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DEFORMATION OF K-THEORETIC CYCLES*

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Abstract. For X a d-dimensional smooth projective variety over a field k of characteristic 0, using higher algebraic K-theory, we study the following two questions asked by Mark Green and Phillip Griffiths in chapter 10 of [9] (page 186-190):

- (1) For each positive integer p satisfying $1 \leq p \leq d$, can one define the tangent space $TZ^p(X)$ to the cycle group $Z^p(X)$?
- (2) Obstruction issues.

The highlight is the appearance of negative K-groups which detect the obstructions to deforming cycles.

Key words. K-theory, algebraic cycles, deformation, tangent spaces, obstructions.

Mathematics Subject Classification. 14C25.

1. Introduction.

For X a d-dimensional smooth projective variety over a field k of characteristic 0, for each positive integer p satisfying $1 \leq p \leq d$, let $Z^p(X)$ denote the cycle group,

$$Z^p(X) = \bigoplus_{y \in X^{(p)}} \mathbb{Z} \cdot \overline{\{y\}}.$$

The following question is posed by Mark Green and Phillip Griffiths:

QUESTION 1.1 (page 186 [9]). *For X a d-dimensional smooth projective variety over a field k of characteristic 0, for each positive integer p satisfying $1 \leq p \leq d$, can one define the tangent space $TZ^p(X)$ to the cycle group $Z^p(X)$?*

Since the abelian group $Z^p(X)$ is not a complex manifold or a scheme, the known deformation theory, such as Kodaira-Spencer theory or the theory of Hilbert schemes, cannot apply to this question directly. We consider $Z^p(-)$ as a functor and attempt to define the tangent space to this functor as usual

$$TZ^p(X) := \mathrm{Ker}\{Z^p(X \times \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))) \xrightarrow{\varepsilon=0} Z^p(X)\},$$

where $k[\varepsilon]/(\varepsilon^2)$ is the ring of dual numbers. Unfortunately, the classical definition of algebraic cycles cannot distinguish nilpotent, $Z^p(X \times \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))) = Z^p(X)$, so this definition is clearly not the desirable one.

Green-Griffiths has answered this question for $p = 1$ (divisors) and $p = \dim(X)$ (0-cycles) and leave the general case as an open question in [9]. Much of their theory was extended by Benjamin Dribus, Jerome W. Hoffman and the author in [6, 17]. To give an example of what tangent spaces to cycle groups are, we recall

DEFINITION 1.2 (page 84-85 and page 141 [9]). For X a smooth projective surface over a field k of characteristic 0, the tangent space $TZ^2(X)$ to the group of 0-cycles on X is defined to be

$$TZ^2(X) := \bigoplus_{x \in X^{(2)}} H_x^2(\Omega^1_{X/\mathbb{Q}}).$$

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The tangent space $TZ^2_{rat}(X)$ to the group of 0-cycles rationally equivalent to zero is defined to be

$$TZ^2_{rat}(X) := \mathrm{Im}(\partial_1^{1,-2}),$$

where $\partial_1^{1,-2}$ is the differential of the Cousin complex of $\Omega^1_{X/\mathbb{Q}}$,

$$0 \to \Omega^1_{k(X)/\mathbb{Q}} \to \bigoplus_{y \in X^{(1)}} H^1_y(\Omega^1_{X/\mathbb{Q}}) \xrightarrow{\partial_1^{1,-2}} \bigoplus_{x \in X^{(2)}} H^2_x(\Omega^1_{X/\mathbb{Q}}) \to 0.$$

It is worth noting that absolute differentials and local cohomology appear in this definition.

Moreover, Green-Griffiths points out that (page 186 [9]):

> *The technical issue that arises in trying straightforwardly extend the definitions given in the text for $p = n^1, 1$ concerns cycles that are linear combinations of irreducible subvarieties*
>
> $$Z = \sum_i n_i Z_i,$$
>
> *where some Z_i may not be the support of a locally Cohen-Macaulay scheme.*

To handle this technical issue, we look at generic points of Z_is and need to use higher algebraic K-theory. In Section 2, we propose a definition of $TZ^p(X)$ in Definition 2.6 for general p, generalizing Green-Griffiths' Definition 1.2 above.

Considering an element $\tau \in TZ^p(X)$ as a first order deformation, Green-Griffiths asks whether we can successively deform τ to infinite order. It is well known that the deformation of a subvariety Y, considered as an element of the Hilbert scheme $\mathrm{Hilb}(X)$, may be obstructed. However, Green-Griffiths predicts that we can eliminate obstructions, by considering Y as an element of $Z^p(X)$,

CONJECTURE 1.3 (page 187-190 [9]). *$TZ^p(X)$ is formally unobstructed, see Conjecture 3.7 and Conjecture 3.8 in Section 3.2.*

We answer this conjecture in Theorem 3.11. The main idea for answering Question 1.1 and Conjecture 1.3 is to use Milnor K-theoretic cycles to replace the classical algebraic cycles. In [3], Balmer defines K-theoretic Chow groups in terms of the derived category $D^{\mathrm{perf}}(X)$ obtained from the exact category of perfect complexes of O_X-modules. His idea is followed by Klein [12] and the author [17]. By modifying Balmer's K-theoretic Chow groups [3], we [17] extend Soulé's variant of Bloch-Quillen identification from X to its infinitesimally trivial deformations. In this note, we use K-theoretic techniques [6, 17] to study deformation of cycles, focusing on the geometry behind the formal definitions of K-theoretic cycles.

This note is organized as follows. We recall Milnor K-theoretic cycles and answer Green-Griffiths' Question 1.1 in Section 2.1, concrete examples of Milnor K-theoretic cycles from geometry (locally complete intersections) are also discussed. In Section 2.2 and Section 2.3, we explain two new aspects of Milnor K-theoretic cycles, which are different from Balmer's [3], featuring negative K-groups and Milnor K-theory.

The relation between obstructions and negative K-groups is discussed in Section 3.1. We discuss obstruction issues and answer Green-Griffiths' Conjecture 1.3 in Section 3.2.

[1] n is the dimension of X.

Notations and conventions.

(1). K-theory used in this note is Thomason-Trobaugh non-connective K-theory, if not stated otherwise.

(2). For any abelian group M, $M_{\mathbb{Q}}$ denotes the image of M in $M \otimes_{\mathbb{Z}} \mathbb{Q}$.

(3). $k[\varepsilon]/(\varepsilon^2)$ is the ring of dual numbers. $X[\varepsilon]$ denotes the first order trivial deformation of X, i.e., $X[\varepsilon] = X \times_k \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$.

2. First order deformation-tangent spaces. In this section, X is a d-dimensional smooth projective variety over a field k of characteristic 0. For each nonnegative integer j, $X_j := X \times_k \mathrm{Spec}(k[\varepsilon]/\varepsilon^{j+1})$ is the j-th order infinitesimally trivial deformation of X. In particular, $X_0 = X$.

Recall that Milnor K-groups with support are rationally defined in terms of eigenspaces of Adams operations in [17].

Definition 2.1 (Definition 3.2 in [17]). Let $x_j \in X_j^{(i)}$, for any integer m, Milnor K-group with support $K_m^M(O_{X_j,x_j} \text{ on } x_j)$ is rationally defined to be

$$K_m^M(O_{X_j,x_j} \text{ on } x_j) := K_m^{(m+i)}(O_{X_j,x_j} \text{ on } x_j)_{\mathbb{Q}},$$

where $K_m^{(m+i)}$ is the eigenspace for $\psi^k = k^{m+i}$ and ψ^k is the Adams operations.

For each positive integer p, there exists the following variant of Gersten complex, see Theorem 3.14 in [17],

$$0 \to \bigoplus_{x_j \in X_j^{(0)}} K_p^M(O_{X_j,x_j}) \to \cdots \to \bigoplus_{x_j \in X_j^{(p-1)}} K_1^M(O_{X_j,x_j} \text{ on } x_j)$$
$$\xrightarrow{d_{1,X_j}^{p-1,-p}} \bigoplus_{x_j \in X_j^{(p)}} K_0^M(O_{X_j,x_j} \text{ on } x_j) \xrightarrow{d_{1,X_j}^{p,-p}} \bigoplus_{x_j \in X_j^{(p+1)}} K_{-1}^M(O_{X_j,x_j} \text{ on } x_j) \to \cdots$$
$$\to \bigoplus_{x_j \in X_j^{(d)}} K_{p-d}^M(O_{X_j,x_j} \text{ on } x_j) \to 0.$$

Definition 2.2 (Definition 3.4 and Definition 3.15 in [17]). Let X be a d-dimensional smooth projective variety over a field k of characteristic 0, for each nonnegative integer j, let X_j be the j-th order infinitesimally trivial deformation of X. For each positive integer p satisfying $1 \leq p \leq d$, the p-th Milnor K-theoretic cycles and Milnor K-theoretic rational equivalence of X_j, denoted $Z_p^M(D^{\mathrm{Perf}}(X_j))$ and $Z_{p,rat}^M(D^{\mathrm{Perf}}(X_j))$, are defined as

$$Z_p^M(D^{\mathrm{Perf}}(X_j)) := \mathrm{Ker}(d_{1,X_j}^{p,-p}),$$

$$Z_{p,rat}^M(D^{\mathrm{Perf}}(X_j)) := \mathrm{Im}(d_{1,X_j}^{p-1,-p}).$$

The p-th Milnor K-theoretic Chow group of X_j is defined to be

$$CH_p^M(D^{\mathrm{perf}}(X_j)) := \frac{\mathrm{Ker}(d_{1,X_j}^{p,-p})}{\mathrm{Im}(d_{1,X_j}^{p-1,-p})}.$$

In Section 2.2, we explain the reason why we take the kernel of $d_{1,X_j}^{p,-p}$ to define $Z_p^M(D^{\mathrm{perf}}(X_j))$. The reason why we use Milnor K-groups with support (certain eigenspaces of K-groups) to define $Z_p^M(D^{\mathrm{perf}}(X_j))$ is explained in Section 2.3.

2.1. Definition of tangent spaces. Let X be a d-dimensional smooth projective variety over a field k of characteristic 0, we use $X[\varepsilon]$ to stand for the first order infinitesimally trivial deformation of X, i.e., $X[\varepsilon] = X \times_k \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$. For $Y \subset X$ a subvariety of codimension p, let $Y^{'}$ be a first order deformation of Y in $X[\varepsilon]$, that is, $Y^{'} \subset X[\varepsilon]$ such that $Y^{'}$ is flat over $\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$ and $Y^{'} \otimes_{\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))} \mathrm{Spec}(k) \cong Y$, then $i_*O_{Y'}$ can be resolved by a bounded complex of vector bundles on $X[\varepsilon]$, where $i : Y^{'} \to X[\varepsilon]$.

Let $D^{\mathrm{perf}}(X[\varepsilon])$ denote the derived category of perfect complexex of $O_{X[\varepsilon]}$-modules, and let $\mathcal{L}_{(i)}(X[\varepsilon]) \subset D^{\mathrm{perf}}(X[\varepsilon])$ be defined as

$$\mathcal{L}_{(i)}(X[\varepsilon]) := \{E \in D^{\mathrm{perf}}(X[\varepsilon]) \mid \mathrm{codim}_{\mathrm{Krull}}(\mathrm{supph(E)}) \geq -i\},$$

where the closed subset $\mathrm{supph(E)} \subset X$ is the support of the total homology of the perfect complex E. The resolution of $i_*O_{Y'}$, which is a perfect complex of $O_X[\varepsilon]$-module supported on Y, defines an element of the Verdier quotient $\mathcal{L}_{(-p)}(X[\varepsilon])/\mathcal{L}_{(-p-1)}(X[\varepsilon])$, denoted $[i_*O_{Y'}]$.

If $Y \subset X$ is a locally complete intersection of codimension p, there exists an open affine $U \subset X$ such that $U \cap Y$ is defined by a regular sequence $f_1, \cdots, f_p$, where $f_i \in O_X(U)$. Locally on U, $Y^{'}$ is given by lifting $f_1, \cdots, f_p$ to $f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p$, where $g_i \in O_X(U)$.

We use $F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$ to denote the Koszul complex associated to the regular sequence $f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p$, which is a resolution of $O_{X[\varepsilon]}(U)/(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$:

$$0 \longrightarrow F_p \xrightarrow{A_p} F_{p-1} \xrightarrow{A_{p-1}} \dots \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0,$$

where each $F_i = \bigwedge^i (O_{X[\varepsilon]}(U))^{\oplus p}$ and $A_i : \bigwedge^i (O_{X[\varepsilon]}(U))^{\oplus p} \to \bigwedge^{i-1} (O_{X[\varepsilon]}(U))^{\oplus p}$ are defined as usual. By using a construction of Angénoil and Lejeune-Jalabert [1], one can define tangent to this Koszul complex, which is given by the following commutative diagram (we assume $g_2 = \cdots = g_p = 0$ for simplicity):

$$\begin{cases} F_\bullet(f_1, f_2, \cdots, f_p) \quad \longrightarrow \quad O_X(U)/(f_1, f_2, \cdots, f_p) \\ F_p(\cong O_X(U)) \xrightarrow{g_1 df_2 \wedge \cdots \wedge df_p} F_0 \otimes \Omega^{p-1}_{O_X(U)/\mathbb{Q}} (\cong \Omega^{p-1}_{O_X(U)/\mathbb{Q}}), \end{cases} \tag{2.1}$$

where $d = d_{\mathbb{Q}}$. See the proof of Theorem 2.13 below or Section 3 of [18] for details.

However, in general, $Y \subset X$ may not be a locally complete intersection and the length of the perfect complex $[i_*O_{Y'}]$, which is the resolution of $i_*O_{Y'}$, may not equal to p. To modify this, instead of considering the perfect complex $[i_*O_{Y'}]$ which is an element of the Verdier quotient $\mathcal{L}_{(-p)}(X[\varepsilon])/\mathcal{L}_{(-p-1)}(X[\varepsilon])$, we consider its image in the idempotent completion $(\mathcal{L}_{(-p)}(X[\varepsilon])/\mathcal{L}_{(-p-1)}(X[\varepsilon]))^{\#}$, denoted $[i_*O_{Y'}]^{\#}$. We have the following result:

Theorem 2.3 ([2]). *For each $i \in \mathbb{Z}$, localization induces an equivalence*

$$(\mathcal{L}_{(i)}(X[\varepsilon])/\mathcal{L}_{(i-1)}(X[\varepsilon]))^{\#} \simeq \coprod_{x[\varepsilon] \in X[\varepsilon]^{(-i)}} D^{\mathrm{perf}}_{x[\varepsilon]}(X[\varepsilon]) \tag{2.2}$$

between the idempotent completion of the Verdier quotient $\mathcal{L}_{(i)}(X[\varepsilon])/\mathcal{L}_{(i-1)}(X[\varepsilon])$ and the coproduct over $x[\varepsilon] \in X[\varepsilon]^{(-i)}$ of the derived category of perfect complexes of

$O_{X[\varepsilon],x[\varepsilon]}$-modules with homology supported on the closed point $x[\varepsilon] \in \operatorname{Spec}(O_{X[\varepsilon],x[\varepsilon]})$. Consequently, one has

$$K_0((\mathcal{L}_{(i)}(X[\varepsilon])/\mathcal{L}_{(i-1)}(X[\varepsilon]))^{\#}) \simeq \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(-i)}} K_0(D^{\mathrm{perf}}_{x[\varepsilon]}(X[\varepsilon])).$$

Let y be the generic point of Y, Y is generically defined by a regular sequence $f_1, \cdots, f_p$ of length p, where $f_1, \cdots, f_p \in O_{X,y}$. Y' is generically given by lifting $f_1, \cdots, f_p$ to $f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p$, where $g_1, \cdots, g_p \in O_{X,y}$. We use $F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$ to denote the Koszul complex associated to the regular sequence $f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p$, which is a resolution of $O_{X,y}[\varepsilon]/(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$.

Under the equivalence (2.2), the localization at the generic point y sends $[i_*O_{Y'}]^{\#}$, which is the image of perfect complex $[i_*O_{Y'}]$ in the idempotent completion, to the Koszul complex $F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$:

$$[i_*O_{Y'}]^{\#} \to F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p).$$

One can define tangent to the Koszul complex $F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$ similarly as (2.1), which defines an element of $H^p_y(\Omega^{p-1}_{X/\mathbb{Q}})$.

Remark 2.4. In general, we do not know whether the above kind of Koszul complexes $F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$ can generate the Grothendieck group $\bigoplus_{y\in X^{(p)}} K_0(O_{X,y}[\varepsilon]$ on $y)$ or not. So we cannot use only these Koszul complexes to define tangent space to cycle groups and have to give a formal approach.

We recall that the Milnor K-theoretic cycles and Chow groups in Definition 2.2 recover the classical ones for X:

Theorem 2.5 (Theorem 3.16 in [17]). *For X a smooth projective variety over a field k of characteristic 0, for each positive integer p satisfying $1 \leq p \leq \dim(X)$, let $Z^p(X)$, $Z^p_{rat}(X)$ and $CH^p(X)$ denote the group of algebraic cycles of codimension p, the group of algebraic cycles of codimension p rationally equivalent to zero and Chow group respectively, then we have the identifications*

$$Z^M_p(D^{\mathrm{perf}}(X)) = Z^p(X)_{\mathbb{Q}},$$

$$Z^M_{p,rat}(D^{\mathrm{perf}}(X)) = Z^p_{rat}(X)_{\mathbb{Q}},$$

$$CH^M_p(D^{\mathrm{perf}}(X)) = CH^p(X)_{\mathbb{Q}}.$$

Using these identifications, we could define the tangent space $TZ^p(X)$ to the group of cycles $Z^p(X)$ to be the tangent space to $Z^M_p(D^{\mathrm{perf}}(X))$. Recall that the tangent space to a functor $\mathcal{F}$, denoted $T\mathcal{F}(X)$, is defined to be

$$T\mathcal{F}(X) := \operatorname{Ker}\{\mathcal{F}(X \times_k \operatorname{Spec}(k[\varepsilon]/(\varepsilon^2))) \xrightarrow{\varepsilon=0} \mathcal{F}(X)\}.$$

Considering $Z^M_p(D^{\mathrm{perf}}(-))$ as a functor, we are guided to the following definition, which answers Green-Griffiths' Question 1.1,

Definition 2.6. For X a smooth projective variety over a field k of characteristic 0, $X[\varepsilon] = X \times_k \operatorname{Spec}(k[\varepsilon]/(\varepsilon^2))$ denotes the first order infinitesimally trivial deformation of X. For each positive integer p satisfying $1 \le p \le \dim(X)$, the tangent space to the group of algebraic cycles $Z^p(X)$, denoted $TZ^p(X)$, is defined to be the tangent space to $Z_p^M(D^{\mathrm{perf}}(X))$

$$TZ^p(X) := TZ_p^M(D^{\mathrm{perf}}(X)) = \mathrm{Ker}\{Z_p^M(D^{\mathrm{perf}}(X[\varepsilon])) \xrightarrow{\varepsilon=0} Z_p^M(D^{\mathrm{perf}}(X))\}.$$

The tangent space to the group of algebraic cycles rationally equivalent to zero $Z^p_{rat}(X)$, denoted $TZ^p_{rat}(X)$, is defined to be the tangent space to $Z^M_{p,rat}(D^{\mathrm{perf}}(X))$

$$TZ^p_{rat}(X) := TZ^M_{p,rat}(D^{\mathrm{perf}}(X)) = \mathrm{Ker}\{Z^M_{p,rat}(D^{\mathrm{perf}}(X[\varepsilon])) \xrightarrow{\varepsilon=0} Z^M_{p,rat}(D^{\mathrm{perf}}(X))\}.$$

To compute the tangent spaces $TZ^p(X)$ and $TZ^p_{rat}(X)$, we recall the following theorem proved in [6, 17].

Theorem 2.7 ([6], Theorem 3.14 in [17]). *Let X be a d-dimensional smooth projective variety over a field k of characteristic 0, we use $X[\varepsilon]$ to denote the first order trivial deformation of X, i.e., $X[\varepsilon] = X \times_k \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$. For each integer $p \geqslant 1$, there exists the following commutative diagram in which the Zariski sheafification of each column is a flasque resolution of $\Omega^{p-1}_{X/\mathbb{Q}}$, $K_p^M(O_{X[\varepsilon]})$ and $K_p^M(O_X)$ respectively, the left arrows are induced by Chern character from K-theory to negative cyclic homology and the right ones are the natural maps sending ε to 0,*

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\Omega^{p-1}_{k(X)/\mathbb{Q}} \arrow[d] & K_p^M(k(X)[\varepsilon]) \arrow[l] \arrow[r] \arrow[d] & K_p^M(k(X)) \arrow[d] \\
\bigoplus_{x\in X^{(1)}} H^1_x(\Omega^{p-1}_{X/\mathbb{Q}}) \arrow[d] & \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(1)}} K^M_{p-1}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) \arrow[l] \arrow[r] \arrow[d] & \bigoplus_{x\in X^{(1)}} K^M_{p-1}(O_{X,x} \text{ on } x) \arrow[d] \\
\dots \arrow[d] & \dots \arrow[l] \arrow[r] \arrow[d] & \dots \arrow[d] \\
\bigoplus_{x\in X^{(p-1)}} H^{p-1}_x(\Omega^{p-1}_{X/\mathbb{Q}}) \arrow[d, "\partial_1^{p-1,-p}"'] & \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(p-1)}} K^M_1(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) \arrow[l] \arrow[r] \arrow[d, "d^{p-1,-p}_{1,X[\varepsilon]}"'] & \bigoplus_{x\in X^{(p-1)}} K^M_1(O_{X,x} \text{ on } x) \arrow[d, "d^{p-1,-p}_{1,X}"'] \\
\bigoplus_{x\in X^{(p)}} H^p_x(\Omega^{p-1}_{X/\mathbb{Q}}) \arrow[d, "\partial_1^{p,-p}"'] & \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(p)}} K^M_0(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) \arrow[l] \arrow[r] \arrow[d, "d^{p,-p}_{1,X[\varepsilon]}"'] & \bigoplus_{x\in X^{(p)}} K^M_0(O_{X,x} \text{ on } x) \arrow[d, "d^{p,-p}_{1,X}"'] \\
\bigoplus_{x\in X^{(p+1)}} H^{p+1}_x(\Omega^{p-1}_{X/\mathbb{Q}}) \arrow[d] & \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(p+1)}} K^M_{-1}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) \arrow[l, "\cong"'] \arrow[r] \arrow[d] & \bigoplus_{x\in X^{(p+1)}} K^M_{-1}(O_{X,x} \text{ on } x) = 0 \arrow[d] \\
\dots \arrow[d] & \dots \arrow[l] \arrow[r] \arrow[d] & \dots \arrow[d] \\
\bigoplus_{x\in X^{(d)}} H^d_x(\Omega^{p-1}_{X/\mathbb{Q}}) \arrow[d] & \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(d)}} K^M_{p-d}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) \arrow[l] \arrow[r] \arrow[d] & \bigoplus_{x\in X^{(d)}} K^M_{p-d}(O_{X,x} \text{ on } x) \arrow[d] \\
0 & 0 & 0.
\end{tikzcd}$$

This diagram enables us to compute $TZ^p(X)$ and $TZ^p_{rat}(X)$. A quick diagram chasing shows

Theorem 2.8. *Let X be a smooth projective variety over a field k of characteristic 0. For each integer p satisfying $1 \le p \le \dim(X)$, we have the following identifications*

$$TZ^p(X) \cong \mathrm{Ker}(\partial_1^{p,-p}),$$

$$TZ^p_{rat}(X) \cong \mathrm{Im}(\partial_1^{p-1,-p}).$$

Evidently, $TZ^p_{rat}(X)$ is a subspace of $TZ^p(X)$. We use the quotient space to define the tangent space to Chow groups:

DEFINITION 2.9. Let X be a smooth projective variety over a field k of characteristic 0. For each integer p satisfying $1 \leq p \leq \dim(X)$, the tangent space to $CH^p(X)$, denoted $TCH^p(X)$, is defined to be

$$TCH^p(X) := \frac{TZ^p(X)}{TZ^p_{rat}(X)}.$$

THEOREM 2.10. *$TCH^p(X)$ agrees with the formal tangent space $T_f CH^p(X)$ defined by Bloch [4], where $T_f CH^p(X) = H^p(X, \Omega^{p-1}_{X/\mathbb{Q}})$.*

Proof. It immediately follows from the fact that the Zariski sheafification of the left column in Theorem 2.7 is a flasque resolution of $\Omega^{p-1}_{X/\mathbb{Q}}$. □

For X a smooth projective surface over a field k of characteristic 0, by taking $p = 2$ in Theorem 2.8, we immediately see that

COROLLARY 2.11. *For X a smooth projective surface over a field k of characteristic 0, the tangent space $TZ^2(X)$ to the group of 0-cycles on X and the tangent space $TZ^2_{rat}(X)$ to the group of 0-cycles rationally equivalent to zero on X defined in Definition 2.6 agree with Green and Griffiths' definitions of $TZ^2(X)$ and $TZ^2_{rat}(X)$, recalled in Definition 1.2.*

Next, we provide concrete examples of Milnor K-theoretic cycles which are from geometry. For X a smooth projective variety over a field k of characteristic 0, let $Y \subset X$ be a locally complete intersection of codimension p. On an open affine $U \subset X$, we assume that $U \cap Y$ is defined by a regular sequence $f_1, \cdots, f_p$, where $f_i \in O_X(U)$.

Let Y' be a first order deformation of Y in $X \times_k \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$, locally on U, Y' is given by lifting $f_1, \cdots, f_p$ to $f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p$, where $g_i \in O_X(U)$.

Let y be the generic point of Y, then $O_{X,y} = (O_X(U))_{(f_1,\cdots,f_p)}$ with maximal ideal $(f_1, \cdots, f_p)$. We use $F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$ to denote the Koszul complex associated to the regular sequence $f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p$, which is a resolution of $O_{X,y}[\varepsilon]/(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$:

$$0 \longrightarrow F_p \xrightarrow{A_p} F_{p-1} \xrightarrow{A_{p-1}} \dots \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0,$$

where each $F_i = \bigwedge^i (O_{X,y}[\varepsilon])^{\bigoplus p}$ and $A_i : \bigwedge^i (O_{X,y}[\varepsilon])^{\bigoplus p} \to \bigwedge^{i-1} (O_{X,y}[\varepsilon])^{\bigoplus p}$ are defined as usual. One sees that $F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p) \in K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])$.

THEOREM 2.12 (Prop. 4.12 of [7]). *The Adams operations ψ^k defined on perfect complexes, defined by Gillet-Soulé in [7], satisfy $\psi^k(F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)) = k^p F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$.*

Hence, $F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$ is of eigenweight p and can be considered as an element of $K_0^{(p)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}}$,

$$F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p) \in K_0^{(p)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}} = K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]).$$

Moreover, we shall show $F_\bullet(f_1+\varepsilon g_1,\cdots,f_p+\varepsilon g_p)$ lies in the kernel of $d^{p,-p}_{1,X[\varepsilon]}$, where

$$d^{p,-p}_{1,X[\varepsilon]}:\bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(p)}} K^M_0(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) \to \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(p+1)}} K^M_{-1}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon])$$

is part of the middle column in the diagram of Theorem 2.7. Hence, $F_\bullet(f_1+\varepsilon g_1,\cdots,f_p+\varepsilon g_p)$ is a Milnor K-theoretic p-cycle in the sense of Definition 2.2.

Theorem 2.13. *Let X be a smooth projective variety over a field k of characteristic 0 and let $Y\subset X$ be a locally complete intersection of codimension p, which is locally on an open affine $U\subset X$ defined by a regular sequence $f_1,\cdots,f_p$. For a first order deformation Y' in $X\times_k \operatorname{Spec}(k[\varepsilon]/(\varepsilon^2))$ which is locally on U given by $f_1+\varepsilon g_1,\cdots,f_p+\varepsilon g_p$, where $g_i\in O_X(U)$, the Koszul complex $F_\bullet(f_1+\varepsilon g_1,\cdots,f_p+\varepsilon g_p)\in \operatorname{Ker}(d^{p,-p}_{1,X[\varepsilon]})$, i.e., $F_\bullet(f_1+\varepsilon g_1,\cdots,f_p+\varepsilon g_p)\in Z^M_p(D^{\mathrm{perf}}(X[\varepsilon]))$.*

The strategy for proving this theorem is to use the commutative diagram (part of the commutative diagram of Theorem 2.7):

$$\begin{array}{ccc}
\bigoplus_{x\in X^{(p)}} H^p_x(\Omega^{p-1}_{X/\mathbb{Q}}) & \longleftarrow & \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(p)}} K^M_0(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) \\
\Big\downarrow \partial_1^{p,-p} & & \Big\downarrow d^{p,-p}_{1,X[\varepsilon]} \\
\bigoplus_{x\in X^{(p+1)}} H^{p+1}_x(\Omega^{p-1}_{X/\mathbb{Q}}) & \xleftarrow{\cong} & \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(p+1)}} K^M_{-1}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]).
\end{array}$$

We describe the image of the Koszul complex $F_\bullet(f_1+\varepsilon g_1,\cdots,f_p+\varepsilon g_p)$ under the map

$$\bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(p)}} K^M_0(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) \to \bigoplus_{x\in X^{(p)}} H^p_x(\Omega^{p-1}_{X/\mathbb{Q}}),$$

and then show this image under the differential

$$\partial_1^{p,-p}:\bigoplus_{x\in X^{(p)}} H^p_x(\Omega^{p-1}_{X/\mathbb{Q}}) \to \bigoplus_{x\in X^{(p+1)}} H^{p+1}_x(\Omega^{p-1}_{X/\mathbb{Q}})$$

is zero.

Proof. Let's assume that y is the generic point of Y. The map (left arrow) induced by Chern character from K-theory to negative cyclic homology in the commutative diagram of Theorem 2.7

$$\mathrm{Ch}: K^M_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \to H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}),$$

can be described by a beautiful construction of Angénoil and Lejeune-Jalabert, see Lemme 3.1.1 on page 24 and Definition 3.4 on page 29 in [1] for details or Section 3 of [18] for a brief summary.

For our purpose, the Ch map on the Koszul complex $F_\bullet(f_1+\varepsilon g_1,\cdots,f_p+\varepsilon g_p)$ can be described as follows. For simplicity, we assume $g_2=\cdots=g_p=0$ in the following. To the Koszul complex,

$$0 \longrightarrow F_p \xrightarrow{A_p} F_{p-1} \xrightarrow{A_{p-1}} \dots \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0,$$

one defines the following class

$$\frac{1}{p!}dA_1 \circ dA_2 \circ \cdots \circ dA_p,$$

where $d = d_{\mathbb{Q}}$ and each dA_i is the matrix of absolute differentials. In other words,

$$dA_i \in \mathrm{Hom}(F_i, F_{i-1} \otimes \Omega^1_{O_{X,y}[\varepsilon]/\mathbb{Q}}).$$

The truncation map $\rfloor \dfrac{\partial}{\partial \varepsilon} \mid_{\varepsilon=0}$ sends $\frac{1}{p!}dA_1 \circ dA_2 \circ \cdots \circ dA_p$ to $g_1 df_2 \wedge \cdots \wedge df_p$. Hence, the image of $F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p)$ under the Ch map in $H^p_y(\Omega^{p-1}_{X/\mathbb{Q}})$ is represented by the following diagram (an element of $Ext^p(O_{X,y}/(f_1, f_2, \cdots, f_p), \Omega^{p-1}_{O_{X,y}/\mathbb{Q}})$),

$$\begin{cases} F_\bullet(f_1, f_2, \cdots, f_p) & \longrightarrow \quad O_{X,y}/(f_1, f_2, \cdots, f_p) \\ F_p(\cong O_{X,y}) & \xrightarrow{g_1 df_2 \wedge \cdots \wedge df_p} F_0 \otimes \Omega^{p-1}_{O_{X,y}/\mathbb{Q}} (\cong \Omega^{p-1}_{O_{X,y}/\mathbb{Q}}). \end{cases} \tag{2.3}$$

By shrinking U, we can assume that $O_X(U)$ is a local ring. The regular sequence $f_1, \cdots, f_p$, in the regular local ring $O_X(U)$, can be extended to be a system of parameter $f_1, \cdots, f_p, f_{p+1}, \cdots, f_n$ in $O_X(U)$. The prime ideals $Q_i := (f_1, \cdots, f_p, f_i)$ define generic points $z_i \in X^{(p+1)}$, where $i = p+1, \cdots, n$. In the following, we consider the prime $Q_{p+1} = (f_1, \cdots, f_p, f_{p+1})$ which defines the generic point z_{p+1}, other cases work similarly.

Let $Q = (f_1, \cdots, f_p)$ be the prime ideal defining the generic point (of Y) $y \in X^{(p)}$, then $O_{X,y} = (O_{X,z_{p+1}})_Q$. The above diagram (2.3) can be rewritten as, denoted $[\alpha]$,

$$\begin{cases} F_\bullet(f_1, f_2, \cdots, f_p) & \longrightarrow \quad (O_{X,z_{p+1}})_Q/(f_1, f_2, \cdots, f_p) \\ F_p(\cong (O_{X,z_{p+1}})_Q) & \xrightarrow{\frac{g_1 f_{p+1}}{f_{p+1}} df_2 \wedge \cdots \wedge df_p} F_0 \otimes \Omega^{p-1}_{(O_{X,z_{p+1}})_Q/\mathbb{Q}} (\cong \Omega^{p-1}_{(O_{X,z_{p+1}})_Q/\mathbb{Q}}). \end{cases} \tag{2.4}$$

Here, $F_\bullet(f_1, f_2, \cdots, f_p)$ is of the form

$$0 \longrightarrow F_p \xrightarrow{A_p} F_{p-1} \xrightarrow{A_{p-1}} \cdots \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0,$$

where each $F_i = \bigwedge^i((O_{X,z_{p+1}})_Q)^{\oplus p}$. Since $f_{p+1} \notin Q = (f_1, \cdots, f_p)$, f^{-1}_{p+1} exists in $(O_{X,z_{p+1}})_Q$, we can write $g_1 df_2 \wedge \cdots \wedge df_p = \dfrac{g_1 f_{p+1}}{f_{p+1}} df_2 \wedge \cdots \wedge df_p$.

The image of the diagram (2.4) under the differential

$$\partial_1^{p,-p} : \bigoplus_{x \in X^{(p)}} H^p_x(\Omega^{p-1}_{X/\mathbb{Q}}) \to \bigoplus_{x \in X^{(p+1)}} H^{p+1}_x(\Omega^{p-1}_{X/\mathbb{Q}})$$

is represented by the following diagram

$$\begin{cases} F_\bullet(f_1, f_2, \cdots, f_p, f_{p+1}) & \longrightarrow \quad O_{X,z_{p+1}}/(f_1, f_2, \cdots, f_p, f_{p+1}) \\ F_{p+1}(\cong O_{X,z_{p+1}}) & \xrightarrow{g_1 f_{p+1} df_2 \wedge \cdots \wedge df_p} F_0 \otimes \Omega^{p-1}_{O_{X,z_{p+1}}/\mathbb{Q}} (\cong \Omega^{p-1}_{O_{X,z_{p+1}}/\mathbb{Q}}). \end{cases}$$

The complex $F_\bullet(f_1, f_2, \cdots, f_p, f_{p+1})$ is of the form

$$0 \longrightarrow \bigwedge^{p+1}(O_{X,z_{p+1}})^{\oplus p+1} \xrightarrow{A_{p+1}} \bigwedge^{p}(O_{X,z_{p+1}})^{\oplus p+1} \longrightarrow \cdots.$$

Let $\{e_1, \cdots, e_{p+1}\}$ be a basis of $(O_{X,z_{p+1}})^{\oplus p+1}$, the map A_{p+1} is

$$e_1 \wedge \cdots \wedge e_{p+1} \to \sum_{j=1}^{p+1}(-1)^j f_j e_1 \wedge \cdots \wedge \hat{e_j} \wedge \cdots e_{p+1},$$

where $\hat{e_j}$ denotes to omit the j-th term.

Noting f_{p+1} appears in A_{p+1},

$$g_1 f_{p+1} df_2 \wedge \cdots \wedge df_p = 0 \in Ext^{p+1}_{O_{X,z_{p+1}}}(O_{X,z_{p+1}}/(f_1, f_2, \cdots, f_p, f_{p+1}), \Omega^{p-1}_{O_{X,z_{p+1}}/\mathbb{Q}}),$$

so $\partial_1^{p,-p}([\alpha]) = 0$. There exists the following commutative diagram, which is part of the commutative diagram in Theorem 2.7,

$$\begin{array}{ccc}
\bigoplus_{x \in X^{(p)}} H^p_x(\Omega^{p-1}_{X/\mathbb{Q}}) & \longleftarrow & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(p)}} K_0^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) \\
\Big\downarrow {\scriptstyle \partial_1^{p,-p}} & & \Big\downarrow {\scriptstyle d^{p,-p}_{1,X[\varepsilon]}} \\
\bigoplus_{x \in X^{(p+1)}} H^{p+1}_x(\Omega^{p-1}_{X/\mathbb{Q}}) & \xleftarrow{\cong} & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(p+1)}} K_{-1}^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]).
\end{array}$$

This gives the following commutative diagram

$$\begin{array}{ccc}
[\alpha] & \xleftarrow{\mathrm{Ch}} & F_\bullet(f_1 + \varepsilon g_1, f_2, \cdots, f_p) \\
\Big\downarrow {\scriptstyle \partial_1^{p,-p}} & & \Big\downarrow {\scriptstyle d^{p,-p}_{1,X[\varepsilon]}} \\
\partial_1^{p,-p}([\alpha]) = 0 & \xleftarrow{\cong} & d^{p,-p}_{1,X[\varepsilon]}(F_\bullet(f_1 + \varepsilon g_1, f_2, \cdots, f_p)),
\end{array}$$

which shows $d^{p,-p}_{1,X[\varepsilon]}(F_\bullet(f_1 + \varepsilon g_1, f_2, \cdots, f_p)) = 0$. $\square$

In general, $Y \subset X$ may not be a locally complete intersection, and the associated Koszul complex $F_\bullet(f_1 + \varepsilon g_1, f_2, \cdots, f_p)$ may not be a Milnor K-theoretic p-cycle. However, we can find another subscheme $Z \subset X$ of codimension p and find a first order deformation $Z' \in \mathrm{T_Z Hilb}^p(X)$ such that the two Koszul complexes associated Y' and Z' defines a Milnor K-theoretic p-cycle

To fix notations, let $W \subset Y$ be a subvariety of codimension 1 in Y, with generic point w. One assumes W is generically defined by $f_1, f_2, \cdots, f_p, f_{p+1}$ and Y is generically defined by $f_1, f_2, \cdots, f_p$. Let y be the generic point of Y, one has $O_{X,y} = (O_{X,w})_P$, where P is the ideal $(f_1, f_2, \cdots, f_p) \subset O_{X,w}$.

Y' is generically given by $(f_1 + \varepsilon g_1, f_2 + \varepsilon g_2, \cdots, f_p + \varepsilon g_p)$, where $g_i \in O_{X,y} = (O_{X,w})_P$. For simplicity, we assume $g_2 = \cdots = g_p = 0$. We can write $g_1 = \dfrac{a}{b}$, where $a, b \in O_{X,w}$ and $b \notin P$. b is either in or not in the maximal idea $(f_1, f_2, \cdots, f_p, f_{p+1}) \subset O_{X,w}$.

THEOREM 2.14 (Theorem 4.7 in [18]). *For $Y' \in \mathrm{T_Y Hilb}^p(X)$ which is generically defined by $(f_1 + \varepsilon g_1, f_2, \cdots, f_p)$, where $g_1 = \dfrac{a}{b} \in O_{X,y} = (O_{X,w})_P$, we use $F_\bullet(f_1 + \varepsilon g_1, f_2, \cdots, f_p)$ to denote the Koszul complex associated to $f_1 + \varepsilon g_1, f_2, \cdots, f_p$,*

- *Case 1: if $b \notin (f_1, f_2, \cdots, f_p, f_{p+1})$, then $F_\bullet(f_1 + \varepsilon g_1, f_2, \cdots, f_p) \in Z_p^M(D^{\mathrm{Perf}}(X[\varepsilon]))$.*

- *Case 2: if $b \in (f_1, f_2, \cdots, f_p, f_{p+1})$, we reduce to considering $b = f_{p+1}$. There exists $Z \subset X$ which is generically defined by $(f_{p+1}, f_2, \cdots, f_p)$ and exists $Z' \in \mathrm{T_Z Hilb}^p(X)$ which is generically defined by $(f_{p+1} + \varepsilon \frac{a}{f_1}, f_2, \cdots, f_p)$ such that*
$$F_{\bullet}(f_1 + \varepsilon \frac{a}{f_{p+1}}, f_2, \cdots, f_p) + F_{\bullet}(f_{p+1} + \varepsilon \frac{a}{f_1}, f_2, \cdots, f_p) \in Z_p^M(D^{\mathrm{Perf}}(X[\varepsilon])).$$

2.2. Why use kernel. In this subsection, we explain the reason why we use the kernel of $d_{1,X_j}^{p,-p}$,

$$d_{1,X_j}^{p,-p} : \bigoplus_{x_j \in X_j^{(p)}} K_0^M(O_{X_j,x_j} \text{ on } x_j) \to \bigoplus_{x_j \in X_j^{(p+1)}} K_{-1}^M(O_{X_j,x_j} \text{ on } x_j),$$

to define $Z_p^M(D^{\mathrm{perf}}(X_j))$ in Definition 2.2.

(1). As explained in the beginning of Section 2.1, in general, the length of the perfect complex $[i_* O_{Y'}]$, which is the resolution of $i_* O_{Y'}$, may not equal to p. To modify this, we need to look at its image $[i_* O_{Y'}]^\#$ in the idempotent completion $(\mathcal{L}_{(-p)}(X[\varepsilon])/\mathcal{L}_{(-p-1)}(X[\varepsilon]))^\#$. From the K-theoretic viewpoint, taking idempotent completion can result in the appearance of negative K-groups. We should include negative K-groups into the study of deformation of cycles, so we use the kernel of $d_{1,X_j}^{p,-p}$ to define $Z_p^M(D^{\mathrm{perf}}(X_j))$. In Section 3.2, negative K-groups will be used to study obstruction issues.

(2). For X a d-dimensional smooth projective variety over a field k of characteristic 0, let $X[\varepsilon] = X \times_k \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$ be the first order trivial deformation of X. From the geometric viewpoint, taking the kernel of

$$d_{1,X[\varepsilon]}^{p,-p} : \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(p)}} K_0^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) \to \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(p+1)}} K_{-1}^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon])$$

to define $Z_p^M(D^{\mathrm{perf}}(X[\varepsilon]))$ can produce the desirable tangent space. This can be explained by the following example.

Let X be a smooth projective surface over a field k of characteristic 0, we consider the group of 1-cycles $Z^1(X)$ on X and study its tangent space $TZ^1(X)$. For simplicity, we look at the sheaf level, that is, the tangent sheaf $\underline{T}Z^1(X)$ to the 1-cycles $Z^1(X)$.

Let $\underline{Z}^1(X)$ be the Zariski sheaf of 1-cycles on X, there exists the following short exact sequence of sheaves

$$0 \to O_X^* \to K(X)^* \xrightarrow{\mathrm{div}} \underline{Z}^1(X) \to 0,$$

where $K(X)$ is the function field of X. It is known that the tangent sheaves to O_X^* and $K(X)^*$ are O_X and $K(X)$ respectively. There exists the following short exact sequence of sheaves:

$$0 \to O_X \to K(X) \to PP_X \to 0, \tag{2.5}$$

where PP_X is the sheaf of principal parts. This suggests that

Definition 2.15 (page 100 [9]). Let X be a smooth projective surface over a field k of characteristic 0. The tangent sheaf $\underline{T}Z^1(X)$ to the 1-cycles $Z^1(X)$ is defined to be the sheaf of principal parts

$$\underline{T}Z^1(X) := PP_X.$$

To relate this definition with the formal Definition 2.6, we note that the Cousin resolution of O_X is

$$0 \to O_X \to K(X) \to \bigoplus_{y \in X^{(1)}} i_{y,*}H^1_y(O_X) \xrightarrow{\partial_1^{1,-1}} \bigoplus_{x \in X^{(2)}} i_{x,*}H^2_x(O_X) \to 0. \tag{2.6}$$

For X a smooth projective surface over a field k of characteristic 0, taking $p = 1$ in Theorem 2.8, we see the tangent space (in Definition 2.6) at the sheaf level to the group of 1-cycles is $\ker(\partial_1^{1,-1})$. The two exact sequences (2.5) and (2.6) show that

$$PP_X \cong K(X)/O_X \cong \ker(\partial_1^{1,-1}).$$

This proves:

Corollary 2.16. *Let X be a smooth projective surface over a field k of characteristic 0. The tangent space $TZ^1(X)$ (in Definition 2.6) at the sheaf level to the group $Z^1(X)$ of 1-cycles agrees with the Definition 2.15 by Green-Griffiths.*

If we do not use the kernel of $d^{1,-1}_{1,X[\varepsilon]}$, but use $\bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])$, to define Milnor K-theoretic 1-cycles, then the tangent sheaf becomes $\bigoplus_{y \in X^{(1)}} H^1_y(O_X)$, which is obviously not the desirable one.

In the next, combining with Green-Griffiths' results in [9], we construct a concrete element of the kernel of $d^{p,-p}_{1,X[\varepsilon]}$.

Theorem 2.17. *Let X be a smooth projective surface over a field k of characteristic 0, we use $X[\varepsilon]$ to denote the first order trivial deformation of X, i.e., $X[\varepsilon] = X \times_k \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$. For $p = 1$ in* Theorem 2.7, *there exists the following commutative diagram, in which the left arrows are induced by Chern character from K-theory to negative cyclic homology and the right ones are the natural maps sending ε to 0:*

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
k(X) \arrow[d] & K_1(k(X)[\varepsilon])_{\mathbb{Q}} \arrow[l] \arrow[r] \arrow[d] & K_1(k(X))_{\mathbb{Q}} \arrow[d] \\
\bigoplus_{y \in X^{(1)}} H^1_y(O_X) \arrow[d, "\partial_1^{1,-1}"'] & \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}} \arrow[l] \arrow[r] \arrow[d, "d^{1,-1}_{1,X[\varepsilon]}"'] & \bigoplus_{y \in X^{(1)}} K_0(O_{X,y} \text{ on } y)_{\mathbb{Q}} \arrow[d] \\
\bigoplus_{x \in X^{(2)}} H^2_x(O_X) \arrow[d] & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(2)}} K_{-1}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon])_{\mathbb{Q}} \arrow[l, "\cong"'] \arrow[r] \arrow[d] & \bigoplus_{x \in X^{(2)}} K_{-1}(O_{X,x} \text{ on } x)_{\mathbb{Q}} = 0 \arrow[d] \\
0 & 0 & 0.
\end{tikzcd}$$

Let's explain why one can use $K_0(O_{X,y} \text{ on } y)_{\mathbb{Q}}$ to replace $K_0^M(O_{X,y} \text{ on } y)$ (defined in Definition 2.1) in the above diagram. By Riemann-Roch without denominators, due to Soulé [16], one notes that $K_0^{(j)}(O_{X,y} \text{ on } y)_{\mathbb{Q}} \cong K_0^{(j-1)}(k(y)) = 0$, except for $j = 1$. That is ,

$$K_0^{(1)}(O_{X,y} \text{ on } y)_{\mathbb{Q}} = K_0(O_{X,y} \text{ on } y)_{\mathbb{Q}}.$$

This says $K_0^M(O_{X,y} \text{ on } y) = K_0(O_{X,y} \text{ on } y)_{\mathbb{Q}}$. Similar arguments work for other K-groups in the middle and right columns in the above diagram.

For X a smooth projective surface over a field k of characteristic 0, let Y_1 and Y_2 be two curves on X with generic point y_1 and y_2 respectively. For simplicity, we work locally in Zariski topology and assume Y_1 and Y_2 intersect transversely at a point x. Around the point x, we can write

$$Y_1 = \operatorname{div}(f_1);\ Y_2 = \operatorname{div}(f_2).$$

Let $X[\varepsilon] = X \times_k \operatorname{Spec}(k[\varepsilon]/(\varepsilon^2))$ be the first order trivial deformation of X. Take $g \in O_{X,x}$, we consider $O_{X,x}[\varepsilon]/(f_1f_2+\varepsilon g)$. The Koszul resolution of $O_{X,x}[\varepsilon]/(f_1f_2+\varepsilon g)$,

$$L^\bullet : 0 \to O_{X,x}[\varepsilon] \xrightarrow{f_1f_2+\varepsilon g} O_{X,x}[\varepsilon],$$

defines an element of $K_0((\mathcal{L}_{-1}(X[\varepsilon])/\mathcal{L}_{-2}(X[\varepsilon]))^\#)$.

Theorem 2.18. *$L^\bullet \in \operatorname{Ker}(d^{1,-1}_{1,X[\varepsilon]})$, i.e., $L^\bullet \in Z_1^M(D^{\mathrm{perf}}(X[\varepsilon]))$.*

Proof. Under the isomorphism in Theorem 2.3

$$K_0((\mathcal{L}_{(-1)}(X[\varepsilon])/\mathcal{L}_{(-2)}(X[\varepsilon]))^\#) \simeq \bigoplus_{y[\varepsilon]\in X[\varepsilon]^{(1)}} K_0(D^{\mathrm{perf}}_{y[\varepsilon]}(X[\varepsilon])),$$

$L^\bullet$ decomposes into the direct sum of

$$L_1^\bullet : 0 \to (O_{X,x})_{(f_1)}[\varepsilon] \xrightarrow{f_1+\varepsilon \frac{g}{f_2}} (O_{X,x})_{(f_1)}[\varepsilon]$$

and

$$L_2^\bullet : 0 \to (O_{X,x})_{(f_2)}[\varepsilon] \xrightarrow{f_2+\varepsilon \frac{g}{f_1}} (O_{X,x})_{(f_2)}[\varepsilon].$$

Noting $O_{X,y_1} = (O_{X,x})_{(f_1)}$, we have $L_1^\bullet \in K_0(O_{X,y_1}[\varepsilon] \text{ on } y_1[\varepsilon])$. Similarly, $L_2^\bullet \in K_0(O_{X,y_2}[\varepsilon] \text{ on } y_2[\varepsilon])$.

As recalled in the proof of Theorem 2.13, the left arrow in Theorem 2.17,

$$\mathrm{Ch} : \bigoplus_{y[\varepsilon]\in X[\varepsilon]^{(1)}} K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}} \to \bigoplus_{y\in X^{(1)}} H^1_y(O_X), \tag{2.7}$$

can be described by a beautiful construction of Angénoil and Lejeune-Jalabert [1]. The following diagram associated to $L_1^\bullet$

$$\begin{cases} (O_{X,x})_{(f_1)} \xrightarrow{f_1} (O_{X,x})_{(f_1)} \longrightarrow (O_{X,x})_{(f_1)}/(f_1) \longrightarrow 0 \\ (O_{X,x})_{(f_1)} \xrightarrow{\frac{g}{f_2}} (O_{X,x})_{(f_1)}, \end{cases}$$

gives an element α in $Ext^1_{O_{X,y_1}}(O_{X,y_1}/(f_1), O_{X,y_1})$, which further defines an element in $H^1_{y_1}(O_X)$ and it is the image of $L_1^\bullet$ under the Ch map above.

Similarly, the following diagram associated to $L_2^\bullet$

$$\begin{cases} (O_{X,x})_{(f_2)} \xrightarrow{f_2} (O_{X,x})_{(f_2)} \longrightarrow (O_{X,x})_{(f_2)}/(f_2) \longrightarrow 0 \\ (O_{X,x})_{(f_2)} \xrightarrow{\frac{g}{f_1}} (O_{X,x})_{(f_2)}, \end{cases}$$

gives an element β in $Ext^1_{O_{X,y_2}}(O_{X,y_2}/(f_2), O_{X,y_2})$, which further defines an element in $H^1_{y_2}(O_X)$ and it is the image of $L^\bullet_2$ under the Ch map.

One notes $\partial_1^{1,-1}$ maps α in $H^2_x(O_X)$ to :

$$\begin{cases} O_{X,x} \xrightarrow{(f_2,-f_1)^T} O_{X,x}^{\oplus 2} \xrightarrow{(f_1,f_2)} O_{X,x} \longrightarrow O_{X,x}/(f_1,f_2) \longrightarrow 0 \\ O_{X,x} \xrightarrow{g} O_{X,x}, \end{cases}$$

where $(-,-)^T$ denotes transpose.

Similarly, $\partial_1^{1,-1}$ maps β in $H^2_x(O_X)$ to :

$$\begin{cases} O_{X,x} \xrightarrow{(f_1,-f_2)^T} O_{X,x}^{\oplus 2} \xrightarrow{(f_2,f_1)} O_{X,x} \longrightarrow O_{X,x}/(f_1,f_2) \longrightarrow 0 \\ O_{X,x} \xrightarrow{g} O_{X,x}. \end{cases}$$

Noting the commutative diagram below

$$\begin{array}{ccccccccc} O_{X,x} & \xrightarrow{(f_2,-f_1)^T} & O_{X,x}^{\oplus 2} & \xrightarrow{(f_1,f_2)} & O_{X,x} & \longrightarrow & O_{X,x}/(f_1,f_2) & \longrightarrow & 0 \\ {\scriptstyle -1}\downarrow & & {\scriptstyle M}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle =}\downarrow & & \\ O_{X,x} & \xrightarrow{(f_1,-f_2)^T} & O_{X,x}^{\oplus 2} & \xrightarrow{(f_2,f_1)} & O_{X,x} & \longrightarrow & O_{X,x}/(f_2,f_1) & \longrightarrow & 0, \end{array}$$

where M stands for the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

Green-Griffiths observes that $\partial_1^{1,-1}(\alpha)$ and $\partial_1^{1,-1}(\beta)$ are negative of each other in $Ext^2_{O_{X,x}}(O_{X,x}/(f_1,f_2), O_{X,x})$. Hence, $\partial_1^{1,-1}(\alpha+\beta)$ is 0 in $H^2_x(O_X)$. This says that $\partial^{1,-1} \circ \mathrm{Ch}(L^\bullet_1 + L^\bullet_2) = \partial_1^{1,-1}(\alpha+\beta) = 0$.

Therefore, $d^{1,-1}_{1,X[\varepsilon]}(L^\bullet) = d^{1,-1}_{1,X[\varepsilon]}(L^\bullet_1 + L^\bullet_2) = 0$ because of the commutative diagram which is part of the diagram in Theorem 2.17:

$$\begin{array}{ccc} \bigoplus_{y\in X^{(1)}} H^1_y(O_X) & \xleftarrow{\mathrm{Ch}} & \bigoplus_{y[\varepsilon]\in X[\varepsilon]^{(1)}} K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \\ {\scriptstyle \partial_1^{1,-1}}\downarrow & & {\scriptstyle d^{1,-1}_{1,X[\varepsilon]}}\downarrow \\ \bigoplus_{x\in X^{(2)}} H^2_x(O_X) & \xleftarrow[\cong]{} & \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(2)}} K_{-1}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]). \end{array}$$

∎

The above argument seems formal, it's useful to intuitively explain the meaning of taking the kernel of $d^{1,-1}_{1,X[\varepsilon]}$. This has been done by using residue by Green-Griffiths [9].

Alternative explanation by using residue, due to Green-Griffiths [9] (page 103-104 and the summary on page 119). To fix notations, let X be a smooth projective surface over the complex number field $\mathbb{C}$ and let Y_1 and Y_2 be two curves on X. It is well known that tangent vectors to the curves Y_1 and Y_2 are given by normal vector fields,

$$v_1 \in H^0(N_{Y_1/X}), v_2 \in H^0(N_{Y_2/X}).$$

For simplicity, we work locally in Zariski topology and assume Y_1 and Y_2 intersect transversely at a point x. Around the point x, we can write

$$Y_1 = \mathrm{div}(f_1);\ Y_2 = \mathrm{div}(f_2).$$

Then v_1 and v_2 can be expressed as

$$v_1 = w_1 \frac{\partial}{\partial f_1}, v_2 = w_2 \frac{\partial}{\partial f_2},$$

for some functions w_1 and w_2. For our purpose, we take $w_1 = \dfrac{g}{f_2}$ and $w_2 = \dfrac{h}{f_1}$, then

$$v_1 = \frac{g}{f_2}\frac{\partial}{\partial f_1}, v_2 = \frac{h}{f_1}\frac{\partial}{\partial f_2}.$$

For $\omega = df_1 \wedge df_2$, we consider the Poincaré residue:

$$\begin{cases} v_1 \rfloor \omega = \mathrm{Res}_{Y_1}(\dfrac{gdf_1 \wedge df_2}{f_1 f_2}) = \dfrac{gdf_2}{f_2} \in \Omega^1_{K(Y_1)/\mathbb{C}}; \\ v_2 \rfloor \omega = \mathrm{Res}_{Y_2}(\dfrac{hdf_1 \wedge df_2}{f_1 f_2}) = -\dfrac{hdf_1}{f_1} \in \Omega^1_{K(Y_2)/\mathbb{C}}. \end{cases}$$

We further take the residue at x:

$$\mathrm{Res}_x(\frac{gdf_2}{f_2}) = g, \mathrm{Res}_x(-\frac{hdf_1}{f_1}) = -h.$$

The sum of the residues is

$$\mathrm{Res}_x(\frac{gdf_2}{f_2}) + \mathrm{Res}_x(-\frac{hdf_1}{f_1}) = g - h.$$

When $g = h$, the sum of the residues is 0.

Conclusion: for $v_1 = \dfrac{g}{f_2}\dfrac{\partial}{\partial f_1}$ and $v_2 = \dfrac{g}{f_1}\dfrac{\partial}{\partial f_2}$,

$$\mathrm{Res}_x(v_1 \rfloor \omega) + \mathrm{Res}_x(v_1 \rfloor \omega) = 0.$$

For normal vectors

$$v_1 = \frac{g}{f_2}\frac{\partial}{\partial f_1}, v_2 = \frac{g}{f_1}\frac{\partial}{\partial f_2},$$

v_1 corresponds to $f_1 + \varepsilon \dfrac{g}{f_2}$ and v_2 corresponds to $f_2 + \varepsilon \dfrac{g}{f_1}$. To connect with K-groups of perfect complexes, v_1 corresponds to the complex

$$L_1^\bullet : 0 \to (O_{X,x})_{(f_1)}[\varepsilon] \xrightarrow{f_1 + \varepsilon \frac{g}{f_2}} (O_{X,x})_{(f_1)}[\varepsilon]$$

and v_2 corresponds to the complex

$$L_2^\bullet : 0 \to (O_{X,x})_{(f_2)}[\varepsilon] \xrightarrow{f_2 + \varepsilon \frac{g}{f_1}} (O_{X,x})_{(f_2)}[\varepsilon].$$

Conclusion: $\mathrm{Res}_x(v_1\rfloor\omega)+\mathrm{Res}_x(v_2\rfloor\omega) = 0$ corresponds to $(L_1^\bullet+L_2^\bullet) \in \mathrm{Ker}(d_{1,X_1}^{1,-1})$ in Theorem 2.18.

REMARK 2.19. One may ask why there is no necessity to take kernel in Quillen's or Soulé 's proofs of Bloch's formula in [15, 16]. That's because negative K-groups are zero in this case, $K_{-1}(k(x)) = 0$. If we take kernel, the cycle group $Z^p(X)$ is still identified with $\bigoplus_{x\in X^{(p)}} K_0(k(x))$.

2.3. Why use Milnor K-theory. In the following, we explain why we use Milnor K-groups with support, i.e., certain eigenspaces of Thomason-Trobaugh K-groups, to define cycles and Chow groups in Definition 2.2.

In private discussions in 2012 fall, Christophe Soulé guided the author to understand Theorem 5 in [16] and advised the author to consider Milnor K-theory. Christophe Soulé's suggestions relate with deformation of cycles as follows. When X is a smooth projective variety over a field k, the Gersten complex has the form of

$$0 \to K_p(k(X)) \to \cdots \to \bigoplus_{x\in X^{(p-1)}} K_1(O_{X,x} \text{ on } x) \to \bigoplus_{x\in X^{(p)}} K_0(O_{X,x} \text{ on } x) \to 0,$$

which agrees with the Gersten complex by Quillen [15] because of Dévissage,

$$0 \to K_p(k(X)) \to \cdots \to \bigoplus_{x\in X^{(p-1)}} K_1(k(x)) \to \bigoplus_{x\in X^{(p)}} K_0(k(x)) \to 0.$$

For $x \in X^{(p)}$, Adams operations can decompose $K_0(O_{X,x} \text{ on } x)$ and $K_0(k(x))$ into direct sums of eigenspaces respectively. Moreover, Riemann-Roch without denominators, due to Soulé [16], says

$$K_0^{(j)}(O_{X,x} \text{ on } x)_{\mathbb{Q}} = K_0^{(j-p)}(k(x))_{\mathbb{Q}}.$$

For $j = p$,

$$K_0^{(p)}(O_{X,x} \text{ on } x)_{\mathbb{Q}} = K_0^{(0)}(k(x))_{\mathbb{Q}} = K_0(k(x))_{\mathbb{Q}},$$

This forces to

$$K_0^{(j)}(O_{X,x} \text{ on } x)_{\mathbb{Q}} = 0, \text{ for } j \neq p.$$

So only $K_0^{(p)}(O_{X,x} \text{ on } x)_{\mathbb{Q}}$ is needed to study $Z^p(X)_{\mathbb{Q}}$.

To give an example, for X a smooth projective three-fold over a field k of characteristic 0 and a point $x \in X^{(3)}$ which is defined by (f, g, h), a first order deformation of x is given by $(f + \varepsilon f_1, g + \varepsilon g_1, h + \varepsilon h_1)$. According to Theorem 2.12, the Koszul complex $F_\bullet(f + \varepsilon f_1, g + \varepsilon g_1, h + \varepsilon h_1)$ associated to $(f + \varepsilon f_1, g + \varepsilon g_1, h + \varepsilon h_1)$ is of weight 3:

$$F_\bullet(f + \varepsilon f_1, g + \varepsilon g_1, h + \varepsilon h_1) \in K_0^{(3)}(O_{X,x}[\varepsilon] \text{ on } x)_{\mathbb{Q}},$$

it is not of weight 2

$$F_\bullet(f + \varepsilon f_1, g + \varepsilon g_1, h + \varepsilon h_1) \notin K_0^{(2)}(O_{X,x}[\varepsilon] \text{ on } x)_{\mathbb{Q}}.$$

We ignore $K_0^{(2)}(O_{X,x}[\varepsilon] \text{ on } x)_{\mathbb{Q}}$ $(\neq 0)$[2], and use only $K_0^{(3)}(O_{X,x}[\varepsilon] \text{ on } x)_{\mathbb{Q}}$ to define Milnor K-theoretic 3-cycles $Z_3^M(D^{\mathrm{Perf}}(X[\varepsilon]))$.

[2]One can compute that $K_0^{(2)}(O_{X,x}[\varepsilon] \text{ on } x)_{\mathbb{Q}} \cong H_x^3(O_X)$, which is not zero because of depth condition.

3. Higher order deformation-obstructions. In this section, X is a smooth projective variety over a field k of characteristic 0. For each nonnegative integer j, $X_j = X \times_k \mathrm{Spec}(k[\varepsilon]/\varepsilon^{j+1})$ is the j-th order infinitesimally trivial deformation of X, let $x \in X^{(i)}$ and $x_j \in X_j^{(i)}$. For any integer m, let $K_m^M(O_{X_j,x_j} \text{ on } x_j, \varepsilon)$ denote the relative K-group, that is, the kernel of the natural projection

$$K_m^M(O_{X_j,x_j} \text{ on } x_j) \xrightarrow{\varepsilon=0} K_m^M(O_{X,x} \text{ on } x).$$

Recall that we have proved the following isomorphisms in [17]:

THEOREM 3.1 (Corollary 3.11 in [17]). *For X a smooth projective variety over a field k of characteristic 0, for each nonnegative integer j, let X_j denote the j-th order infinitesimally trivial deformation of X. Let $x \in X^{(i)}$ and $x_j \in X_j^{(i)}$, Chern character induces the following isomorphisms between relative K-groups and local cohomology groups, where m is any integer,*

$$K_m^M(O_{X_j,x_j} \text{ on } x_j, \varepsilon) \cong H_x^i((\Omega_{X/\mathbb{Q}}^{m+i-1})^{\oplus j}).$$

The split exact sequence

$$0 \to K_m^M(O_{X_j,x_j} \text{ on } x_j, \varepsilon) \to K_m^M(O_{X_j,x_j} \text{ on } x_j) \xrightarrow{\varepsilon=0} K_m^M(O_{X,x} \text{ on } x) \to 0,$$

gives rise to

$$\begin{aligned} K_m^M(O_{X_j,x_j} \text{ on } x_j) &= K_m^M(O_{X,x} \text{ on } x) \oplus K_m^M(O_{X_j,x_j} \text{ on } x_j, \varepsilon) \\ &\cong K_m^M(O_{X,x} \text{ on } x) \oplus H_x^i((\Omega_{X/\mathbb{Q}}^{m+i-1})^{\oplus j}). \end{aligned}$$

Moreover, from the computation of Hochschild (cyclic) homology of truncated polynomials, it is known that the relative K-group $K_m^M(O_{X_j,x_j} \text{ on } x_j, \varepsilon)$, which is isomorphic to $H_x^i((\Omega_{X/\mathbb{Q}}^{m+i-1})^{\oplus j})$, carries an additional structure:

$$K_m^M(O_{X_j,x_j} \text{ on } x_j, \varepsilon) \cong \varepsilon H_x^i(\Omega_{X/\mathbb{Q}}^{m+i-1}) \oplus \cdots \oplus \varepsilon^j H_x^i(\Omega_{X/\mathbb{Q}}^{m+i-1}). \tag{3.1}$$

This implies that

$$H_x^i((\Omega_{X/\mathbb{Q}}^{m+i-1})^{\oplus j}) \cong \varepsilon H_x^i(\Omega_{X/\mathbb{Q}}^{m+i-1}) \oplus \cdots \oplus \varepsilon^j H_x^i(\Omega_{X/\mathbb{Q}}^{m+i-1}). \tag{3.2}$$

To simplify the notations, we use A to denote $K_m^M(O_{X,x} \text{ on } x)$ and use B to denote $H_x^i(\Omega_{X/\mathbb{Q}}^{m+i-1})$, then we have

$$K_m^M(O_{X_j,x_j} \text{ on } x_j) \cong A \oplus \varepsilon B \oplus \cdots \varepsilon^j B.$$

The natural map

$$f_j : X_j \to X_{j+1},$$

induces $f_j^* : K_m^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \to K_m^M(O_{X_j,x_j} \text{ on } x_j)$. Moreover, there exists the commutative diagram

$$\begin{array}{ccc} K_m^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) & \xrightarrow{f_j^*} & K_m^M(O_{X_j,x_j} \text{ on } x_j) \\ \cong \downarrow & & \cong \downarrow \\ A \oplus \varepsilon B \oplus \cdots \varepsilon^j B \oplus \varepsilon^{j+1} B & \xrightarrow{\varepsilon^{j+1}=0} & A \oplus \varepsilon B \oplus \cdots \varepsilon^j B, \end{array}$$

and there exists the short exact sequence of abelian groups

$$0 \to B \to A \oplus \varepsilon B \oplus \cdots \varepsilon^j B \oplus \varepsilon^{j+1} B \xrightarrow{\varepsilon^{j+1}=0} A \oplus \varepsilon B \oplus \cdots \varepsilon^j B \to 0.$$

This shows that

LEMMA 3.2. *For X a smooth projective variety over a field k of characteristic 0, for each nonnegative integer j, let X_j denote the j-th order infinitesimally trivial deformation of X. Let $x \in X^{(i)}$ and $x_j \in X_j^{(i)}$, there exists the following short exact sequence of abelian groups, where m is any integer,*

$$0 \to H_x^i(\Omega_{X/\mathbb{Q}}^{m+i-1}) \to K_m^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \xrightarrow{f_j^*} K_m^M(O_{X_j,x_j} \text{ on } x_j) \to 0.$$

3.1. Obstructions and negative K-groups. The natural map $f_j : X_j \to X_{j+1}$ induces the following commutative diagram:

$$\begin{array}{ccc}
\bigoplus_{x_{j+1}\in X_{j+1}^{(p)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) & \xrightarrow{f_j^*} & \bigoplus_{x_j\in X_j^{(p)}} K_0^M(O_{X_j,x_j} \text{ on } x_j) \\
\Big\downarrow d^{p,-p}_{1,X_{j+1}} & & \Big\downarrow d^{p,-p}_{1,X_j} \\
\bigoplus_{x_{j+1}\in X_{j+1}^{(p+1)}} K_{-1}^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) & \xrightarrow{f_j^*} & \bigoplus_{x_j\in X_j^{(p+1)}} K_{-1}^M(O_{X_j,x_j} \text{ on } x_j),
\end{array}$$

so it further induces $f_j^* : Z_p^M(D^{\mathrm{perf}}(X_{j+1})) \to Z_p^M(D^{\mathrm{perf}}(X_j))$, recall that $Z_p^M(D^{\mathrm{perf}}(X_j))$ is defined as the kernel of $\mathrm{Ker}(d^{p,-p}_{1,X_j})$, see Definition 2.2.

DEFINITION 3.3. Given $\xi_j \in Z_p^M(D^{\mathrm{perf}}(X_j))$, an element $\xi_{j+1} \in Z_p^M(D^{\mathrm{perf}}(X_{j+1}))$ is called a deformation of ξ_j, if $f_j^*(\xi_{j+1}) = \xi_j$.

ξ_j and ξ_{j+1} can be formally written as finite sums

$$\sum_{x_j} \lambda_j \cdot \overline{\{x_j\}}_{\mathrm{red}} \text{ and } \sum_{x_{j+1}} \lambda_{j+1} \cdot \overline{\{x_{j+1}\}}_{\mathrm{red}},$$

where $\sum_{x_j} \lambda_j \in \mathrm{Ker}(d^{p,-p}_{1,X_j}) \subset \bigoplus_{x_j\in X_j^{(p)}} K_0^M(O_{X_j,x_j} \text{ on } x_j)$ and $\overline{\{x_j\}}_{\mathrm{red}}$ is the closed reduced scheme associated to $\overline{\{x_j\}}$. Since $\overline{\{x_j\}}_{\mathrm{red}} = \overline{\{x_{j+1}\}}_{\mathrm{red}}$, when we deform from ξ_j to ξ_{j+1}, we deform the **coefficients**, i.e., we deform from $\sum_{x_j} \lambda_j$ to $\sum_{x_{j+1}} \lambda_{j+1}$.

Since

$$f_j^* : \bigoplus_{x_{j+1}\in X_{j+1}^{(p)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \to \bigoplus_{x_j\in X_j^{(p)}} K_0^M(O_{X_j,x_j} \text{ on } x_j)$$

is surjective, see lemma 3.2, given any $\xi_j \in Z_p^M(D^{\mathrm{perf}}(X_j))$, there exists

$$\xi_{j+1} \in \bigoplus_{x_{j+1}\in X_{j+1}^{(p)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1})$$

such that $f_j^*(\xi_{j+1}) = \xi_j$. We would like to know whether $\xi_{j+1} \in Z_p^M(D^{\mathrm{perf}}(X_{j+1}))$.

An easy diagram chasing shows $f_j^* d_{1,X_{j+1}}^{p,-p}(\xi_{j+1}) = 0$, so $d_{1,X_{j+1}}^{p,-p}(\xi_{j+1}) \in \mathrm{Ker}(f_j^*) = \bigoplus_{x \in X^{(p+1)}} H_x^{p+1}(\Omega_{X/\mathbb{Q}}^{p-1})$, see lemma 3.2 (take $m = -1$ and $i = p+1$). If $d_{1,X_{j+1}}^{p,-p}(\xi_{j+1}) = 0$, then ξ_{j+1} is a deformation of ξ_j in the sense of Definition 3.3.

Definition 3.4. The obstruction space for lifting elements in $Z_p^M(D^{\mathrm{perf}}(X_j))$ to $Z_p^M(D^{\mathrm{perf}}(X_{j+1}))$ is defined to be $\bigoplus_{x \in X^{(p+1)}} H_x^{p+1}(\Omega_{X/\mathbb{Q}}^{p-1})$.

3.2. Obstruction issues-versus Hilbert scheme. For each positive integer j, let X_j denote the j-th trivial deformation of X. Let $Y \subset X$ be a subvariety of codimension p. Obstruction can occur when trying to lift Y to Y_j successively, where $Y_j \subset X_j$ is with suitable assumptions.

It is a common phenomenon that obstructions can occur in deformation, though the deformation of X is trivial. It is well known that, considering Y as an element of $\mathrm{Hilb}(X)$, the tangent space $T_Y\mathrm{Hilb}(X)$ may be obstructed.

However, Green-Griffiths predicts that, considering Y as an element of the cycle group $Z^p(X)$, we can eliminate obstructions in their program [9].

Obstruction issues (page 187-190 in [9]). Let X be a d-dimensional smooth projective variety over a field k of characteristic 0. For each positive integer p satisfying $1 \le p \le d$, let $TZ^p(X)$ denote the tangent space to the cycle group $Z^p(X)$. There are essentially four (not mutually exclusive) possibilities:

- (i) $TZ^p(X)$ may be obstructed. That is, there exists some $\tau \in TZ^p(X)$ such that, thinking of τ as a map
$$\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2)) \to Z^p(X),$$
this map cannot be lifted to a map
$$\mathrm{Spec}(k[\varepsilon]/(\varepsilon^{k+1})) \to Z^p(X)$$
for some $k \ge 2$.
- (ii) $TZ^p(X)$ is formally unobstructed. That is, for any $\tau \in TZ^p(X)$, τ may be lifted to a map
$$\lim(\mathrm{Spec}(k[\varepsilon]/(\varepsilon^{k+1}))) \to Z^p(X).$$
- (iiii) $TZ^p(X)$ is formally unobstructed, but there exists $\tau \in TZ^p(X)$ which is not the tangent to a geometric arc in $Z^p(X)$.
- (iv) Every $\tau \in TZ^p(X)$ is the tangent to a geometric arc in $Z^p(X)$.

In the expressions $\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2)) \to Z^p(X)$ and $\mathrm{Spec}(k[\varepsilon]/(\varepsilon^{k+1})) \to Z^p(X)$, the left-hand sides are schemes, whereas the right-hand side $Z^p(X)$ has no scheme structure. These expressions only have heuristic meaning.

For $p = 1$, this question was solved by TingFai Ng in his Ph.D thesis,

Theorem 3.5 (Theorem 1.3.3 in [14]). *Every $\tau \in TZ^1(X)$ is the tangent to a geometric arc in $Z^1(X)$.*

For $p \ge 2$, Green-Griffiths observes that

Proposition 3.6 ((10.11) on page 189 [9]). *For $p \ge 2$, there exits a smooth projective variety X and $\tau \in TZ^p(X)$ which is not the tangent to a geometric arc in $Z^p(X)$.*

This means only possibilities (i)-(iii) can occur for $p \geq 2$. Green-Griffiths conjectures that

Conjecture 3.7 (page 190 [9]). *(ii) and (iii) above are the only possibilities that actually occur for $p \geq 2$.*

Because of the Proposition 3.6 above, all we need is to show that the tangent space $TZ^p(X)$ to the cycle group $Z^p(X)$ is formally unobstructed. Using the tangent space $TZ^p(X)$ defined in Definition 2.6, we restate conjecture 3.7 as follows.

Conjecture 3.8 ([9]). *Let X be a smooth projective variety over a field k of characteristic 0. For each positive integer p satisfying $1 \leq p \leq \dim(X)$, the tangent space $TZ^p(X)$ defined in Definition 2.6 is formally unobstructed. To be precise, for each nonnegative integer j, let X_j denote the j-th order infinitesimally trivial deformation of X, for any $\tau \in TZ^p(X)$, τ can be lifted to $\tau_j \in Z_p^M(D^{\mathrm{perf}}(X_j))$ successively, where $j = 1, 2, \cdots$.*

To get a feeling of how to eliminate obstructions to deforming cycles, we first look at locally complete intersections.

For X a smooth projective variety over a field k of characteristic 0 and $Y \subset X$ a subvariety, which is a locally complete intersection of codimension p. We assume that, on an open affine $U_i \subset X$, $Y \cap U_i$ is defined by a regular sequence $f_1^i, \cdots, f_p^i$, on another open affine $U_j \subset X$, $Y \cap U_j$ is defined by a regular sequence $f_1^j, \cdots, f_p^j$.

Let Y' be a first order deformation of Y in $X \times_k \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$, then $Y' \cap U_i$ is given by lifting $f_1^i, \cdots, f_p^i$ to $f_1^i + \varepsilon g_1^i, \cdots, f_p^i + \varepsilon g_p^i$, where $g_*^i \in O_X(U_i)$. $Y' \cap U_j$ is given by lifting $f_1^j, \cdots, f_p^j$ to $f_1^j + \varepsilon g_1^j, \cdots, f_p^j + \varepsilon g_p^j$, where $g_*^j \in O_X(U_j)$.

On the intersection $U_{ij} = U_i \cap U_j$, there exists two liftings which defines an element of $\alpha_{ij} \in \Gamma(U_{ij}, \mathcal{N}_{Y/X})$, where $\mathcal{N}_{Y/X}$ is the normal sheaf. On the intersection $U_{ijk} = U_i \cap U_j \cap U_k$ of three open affine subschemes, there are three liftings which defines α_{ij}, α_{jk} and α_{ik}. One checks (α_{ij}) is a Čech 1-cocyle, which is the obstruction to finding a global lifting Y', see Theorem 6.2 (page 47) of [11] for details.

Let $y \in Y$ be the generic point, then $y \in U_i$. One has $O_{X,y} = O_{U_i,y} = O_X(U_i)_{(f_1^i,\cdots,f_p^i)}$ with maximal ideal $(f_1^i, \cdots, f_p^i)$. Y is generically generated by $f_1^i, \cdots, f_p^i$ and the Koszul complex $F_\bullet(f_1^i, \cdots, f_p^i)$ is a Milnor K-theoretic cycle in the sense of Definition 2.2 , $F_\bullet(f_1^i, \cdots, f_p^i) \in K_0^{(p)}(O_{X,y} \text{ on } y) \subset Z_p^M(D^{\mathrm{perf}}(X))$.

To lift the Milnor K-theoretic cycle $F_\bullet(f_1^i, \cdots, f_p^i)$ in the sense of Definition 3.3, we need to lift it to an element of $Z_p^M(D^{\mathrm{perf}}(X \times_k \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))))$ which is a perfect complex of $O_{X,y}[\varepsilon]$-module. We have shown that, in Theorem 2.13, the Koszul complex $F_\bullet(f_1^i + \varepsilon g_1^i, \cdots, f_p^i + \varepsilon g_p^i) \in Z_p^M(D^{\mathrm{perf}}(X \times_k \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))))$. It is obvious that $F_\bullet(f_1^i + \varepsilon g_1^i, \cdots, f_p^i + \varepsilon g_p^i)$ lifts $F_\bullet(f_1^i, \cdots, f_p^i)$ in the sense of Definition 3.3.

By mimicking the proof of Theorem 2.13, we can show that the Koszul complex $F_\bullet(f_1^i + \varepsilon g_1^i + \varepsilon^2 h_1^i, \cdots, f_p^i + \varepsilon g_p^i + \varepsilon^2 h_p^i) \in Z_p^M(D^{\mathrm{perf}}(X \times_k \mathrm{Spec}(k[\varepsilon]/(\varepsilon^3))))$, where $h_*^i \in O_X(U_i)$. This says that $F_\bullet(f_1^i + \varepsilon g_1^i + \varepsilon^2 h_1^i, \cdots, f_p^i + \varepsilon g_p^i + \varepsilon^2 h_p^i)$ is a Milnor K-theoretic p-cycle and it lifts $F_\bullet(f_1^i + \varepsilon g_1^i, \cdots, f_p^i + \varepsilon g_p^i)$ in the sense of Definition 3.3. Furthermore, we can lift $F_\bullet(f_1^i + \varepsilon g_1^i + \varepsilon^2 h_1^i, \cdots, f_p^i + \varepsilon g_p^i + \varepsilon^2 h_p^i)$ to higher order successively. In summary, we have shown that

Lemma 3.9. *For X a smooth projective variety over a field k of characteristic 0 and $Y \subset X$ a subvariety which is locally defined by a regular sequence $f_1, \cdots, f_p$, let $F_\bullet(f_1, \cdots, f_p)$ denote the associated Koszul complex which defines a K-theoretic cycle* *in $Z_p^M(D^{\mathrm{perf}}(X))$, one can lift (in the sense of Definition 3.3) this K-theoretic cycle $F_\bullet(f_1,\cdots,f_p)$ to higher order successively.*

In general, $Y \subset X$ may not be a locally complete intersection. To eliminate the obstructions to lifting Y to higher order, we use the following strategy which has been known to Green-Griffiths [9] (page 187-189) and Ng [14] for the divisor case. When the deformation of Y is obstructed, we could find another cycle Z to help Y to eliminate obstructions. As an algebraic cycle,

$$Y = (Y + Z) - Z,$$

and the cycle Z should satisfy that

- (1) One can lift $(Y + Z)$ to higher order successively, i.e., Z helps Y to eliminate obstructions.
- (2) Z doesn't introduce new obstructions.

To illustrate the idea, we sketch an example of curves on a three-fold and refer the readers to [19] for details. For X a nonsingular projective 3-fold over a field k of characteristic 0, let $Y \subset X$ be a curve with generic point y. For a point $x \in Y \subset X$ which is defined by (f, g, h), we assume Y is generically defined by (f, g). The Koszul complex $F_\bullet(f, g)$ is a K-theoretic 2-cycle in the sense of Definition 2.2:

$$F_\bullet(f,g) \in K_0(O_{X,y} \text{ on } y) \subset Z_2^M(D^{\mathrm{perf}}(X)).$$

For a first order deformation Y' which is generically given by $(f+\varepsilon\dfrac{1}{h}, g)$, the Koszul complex $F_\bullet(f+\varepsilon\dfrac{1}{h}, g)$ associated to $(f+\varepsilon\dfrac{1}{h}, g)$ is in $K_0(O_{X,y}[\varepsilon] \text{ on } y)$, but we can show it is not in $Z_p^M(D^{\mathrm{perf}}(X \times_k \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))))$, see Example 4.4 in [18]. So $F_\bullet(f+\varepsilon\dfrac{1}{h}, g)$ is not a first order deformation of $F_\bullet(f, g)$ in the sense of Definition 3.3.

To modify this, we consider the curve Z on X which is generically defined by (h, g). As an algebraic cycle,

$$Y = (Y + Z) - Z \in Z^p(X).$$

As a K-theoretic cycle,

$$F_\bullet(f,g) = (F_\bullet(f,g) + F_\bullet(h,g)) - F_\bullet(h,g) \in \bigoplus_{y \in X^{(2)}} K_0(O_{X,y} \text{ on } y).$$

To lift $F_\bullet(f, g)$ is equivalent to lifting $(F_\bullet(f, g) + F_\bullet(h, g))$ and $F_\bullet(h, g)$ respectively. We can show that $(F_\bullet(f+\varepsilon\dfrac{1}{h}, g) + F_\bullet(h+\varepsilon\dfrac{1}{f}, g))$ is in $Z_p^M(D^{\mathrm{perf}}(X \times_k \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))))$, and that it is a first order deformation of $(F_\bullet(f, g) + F_\bullet(h, g))$ in the sense of Definition 3.3. Moreover, $(F_\bullet(f+\varepsilon\dfrac{1}{h}, g) + F_\bullet(h+\varepsilon\dfrac{1}{f}, g))$ can be lifted to higher order successively. On the other hand, $F_\bullet(h, g)$ is always a first order deformation of itself which means we fix $F_\bullet(h, g)$, so it doesn't introduce new obstructions. Consequently,

$$(F_\bullet(f+\varepsilon\frac{1}{h}, g) + F_\bullet(h+\varepsilon\frac{1}{f}, g)) - F_\bullet(h,g)$$

is a first order deformation of $F_\bullet(f,g)$, and can be lifted to higher order successively.

However, as pointed out in Remark 2.4, in general, we don't know whether the Milnor K-theoretic cycles $Z_p^M(D^{\mathrm{perf}}(X_j))$ are generated by these Koszul complexes or not. To answer Green-Griffiths' Conjecture 3.8, we have to give a formal argument which relies on the following theorem.

Theorem 3.10 ([6], Theorem 3.14 in [17]). *For X a d-dimensional smooth projective variety over a field k of characteristic 0, let $X_j = X \times_k \mathrm{Spec}(k[\varepsilon]/(\varepsilon^{j+1}))$ denote the j-th order infinitesimally trivial deformation of X, where j is any positive integer. For each integer $p \geqslant 1$, there exists the following commutative diagram in which the Zariski sheafification of each column is a flasque resolution of $(\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}$, $K_p^M(O_{X_j})$ and $K_p^M(O_X)$ respectively, the left arrows are induced by Chern character from K-theory to negative cyclic homology and the right ones are the natural maps sending ε to 0,*

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
(\Omega^{p-1}_{k(X)/\mathbb{Q}})^{\oplus j} \arrow[d] & K_p^M(k(X_j)) \arrow[l] \arrow[r] \arrow[d] & K_p^M(k(X)) \arrow[d] \\
\bigoplus\limits_{x\in X^{(1)}} H_x^1((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}) \arrow[d] & \bigoplus\limits_{x_j\in X_j^{(1)}} K_{p-1}^M(O_{X_j,x_j} \text{ on } x_j) \arrow[l] \arrow[r] \arrow[d] & \bigoplus\limits_{x\in X^{(1)}} K_{p-1}^M(O_{X,x} \text{ on } x) \arrow[d] \\
\cdots \arrow[d] & \cdots \arrow[l] \arrow[r] \arrow[d] & \cdots \arrow[d] \\
\bigoplus\limits_{x\in X^{(p-1)}} H_x^{p-1}((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}) \arrow[d, "\partial_{1,j}^{p-1,-p}"'] & \bigoplus\limits_{x_j\in X_j^{(p-1)}} K_1^M(O_{X_j,x_j} \text{ on } x_j) \arrow[l] \arrow[r] \arrow[d, "d_{1,X_j}^{p-1,-p}"'] & \bigoplus\limits_{x\in X^{(p-1)}} K_1^M(O_{X,x} \text{ on } x) \arrow[d, "d_{1,X}^{p-1,-p}"'] \\
\bigoplus\limits_{x\in X^{(p)}} H_x^{p}((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}) \arrow[d, "\partial_{1,j}^{p,-p}"'] & \bigoplus\limits_{x_j\in X_j^{(p)}} K_0^M(O_{X_j,x_j} \text{ on } x_j) \arrow[l] \arrow[r] \arrow[d, "d_{1,X_j}^{p,-p}"'] & \bigoplus\limits_{x\in X^{(p)}} K_0^M(O_{X,x} \text{ on } x) \arrow[d, "d_{1,X}^{p,-p}"'] \\
\bigoplus\limits_{x\in X^{(p+1)}} H_x^{p+1}((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}) \arrow[d] & \bigoplus\limits_{x_j\in X_j^{(p+1)}} K_{-1}^M(O_{X_j,x_j} \text{ on } x_j) \arrow[l, "\cong"'] \arrow[r] \arrow[d] & \bigoplus\limits_{x\in X^{(p+1)}} K_{-1}^M(O_{X,x} \text{ on } x) = 0 \arrow[d] \\
\cdots \arrow[d] & \cdots \arrow[l] \arrow[r] \arrow[d] & \cdots \arrow[d] \\
\bigoplus\limits_{x\in X^{(d)}} H_x^{d}((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}) \arrow[d] & \bigoplus\limits_{x_j\in X_j^{(d)}} K_{p-d}^M(O_{X_j,x_j} \text{ on } x_j) \arrow[l] \arrow[r] \arrow[d] & \bigoplus\limits_{x\in X^{(d)}} K_{p-d}^M(O_{X,x} \text{ on } x) \arrow[d] \\
0 & 0 & 0.
\end{tikzcd}$$

Using this theorem, we answer Green-Griffiths' Conjecture 3.8 affirmatively:

Theorem 3.11. *The Conjecture 3.8 is true, that is, the tangent space $TZ^p(X)$ defined in Definition 2.6 is formally unobstructed.*

Proof. For any positive integer j and given any $\xi_j \in Z_p^M(D^{\mathrm{perf}}(X_j))(:= \mathrm{Ker}(d_{1,X_j}^{p,-p}))$, we need to show ξ_j can be lifted to an element of $Z_p^M(D^{\mathrm{perf}}(X_{j+1}))(:= \mathrm{Ker}(d_{1,X_{j+1}}^{p,-p}))$. There exists the commutative diagram (part of the diagram in Theorem 3.10),

$$\begin{CD}
\bigoplus_{x\in X^{(p)}} H_x^p((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}) @<{\mathrm{Ch}}<< \bigoplus_{x_j\in X_j^{(p)}} K_0^M(O_{X_j,x_j} \text{ on } x_j)\\
@V{\partial_{1,j}^{p,-p}}VV @VV{d_{1,X_j}^{p,-p}}V\\
\bigoplus_{x\in X^{(p+1)}} H_x^{p+1}((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}) @<<{\cong}< \bigoplus_{x_j\in X_j^{(p+1)}} K_{-1}^M(O_{X_j,x_j} \text{ on } x_j),
\end{CD}$$

where the map Ch are induced by Chern character from K-theory to negative cyclic homology. It is obvious that $\mathrm{Ch}(\xi_j)\in \mathrm{Ker}(\partial_{1,j}^{p,-p})$.

There exists a similar commutative diagram for j+1:

$$\begin{CD}
\bigoplus_{x\in X^{(p)}} H_x^p((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j+1}) @<{\mathrm{Ch}}<< \bigoplus_{x_{j+1}\in X_{j+1}^{(p)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1})\\
@V{\partial_{1,j+1}^{p,-p}}VV @VV{d_{1,X_{j+1}}^{p,-p}}V\\
\bigoplus_{x\in X^{(p+1)}} H_x^{p+1}((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j+1}) @<<{\cong}< \bigoplus_{x_{j+1}\in X_{j+1}^{(p+1)}} K_{-1}^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}).
\end{CD}$$

As explained on page 21 (isomorphisms (3.1) and (3.2)), $\bigoplus_{x\in X^{(p)}} H_x^p((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j})$ carries an additional structure:

$$\bigoplus_{x\in X^{(p)}} H_x^p((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}) \cong (\varepsilon \bigoplus_{x\in X^{(p)}} H_x^p(\Omega^{p-1}_{X/\mathbb{Q}}))\bigoplus\cdots\bigoplus(\varepsilon^j \bigoplus_{x\in X^{(p)}} H_x^p(\Omega^{p-1}_{X/\mathbb{Q}})).$$

The differential

$$\partial_{1,j}^{p,-p}: \bigoplus_{x\in X^{(p)}} H_x^p((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}) \to \bigoplus_{x\in X^{(p+1)}} H_x^{p+1}((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j})$$

is $\varepsilon\partial_1^{p,-p}\oplus\cdots\oplus\varepsilon^j\partial_1^{p,-p}$:

$$\begin{CD}
\bigoplus_{x\in X^{(p)}} H_x^p((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}) @>{\cong}>> (\varepsilon \bigoplus_{x\in X^{(p)}} H_x^p(\Omega^{p-1}_{X/\mathbb{Q}}))\bigoplus\cdots\bigoplus(\varepsilon^j \bigoplus_{x\in X^{(p)}} H_x^p(\Omega^{p-1}_{X/\mathbb{Q}}))\\
@V{\partial_{1,j}^{p,-p}}VV @VV{\varepsilon\partial_1^{p,-p}\oplus\cdots\oplus\varepsilon^j\partial_1^{p,-p}}V\\
\bigoplus_{x\in X^{(p+1)}} H_x^{p+1}((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}) @>{\cong}>> (\varepsilon \bigoplus_{x\in X^{(p+1)}} H_x^{p+1}(\Omega^{p-1}_{X/\mathbb{Q}}))\bigoplus\cdots\bigoplus(\varepsilon^j \bigoplus_{x\in X^{(p+1)}} H_x^{p+1}(\Omega^{p-1}_{X/\mathbb{Q}})),
\end{CD}$$

where $\partial_1^{p,-p}: \bigoplus_{x\in X^{(p)}} H_x^p(\Omega^{p-1}_{X/\mathbb{Q}}) \to \bigoplus_{x\in X^{(p+1)}} H_x^{p+1}(\Omega^{p-1}_{X/\mathbb{Q}})$.

Under these isomorphisms, $\mathrm{Ch}(\xi_j)$ can be written as $\varepsilon a_1+\cdots+\varepsilon^j a_j$, where each $a_i\in \bigoplus_{x\in X^{(p)}} H_x^p(\Omega^{p-1}_{X/\mathbb{Q}})$ and $\partial_1^{p,-p}(a_i)=0$. There exists a similar isomorphism for $j+1$:

$$\bigoplus_{x\in X^{(p)}} H_x^p((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j+1}) \cong (\varepsilon \bigoplus_{x\in X^{(p)}} H_x^p(\Omega^{p-1}_{X/\mathbb{Q}}))\bigoplus\cdots\bigoplus(\varepsilon^{j+1} \bigoplus_{x\in X^{(p)}} H_x^p(\Omega^{p-1}_{X/\mathbb{Q}})).$$

The differential

$$\partial_{1,j+1}^{p,-p}: \bigoplus_{x\in X^{(p)}} H_x^p((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j+1}) \to \bigoplus_{x\in X^{(p+1)}} H_x^{p+1}((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j+1})$$

is $\varepsilon\partial_1^{p,-p} \oplus \cdots \oplus \varepsilon^{j+1}\partial_1^{p,-p}$.

For $\mathrm{Ch}(\xi_j) = \varepsilon a_1 + \cdots + \varepsilon^j a_j$, where each $a_i \in \bigoplus_{x \in X^{(p)}} H_x^p(\Omega_{X/\mathbb{Q}}^{p-1})$ and $\partial_1^{p,-p}(a_i) = 0$, there exists $\eta_{j+1} := \varepsilon a_1 + \cdots + \varepsilon^j a_j + \varepsilon^{j+1} a_{j+1}$ (note $\varepsilon^{j+1} \neq 0$ and $\varepsilon^{j+2} = 0$ here), where $a_{j+1} \in \mathrm{Ker}(\partial_1^{p,-p})$. So $\eta_{j+1} \in \mathrm{Ker}(\partial_{1,j+1}^{p,-p})$. Hence, we can always lift $\mathrm{Ch}(\xi_j)$ to $\eta_{j+1} \in \mathrm{Ker}(\partial_{1,j+1}^{p,-p})$.

Since the map

$$\mathrm{Ch}: \bigoplus_{x_{j+1} \in X_{j+1}^{(p)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \to \bigoplus_{x \in X^{(p)}} H_x^p((\Omega_{X/\mathbb{Q}}^{p-1})^{\oplus j+1})$$

is surjective, see Theorem 3.1, there exists $\xi_{j+1} \in \bigoplus_{x_{j+1} \in X_{j+1}^{(p)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1})$ such that $\mathrm{Ch}(\xi_{j+1}) = \eta_{j+1}$.

By the naturality of Chern character, there exists the following commutative diagram:

$$\begin{array}{ccc}
\bigoplus_{x \in X^{(p)}} H_x^p((\Omega_{X/\mathbb{Q}}^{p-1})^{\oplus j+1}) & \xleftarrow{\mathrm{Ch}} & \bigoplus_{x_{j+1} \in X_{j+1}^{(p)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \\
{\scriptstyle \varepsilon^{j+1}=0}\downarrow & & {\scriptstyle \varepsilon^{j+1}=0}\downarrow \\
\bigoplus_{x \in X^{(p)}} H_x^p((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j}) & \xleftarrow{\mathrm{Ch}} & \bigoplus_{x_j \in X_j^{(p)}} K_0^M(O_{X_j,x_j} \text{ on } x_j).
\end{array}$$

So there exists the commutative diagram:

$$\begin{array}{ccc}
\eta_{j+1} = \mathrm{Ch}(\xi_{j+1}) & \xleftarrow{\mathrm{Ch}} & \xi_{j+1} \\
{\scriptstyle \varepsilon^{j+1}=0}\downarrow & & {\scriptstyle \varepsilon^{j+1}=0}\downarrow \\
\eta_{j+1}|_{\varepsilon^{j+1}=0} & \xleftarrow{\mathrm{Ch}} & \xi_{j+1}|_{\varepsilon^{j+1}=0}.
\end{array}$$

This says $\eta_{j+1}|_{\varepsilon^{j+1}=0} = \mathrm{Ch}(\xi_{j+1}|_{\varepsilon^{j+1}=0})$. On the other hand, since η_{j+1} lifts $\mathrm{Ch}(\xi_j)$, $\eta_{j+1}|_{\varepsilon^{j+1}=0} = \mathrm{Ch}(\xi_j)$. Hence, $\mathrm{Ch}(\xi_{j+1}|_{\varepsilon^{j+1}=0}) = \mathrm{Ch}(\xi_j)$. This implies that $\xi_{j+1}|_{\varepsilon^{j+1}=0} - \xi_j$ is in the kernel of the map

$$\mathrm{Ch}: \bigoplus_{x_j \in X_j^{(p)}} K_0^M(O_{X_j,x_j} \text{ on } x_j) \to \bigoplus_{x \in X^{(p)}} H_x^p((\Omega_{X/\mathbb{Q}}^{p-1})^{\oplus j}),$$

which is $\bigoplus_{x \in X^{(p)}} K_0^M(O_{X,x} \text{ on } x)$. Therefore, there exists some $W \in \bigoplus_{x \in X^{(p)}} K_0^M(O_{X,x} \text{ on } x)$ such that

$$\xi_{j+1}|_{\varepsilon^{j+1}=0} = \xi_j + W. \tag{3.3}$$

As a **cycle**, ξ_j can be written as a formal sum

$$\xi_j = (\xi_j + W) - W.$$

Here, since $\bigoplus_{x \in X^{(p)}} K_0^M(O_{X,x} \text{ on } x)$ is a direct summand of $\bigoplus_{x_j \in X_j^{(p)}} K_0^M(O_{X_j,x_j} \text{ on } x_j)$, both W and $\xi_j + W$ are in $\bigoplus_{x_j \in X_j^{(p)}} K_0^M(O_{X_j,x_j} \text{ on } x_j)$.

Similarly, since $\bigoplus_{x \in X^{(p)}} K_0^M(O_{X,x} \text{ on } x)$ is also a direct summand of $\bigoplus_{x_{j+1} \in X_{j+1}^{(p)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1})$, $W \in \bigoplus_{x_{j+1} \in X_{j+1}^{(p)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1})$. The cycle $\xi_{j+1} - W \in \bigoplus_{x_{j+1} \in X_{j+1}^{(p)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1})$ satisfies

$$(\xi_{j+1} - W)|_{\varepsilon^{j+1}=0} = \xi_{j+1}|_{\varepsilon^{j+1}=0} - W = (\xi_j + W) - W = \xi_j,$$

where $\xi_{j+1}|_{\varepsilon^{j+1}=0} = \xi_j + W$ is from (3.3).

Moreover, $\mathrm{Ch}(\xi_{j+1} - W) = \mathrm{Ch}(\xi_{j+1}) = \eta_{j+1} \in \mathrm{Ker}(\partial_{1,j+1}^{p,-p})$, hence, $\xi_{j+1} - W \in Z_p^M(D^{\mathrm{perf}}(X_{j+1}))(:= \mathrm{Ker}(d_{1,X_{j+1}}^{p,-p}))$ because of the commutative diagram

$$\begin{array}{ccc}
\bigoplus_{x \in X^{(p)}} H_x^p((\Omega_{X/\mathbb{Q}}^{p-1})^{\oplus j+1}) & \xleftarrow{\mathrm{Ch}} & \bigoplus_{x_{j+1} \in X_{j+1}^{(p)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \\
\Big\downarrow \partial_{1,j+1}^{p,-p} & & \Big\downarrow d_{1,X_{j+1}}^{p,-p} \\
\bigoplus_{x \in X^{(p+1)}} H_x^{p+1}((\Omega_{X/\mathbb{Q}}^{p-1})^{\oplus j+1}) & \xleftarrow[\cong]{} & \bigoplus_{x_{j+1} \in X_{j+1}^{(p+1)}} K_{-1}^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}).
\end{array}$$

In conclusion, $\xi_{j+1} - W \in Z_p^M(D^{\mathrm{perf}}(X_{j+1}))$ can lift ξ_j. $\square$

In Section 4 of [8], Green-Griffiths conjectures that

CONJECTURE 3.12 ((4.7) on page 506 [8]). *Let X be a smooth projective variety over a field k of characteristic 0, for each positive integer p satisfying $1 \leq p \leq \dim(X)$, the tangent space $TZ_{rat}^p(X)$ to the group of algebraic cycles rationally equivalent to zero is formally unobstructed.*

Using the tangent space $TZ_{rat}^p(X)$ defined in Definition 2.6, we answer this question as follows.

For any nonnegative integer j, X_j denotes the j-th order infinitesimally trivial deformation of X. The natural map $f_j : X_j \to X_{j+1}$ induces the commutative diagram:

$$\begin{array}{ccc}
\bigoplus_{x_{j+1} \in X_{j+1}^{(p-1)}} K_1^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) & \xrightarrow{f_j^*} & \bigoplus_{x_j \in X_j^{(p-1)}} K_1^M(O_{X_j,x_j} \text{ on } x_j) \\
\Big\downarrow d_{1,X_{j+1}}^{p-1,-p} & & \Big\downarrow d_{1,X_j}^{p-1,-p} \\
\bigoplus_{x_{j+1} \in X_{j+1}^{(p-1)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) & \xrightarrow{f_j^*} & \bigoplus_{x_j \in X_j^{(p-1)}} K_0^M(O_{X_j,x_j} \text{ on } x_j).
\end{array}$$

Given any $\eta_j \in Z_{p,rat}^M(D^{\mathrm{perf}}(X_j))(:= \mathrm{Im}(d_{1,X_j}^{p-1,-p}))$, we want to know whether η_j can be lifted to $\eta_{j+1} \in Z_{p,rat}^M(D^{\mathrm{perf}}(X_{j+1}))$.

By definition, $\eta_j = d_{1,X_j}^{p-1,-p}(\xi_j)$, for some $\xi_j \in \bigoplus_{x_j \in X_j^{(p-1)}} K_1^M(O_{X_j,x_j} \text{ on } x_j)$. According to Lemma 3.2,

$$f_j^* : \bigoplus_{x_{j+1} \in X_{j+1}^{(p-1)}} K_1^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \to \bigoplus_{x_j \in X_j^{(p-1)}} K_1^M(O_{X_j,x_j} \text{ on } x_j)$$

is surjective, so we can always lift ξ_j to $\xi_{j+1} \in \bigoplus_{x_{j+1} \in X_{j+1}^{(p-1)}} K_1^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1})$.
Hence, $d_{1,X_{j+1}}^{p-1,-p}(\xi_{j+1})$ lifts η_j because of the above commutative diagram.

This proves that the deformation from $Z_{p,rat}^M(D^{\text{perf}}(X_j))$ to $Z_{p,rat}^M(D^{\text{perf}}(X_{j+1}))$ is unobstructed. So we have

THEOREM 3.13. *The conjecture 3.12 is true, i.e., the tangent space $TZ^p_{rat}(X)$ to the group of algebraic cycles rationally equivalent to zero, defined in Definition 2.6, is formally unobstructed.*

Recall that in Definition 2.2, the p-th Milnor K-theoretic Chow group is defined to be

$$CH_p^M(D^{\text{perf}}(X_j)) := \frac{Z_p^M(D^{\text{Perf}}(X_j))}{Z_{p,rat}^M(D^{\text{Perf}}(X_j))}.$$

The proof of Theorem 3.11 says, for any nonnegative integer j and for any given $[\xi_j] \in CH_p^M(D^{\text{perf}}(X_j))$, we can lift $[\xi_j]$ to $[\xi_{j+1}] \in CH_p^M(D^{\text{perf}}(X_{j+1}))$.

Recall that we [17] have shown that, $CH_p^M(D^{\text{perf}}(X_j))$ satisfies Soulé's variant of Bloch-Quillen identification:

$$CH_p^M(D^{\text{perf}}(X_j)) = H^p(X, K_p^M(O_{X_j}))_{\mathbb{Q}}.$$

So we have proved the following fact, which is already known to Green-Griffiths and can be deduced from Proposition 2.6 of [8] (recalled below),

COROLLARY 3.14 ([8]). *For each positive integer j, $X_j = X \times_k \operatorname{Spec}(k[t]/(t^{j+1}))$, for any given $[\xi_j] \in H^p(X, K_p^M(O_{X_j}))_{\mathbb{Q}}$, we can lift it to $[\xi_{j+1}] \in H^p(X, K_p^M(O_{X_{j+1}}))_{\mathbb{Q}}$.*

We briefly explain how Green-Griffiths [8] prove this Corollary. To study deformation of algebraic cycle classes, one considers a smooth projective morphism $f : \mathcal{X} \to S$, where $S = \operatorname{Spec}(k[[t]])$ and k is a field of characteristic 0. Let $X_j = \mathcal{X} \times_S S_j$, where $S_j = \operatorname{Spec}(k[t]/t^{j+1})$. We use X to denote X_0, and call the family $\{X_j\}_j$ a deformation of X, where X_j is called the j-th infinitesimal thickening of X.

It is known that , see [8] (page 498) or (2.8) of Proposition 2.3 of [5], there exists the short exact sequence of sheaves

$$0 \to \Omega^{p-1}_{X/\mathbb{Q}} \to K_p^M(O_{X_{j+1}}) \to K_p^M(O_{X_j}) \to 0.$$

The associated long exact sequence is of the form

$$\cdots \to H^p(X, K_p^M(O_{X_{j+1}}))_{\mathbb{Q}} \to H^p(X, K_p^M(O_{X_j}))_{\mathbb{Q}} \xrightarrow{\delta} H^{p+1}(X, \Omega^{p-1}_{X/\mathbb{Q}}) \to \cdots. \quad (3.4)$$

The dlog map

$$K_p^M(O_{X_j}) \to \Omega^p_{X_j/\mathbb{Q}}$$

$$\{r_1, \cdots, r_p\} \to \frac{dr_1}{r_1} \wedge \cdots \wedge \frac{dr_p}{r_p},$$

where $d = d_{\mathbb{Q}}$, induces the arithmetic cycle mapping

$$\eta : H^p(X, K_p^M(O_{X_j})) \to H^p(X, \Omega^p_{X_j/\mathbb{Q}}).$$

Let θ_j denote the j-th Kodaira-Spencer class, see Section 3.1 (page 492) of [8] for the definition.

LEMMA 3.15 (Proposition 2.6 of [8] (page 502)). *The coboundary map δ in the above long exact sequence (3.4) is given by*

$$\delta(\xi_j) = \theta_j \rfloor \eta(\xi_j).$$

In other words, the obstruction to lifting $\xi_j \in H^p(X, K_p^M(O_{X_j}))$ to $H^p(X, K_p^M(O_{X_{j+1}}))$ is given by

$$\delta(\xi_j) = \theta_j \rfloor \eta(\xi_j),$$

where $\eta(\xi_j)$ is the arithmetic cycle class of ξ_j.

When the family $\{X_j\}_j$ is a trivial deformation of X which is the case considered in this note, that is, for each j, $X_j = X \times \mathrm{Spec}(k[t]/(t^{j+1}))$, the Kodaira-Spencer class $\theta_j = 0$ (see page 492 of [8]), so the coboundary map $\delta = 0$. This proves Corollary 3.14 above.

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