

PROOF OF A GROMOV CONJECTURE ON THE INFINITESIMAL INVERTIBILITY OF THE METRIC-INDUCING OPERATORS*

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Abstract. We prove a conjecture of Gromov about non-free isometric immersions.

Key words. Isometric Immersions, Free Maps, h-Principle; Undetermined Linear Partial Differential Operators.

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1. Introduction, main result and notations. Let M be a n -dimensional manifold, $\text{Imm}^1(M, \mathbb{R}^q)$ the set of C^1 immersions of M into \mathbb{R}^q , $\mathcal{G}^0(M)$ the bundle of C^0 Riemannian metrics over M and e_q the Euclidean metric on \mathbb{R}^q . Recall that a C^2 map f is *free* when, with respect to any coordinate system on M and any frame on \mathbb{R}^q , the $n \times q$ matrix D^2f of the first and second partial derivatives of f has rank $n + s_n$, $s_n = n(n + 1)/2$, at every point. More generally, we call *2-rank* of f at x the rank of the matrix D^2f at x – e.g. a free map has 2-rank $n + s_n$ at every point.

It is well known (e.g. see [Gro86], Sec. 2.3.1) that the metric-inducing operator $\mathcal{D}_{M,q} : \text{Imm}^1(M, \mathbb{R}^q) \rightarrow \mathcal{G}^0(M)$, defined by $\mathcal{D}_{M,q}(f) = f^*e_q$, is an open map, in the Whitney strong topology (throughout the present article we will always use this topology for our functional spaces), over the (open) set of smooth free maps $\text{Free}^\infty(M, \mathbb{R}^q)$. For example this means that, if $f_0 \in \text{Free}^\infty(M, \mathbb{R}^q)$ and $g_0 = \mathcal{D}_{M,q}(f_0)$ then, for every $g \in \mathcal{G}^0(M)$ close enough to g_0 , there is a map f close enough to f_0 such that $g = \mathcal{D}_{M,q}(f)$. In other words, the solvability of the Partial Differential Equation (PDE)

$$\mathcal{D}_{M,q}(f) = g \tag{1}$$

is stable with respect to small smooth perturbations over the (open) subset of the metrics induced by smooth free maps.

In a recent review article ([Gro17], p. 42) Gromov formulated the following conjecture:

CONJECTURE. *If $q > s_n$, the operator $\mathcal{D}_{M,q}$ is open over an open dense subset of $C^\infty(M, \mathbb{R}^q)$.*

In [Gro86], Sec. 2.3.8 (E), Gromov suggested to prove his conjecture when $q \geq n + s_n - \sqrt{n/2}$ by using the same line of argument of his proof that a generic underdetermined linear Partial Differential Operator (PDO) admits a right inverse. In [DL07], G. D’Ambra and A. Loi made a first step in this direction by showing that $\mathcal{D}_{\mathbb{R}^2,4}$ is open over a non-empty subset of the (open) subset of maps whose 2-rank is full at every point by using ideas from the Lie systems results in [Gro86], Sec. 2.3.8 (C). In [DeL10] we extended this result to $\mathcal{D}_{\mathbb{R}^n, n+s_n-1}$ as a consequence of a general theorem by Duistermaat and Hormander. Here we go back to Gromov’s suggestion and we prove the following result (see Theorems 7 and 8 and Corollary 2 in Sec. 6 for more precise and slightly stronger related statements):

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THEOREM. *If $q \geq n + s_n - \sqrt{n/2} + 1/2$, the operator $\mathcal{D}_{M,q}$ is open over an open dense subset of the set of C^∞ maps $M \rightarrow \mathbb{R}^q$ of full 2-rank.*

Our proof follows closely the Gromov proof on the right invertibility of generic undetermined linear PDOs in Section 2.3.8 (E) of [Gro86], so we will present in full detail only the parts of the proof that we needed to modify or that play some role in our construction.

Throughout the article we will use the following notations for indices: $i = 1, \dots, q$; $a = 1, \dots, m$; $\alpha, \beta = 1, \dots, n$. Whenever there will be no ambiguity, we will use the Einstein summation convention over repeated indices. Our base manifold will be always denoted by M , its points by x and its dimension by n .

2. The compatibility PDO \mathcal{L}_f . In local coordinates (x^α) on M and y^i on \mathbb{R}^q , Equation (1) writes as the following second order quadratic PDE of s_n equations in the q variables f^i :

$$\delta_{ij} \partial_\alpha f^i \partial_\beta f^j = g_{\alpha\beta}. \tag{2}$$

By considering a 1-parametric smooth family of metrics $(g_t)_{\alpha\beta}$ and a corresponding smooth family of solutions f_t^i , by differentiating with respect to the parameter t we obtain the following *linear* first order PDE of s_n equations in the q variables δf^i :

$$2\delta_{ij} \partial_\alpha f_0^i \partial_\beta \delta f^j = (\delta g_0)_{\alpha\beta}, \tag{3}$$

where we used the standard notation $\delta = d/dt|_{t=0}$. The δf^j are components of a map $\delta f \in C^1(M, \mathbb{R}^q)$ that can be thought as a vector tangent to the curve $\{f_t\} \subset C^1(M, \mathbb{R}^q)$ at f_0 .

DEFINITION 1. *Let $\Gamma^0(S_2^0(M))$ be the set of C^0 sections of the tensor bundle of symmetric $(0, 2)$ tensors over M . The PDO*

$$T_{f_0} \mathcal{D}_{M,q} : T_{f_0} C^1(M, \mathbb{R}^q) \simeq C^1(M, \mathbb{R}^q) \rightarrow T_{g_0} \mathcal{G}^0(M) \simeq \Gamma^0(S_2^0(M))$$

defined by

$$T_{f_0} \mathcal{D}_{M,q}(\delta f) = 2\delta_{ij} \partial_\alpha f_0^i \partial_\beta \delta f^j$$

is the tangent map (linearization) of $\mathcal{D}_{M,q}$ at f_0 . We say that $\mathcal{D}_{M,q}$ is infinitesimally invertible over $\mathcal{A} \subset C^1(M, \mathbb{R}^q)$ if there exists a family \mathcal{E} of linear PDOs $\mathcal{E}_f : \Gamma^s(S_2^0 M) \rightarrow T_f C^0(M, \mathbb{R}^q)$, $f \in \mathcal{A}$, of some order s such that:

- 1. there is an integer $d \geq 1$, called defect of \mathcal{E} , such that $\mathcal{A} = \mathcal{A}^d \subset C^d(M, \mathbb{R}^q)$ and \mathcal{A} is defined by some open differential relation;*
- 2. the map $\mathcal{E} : \mathcal{A}^d \times \Gamma^s(S_2^0 M) \rightarrow TC^0(M, \mathbb{R}^q)$ defined by $\mathcal{E}(f, \eta) \rightarrow \mathcal{E}_f(\eta)$ is a PDO of order d in the first variable and order s in the second;*
- 3. $T_f \mathcal{D}_{M,q}(\mathcal{E}_f(\delta g)) = \delta g$ for all $f \in \mathcal{A}^d \cap C^{d+1}(M, \mathbb{R}^q)$ and $\delta g \in \Gamma^{s+1}(S_2^0 M)$.*

Among the fundamental results of Nash in his celebrated article [Nas56] is that $\mathcal{D}_{M,q}$ admits an infinitesimal inverse of order 0 (namely *algebraic*) and defect 2 over the set of free maps. This fact is an immediate consequence of the product rule for derivatives: indeed, since $\partial_\beta (f^i \delta f^j) = \delta f^j \partial_\beta f^i + f^i \partial_\beta \delta f^j$, the PDE system (3) is equivalent to the following *algebraic* system of $n + s_n$ equation in the q unknowns δf^i :

$$\begin{cases} \delta_{ij} \partial_\alpha f_0^i \delta f^j = h_\alpha, \\ 2\delta_{ij} \partial_{\alpha\beta}^2 f_0^i \delta f^j = \partial_\alpha h_\beta + \partial_\beta h_\alpha - (\delta g_0)_{\alpha\beta}, \end{cases} \tag{4}$$

where the h_α are n auxiliary functions. When f_0 is a free map, by definition the matrix of its first and second derivative is surjective at every point $x \in M$ and so system (4) is solvable *independently on the choice of the h_α* (which, therefore, are usually all set equal to zero).

The following Implicit Function Theorem in infinite dimension, a powerful Gromov’s generalization of the celebrated work of Nash on isometric immersions, allows us to focus, rather than on the original quadratic PDE system (2), on the (much simpler) linear system (4):

THEOREM 1 (Gromov, 1986). *Let $F \rightarrow E$ be a fiber bundle, $G \rightarrow E$ a vector bundle and $\Gamma^r F$ and $\Gamma^s G$, respectively, the sets of their C^r and C^s sections. If a PDO of order r , $\mathcal{D}_r : \Gamma^r F \rightarrow \Gamma^0 G$, admits an infinitesimal inversion of order s and defect d over $\mathcal{A}^d \subset \Gamma^d F$, then for every $f_0 \in \mathcal{A}^d \cap \Gamma^\infty F$ there exists a neighborhood of zero $\mathcal{U} \subset \Gamma^{\bar{s}+s+1} G$, $\bar{s} = \max\{d, 2r + s\}$, such that, for every $g \in \mathcal{U} \cap \Gamma^{\sigma+s} G$, $\sigma \geq \bar{s} + 1$, the equation $\mathcal{D}(f) = \mathcal{D}(f_0) + g$ has a C^σ solution.*

COROLLARY 1 (Gromov, 1986). *Under the hypotheses of the theorem above, the restriction of \mathcal{D}_r to $\mathcal{A}^\infty = \mathcal{A}^d \cap \Gamma^\infty E$ is an open map.*

In case of the metric-inducing operator $\mathcal{D}_{M,q}$, this implies the following celebrated results of Nash:

THEOREM 2 (Nash, 1956). *If $f_0 \in Free^\infty(M, \mathbb{R}^q)$ and $g_0 = \mathcal{D}_{M,q}(f_0)$, then the C^σ metric $g_0 + g$ can be induced by a C^σ immersion of M into \mathbb{R}^q for all g that are “ C^σ -small enough” and every $\sigma \geq 3$.*

Clearly this result is void when $Free^\infty(M, \mathbb{R}^q)$ is empty, in particular for $q < n + s_n$. Nevertheless, in the interval $n + s_n > q > s_n$, although no free map can arise, there can still be maps whose 2-rank is full at every point.

EXAMPLE 1. *Recall that the maps $f_n : \mathbb{R}^n \rightarrow \mathbb{R}^{n+s_n}$ defined by*

$$f_n(x^1, \dots, x^n) = (x^1, \dots, x^n, (x^1)^2, x^1 x^2, \dots, (x^n)^2)$$

are free for all $n = 1, 2, \dots$. The maps $f_{n,m} : \mathbb{R}^n \rightarrow \mathbb{R}^{n+s_n-m}$, $m = 1, \dots, n$, obtained by composing f_n with any projection $\pi_{n,m} : \mathbb{R}^{n+s_n} \rightarrow \mathbb{R}^{n+s_n-m}$ that “drops” any m of the components of index larger than n are all maps of full 2-rank.

EXAMPLE 2. *The maps $f_n : \mathbb{T}^n \rightarrow \mathbb{R}^{2n}$ defined by*

$$f_n(x^1, \dots, x^n) = (\sin x^1, \cos x^1, \dots, \sin x^n, \cos x^n)$$

are all full 2-rank maps, with $m = s_n - n$. In particular there are full 2-rank maps from \mathbb{T}^2 to \mathbb{R}^4 . We recall that, on the contrary, it is still unknown whether there exist free maps from \mathbb{T}^2 to \mathbb{R}^5 (see [Gro17], Sec. 2.2).

Now let us set $m = n + s_n - q$ and let f_0 be a full 2-rank map $M \rightarrow \mathbb{R}^q$. Then, among the $n + s_n$ vectors $(\partial_\alpha f_0, \partial_{\alpha\beta}^2 f_0)$, $\alpha \leq \beta$, at every point $x \in M$ hold exactly m non-trivial linear relations

$$\lambda_a^\alpha(f_0, x) \partial_\alpha f_0(x) + \sum_{\alpha \leq \beta} \lambda_a^{\alpha\beta}(f_0, x) \partial_{\alpha\beta}^2 f_0(x) = 0, \quad a = 1, \dots, m,$$

where the $m(n + s_n)$ coefficients $(\lambda_a^\alpha(f_0), \lambda_a^{\alpha\beta}(f_0))$, $\alpha \leq \beta$, can be chosen as polynomial in the components of $(\partial_\alpha f_0, \partial_{\alpha\beta}^2 f_0)$.

A solution of system (4) therefore exists if and only if the same relation holds on the right hand side of the system, namely if the PDE

$$\lambda_a^\alpha(f_0) h_\alpha + \frac{1}{2} \sum_{\alpha \leq \beta} \lambda_a^{\alpha\beta}(f_0) (\partial_\beta h_\alpha + \partial_\alpha h_\beta) = \frac{1}{2} \sum_{\alpha \leq \beta} \lambda_a^{\alpha\beta}(f_0) (\delta g_0)_{\alpha\beta}. \tag{5}$$

in the n unknowns h_α does admit a solution. Therefore, a sufficient condition for the solvability of system (4) is the surjectivity of the *compatibility 1-st order linear PDO* $\mathcal{L}_f : C^1(M, \mathbb{R}^n) \rightarrow C^0(M, \mathbb{R}^m)$ defined by

$$\mathcal{L}_f(h_1, \dots, h_n) = \lambda_a^\alpha(f) h_\alpha + \hat{\lambda}_a^{\alpha\beta}(f) \partial_\beta h_\alpha,$$

where $\hat{\lambda}_a^{\alpha\alpha} = \lambda_a^{\alpha\alpha}$ and $\hat{\lambda}_a^{\alpha\beta} = \hat{\lambda}_a^{\beta\alpha} = \frac{1}{2} \lambda_a^{\alpha\beta}$ for $\alpha < \beta$.

In the next sections we study general conditions that ensure the surjectivity of this class of PDOs.

3. Surjectivity of generic underdetermined PDOs. In [Gro86], Sec. 2.3.8, Gromov generalizes Nash’s technique of purely algebraic inversion of the linear PDO $T_f \mathcal{D}_{M, \mathbb{R}^q}$ to any underdeterminate linear PDO. The price of this generalization is that the order of the inverse is usually far from being zero.

In order to state the main results, we need first to introduce a series of definitions and notations. Let $F \rightarrow E$ and $G \rightarrow E$ be two vector bundles over E with fiber dimension equal to q and q' respectively. We denote by $\Gamma^r F$ the set of its C^r sections and by $J^r F$ the affine bundle of the r -jets of its sections, and similarly for G .

DEFINITION 2. *A linear PDO of order r on F with values in G is given by either of these equivalent definitions:*

1. a linear map $\mathcal{L}_r : \Gamma^r F \rightarrow \Gamma^0 G$ given, in coordinates, by

$$\mathcal{L}_r(f) = \left(\sum_{|A| \leq r} \Lambda_i^{aA} \partial_A f^i \right) = (\Lambda_i^a f^i + \Lambda_i^{a\alpha} \partial_\alpha f^i + \dots + \Lambda_i^{a\alpha_1 \dots \alpha_r} \partial_{\alpha_1 \dots \alpha_r} f^i)$$

2. a smooth section Λ_r of the bundle $\text{Hom}(J^r F, G) \rightarrow E$ of all linear homomorphisms between $J^r F$ and G , namely

$$\Lambda_r(x) = (x, \Lambda_i^a(x), \dots, \Lambda_i^{a\alpha_1 \dots \alpha_r}(x))$$

The relation between the two definitions is that $(j^r f)^* \Lambda_r = \mathcal{L}_r(f)$ for every $f \in \Gamma^r F$.

EXAMPLE 3. *The most natural linear PDOs over a manifold M are associated to vector fields $\xi \in \Gamma^\infty(TM)$, i.e. smooth sections of the tangent bundle $TM \xrightarrow{\tau_M} M$. We denote the corresponding first-order linear homogeneous PDOs by $L_\xi : C^1(M) \rightarrow C^0(M)$ (Lie derivative in the ξ direction), namely here $E = M$ and $F = G = M \times \mathbb{R}$. In coordinates $L_\xi = \xi^\alpha \partial_\alpha$ and the corresponding section $\Lambda_\xi : M \rightarrow \text{Hom}(J^1(M, \mathbb{R}), M)$ is defined as $\Lambda_\xi(x^\alpha) = (x^\alpha, 0, \xi^\alpha)$, namely $(\Lambda_\xi)_1^{1\alpha} = 0$, $(\Lambda_\xi)_1^{1\alpha} = \xi^\alpha$. The corresponding PDE $L_\xi(f) = g$ is called cohomological equation, whose solvability has been recently studied in several contexts (see [For99, Nov08, DeL15]).*

REMARK 1. *It was shown in [DeL11] that, in case of vector fields ξ with no zeros on \mathbb{R}^2 , the PDO L_ξ is surjective if and only if ξ is conjugate to a constant vector*

field (recall that on the 2-torus even this condition is not enough for the surjectivity, e.g. see[For95]). On the contrary, in [Gro86], Sec 2.3.8 (C), Gromov has shown that, for any manifold M , the operator $C^1(M, \mathbb{R}^2) \rightarrow C^0(M)$ that maps $(f_1, f_2) \mapsto L_{\xi_1}(f_1) + L_{\xi_2}(f_2)$ is surjective for any pair of vector fields $\xi_1, \xi_2 \in \Gamma^\infty(TM)$ in general position. In other words, topology affects the solvability of linear first order PDEs only in the equal dimension case $q = q'$

Now consider an open subset $\mathcal{H} \subset J^{r+s}(Hom(J^r F, G))$ for some $s \geq 0$.

DEFINITION 3. Let \mathcal{L}_r be a linear PDO of order r such that $\Lambda_r(E) \subset \mathcal{H}$. A \mathcal{H} -universal right inverse for \mathcal{L}_r is a PDO $\mathcal{M}_s : \Gamma^\infty \mathcal{H} \times \Gamma^\infty G \rightarrow \Gamma^\infty F$ of order $s + r$ in the first component and linear of order s in the second such that

$$\mathcal{L}_r(\mathcal{M}_s(\mathcal{L}_r, g)) = g, \text{ for all } g \in \Gamma^\infty G.$$

EXAMPLE 4. Let $\xi_1, \dots, \xi_q \in \Gamma^\infty(TM)$, $f \in C^1(M, \mathbb{R}^q)$, $q > n$, and define the linear PDO $\Xi : C^1(M, \mathbb{R}^q) \rightarrow C^0(M)$ as $\Xi(f) = L_{\xi_1}(f^1) + \dots + L_{\xi_q}(f^q)$, where $L_\xi(f)$ is the Lie derivative of f with respect to ξ . We say that Ξ is large if the $q \times (n + 1)$ matrix

$$\begin{pmatrix} \xi_1^1 & \dots & \xi_1^n & -\partial_\alpha \xi_1^\alpha \\ \vdots & \vdots & \vdots & \vdots \\ \xi_q^1 & \dots & \xi_q^n & -\partial_\alpha \xi_q^\alpha \end{pmatrix} \tag{6}$$

has full rank at every point in any (and so every) coordinate system. Now, let \mathcal{H} be the open subbundle of $J^1(Hom(J^1(M, \mathbb{R}^q), J^0(M, \mathbb{R})))$ spanned by the 1-jets of sections of large PDOs on $C^1(M, \mathbb{R}^q)$ and let $(\lambda^1(\Xi), \dots, \lambda^q(\Xi))$ be the solution of the system $\lambda^i \xi_i^\alpha = 0, \lambda^i \partial_\alpha \xi_i^\alpha = -1$ closest to the origin. Then the PDO $\mathcal{M}_0 : \Gamma^\infty \mathcal{H} \times C^\infty(M) \rightarrow C^\infty(M, \mathbb{R}^q)$ defined by $\mathcal{M}_0(\Xi, g) = (\lambda^i(\Xi)g)$ is a \mathcal{H} -universal right inverse for Ξ of order 0 since it depends only algebraically on g and $\Xi(\lambda^i(\Xi)g) = \xi_i(\lambda^i(\Xi)g) = \xi_i^\alpha \partial_\alpha \lambda^i(\Xi)g + \lambda^i(\Xi) \xi_i^\alpha \partial_\alpha g = g$.

EXAMPLE 5. The technique of Nash to solve the PDE system (3) through the algebraic system (4) shows that the linearization of the isometric operator $T_f \mathcal{D}_{M,q}$ admits a 0-order inverse $\mathcal{M}_0 : \Gamma^\infty \mathcal{H} \times \Gamma^\infty(S_2^0 M) \rightarrow C^\infty(M, \mathbb{R}^q)$, where \mathcal{H} is the subbundle of $J^1(Hom(J^1(M, \mathbb{R}^q), J^0(M, \mathbb{R})))$ spanned by the 1-jets of the operators $T_f \mathcal{D}_{M,q}$ with f free. In this case, indeed, the matrix of first and second partial derivatives of f is surjective and the value of $\mathcal{M}_0(f, \delta g)$ can be defined, for example, as the solution of (4) closer to the origin. Note that, when f is free, the h_α play no role and we can set all of them to zero.

DEFINITION 4. The formal adjoint of a linear PDO $\mathcal{L}_r : \Gamma^r F \rightarrow \Gamma^0 G$ of order r is the linear PDO $\overline{\mathcal{L}}_r : \Gamma^r G \rightarrow \Gamma^0 F$ of order r defined by

$$\overline{\mathcal{L}}_r(g) = \left(\sum_{|A| \leq r} \overline{\Lambda}_a^{iA} \partial_A g^a \right) \stackrel{\text{def}}{=} \left(\sum_{|A| \leq r} (-1)^{|A|} \partial_A \left(\left(\overline{\Lambda}_a^i \right)^A g^a \right) \right),$$

where, for every multi-index A , the matrix $\left(\overline{\Lambda}_a^i \right)^A$ is the transpose of the matrix (Λ_i^{aA}) .

EXAMPLE 6. *The formal adjoint of the linear first-order PDO $\Xi(f^1, \dots, f^q) = \xi_i^\alpha \partial_\alpha f^i$ in Example 4 is given by $\bar{\Xi}(g) = -\partial_\alpha (\xi_i^\alpha g) = -(\partial_\alpha \xi_i^\alpha)g - \xi_i^\alpha \partial_\alpha g$.*

Note that the adjunction itself is a PDO $J^r(\text{Hom}(J^r F, G)) \rightarrow \text{Hom}(J^r G, F)$ that satisfies $\overline{\mathcal{L}_r} = \mathcal{L}_r$ and $\overline{\mathcal{L}_r \mathcal{M}_s} = \overline{\mathcal{M}_s} \overline{\mathcal{L}_r}$. Gromov’s crucial observation, generalizing Nash’s method of solving (3) through (4), is that, while the problem of finding an \mathcal{H} -universal right inverse $\mathcal{M}_s : \Gamma^\infty \mathcal{H} \times \Gamma^\infty G \rightarrow \Gamma^\infty F$ for $\mathcal{L}_r : \Gamma^r F \rightarrow \Gamma^0 G$ requires studying the solvability of a (usually non-trivial) linear PDE of order r in the components of \mathcal{M}_s , namely

$$\sum_{|A| \leq r} \Lambda_i^{aA} \partial_A \left(\sum_{|B| \leq s} M_b^{iB} \partial_B \right) = \delta_b^a, \tag{7}$$

the equivalent problem of finding a *left* \mathcal{H} -universal inverse $\overline{\mathcal{M}_s} : \Gamma^\infty \mathcal{H} \times \Gamma^\infty F \rightarrow \Gamma^\infty G$ for the adjoint PDO $\mathcal{L}_r : \Gamma^r G \rightarrow \Gamma^0 F$, namely

$$\sum_{|B| \leq s} \overline{M}_i^{aB} \partial_B \left(\sum_{|A| \leq r} \overline{\Lambda}_b^{iA} \partial_A \right) = \delta_b^a, \tag{8}$$

involves just solving a purely *algebraic* linear system in the s -jets of $\overline{\Lambda}_r$, namely in the $(s + r)$ -jets of Λ_r , and therefore it can be solved by purely combinatorial and transversality arguments. Following this idea, in [Gro86] Gromov was able to prove the following:

THEOREM 3 (Gromov, 1986). *If $q > q'$, a generic linear PDO $\mathcal{L}_r : \Gamma^r F \rightarrow \Gamma^0 G$ is surjective.*

4. Surjectivity of Upper Totally Symmetric PDOs. As it often happens for very general theorems, the proof of Theorem 3 can be used, with minor changes, to prove several particular cases not covered by the general statement. In this section we adapt Gromov’s proof to a class of PDOs that plays a crucial role in the proof of our main theorem.

DEFINITION 5. *We call a linear PDO $\mathcal{L}_r : C^r(M, \mathbb{R}^{m+1}) \rightarrow C^0(M, \mathbb{R}^m)$, with*

$$\mathcal{L}_r(h_1, \dots, h_{m+1}) = (\Lambda_a^\alpha h_\alpha + \Lambda_a^{\alpha\beta_1} \partial_{\beta_1} h_\alpha + \dots + \Lambda_a^{\alpha\beta_1 \dots \beta_r} \partial_{\beta_1 \dots \beta_r} h_\alpha),$$

upper totally symmetric (UTS) if its highest order components $\Lambda_a^{\alpha\beta_1 \dots \beta_r}$ are symmetric with respect to all permutations of the indices $\alpha\beta_1 \dots \beta_r$ that keep the first one not larger than $m + 1$. We denote by $\mathcal{U}_m^r(M)$ the subbundle of $\text{Hom}(J^r(M, \mathbb{R}^{m+1}), J^0(M, \mathbb{R}^m))$ whose fibers are spanned, at every $x \in M$, by the images of germs of UTS PDOs about x .

Note that, in a UTS PDO, the dimension of the set of terms of highest order decreases from $m(m + 1) \binom{n}{r}$ to $m \sum_{i=0}^m \binom{n-i}{r}$.

We first show, with a purely combinatorial argument, that \mathcal{L}_r does admit a *formal* universal right inverse:

LEMMA 1. *There exists an integer $s_{n,m}$ such that, for all $s \geq s_{n,m}$, there is an open dense algebraic subbundle $\mathcal{H}_m^{r,s}(M) \subset J^{r+s} \mathcal{U}_m^r(M)$ with the property that every UTS PDO $\Lambda_r : M \rightarrow \text{Hom}(J^r(M, \mathbb{R}^{m+1}), J^0(M, \mathbb{R}^m))$ satisfying $j^{r+s} \Lambda_r(M) \subset \mathcal{H}_m^{r,s}(M)$ admits a $\mathcal{H}_m^{r,s}(M)$ -universal right inverse.*

Proof. As mentioned in the previous section, finding a right inverse for \mathcal{L}_r is equivalent to solving the algebraic system (8). In every fiber of $J^s \mathcal{U}_m^r(M)$, system (8) writes as

$$\left\{ \begin{array}{l} \sum_{|A| \leq s} \overline{M}_\alpha^{aA} \overline{\Lambda}_{bA}^{-\alpha} = \delta_b^a \\ \sum_{|A| \leq s} \overline{M}_\alpha^{aA} \overline{\Lambda}_{bA}^{-\alpha\beta_1} + \sum_{|A| \leq s-1} \overline{M}_\alpha^{a\beta_1 A} \overline{\Lambda}_{bA}^{-\alpha} = 0 \\ \sum_{|A| \leq s} \overline{M}_\alpha^{aA} \overline{\Lambda}_{bA}^{-\alpha\beta_1\beta_2} + \sum_{|A| \leq s-1} \overline{M}_\alpha^{a\beta_1 A} \overline{\Lambda}_{bA}^{-\alpha\beta_2} + \sum_{|A| \leq s-2} \overline{M}_\alpha^{a\beta_1\beta_2 A} \overline{\Lambda}_{bA}^{-\alpha} = 0 \\ \vdots \\ \sum_{|A| \leq 1} \overline{M}_\alpha^{a\beta_1 \dots \beta_{s-1} A} \overline{\Lambda}_{bA}^{-\alpha\beta_s \dots \beta_{s+r-1}} + \overline{M}_\alpha^{a\beta_1 \dots \beta_s} \overline{\Lambda}_b^{-\alpha\beta_{s+1} \dots \beta_{s+r-1}} = 0 \\ \overline{M}_\alpha^{a\beta_1 \dots \beta_s} \overline{\Lambda}_b^{-\alpha\beta_{s+1} \dots \beta_{s+r}} = 0 \end{array} \right. \tag{9}$$

where $1 \leq a, b \leq m$, $1 \leq \alpha \leq m + 1$ and $1 \leq \beta_\ell \leq n$ for all $1 \leq \ell \leq s + r$.

Note that (9) naturally splits in m independent systems, one for each value of a , of $(m + 1) \binom{n+s}{s}$ unknowns \overline{M}_α^{aA} in $m \binom{n+r+s}{r+s}$ equations. All rows but one in each of these independent systems are homogeneous and so each of them admits a formal algebraic solution $\overline{M}_\alpha^{aA}(\overline{\Lambda}_{bA}^{-\alpha B})$ if and only if the non-homogeneous row is not a linear combination of the homogeneous ones with coefficient in the ring of rational functions in the fiber coordinates. Since the system is *triangular*, in the sense that in the column of each \overline{M}_α^{aA} only one coefficient $\overline{\Lambda}_{bA}^{-\alpha}$ of zero order appears (and it does it in the non-homogeneous row), we can recursively substitute the expressions of such coefficients in the other columns so that in the end no $\overline{\Lambda}_{bA}^{-\alpha}$ appears anywhere except in the non-homogeneous row. When that row is a linear combination of the homogeneous ones, therefore, we can express the coefficients $\overline{\Lambda}_{bA}^{-\alpha}$ as rational functions of all other coefficients $\overline{\Lambda}_{bA}^{-\alpha B}$, $|B| \geq 1$, and of a number of rational functions of all $\overline{\Lambda}_{bA}^{-\alpha B}$ (the coefficients of the linear combination) equal to the number of homogeneous rows. Since the $(m + 1) \binom{n+s}{s}$ coefficients are independent coordinates, such combination clearly cannot exist if we choose s big enough that $(m + 1) \binom{n+s}{s} > m \binom{n+r+s}{r+s}$. Note that this is always possible since

$$(m + 1) \binom{n + s}{s} > m \binom{n + r + s}{r + s} \quad \text{iff} \quad \frac{m + 1}{m} > \prod_{\ell=1}^r \left(1 + \frac{n}{s + \ell} \right)$$

and the r.h.s. term converges to 1 for $s \rightarrow \infty$.

The \overline{M}_α^{aA} solving (9) are algebraic functions of the coordinates $\overline{\Lambda}_{bA}^{-\alpha B}$. Let P be the polynomial obtained as the product of all denominators of such functions and define $\mathcal{H} = \{P \neq 0\}$. Then $\mathcal{H}_m^{r,s}(M)$ is an algebraic subbundle of $J^{r+s} \mathcal{U}_m^r(M)$ satisfying the claim of the theorem. \square

Of course such inverse remains purely formal until we are able to prove the existence of global sections Λ_r whose $(r + s)$ -jets images are contained in $\mathcal{H}_m^{r,s}(M)$. Following Gromov, we prove this by showing that the codimension of $\mathcal{H}_m^{r,s}(M)$ grows with s large enough that the image of the $(r + s)$ -jets of *generic* sections must be entirely contained into it.

DEFINITION 6. A linear PDO $\mathcal{L}_r : \Gamma^r F \rightarrow \Gamma^0 G$, $\mathcal{L}_r(f) = \left(\sum_{|A| \leq r} \Lambda_i^{\alpha A} \partial_A f^i\right)$, is transversal to a codimension- k submanifold $E_0 \subset E$ at $x_0 \in E_0$, locally defined by $H(x) = 0$ for some smooth map $H : E \rightarrow \mathbb{R}^k$, if the k matrices $q \times q'$

$$T^\ell(x_0) = (\Lambda_i^{\alpha \alpha_1 \dots \alpha_r}(x_0) \partial_{\alpha_1} H^\ell(x_0) \dots \partial_{\alpha_r} H^\ell(x_0))$$

have all maximal rank, i.e. $\text{rk } T^\ell(x_0) = q'$ for all $\ell = 1, \dots, k$. If \mathcal{L}_r is not transversal to E_0 , we say that it is tangent to it. We say that E_0 is a characteristic submanifold for \mathcal{L}_r if \mathcal{L}_r is tangent to E_0 at every point.

EXAMPLE 7. The linear 1-st order PDO Ξ in Example 4 is transversal to the hypersurface $E_0 = \{H(x) = 0\}$ at $x = x_0$ if $L_{\xi_i} H(x_0) \neq 0$ for at least one index $1 \leq i \leq q$, i.e. if at least one vector field ξ_i is transversal to E_0 at x_0 in the usual sense (namely that $\xi_i(x_0)$ is not contained in $T_{x_0} E_0$). On the contrary, E_0 is characteristic for Ξ if the characteristics (i.e. the integral trajectories) of all vector fields ξ_i are entirely contained in E_0 .

EXAMPLE 8. Consider a (pseudo-)Riemannian constant metric $g = g_{\alpha\beta} dx^\alpha dx^\beta$ on \mathbb{R}^n , so that the Laplacian PDO is given by $\Delta_g(f) = g^{\alpha\beta} \partial_{\alpha\beta}^2 f$, where $(g^{\alpha\beta})$ is the inverse matrix of $(g_{\alpha\beta})$. Then Δ_g is transversal to the hypersurface $E_0 = \{H(x) = 0\}$ at $x = x_0$ iff $g(dH, dH) = 0$ is zero at $x = x_0$. For example, if g is Riemannian then it has no characteristic submanifold of codimension 1 while, if its signature is $(1, n - 1)$, its only characteristic codimension-1 submanifolds are its light-cones.

LEMMA 2. Generic UTS PDOs have no characteristic submanifold of positive codimension.

Proof. Assume that $E_0 = \{H(x) = 0\}$, $H : E \rightarrow \mathbb{R}^k$, is a codimension- k characteristic submanifold for $\mathcal{L}_r : C^r(M, \mathbb{R}^{m+1}) \rightarrow C^0(M, \mathbb{R}^m)$. Then E_0 is contained in the set $C_k = \{\text{rk } T^\ell(x) < m, \ell = 1, \dots, k\}$. The number of such functionally independent relations can be evaluated by considering the case $H^\ell(x^1, \dots, x^n) = x^\ell$, $1 \leq \ell \leq k$. When $r > 1$, the $m \times (m + 1)$ matrices

$$T^\ell(x) = (\Lambda_a^{\alpha \overbrace{\ell \dots \ell}^r}(x))$$

have no element in common for all ℓ and so $\text{codim } C_k = 2k$, which is incompatible with the facts that $\text{codim } E_0 = k$ and $E_0 \subset C_k$. When $r = 1$ and $k > m + 1 \geq 2$ then the number of independent relations is $2k - 1$, so we get again an absurd, while for $k \leq m + 1$ we have again $\text{codim } C_k = 2k$. \square

The proof of the following proposition is the same as the one of Lemma 4 in Section 2.3.8 (E) in [Gro86].

THEOREM 4. There exists an integer $\bar{s} = s(n, m, r) \geq \frac{n}{(1+1/m)^{1/r-1}}$ such that, for every $s \geq \bar{s}$, there is a maximal algebraic subbundle $\mathcal{H}_m^{r,s}(M) \subset J^{r+s} \mathcal{U}_m^r(M)$ whose complement has codimension larger than n and with the property that every UTS PDO $\Lambda_r : M \rightarrow \text{Hom}(J^r(M, \mathbb{R}^{m+1}), J^0(M, \mathbb{R}^m))$ satisfying $j^{r+s} \Lambda_r(M) \subset \mathcal{H}_m^{r,s}(M)$ admits a $\mathcal{H}_m^{r,s}(M)$ -universal right inverse.

As an immediate corollary we get the surjectivity of generic UTS PDOs:

THEOREM 5. *A generic upper totally symmetric PDO $\mathcal{L}_r : C^r(M, \mathbb{R}^{m+1}) \rightarrow C^0(M, \mathbb{R}^m)$ is surjective.*

REMARK 2. *Note that in no part of the proofs of the Theorems and Lemmas of this section that lead to Theorem 5 were ever explicitly used the components of the terms of intermediate degree $0 < r' < r$. Hence all properties proved in this section hold, mutatis mutandis, for PDOs that are sections of any subbundle of $\mathcal{U}_m^r(M)$ defined by any number of closed relations among such components (the same is true, indeed, also for Gromov's Theorem 3).*

5. Surjectivity of the compatibility operators. Now we go back, in a slightly more general setting, to the compatibility operators we mentioned in Section 1 and provide condition for their surjectivity.

DEFINITION 7. *We call r -rank of a map $f : M \rightarrow \mathbb{R}^q$ at $x \in M$ the rank of the matrix $D_x^r f$ of the partial derivatives of f at x up to order r . A map is r -free if $\text{rk } D_x^r f = s_{n,r} \stackrel{\text{def}}{=} \binom{n+r}{r} - 1$ for every $x \in M$, namely if the vectors of all its partial derivatives up to order r are linearly independent at every point. Finally, we say that a r -free map $f : M \rightarrow \mathbb{R}^q$ has full $r+1$ -rank when $s_{n,r} < q \leq s_{n,r+1}$ and $\text{rk } D_x^{r+1} f = q$ for all $x \in M$, namely when, at every x , its $s_{n,r}$ vectors of its partial derivatives up to order r are mutually linearly independent and exactly $q - s_{n,r}$ of the $s_{n,r+1} - s_{n,r}$ vectors of its derivatives of order $r+1$ are linearly independent from them.*

Note that clearly r -free maps are also r' -free for all $r' \leq r$ and can only arise when $q \geq s_{n,r}$.

EXAMPLE 9. *A 1-free map is an immersion, a 2-free map is a standard free map. A full 2-rank map $f : M \rightarrow \mathbb{R}^q$, $n < q \leq n + s_n$ is an immersion such that the $n + s_n$ vectors $\{f_\alpha(x), f_{\alpha\beta}(x)\}$ span at every point the whole \mathbb{R}^q . A full $(r+1)$ -rank map with $q = s_{n,r+1}$ is a $(r+1)$ -free map.*

REMARK 3. *Except for the case $r = 1$, this definition is not invariant with respect to the whole group $\text{Diff}(\mathbb{R}^m)$ but only with respect to its subgroup of affine transformation. Geometrically, this corresponds to the fact that the only tensor bundle among the $J^r(M, \mathbb{R}^m) \rightarrow J^{r-1}(M, \mathbb{R}^m)$ is the one with $r = 1$, while all others are affine. These definitions, though, can be made invariant if we replace \mathbb{R}^m with a Riemannian manifold and partial derivatives with the corresponding covariant derivatives.*

We are interested in the following natural map of full $(r+1)$ -rank maps into PDOs of order r . Given a full $(r+1)$ -rank map $f : M \rightarrow \mathbb{R}^q$, between the $s_{n,r+1}$ vectors of its partial derivatives up to order $r+1$ hold at every point $m = s_{n,r+1} - q$ linear relations

$$\sum_{|A| \leq r+1} \lambda_a^A(f) \partial_A f^i = 0, \quad a = 1, \dots, m$$

where the $m \cdot s_{n,r+1}$ coefficients $\lambda_a^A(f) \in C^0(M)$, defined modulo a factor $\pm e^p$, $p \in C^0(M)$, are not all zero at the same time at any point.

DEFINITION 8. *We call compatibility operator associated to the full $(r+1)$ -rank map $f : M \rightarrow \mathbb{R}^q$ the PDO of order r $\mathcal{L}_f : C^r(M, \mathbb{R}^n) \rightarrow C^0(M, \mathbb{R}^m)$ defined as*

$$\mathcal{L}_f(h) = \sum_{|A| \leq r} \lambda_a^{\alpha A}(f) \partial_A h_\alpha.$$

The elementary linear algebra lemma below shows that the $\lambda_a^A(f)$ can all be chosen as homogeneous fully antisymmetric polynomials of degree q .

DEFINITION 9. Given a set of N vectors $v = \{v_1, \dots, v_N\} \subset \mathbb{R}^{N-m}$, we denote by V_ℓ , $\ell = 1, \dots, m$, the $(N - m + 1) \times (N - m)$ matrix whose $N - m + 1$ columns are the components of the vectors $\{v_1, \dots, v_{N-m}, v_{N-m+\ell}\}$, by $V_{\ell,i}$, $i = 1, \dots, N - m + 1$ the $(N - m) \times (N - m)$ matrix obtained from V_ℓ by removing the i -th column and by V_0 the $(N - m) \times (N - m)$ matrix whose columns are the components of the first $N - m$ vectors.

LEMMA 3. Let $\{v_1, \dots, v_N\}$ be a set of N vectors in \mathbb{R}^{N-m} spanning the whole vector space and ordered so that the first $N - m$ are linearly independent. Let λ_a^i be non-trivial coefficients expressing the linear dependency conditions between the v_i , namely $\lambda_a^i v_i = 0$ for $a = 1, \dots, m$. Then, for every a , these coefficients can be chosen as: $\lambda_a^i = (-1)^i \det V_{a,i}$ for $i = 1, \dots, N - m$; $\lambda_a^{N-m+a} = (-1)^{N-m+1} \det V_0$; $\lambda_a^i = 0$ otherwise.

In order to ensure the existence of an inverse for \mathcal{L}_f we need to ensure the differential independence of its components. Clearly such independence cannot take place when there are more components $\lambda_a^{\alpha A}(f)$ than independent variables f^i .

EXAMPLE 10. Consider a map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ of full 1-rank (i.e. an immersion), so that the three vectors f_x, f_y, f_z satisfy at every point a non-trivial linear dependence condition $\lambda f_x + \mu f_y + \nu f_z = 0$, where

$$\lambda = \det \begin{pmatrix} \partial_y f^1 & \partial_z f^1 \\ \partial_y f^2 & \partial_z f^2 \end{pmatrix}, \mu = \det \begin{pmatrix} \partial_x f^1 & \partial_z f^1 \\ \partial_x f^2 & \partial_z f^2 \end{pmatrix}, \nu = \det \begin{pmatrix} \partial_x f^1 & \partial_y f^1 \\ \partial_x f^2 & \partial_y f^2 \end{pmatrix}.$$

The three functions λ, μ, ν are mutually functionally independent, in the sense that there exists no $F \in C^0(\mathbb{R}^3)$ such that $F(\lambda, \mu, \nu) = 0$ identically on any open set, but they are not differentially independent since $\partial_x \lambda + \partial_y \mu + \partial_z \nu = 0$.

LEMMA 4. Let $f : M \rightarrow \mathbb{R}^q$, $s_{n,r} < q < s_{n,r+1}$, be a generic map of full $(r + 1)$ -rank and set the $\lambda_a^{\alpha A}(f)$ as in Lemma 3. Then, for any $s = 0, 1, \dots$, any q distinct non-zero functions $\lambda_a^{\alpha A}(f)$ and their derivatives up to order s are functionally independent.

Proof. To prove the claim of the lemma we need to consider the $j^s \lambda_a^{\alpha A}(f)$ as functions on $J^{s+r+1}(M, \mathbb{R}^q)$ and show that they are independent¹ for a generic map f .

First of all consider a single coefficient $\lambda_a^{\alpha A}(f)$ and let $\partial_{A'} f^{i_0}$ be one of the terms in it with derivative of highest order (it can be either $|A'| = r$ or $|A'| = r + 1$). Note that i_0 can be chosen as any integer between 1 and q since for every derivative order all components of f with the same order enter in the expression of $\lambda_a^{\alpha A}(f)$. Then $\partial_{A'B} f^{i_0}$ is one of the derivatives of highest order appearing in $\lambda_a^{\alpha A}(f)$ and does not appear in any other term $\lambda_a^{\alpha A}(f)$ with $|B'| \leq |B|$. This means that, in the expansion of $\bigwedge_{|B| \leq s} d\lambda_a^{\alpha A}(f)$, it appears a single term proportional to $\bigwedge_{|B| \leq s} df_{A'B}^{i_0}$. The coefficient of this term is the product of some number of $f_A^{i_0}$ coordinates and therefore it is not identically zero on any open set, i.e. there is no differential relation between any $\lambda_a^{\alpha A}(f)$ and its derivatives.

¹Recall that some set of functions on a manifold is functionally independent if and only if their wedge product is not identically zero on any open set.

Now consider q distinct non-zero coefficients $\{\lambda_{a_1}^{\alpha_1 A_1}(f), \dots, \lambda_{a_q}^{\alpha_q A_q}(f)\}$ and their derivatives up to order B . Then, for each $i = 1, \dots, q$, there is a multi-index A'_i such that $\bigwedge_{|B| \leq s} d\lambda_{a_i B}^{\alpha_i A_i}(f)$ contains a non-zero term proportional to $\bigwedge_{|B| \leq s} df_{A'_i B}^i$. Moreover, since the $\lambda_{a_i}^{\alpha_i A_i}(f)$ are determinants of matrices that differ by at least one column and whose columns are partial derivatives of f , each $\bigwedge_{|B| \leq s} df_{A'_i B}^i$ does not appear in $\lambda_{a_{i'}}^{\alpha_{i'} A_{i'}}(f)$ for $i' \neq i$. Hence $\bigwedge_{i=1, \dots, q} \left(\bigwedge_{|B| \leq s} d\lambda_{a_i B}^{\alpha_i A_i}(f) \right)$ is not zero either because it contains $\bigwedge_{i=1, \dots, q} \bigwedge_{|B| \leq s} df_{A'_i B}^i \neq 0$. \square

THEOREM 6. *Let $f : M \rightarrow \mathbb{R}^q$ be a generic map of full $r+1$ -rank, $m = s_{n,r+1} - q$ and assume that $q \geq m \sum_{i=0}^m \binom{n-i}{r}$ and $m \leq (n-1)/(r+1)$. Then the associated order- r compatibility PDO \mathcal{L}_f is surjective.*

Proof. Since f is of full $r+1$ -rank, at every point of M are linearly independent all vectors $\partial_A f$, $|A| \leq r$ and $s_{n,r+1} - s_{n,r} - m$ of the $\partial_A f$, $|A| = r+1$, namely at every point hold m linear dependence relations $\sum_{|A| \leq r+1} \lambda_a^A \partial_A f$. Moreover, since $n \geq m+1+rm$, in some neighborhood of any point we can always rename coordinates so that the components $\lambda_{a_1 \dots a_{r+1}}(f)$ equal to zero have $\alpha_\ell > m+1$ for all $\ell = 1, \dots, r+1$.

At this point, if we restrict the corresponding compatibility PDO $\mathcal{L}_f : C^r(M, \mathbb{R}^n) \rightarrow C^0(M, \mathbb{R}^m)$ to the linear subspace $h_{m+2} = \dots = h_n = 0$, we get a UTS PDO (that we will keep calling \mathcal{L}_f to keep the notation light) of special kind, namely such that the UTS condition is true for its components of every order, and whose $m \sum_{i=0}^m \binom{n-i}{r}$ components of order r are all non-trivial. As explained in Remark 2, the statements of Theorem 4 and 5 also hold for this more special kind of operators and so there is a $s = \hat{s}(n, m, r)$ and a maximal algebraic subbundle $\mathcal{H}_m^{r,s}(M)$ of $J^{r+s}(Hom(J^r(M, \mathbb{R}^q), J^0(M, \mathbb{R}^q)))$ such that the codimension of its complement is larger than n and \mathcal{L}_f is surjective if $j^{r+s} \Lambda_f(M) \subset \mathcal{H}_m^{r,s}(M)$. By hypothesis the components of \mathcal{L}_f are all independent and so for, a generic f , \mathcal{L}_f is surjective. \square

6. Full rank isometries. In the particular case of full 2-rank maps, Theorem 6 has the following important consequences about isometric immersions. Given $n = 1, 2, \dots$, denote by m_n the middle root of $n + s_n - m = m(m+1)(n+1) - ms_m$.

DEFINITION 10. *Let $s(n, m, r)$ and*

$$\mathcal{H}_m^{1,s}(M) \subset J^{1+s}(Hom(J^1(M, \mathbb{R}^q), J^0(M, \mathbb{R}^q))), \quad q = n + s_n - m,$$

be the function and the subbundle in the claim of Theorem 4. For every $s \geq s(n, m, 1)$, we denote by $F^{s+3}(M, \mathbb{R}^q) \subset C^{s+3}(M, \mathbb{R}^q)$ the open set of full 2-rank maps f such that $j^{s+1} \Lambda_f(M) \subset \mathcal{H}_m^{1,s}(M)$.

PROPOSITION 1. *The set $F^{s+3}(M, \mathbb{R}^q)$ is dense in the set of C^{s+3} full 2-rank maps for $q \geq n + s_n - m_n$.*

Proof. The condition $q \geq n + s_n - m_n$ is equivalent to $q \geq m \sum_{i=0}^m \binom{n-i}{1}$ and $m_n \leq (n-1)/2$ for all $n \geq 2$. Hence the proof of Theorem 6 shows that, under such hypotheses, a generic f will induce a compatibility UTS PDO \mathcal{L}_f such that $j^{1+s} \Lambda_f(M)$ will be entirely contained in $\mathcal{H}_m^{1,s}(M)$. \square

THEOREM 7. *For $q \geq n + s_n - m_n$, there is a $d = d(n, m) \geq nm+3$, $m = n + s_n - q$, such that the metric-inducing operator $\mathcal{D}_{M,q}$ admits an infinitesimal inversion of order 0 and defect d over $F^d(M, \mathbb{R}^q)$.*

Proof. By Proposition 1, \mathcal{L}_f is surjective for every $f \in F^d(M, \mathbb{R}^q)$. By Theorem 4, the order of its $\mathcal{H}_m^{1,s}(M)$ -universal right inverse \mathcal{M}_s is at least $\bar{s} = s(n, m, 1) \geq mn$. Since under such conditions \mathcal{L}_f is surjective, we can choose the h_α in system (4) so that the compatibility condition (5) is satisfied. In order to solve (4) this way, we need the full 2-rank map f to be at least of class $C^{\bar{s}+1+2}$, since we must be able to evaluate $j^{s+1}\Lambda_f(M)$ and the components of \mathcal{L}_f are polynomials in the components of j^2f . Hence the infinitesimal inverse operator of $\mathcal{D}_{M,q}$ has order 0 and defect $d = \bar{s} + 3 \geq nm + 3$ over $F^d(M, \mathbb{R}^q)$. \square

From this last result and Gromov’s Theorem 1 we get the following two results:

THEOREM 8. *Let $q \geq n + s_n - m_n$ and $d = d(n, m)$. If $f_0 \in F^d(M, \mathbb{R}^q) \cap C^\infty(M, \mathbb{R}^q)$ and $g_0 = \mathcal{D}_{M,q}f_0$, then there is a C^{d+1} -neighborhood of zero $U \subset \Gamma^{d+1}(S_2^0(M))$ such that, for all $g \in U \cap \Gamma^\sigma(S_2^0(M))$, $\sigma \geq d + 1$, the metric $g_0 + g$ can be induced by an immersion $f \in C^\sigma(M, \mathbb{R}^q)$.*

COROLLARY 2. *For $q \geq n + s_n - m_n$, the metric-inducing operator $\mathcal{D}_{M,q}$ is open over the set $F^\infty(M, \mathbb{R}^q) = F^d(M, \mathbb{R}^q) \cap C^\infty(M, \mathbb{R}^q)$, $d = d(m, n)$.*

REMARK 4. *Notice that the condition $q \geq n + s_n - \sqrt{n/2} - 1/2$ is weaker but almost equivalent to $q \geq n + s_n - m_n$ since $m_n \leq \sqrt{n/2} \leq m_n + 1/2$.*

In Examples 1 and 2 we pointed out that full-rank maps arise on Euclidean spaces for all allowed combinations of n , m and q and in some particular case on tori. We conclude the article by briefly discussing the existence of maps of full rank on general open manifolds (namely manifolds without compact components).

THEOREM 9 (Gromov, 1986). *Let E be an open smooth manifold, $F \rightarrow E$ be a smooth fibration over it and $\mathcal{R} \subset J^r F$ an open subset invariant by the action of $\text{Diff}(E)$. Denote by $\text{Hol}^r(\mathcal{R}) \subset \Gamma^0 \mathcal{R}$ the set of holonomic sections of $\mathcal{R} \rightarrow E$, namely sections $\phi = j^r f$ for some $f : E \rightarrow F$. Then \mathcal{R} satisfies the parametric h-principle, namely the map $j^r : \text{Hol}^r(\mathcal{R}) \rightarrow \Gamma^0 \mathcal{R}$ is a weak homotopy equivalence.*

REMARK 5. *We recall that, in particular, this means that j^r induces an isomorphism between the first homotopy groups $\pi_0(\text{Hol}^r(\mathcal{R}))$ and $\pi_0(\Gamma^0 \mathcal{R})$. The surjectivity expresses the h-principle for \mathcal{R} , namely that every section $E \rightarrow \mathcal{R}$ can be homotoped, in $\Gamma^0 \mathcal{R}$, to the r -jet of some section $f : E \rightarrow F$. The injectivity expresses the 1-parametric h-principle, namely that every two holonomic sections $j^r f_1, j^r f_2$ that are homotopic in $\Gamma^0 \mathcal{R}$ are also homotopic in $\text{Hol}^r(\mathcal{R})$. See [Gro86], Sec. 1.2.1 (C), and [EM02], Sec. 6.2, for more details about multi-parametric and other flavors of h-principles.*

Consider the subbundle $\mathcal{F}^r(M, \mathbb{R}^q) \subset J^r(M, \mathbb{R}^q)$, with $s_{n,r-1} < q < s_{n,r}$, whose fibers over each point x are the subset such that $(f_A^i), |A| \leq r$, seen as a $q \times s_{n,r}$ matrix, has rank q . Then $\text{Hol}^r(\mathcal{F}^r(M, \mathbb{R}^q)) = F^r(M, \mathbb{R}^q)$. Clearly $\mathcal{F}^r(M, \mathbb{R}^q)$ is open and invariant by all diffeomorphisms of the base, so that we can claim the following result:

COROLLARY 3. *Let M be an open manifold. Then $F^r(M, \mathbb{R}^q) \neq \emptyset$ if and only if there exist C^0 sections $M \rightarrow \mathcal{F}^r(M, \mathbb{R}^q)$. In particular full r -rank maps $M \rightarrow \mathbb{R}^q$ arise on any parallelizable open manifold M for all $s_{n,r-1} < q < s_{n,r}$.*

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