} \|u\|_{\mathcal{Q}_\alpha(\Omega)}. \end{aligned}$$

Consquently,

$$\begin{aligned} H &\equiv \ell(I)^{2\alpha - n} \int_I \int_I \frac{|Eu(x) - Eu(y)|^2}{|x-y|^{n+2\alpha}}\, dx\, dy \\ &\lesssim \|u\|^2_{\mathcal{Q}_\alpha(\Omega)} \ell(I)^{2\alpha - n} \int_I \int_I \frac{|x-y|^{2-n-2\alpha}}{d(I,\, \Omega)^2}\, dx\, dy. \end{aligned}$$

Since

$$\begin{aligned}\ell(I)^{2\alpha-n}\int_I\int_I \frac{dxdy}{|x-y|^{n+2\alpha-2}} &\le \int_I\int_{B(y,\,\sqrt{n}\ell(I))}\frac{dxdy}{|x-y|^{n+2\alpha-2}}\\ &\lesssim \ell(I)^{2\alpha-n}\int_I (\ell(I))^{2-2\alpha}dy\\ &\lesssim \ell(I)^2,\end{aligned}$$

we use $\ell(I)\lesssim d(I,\Omega)$ to obtain

$$H\lesssim\left(\frac{\ell(I)}{d(I,\,\Omega)}\right)^2\|u\|^2_{\mathscr{Q}_\alpha(\Omega)}\lesssim\|u\|^2_{\mathscr{Q}_\alpha(\Omega)},$$

as desired.

Case: $\operatorname{diam}\Omega<\infty$.

Without loss of generality, we may assume that $\operatorname{diam}\Omega=1$ and $\Omega\subset I(0,1)$, then $\operatorname{vol}(\Omega)\approx 1$. Two situations are handled below.

The first is $\ell(I)\le 1000\sqrt{n}$.

- If $I\cap\Omega\ne\emptyset$, then the same argument as in checking the subcase $I\cap\Omega\ne\emptyset$ of *Case:* $\operatorname{diam}\Omega=\infty$ gives (2.4).
- But, if $I\cap\Omega=\emptyset$, then two more subcases are taken into account: when $d(I,\Omega)\le 100\sqrt{n}\ell(I)$, considering the cube $200I$ and by $200I\cap\Omega\ne\emptyset$, we have (2.4) as well; when $d(I,\Omega)\ge 100\sqrt{n}\ell(I)$, a similar argument for the subcase of $d(I,\Omega)\ge 100\sqrt{n}\ell(I)$ of *Case:* $\operatorname{diam}\Omega=\infty$ derives (2.4).

The second is $\ell(I)\ge 1000\sqrt{n}$.

- If $\ell(I)\ge 1000\sqrt{n}$ and $d(I,\Omega)\ge 100\sqrt{n}$, then for $x\in I$ and $i\in\Lambda_x$ (ensuring $x\in 2I_i$), we claim

  $$Eu(x)=\fint_\Omega u(z)\,dz\quad\forall\quad x\in I.$$

  Obviously, the claim gives the desired result. To validate the claim, for every $x\in I$ and $i\in\Lambda_x$, by Lemma 2.1, we obtain

  $$\begin{cases}d(2I_i,\,\Omega)\le d(I_i,\,\Omega)+2\sqrt{n}\ell_i\le 18\sqrt{n}\ell_i;\\ d(I,\,\Omega)\le d(x,\,\Omega)\le d(2I_i,\,\Omega)+2\sqrt{n}\ell_i\le 20\sqrt{n}\ell_i,\end{cases}$$

  whence $100\sqrt{n}\le 20\sqrt{n}\ell_i$; i.e., $\ell_i\ge 5$. By $x_i^*\in\Omega$ and $\operatorname{diam}\Omega=1$, we get $I({x_i}^*,\ell_i)\cap\Omega=\Omega$. So, for $i\in\Lambda_x$, we use (2.2) to achieve

  $$Eu(x)=\sum_{i\in\Lambda_x}\varphi_i(x)\fint_{I({x_i}^*,\ell_i)\cap\Omega}u(z)\,dz=\sum_{i\in\Lambda_x}\varphi_i(x)\fint_\Omega u(z)\,dz=\fint_\Omega u(z)\,dz,$$

  thereby verifying the above claim.
- If $\ell(I)\ge 1000\sqrt{n}$ and $I\cap\Omega\ne\emptyset$, then

  $$\begin{aligned}&\int_I\int_I\frac{|Eu(x)-Eu(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy\\ &=\int_{I\setminus I(0,\,100\sqrt{n})}\int_{I\setminus I(0,\,100\sqrt{n})}\frac{|Eu(x)-Eu(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy\\ &\quad+2\int_{I\setminus I(0,\,200\sqrt{n})}\int_{I(0,\,100\sqrt{n})}\cdots+\int_{I(0,\,200\sqrt{n})}\int_{I(0,\,200\sqrt{n})}\cdots\\ &\equiv H_{4,1}+H_{4,2}+H_{4,3}.\end{aligned}$$

From $\mathrm{vol}(I(0,\,200\sqrt{n}))\approx 1$ and the treatment of $\ell(I)\le 1000\sqrt{n}$ and $I\cap\Omega\neq\emptyset$ it follows that

$$H_{4,3}\lesssim \|u\|^2_{\mathscr{Q}_\alpha(\Omega)}\lesssim \mathrm{vol}(I)^{(n-2\alpha)/n}\|u\|^2_{\mathscr{Q}_\alpha(\Omega)}.$$

For $H_{4,1}$, an application of the argument for the setting $\ell(I)\ge 1000\sqrt{n}$ and $d(I,\Omega)\ge 100\sqrt{n}$ derives

$$Eu(x)=\fint_\Omega u(z)\,dz\quad\forall\quad x\in I\setminus I(0,\,200\sqrt{n}).$$

So, the above claim gives

$$H_{4,1}\lesssim \mathrm{vol}^{(n-2\alpha)/n}\|u\|^2_{\mathscr{Q}_\alpha(\Omega)}.$$

For $H_{4,2}$, we notice that if $x\in I\setminus I(0,\,200\sqrt{n})$ and $y\in I(0,\,100\sqrt{n})$, then

$$\begin{cases}|x-y|\ge 2^{-1}|x| \;\;\&\;\; Eu(x)=\fint_\Omega u(z)\,dz;\\ |Eu(x)-Eu(y)|^2\lesssim \fint_\Omega |u(z)-Eu(y)|^2\,dz\approx\int_\Omega |u(z)-Eu(y)|^2\,dz,\end{cases}$$

and hence

$$\begin{aligned}&\int_{I(0,\,100\sqrt{n})}\frac{|Eu(x)-Eu(y)|^2}{|x-y|^{n+2\alpha}}\,dy\\ &\lesssim |x|^{-n-2\alpha}\int_{I(0,\,100\sqrt{n})}\int_\Omega |u(z)-Eu(y)|^2\,dz\,dy\\ &\lesssim |x|^{-n-2\alpha}\int_{I(0,\,100\sqrt{n})}\int_{I(0,\,100\sqrt{n})}\frac{|Eu(z)-Eu(y)|^2}{|y-z|^{n+2\alpha}}\,dz\,dy\\ &\lesssim |x|^{-n-2\alpha}\|u\|^2_{\mathscr{Q}_\alpha(\Omega)}.\end{aligned}$$

Here the last inequality follows from the result in the case $\ell(I)\le 1000\sqrt{n}$ and $I\cap\Omega\neq\emptyset$. Thus, we obtain

$$H_{4,2}\lesssim \|u\|^2_{\mathscr{Q}_\alpha(\Omega)}\int_{I\setminus I(0,\,200\sqrt{n})}|x|^{-n-2\alpha}\,dx\lesssim \|u\|^2_{\mathscr{Q}_\alpha(\Omega)}\lesssim \mathrm{vol}(I)^{(n-2\alpha)/n}\|u\|^2_{\mathscr{Q}_\alpha(\Omega)}.$$

Putting together estimates for $H_{4,1}, H_{4,2}$ and $H_{4,3}$ derives (2.4).

- If $\ell(I)\ge 1000\sqrt{n}$ and $d(I,\Omega)\le 100\sqrt{n}$, then a consideration of the cube $2\sqrt{n}I$ and by $2\sqrt{n}I\cap\Omega\neq\emptyset$, yields (2.4) as well.

A combination of the previous cases completes the proof of (i)$\Longrightarrow$(ii).

Proof of (ii)$\Longrightarrow$(iii). We need the John-Nirenberg inequality (cf. [7, pp.524-527] for (i)) and the imbedding of Q_α into BMO (cf. [3] for (ii)):

LEMMA 2.2. *Let* $\alpha\in[0,1)$.

(i) *For every* $u\in BMO$ *and any cube* $I\subset\mathbb{R}^n$ *with edge parallel to the coordinate axes, one has*

$$vol(\{x\in I:|u-u_I|>\lambda\})\le e^{2^n e}vol(I)\exp\left(-\frac{\lambda}{2^n e\|u\|_{BMO}}\right).$$

(ii) *For every* $u\in\mathscr{Q}_\alpha(\mathbb{R}^n)$ *there exists a constant* $C_3>1$ *such that* $\|u\|_{BMO}\le C_3\|u\|_{\mathscr{Q}_\alpha(\mathbb{R}^n)}$.

Now, if (ii) of Theorem 1.1 is valid; i.e., Ω is a $\mathscr{Q}_\alpha$-extension domain, then for every function $u \in \mathscr{Q}_\alpha(\Omega)$ we can find a function $\tilde{u} = Eu \in \mathscr{Q}_\alpha(\mathbb{R}^n)$ such that

$$Eu(x) = u(x) \ \ \forall \ \ x \in \Omega \ \ \& \ \ \|Eu\|_{\mathscr{Q}_\alpha(\mathbb{R}^n)} \le C_4 \|u\|_{\mathscr{Q}_\alpha(\Omega)},$$

where C_4 is a constant independent of u. Using Lemma 2.2 and the layer-cake formula, we can choose a positive constant $C_1 < 1/(2^n e)$ to enjoy

$$\begin{aligned}\inf_{c\in\mathbb{R}} \int_{I\cap\Omega} \exp\left(\frac{C_1}{C_3C_4}\frac{|u(x)-c|}{\|u\|_{\mathscr{Q}_\alpha(\Omega)}}\right) dx &\le \inf_{c\in\mathbb{R}} \int_{I\cap\Omega} \exp\left(C_1 \frac{|u(x)-c|}{\|\tilde{u}\|_{BMO}}\right) dx \\ &\le \int_I \exp\left(C_1 \frac{|\tilde{u}(x)-\tilde{u}_I|}{\|\tilde{u}\|_{BMO}}\right) dx \\ &\le \int_0^\infty \mathrm{vol}\left(\{x\in I : C_1 \frac{|\tilde{u}(x)-\tilde{u}_I|}{\|\tilde{u}\|_{BMO}} > t\}\right) de^t \\ &\le \mathrm{vol}(I) \int_0^\infty \exp\left(-t\Big(\frac{1}{2^n e C_1} - 1\Big)\right) dt \\ &\lesssim \mathrm{vol}(I),\end{aligned}$$

as desired.

Proof of (iii)$\Longrightarrow$(i). Suppose that (iii) of Theorem 1.1 is valid. A slight modification of the argument for [20, Theorem 4.1] shows that

$$\|u\|_{\mathscr{Q}_\alpha(\Omega)} \lesssim \|u\|_{W^{1,n}(\Omega)}.$$

So, there are constants $C_1, C_2 > 0$ such that

$$\inf_{c\in\mathbb{R}} \sup_{I\subseteq\mathbb{R}^n} \mathrm{vol}(I)^{-1} \int_{I\cap\Omega} \exp\left(C_1 \frac{|u(x)-c|}{\|u\|_{W^{1,n}(\Omega)}}\right) dx \le C_2.$$

According to [8, Theorem 1(b)] (with $p = n$), we see that Ω enjoys the Ahlfors n-regular property; i.e., (i) of Theorem 1.1 holds.

2.2. Proof of Theorem 1.2. It follows from taking $\Omega = \mathbb{R}^n$ in Theorem 1.1, the argument for Lemma 2.2, and the following inequality

$$\begin{aligned}&\int_{B(x_0,r_0)} \exp\left(C\frac{|u(x)-\bar{u}_{x_0,r_0}|}{[u]_{Q_\alpha}}\right) dx \\ &\ge \mathrm{vol}\big(\{x \in B(x_0,r_0) : \ |u(x)-\bar{u}_{x_0,r_0}| > t\}\big) \exp\left(\frac{Ct}{[u]_{Q_\alpha}}\right) \ \forall \ t > 0.\end{aligned}$$

3. Relationships between both $\mathscr{Q}$-spaces and Q-spaces.

3.1. $\mathscr{Q}_\alpha(\Omega) = Q_\alpha(\Omega)$ **under** (1.1)**.** Working on this equality, we gain

THEOREM 3.1. *Let* $\alpha \in [0,1)$ *and* $\Omega \subset \mathbb{R}^n$ *be a uniform domain. If*

$$n > \begin{cases} 2\alpha + \overline{\dim}_{\mathscr{L}}\partial\Omega & \text{when } \partial\Omega \text{ is bounded;} \\ 2\alpha + \overline{\dim}_{\mathscr{G}}\partial\Omega & \text{when } \partial\Omega \text{ is unbounded,} \end{cases}$$

then $\mathscr{Q}_\alpha(\Omega) = Q_\alpha(\Omega)$.

In order to verify this theorem, we recall that any uniform domain Ω admits a Whitney decomposition (which may be treated as a dyadic version of Lemma 2.1 with a different constant) as described below.

LEMMA 3.1. *There exists a collection $W_\Omega = \{S_j\}_{j\in\mathbb{N}}$ of countably many dyadic (closed) cubes such that*

(i) $\Omega = \cup_{j\in\mathbb{N}} S_j$ *and* $(S_i)^\circ \cap (S_j)^\circ = \emptyset$ *for all* $j, i \in \mathbb{N}$ *with* $j \neq i$;
(ii) $2^7\sqrt{n}\ell(S_j) \leq d(S_j, \partial\Omega) \leq 2^9\sqrt{n}\ell(S_j)$;
(iii) $4^{-1}\ell(S_i) \leq \ell(S_j) \leq 4\ell(S_i)$ *whenever* $S_i \cap S_j \neq \emptyset$.

For disjoint Whitney cubes S_i and S_j, a family of Whitney cubes $\{S_{i_s}\}_{s=1}^t$ is called a Whitney chain joining S_i and S_j if $S_{i_1} = S_i$, $S_{i_t} = S_j$, and $S_{i_s} \cap S_{i_{s+1}}$ consists of an $(n-1)$-dimensional (closed) cube for all $s = 1, \cdots, t-1$. The constant t is called the length of the chain. The following result was established in [11, 5].

LEMMA 3.2. *For every pair of disjoint Whitney cubes S_i and S_j, there exists a Whitney chain $F(S_i, S_j) = \{S_{i_s}\}_{s=1}^t$ joining S_i and S_j with its length*

$$t \leq C\left|\log_2 \frac{\ell(S_i)}{\ell(S_j)}\right| + C\log_2\left(\frac{d(S_i, S_j)}{\ell(S_i) + \ell(S_j)} + 2\right),$$

where C is a constant independent of S_i and S_j.

Moreover, for every cube I with edges parallel to the coordinate axes and every $k \in \mathbb{Z}$, set

$$\mathscr{S}_k(I) = \{S_j \in W_\Omega : \ S_j \cap I \neq \emptyset, \ell(S_j) = 2^{-k}\} \equiv \{S_{k,i}\}_i. \tag{3.1}$$

Then we have the following lemma.

LEMMA 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain and $\epsilon > 0$ be a small number such that*

$$n - 2\alpha - \epsilon > \begin{cases} \overline{\dim}_{\mathscr{L}}\partial\Omega & \text{when } \partial\Omega \text{ is bounded;} \\ \overline{\dim}_{\mathscr{G}}\partial\Omega & \text{when } \partial\Omega \text{ is unbounded.} \end{cases}$$

(i) *If* $\operatorname{diam}\Omega < \infty$, *then there exists a constant C such that for every cube I with $x_I \in \Omega$, $\ell(I) \leq 2$ and $I \cap \Omega^{\complement} \neq \emptyset$ one has*

$$\begin{cases} \sharp\mathscr{S}_k(I) = 0 \quad \text{if} \quad k < -\log_2 \ell(I) + 7; \\ \sharp\mathscr{S}_k(I) \leq C[2^k\ell(I)]^{\overline{\dim}_{\mathscr{L}}\partial\Omega+\epsilon} \quad \text{if} \quad k \geq -\log_2 \ell(I) + 7. \end{cases}$$

(ii) *If* $\operatorname{diam}\Omega = \infty$, *then there exists a constant C such that for every cube I with $x_I \in \Omega$ and $I \cap \Omega^{\complement} \neq \emptyset$ one has*

$$\sharp\mathscr{S}_k(I) \leq C[2^k\ell(I)]^{\overline{\dim}_{\mathscr{G}}\partial\Omega+\epsilon}.$$

Proof. The proof of (ii) is similar to but easier than (i). So, we only prove (i) and leave the proof of (ii) to the reader.

To prove (i), up to dividing I into 2^n subcubes we may assume that $\ell(I) \leq 1$. If

$$S_j \cap I \neq \emptyset \ \ \& \ \ I \cap \Omega^{\complement} \neq \emptyset,$$

then

$$d(S_j, \Omega^{\complement}) \leq \operatorname{diam} I \leq \sqrt{n}\ell(I),$$

which together with (ii) of Lemma 3.1 yields that $\ell(S_j) \leq 2^{-7}\ell(I)$. Consequently, if $k < -\log_2 \ell(I) + 7$, then $\mathscr{S}_k(I) = \emptyset$.

When $k \geq -\log_2 \ell(I)+7$, for every $\delta > 0$, denote by $\mathscr{N}_{\text{cov}}(\delta,\, \partial\Omega\cap J)$ the collection of cubes of edge length δ required to cover $\partial\Omega \cap J$. Then

$$\sharp\mathscr{N}_{\text{cov}}(\delta,\, \partial\Omega \cap J) = N_{\text{cov}}(\delta,\, \partial\Omega \cap J).$$

For $S_{k,i} \in \mathscr{S}_k(I)$, we have $2^{11}\sqrt{n}S_{k,i} \cap \partial\Omega \neq \emptyset$, whence $2^{11}\sqrt{n}S_{k,i}$ intersects some cube $R \in N_{\text{cov}}(2^{-k},\, \partial\Omega \cap J)$, which implies $S_{k,i} \subset 2^{13}nR$. Notice also that for each cube $R \in \mathscr{N}_{\text{cov}}(2^{-k}, \partial\Omega \cap J)$, the cube $2^{13}nR$ can only contain a uniformly bounded number of $S_{k,i} \in \mathscr{S}_k(J)$. So, we conclude that if

$$k \geq -\log_2 \ell(I) + 7$$

then

$$\sharp\mathscr{S}_k(J) \lesssim N_{\text{cov}}(2^{-k}, \partial\Omega \cap J).$$

Let $0 < \epsilon < n - 2\alpha - \overline{\dim}_{\mathscr{L}}\partial\Omega$ be small enough. It suffices to prove that for every cube I with $\ell(J) \leq 1$ and $k > -\log_2 \ell(J)$ one has

$$N_{\text{cov}}(2^{-k}, \partial\Omega \cap J) \lesssim [2^k\ell(J)]^{\overline{\dim}_{\mathscr{L}}\partial\Omega+\epsilon}. \tag{3.2}$$

Indeed, by the definition of $\overline{\dim}_{\mathscr{L}}\partial\Omega$ and $k \geq -\log_2 \ell(J)$, for sufficiently small ϵ there exist constants $N_1 \geq 8$ and $k_1 \in \mathbb{N}$ such that $k \geq k_1$ for all cube J with $\ell(J) \in [2^{-k+N_1}, 1]$, we obtain

$$\frac{\log_2 N_{\text{cov}}(2^{-k},\, \partial\Omega \cap J)}{\log_2(\ell(J)/2^{-k})} \leq \overline{\dim}_{\mathscr{L}}\partial\Omega + \epsilon,$$

which implies

$$N_{\text{cov}}(2^{-k}, \partial\Omega \cap J) \leq [2^k\ell(J)]^{\overline{\dim}_{\mathscr{L}}\partial\Omega+\epsilon}. \tag{3.3}$$

Therefore, given a cube I with $\ell(J) \leq 1$ and $k > -\log_2 \ell(J)$, if $k \geq k_1$ and $k \geq -\log_2 \ell(J) + N_1$, then (3.2) holds. If $-\log_2 \ell(J) \leq k < k_1$ or $-\log_2 \ell(J) \leq k < -\log_2 \ell(J) + N_1$, then $2^k\ell(J) \lesssim 1$ and hence

$$N_{\text{cov}}(2^{-k}, \partial\Omega \cap I) \leq [2^k\ell(I)]^n \lesssim [2^k\ell(I)]^{\overline{\dim}_{\mathscr{L}}\partial\Omega+\epsilon}.$$

□

Corollary 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain.*

(i) *If $\operatorname{diam}\Omega < \infty$, then there exists a constant $C > 0$ such that for every Whitney cube S_j with $\ell(S_j) \leq 1$, $m \geq -\log_2 S_j$ and $1 \leq \delta \leq -\log_2 S_j + 4$ one has*

$$\sharp\mathscr{S}_m(2^\delta S_j) \lesssim [2^{m+\delta}\ell(S_j)]^{\overline{\dim}_{\mathscr{L}}\partial\Omega+\epsilon}.$$

(ii) *If* $\operatorname{diam}\Omega = \infty$*, then there exists a constant* $C > 0$ *such that for every Whitney cube* S_j*,* $m \geq -\log_2 S_j$ *and* $\delta \geq 1$*, one has*

$$\sharp\mathscr{S}_m(2^\delta S_j) \lesssim [2^{m+\delta}\ell(S_j)]^{\overline{\dim}_{\mathscr{G}}\partial\Omega+\epsilon}.$$

Proof of Theorem 3.1. Since $\mathscr{Q}_\alpha(\Omega) \subset Q_\alpha(\Omega)$, it suffices to prove that $Q_\alpha(\Omega) \subset \mathscr{Q}_\alpha(\Omega)$. We need to consider three cases:

$$\begin{cases} \operatorname{diam}\partial\Omega < \infty \ \& \ \operatorname{diam}\Omega < \infty; \\ \operatorname{diam}\partial\Omega < \infty \ \& \ \operatorname{diam}\Omega = \infty; \\ \operatorname{diam}\partial\Omega = \infty. \end{cases}$$

<u>*Case:* $\operatorname{diam}\partial\Omega < \infty \ \& \ \operatorname{diam}\Omega < \infty$.</u>

Without loss of generality, we may assume that $\operatorname{diam}\partial\Omega = 1$ and $\Omega \subset I(0,1)$. We need to show

$$\Psi_\alpha(u, I\cap\Omega) \equiv \int_{I\cap\Omega}\int_{I\cap\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dy\,dx \lesssim \ell(I)^{n-2\alpha}\|u\|^2_{Q_\alpha(\Omega)} \tag{3.4}$$

for all cube I. If $I \subset \Omega$ or $I \subset \Omega^{\complement}$, then the estimate is known. In the sequel we assume that $I\cap\Omega \neq \emptyset$ and $I \subset \Omega^{\complement} \neq \emptyset$. Up to taking a new cube centered in $I\cap\Omega$ with side length $2\ell(I)$ so that it contains I, we may assume that the center of I belongs to Ω. We may also assume $\ell(I) \leq 1$. Indeed, if $\ell(I) \geq 1$, then $\Omega \subset I(0,1) \subset 2I$. Then

$$\Psi_\alpha(u, I\cap\Omega) \leq \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dy\,dx = \Psi_\alpha(u, I(0,1)\cap\Omega).$$

If

$$\Psi_\alpha(u, I(0,1)\cap\Omega) \lesssim \|u\|^2_{Q_\alpha(\Omega)},$$

then by $\ell(I) \geq 1$ we have

$$\Psi_\alpha(u, I\cap\Omega) \lesssim \ell(I)^{n-2\alpha}\|u\|^2_{Q_\alpha(\Omega)},$$

as desired.

Given a cube I with its center being in Ω, $\ell(I) \leq 1$ and $I\cap\Omega^{\complement} \neq \emptyset$. Let $\mathscr{S}_k(I) = \{S_{k,i}\}_i$ be as in (3.1). Write

$$\begin{aligned}
\Psi_\alpha(u, I\cap\Omega) &= 2\sum_{k\geq -\log_2\ell(I)+7}\ \sum_{i\in\mathscr{S}_k(I)}\ \sum_{m\geq k}\ \sum_{j\in\mathscr{S}_m(I)} \int_{S_{ki}}\int_{S_{mj}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dy\,dx \\
&= 2\sum_{k\geq -\log_2\ell(I)+7}\ \sum_{i\in\mathscr{S}_k(I)}\ \sum_{m\geq k}\ \sum_{\substack{j\in\mathscr{S}_m(I)\\ S_{mj}\cap S_{ki}\neq\emptyset}} \int_{S_{ki}}\int_{S_{mj}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dy\,dx \\
&\quad +2\sum_{k\geq -\log_2\ell(I)+7}\ \sum_{i\in\mathscr{S}_k(I)}\ \sum_{m\geq k}\ \sum_{\substack{j\in\mathscr{S}_m(I)\\ S_{mj}\cap S_{ki}=\emptyset}} \int_{S_{ki}}\int_{S_{mj}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dy\,dx \\
&\equiv H_1 + H_2.
\end{aligned}$$

Observe that if

$$m \geq k, \quad j \in \mathscr{S}_m(I) \ \ \& \ \ S_{mj}\cap S_{ki} \neq \emptyset,$$

then

$$|k-m|\le 2 \ \ \& \ \ S_{mj}\subset 4S_{ki}\subset\Omega,$$

and hence

$$\begin{aligned}&\sum_{m\ge k}\sum_{\substack{j\in\mathscr{S}_m(I)\\ S_{mj}\cap S_{ki}\ne\emptyset}}\int_{S_{ki}}\int_{S_{mj}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dy\,dx\\ &\lesssim\int_{4S_{ki}}\int_{4S_{ki}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dy\,dx\lesssim \mathrm{vol}(S_{ki})^{(n-2\alpha)/n}\|u\|^2_{Q_\alpha(\Omega)}.\end{aligned}$$

So, by Lemma 3.3 we get

$$\begin{aligned}H_1&\lesssim\sum_{k\ge-\log_2\ell(I)+7}\sum_{i\in\mathscr{S}_k(I)}\big(\mathrm{vol}(S_{ki})\big)^{(n-2\alpha)/n}\|u\|^2_{Q_\alpha(\Omega)}\\ &\lesssim\sum_{k\ge-\log_2\ell(I)+7}(2^k\ell(I))^{\overline{\dim}_{\mathscr{L}}\partial\Omega+\epsilon}2^{-k(n-2\alpha)}\|u\|^2_{Q_\alpha(\Omega)}.\end{aligned}$$

Since $n-2\alpha-\overline{\dim}_{\mathscr{L}}\partial\Omega-\epsilon>0$, we arrive at

$$H_1\lesssim\ell(I)^{n-2\alpha}\|u\|^2_{Q_\alpha(\Omega)}\lesssim\mathrm{vol}(I)^{(n-2\alpha)/n}\|u\|^2_{Q_\alpha(\Omega)}.$$

Now we turn to estimate H_2. For

$$m\ge k,\ i\in\mathscr{S}_k(I),\ j\in\mathscr{S}_m(I)\ \&\ S_{mj}\cap S_{ki}=\emptyset,$$

we compute

$$\begin{aligned}&\int_{S_{ki}}\int_{S_{mj}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dy\,dx\\ &\lesssim\big(d(S_{ki},S_{mj})\big)^{-2\alpha-n}\mathrm{vol}(S_{mj})\mathrm{vol}(S_{ki})⨍_{S_{ki}}⨍_{S_{mj}}|u(x)-u(y)|^2\,dy\,dx.\end{aligned}$$

By Lemma 3.2, there exists a Whitney chain $F(S_{ki},S_{mj})=\{S_{k_1,i_1},\cdots,S_{k_tj_t}\}\subset W_\Omega$ joining S_{ki} and S_{mj} with its length

$$\begin{aligned}t&\lesssim\log_2\frac{\ell(S_{mj})}{\ell(S_{ki})}+\log_2\left(\frac{d(S_{ki},S_{mj})}{\ell(S_{ki})+\ell(S_{mj})}+2\right)\\ &\lesssim\log_2\frac{\ell(S_{mj})}{\ell(S_{ki})}+\log_2\left(\frac{d(S_{ki},S_{mj})}{\ell(S_{mj})}+2\right)\\ &\lesssim\log_2\left(\frac{d(S_{ki},S_{mj})+2\ell(S_{mj})}{\ell(S_{ki})}\right).\end{aligned}$$

Since

$$S_{ki}\cap S_{mj}=\emptyset\Longrightarrow d(S_{ki},S_{mj})\ge4^{-1}\ell(S_{mj})\ge4^{-1}\ell(S_{ki}),$$

we have

$$t\lesssim3+\log_2\left(\frac{d(S_{ki},S_{mj})}{\ell(S_{ki})}\right). \tag{3.5}$$

With the aid of $F(S_{ki}, S_{mj})$, we obtain

$$\begin{aligned}
&\fint_{S_{ki}} \fint_{S_{mj}} |u(x) - u(y)|^2 \, dy \, dx \\
&\lesssim \fint_{S_{ki}} |u(x) - u_{S_{ki}}|^2 \, dy + |u_{S_{ki}} - u_{S_{mj}}|^2 + \fint_{S_{mj}} |u(y) - u_{S_{mj}}|^2 \, dy \\
&\lesssim \fint_{S_{ki}} |u(x) - u_{S_{ki}}|^2 \, dy + \left(\sum_{s=1}^{t-1} |u_{S_{k_s i_s}} - u_{S_{k_{s+1} i_{s+1}}}| \right)^2 + \fint_{S_{mj}} |u(y) - u_{S_{mj}}|^2 \, dy.
\end{aligned}$$

By the fractional Poincaré inequality (see [1]), we have

$$\fint_{S_{ki}} |u(x) - u_{S_{ki}}|^2 \, dy \lesssim \ell(S_{ki})^{2\alpha - n} \fint_{S_{ki}} \fint_{S_{ki}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}} \, dx \, dy \lesssim \|u\|^2_{Q_\alpha(\Omega)}.$$

Since

$$S_{k_{s+1} i_{s+1}} \subset 16 S_{k_s i_s} \subset \Omega \quad \& \quad \mathrm{vol}\big(S_{k_{s+1} i_{s+1}}\big) \approx \mathrm{vol}\big(S_{k_s i_s}\big),$$

we also have

$$\begin{aligned}
|u_{S_{k_s i_s}} - u_{S_{k_{s+1} i_{s+1}}}| &\lesssim |u_{S_{k_s i_s}} - u_{16 S_{k_s i_s}}| + |u_{16 S_{k_s i_s}} - u_{S_{k_{s+1} i_{s+1}}}| \\
&\lesssim \fint_{16 S_{k_s i_s}} |u - u_{16 S_{k_s i_s}}| \lesssim \|u\|_{Q_\alpha(\Omega)}.
\end{aligned}$$

So by (3.5), we obtain

$$\fint_{S_{ki}} \fint_{S_{mj}} |u(x) - u(y)|^2 \, dy \, dx \lesssim \left[3 + \log_2 \left(\frac{d(S_{ki}, S_{mj})}{\ell(S_{ki}))} \right) \right]^2 \|u\|^2_{Q_\alpha(\Omega)},$$

and thus

$$\begin{aligned}
&\int_{S_{ki}} \int_{S_{mj}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}} \, dy \, dx \\
&\lesssim \left[3 + \log_2 \left(\frac{d(S_{ki}, S_{mj})}{\ell(S_{ki}))} \right) \right]^2 \frac{\mathrm{vol}(S_{mj}) \mathrm{vol}(S_{ki})}{d(S_{ki}, S_{mj})^{n+2\alpha}} \|u\|^2_{Q_\alpha(\Omega)}.
\end{aligned}$$

Notice that for all $k > -\log_2 \ell(I) + 7$ and all $S_{k,i} \in \mathscr{S}_k(I)$. So we have $I \subset 2^{k+2} \ell(I) S_{ki}$. For each $m \geq k$ and $j \in \mathscr{S}_m(I)$ with $S_{mj} \cap S_{ki} \neq \emptyset$, there exists exactly one $\delta = 1, \cdots, k + 2 + \log_2 \ell(I)$ such that $S_{mj} \cap (2^\delta S_{ki} \setminus 2^{\delta - 1} S_{ki}) \neq \emptyset$. For such δ, we have $d(S_{ki}, S_{mj}) \lesssim 2^{\delta-k}$. Thus

$$\begin{aligned}
&\sum_{m\geq k} \sum_{\substack{S_{mj}\in\mathscr{S}_m(I)\\ S_{mj}\cap S_{ki}=\emptyset}} \int_{S_{ki}}\int_{S_{mj}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dy\,dx \\
&\lesssim \mathrm{vol}(S_{ki})\|u\|^2_{Q_\alpha(\Omega)} \sum_{\delta=1}^{k+2+\log_2\ell(I)} \sum_{m\geq k} \sum_{\substack{S_{mj}\in\mathscr{S}_m(I)\\ S_{mj}\cap(2^\delta S_{ki}\setminus 2^{\delta-1}S_{ki})\neq\emptyset}} \\
&\quad\times \left[3+\log_2\frac{d(S_{ki},S_{mj})}{\ell(S_{mj})}\right]^2 [d(S_{ki},S_{mj})]^{-2\alpha-n}\,\mathrm{vol}(S_{mj}) \\
&\lesssim \mathrm{vol}(S_{ki})\|u\|^2_{Q_\alpha(\Omega)} \sum_{\delta=1}^{k+2+\log_2\ell(I)} \sum_{m\geq k} \sum_{\substack{S_{mj}\in\mathscr{S}_m(I)\\ S_{mj}\cap(2^\delta S_{ki}\setminus 2^{\delta-1}S_{ki})\neq\emptyset}} \\
&\quad\times [3+\delta+m-k]^2\, 2^{(-\delta+k)(2\alpha+n)} 2^{-mn} \\
&\lesssim \mathrm{vol}(S_{ki})\|u\|^2_{Q_\alpha(\Omega)} \sum_{m\geq k} \sum_{\delta=1}^{k+2+\log_2\ell(I)} \sharp\{S_{mj}\in\mathscr{S}_m(I) : S_{mj}\cap(2^\delta S_{ki}\setminus 2^{\delta-1}S_{ki})\neq\emptyset\} \\
&\quad\times \frac{(1+m-k+\delta)^2}{2^{(2\alpha+n)(m-k+\delta)-2m\alpha}}.
\end{aligned}$$

Applying Corollary 3.1, for $m\geq k$ and $\delta = 1,\cdots,k+2+\log_2\ell(I)$ we also have

$$\sharp\{S_{mj}\in\mathscr{S}_m(I) : S_{mj}\cap(2^\delta S_{ki}\setminus 2^{\delta-1}S_{ki})\neq\emptyset\} = \sharp\mathscr{S}_m(2^\delta S_{ki}) \lesssim 2^{(m-k+\delta)(\overline{\dim}_{\mathscr{L}}\partial\Omega+\epsilon)}.$$

So by $\dim_{\mathscr{L}}\partial\Omega+\epsilon < n-2\alpha$, we arrive at

$$\begin{aligned}
&\sum_{m\geq k} \sum_{\substack{S_{mj}\in\mathscr{S}_m(I)\\ S_{mj}\cap 2S_{ki}=\emptyset}} \int_{S_{ki}}\int_{S_{mj}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dy\,dx \\
&\lesssim \mathrm{vol}(S_{ki})\|u\|^2_{Q_\alpha(\Omega)} \sum_{m\geq k} \sum_{\delta=1}^{k+2+\log_2\ell(I)} 2^{(m-k+\delta)(\overline{\dim}_{\mathscr{L}}\partial\Omega+\epsilon)} \\
&\quad\times \left[(1+m-k)^2+\delta^2\right] 2^{(-2\alpha-n)(m-k+\delta)} 2^{2\alpha m}.
\end{aligned}$$

Since $\overline{\dim}_{\mathscr{L}}\partial\Omega+\epsilon < n-2\alpha$, we obtain

$$\begin{aligned}
&\sum_{m\geq k} \sum_{\substack{S_{mj}\in\mathscr{S}_m(I)\\ S_{mj}\cap 2S_{ki}=\emptyset}} \int_{S_{ki}}\int_{S_{mj}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dy\,dx \\
&\lesssim \mathrm{vol}(S_{ki})\|u\|^2_{Q_\alpha(\Omega)} \sum_{m\geq k} 2^{(m-k)(\overline{\dim}_{\mathscr{L}}\partial\Omega+\epsilon)} [1+m-k]^2\, 2^{(-2\alpha-n)(m-k)} 2^{2\alpha m} \\
&\lesssim \mathrm{vol}(S_{ki})\|u\|^2_{Q_\alpha(\Omega)} 2^{2\alpha k}.
\end{aligned}$$

Therefore, by $\overline{\dim}_{\mathscr{L}}\partial\Omega+\epsilon<n-2\alpha$ and $\ell(I)\le 1$, we have

$$\begin{aligned}
H_2 &\lesssim \|u\|^2_{Q_\alpha(\Omega)} \sum_{k\ge -\log_2 \ell(I)+7}\ \sum_{i\in\mathscr{S}_k(I)} |S_{ki}|2^{2\alpha k}\\
&\lesssim \|u\|^2_{Q_\alpha(\Omega)} \sum_{k\ge -\log_2 \ell(I)+7} [2^k\ell(I)]^{\overline{\dim}_{\mathscr{L}}\partial\Omega+\epsilon}2^{-(n-2\alpha)k}\\
&\lesssim 2^{(n-2\alpha-\overline{\dim}_{\mathscr{L}}\partial\Omega-\epsilon)\log_2\ell(I)}\ell(I)^{\overline{\dim}_{\mathscr{L}}\partial\Omega+\epsilon}\|u\|^2_{Q_\alpha(\Omega)}\\
&\approx \ell(I)^{n-2\alpha}\|u\|^2_{Q_\alpha(\Omega)},
\end{aligned}$$

as desired.

Case: $\operatorname{diam}\partial\Omega<\infty$ & $\operatorname{diam}\Omega=\infty$.

In this case, we need to estimate $\Psi_\alpha(u,I\cap\Omega)$ as in (3.4) for all I, and hence for all I with its center in Ω and $I\cap\Omega^{\complement}\ne\emptyset$. Without loss of generality we may assume that $\partial\Omega\subset B(0,1)$. If $\ell(I)\le 8$, by an argument similar as above, we have the desired estimate for $\Psi_\alpha(u,I\cap\Omega)$. If $\ell(I)>8$, then we may assume that $2^{k_0}-1\le\ell(I)\le 2^{k_0}$ for some $k_0\ge 4$. Then $I\subset I(0,2^{k_0+1})$, and hence we may further assume that $I=I(0,2^{k_0+1})$ for some $k_0\ge 4$. Write

$$\Psi_\alpha(u,I\cap\Omega)=\int_{I\cap\Omega}\int_{I\cap\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dy\,dx\equiv\Psi_\alpha(u,I(0,8)\cap\Omega)+H_3+H_4+H_5,$$

where

$$\begin{cases}
H_3=\int_{I(0,4)\cap\Omega}\int_{I\setminus I(0,8)}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dy\,dx;\\
H_4=\int_{I\setminus I(0,4)}\int_{I(0,2)\cap\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dy\,dx;\\
H_5=\int_{I\setminus I(0,4)}\int_{I\setminus I(0,2)}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dy\,dx.
\end{cases}$$

Because we already have

$$\Psi_\alpha(u,I(0,8)\cap\Omega)\lesssim\|u\|^2_{Q_\alpha(\Omega)}\lesssim\ell(I)^{n-2\alpha}\|u\|^2_{Q_\alpha(\Omega)},$$

it suffices to control $H_3;H_4;H_5$. Regarding H_3, we find a Whitney cube $J\subset I(0,7)\setminus I(0,5)$. Since $2^{-9}/\sqrt{n}\ell(J)<1$, one has

$$\begin{aligned}
H_3 &\le \int_{I(0,4)\cap\Omega}\int_{I\setminus I(0,8)}\frac{|u(x)-u_J|^2}{|x-y|^{n+2\alpha}}\,dy\,dx+\int_{I(0,4)\cap\Omega}\int_{I\setminus I(0,8)}\frac{|u_J-u(y)|^2}{|x-y|^{n+2\alpha}}\,dy\,dx\\
&\lesssim \ell(I)^{-2\alpha}\int_{I(0,4)\cap\Omega}|u(x)-u_J|^2\,dx+\int_{I\setminus I(0,8)}\frac{|u_J-u(y)|^2}{|y|^{n+2\alpha}}\,dy\\
&\equiv H_{3,1}+H_{3,2}.
\end{aligned}$$

Obviously,

$$\begin{aligned}
H_{3,1} &\lesssim \ell(I)^{-2\alpha}\int_{I(0,4)\cap\Omega}\int_J|u(x)-u(y)|^2\,dy\,dx\\
&\lesssim \ell(I)^{-2\alpha}\Psi_\alpha(u,I(0,4)\cap\Omega)\\
&\lesssim \ell(I)^{-2\alpha}\|u\|^2_{Q_\alpha(\Omega)}\\
&\lesssim \ell(I)^{n-2\alpha}\|u\|^2_{Q_\alpha(\Omega)}.
\end{aligned}$$

Suppose

$$2^{2^{k_0}-1} \le \ell(I) \le 2^{2^{k_0}} \quad \text{for some} \quad k_0 \ge 4.$$

Then $I \subset I(0, 2^{k_0+1})$ and hence

$$\begin{aligned} H_{3,2} &\le \sum_{k=3}^{k_0+2} \int_{I(0,2^{k+1})\setminus I(0,2^k)} \frac{|u_J - u(y)|^2}{|y|^{n+2\alpha}}\, dy \\ &\lesssim \sum_{k=3}^{k_0+2} 2^{-k(n+2\alpha)} \int_{I(0,2^{k+1})\setminus I(0,2^k)} |u_J - u(y)|^2\, dy. \end{aligned}$$

Denote by $\widetilde{\mathscr{S}_k}$ as the collection of S_i such that $S_i \cap (I(0, 2^{k+1}) \setminus I(0, 2^k)) \ne \emptyset$. Then for each $S_i \in \widetilde{\mathscr{S}_k}$ we have $2^{-k+7} \le \ell(S_i) \le 2^{-k+2}$, and hence $\sharp\widetilde{\mathscr{S}_k} \lesssim 2^{k(n-1)}$. This in turns implies

$$H_{3,2} \le \sum_{k=3}^{k_0+2} 2^{-2k\alpha} \sum_{S_i\in\widetilde{\mathscr{S}_k}} ⨍_{S_i} |u_J - u(y)|^2\, dy.$$

For each $S_i \in \widetilde{\mathscr{S}_k}$, by Lemma 3.1, there exists a Whitney chain joining J and S_i with the length less than Ck. Therefore, by the fractional Poincaré inequality we have

$$⨍_{S_i} |u_J - u(y)|^2\, dy \lesssim k\|u\|^2_{Q_\alpha(\Omega)},$$

whence

$$H_{3,2} \lesssim \sum_{k=3}^{k_0+2} 2^{-2k\alpha} k\|u\|^2_{Q_\alpha(\Omega)} \lesssim \|u\|^2_{Q_\alpha(\Omega)} \lesssim \ell(I)^{n-2\alpha}\|u\|^2_{Q_\alpha(\Omega)}.$$

Since the estimate for H_4 is similar to H_3, we hereby omit its demonstration.

Regarding H_5, let $J \subset I(0,\, 2^{k_0+4})$ be a cube with side length 2^{k_0+2} and $J \cap I(0, 2^{k_0+3}) = \emptyset$. Then

$$H_5 \lesssim \int_{I\setminus I(0,2)} \int_{I\setminus I(0,2)} \frac{|u(x) - u_J|^2}{|x-y|^{n+2\alpha}}\, dy\, dx \lesssim \ell(I)^{-2\alpha} \int_{I\setminus I(0,2)} |u(x) - u_J|^2\, dx.$$

Covering $I \setminus I(0, 2)$ by 2^n many cubes R_j with side length 2^{k_0+2} and $R_j \cap I(0,\, 2) = \emptyset$, we arrive at

$$H_5 \lesssim \ell(I)^{n-2\alpha} \sum_{j=1}^{2^n} ⨍_{R_j} |u(x) - u_J|^2\, dx.$$

Also, for each R_j we can find cubes $\{R_{js}\}_{s=1}^M$ such that

$$\begin{cases} M \lesssim 1; \\ R_{j1} = J; \\ R_{jM} = R_j; \\ R_{js} \cap R_{j(s+1)} \ne \emptyset \text{ for } \ 1 < s < M-1; \\ 2R_{j(s+1)} \cap I(0,2) = \emptyset \ \text{ for } \ 1 < s < M-1. \end{cases}$$

So, under this situation we have

$$\fint_{R_j} |u(x) - u_J|^2\, dx \lesssim M^2 \sum_{s=1}^{M} \fint_{2R_{js}} |u(x) - u_{2R_{js}}|^2\, dx \lesssim \|u\|^2_{Q_\alpha(\Omega)},$$

whence

$$H_5 \lesssim \ell(I)^{n-2\alpha} \|u\|^2_{Q_\alpha(\Omega)}.$$

Case: $\operatorname{diam} \partial\Omega = \infty$.

Since there is no restriction on the side length of I or δ in Lemma 3.2 (ii) and Corollary 3.1 (ii), applying the argument similar to *Case:* $\operatorname{diam} \partial\Omega < \infty$ & $\operatorname{diam} \Omega < \infty$, we can get the desired estimates. □

3.2. $\mathscr{Q}_\beta(\Omega_\alpha) \neq Q_\beta(\Omega_\alpha)$ **under** (1.2)**.** Working on this inequality, we have the forthcoming Theorems 3.2-3.3.

Theorem 3.2. *For each* $\alpha \in (0,1)$ *there is a uniform domain* $\Omega_\alpha \subsetneq \mathbb{R}^n$ *such that*

$$n = \begin{cases} 2\alpha + \overline{\dim}_{\mathscr{L}} \partial\Omega_\alpha & \text{when } \partial\Omega_\alpha \text{ is bounded;} \\ 2\alpha + \overline{\dim}_{\mathscr{G}} \partial\Omega_\alpha & \text{when } \partial\Omega_\alpha \text{ is unbounded,} \end{cases}$$

but $Q_\beta(\mathbb{R}^n)|_{\Omega_\alpha} \neq Q_\beta(\Omega_\alpha)\ \forall\ \beta \in [\alpha, 1)$.

Proof. Step 1. Construction of the set Ω_α.

Let I_0 be unit cube $[0,1]^n$ and $a \in (0,1)$. Denote by E_a the Cantor dust with parameter a obtained as follows. Set $L_0 = [0,1]$. Let L_i, $i = 1, 2$, be the two closed intervals obtained by removing the middle open interval of length a from $L_0 = [0,1]$ ordered from left to right; when $m \geq 2$, the subintervals $L_{i_1\cdots i_m}$, $i_m = 1, 2$, are the two closed intervals obtained by removing the middle open intervals of length $a[(1-a)/2]^{m-1}$ from $L_{i_1\cdots i_{m-1}}$ ordered from left to right. Notice that

$$\ell(L_{i_1\cdots i_m}) = [(1-a)/2]^m \quad \forall \quad m \geq 1.$$

So, for each $m \geq 1$ set

$$L_a^m = \bigcup_{i_1, \cdots, i_m \in \{1,2\}} L_{i_1\cdots i_m} \quad \& \quad E_a^m = (L_a^m)^n = L_a^m \times \cdots \times L_a^m.$$

For each $m \geq 1$, we order all cubes consisting E_a^m with side length $(\frac{1-a}{2})^m$ by $I_{m,j}$, $j = 1, \cdots, 2^{mn}$, and write the center of $I_{m,j}$ as $z_{I_{mj}}$ for all possible m and j. For $I_0 = L_0^n$, write its center as z_{I_0}. Define the Cantor dust as

$$E_a = \bigcap_{m=1}^{\infty} E_a^m.$$

Set

$$\mathscr{E}_a = \bigcup_{k\geq 0} \left\{ \left(\frac{2}{1-a}\right)^k x :\ x \in E_a \right\}.$$

We have the following lemma whose proof is postponed after proving Theorem 3.2.

Lemma 3.4. *Let* $a \in (0,1)$. *Then*

(i) $\overline{\dim}_{\mathscr{L}} E_a = \overline{\dim}_{\mathscr{G}} \mathscr{E}_a = \frac{n}{\log_2[2/(1-a)]}$.
(ii) *Both* $\mathbb{R}^n \setminus E_a$ *and* $\mathbb{R}^n \setminus \mathscr{E}_a$ *are uniform domains.*

In particular, choosing $a = 1 - 2^{-2\alpha/(n-2\alpha)}$, we have $a \in (0,1)$ (due to $0 < 2\alpha < 2 \le n$) and take $\Omega_a = \mathbb{R}^n \setminus E_a$. Then $\overline{\dim}_{\mathscr{L}} \partial\Omega_a = n - 2\alpha$. We will show below that Ω_a is exactly the critical domain Ω_α with bounded boundary required by Theorem 3.2.

The critical domain Ω_α with unbounded boundary required by Theorem 3.2 will be given by $\mathbb{R}^n \setminus \mathscr{E}_a$. The proof is similar to the bounded case; and hence will be omitted.

Step 2. Construction of function u.
To each cube $I = I_{mj}$ we assign a function u_I, which is supported in $\frac{a}{8}I$ and $u_I = 1$ on $\frac{a}{16}I$, $0 \le u_I \le 1$ on $\mathbb{R}^n$ and $|\nabla u_I| \lesssim \ell(I)^{-1}$. Define

$$u = \sum_{m=1}^{\infty} \sum_{j=1}^{2^{mn}} u_{I_{mj}}.$$

Step 3. $u \notin \mathscr{Q}_\beta(\Omega)$ *when* $1 > \beta \ge \alpha$. For $I = I_0$, we have

$$\begin{aligned}\int_{I\cap\Omega}\int_{I\cap\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\,dx\,dy &\ge \sum_{m=1}^{\infty}\sum_{j=1}^{2^{mn}} \int_{\frac{a}{4}I_{mj}}\int_{\frac{a}{4}I_{mj}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\beta}}\,dx\,dy \\ &\ge \sum_{m=1}^{\infty}\sum_{j=1}^{2^{mn}} \int_{\frac{a}{16}I_{mj}}\int_{\frac{a}{4}I_{mj}\setminus\frac{a}{8}I_{mj}} \frac{|u_{I_{mj}}(x)-u_{I_{mj}}(y)|^2}{|x-y|^{n+2\beta}}\,dx\,dy \\ &\gtrsim \sum_{m=1}^{\infty}\sum_{j=1}^{2^{mn}} \Big(\frac{1-a}{2}\Big)^{-(n+2\beta)m}\Big(\frac{1-a}{2}\Big)^{2mn} \\ &\gtrsim \sum_{m=1}^{\infty} 2^{mn}\Big(\frac{1-a}{2}\Big)^{(n-2\beta)m}.\end{aligned}$$

Since $a = 1 - 2^{-2\alpha/(n-2\alpha)}$, we have

$$\frac{1-a}{2} = 2^{-2\alpha/(n-2\alpha)-1} = 2^{-n/(n-2\alpha)},$$

thereby getting

$$\alpha \le \beta \Longrightarrow 2^{mn}\Big(\frac{1-a}{2}\Big)^{(n-2\beta)m} = 2^{mn}2^{-mn(n-2\beta)/(n-2\alpha)} = 2^{2mn(\beta-\alpha)/(n-2\alpha)} \ge 1$$

and

$$\sum_{m=1}^{\infty} 2^{mn}\Big(\frac{1-a}{2}\Big)^{(n-2\beta)m} = \infty.$$

This in turn implies

$$\|u\|^2_{\mathscr{Q}_\beta(\Omega_a)} \ge \int_{I\cap\Omega_a}\int_{I\cap\Omega_a} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\,dx\,dy = \infty.$$

Setp 4. $u \in Q_\beta(\Omega)$ *when* $0 \le \beta < 1$.

To treat this situation, we observe that for each $I \subset \Omega_a$, there exists at most one I_{mj} for some $m \geq 1$ and $j = 1, \cdots, 2^{mn}$ such that $I \cap \frac{a}{8} I_{mj} \neq \emptyset$. Then

$$\|u\|_{Q_\alpha(\Omega_a)} = \|u_{I_{mj}}\|_{Q_\alpha(\mathbb{R}^n)} \lesssim \|u_{I_{mj}}\|_{W^{1,n}(\mathbb{R}^n)} \lesssim 1.$$

$\square$

Proof of Lemma 3.4. (i) The proof is similar to validating [14, Lemma 2.3], and hence omitted.

(ii) We only verify the result with $n = 2$ since the argument under $n \geq 3$ is analogous. For every pair of points $x, y \in \mathbb{R}^2 \setminus E_a$, we need to find a curve $\gamma \in \mathbb{R}^2 \setminus E_a$ joining y, x such that $l(\gamma) \lesssim |x - y|$ and

$$d(\gamma(t), \mathbb{R}^2 \setminus E_a) \gtrsim \frac{|x - \gamma(t)||y - \gamma(t)|}{|x - y|} \quad \forall t \in [0, l(\gamma)].$$

If $x, y \in \mathbb{R}^2 \setminus I_0$, then it is easy to find such a γ since $\mathbb{R}^2 \setminus I_0$ is uniform and

$$d(z, \mathbb{R}^2 \setminus I_0) \geq d(z, \mathbb{R}^2 \setminus E_a) \;\; \forall \;\; z \in \mathbb{R}^2 \setminus I_0.$$

Now we assume that $y \in I_0 \setminus E_a$. It suffices to find the curve required γ joining y and x where:

$$x \in \frac{1+3a}{1-a} I_0 \setminus I_0; \quad x \in I_0 \setminus E_a \;\; \& \;\; x \notin \frac{1+3a}{1-a} I_0.$$

Before finding the curve γ, we observe that there exists a minimal $m_y \geq 1$ such that $y \in I_0 \setminus E_a^{m_y}$, denote by I_y the building-up cube of $E_a^{m_y - 1}$ (letting $E_a^0 = I_0$) that contains y. For $0 \leq k \leq m_y - 1$ there exist cubes $I_y^{(k)}$, the building-up cubes of $E_a^{m_y - 1 - k}$, such that $I_y^{(0)} = I_y \subset I_y^{(1)} \subset \cdots \subset I_y^{(m_y - 1)} = I_0$.

Moreover, for each cube I and $m \geq 1$, denote by $E_{a,I}^m$ as image of E_a^m under map T, where T is linear map from $[0,1]^2$ onto I. Then we have the following observations.

Observation 1. $\frac{1+3a}{1-a} I \setminus E_{a,I}^1$ is a uniform domain for some constant $C_1(a)$ independent of I.

Observation 2. For each $x \in \frac{1+3a}{1-a} I \setminus E_{a,I}^1$, there exists a curve $\gamma \subset \frac{1+3a}{1-a} I \setminus E_{a,I}^1$ joining x and the centre of I such that

$$l(\gamma) \leq C_2(a)\ell(I) \;\; \& \;\; (\gamma(t), \frac{1+3a}{1-a} I \setminus E_{a,I}^1) \geq \frac{t}{C_2(a)} \quad \forall t \in [0, l(\gamma)].$$

Observation 3. For I and any building-up cube J of $E_{a,I}^1$, there exists a curve $\gamma \subset I \setminus E_{a,I}^2$ joining the centers of I and J such that

$$l(\gamma) \leq C_3(a)\ell(I) \;\; \& \;\; d(\gamma(t), I \setminus E_{a,I}^2) \geq \frac{\ell(I)}{C_3(a)} \quad \forall t \in [0, l(\gamma)].$$

Now we are going to find the curve γ joining y and x.

Case: $x \in \frac{1+3a}{1-a} I_0 \setminus E_a^1$. Let k_x be the minimal $0 < k \leq m_y - 1$ such that $x \in \frac{1+3a}{1-a} I_y^{(k)}$. If $x \in \frac{1+3a}{1-a} I_y$, then the desired curve is given by the curve in Observation 1. Assume that $x \notin \frac{1+3a}{1-a} I_y$. Obviously, $m_y > 1$, otherwise, $E_a^{m_y - 1} = I_0$ and $x \in \frac{1+3a}{1-a} I_y$.

Note that $|x-y| \approx (\frac{1-a}{2})^{m_y-k_x}$. So, by Observation 2 we can find a curve $\gamma_1 \subset \frac{1+3a}{1-a} I_y^{(0)} \setminus I_y^{(0)}$ joining y and the center of $I_y^{(1)}$ since it appears in $\frac{1+3a}{1-a} I_y^{(0)}$ such that

$$l(\gamma_1) \lesssim 2^{-m_y} \ \& \ d(\gamma_1(t), \mathbb{R}^2 \setminus E_a) \geq d(\gamma_1(t), \frac{1+3a}{1-a} I_y^{(0)} \setminus I_y^{(0)}) \geq \frac{t}{C_2(a)} \quad \forall t \in [0, l(\gamma_1)].$$

By Observation 3, for $2 \leq j \leq k_x - 1$, we can find $\gamma_j \in I \setminus E_a$ joining the center of $I_y^{(j-1)}$ and centre of $I_y^{(j)}$ such that

$$l(\gamma_j) \leq C_3(a)(\frac{1-a}{2})^{m_y-j} \ \& \ d(\gamma_j(t), \mathbb{R}^2 \setminus E_a) \geq C_3(a)^{-1} 2^{-m_y+j} \quad \forall t \in [0, l(\gamma_j)].$$

Let γ_y be the union of $\gamma_1, \cdots, \gamma_{k_x}$. Then

$$l(\gamma_y) \lesssim \sum_{j=1}^{k_x} (\frac{1-a}{2})^{m_y-j} \lesssim (\frac{1-a}{2})^{m_y-k_x} \approx |x-y|,$$

and if

$$l(\gamma_1) + \cdots + l(\gamma_j) \leq t \leq l(\gamma_1) + \cdots + l(\gamma_{j+1}) \ \ \forall \ \ j = 0, \cdots, k_x - 1,$$

then

$$\begin{aligned} t \approx (\frac{1-a}{2})^{m_y-j} \ \ \& \ \ d(\gamma_y(t), \mathbb{R}^n \setminus E_a) &\geq d(\gamma_{j+1}(t - [l(\gamma_1) + \cdots + l(\gamma_j)]), \mathbb{R}^n \setminus E_a) \\ &\gtrsim (\frac{1-a}{2})^{m_y-j} \gtrsim t. \end{aligned}$$

Moreover, there is a curve $\gamma_{k_x+1} \subset \frac{1+3a}{1-a} I_y^{(k_x)} \setminus E^1_{a, I_y^{(k_x)}}$ joining x and the center of $I_y^{(k_x)}$ such that

$$l(\gamma_{k_x+1}) \leq C_3(a)(\frac{1-a}{2})^{m_y-j} \ \ \& \ \ d(\gamma_{k_x+1}(t), \mathbb{R}^2 \setminus E_a) \geq \frac{t}{C_3(a)} \quad \forall \quad t \in [0, l(\gamma_{k_x+1})].$$

Let γ be the union of γ_y and $\overline{\gamma}_{k_x+1}$, where

$$\overline{\gamma}_{k_x+1}(t) = \gamma_{k_x+1}(l(\gamma_{k_x+1}) - t).$$

It is easy to see that γ is the required curve to joining y, x. Indeed,

$$l(\gamma) \lesssim l(\gamma_y) + l(\gamma_{k_x+1}) \approx |x-y|;$$

and if $0 \leq t \leq l(\gamma_y)$ for $j = 0, \cdots, k_x - 1$, then

$$d(\gamma(t), \mathbb{R}^n \setminus E_a) \geq d(\gamma_y(t), \mathbb{R}^n \setminus E_a) \gtrsim (\frac{1-a}{2})^{m_y-j} \gtrsim t;$$

and if $l(\gamma_y) \leq t \leq l(\gamma)$, then

$$d(\gamma(t), \mathbb{R}^n \setminus E_a) \geq d(\overline{\gamma}_{k_x+1}(t - l(\gamma_y)), \mathbb{R}^n \setminus E_a) = d(\gamma_{k_x+1}(l(\gamma) - t, \mathbb{R}^n \setminus E_a) \gtrsim l(\gamma) - t.$$

<u>*Case:* $x \in I_0 \setminus E_a$.</u> There exists a minimal $m_x \geq 1$ such that $x \in I_0 \setminus E_a^{m_x}$. Without loss of generality, we may assume that $m_x \leq m_y$; otherwise we change the role of x, y.

If $I_y = I_x$, then by Observation 1, we have the desired curve.

If $I_y \subsetneq I_x$, then upon letting k_x be the minimal integer k such that $I_y^{(k)} \subset I_x$ and viewing $I_y^{(k-1)}$ as I_0, we get the desired curve in a way similar to the case $x \in \frac{1+3a}{1-a} I_0 \setminus E_a^1$.

If $I_x \cap I_y = \emptyset$, then there exists a minimal $m_{xy} < \min\{m_x, m_y\}$ such that

$$I_x, I_y \subset I_{xy} \quad \text{with} \quad \ell(I_{xy}) = (\frac{1-a}{2})^{m_{xy}}.$$

Therefore,

$$|x - y| \approx (\frac{1-a}{2})^{m_{xy}} \approx |x - z_{xy}| + |y - z_{xy}|,$$

where z_{xy} is the center of I_{xy}. Repeating the argument in the case $x \in \frac{1+3a}{1-a} I_0 \setminus E_a^1$, we can find a curve γ_y joining y and z_{xy}, and γ_x joining x and z_{xy} such that

$$\begin{cases} l(\gamma_y) \le C_2(a)(\frac{1-a}{2})^{m_{xy}} \lesssim |x-y|; \\ d(\gamma_y(t), \mathbb{R}^2 \setminus E_a) \ge \frac{1}{C_2(a)} t \quad \forall \quad t \in [0, l(\gamma_y)]; \\ l(\gamma_x) \lesssim |x-y|; \\ d(\gamma_x(t), \mathbb{R}^2 \setminus E_a) \ge \frac{1}{C_2(a)} t \quad \forall \quad t \in [0, l(\gamma_x)]. \end{cases}$$

Letting $\gamma = \gamma_y \cup \overline{\gamma}_x$ (where $\overline{\gamma}_x(t) = \gamma_x(l(\gamma_x) - t)$ for $t \in [0, l(\gamma_x))$ gives the desired curve.

Case: $x \notin \mathbb{R}^2 \setminus \frac{1+3a}{1-a} I_0$. Then $|x-y| \gtrsim 1$. In this case, by an argument similar to the case $x \in \frac{1+3a}{1-a} I_0 \setminus E_a^1$, there exists a curve γ_y joining y and $z_0 \in \frac{1+a}{1-a} I_0$ such that

$$l(\gamma_y) \lesssim 1 \;\; \& \;\; d(\gamma_y(t), \mathbb{R}^2 \setminus E_a) \gtrsim t \quad \forall \quad t \in [0, l(\gamma_y)].$$

Moreover we can find a curve γ_x joining z_0 and x such that

$$l(\gamma_x) \lesssim 1 \;\; \& \;\; d(\gamma_x(t), \mathbb{R}^2 \setminus I_0) \gtrsim t + 1 \quad \forall \quad t \in [0, l(\gamma_x)].$$

The union of γ_y and $\overline{\gamma}_x$ gives the desired curve joining y and x, and then completes the argument. □

THEOREM 3.3. *For each $\alpha \in (0, \frac{1}{2}]$ there is a simply connected uniform domain $\Omega_\alpha \subset \mathbb{R}^2$ such that*

$$2 = \begin{cases} 2\alpha + \overline{\dim}_{\mathscr{L}} \partial\Omega_\alpha \text{ when } \partial\Omega_\alpha \text{ is bounded;} \\ 2\alpha + \overline{\dim}_{\mathscr{G}} \partial\Omega_\alpha \text{ when } \partial\Omega_\alpha \text{ is unbounded,} \end{cases}$$

but $Q_\beta(\mathbb{R}^2)|_{\Omega_\alpha} \neq Q_\beta(\Omega_\alpha) \;\; \forall \;\; \beta \in [\alpha, 1)$.

Proof. We consider two cases: $\alpha = \frac{1}{2}$ and $\alpha \in (0, \frac{1}{2})$.

Case: $\alpha = \frac{1}{2}$.

In the sequel, let $\Omega \subset \mathbb{R}^2$ be the triangle domain

$$\Omega_\alpha = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1 - x\}.$$

Obviously $\overline{\dim}_{\mathscr{L}}\Omega = 1$. We are going to construct a function $u \in Q_\beta(\Omega)$ but $u \notin \mathscr{Q}_\beta(\Omega)$ whenever $\beta \in [\frac{1}{2}, 1)$. We can show that the similar result holds for the unbounded domain

$$\Omega_\alpha = \{(x, y) \in \mathbb{R}^2 : 0 < y < 1 - x\}$$

which has the global Minkowski type dimension 1.

Setp 1. Construction of function u.

By a dyadic cube we mean a cube $2^{-j}(k_1, k_2) + 2^{-j}[0, 1]^2$ for some $k_1, k_2, j \in \mathbb{Z}$. Observe that when $k = 1$, there are exactly 1 dyadic cube I_1 with side length 2^{-1} contained in Ω; when $k = 2$, there are exactly 2 dyadic cubes I_{ki} for $i = 1, 2$ with side length 2^{-2} contained in $\Omega \setminus I_1$; generally, when $k \geq 2$, there exist exactly 2^{k-1} dyadic cubes $I_{k1}, \cdots, I_{k2^{-k+1}}$ (with side length 2^{-k}) contained in $\Omega \setminus \cup_{\ell=1}^{k} \cup_{j=1}^{2^{k-2}} I_{\ell j}$. Obviously,

$$\Omega = \bigcup_{k\geq 1} \bigcup_{j=1}^{2^{k-1}} I_{kj} \quad \& \quad I^\circ_{k,j} \cap I^\circ_{\ell,i} = \emptyset \ \ \forall \ \ (k, j) \neq (\ell, i).$$

For each possible pair (k, j), let $u_{k,j}$ be a smooth function satisfying

$$\begin{cases} \mathrm{supp} u_{k,j} \subset 4^{-1}I_{k,j}; \\ u_{k,j} = 1 \text{ on } \ 8^{-1}I_{k,j}; \\ |\nabla u_{k,j}| \approx 2^k \ \text{ on } \ 4^{-1}I_{k,j} \setminus 8^{-1}I_{k,j}. \end{cases}$$

Then

$$\|u_{k,j}\|_{Q_\alpha(\Omega)} \lesssim \|\nabla u_{k,j}\|_{L^2(\Omega)} \lesssim 1.$$

Define

$$u = \sum_{k=1}^{\infty} \sum_{j=1}^{2^{k-1}} u_{k,j}.$$

Setp 2. $u \notin \mathscr{Q}_\beta(\Omega)$ when $1 > \beta \geq \frac{1}{2}$.

For $I = [0, 1]^2$, we have

$$\begin{aligned} \int_{I\cap\Omega} \int_{I\cap\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2\beta}} \, dx \, dy &\geq \sum_{k=1}^{\infty} \sum_{j=1}^{2^{k-1}} \int_{I_{kj}} \int_{I_{kj}} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2\beta}} \, dx \, dy \\ &\geq \sum_{k=1}^{\infty} \sum_{j=1}^{2^{k-1}} \int_{\frac{1}{8}I_{kj}} \int_{I_{kj}\setminus 4^{-1}I_{kj}} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2\beta}} \, dx \, dy \\ &\geq \sum_{k=1}^{\infty} \sum_{j=1}^{2^{k-1}} 2^{-4k} 2^{(2+2\beta)k} \\ &\geq \sum_{k=1}^{\infty} 2^{-k} 2^{2\beta k} \\ &= \infty. \end{aligned}$$

Setp 3. $u \in Q_\beta(\Omega)$ for all $\beta \in [0,1)$.

If $I \subset I_{kj}$ for some k, j, then we already have

$$\ell(I)^{2\beta-2} \int_I \int_I \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\, dx\, dy \le \|u_{kj}\|^2_{Q_\beta(\mathbb{R}^n)} \lesssim 1.$$

Generally, given an open cube $I \subset \Omega$, for each k, there exists at most one $I_{k,j}$, denoted by I_k, such that $I \cap I_{k,j} \neq \emptyset$. Thus

$$\begin{aligned}
&\int_I \int_I \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\, dx\, dy \\
&\lesssim \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \int_{I_k \cap I} \int_{I_\ell \cap I} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\, dx\, dy + \sum_{k=1}^{\infty} \int_{I_k \cap I} \int_{I \setminus (\cup I_\ell)} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\, dx\, dy \\
&\lesssim H_1 + H_2 + H_3,
\end{aligned}$$

where

$$\begin{cases}
H_1 = \sum_{k=1}^{\infty} \int_{I_k \cap I} \int_{I_k \cap I} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\, dx\, dy; \\
H_2 = \sum_{k=1}^{\infty} \sum_{\ell > k}^{\infty} \int_{I_k \cap I} \int_{I_\ell \cap I} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\, dx\, dy; \\
H_3 = \sum_{k=1}^{\infty} \int_{4^{-1} I_k \cap I} \int_{I \setminus I_k} \frac{|u(x)|^2}{|x-y|^{2+2\beta}}\, dx\, dy.
\end{cases}$$

Denote by k_I the minimal k such that $I \cap 4^{-1} I_k \neq \emptyset$. Observe that if $k_I = \infty$, then $u = 0$ on I. So we may assume $k_I < \infty$. Then $\ell(I) \gtrsim 2^{-k_I}$, and hence

$$\begin{aligned}
H_1 &\le \sum_{k=k_I}^{\infty} \int_{I_k} \int_{I_k} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\, dx\, dy \\
&\lesssim \sum_{k=k_I}^{\infty} 2^{-k(2-2\beta)} \|u_{k,j}\|^2_{Q_\beta(\Omega)} \lesssim 2^{-k_I(2-2\beta)} \lesssim \ell(I)^{2-2\beta}.
\end{aligned}$$

Regarding H_2, for $\ell > k$, we write

$$H_{2k\ell} = \int_{I_k \cap I} \int_{I_\ell \cap I} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\, dx\, dy$$

If $k, \ell < k_I$, then $H_{2k\ell} = 0$. If $k < k_I$ and $\ell \ge k_I$, then by the Poincaré inequality we get

$$\begin{aligned}
\sum_{k<k_I} H_{2k\ell} &= \sum_{k<k_I} \int_{I_k \cap I} \int_{4^{-1} I_\ell \cap I} \frac{|u(y)|^2}{|x-y|^{2+2\beta}}\, dy\, dx \\
&\lesssim \int_{4^{-1} I_\ell \cap I} |u(y)|^2 \sum_{k<k_I} \int_{I_k \cap I} \frac{dxdy}{|x-y|^{2+2\beta}} \\
&\lesssim \int_{4^{-1} I_\ell \cap I} |u(y)|^2 \int_{|x-y| \ge 2^{-\ell-1}} \frac{dxdy}{|x-y|^{2+2\beta}} \\
&\lesssim 2^{2\beta\ell} \int_{4^{-1} I_\ell} |u(y)|^2 dy \\
&\lesssim 2^{-(2-2\beta)\ell} \int_{4^{-1} I_\ell} |\nabla u(y)|^2 dy \\
&\lesssim 2^{-(2-2\beta)\ell}.
\end{aligned}$$

If $k \geq k_I$ and $\ell > k_I$, then we write

$$\begin{aligned} H_{2k\ell} &= \int_{I_k \cap I} \int_{I_\ell \cap I} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\,dx\,dy \\ &= \int_{4^{-1}I_k \cap I} \int_{I_\ell \cap I} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\,dx\,dy + \int_{(I_k \setminus 4^{-1}I_k)\cap I} \int_{I_\ell \cap I} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\,dx\,dy \\ &\equiv H_{2k\ell 1} + H_{2k\ell 2}. \end{aligned}$$

Observe that by the Poincaré inequality,

$$\begin{aligned} H_{2k\ell 1} &\lesssim 2^{k(2+2\beta)} \int_{4^{-1}I_k \cap I} \int_{I_\ell \cap I} (|u(x)|^2 + |u(y)|^2)\,dx\,dy \\ &\lesssim 2^{k(2+2\alpha)} 2^{-2\ell} \int_{4^{-1}I_k} |u(x)|^2\,dx + 2^{2\beta k} \int_{I_\ell} |u(y)|^2\,dy \\ &\lesssim 2^{2\beta k} 2^{-2\ell} \end{aligned}$$

and that

$$\begin{aligned} H_{2k\ell 2} &= \int_{(I_k \setminus 4^{-1}I_k)\cap I} \int_{4^{-1}I_\ell \cap I} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\,dx\,dy \\ &\lesssim \int_{I_k \cap I} \int_{4^{-1}I_\ell \cap I} \frac{|u(y)|^2}{|x-y|^{2+2\beta}}\,dy\,dx \lesssim 2^{-(2-2\beta)\ell}. \end{aligned}$$

Therefore,

$$\begin{aligned} H_2 &\lesssim \sum_{\ell \geq k_I} \sum_{k<k_I} H_{2k\ell} + \sum_{k\geq k_I} \sum_{\ell > k} H_{2k\ell} \\ &\lesssim \sum_{\ell \geq k_I} 2^{-(2-2\beta)\ell} + \sum_{k \geq k_I} \sum_{\ell > k} [2^{2\beta k} 2^{-2\ell} + 2^{-(2-2\beta)\ell}] \lesssim \ell(I)^{2-2\beta}. \end{aligned}$$

Concerning H_3, we have

$$\begin{aligned} H_3 &\approx \sum_{k=k_I}^{\infty} \int_{4^{-1}I_k \cap I} \int_{I \setminus I_k} \frac{|u(x)|^2}{|x-y|^{2+2\beta}}\,dx\,dy \\ &\lesssim \sum_{k=k_I}^{\infty} |I_k|^{-\beta} \int_{4^{-1}I_k} |u(x)|^2\,dx \\ &\lesssim \sum_{k=k_I}^{\infty} |I_k|^{1-\beta} \int_{4^{-1}I_k} |\nabla u(x)|^2\,dx \\ &\lesssim \sum_{k=k_I}^{\infty} |I_k|^{1-\beta} \\ &\lesssim \operatorname{vol}(I)^{1-\beta}. \end{aligned}$$

Combining the estimates for H_1 through H_3, we have

$$\ell(I)^{2\beta-2} \int_I \int_I \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\,dx\,dy \lesssim 1.$$

So $u \in Q_\beta(\Omega)$ follows.

Case: $\alpha \in (0, \frac{1}{2})$.

For $t \in (1/4, \frac{1}{2})$ let us recall the construction of von Koch type curve γ_t based on $[0,1]$ (cf. [13]): if

$$a_1 = 0;\ a_2 = t;\ a_3 = 2^{-1} + i(t - 1/4)^{\frac{1}{2}};\ a_4 = 1 - t;\ a_5 = 1,$$

then the length of each closed line segment $a_j a_{j+1}$ is t for $j = 1, 2, 3, 4$. Let σ_j be similarities such that

$$\sigma_j(a_1 a_5) = a_j a_{j+1}\ \ \&\ \ \sigma_j(a_1) = a_j\ \ \forall\ \ j = 1, 2, 3, 4.$$

For set A, write

$$\Sigma(A) = \Sigma^1(A) = \cup_{i=1}^4 \sigma_j(A).$$

For each integer $k \geq 1$, set

$$\Sigma^k(a_1 a_5) = \Sigma(\Sigma^{k-1}(a_1 a_5)).$$

Let γ_t be the limit set of $\Sigma^k(a_1 a_5)$, which is also the unique compact set such that $\Sigma(\gamma_t) = \gamma_t$. Notice that $t\gamma_t$ is a subarc of γ_t and γ_t is a subarc of $t^{-1}\gamma_t$. Recall that $\gamma_{1/3}$ is the von Koch curve. We also set the following arc

$$\Gamma_t = \cup_{m \geq 0} t^{-m}(\gamma_t \cup (-\gamma_t)),$$

which serves as the self-similar extension of γ_t to $\mathbb{R}^2$.

The bounded domain Ω_t is obtained by doing the same construction for each side of $[0,1] \times [-1,0]$ outward. The unbounded domain $\widetilde{\Omega}_t$ is given by the lower half of $\mathbb{R}^2 \setminus \Gamma_t$.

Lemma 3.5. *For every $t \in (1/4, \frac{1}{2})$, both $\partial\Omega_t$ and $\Gamma_t = \partial\widetilde{\Omega}_t$ are quasicircles with*

$$\overline{\dim}_{\mathscr{L}} \gamma_t = \overline{\dim}_{\mathscr{G}} \Gamma_t = \frac{\log_2 4}{\log_2(1/t)}.$$

Proof. From [13] it follows that $\partial\Omega_t$ and $\Gamma_t = \partial\widetilde{\Omega}_t$ are quasicircles. So, the proof of the Minkowski type dimensions of γ_t and Γ_t can be given by modifying the argument for [14, Lemma 2.4]. □

Let $t = 2^{-1/(1-\alpha)}$. Then $2 - 2\alpha = \frac{\log_2 4}{\log_2 1/t}$. We are about to show that Ω_t is the desired bounded domain Ω_α so that $Q_\beta(\Omega_\alpha) \neq \mathscr{Q}_\beta(\Omega_\alpha)\ \ \forall\ \ \beta \in [\alpha, 1)$. It is known by Theorem 3.1 that $Q_\beta(\Omega_t) = \mathscr{Q}_\beta(\Omega_t)\ \ \forall\ \ \beta \in [0, \alpha)$. For the unbounded case, a similar proof will validate $Q_\beta(\widetilde{\Omega}_t) \neq \mathscr{Q}_\beta(\widetilde{\Omega}_t)\ \ \forall\ \ \beta \in [\alpha, 1)$.

Setp 1. Construction of function u. For the curve γ_t based on $[0,1]$, for every $m \geq 1$ there are 4^m triangles, each of which has two sides belonging to γ_t and are of length t^m. We denote by these triangles as $\triangle_{m,j}$ for $j = 1, \cdots, 4^m$ and $m \geq 1$. Inside each $\triangle_{m,j}$, there is the largest cube $I_{m,j}$ and its radius is comparable to t^m. Set $u_{m,j}$ be a smooth function satisfying

$$\begin{cases} \operatorname{supp} u_{m,j} \subset 4^{-1} I_{m,j}; \\ u_{m,j} = 1 \ \text{ on } \ 8^{-1} I_{m,j}; \\ |\nabla u_{m,j}| \approx t^{-m} \ \text{ on } \ 4^{-1} I_{m,j} \setminus 8^{-1} I_{m,j}. \end{cases}$$

Then

$$\|u_{m,j}\|_{Q_\alpha(\Omega)} \lesssim \|\nabla u_{m,j}\|_{L^2(\Omega)} \lesssim 1.$$

Define

$$u = \sum_{m=1}^{\infty}\sum_{j=1}^{4^m} u_{m,j}.$$

Setp 2. $u \notin \mathscr{Q}_\beta(\Omega)$ *when* $1 > \beta \geq \alpha$.

Let $I = [-1, 2] \times [-2, 1]$. Then

$$2 - 2\alpha = \frac{\log_2 4}{\log_2 1/t} \Rightarrow 4t^{(2-2\beta)} \geq 4t^{2-2\alpha} = 4 \cdot 2^{2(\log_2 t)/\log_2(1/t)} = 1.$$

Consequently,

$$\begin{aligned}\int_{I\cap\Omega_t}\int_{I\cap\Omega_t} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\,dx\,dy &\geq \sum_{k=1}^{\infty}\sum_{j=1}^{4^k}\int_{I_{kj}}\int_{I_{kj}} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\,dx\,dy \\ &\geq \sum_{k=1}^{\infty}\sum_{j=1}^{4^k}\int_{\frac{1}{8}I_{kj}}\int_{I_{kj}\setminus 4^{-1}I_{kj}} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\beta}}\,dx\,dy \\ &\geq \sum_{k=1}^{\infty}\sum_{j=1}^{4^k} t^{4k}t^{-(2+2\beta)k} \\ &\geq \sum_{k=1}^{\infty} 4^k t^{(2-2\beta)k} \\ &= \infty,\end{aligned}$$

Setp 3. $u \in Q_\beta(\Omega_t)$ *for all* $\beta \in [0, 1)$. Notice that for each cube $I \subset \Omega_t$, there are at most three $I'_{mj}s$ such that $I \cap I_{mj} \neq \emptyset$. Moreover, the side lengths of these I_{mj} are comparable. Now, it is easy to achieve $\|u\|_{Q_\alpha(\Omega_t)} < \infty$ and then conclude the entire argument. □

4. Demonstration of Theorem 1.3.

4.1. Proof of Theorem 1.3(i). It follows easily from Theorem 1.1 and Theorems 3.1-3.2-3.3.

4.2. Proof of Theorem 1.3(ii). Recall that a domain Ω is linearly locally connected (l.l.c.) provided that there exists a constant $b \in (0, 1]$ such that: for any point $(z, r) \in \mathbb{R}^n \times (0, \infty)$,

- l.l.c.(1) – points in $\Omega \cap B(z, r)$ can be joined by a rectifiable curve in $\Omega \cap B(z, r/b)$;
- l.l.c.(2) – points in $\Omega \setminus B(z, r)$ can be joined by a rectifiable curve in $\Omega \setminus B(z, br)$.

The concept of l.l.c.-domains can be employed to characterize a uniform domain in the following sense (cf. [4]).

Lemma 4.1. *Let* $\Omega \subset \mathbb{R}^n$ *be simply connected under* $n = 2$ *or quasiconformally equivalent to a uniform domain under* $n \geq 3$. *Then* Ω *is a uniform domain if and only if* Ω *is l.l.c..*

With the aid of Lemma 4.1, Theorem 3.1(ii) follows readily from the next result.

THEOREM 4.1. *Let* $\alpha \in [0,1)$. *If* $Q_\alpha|_\Omega = Q_\alpha(\Omega)$, *then* Ω *is l.l.c..*

In order to demonstrate Theorem 4.1, we need two lemmas. The first is related to capacity. Given a pair of disjoint sets $E,\, F \subset \Omega$, define the Q_α-capacity as

$$\mathrm{Cap}_{Q_\alpha}(E,\, F,\, \Omega) \equiv \inf \Big\{ \|u\|^2_{Q_\alpha(\Omega)} :\ u \in \Delta(E,\, F,\, \Omega) \Big\},$$

where $\Delta(E,\, F,\, \Omega)$ denotes the class of all continuous functions $u \in Q_\alpha(\Omega)$ such that $0 \le u \le 1$ on Ω, $u = 0$ on E, and $u = 1$ on F. Obviously, if $\widetilde{E},\, \widetilde{F} \subset \mathbb{R}^n$ are disjoint sets, $E \subset \widetilde{E}$ and $F \subset \widetilde{F}$, then

$$\mathrm{Cap}_{Q_\alpha}(E,\, F,\, \Omega) \le \mathrm{Cap}_{Q_\alpha}(\widetilde{E},\, \widetilde{F},\, \mathbb{R}^n).$$

LEMMA 4.2. *For* $\alpha \in [0,1)$ *let* $\Omega \subset \mathbb{R}^n$ *be a* Q_α*-extension domain.*

(i) *There exists a positive constant* C *such that for every pair of disjoint sets* $E,\, F \subset \Omega$,

$$\mathrm{Cap}_{Q_\alpha}(E,\, F,\, \Omega) \le \mathrm{Cap}_{Q_\alpha}(E,\, F,\, \mathbb{R}^n) \le C\mathrm{Cap}_{Q_\alpha}(E,\, F,\, \Omega).$$

(ii) *For any* $\delta > 0$ *there exists a positive constant* C *depending only on* n *and* δ *such that for every pair of disjoint sets* $E,\, F \subset \Omega \cap B(x_0,\, r)$ *for some* $r > 0$ *one has*

$$\min\{vol(E),\, vol(F)\} \ge \delta r^n \Rightarrow \mathrm{Cap}_{Q_\alpha}(E,\, F,\, \Omega) \ge C.$$

Proof. (i) follows readily from the definition of a Q_α-extension domain. To check (ii), without loss of generality we may assume that $\Omega = \mathbb{R}^n$. Let $u \in \Delta(E, F, \mathbb{R}^n)$. Assume first that $u_{B(x_0,\, r)} \ge \frac{1}{2}$. Then, by the fractional Poincaré inequality,

$$\begin{aligned}\frac{\mathrm{vol}(E)}{2\mathrm{vol}\big(I(x_0, r)\big)} &\le \fint_{I(x_0,\, r)} |u(x) - u_{I(x_0,\, r)}|\, dx \\ &\lesssim \left(r^{2\alpha - n} \int_{I(x_0,\, r)} \int_{I(x_0, r)} \frac{|u(x) - u(y)|^2}{|x-y|^{n-2\alpha}}\, dxdy \right)^{\frac{1}{2}},\end{aligned}$$

and the desired estimate follows. If $u_{B(x_0,\, r)} < \frac{1}{2}$, we obtain our estimate by replacing E in the above string of inequalities with F. □

The second is about volume.

LEMMA 4.3. *Let* $\alpha \in [0,1)$, K *be a closed subset of* $\mathbb{R}^n$ *and* Ω *be a* Q_α*-extension domain in* $\mathbb{R}^n$. *Then there exists a positive constant* C *such that for every component* Ω_0 *of* $\Omega \setminus K$ *and for all* $x \in \Omega_0$ *and* $0 < r \le \min\{\,\mathrm{dist}\,(x,\, \partial\Omega_0 \setminus \partial\Omega),\ \mathrm{diam}\,\Omega\}$ *one has*

$$vol\big(\Omega_0 \cap I(x,\, r)\big) \ge Cr^n.$$

Proof. Without loss of generality, we may assume that $r < \infty$. Let $b_0 = 1$ and $b_j \in (0,1)$ for $j \in \mathbb{N}$ be such that

$$\mathrm{vol}\big(I(x,\, b_j r) \cap \Omega_0\big) = 2^{-1}\mathrm{vol}\big(I(x,\, b_{j-1} r) \cap \Omega_0\big) = 2^{-j}\mathrm{vol}\big(I(x,\, r) \cap \Omega_0\big). \qquad (4.1)$$

Define a function u on Ω by setting

$$u(y) \equiv \begin{cases} 1 & \text{as } \ y \in I(x,\, b_2 r) \cap \Omega_0, \\ \dfrac{b_1 r - |y-x|}{b_1 r - b_2 r} & \text{as } \ y \in (I(x,\, b_1 r) \setminus I(x,\, b_2 r)) \cap \Omega_0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $I(x,\, r) \cap \Omega_0 \cap K = \emptyset$ and $I(x,\, r)$ has empty intersection with other components of $\Omega \setminus K$, we know that u is well defined. Then

$$\|u\|_{W^{1,n}(\Omega)} \lesssim \frac{\mathrm{vol}\big(I(x,\, b_1 r) \setminus I(x,\, b_2 r)) \cap \Omega_0\big)^{\frac{1}{n}}}{b_1 r - b_2 r} \lesssim \frac{\mathrm{vol}\big(I(x,\, r) \cap \Omega_0\big)^{\frac{1}{n}}}{b_1 r - b_2 r}.$$

Since Ω is a Q_α-extension domain, we have

$$\|u\|_{\mathscr{Q}_\alpha(\Omega)} \lesssim \|u\|_{Q_\alpha(\mathbb{R}^n)} \lesssim \|u\|_{Q_\alpha(\Omega)} \lesssim \|u\|_{W^{1,n}(\Omega)} \lesssim \frac{\mathrm{vol}\big(I(x,\, r) \cap \Omega_0\big)^{\frac{1}{n}}}{b_1 r - b_2 r}.$$

By Theorem 1.1 and (4.1) we have

$$\inf_{c \in \mathbb{R}} \int_{I(x,\, r) \cap \Omega_0} \exp\left(C \frac{|u(y) - c|(b_1 r - b_2 r)}{\mathrm{vol}\big(I(x,\, b_1 r) \cap \Omega_0\big)^{\frac{1}{n}}} \right) dy \lesssim r^n.$$

Observe that $c \in \mathbb{R}$ ensures $|u - c| \geq \frac{1}{2}$ either on $(I(x,\, r) \setminus I(x,\, b_1 r)) \cap \Omega_0$ or on $I(x,\, b_2 r) \cap \Omega_0$. By (4.1) we conclude

$$\mathrm{vol}(I(x,\, b_1 r) \cap \Omega_0) \exp\left(\frac{C(b_1 r - b_2 r)}{\mathrm{vol}\big(I(x,\, b_1 r) \cap \Omega_0\big)^{\frac{1}{n}}} \right) \lesssim r^n,$$

which implies

$$b_1 r - b_2 r \leq \mathrm{vol}(I(x,\, b_1 r) \cap \Omega_0)^{\frac{1}{n}} \log_2 \left(\frac{C r^n}{\mathrm{vol}\big(I(x,\, b_1 r) \cap \Omega_0\big)} \right).$$

Similar inequalities also hold for $b_j r - b_{j+1} r$ with $j \geq 2$. This leads to

$$\begin{aligned} b_1 r &= \sum_{j \in \mathbb{N}} (b_j r - b_{j+1} r) \\ &\lesssim \sum_{j \in \mathbb{N}} \mathrm{vol}\big(I(x,\, b_j r) \cap \Omega_0\big)^{\frac{1}{n}} \log_2 \left(\frac{C (b_{j-1} r)^n}{\mathrm{vol}\big(B(z,\, b_j r) \cap \Omega_0\big)} \right) \\ &\lesssim \sum_{j \in \mathbb{N}} 2^{-j/n} \mathrm{vol}\big(I(x,\, r) \cap \Omega_0\big)^{\frac{1}{n}} \log_2 \left(2^j \frac{C r^n}{\mathrm{vol}\big(I(x,\, r) \cap \Omega_0\big)} \right) \\ &\lesssim \mathrm{vol}\big(I(x,\, r) \cap \Omega_0\big)^{\frac{1}{n}} \log_2 \left(\frac{C r^n}{\mathrm{vol}\big(I(x,\, r) \cap \Omega_0\big)} \right). \end{aligned}$$

If $b_1 \geq 1/10$, then observing

$$t \log_2 t^{-1} \geq 1 \Longrightarrow t \gtrsim 1,$$

we have

$$\mathrm{vol}\big(I(x,\, r)\cap\Omega_0\big)^{\frac{1}{n}} \gtrsim r.$$

If $b_1 \le 1/10$, then choosing $R = 2r/5$ and a point $y \in I(x,\, r)\cap\Omega_0$ such that $|y-x| = b_1 r + R/2$ we obtain

$$I(x,\, b_1 r) \subset B(y,\, R) \subset I(x,\, r) \ \text{ but } \ B(y,\, R/2)\cap I(x,\, b_1 r) = \emptyset.$$

Therefore, if

$$\mathrm{vol}\big(I(y,\, \widetilde{b}_1 R)\cap\Omega_0\big) = 2^{-1}\mathrm{vol}\big(I(y,\, R)\cap\Omega_0\big),$$

then by

$$\mathrm{vol}\big(I(x,\, b_1 r)\cap\Omega_0\big) \ge 2^{-1}\mathrm{vol}\big(I(y,\, R)\cap\Omega_0\big),$$

we have $\widetilde{b}_1 \ge \frac{1}{2}$. Applying the result when $b_1 \ge 1/10$, we conclude

$$\mathrm{vol}\big(I(y,\, R)\cap\Omega_0\big) \gtrsim R^n,$$

thereby reaching

$$\mathrm{vol}\big(I(x,\, r)\cap\Omega_0\big) \gtrsim r^n.$$

□

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. With the help of Lemmas 4.2-4.3, Theorem 4.1 can be proved by using the idea from [4, 25]. Here we give the details for the reader's convenience.

Case: l.l.c.(2). Let $x_1,\, x_2 \in \Omega\setminus B(x_0,\, r)$ for some $x_0\in\Omega$ and $r>0$. Without loss of generality, we may assume that $x_1, x_2 \in \partial B(x_0,\, r)\cap\Omega$. Indeed, we can find a rectifiable curve $\gamma\subset\Omega$ joining x_1 and x_2. Let $\widetilde{x}_i = \gamma(t_i)$ for $i = 1,\, 2$ be the points in $\partial B(x_0,\, r)$ such that $\gamma_1 = \gamma([0,\, t_1])$ and $\gamma_2 = \gamma([t_2, 1])$ intersect with $\partial B(x_0,\, r)$ at $\widetilde{x}_1$ and $\widetilde{x}_2$ respectively. If we can find a rectifiable $\gamma_3 \subset \Omega\setminus B(x_0,\, cr)$ for some $c\in(0,1)$ joining $\widetilde{x}_1$ and $\widetilde{x}_2$, then the rectifiable curve $\gamma_1\cup\gamma_3\cup\gamma_2 \subset \Omega\setminus B(x_0,\, cr)$, joins $x_1,\, x_2$.

Assume that x_1 and x_2 are not in the same component of $\Omega\setminus B(x_0,\, br)$ but in the same component of $\Omega\setminus B(x_0,\, br/2)$ for some $b\in(0,1)$. If $b\ge 1/16$, then we can find a curve such that $\gamma\subset\Omega\setminus B(x_0,\, r/32)$ joining $x_1,\, x_2$. So we may assume that $b<1/16$. It then suffices to prove that b is bounded away from 0 uniformly for any $x_1,\, x_2,\, x_0,\, r$. Denote by Ω_i the component of $\Omega\setminus\overline{B(x_0,\, br)}$ containing x_i for $i = 1, 2$. Then $\Omega_1\cap\Omega_2 = \emptyset$.

Let $F_i = \Omega_i \cap B(x_i,\, r/2)$ for $i = 1,\, 2$. Obviously, we have

$$\begin{cases} F_1,\, F_2 \subset B(x_0,\, 2r);\\ F_1\cap F_2 = \emptyset;\\ d(x_i, \partial\Omega_i\setminus\partial\Omega) \ge r - br > r/2;\\ F_i\cap B(x_0, r/2) = \emptyset. \end{cases}$$

By Lemma 4.3 we find $|F_i| \gtrsim r^n$ for $i = 1,\, 2$. Applying Lemma 4.2, we obtain

$$\mathrm{Cap}_{Q_\alpha}(F_1,\, F_2,\, \Omega) \gtrsim 1.$$

The lower bound of b will follow from the upper bound

$$\mathrm{Cap}_{Q_\alpha}(F_1,\, F_2,\, \Omega) \lesssim \left(\log_2 \frac{1}{2b}\right)^{\frac{1-n}{n}}, \tag{4.2}$$

which actually implies $\left(\log_2 \frac{1}{b}\right)^{\frac{1}{n}-1} \gtrsim 1$ and so $b \gtrsim 1$.

In order to check (4.2), define

$$u(x) \equiv \begin{cases} 1 & \text{as } \; x \in \Omega_2 \setminus B(x_0,\, r/2); \\ \left(\log_2 \frac{1}{2b_0}\right)^{-1} \log_2 \frac{|x-x_0|}{b_0 r} & \text{as } \; x \in \Omega_2 \cap (B(x_0,\, r/2) \setminus B(x_0,\, b_0 r)); \\ 0 & \text{otherwise.} \end{cases}$$

Obviously this function is well defined. Then

$$|\nabla u| = \left(\log_2 \frac{1}{2b_0}\right)^{-1} |x-x_0|^{-1} \chi_{\Omega_2 \cap (B(x_0,\, r/2) \setminus B(x_0,\, b_0 r))}$$

and hence $u \in Q_\alpha(\Omega)$ with

$$\begin{aligned} \|u\|_{Q_\alpha(\Omega)} &\le \|u\|_{W^{1,\,n}(\Omega)} \\ &\lesssim \left(\log_2 \frac{1}{2b_0}\right)^{-1} \left\{ \int_{B(x_0,\, r/2) \setminus B(x_0,\, b_0 r)} |z-x_0|^{-n}\, dz \right\}^{\frac{1}{n}} \\ &\lesssim \left(\log_2 \frac{1}{2b_0}\right)^{\frac{1-n}{n}}. \end{aligned}$$

Obviously, $u \in \Delta(F_1,\, F_2,\, \Omega)$, and hence (4.2) holds.

Case: l.l.c.(1). Let $x_1,\, x_2$ be an pair of points in Ω and $x_1, x_2 \in B(z,\, r)$ for some $z \in \mathbb{R}^n$ and $r > 0$. Then $r > |x_1 - x_2|/2$. We are about to show the existence of a rectifiable curve $\gamma \subset B(z,\, r/b)$ joining x_1 and x_2 for some constant $b < 1$.

If $|x_1 - x_2| \le \max\{d(x_1, \Omega^{\complement}),\, d(x_2, \Omega^{\complement})\}$, then the line segment joining x_1 and x_2 gives the desired curve. In the sequel we assume that $|x_1 - x_2| > \max\{d(x_1, \Omega^{\complement}),\, d(x_2, \Omega^{\complement})\}$.

Let Ω_i be the connected component of

$$\Omega \cap B(x_i,\, |x_1 - x_2|/8) = \Omega \setminus B(x_i,\, |x_1 - x_2|/8)^{\complement}$$

containing x_i for $i = 1,\, 2$. By Theorem 3.2, we have

$$\mathrm{vol}(\Omega_i) \gtrsim |x_1 - x_2|^n \quad \text{for} \quad i = 1,\, 2.$$

Obviously, $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega_1,\, \Omega_2 \subset B(x_1,\, 2|x_1 - x_2|)$. Applying Lemma 3.5, we obtain

$$\mathrm{Cap}_{Q_\alpha}(\Omega_1,\, \Omega_2,\, \Omega) \gtrsim 1.$$

Now we claim that there exists a positive constant $N_0 \ge 2$ independent of $x_1,\, x_2,\, \Omega_1$ and Ω_2 such that $\Omega_1,\, \Omega_2$ are in the same component of $\Omega \cap B(x_1,\, N_0|x_1 - x_2|)$. To see this, assume that Ω_1, Ω_2 are not in the same component of $\Omega \cap$

$B(x_1, N|x_1-x_2|)$ for some $N > 2$, say, in components $\widetilde{\Omega}_1$, $\widetilde{\Omega}_2$ of $\Omega\cap B(x_1, N|x_1-x_2|)$ respectively. Define a function u on Ω by setting

$$u(y) \equiv \begin{cases} 1 & \text{as } \ y \in B(x_1, |x_1-x_2|/8) \cap \widetilde{\Omega}_1; \\ \frac{1}{\log_2 8N} \log_2 \frac{N|x_1-x_2|}{|y-x_1|} & \text{as } \ y \in (B(x_1, N|x_1-x_2|) \setminus B(x_1, |x_1-x_2|/8)) \cap \widetilde{\Omega}_1; \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\|u\|_{Q_\alpha(\Omega)} \lesssim \|u\|_{W^{1,n}(\Omega)} \lesssim (\log_2 N)^{\frac{1-n}{n}} .$$

Upon observing

$$u = \begin{cases} 1 \ \text{ on } \ \Omega_1; \\ 0 \ \text{ on } \ \Omega_2, \end{cases}$$

we have

$$\mathrm{Cap}_{Q_\alpha}(E, F, \Omega) \lesssim (\log_2 N)^{\frac{1-n}{n}}$$

which means $N \lesssim 1$ and hence shows the existence of N_0 as claimed.

As a consequence, there exists a rectifiable curve $\gamma_0 \subset \Omega \cap B(x_1, N_0|x_1-x_2|)$ joining Ω_1 and Ω_2. Let $\widetilde{x}_i \in \Omega_i \cap \gamma_0$ for $i = 1, 2$. Since $x_i, \widetilde{x}_i \in \Omega_i$ and Ω_i is connected, we can find a rectifiable curve $\gamma_i \subset \Omega_i \subset \Omega\cap B(x_1, N_0|x_1-x_2|)$ connecting x_i and $\widetilde{x}_i$ for $i = 1, 2$. Set $\gamma = \gamma_0 \cup \gamma_1 \cup \gamma_2$. Then γ joins x_1 and x_2 and $\gamma \subset \Omega \cap B(x_1, N_1|x_1-x_2|)$. Noticing

$$B(x_1, N_0|x_1-x_2|) \subset B(z, N_0|x_1-x_2|+|z-x_1|) \subset B(z, (N_0+1)|x_1-x_2|) \subset B(z, 2(N_0+1)r),$$

we have $\gamma \subset \Omega \cap B(z, r/b)$ with $b = \frac{1}{2}(N_0 + 1)$, as desired. □

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