OUTERMOST APPARENT HORIZONS DIFFEOMORPHIC TO UNIT NORMAL BUNDLES[*]

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Abstract. Given a submanifold $S \subset \mathbb{R}^n$ of codimension at least three, we construct an asymptotically Euclidean Riemannian metric on $\mathbb{R}^n$ with nonnegative scalar curvature for which the outermost apparent horizon is diffeomorphic to the unit normal bundle of S.

Key words. asymptotically Euclidean manifolds, nonnegative scalar curvature, outermost apparent horizons.

Mathematics Subject Classification. Primary 53C21; Secondary 53A10, 53C80, 83C57.

1. Introduction. An asymptotically Euclidean manifold is a Riemannian manifold with an end on which the metric approaches the Euclidean metric. In such a manifold, an outermost apparent horizon is a bounding minimal hypersurface which encloses all other bounding minimal hypersurfaces. An asymptotically Euclidean manifold can be interpreted as a time symmetric slice of an asymptotically Minkowskian spacetime describing an isolated gravitational system. The dominant energy condition on the spacetime then means that the time symmetric slice has nonnegative scalar curvature. By the Hawking–Penrose singularity theorem the outermost apparent horizon must be located inside the event horizon in the spacetime. The outermost apparent horizon, which can be found with no further data than a time symmetric slice, may therefore serve as a substitute for the event horizon, which depends on the entire spacetime structure. The question which motivates the work in this paper is the following: Which smooth manifolds can be found as outermost apparent horizons in asymptotically Euclidean manifolds with nonnegative scalar curvature?

Outermost apparent horizons. We use the convention that the mean curvature of an oriented hypersurface in a Riemannian manifold is the trace of its second fundamental form, with positive sign if a variation in the direction of the oriented normal ν increases area. In other words, the k-dimensional sphere of radius r in $(k+1)$-dimensional Euclidean space has positive mean curvature k/r with respect to the outward direction.

For spacetime initial data sets, the concepts of weakly outer trapped surfaces, trapped regions, and outermost apparent horizons are defined using null expansions. In our setting of Riemannian manifolds these definitions reduce to the following. Let (M, g) be an n-dimensional connected Riemannian manifold with an asymptotically Euclidean end. A compact hypersurface in M which separates the asymptotically Euclidean end from the rest of the manifold is called *weakly outer trapped* if its mean curvature with respect to the normal directed towards the asymptotically Euclidean end is non-positive. The *trapped region* $\mathcal{T}$ is the union of compact sets with smooth boundary for which the boundary is weakly outer trapped. The *outermost apparent horizon* of (M, g) is the boundary $\partial \mathcal{T}$ of the trapped region. Note that the trapped region may be empty.

For asymptotically Euclidean manifolds with nonnegative scalar curvature and smooth outermost apparent horizon, the induced metric on the horizon is conformal to a metric with positive scalar curvature. The first result in this direction, when the manifold has dimension three, is Hawking's black hole topology theorem [13, Proposition 9.3.2]. The general result, in higher dimensions, can be found in work by Cai, Galloway and Schoen [3], [11], [10]. The proofs use the fact that the horizon is necessarily stable and outer area minimizing. Hence our motivating question may be refined: Is the existence of a positive scalar curvature metric sufficient for a compact, bounding manifold to be the outermost apparent horizon in an asymptotically Euclidean manifold of nonnegative scalar curvature? Note that there are bounding manifolds which do not admit positive scalar curvature metrics, for instance the n-dimensional torus. In the present paper we construct many new examples of outermost apparent horizons, but we are far from answering the question completely.

Obvious examples of outermost apparent horizons are spheres, which appear in constant time slices of the Schwarzschild spacetime. Chruściel and Mazzeo [5] construct asymptotically Euclidean metrics for which the outermost apparent horizon is diffeomorphic to a number of spheres. See also work by Corvino [6]. In the work [16], Schwartz has a construction of outermost apparent horizons diffeomorphic to a product of spheres. Unfortunately, there is a gap in Schwartz's argument where the horizon is implicitly assumed to be connected, and it does not seem to be possible to repair this gap with the methods of that paper. An inspiration when searching for horizons with nontrivial topology are examples of non-spherical black holes. The first such example is the construction by Emparan and Reall of a black ring spacetime [9]. Another important example is the "Black Saturn" found by Elvang and Figueras [8].

In this paper we will construct examples of outermost apparent horizons which are diffeomorphic to unit normal bundles of submanifolds of Euclidean space. Our result is inspired by Schwartz [16], Chruściel and Mazzeo [5], as well as by Carr [4, Theorem 1]. In [4], it is proved that boundaries of regular neighborhoods of embedded cell complexes of codimension at least three admit metrics of positive scalar curvature. For compact submanifolds, the boundary of a regular neighborhood is diffeomorphic to the unit normal bundle.

Statement of results. Let $S \subset \mathbb{R}^n$ be a smooth submanifold. We denote the unit normal bundle of S by UNS, so that $UN_x S$ is the sphere of unit vectors in the fiber $N_x S$ of the normal bundle NS.

The main theorem of this paper is the following.

THEOREM 1.1. *Let $n \geq 3$ and let $S \subset \mathbb{R}^n$ be a compact embedded smooth submanifold of dimension m. Assume that $n - m \geq 3$. For $\epsilon > 0$ and $x \in \mathbb{R}^n \setminus S$, let*

$$u_\epsilon(x) := 1 + \epsilon^{n-m-2} \int_S |x - y|^{-(n-2)} \, dy$$

and let g_ϵ be the Riemannian metric on $\mathbb{R}^n \setminus S$ defined by

$$g_\epsilon := u_\epsilon^{4/(n-2)} \delta$$

where δ is the Euclidean metric on $\mathbb{R}^n$. For sufficiently small $\epsilon > 0$ it holds that the outermost apparent horizon Σ_ϵ of $(\mathbb{R}^n \setminus S, g_\epsilon)$ is diffeomorphic to the unit normal bundle UNS. In fact, Σ_ϵ is the graph of a smooth function on UNS in normal coordinates for S.

The integrand in the definition of u_ϵ is the Green's function of the Euclidean Laplacian, and hence u_ϵ is harmonic and the scalar curvature of $(\mathbb{R}^n \setminus S, g_\epsilon)$ is zero (see [2, Theorem 1.159]). For large x we have

$$u_\epsilon(x) = 1 + \epsilon^{n-m-2} \mathcal{H}_m(S)|x|^{-(n-2)} + O(|x|^{-(n-1)}),$$

where $\mathcal{H}_m$ denotes the m-dimensional Hausdorff measure with respect to the Euclidean metric δ. Thus $(\mathbb{R}^n \setminus S, g_\epsilon)$ has an asymptotically Euclidean end.

A neighborhood of S is another end of the manifold $(\mathbb{R}^n \setminus S, g_\epsilon)$, and the function u_ϵ tends to infinity at S. The proof will show that the outermost apparent horizon Σ_ϵ encloses a tubular hypersurface of Euclidean radius proportional to ϵ around S. It is possible to modify the harmonic function u_ϵ inside this tubular hypersurface to give a smooth superharmonic function on all of $\mathbb{R}^n$, thus removing the end at S. The modification of u_ϵ can be done by appropriately cutting off the singularity at zero of the function $r \mapsto |r|^{-(n-2)}$ appearing in the integral in the definition of u_ϵ. This modification will not change the location of the outermost apparent horizon, and we get the same type of examples of horizons in asymptotically flat manifolds but with non-negative scalar curvature and no end near S. We conclude the following.

THEOREM 1.2. *Let $S \subset \mathbb{R}^n$ be a compact embedded smooth submanifold of codimension at least 3. Then there is an asymptotically Euclidean metric on $\mathbb{R}^n$ with non-negative scalar curvature for which the outermost apparent horizon is diffeomorphic to the unit normal bundle UNS. In fact, the horizon is the graph of a smooth function on UNS in normal coordinates for S.*

The basic idea of the proof of Theorem 1.1 is this: Since the conformal factor is close to one outside of a neighborhood of S which shrinks as ϵ tends to zero, the horizon (if it exists) is expected to approach S in this limit. Close to S, the difference between S and its tangent space should be negligible, so the horizon Σ_ϵ should be close to the one we would have if S were a linear subspace, in which case Σ_ϵ would be a cylinder. If the horizon is locally close to a cylinder around the tangent space of S, then it should be globally diffeomorphic to the unit normal bundle of S.

We now present some examples of outermost apparent horizons that can be found with Theorem 1.1.

- Horizons diffeomorphic to products of spheres: Let n and m be positive integers such that $n \geq m + 3$. Let $S = S^m$ be the unit sphere in $\mathbb{R}^{m+1} \subset \mathbb{R}^n$. Then Theorem 1.1 gives an outermost apparent horizon diffeomorphic to $S^m \times S^{n-m-1}$.
- Horizons can have many components: Let $n \geq 3$. Let S be a set of k points in $\mathbb{R}^n$. Then Theorem 1.1 gives an outermost apparent horizon Σ diffeomorphic to the disjoint union of k spheres of dimension $(n-1)$. In general, we can find examples of disconnected horizons using any choice of a disconnected submanifold S. Refining the proof of Theorem 1.1 one can probably allow the components of S to have different dimensions, which would allow examples like the "Black Saturn".
- Horizons can have any fundamental group: Let π be a finitely presented group. Using the generators and relations of π, it is not difficult to construct a compact hypersurface $S \subset \mathbb{R}^5$ such that $\pi_1(S) = \pi$. Embed $\mathbb{R}^5$ in $\mathbb{R}^7$. Theorem 1.1 gives us an apparent horizon which is diffeomorphic to the unit normal bundle of $S \subset \mathbb{R}^7$. The fiber of this bundle is S^2 so the long exact sequence for the homotopy groups of a fibration tells us that $\pi_1(UNS) = \pi_1(S) = \pi$.

This shows that every finitely presented group is the fundamental group of an outermost apparent horizon in an asymptotically flat 7-dimensional scalar flat Riemannian manifold.

Overview of the paper. In Section 2 we prove that the function u_ϵ near a point of S is close to the function u_∞ obtained by replacing S by its tangent space at the point. This is proved by explicit computations involving rescaled versions of the function u_ϵ.

In Section 3 we determine the mean curvature in g_ϵ of tubular hypersurfaces around S. We find constants C_{inner}, C_{outer} and R_{outer} such that the mean curvature is negative for tubular hypersurfaces with radius smaller than $C_{\text{inner}}\epsilon$ and positive for tubular hypersurfaces with radius between $C_{\text{outer}}\epsilon$ and R_{outer}.

In Section 4 we use these tubular hypersurfaces, the maximum principle of Solomon and White, and a convergence argument to determine the location of outer area minimizing stationary hypersurfaces. The conclusion is that any such hypersurface must be located between the tubular hypersurfaces of radii $C_{\text{inner}}\epsilon$ and $C_{\text{outer}}\epsilon$.

In Section 5 we apply a convergence argument to prove that the outer area minimizing stationary hypersurfaces are graphs of smooth functions on UNS in normal coordinates for S.

Finally, in Section 6, we combine the previous results and prove that there is a unique outer area minimizing stationary hypersurface. This hypersurface is then shown to coincide with the outermost apparent horizon, which proves Theorem 1.1.

2. Rescaled metrics and their convergence. The purpose of this section is to prove that, after suitable rescaling and change of coordinates, the metrics g_ϵ converge as ϵ tends to zero. This is done in Corollary 2.3. The computations needed are mainly contained in Lemma 2.1, where rescalings of the integral in the definition of u_ϵ are studied.

For $(x_\infty, x, \epsilon) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+$ define the map $\Pi^\epsilon_{x_\infty,x}\colon T_{x_\infty}\mathbb{R}^n \to \mathbb{R}^n$ to be the composition $T_{x_\infty}\mathbb{R}^n \to T_x\mathbb{R}^n \to \mathbb{R}^n$ of the isomorphism $T_{x_\infty}\mathbb{R}^n \to T_x\mathbb{R}^n$ of tangent spaces given by parallel transport with respect to the Euclidean metric δ, and the map $\zeta \mapsto \exp^\delta(\epsilon\zeta)$. Here $\exp^\delta$ is the exponential map for the Euclidean metric δ. This map is used to focus attention on an ϵ-neighborhood of the point x; the pullback by $\Pi^\epsilon_{x_\infty,x}$ of the ball of radius ϵ around x is the ball of radius 1 around the origin in $T_{x_\infty}\mathbb{R}^n$.

In the following it is helpful to keep in mind that $\Pi^\epsilon_{x_\infty,x}(\zeta)$ is the same as $x + \epsilon\zeta$ after natural identifications of points and vectors.

LEMMA 2.1. *Fix $\gamma > m$ and $\beta_1, \beta_2 > 0$. Let $x_k \in S$ and $\epsilon_k > 0$ be sequences with $x_k \to x_\infty$ and $\epsilon_k \to 0$.*

(i) *There are constants R and C depending on $(S, x_\infty, \gamma, \beta_1, \beta_2)$ such that if k is such that $|x_k - x_\infty| + \epsilon_k \beta_1 \leq R/2$, and $\zeta \in T_{x_\infty} \mathbb{R}^n$ satisfies $|\zeta| \leq \beta_1$ and $\mathrm{dist}^\delta(\zeta, T_{x_\infty} S) \geq \beta_2$ then*

$$\epsilon_k^{\gamma - m} \int_S |\Pi_{x_\infty, x_k}^{\epsilon_k}(\zeta) - y|^{-\gamma} \, dy < C.$$

(ii) *For all $\zeta \notin T_{x_\infty} S$ it holds that*

$$\lim_{k \to \infty} \epsilon_k^{\gamma - m} \int_S |\Pi_{x_\infty, x_k}^{\epsilon_k}(\zeta) - y|^{-\gamma} \, dy = \int_{T_{x_\infty} S} |\zeta - \eta|^{-\gamma} \, d\eta.$$

In other words, the sequence of functions

$$F_k(\zeta) := \epsilon_k^{\gamma - m} \int_S |\Pi_{x_\infty, x_k}^{\epsilon_k}(\zeta) - y|^{-\gamma} \, dy$$

is uniformly bounded for $\zeta \in T_{x_\infty} \mathbb{R}^n$ with $|\zeta| \leq \beta_1$ and $\mathrm{dist}^\delta(\zeta, T_{x_\infty} S) \geq \beta_2$, and it converges pointwise to the function

$$F_\infty(\zeta) := \int_{T_{x_\infty} S} |\zeta - \eta|^{-\gamma} \, d\eta$$

on $T_{x_\infty} \mathbb{R}^n \setminus T_{x_\infty} S$.

Proof. Let $\pi \colon T_{x_\infty} \mathbb{R}^n \to T_{x_\infty} S$ denote the orthogonal projection and recall that $N_{x_\infty} S$ denotes the linear subspace of $T_{x_\infty} \mathbb{R}^n$ orthogonal to $T_{x_\infty} S$. Choose $R > 0$ such that there is an open neighborhood $A \subset T_{x_\infty} S$ of 0, and a smooth function $\sigma \colon A \to N_{x_\infty} S$ with

$$S \cap B_R(x_\infty) = \exp^\delta(\mathrm{graph}\, \sigma)$$

and such that σ can be extended smoothly to a neighborhood of $\overline{A}$, with Lipschitz constant smaller than $\beta_2/(4\beta_1)$. We will consider integrals over $S \setminus B_R(x_\infty)$ and $S \cap B_R(x_\infty)$ separately.

Part (i): Uniform boundedness. Note that

$$|\Pi_{x_\infty, x_k}^{\epsilon_k}(\zeta) - x_\infty| \leq |x_k - x_\infty| + \epsilon_k |\zeta|$$
$$\leq |x_k - x_\infty| + \epsilon_k \beta_1.$$

Let k be such that $|x_k - x_\infty| + \epsilon_k \beta_1 \leq R/2$. Then

$$|\Pi_{x_\infty, x_k}^{\epsilon_k}(\zeta) - y| \geq |y - x_\infty| - |\Pi_{x_\infty, x_k}^{\epsilon_k}(\zeta) - x_\infty|$$
$$\geq R - R/2$$
$$= R/2$$

for $y \in S \setminus B_R(x_\infty)$, so

$$\int_{S \setminus B_R(x_\infty)} |\Pi_{x_\infty, x_k}^{\epsilon_k}(\zeta) - y|^{-\gamma} \, dy \leq \int_{S \setminus B_R(x_\infty)} (R/2)^{-\gamma} \, dy$$
$$\leq (R/2)^{-\gamma} \mathcal{H}_m(S).$$

Therefore

$$\epsilon_k^{\gamma-m} \int_{S \setminus B_R(x_\infty)} |\Pi_{x_\infty,x_k}^{\epsilon_k}(\zeta) - y|^{-\gamma} \, dy \to 0 \tag{1}$$

as $k \to \infty$, since $\gamma > m$ and $\epsilon_k \to 0$.

We parametrize $S \cap B_R(x_\infty)$ by $\hat{\eta}^\| \mapsto \exp^\delta(\hat{\eta}^\| + \sigma(\hat{\eta}^\|))$ where $\hat{\eta}^\| \in A$. The Jacobian of this map is represented by the matrix

$$\begin{bmatrix} I \\ J\sigma(\hat{\eta}^\|) \end{bmatrix}.$$

With this parametrization, the integral

$$\epsilon_k^{\gamma-m} \int_{S \cap B_R(x_\infty)} |\Pi_{x_\infty,x_k}^{\epsilon_k}(\zeta) - y|^{-\gamma} \, dy$$

can be expressed as

$$\epsilon_k^{\gamma-m} \int_A \left| \Pi_{x_\infty,x_k}^{\epsilon_k}(\zeta) - \exp^\delta\left(\hat{\eta}^\| + \sigma(\hat{\eta}^\|)\right) \right|^{-\gamma} \mathcal{J}(\hat{\eta}^\|) \, d\hat{\eta}^\|$$

where

$$\mathcal{J}(\hat{\eta}^\|) = \sqrt{\det\left(\begin{bmatrix} I \\ J\sigma(\hat{\eta}^\|) \end{bmatrix}^T \begin{bmatrix} I \\ J\sigma(\hat{\eta}^\|) \end{bmatrix}\right)} = \sqrt{\det\left(I + J\sigma(\hat{\eta}^\|)^T J\sigma(\hat{\eta}^\|)\right)}.$$

If $x_k \in S \cap B_R(x_\infty)$ then $x_k = \exp^\delta(\xi_k^\| + \sigma(\xi_k^\|))$ for some $\xi_k^\| \in A$. It holds that

$$\Pi_{x_\infty,x_k}^{\epsilon_k}(\zeta) = \exp^\delta\left(\xi_k^\| + \sigma(\xi_k^\|) + \epsilon_k\zeta\right),$$

so

$$\epsilon_k^{\gamma-m} \int_{S \cap B_R(x_\infty)} |\Pi_{x_\infty,x_k}^{\epsilon_k}(\zeta) - y|^{-\gamma} \, dy$$

$$= \epsilon_k^{\gamma-m} \int_A \left| \exp^\delta\left(\xi_k^\| + \sigma(\xi_k^\|) + \epsilon_k\zeta\right) - \exp^\delta\left(\hat{\eta}^\| + \sigma(\hat{\eta}^\|)\right) \right|^{-\gamma} \mathcal{J}(\hat{\eta}^\|) \, d\hat{\eta}^\|$$

$$= \epsilon_k^{\gamma-m} \int_A \left| \xi_k^\| + \sigma(\xi_k^\|) + \epsilon_k\zeta - \hat{\eta}^\| + \sigma(\hat{\eta}^\|) \right|^{-\gamma} \mathcal{J}(\hat{\eta}^\|) \, d\hat{\eta}^\|$$

$$= \epsilon_k^{-m} \int_A \left| \zeta - \frac{\hat{\eta}^\| - \xi_k^\|}{\epsilon_k} - \frac{\sigma(\hat{\eta}^\|) - \sigma(\xi_k^\|)}{\epsilon_k} \right|^{-\gamma} \mathcal{J}(\hat{\eta}^\|) \, d\hat{\eta}^\|.$$

By a change of variables $\hat{\eta}^\| = \xi_k^\| + \epsilon_k\eta^\|$ we see that

$$\epsilon_k^{\gamma-m} \int_{S \cap B_R(x_\infty)} |\Pi_{x_\infty,x_k}^{\epsilon_k}(\zeta) - y|^{-\gamma} \, dy$$

$$= \int_{(A-\xi_k^\|)/\epsilon_k} \left| \zeta - \eta^\| - \frac{\sigma(\xi_k^\| + \epsilon_k\eta^\|) - \sigma(\xi_k^\|)}{\epsilon_k} \right|^{-\gamma} \mathcal{J}(\xi_k^\| + \epsilon_k\eta^\|) \, d\eta^\|. \tag{2}$$

We will now bound the integrand by an integrable function of $\eta^\|$. Let L denote the Lipschitz constant of σ and recall that

$$L < \beta_2/(4\beta_1).$$

Then

$$\left| \frac{\sigma(\xi_k^\| + \epsilon_k \eta^\|) - \sigma(\xi_k^\|)}{\epsilon_k} \right| \leq \frac{L\epsilon_k|\eta^\||}{\epsilon_k} = L|\eta^\|| < \frac{\beta_2}{4\beta_1}|\eta^\||. \tag{3}$$

By orthogonal decomposition of $\zeta \in T_{x_\infty}S \oplus N_{x_\infty}S$ we get

$$\begin{aligned}
&\left| \zeta - \eta^\| - \frac{\sigma(\xi_k^\| + \epsilon_k \eta^\|) - \sigma(\xi_k^\|)}{\epsilon_k} \right|^2 \\
&\qquad = |\pi(\zeta) - \eta^\||^2 + \left| (\zeta - \pi(\zeta)) - \frac{\sigma(\xi_k^\| + \epsilon_k \eta^\|) - \sigma(\xi_k^\|)}{\epsilon_k} \right|^2.
\end{aligned} \tag{4}$$

By the triangle inequality and (3) we further have

$$\begin{aligned}
\left| (\zeta - \pi(\zeta)) - \frac{\sigma(\xi_k^\| + \epsilon_k \eta^\|) - \sigma(\xi_k^\|)}{\epsilon_k} \right| &\geq |\zeta - \pi(\zeta)| - \left| \frac{\sigma(\xi_k^\| + \epsilon_k \eta^\|) - \sigma(\xi_k^\|)}{\epsilon_k} \right| \\
&\geq \operatorname{dist}^\delta(\zeta, T_{x_\infty}S) - \frac{\beta_2}{4\beta_1}|\eta^\|| \\
&\geq \beta_2 - \frac{\beta_2}{4\beta_1}|\eta^\||
\end{aligned}$$

so if $|\eta^\|| < 2\beta_1$ then

$$\left| (\zeta - \pi(\zeta)) - \frac{\sigma(\xi_k^\| + \epsilon_k \eta^\|) - \sigma(\xi_k^\|)}{\epsilon_k} \right| > \frac{\beta_2}{2}.$$

This gives a good bound for small $|\eta^\||$. For large $|\eta^\||$ we will use the following bound instead. The orthogonal decomposition (4) tells us that

$$\begin{aligned}
\left| \zeta - \eta^\| - \frac{\sigma(\xi_k^\| + \epsilon_k \eta^\|) - \sigma(\xi_k^\|)}{\epsilon_k} \right| &\geq |\pi(\zeta) - \eta^\|| \\
&\geq |\eta^\|| - |\pi(\zeta)| \\
&\geq |\eta^\|| - |\zeta| \\
&\geq |\eta^\|| - \beta_1.
\end{aligned}$$

Finally note that $\mathcal{J}(\hat{\eta}^\|)$ is bounded by some constant $C' = C'(S, x_\infty, \beta_1, \beta_2)$ on A since σ can be extended smoothly beyond $\overline{A}$. Hence

$$\left| \frac{\zeta - \eta^\| - (\sigma(\xi_k^\| + \epsilon_k \eta^\|) - \sigma(\xi_k^\|))}{\epsilon_k} \right|^{-\gamma} \mathcal{J}(\xi_k^\| + \epsilon_k \eta^\|)$$

is bounded from above by the function

$$\Phi(\eta^{\|}) := \begin{cases} C' \left(\frac{\beta_2}{2}\right)^{-\gamma} & \text{if } |\eta^{\|}| < 2\beta_1, \\ C'(|\eta^{\|}| - \beta_1)^{-\gamma} & \text{if } |\eta^{\|}| \geq 2\beta_1, \end{cases}$$

which is integrable on all of $T_{x_\infty} S$ since $\gamma > m$. This means that we have now shown that

$$\epsilon_k^{\gamma-m} \int_S |\Pi_{x_\infty, x_k}^{\epsilon_k}(\zeta) - y|^{-\gamma} \, dy$$
$$\leq (R/2)^{-m} \mathcal{H}_m(S)$$
$$+ \int_{(A-\xi_k^{\|})/\epsilon_k} \left| \zeta - \eta^{\|} - \frac{\sigma(\xi_k^{\|} + \epsilon_k \eta^{\|}) - \sigma(\xi_k^{\|})}{\epsilon_k} \right|^{-\gamma} \mathcal{J}(\xi_k^{\|} + \epsilon_k \eta^{\|}) \, d\eta^{\|}$$
$$\leq (R/2)^{-m} \mathcal{H}_m(S) + \int_{(A-\xi_k^{\|})/\epsilon_k} \Phi(\eta^{\|}) \, d\eta^{\|}$$
$$\leq (R/2)^{-m} \mathcal{H}_m(S) + \int_{T_{x_\infty} S} \Phi(\eta^{\|}) \, d\eta^{\|}$$

for $x_k \in \mathbb{R}^n$ and $\epsilon_k > 0$ such that $|x_k - x_\infty| + \epsilon_k \beta_1 \leq R/2$. Choosing

$$C(S, x_\infty, \gamma, \beta_1, \beta_2) := (R/2)^{-m} \mathcal{H}_m(S) + \int_{T_{x_\infty} S} \Phi(\eta^{\|}) \, d\eta^{\|}$$

completes the proof of the boundedness statement in part (i).

Part (ii): Pointwise convergence. We will now apply the dominated convergence theorem to pass to the limit in (2). As preparation we first note that

$$\lim_{k \to \infty} \frac{\sigma(\xi_k^{\|} + \epsilon_k \eta^{\|}) - \sigma(\xi_k^{\|})}{\epsilon_k} = \lim_{\substack{\xi_k^{\|} \to x_\infty \\ \epsilon_k \to 0}} \frac{\sigma(\xi_k^{\|} + \epsilon_k \eta^{\|}) - \sigma(\xi_k^{\|})}{\epsilon_k} = 0$$

since $J\sigma(x_\infty) = 0$. For the same reason,

$$\lim_{k \to \infty} \mathcal{J}(\xi_k^{\|} + \epsilon_k \eta^{\|}) = \lim_{\substack{\xi_k^{\|} \to x_\infty \\ \epsilon_k \to 0}} \sqrt{\det(I + J\sigma(\xi_k^{\|} + \epsilon_k \eta^{\|})^T J\sigma(\xi_k^{\|} + \epsilon_k \eta^{\|}))} = 1.$$

Let $\chi_{(A-\xi_k^{\|})/\epsilon_k}$ denote the characteristic function of the set $(A - \xi_k^{\|})/\epsilon_k$. Using the existence of the integrable function Φ we can apply the dominated convergence theorem

to see that

$$\lim_{k\to\infty} \int_{(A-\xi_k^\|)/\epsilon_k} \left| \zeta - \eta^\| - \frac{\sigma(\xi_k^\| + \epsilon_k \eta^\|) - \sigma(\xi_k^\|)}{\epsilon_k} \right|^{-\gamma} \mathcal{J}(\xi_k^\| + \epsilon_k \eta^\|)\, d\eta^\|$$

$$= \lim_{k\to\infty} \int_{T_{x_\infty} S} \chi_{\frac{A-\xi_k^\|}{\epsilon_k}}(\eta^\|) \left| \zeta - \eta^\| - \frac{\sigma(\xi_k^\| + \epsilon_k \eta^\|) - \sigma(\xi_k^\|)}{\epsilon_k} \right|^{-\gamma}$$

$$\cdot \mathcal{J}(\xi_k^\| + \epsilon_k \eta^\|)\, d\eta^\|$$

$$= \int_{T_{x_\infty} S} \lim_{k\to\infty} \left| \zeta - \eta^\| - \frac{\sigma(\xi_k^\| + \epsilon_k \eta^\|) - \sigma(\xi_k^\|)}{\epsilon_k} \right|^{-\gamma} \mathcal{J}(\xi_k^\| + \epsilon_k \eta^\|)\, d\eta^\|$$

$$= \int_{T_{x_\infty} S} |\zeta - \eta^\||^{-\gamma}\, d\eta^\|.$$

This together with (2) means that we have now shown that

$$\lim_{k\to\infty} \epsilon_k^{\gamma-m} \int_{S\cap B_R(x_\infty)} |\Pi_{x_\infty, x_k}^{\epsilon_k}(\zeta) - y|^{-\gamma}\, dy = \int_{T_{x_\infty} S} |\zeta - \eta^\||^{-\gamma}\, d\eta^\|,$$

which together with (1) completes the proof of part (ii). □

In the next proposition we prove that the functions F_k from Lemma 2.1 converge in $C^r(K)$ for compact domains K.

PROPOSITION 2.2. *Fix $\gamma > m$. Let $x_k \in S$ and $\epsilon_k > 0$ be sequences with $x_k \to x_\infty$ and $\epsilon_k \to 0$. Let $K \subset T_{x_\infty}\mathbb{R}^n \setminus T_{x_\infty}S$ be compact. Then for $r \in \mathbb{N}$ the functions F_k converge to F_∞ in $C^r(K)$ as $k \to \infty$.*

Proof. Fix $r \in \mathbb{N}$. We will apply Lemma 2.1. Let

$$\beta_1 := \max_{\zeta \in K} |\zeta|,$$

$$\beta_2 := \min_{\zeta \in K} \text{dist}^\delta(\zeta, T_{x_\infty}S).$$

For each $l \in \mathbb{N}$ let

$$D_l := \prod_{i=0}^{l-1} (\gamma + i).$$

We have

$$\left| \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_l}} |x - y|^{-\gamma} \right| \le D_l |x - y|^{-(\gamma+l)}$$

so

$$\left| \frac{\partial}{\partial \zeta^{i_1}} \cdots \frac{\partial}{\partial \zeta^{i_l}} \epsilon_k^{\gamma-m} |\Pi_{x_\infty, x_k}^{\epsilon_k}(\zeta) - y|^{-\gamma} \right|$$

$$\le D_l \epsilon_k^{\gamma+l-m} |\Pi_{x_\infty, x_k}^{\epsilon_k}(\zeta) - y|^{-(\gamma+l)}.$$

By applying Lemma 2.1 for all exponents $\{\gamma + l \mid 0 \leq l \leq r + 1\}$ we get constants C_l such that

$$D_l \epsilon_k^{\gamma+l-m} \int_S |\Pi_{x_\infty,x_k}^{\epsilon_k}(\zeta) - y|^{-(\gamma+l)} \, dy \leq C_l.$$

This means that

$$\frac{\partial}{\partial \zeta^{i_1}} \cdots \frac{\partial}{\partial \zeta^{i_l}} F_k(\zeta) = \frac{\partial}{\partial \zeta^{i_1}} \cdots \frac{\partial}{\partial \zeta^{i_l}} \epsilon_k^{\gamma-m} \int_S |\Pi_{x_\infty,x_k}^{\epsilon_k}(\zeta) - y|^{-\gamma} \, dy$$

$$= \epsilon_k^{\gamma-m} \int_S \frac{\partial}{\partial \zeta^{i_1}} \cdots \frac{\partial}{\partial \zeta^{i_l}} |\Pi_{x_\infty,x_k}^{\epsilon_k}(\zeta) - y|^{-\gamma} \, dy$$

$$\leq D_l \epsilon_k^{\gamma+l-m} \int_S |\Pi_{x_\infty,x_k}^{\epsilon_k}(\zeta) - y|^{-(\gamma+l)} \, dy$$

$$\leq C_l.$$

Hence we have uniform bounds for $\|F_k\|_{C^{r+1}(K)}$.

From the uniform bounds on $\|F_k\|_{C^{r+1}(K)}$ and the pointwise convergence of F_k to F_∞ it follows that $F_k \to F_\infty$ in $C^r(K)$. $\square$

As a corollary we can determine what the metrics g_{ϵ_k} converge to after appropriate rescaling and change of coordinates.

COROLLARY 2.3. *Let $K \subset T_{x_\infty}\mathbb{R}^n \setminus T_{x_\infty}S$ be compact. Let $x_k \in S$ and $\epsilon_k > 0$ be sequences with $x_k \to x_\infty$ and $\epsilon_k \to 0$. Then the rescaled pullback metrics $\epsilon_k^{-2}(\Pi_{x_\infty,x_k}^{\epsilon_k})^*(g_{\epsilon_k})$ on K converge in every $C^r(K)$ to*

$$u_\infty^{4/(n-2)}\delta$$

where δ is the Euclidean metric on $K \subset T_{x_\infty}\mathbb{R}^n$ and

$$u_\infty(\zeta) := 1 + \int_{T_{x_\infty}S} |\zeta - \eta|^{-(n-2)} \, d\eta.$$

Proof. With $\gamma = n-2$ we have $u_{\epsilon_k}(\Pi_{x_\infty,x_k}^{\epsilon_k}(\zeta)) = 1 + F_k(\zeta)$ and $u_\infty(\zeta) = 1 + F_\infty(\zeta)$. Since $n - 2 > m$ it follows from Proposition 2.2 that

$$u_{\epsilon_k}(\Pi_{x_\infty,x_k}^{\epsilon_k}(\zeta)) \to u_\infty(\zeta)$$

in $C^r(K)$ as $k \to \infty$. The corollary follows since $\epsilon_k^{-2}(\Pi_{x_\infty,x_k}^{\epsilon_k})^*(\delta) = \delta$. $\square$

3. The mean curvature of tubular hypersurfaces. By *the tubular hypersurface of radius a around S*, denoted $\mathrm{Tub}(S,a)$, we mean the image of UNS under the map $\zeta \mapsto \exp^\delta(a\zeta)$. Recall that $\exp^\delta$ is the exponential map for the Euclidean metric δ.

We need to understand the behaviour of S and $\mathrm{Tub}(S,a)$ under the rescaling studied in the previous section. By writing S as a graph over $T_{x_\infty}S$ in normal coordinates, we conclude the following.

LEMMA 3.1. *Let $x_k \in S$ and $\epsilon_k > 0$ be sequences with $x_k \to x_\infty$ and $\epsilon_k \to 0$. Then*

$$(\Pi_{x_\infty,x_k}^{\epsilon_k})^{-1}(S) \to T_{x_\infty}S$$

and

$$(\Pi^{\epsilon_k}_{x_\infty, x_k})^{-1}(\mathrm{Tub}(S, a\epsilon_k)) \to \mathrm{Tub}(T_{x_\infty}S, a)$$

smoothly on compact subsets of $T_{x_\infty}\mathbb{R}^n$ as $k \to \infty$.

Here, smooth convergence of submanifolds N_k means that if k is sufficiently large then in normal coordinates for the limit submanifold N_∞, each N_k is the graph of a smooth function on N_∞, and these functions converge smoothly to 0 as $k \to \infty$.

The cylinder $\mathrm{Tub}(T_{x_\infty}S, a)$ has mean curvature $(n - m - 1)a^{-1}$ in the metric δ. Thus, we can use Lemma 3.1 to compute the mean curvature of $\mathrm{Tub}(S, a)$ in the Euclidean metric δ when the radius a is small. See also for instance [14, Section 4].

LEMMA 3.2. *The mean curvature of $\mathrm{Tub}(S, a)$ in the Euclidean metric δ is*

$$(n - m - 1)a^{-1} + O(1)$$

as $a \to 0$, where the bounded term $O(1)$ depends only on the geometry of S.

The tubular hypersurfaces $\mathrm{Tub}(T_{x_\infty}S, a)$ with varying a give a foliation of $T_{x_\infty}\mathbb{R}^n \setminus T_{x_\infty}S$ by cylinders with constant mean curvature in the metric $u_\infty^{4/(n-2)}\delta$. In the next lemma we see that this foliation contains exactly one minimal hypersurface.

LEMMA 3.3. *Let*

$$\hat{a} := \left(D_{n,m} \frac{1 - C_{n,m}}{C_{n,m}} \right)^{1/(n-m-2)}$$

where

$$C_{n,m} := \frac{(n-2)(n-m-1)}{2(n-1)(n-m-2)}$$

and

$$D_{n,m} := \int_{\mathbb{R}^m} (1 + |\eta|^2)^{-(n-2)/2}\, d\eta.$$

Then the mean curvature of $\mathrm{Tub}(T_{x_\infty}S, a)$ in the metric $u_\infty^{4/(n-2)}\delta$ is zero for $a = \hat{a}$, negative for $0 < a < \hat{a}$ and positive for $a > \hat{a}$.

Proof. For $\zeta \in \mathrm{Tub}(T_{x_\infty}S, a)$ it holds that

$$
\begin{aligned}
u_\infty(\zeta) &= 1 + \int_{T_{x_\infty}S} |\zeta - \eta|^{-(n-2)}\, d\eta \\
&= 1 + \int_{T_{x_\infty}S} (a^2 + |\eta|^2)^{-(n-2)/2}\, d\eta \\
&= 1 + a^{-(n-m-2)} \int_{\mathbb{R}^m} (1 + |\eta|^2)^{-(n-2)/2}\, d\eta \\
&= 1 + a^{-(n-m-2)} D_{n,m}.
\end{aligned}
$$

Since the mean curvature of $\mathrm{Tub}(T_{x_\infty}S, a)$ in the metric δ is $(n - m - 1)a^{-1}$ we find that its mean curvature in the metric $u_\infty^{4/(n-2)}\delta$ is

$$
u_\infty^{-2/(n-2)} \left(\frac{n - m - 1}{a} + 2\frac{n-1}{n-2}\frac{d}{da} \ln u_\infty \right)
$$

$$
= (1 + a^{-\gamma} D_{n,m})^{-\frac{2}{n-2}} \frac{1}{a} \left((\gamma + 1) - 2\gamma \frac{n-1}{n-2}\frac{a^{-\gamma} D_{n,m}}{1 + a^{-\gamma} D_{n,m}} \right)
$$

where we have set $\gamma \coloneqq n - m - 2$. From this it is not complicated to verify the claim of the lemma. $\square$

In the following proposition we show that a tubular hypersurface around S with radius in a certain interval has positive mean curvature in $(\mathbb{R}^n \setminus S, g_\epsilon)$.

PROPOSITION 3.4. *There are constants $C_{\mathrm{outer}} > 0$ and $R_{\mathrm{outer}} > 0$ such that for all sufficiently small $\epsilon > 0$ it holds for all $C_{\mathrm{outer}}\epsilon \le a \le R_{\mathrm{outer}}$ that the tubular hypersurface $\mathrm{Tub}(S, a)$ has positive mean curvature in $(\mathbb{R}^n \setminus S, g_\epsilon)$.*

Proof. Suppose that the result does not hold. Then for every choice of C and R there is a sequence $\epsilon_k \to 0$ and a sequence a_k with $C\epsilon_k \le a_k \le R$ such that $\mathrm{Tub}(S, a_k)$ contains at least one point x_k where the mean curvature is nonpositive.

We first consider the case where $a_k \to 0$ and there is a subsequence for which ϵ_k / a_k converges to a positive number L. Note that then $L \le 1/C$. We rescale by La_k around the point x_k using the map $\Pi_{x_\infty, x_k}^{La_k}$. Then by Lemma 3.1 we have

$$(\Pi_{x_\infty, x_k}^{La_k})^{-1}(\mathrm{Tub}(S, a_k)) \to \mathrm{Tub}(T_{x_\infty}S, 1/L).$$

It holds that

$$(La_k)^{-2}(\Pi_{x_\infty, x_k}^{La_k})^*(g_{\epsilon_k})(\zeta)$$
$$= \left(1 + \epsilon_k^{n-m-2}\int_S |\Pi_{x_\infty, x_k}^{La_k}(\zeta) - y|^{-(n-2)}\, dy\right)^{4/(n-2)}\delta$$
$$= \left(1 + \left(\frac{\epsilon_k}{La_k}\right)^{n-m-2} F_k(\zeta)\right)^{4/(n-2)}\delta$$

where

$$F_k(\zeta) = (La_k)^{n-m-2}\int_S |\Pi_{x_\infty, x_k}^{La_k}(\zeta) - y|^{-(n-2)}\, dy.$$

We conclude from Proposition 2.2 applied with $\epsilon_k = La_k$ that

$$\lim_{k\to\infty}(La_k)^{-2}(\Pi_{x_\infty, x_k}^{La_k})^*(g_{\epsilon_k})(\zeta) = (1 + F_\infty(\zeta))^{4/(n-2)}\delta$$
$$= (u_\infty(\zeta))^{4/(n-2)}\delta$$

where

$$F_\infty(\zeta) = \int_{T_{x_\infty}S} |\zeta - \eta|^{-(n-2)}\, d\eta.$$

Choose $C > \hat{a}$. Then $1/L \ge C > \hat{a}$ so that by Lemma 3.3 the limit cylinder $\mathrm{Tub}(T_{x_\infty}S, 1/L)$ has positive mean curvature in the limit metric $u_\infty^{4/(n-2)}\delta$. This contradicts the assumption that the mean curvature of $\mathrm{Tub}(S, a_k)$ in the metric g_{ϵ_k} is nonpositive at x_k for every k.

Second, we consider the case where $a_k \to 0$ and $\epsilon_k / a_k \to 0$. In this case we rescale by a_k around the point x_k using the map $\Pi_{x_\infty, x_k}^{a_k}$, in which case

$$(\Pi_{x_\infty, x_k}^{a_k})^{-1}(\mathrm{Tub}(S, a_k)) \to \mathrm{Tub}(T_{x_\infty}S, 1).$$

From a computation like in the previous case we find that

$$\lim_{k \to \infty} a_k^{-2} (\Pi_{x_\infty, x_k}^{a_k})^* (g_{\epsilon_k}) = \delta.$$

Since $\mathrm{Tub}(T_{x_\infty} S, 1)$ has positive mean curvature in the metric δ we again find a contradiction.

Finally, we consider the case where there is a subsequence for which $a_k \to a_\infty \neq 0$. Then $a_\infty \leq R$. If R is sufficiently small, the tubular hypersurfaces $\mathrm{Tub}(S, a_k)$ stay in a compact set bounded away from S and the metrics g_{ϵ_k} tend uniformly to the Euclidean metric δ on this compact set. Choosing R sufficiently small compared to the geometry of S we find from Lemma 3.2 that the limit surface $\mathrm{Tub}(S, a_\infty)$ has positive mean curvature in the limit metric δ, which contradicts the assumption of nonpositive mean curvature of $\mathrm{Tub}(S, a_k)$ in g_{ϵ_k} at x_k. $\square$

The following proposition states that any tubular hypersurface around S with sufficiently small radius has negative mean curvature in $(\mathbb{R}^n \setminus S, g_\epsilon)$. The proof is similar to the proof of the previous proposition, and we omit it.

PROPOSITION 3.5. *There is a constant $C_{\mathrm{inner}} > 0$ such that for all sufficiently small $\epsilon > 0$ it holds for all $0 < a \leq C_{\mathrm{inner}} \epsilon$ that the tubular hypersurface $\mathrm{Tub}(S, a)$ has negative mean curvature in $(\mathbb{R}^n \setminus S, g_\epsilon)$.*

4. The location of outer area minimizing stationary hypersurfaces. The purpose of this section is to prove that for small ϵ, all outer area minimizing stationary bounding hypersurfaces in $(\mathbb{R}^n \setminus S, g_\epsilon)$ are contained in the region between $\mathrm{Tub}(S, C_{\mathrm{inner}} \epsilon)$ and $\mathrm{Tub}(S, C_{\mathrm{outer}} \epsilon)$. In this paper, we use the word "hypersurface" to mean a smooth hypersurface. Note that this means that the closure of a hypersurface might contain singular points. We begin by introducing the relevant terminology.

Let (M, g) be an asymptotically Euclidean Riemannian manifold. A hypersurface $\Sigma \subset M$ is called *bounding* if its closure $\overline{\Sigma}$ is compact and Σ is the reduced boundary of a Caccioppoli set (see for instance [12], in particular [12, Chapter 3 and Remark 3.2]), the complement of which is a neighborhood of the asymptotically Euclidean end. We call this Caccioppoli set *the region inside Σ*. The bounding hypersurface Σ *encloses* a bounding hypersurface Σ' if the region inside Σ contains the region inside Σ'. We say that a bounding hypersurface in $(\mathbb{R}^n \setminus S, g_\epsilon)$ *encloses the end at S* if it encloses $\mathrm{Tub}(S, a)$ for all sufficiently small $a > 0$. We define the *regular* and *singular* sets of a hypersurface Σ as follows: The regular set $\mathrm{reg}(\Sigma)$ is the set of points in $\overline{\Sigma}$ which have a neighborhood in which $\overline{\Sigma}$ is a connected hypersurface. The singular set is $\mathrm{sing}(\Sigma) = \overline{\Sigma} \setminus \mathrm{reg}(\Sigma)$. We say that Σ is *nonsingular* if $\mathrm{sing}(\Sigma) = \emptyset$. We only consider hypersurfaces Σ for which $\Sigma = \mathrm{reg}(\Sigma)$, in other words that no points where Σ is smooth have been left out.

Let Σ be a bounding hypersurface in (M, g), and let E be the region inside Σ. Then Σ is *outer area minimizing* if its area is not greater than the perimeter of any Caccioppoli set containing E. A hypersurface Σ in a Riemannian manifold (M, g) is *stationary* if the first variation of its area vanishes when Σ is deformed in the direction of any vector field with compact support in M. A hypersurface is *stable* if, in addition, the second variation of area is nonnegative for such vector fields. We will use the concept of stationarity for Σ in $M = \mathbb{R}^n \setminus S$ as well as in $M = \mathbb{R}^n$. Note that every outer area minimizing stationary hypersurface is stable.

The outer area minimizing property gives local area bounds, as expressed in the following lemma. We denote k-dimensional Hausdorff measure with respect to a metric g by $\mathcal{H}_k^g$.
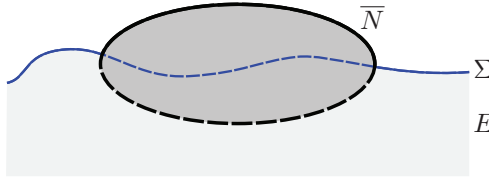
Fig. 1: $\partial(E \cup N)$ is an outward variation of Σ and hence has at least the same area. This gives a bound for the area of $\Sigma \cap N$.

LEMMA 4.1. *Let Σ be an outer area minimizing hypersurface in an asymptotically Euclidean Riemannian manifold (M, g). Let $N \subset M$ be an open set with compact closure and piecewise smooth boundary. Then*

$$\mathcal{H}^g_{n-1}(\Sigma \cap N) \leq \mathcal{H}^g_{n-1}(\partial N).$$

Proof. Let E be the region inside Σ. The set $\partial(E \cup N)$ is an outward variation of Σ, as shown in Figure 1, and hence

$$\mathcal{H}^g_{n-1}(\Sigma) \leq \mathcal{H}^g_{n-1}(\partial(E \cup N)).$$

Splitting this inequality into the part inside $\overline{N}$ and the part outside $\overline{N}$ we have

$$\mathcal{H}^g_{n-1}\left(\Sigma \cap \overline{N}\right) + \mathcal{H}^g_{n-1}\left(\Sigma \setminus \overline{N}\right)$$
$$\leq \mathcal{H}^g_{n-1}\left(\partial(E \cup N) \cap \overline{N}\right) + \mathcal{H}^g_{n-1}\left(\partial(E \cup N) \setminus \overline{N}\right).$$

Since $\Sigma \setminus \overline{N} = \partial(E \cup N) \setminus \overline{N}$ the above inequality reduces to

$$\mathcal{H}^g_{n-1}(\Sigma \cap \overline{N}) \leq \mathcal{H}^g_{n-1}(\partial(E \cup N) \cap \overline{N}).$$

Since $\Sigma \cap N \subseteq \Sigma \cap \overline{N}$ and $\partial(E \cup N) \cap \overline{N} \subseteq \partial N$ we obtain the desired bound. $\square$

For the proof of Proposition 4.5 and Proposition 5.1 we will use a convergence result for stable hypersurfaces. We have not been able to find a proof of this result in the exact form we need, but it follows from the work of Schoen and Simon in [15] and we sketch a proof.

THEOREM 4.2 (Schoen–Simon [15]). *Let Ω be an orientable smooth manifold of dimension n and let $K \subset \Omega$ be compact. Let h_∞ be a Riemannian metric on Ω, let h_k be a sequence of smooth Riemannian metrics on Ω, and let Γ_k be a sequence of hypersurfaces in Ω. Assume that*
- *$\overline{\Gamma}_k \cap K \neq \emptyset$ for all k,*
- *each Γ_k is a stable hypersurface for the metric h_k,*
- *$\mathcal{H}^{h_\infty}_{n-3}(\mathrm{sing}(\Gamma_k)) = 0$,*
- *for every open set Ω' with compact closure $\overline{\Omega'} \subseteq \Omega$*
  - *the metrics h_k converge to h_∞ uniformly in $C^r(\Omega')$ for every r, and*
  - *$\limsup_{k \to \infty} \mathcal{H}^{h_k}_{n-1}(\Gamma_k \cap \Omega') < \infty$.*

Then there is
- *a subsequence of Γ_k, still denoted Γ_k, and*
- *a hypersurface $\Gamma_\infty \subset \Omega$*

such that

- $\Gamma_k \to \Gamma_\infty$ *as varifolds,*
- $\mathcal{H}_\alpha^{h_\infty}(\mathrm{sing}(\Gamma_\infty)) = 0$ *if $\alpha > n - 8$ and $\alpha \geq 0$,*
- $\Gamma_\infty \cap K \neq \emptyset$,
- Γ_∞ *is a stable hypersurface in the metric h_∞,*
- *for every open set Ω' with compact closure $\overline{\Omega'} \subseteq \Omega \setminus \mathrm{sing}(\Gamma_\infty)$*
    - $\mathcal{H}_{n-1}^{h_\infty}(\Gamma_\infty \cap \Omega') \leq \limsup_{k\to\infty} \mathcal{H}_{n-1}^{h_k}(\Gamma_k \cap \Omega')$, *and*
    - *the subsequence Γ_k converges smoothly to Γ_∞ (disregarding multiplicity) on Ω'.*

The smooth convergence of Γ_k to Γ_∞ on Ω', disregarding multiplicity, means that if k is sufficiently large then in normal coordinates for Γ_∞, each Γ_k is the union of graphs of a finite number of smooth functions on Γ_∞, and these functions converge in $C^\infty(\Gamma_\infty \cap \Omega')$ to 0 as $k \to \infty$.

Proof. In this proof, we let $\mathrm{sing}_{C^2}(\Gamma)$ denote the singular part of Γ in the C^2 sense. Cover Ω by balls of a fixed small radius. Applying [15, Theorem 2] in each ball we obtain a (possibly empty) stable C^2 hypersurface Γ_∞, with $\mathcal{H}_\alpha^{h_\infty}(\mathrm{sing}_{C^2}(\Gamma_\infty)) = 0$ if $\alpha > n - 8$ and $\alpha \geq 0$, to which a subsequence Γ_k converges in the sense of varifolds. Moreover, there is at least one ball in which $\Gamma_\infty \cap K \neq \emptyset$. Taking a diagonal subsequence we can patch these locally defined hypersurfaces together to a hypersurface $\Gamma_\infty \subset \Omega$ with $\Gamma_\infty \cap K \neq \emptyset$.

Let Ω' be an open set with compact closure $\overline{\Omega'} \subseteq \Omega \setminus \mathrm{sing}_{C^2}(\Gamma_\infty)$. Since $\Gamma_k \to \Gamma_\infty$ as varifolds, we have $\mathcal{H}_{n-1}^{h_\infty}(\Gamma_\infty \cap \Omega') \leq \limsup_{k\to\infty} \mathcal{H}_{n-1}^{h_k}(\Gamma_k \cap \Omega')$.

Using [15, Inequality (1.18)] one can prove that varifold convergence implies convergence in Hausdorff distance (see also [15, Remark 10 on p. 779]). Hence [15, Theorem 1] is applicable at every point of $\Gamma_\infty \cap \Omega'$, telling us that the hypersurfaces $\Gamma_k \cap \Omega'$ eventually become graphs of functions over the tangent space of $\Gamma_\infty \cap \Omega'$ at the chosen point. This theorem also gives uniform C^2 bounds for these functions, which implies subsequential convergence in $C^{1,\alpha}$ for all $\alpha \in (0,1)$. The hypersurfaces $\Gamma_k \cap \Omega'$ are graphs of functions satisfying the minimal surface equations for a converging sequence of metrics. By ellipticity of the minimal surface equations, we conclude C^∞ convergence and that $\Gamma_\infty \cap \Omega'$ is smooth, which implies that $\mathrm{sing}(\Gamma_\infty) = \mathrm{sing}_{C^2}(\Gamma_\infty)$. $\square$

Having completed these preliminaries, we continue the study of the location of outer area minimizing stationary bounding hypersurfaces in $(\mathbb{R}^n \setminus S, g_\epsilon)$. Since the metrics g_ϵ converge uniformly to the Euclidean metric outside of a neighborhood of S as $\epsilon \to 0$ we conclude the following.

PROPOSITION 4.3. *Let $S^{n-1}(R)$ denote the sphere of radius R around the origin in $\mathbb{R}^n$. There is a constant R_{end} such that, for small ϵ, it holds that $S^{n-1}(R)$ has positive mean curvature in $(\mathbb{R}^n, g_\epsilon)$ if $R \geq R_{\mathrm{end}}$.*

Using the maximum principle of Solomon and White [18] we can prove the following proposition.

PROPOSITION 4.4. *Every stationary bounding hypersurface Σ in $(\mathbb{R}^n \setminus S, g_\epsilon)$ is contained in the region between $\mathrm{Tub}(S, C_{\mathrm{inner}}\epsilon)$ and $S^{n-1}(R_{\mathrm{end}})$.*

Proof. Let $a := \inf_{x \in \Sigma} \mathrm{dist}^\delta(x, S)$. It holds that $0 < a$ since $\overline{\Sigma} \subset \mathbb{R}^n \setminus S$ is compact. Let N denote the region outside $\mathrm{Tub}(S, a)$. Suppose for contradiction that $a \leq C_{\mathrm{inner}}\epsilon$. By Proposition 3.5 the hypersurface $\partial N = \mathrm{Tub}(S, a)$ has negative mean curvature. Since Σ is a stationary hypersurface with $\overline{\Sigma} \cap \partial N \neq \emptyset$, the maximum

principle of Solomon and White [18, Theorem, p. 686 and Remarks 1 and 2, pp. 690-691] and [19, Theorem 4] implies that the mean curvature of at least one connected component of ∂N is zero, which is a contradiction. Hence Σ is contained in the region outside $\mathrm{Tub}(S, C_{\mathrm{inner}}\epsilon)$.

A similar argument using $a := \sup_{x \in \Sigma} \mathrm{dist}^\delta(x, 0)$ and Proposition 4.3 proves that Σ is contained in the region inside $S^{n-1}(R_{\mathrm{end}})$. $\square$

The previous result restricts the location of outer area minimizing stationary hypersurfaces to within a large coordinate sphere. We will now prove that they actually collapse to S as $\epsilon \to 0$. The proof is inspired by the proof of [5, Theorem 3.2].

PROPOSITION 4.5. *Let $R > 0$. For sufficiently small $\epsilon > 0$, every outer area minimizing stationary hypersurface Σ in $(\mathbb{R}^n \setminus S, g_\epsilon)$ with $\mathcal{H}_{n-3}(\mathrm{sing}(\Sigma)) = 0$ is enclosed by $\mathrm{Tub}(S, R)$.*

Proof. Suppose for contradiction that there is a sequence $\epsilon_k \to 0$, a sequence Σ_k of outer area minimizing stationary hypersurfaces in $(\mathbb{R}^n \setminus S, g_{\epsilon_k})$, and a sequence of points $x_k \in \overline{\Sigma}_k$ such that $\mathrm{dist}^\delta(x_k, S) > R$. Note that g_{ϵ_k} converges to the Euclidean metric δ uniformly on compact subsets in $\mathbb{R}^n \setminus S$. Moreover, since Σ_k is outer area minimizing we have $\mathcal{H}_{n-1}^{g_{\epsilon_k}}(\Sigma_k) \leq \mathcal{H}_{n-1}^{g_{\epsilon_k}}(S^{n-1}(R_{\mathrm{end}}))$ which is uniformly bounded. By Theorem 4.2, applied with Ω equal to $\mathbb{R}^n \setminus S$ and K being the closure of the region between $\mathrm{Tub}(S, R)$ and $S^{n-1}(R_{\mathrm{end}})$, there is then a nonempty stable hypersurface Σ_∞ in $(\mathbb{R}^n \setminus S, \delta)$ such that $\mathcal{H}_\alpha^\delta(\mathrm{sing}(\Sigma_\infty) \setminus S) = 0$ for all nonnegative $\alpha > n - 8$ and

$$\mathcal{H}_{n-1}^\delta(\Sigma_\infty \cap \Omega') \leq \limsup_{k \to \infty} \mathcal{H}_{n-1}^{g_{\epsilon_k}}(\Sigma_k \cap \Omega')$$

for all compact sets $\Omega' \subset \mathbb{R}^n \setminus S$. Moreover, Σ_∞ is enclosed by $S^{n-1}(R_{\mathrm{end}})$ since this holds for each Σ_k by Proposition 4.4.

The hypersurface Σ_∞ is stationary in $(\mathbb{R}^n \setminus S, \delta)$. We now prove that it is stationary as a hypersurface in $(\mathbb{R}^n, \delta)$. We need to prove that

$$\int_{\Sigma_\infty} \mathrm{div}_{\Sigma_\infty}^\delta(X) \, d\mathcal{H}_{n-1}^\delta = 0$$

for all compactly supported vector fields X on $\mathbb{R}^n$. Here $\mathrm{div}_{\Sigma_\infty}^\delta(X)$ is the divergence of X along the hypersurface Σ_∞ in the Euclidean metric δ. For this, we define a family of smooth cut-off functions η_ρ for all sufficiently small $\rho > 0$ by

$$\eta_\rho(x) := \eta\left(\frac{\mathrm{dist}^\delta(x, S)}{\rho}\right)$$

where $\eta: [0, \infty) \to [0, 1]$ is a smooth increasing function such that $\eta(1) = 0$ and $\eta(2) = 1$, with derivative $\eta'(t) < 2$ for all t. Since $\Sigma_k \to \Sigma_\infty$, and each Σ_k is outer area minimizing so that Lemma 4.1 is applicable, there are constants C_1, C_2, and C_3

such that

$$
\begin{aligned}
\int_{\Sigma_\infty} |d\eta_\rho|\, d\mathcal{H}^\delta_{n-1} &\leq \frac{2}{\rho}\mathcal{H}^\delta_{n-1}(\Sigma_\infty \cap \mathrm{supp}(d\eta_\rho)) \\
&= \frac{2}{\rho}\mathcal{H}^\delta_{n-1}(\Sigma_\infty \cap \{x \in \mathbb{R}^n \mid \rho < \mathrm{dist}^\delta(x,S) < 2\rho\}) \\
&\leq \frac{C_1}{\rho}\mathcal{H}^{g_{\epsilon_k}}_{n-1}(\Sigma_k \cap \{x \in \mathbb{R}^n \mid \rho < \mathrm{dist}^\delta(x,S) < 2\rho\}) \\
&\leq \frac{C_1}{\rho}\mathcal{H}^{g_{\epsilon_k}}_{n-1}(\mathrm{Tub}(S,\rho) \cup \mathrm{Tub}(S,2\rho)) \\
&\leq \frac{C_2}{\rho}\mathcal{H}^\delta_{n-1}(\mathrm{Tub}(S,\rho) \cup \mathrm{Tub}(S,2\rho)) \\
&= C_3\rho^{n-m-2},
\end{aligned}
$$

where the intermediate inequalities hold for all sufficiently large k. Since Σ_∞ is stationary as a hypersurface in $(\mathbb{R}^n \setminus S, \delta)$ it holds that

$$
\int_{\Sigma_\infty} \mathrm{div}^\delta_{\Sigma_\infty}(\eta_\rho X)\, d\mathcal{H}^\delta_{n-1} = 0.
$$

Let π denote the δ-orthogonal projection of $T\mathbb{R}^n$ onto $T\Sigma_\infty$. Since $\mathrm{div}^\delta_{\Sigma_\infty}(\eta_\rho X) = \eta_\rho\, \mathrm{div}^\delta_{\Sigma_\infty}(X) + \pi(X)(\eta_\rho)$ and $n - m - 2 \geq 1$ we have

$$
\begin{aligned}
\left| \int_{\Sigma_\infty} \mathrm{div}^\delta_{\Sigma_\infty}(X)\, d\mathcal{H}^\delta_{n-1} \right| &= \lim_{\rho \to 0} \left| \int_{\Sigma_\infty} \eta_\rho\, \mathrm{div}^\delta_{\Sigma_\infty}(X)\, d\mathcal{H}^\delta_{n-1} \right| \\
&= \lim_{\rho \to 0} \left| \int_{\Sigma_\infty} \pi(X)(\eta_\rho)\, d\mathcal{H}^\delta_{n-1} \right| \\
&\leq \left(\sup_{\Sigma_\infty} |X| \right) \lim_{\rho \to 0} \int_{\Sigma_\infty} |d\eta_\rho|\, d\mathcal{H}^\delta_{n-1} \\
&\leq \left(\sup_{\Sigma_\infty} |X| \right) \lim_{\rho \to 0} C_3\rho^{n-m-2} \\
&= 0.
\end{aligned}
$$

Hence Σ is stationary as a hypersurface in $(\mathbb{R}^n, \delta)$.

Let $L\colon \mathbb{R}^n \to \mathbb{R}$ be a nonzero linear function. Let $a := \sup_{x \in \Sigma} L(x)$. Since $\overline{\Sigma}$ is compact, $a < \infty$. The hyperplane $L^{-1}(a)$ is a connected minimal hypersurface in $(\mathbb{R}^n, \delta)$ and Σ is stationary. According to the maximum principle of Solomon and White [18, Theorem, p. 686 and Remarks 1 and 2, pp. 690-691] and [19, Theorem 4], the hyperplane $L^{-1}(a)$ is contained in $\overline{\Sigma}$ which contradicts compactness of $\overline{\Sigma}$. □

We summarize the results of this section in the following proposition.

PROPOSITION 4.6. *For all sufficiently small $\epsilon > 0$, every outer area minimizing stationary hypersurface Σ in $(\mathbb{R}^n \setminus S, g_\epsilon)$ with $\mathcal{H}_{n-3}(\mathrm{sing}(\Sigma)) = 0$ must be contained in the region between $\mathrm{Tub}(S, C_{\mathrm{inner}}\epsilon)$ and $\mathrm{Tub}(S, C_{\mathrm{outer}}\epsilon)$.*

Proof. By Proposition 4.4 and Proposition 4.5 it is sufficient to prove that Σ does not intersect the region between $\mathrm{Tub}(S, C_{\mathrm{outer}}\epsilon)$ and $\mathrm{Tub}(S, R_{\mathrm{outer}})$. Proposition 3.4 tells us that this region is foliated by hypersurfaces with positive mean curvature, which means that Σ cannot intersect it without contradicting the maximum principle of Solomon and White, as in Proposition 4.4. □

5. Tubularity of outer area minimizing stationary hypersurfaces. In this section we will prove Proposition 5.2, which states that if $\epsilon > 0$ is sufficiently small, then every outer area minimizing stationary hypersurface Σ in $(\mathbb{R}^n \setminus S, g_\epsilon)$ which encloses the end at S and has $\mathcal{H}_{n-3}(\mathrm{sing}(\Sigma)) = 0$ is diffeomorphic to the unit normal bundle of S. The idea of the proof is the following. By using a contradiction argument it is sufficient to study a sequence of such hypersurfaces Σ_k close to x_k for a convergent sequence $(x_k, \epsilon_k) \to (x_\infty, 0)$. We prove that, after rescaling around the points x_k, the hypersurfaces Σ_k converge smoothly to a cylinder around $T_{x_\infty} S$. For this convergence we use the area bounds from Lemma 4.1, which we obtained by using the fact that the hypersurfaces are outer area minimizing. Since the rescaled hypersurfaces converge smoothly to a cylinder, they must be the union of graphs of a number of smooth functions on the cylinder provided k is sufficiently large. By using the outer area minimizing property again we can prove that, near x_k, each hypersurface is the graph of a single smooth function, thereby proving the proposition.

The contradiction part of the argument is contained in Proposition 5.2, while the local result is the following proposition.

PROPOSITION 5.1. *Let $x_k \to x_\infty$ be a convergent sequence of points in S and let $\epsilon_k \to 0$. For each k, let Σ_k be an outer area minimizing stationary hypersurface in $(\mathbb{R}^n \setminus S, g_{\epsilon_k})$ which encloses the end at S and has $\mathcal{H}_{n-3}(\mathrm{sing}(\Sigma_k)) = 0$. Then for all sufficiently large k there is an open set $U_k \subseteq S$ containing x_k such that, in normal coordinates, the set $\overline{\Sigma}_k \cap \exp^\delta((C_{\mathrm{inner}}\epsilon_k, C_{\mathrm{outer}}\epsilon_k)UNU_k)$ is the graph of a smooth function on the unit normal bundle UNU_k of U_k.*

Proof.   Throughout the proof, $\exp^\delta$ denotes the normal exponential map of $S \subset \mathbb{R}^n$ in the Euclidean metric δ, and we consider only $\epsilon > 0$ which are sufficiently small for $\exp^\delta \colon [C_{\mathrm{inner}}\epsilon, C_{\mathrm{outer}}\epsilon]UNS \to \mathbb{R}^n$ to be an embedding.

Step 1: Rescaling. Consider the metrics

$$h_k := \epsilon_k^{-2} \left(\Pi_{x_\infty, x_k}^{\epsilon_k}\right)^* (g_{\epsilon_k})$$

on

$$\Omega := T_{x_\infty}\mathbb{R}^n \setminus T_{x_\infty} S.$$

By Corollary 2.3 these metrics converge in $C^r(\Omega')$ for every $r \geq 0$ and every open set Ω' with compact closure $\overline{\Omega'} \subseteq \Omega$ to

$$h_\infty := u_\infty^{4/(n-2)}\delta$$

where

$$u_\infty(\zeta) = 1 + \int_{T_{x_\infty} S} |\zeta - \eta|^{-(n-2)}\, d\eta.$$

Define

$$\Gamma_k := \left(\Pi_{x_\infty, x_k}^{\epsilon_k}\right)^{-1} (\Sigma_k).$$

Then Γ_k is an outer area minimizing stable hypersurface for the metric h_k.

Step 2: Convergence. We are now going to apply Theorem 4.2 with Ω, h_k, h_∞, and Γ_k as defined in Step 1, and with $K := [C_{\mathrm{inner}}/2, 2C_{\mathrm{outer}}]UN_{x_\infty}S$. Proposition 4.6

tells us that the intersection $\Gamma_k \cap K$ is nonempty for sufficiently large k. By Lemma 4.1 it holds that

$$\mathcal{H}^{h_k}_{n-1}(\Gamma_k \cap \Omega') \leq \mathcal{H}^{h_k}_{n-1}(\partial \Omega')$$

for every open set Ω' with compact closure $\overline{\Omega'} \subseteq \Omega$ and piecewise smooth boundary. This, together with the smooth convergence $h_k \to h_\infty$, tells us that

$$\limsup_{k \to \infty} \mathcal{H}^{h_k}_{n-1}(\Gamma_k \cap \Omega') \leq \limsup_{k \to \infty} \mathcal{H}^{h_k}_{n-1}(\partial \Omega') = \mathcal{H}^{h_\infty}_{n-1}(\partial \Omega').$$

With these choices, Theorem 4.2 is applicable, giving a stable hypersurface Γ_∞ and a subsequence of Γ_k, still denoted Γ_k, which are such that
- $\Gamma_k \to \Gamma_\infty$ as varifolds,
- $\mathcal{H}^{h_\infty}_\alpha(\text{sing}(\Gamma_\infty)) = 0$ if $\alpha > n - 8$ and $\alpha \geq 0$,
- $\Gamma_\infty \cap K \neq \emptyset$,
- Γ_∞ is a stable hypersurface in the metric h_∞,
- for every open set Ω' with compact closure $\overline{\Omega'} \subseteq \Omega \setminus \text{sing}(\Gamma_\infty)$
  - $\mathcal{H}^{h_\infty}_{n-1}(\Gamma_\infty \cap \Omega') \leq \limsup_{k \to \infty} \mathcal{H}^{h_k}_{n-1}(\Gamma_k \cap \Omega')$, and
  - the subsequence Γ_k converges smoothly to Γ_∞ (disregarding multiplicity) on Ω'.

By the above,

$$\mathcal{H}^{h_\infty}_{n-1}(\Gamma_\infty \cap \Omega') \leq \mathcal{H}^{h_\infty}_{n-1}(\partial \Omega'). \tag{5}$$

Step 3: Identifying the limit. We will now show that $\Gamma_\infty = \text{Tub}(T_{x_\infty} S, \hat{a})$ with $\hat{a}$ as in Lemma 3.3. Suppose for contradiction that $\Gamma_\infty \setminus \text{Tub}(T_{x_\infty} S, \hat{a}) \neq \emptyset$. Then either Γ_∞ contains points strictly outside of $\text{Tub}(T_{x_\infty} S, \hat{a})$, or it contains points strictly inside of $\text{Tub}(T_{x_\infty} S, \hat{a})$. We derive a contradiction only in the former case; the argument for a contradiction in the latter case is analogous. Let

$$a := \sup_{\xi \in \Gamma_\infty} \text{dist}(\xi, T_{x_\infty} S) > \hat{a}$$

as illustrated in Figure 2. We will use a sequence of translations and an application of Theorem 4.2 to reduce to the case where the supremum is attained.

Let $\xi_i \in \Gamma_\infty$ be a sequence of points such that $\text{dist}(\xi_i, T_{x_\infty} S) \to a$. As above, let $K := [C_{\text{inner}}/2, 2C_{\text{outer}}]UN_{x_\infty} S$. Let Ω_K be an open neighborhood of K with compact closure $\overline{\Omega_K} \subset \Omega$ and piecewise smooth boundary. Note that the metric h_∞ has translational symmetry along $T_{x_\infty} S$. For each i let τ_i be a translation along $T_{x_\infty} S$ with $\tau_i(\xi_i) \in K$. Let $\Gamma^i_\infty := \tau_i(\Gamma_\infty) \cap \Omega_K$. Then, by the translation symmetry of h_∞ and by inequality (5),

$$\begin{aligned}
\mathcal{H}^{h_\infty}_{n-1}(\Gamma^i_\infty) &= \mathcal{H}^{h_\infty}_{n-1}(\tau_i(\Gamma_\infty) \cap \Omega_K) \\
&= \mathcal{H}^{h_\infty}_{n-1}(\Gamma_\infty \cap \tau_i^{-1}(\Omega_K)) \\
&\leq \mathcal{H}^{h_\infty}_{n-1}(\partial(\tau_i^{-1}(\Omega_K))) \\
&= \mathcal{H}^{h_\infty}_{n-1}(\partial \Omega_K).
\end{aligned}$$

This means that

$$\mathcal{H}^{h_\infty}_{n-1}(\Gamma^i_\infty \cap \Omega') \leq \mathcal{H}^{h_\infty}_{n-1}(\partial \Omega_K)$$
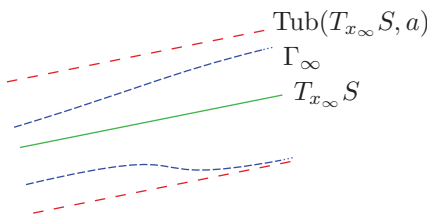
Fig. 2: *The surface Γ_∞ approaches* $\mathrm{Tub}(T_{x_\infty}S, a)$ *from the inside.*
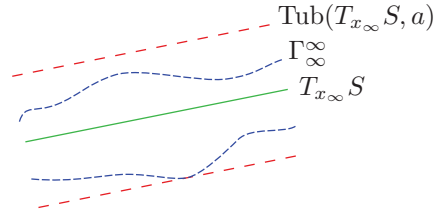
Fig. 3: *After translations, the surface Γ_∞^∞ is tangent to $\mathrm{Tub}(T_{x_\infty}S, a)$ from the inside.*

for every open set Ω' with compact closure $\overline{\Omega'} \subseteq \Omega_K$. The hypersurfaces Γ_∞^i all intersect the compact set K. By Theorem 4.2, a subsequence of Γ_∞^i converges to a stable hypersurface Γ_∞^∞.

Let ξ_∞ be a limit point of the sequence $\tau_i(\xi_i)$. Since $\Gamma_\infty^i \to \Gamma_\infty^\infty$ in Hausdorff distance (see [15, Remark 10 on p. 779]) it holds that $\xi_\infty \in \overline{\Gamma_\infty^\infty}$. This proves that the supremum a of $\mathrm{dist}(\cdot, T_{x_\infty}S)$ on $\overline{\Gamma_\infty^\infty}$ is attained at $\xi_\infty \in K \subset \Omega_K$. The stationary hypersurface $\Gamma_\infty^\infty \subset \Omega_K$ is contained in the region inside $\mathrm{Tub}(T_{x_\infty}S, a)$, and $\xi_\infty \in \overline{\Gamma_\infty^\infty} \cap \mathrm{Tub}(T_{x_\infty}S, a)$ as in Figure 3. Since we have assumed for contradiction that $a > \hat{a}$, Lemma 3.3 tells us that $\mathrm{Tub}(T_{x_\infty}S, a)$ has positive mean curvature. As in Proposition 4.4, this contradicts the maximum principle of Solomon and White. Hence $\sup_{\xi \in \Gamma_\infty} \mathrm{dist}(\xi, T_{x_\infty}S) = \hat{a}$. Similarly, $\inf_{\xi \in \Gamma_\infty} \mathrm{dist}(\xi, T_{x_\infty}S) = \hat{a}$, so that $\Gamma_\infty \subseteq \mathrm{Tub}(T_{x_\infty}S, \hat{a})$.

The hypersurfaces Σ_k enclose the end at S, so each $\overline{\Gamma}_k$ separates $\left(\Pi_{x_\infty, x_k}^{\epsilon_k}\right)^{-1}(S)$ from faraway points in $\mathbb{R}^n$, in the sense that every continuous curve from $\left(\Pi_{x_\infty, x_k}^{\epsilon_k}\right)^{-1}(S)$ from faraway points in $\mathbb{R}^n$ intersects $\overline{\Gamma}_k$. This means that the hypersurface $\overline{\Gamma}_\infty$ separates $T_{x_\infty}S$ from faraway points in the normal space $N_{x_\infty}S$. The only subset of $\mathrm{Tub}(T_{x_\infty}S, \hat{a})$ which separates $T_{x_\infty}S$ from such points is $\mathrm{Tub}(T_{x_\infty}S, \hat{a})$ itself, so $\overline{\Gamma}_\infty = \mathrm{Tub}(T_{x_\infty}S, \hat{a})$. By our convention, $\Gamma_\infty = \mathrm{reg}\,\Gamma_\infty$, so it holds that Γ_∞ is nonsingular.

Step 4: Determining the multiplicity. To simplify notation we introduce a *rescaling in the normal direction* defined for $r \in \mathbb{R}$ and $\zeta \in T_{x_\infty}\mathbb{R}^n$ by

$$r * \zeta = \pi(\zeta) + r(\zeta - \pi(\zeta)).$$

Recall that π denotes the orthogonal projection $\pi \colon T_{x_\infty}\mathbb{R}^n \to T_{x_\infty}S$.

In the previous step we found that the hypersurfaces Γ_k converge smoothly to $\mathrm{Tub}(T_{x_\infty}S, \hat{a})$ on each compact set, if we disregard multiplicities. We will now prove that the multiplicity is one. Choose an open neighborhood $U \subset T_{x_\infty}S$ of 0 with compact closure. The smooth convergence of Γ_k to $\mathrm{Tub}(T_{x_\infty}S, \hat{a})$ on $(C_{\mathrm{inner}}, C_{\mathrm{outer}}) * \mathrm{Tub}(U, 1)$ implies for any $\rho > 0$ that for all sufficiently large k, there is a finite set of smooth functions

$$\Psi_k^1, \ldots, \Psi_k^N \colon \mathrm{Tub}(U, 1) \to (C_{\mathrm{inner}}, C_{\mathrm{outer}})$$

such that

$$\sup_{\substack{\zeta \in \mathrm{Tub}(U,1) \\ i \in \{1, \ldots, N\}}} |\Psi_k^i(\zeta) - \hat{a}| < \rho$$
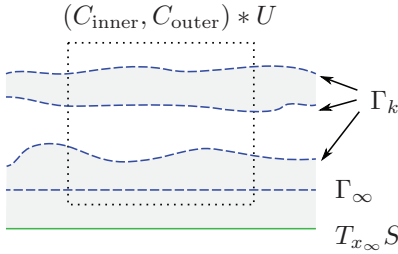
Fig. 4: *The hypersurface Γ_k is the union of graphs of functions Ψ_k^i. The set E_k is shown in light grey.*
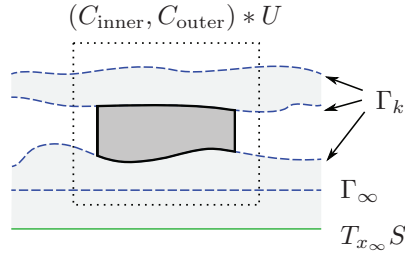
Fig. 5: *The set D_k is shown in dark grey.*

and such that the maps $\omega \mapsto \Psi_k^i(\omega) * \omega$ together give a surjective map

$$\bigsqcup_{i \in \{1,\ldots,N\}} \mathrm{Tub}(U, 1) \to \Gamma_k \cap (C_{\mathrm{inner}}, C_{\mathrm{outer}}) * \mathrm{Tub}(U, 1),$$

see Fig. 4. The functions $(\Psi_k^i)_{1 \le i \le N}$ may agree at some points, but since we assume that $\mathcal{H}_{n-3}(\mathrm{sing}(\Sigma_k)) = 0$, this intersection set has zero $(n-3)$-dimensional Hausdorff measure. Thus, we may order the functions so that $\Psi_k^1 < \Psi_k^2 < \cdots < \Psi_k^N$ except on a set with zero $(n-3)$-dimensional Hausdorff measure.

Our goal is now to show that $N = 1$. To do this, we will use an argument similar to the one we used in Lemma 4.1 to obtain area bounds. Since Γ_k is bounding, it encloses a region E_k. Choose an open neighborhood $V \subset T_{x_\infty} S$ of 0 with smooth boundary such that $\overline{V} \subset U$. Let

$$D_k := \{r * \omega \mid \omega \in \mathrm{Tub}(V, 1), \Psi_k^1(\omega) \le r \le \Psi_k^{N-1}(\omega)\},$$

as shown in Figure 5. The boundary of D_k has a "vertical" component

$$\partial^\perp D_k := \{r * \omega \mid \omega \in \partial \mathrm{Tub}(V, 1), \Psi_k^1(\omega) \le r \le \Psi_k^{N-1}(\omega)\}.$$

Of course $E_k \subseteq E_k \cup D_k$ so $\partial(E_k \cup D_k)$ is an outward variation of Γ_k. In this variation the boundary is changed by adding the hypersurface $\partial(E_k \cup D_k) \setminus \partial E_k$ and removing the hypersurface $\partial E_k \setminus \partial(E_k \cup D_k)$. To estimate the area added and removed, we note that

$$\partial(E_k \cup D_k) \setminus \partial E_k \subseteq \partial^\perp D_k$$

and

$$\partial E_k \setminus \partial(E_k \cup D_k) = \bigcup_{i=1}^{N-1} \{\Psi_k^i(\omega) * \omega \mid \omega \in \mathrm{Tub}(V, 1)\},$$

up to sets of zero $(n-3)$-dimensional Hausdorff measure. On the set $(C_{\mathrm{inner}}, C_{\mathrm{outer}}) * \mathrm{Tub}(U, 1)$ all metrics h_k and h_∞ are conformal to the Euclidean metric with uniformly bounded conformal factors. Therefore there are constants C^+ and C^- depending on U, C_{inner}, and C_{outer}, but not depending on ρ and k, such that

$$\mathcal{H}_{n-1}^{h_k}(\partial^\perp D_k) \le \rho C^+ \mathcal{H}_{n-2}^{h_\infty}(\partial \mathrm{Tub}(V, 1))$$

and

$$\mathcal{H}_{n-1}^{h_k}\left(\{\Psi_k^i(\omega) * \omega \mid \omega \in \mathrm{Tub}(V,1)\}\right) \geq C^- \mathcal{H}_{n-1}^{h_\infty}(\mathrm{Tub}(V,1))$$

for all $1 \leq i \leq N-1$.

The hypersurface Γ_k is outer area minimizing, since Σ_k is outer area minimizing by assumption. Hence $\partial(E_k \cup D_k)$ has at least the same area as Γ_k in h_k. This means that

$$\mathcal{H}_{n-1}^{h_k}(\partial E_k \setminus \partial(E_k \cup D_k)) \leq \mathcal{H}_{n-1}^{h_k}(\partial(E_k \cup D_k) \setminus \partial E_k).$$

With the above estimates we get

$$N - 1 \leq \frac{\rho C^+ \mathcal{H}_{n-2}^{h_\infty}(\partial \mathrm{Tub}(V,1))}{C^- \mathcal{H}_{n-1}^{h_\infty}(\mathrm{Tub}(V,1))}.$$

Choosing ρ such that the right hand side is less than 1 we conclude that $N = 1$. We have now proved that, for all sufficiently large k, there is a single function $\Psi_k := \Psi_k^1 \colon \mathrm{Tub}(U,1) \to (C_{\mathrm{inner}}, C_{\mathrm{outer}})$ such that the map $\omega \mapsto \Psi_k(\omega) * \omega$ is a diffeomorphism

$$\mathrm{Tub}(U,1) \to \Gamma_k \cap (C_{\mathrm{inner}}, C_{\mathrm{outer}}) * \mathrm{Tub}(U,1).$$

We will now use the function Φ_k to write Σ_k as a graph. Let $P_k := \Pi_{x_\infty, x_k}^{\epsilon_k}(T_{x_\infty} S)$ of $\mathbb{R}^n$. This is the tangent space of S at x_∞ considered as an affine subspace of $\mathbb{R}^n$. Define the map

$$\widehat{\Psi}_k \colon UNP_k \supset UN\widehat{U}_k \to \mathrm{Tub}(P_k, \epsilon_k) \to \mathrm{Tub}(T_{x_\infty} S, 1) \to (C_{\mathrm{inner}}\epsilon_k, C_{\mathrm{outer}}\epsilon_k)$$

by

$$\widehat{\Psi}_k(\omega) := \epsilon_k \Phi_k \left(\left(\Pi_{x_\infty, x_k}^{\epsilon_k} \right)^{-1} \left(\exp^\delta(\epsilon_k \omega) \right) \right)$$

where $\widehat{U}_k := \Pi_{x_\infty, x_k}^{\epsilon_k}(U) \subset P_k$. Let

$$\widehat{\Omega}_k := \exp^\delta((C_{\mathrm{inner}}\epsilon_k, C_{\mathrm{outer}}\epsilon_k)UN\widehat{U}_k).$$

With these choices, $\overline{\Sigma}_k$ is the graph of $\widehat{\Psi}_k$ over $\widehat{U}_k$ in the sense that

$$\overline{\Sigma}_k \cap \widehat{\Omega}_k = \exp^\delta \left(\{ \widehat{\Psi}_k(\omega)\omega \mid \omega \in UN\widehat{U}_k \} \right).$$

If $|d\widehat{\Psi}_k|$ is bounded and there is an open set $\widetilde{\Omega}_k \subseteq \widehat{\Omega}_k$ containing x_k in which the tangent spaces to S are sufficiently close to being parallel to P_k, then $\overline{\Sigma}_k$ is the graph of a smooth function $\widetilde{\Psi}_k$ over some open set $\widetilde{U}_k \subset S$ containing x_k in the sense that

$$\overline{\Sigma}_k \cap \widetilde{\Omega}_k = \exp^\delta \left(\{ \widetilde{\Psi}_k(\omega)\omega \mid \omega \in UN\widetilde{U}_k \} \right),$$

as shown in Figure 6. By letting k be sufficiently large, and letting $\widetilde{\Omega}_k \subseteq \widehat{\Omega}_k$ be sufficiently small, this condition on the tangent spaces of S is satisfied. Moreover, $|d\widehat{\Psi}_k|$ is bounded since $|d\Psi_k| \to 0$ when $k \to 0$. Hence, in normal coordinates, the set $UN\widetilde{U}_k \cap \overline{\Sigma}_k$ is the graph of a smooth function on the unit normal bundle $UN\widetilde{U}_k$ of $\widetilde{U}_k$. Choosing $U_k = \widetilde{U}_k$ completes the proof. $\square$
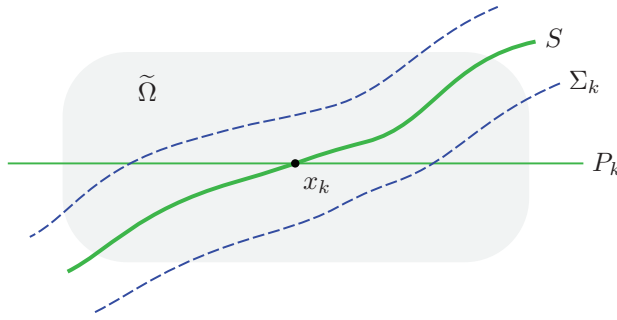
Fig. 6: *Since Σ_k is the graph of a function on UNP_k with bounded gradient and S is close to P_k, it follows that it is also the graph of a function on $UN\widetilde{U}_k$ for some $\widetilde{U}_k \subset S$.*

Proposition 5.1 globalizes in the following manner.

PROPOSITION 5.2. *For sufficiently small $\epsilon > 0$ it holds that every outer area minimizing stationary hypersurface Σ in $(\mathbb{R}^n \setminus S, g_\epsilon)$ which encloses the end at S and has $\mathcal{H}_{n-3}(\mathrm{sing}(\Sigma)) = 0$ is diffeomorphic to the unit normal bundle UNS. In fact, each such hypersurface Σ is the graph of a smooth function on UNS in normal coordinates for S.*

Proof. Suppose for contradiction that this is not true. Then there is a sequence $\epsilon_k \to 0$ and outer area minimizing stationary hypersurfaces Σ_k in $(\mathbb{R}^n \setminus S, g_{\epsilon_k})$ such that Σ_k is not the graph of a smooth function on UNS in normal coordinates for S. From Proposition 4.6 we know that $\overline{\Sigma}$ is contained in the image of $(C_{\mathrm{inner}}\epsilon_k, C_{\mathrm{outer}}\epsilon_k)UNS$ in normal coordinates. Hence, for each k there is a point $x_k \in S$ such that if $U_k \subseteq S$ is an open neighborhood of x_k then $\overline{\Sigma}_k \cap \exp^\delta((C_{\mathrm{inner}}\epsilon_k, C_{\mathrm{outer}}\epsilon_k)UNU_k)$ is not the graph of a smooth function on UNU_k in normal coordinates. Taking a convergent subsequence $x_k \to x_\infty$, this contradicts Proposition 5.1. $\square$

6. Proof of the main theorem. In this section we prove existence and uniqueness of an *outermost* outer area minimizing stationary hypersurface Σ_ϵ which encloses the end at S and has $\mathcal{H}_{n-3}(\mathrm{sing}(\Sigma_\epsilon)) = 0$, in other words one which is not enclosed by any other such hypersurface. Further, we prove that this hypersurface is the outermost apparent horizon, which by definition is the boundary of the trapped region. This, combined with the conclusion of the previous section proves our main theorem.

There are general results in dimensions $3 \leq n \leq 7$ stating that the boundary of the trapped region is smooth, and hence is the unique outermost outer area minimizing stationary hypersurface; see [7, Theorem 5.1] and [1, Theorem 4.6]. In our special case, the uniqueness can be deduced by a simpler argument which does not need the dimensional restriction.

First, we prove that there are outer area minimizing stationary hypersurfaces outside any obstacle in the form of a bounding hypersurface with negative mean curvature.

PROPOSITION 6.1. *Let T be a bounding nonsingular hypersurface in $(\mathbb{R}^n \setminus S, g_\epsilon)$ which encloses the end at S and has negative mean curvature. There is at least one outer area minimizing stationary hypersurface Σ in $(\mathbb{R}^n \setminus S, g_\epsilon)$ which encloses T and is such that $\mathcal{H}_\alpha^{h\infty}(\mathrm{sing}(\Sigma)) = 0$ if $\alpha > n - 8$ and $\alpha \geq 0$.*

Proof. Let $L \subset \mathbb{R}^n$ be the closure of the region inside T. Let $\Omega \subset \mathbb{R}^n$ be the open region between T and $S^{n-1}(R_{\text{end}})$. Consider the Caccioppoli sets F such that $L \subseteq F \subseteq L \cup \Omega$. By the existence theorem for minimal sets (see for instance [12, Theorem 1.20]) there is a set E which minimizes perimeter among all such sets. This minimizer E may be chosen such that $\partial E = \overline{\partial^* E}$; see [12, Theorem 4.4]. Here $\partial^* E$ is the reduced boundary of E; see [12, Definition 3.3].

The reduced boundary $\partial^* E$ is a rectifiable varifold (see [17, Theorem 14.3]) which is area minimizing with respect to T and $S(R_{\text{end}})$ as obstacles, since E minimizes perimeter with respect to such variations. By Proposition 4.3 the sphere $S^{n-1}(R_{\text{end}})$ has positive mean curvature, and by hypothesis T has negative mean curvature. Hence it follows from the maximum principle of Solomon and White [18, Theorem, p. 686 and Remarks 1 and 2, pp. 690-691] and [19, Theorem 4] that $\overline{\partial^* E}$ does not intersect $\partial \Omega$. This means that $\partial^* E$ is actually a solution to an area minimization problem without obstacles, and hence smooth (compare [12, Theorem 8.4]). Let $\Sigma = \partial^* E$. Now $\operatorname{sing} \Sigma \subseteq \partial E \setminus \partial^* E$, and it holds by [12, Theorem 11.8] that $\mathcal{H}_\alpha(\operatorname{sing}(\partial E \setminus \partial^* E)) = 0$ for all $\alpha > n - 8$ and $\alpha \geq 0$.

By construction, Σ encloses T. It coincides with the global area minimizer outside of T since the global area minimizer cannot intersect the region outside $S^{n-1}(R_{\text{end}})$ (which by Proposition 4.3 can be foliated by positive mean curvature spheres) without contradicting the maximum principle of Solomon and White. Hence Σ is also outer area minimizing. $\square$

By Proposition 3.5, the tubular hypersurface $\operatorname{Tub}(S, C_{\text{inner}}\epsilon)$ has negative mean curvature if $\epsilon > 0$ is sufficiently small, which yields the following.

COROLLARY 6.2. *For all sufficiently small $\epsilon > 0$ there is at least one outer area minimizing stationary hypersurface Σ in $(\mathbb{R}^n \setminus S, g_\epsilon)$ which encloses the end at S and is such that $\mathcal{H}_\alpha^{h\infty}(\operatorname{sing}(\Sigma)) = 0$ if $\alpha > n - 8$ and $\alpha \geq 0$.*

Note that it follows from Proposition 5.2 that the hypersurface Σ from Corollary 6.2 is nonsingular.

To prove that there is a unique outermost outer area minimizing stationary hypersurface, we also need to be able to find such hypersurfaces outside the union of two obstacles with possibly nonempty singular parts. However, in this case, we only need to consider obstacles which are outer area minimizing and stationary.

PROPOSITION 6.3. *Let Σ_1 and Σ_2 be outer area minimizing stationary hypersurfaces in $(\mathbb{R}^n \setminus S, g_\epsilon)$ which enclose the end at S and have $\mathcal{H}_{n-3}(\operatorname{sing}(\Sigma_1)) = \mathcal{H}_{n-3}(\operatorname{sing}(\Sigma_2)) = 0$. If $\epsilon > 0$ is sufficiently small, there is an outer area minimizing stationary hypersurface Σ which encloses both Σ_1 and Σ_2, and is such that $\mathcal{H}_\alpha^{h\infty}(\operatorname{sing}(\Sigma)) = 0$ if $\alpha > n - 8$ and $\alpha \geq 0$.*

Proof. Assume without loss of generality that Σ_1 and Σ_2 have no common connected components.

It follows from Proposition 5.2 that Σ_1 and Σ_2 are nonsingular. Let E_1 denote the closure of the region inside Σ_1 and let E_2 denote the closure of the region inside Σ_2. Let $L := E_1 \cup E_2 \subset \mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be the open region between ∂L and $S^{n-1}(R_{\text{end}})$. As in Proposition 6.1 we obtain a Caccioppoli set E with $\partial E = \overline{\partial^* E}$ which minimizes perimeter among those which contain $E_1 \cup E_2$. Let $\Sigma = \partial^* E$. By the maximum principle of Solomon and White, $\overline{\Sigma}$ cannot intersect the outer boundary $S^{n-1}(R_{\text{end}})$.

We begin by proving that $\overline{\Sigma}$ cannot intersect $\Sigma_1 \cap \Sigma_2$. Suppose for contradiction that there is a point $x \in \overline{\Sigma} \cap \Sigma_1 \cap \Sigma_2$. Let ν_i denote the outward-directed unit normal

vector field of Σ_i. It cannot hold that $\nu_1 = -\nu_2$ anywhere, since Proposition 5.2 tells us that Σ_1 and Σ_2 are graphs over UNS in normal coordinates. Let $(\nu_*)_x$ be the unit vector in the direction $(\nu_1)_x + (\nu_2)_x$, so that $g_\epsilon((\nu_*)_x, (\nu_i)_x) > 0$.

For the proof we need, in a neighborhood of x, a stationary nonsingular hypersurface Σ_* with $x \in \Sigma_* \subset \overline{E_1 \cup E_2}$, and normal vector $(\nu_*)_x$ at x. If $\nu_1 = \nu_2$ at x, we choose Σ_* to be an open neighborhood of x in Σ_1. If not, then the intersection of Σ_1 and Σ_2 is transverse, in which case $I := \Sigma_1 \cap \Sigma_2$ is a smooth submanifold of codimension 2 in $\mathbb{R}^n$. Extend ν_* be the unit vector field on I in direction $\nu_1 + \nu_2$. Let Σ_* be a hypersurface which contains I, is orthogonal to ν_* along I, and has nonpositive mean curvature. This exists since the mean curvature of I in direction ν_* can be compensated by the curvature in the direction orthogonal to I and ν_*. After shrinking Σ_* if necessary it holds that $\Sigma_* \subset \overline{E_1 \cup E_2}$. Extend ν_* to a unit normal vector field on all of Σ_*.

If $\Sigma_* \subseteq \overline{\Sigma}$, it holds that Σ_*, Σ_1 and Σ_2 all agree in a neighborhood of x since Σ encloses Σ_1 and Σ_2, which in turn enclose Σ_*. This proves that $\overline{\Sigma} \cap \Sigma_1 \cap \Sigma_2$ is an open subset of both Σ_1 and Σ_2. It is also a closed subset since $\overline{\Sigma}$, Σ_1 and Σ_2 are closed sets. Since we assumed that Σ_1 and Σ_2 have no common connected component, this is a contradiction.

Suppose instead that Σ_* is not a subset of $\overline{\Sigma}$. The proof of [18, Theorem, p. 686] then gives a vector field v which defines a variation which strictly decreases the area of $\overline{\Sigma}$. It is not obvious that this vector field is outward-directed along Σ_1 and Σ_2. However, by choosing ϵ and s in [18, pp. 687-690] sufficiently small, we may arrange that the functions $u_{s,t,\epsilon}$ in that proof are C^1-close to the function u for all t sufficiently close to 0. This means that τ may be chosen arbitrarily close to 0, so that the function $u_{s,\tau}$ in the proof is arbitrarily close to u in C^1 norm. This, in turn, means that the vector field v can be chosen arbitrarily close to our vector field ν_*, so that v is outward-directed along Σ_1 and Σ_2 in a neighborhood of x. Since Σ minimizes area outside of $E_1 \cup E_2$ and the variation along v decreases area, we have a contradiction. This completes the proof that $\overline{\Sigma} \cap \Sigma_1 \cap \Sigma_2 = \emptyset$.

Let $x \in \overline{\Sigma} \cap \Sigma_1$. Then $x \notin \Sigma_2$, so we can apply the maximum principle of Solomon and White to conclude that x is an interior point of $\overline{\Sigma} \cap \Sigma_1$ in Σ_1. The set $\overline{\Sigma} \cap \Sigma_1$ is closed in Σ_1. Hence every connected component of Σ_1 is either contained in $\overline{\Sigma}$ or disjoint from $\overline{\Sigma}$. Similarly, every connected component of Σ_2 is either contained in $\overline{\Sigma}$ or disjoint from $\overline{\Sigma}$.

Fix $i \in \{1, 2\}$. We will show, using the argument from [19, Theorem 4], that every connected component of $\overline{\Sigma}$ either coincides with a connected component of Σ_i or is disjoint from Σ_i. Suppose for contradiction that there is a point $x \in \Sigma_i \cap (\overline{\Sigma} \setminus \Sigma_i)$. Let W be the connected component of Σ_1 containing x. Let $W' = \Sigma - W$, where we view Σ and W as unit density rectifiable varifolds. Since W is stationary and Σ minimizes area to first order in the complement of $E_1 \cup E_2$, it holds that W' minimizes area to first order in this region. We can then apply the maximum principle of Solomon and White in a neighborhood of x to see that the support of W' includes W, which is a contradiction. Hence every connected component of $\overline{\Sigma}$ either coincides with a connected component of Σ_i or is disjoint from Σ_i.

The connected components which coincide with connected components of Σ_1 or Σ_2 are stationary, since the hypersurfaces Σ_1 and Σ_2 are stationary. The connected components which are disjoint from Σ_1 and Σ_2 are solutions to an area minimization problem without obstacles, and hence stationary. This means that Σ is stationary. Since it is the global area minimizer in the complement of $E_1 \cup E_2$ it is also outer area

minimizing. As in Proposition 6.1 it follows that $\mathcal{H}_\alpha^{h_\infty}(\mathrm{sing}(\Sigma)) = 0$ if $\alpha > n - 8$ and $\alpha \geq 0$. □

Having proved that there is an outer area minimizing stationary hypersurface enclosing the end at S, we turn our attention to proving that there is an outermost such hypersurface, and that this outermost hypersurface is unique.

PROPOSITION 6.4. *For all sufficiently small $\epsilon > 0$, there is a unique outermost outer area minimizing stationary hypersurface Σ_ϵ in $(\mathbb{R}^n \setminus S, g_\epsilon)$ which encloses the end at S and has $\mathcal{H}_{n-3}(\mathrm{sing}(\Sigma_\epsilon)) = 0$.*

Proof. We begin by proving that there is at least one outermost outer area minimizing stationary hypersurface. The set of outer area minimizing stationary hypersurfaces Σ which enclose the end at S and have $\mathcal{H}_{n-3}(\mathrm{sing}(\Sigma)) = 0$ is nonempty by Corollary 6.2 and forms a partially ordered set under the relation that $\Sigma_1 \leq \Sigma_2$ if Σ_2 encloses Σ_1. We want to prove that there is a maximal element under this partial order, since such an element is outermost. By Zorn's lemma it is sufficient to verify that every nonempty chain has an upper bound. Let A be a chain. For $\Sigma \in A$, let Ω_Σ be the interior of the region inside Σ. Then $\bigcup_{\Sigma \in A} \Omega_\Sigma$ is an open cover of itself, and since it is a subset of $\mathbb{R}^n$ it has a countable subcover $\bigcup_{i=1}^\infty \Omega_{\Sigma_i}$. The sequence $\Omega_{\Sigma_1} \subseteq \Omega_{\Sigma_2} \subseteq \ldots$ is increasing since A is a chain. By Theorem 4.2 there is a subsequence of Σ_k converging as varifolds to a stationary hypersurface Σ_∞ with $\mathcal{H}_\alpha^{g_\epsilon}(\mathrm{sing}(\Sigma_\infty)) = 0$ if $\alpha > n - 8$ and $\alpha \geq 0$. Then $\mathcal{H}_{n-1}^{g_\epsilon}(\Sigma_\infty) = \lim_{k \to \infty} \mathcal{H}_{n-1}^{g_\epsilon}(\Sigma_k)$, so since each Σ_k is outer area minimizing it holds that Σ_∞ is outer area minimizing. Hence Σ is an upper bound for the chain. Hence Zorn's lemma tells us that there is an outermost outer area minimizing stationary hypersurface Σ which encloses the end at S and has $\mathcal{H}_{n-3}(\mathrm{sing}(\Sigma)) = 0$.

Suppose that there were two different such hypersurfaces Σ_1 and Σ_2. Then the previous proposition produces one which encloses both of them, which is a contradiction. Hence there is a unique outermost outer area minimizing stationary hypersurface Σ_ϵ which encloses the end at S and has $\mathcal{H}_{n-3}(\mathrm{sing}(\Sigma_\epsilon)) = 0$. □

The uniqueness of Σ_ϵ implies that it coincides with the outermost apparent horizon, as shown in the following proposition.

PROPOSITION 6.5. *If $\epsilon > 0$ is sufficiently small, the hypersurface Σ_ϵ in $(\mathbb{R}^n \setminus S, g_\epsilon)$ is the boundary of the trapped region.*

Proof. We know from Proposition 5.2 that Σ_ϵ is nonsingular and hence contained in the trapped region $\mathcal{T}_\epsilon$ of $(\mathbb{R}^n \setminus S, g_\epsilon)$.

Let $x \in \partial \mathcal{T}_\epsilon$. We want to show that $x \in \Sigma_\epsilon$. Choose a sequence $x_k \to x$ contained in the interior of the trapped region. Then for each k there is a weakly outer trapped surface T_k enclosing the point x_k. Since T_k is weakly outer trapped, it also encloses the end at S. Let Σ_k denote an outer area minimizing stationary hypersurface enclosing T_k obtained from Proposition 6.1. Using Theorem 4.2 as in the proof of Proposition 6.4 we obtain a subsequential limit hypersurface Σ' which is still outer area minimizing, encloses the end at S, and encloses or contains x. Since Σ_ϵ is an outermost outer area minimizing stationary hypersurface, it encloses Σ' by Proposition 6.3. Hence x is enclosed by or contained in Σ_ϵ. This completes the proof. □

Finally, we prove the main theorem of this paper.

Proof of Theorem 1.1. There is a unique outermost outer area minimizing stationary hypersurface Σ_ϵ with $\mathcal{H}_{n-3}(\mathrm{sing}(\Sigma_\epsilon)) = 0$ by Proposition 6.4, and this hyper-

surface is a smooth graph over UNS in normal coordinates by Proposition 5.2. From Proposition 6.5 we know that Σ_ϵ is the outermost apparent horizon of $(\mathbb{R}^n \setminus S, g_\epsilon)$. □

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