Asian J. Math.
Vol. 24, No. 3, pp. 501–532, June 2020

006

EQUIVARIANT ASYMPTOTICS OF SZEGÖ KERNELS UNDER HAMILTONIAN $SU(2)$-ACTIONS*

ANDREA GALASSO† AND ROBERTO PAOLETTI‡

Abstract. Let M be complex projective manifold, and A a positive line bundle on it. Assume that $G = SU(2)$ acts on M in a Hamiltonian manner, with nowhere vanishing moment map, and that this action linearizes to A. Then there is an associated unitary representation of G on the associated algebro-geometric Hardy space, and the isotypical components are all finite dimensional. We consider the local and global asymptotic properties of the equivariant projector associated to a weight $k\boldsymbol{\nu}$, when $\boldsymbol{\nu}$ is fixed and $k \to +\infty$.

Key words. Hamiltonian action, Szegö kernel, Hardy space, equivariant asymptotics.

Mathematics Subject Classification. 30H10, 32M05, 41A60, 53D20, 53D35 ,53D50, 57S15.

1. Introduction. Let M be a connected complex d-dimensional projective manifold, and let ω be a Hodge form on it. Thus M has a natural choice of a volume form, given by $\mathrm{d}V_M := (1/d!)\,\omega^{\wedge d}$.

Suppose given, in addition, an action $\mu : G\times M \to M$ of a compact and connected Lie group G, which is holomorphic (meaning that each diffeomorphism $\mu_g : M \to M$, $g \in G$, is holomorphic), and Hamiltonian with respect to $2\,\omega$, with moment map $\Phi : M \to \mathfrak{g}^\vee$ ($\mathfrak{g}$ being of course the Lie algebra of G).

Let (A, h) be a positive line bundle on M, such that unique compatible connection on A has curvature form $-2\pi\,\imath\,\omega$; let $A^\vee$ be the dual line bundle, and $X \subset A^\vee$ the unit circle bundle, with projection $p : X \to M$. Then X is naturally a contact and CR manifold by positivity of A; if α is the contact form, X inherits the volume form $\mathrm{d}V_X := (2\pi)^{-1}\,\alpha \wedge p^*(\mathrm{d}V_M)$.

Furthermore, X has a natural Riemannian structure g_X. The latter is uniquely determined by the following conditions: 1): the vector sub-bundles $\mathcal{V}(X/M) := \ker(\mathrm{d}p)$, $\mathcal{H}(X/M) := \ker(\alpha) \subset TX$ are mutually orthogonal; 2): $p : X \to M$ is a Riemannian submersion; 3): S^1 (under the standard action) acts on X by isometries; 4): the fibers of p have unit length. Hence there is on $X \times X$ a Riemannian distance function, that will be denoted dist_X.

We shall henceforth identify densities and half-densities on X, and accordingly use the abridged notation $L^2(X)$ for the space of square summable half-densities on X.

The Hamiltonian action μ naturally induces an infinitesimal contact action of $\mathfrak{g}$ on X [Ko]; explicitly, if $\xi \in \mathfrak{g}$ and ξ_M is the corresponding Hamiltonian vector field on M, then its contact lift ξ_X is as follows. Let $\upsilon^\sharp$ denote the horizontal lift on X of a vector field υ on M, and denote by ∂_θ the generator of the structure circle action on X. Then

$$\xi_X := \xi_M^\sharp - \langle \Phi_G \circ p, \xi\rangle\,\partial_\theta. \tag{1}$$

*Received June 21, 2018; accepted for publication October 4, 2019.

†Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Via R. Cozzi 55, 20125 Milano, Italy (andrea.galasso@unimib.it). Current address: Room 407, Chee-Chun Leung Cosmology Hall, National Taiwan University (andrea.galasso@ncts.ntu.edu.tw).

‡Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Via R. Cozzi 55, 20125 Milano, Italy (roberto.paoletti@unimib.it).

We shall make the stronger assumption that μ lifts to a contact action $\widetilde{\mu} : G \times X \to X$ lifting μ, and preserving the CR structure (in other words, μ linearizes to a metric preserving action on A). This assumption is not always fullfilled and the obstruction is of a topological nature; in particular, the lifting always exists if G is compact and simply connected (as in the case of $SU(n)$), but in general the condition is not trivial, and involves certain integrality conditions on Φ. In particular, an Hamiltonian action of $\mathbb{R}$ can always be lifted to a contact action, but this need not happen for $G = S^1$. As an explicit example, as is apparent from (1) in the case of the trivial action of S^1 on M with constant moment map $\Phi = \lambda \in \mathbb{R}$ the lifting exists if and only if $\lambda \in \mathbb{Z}$. For an exhaustive treatment of this point, the reader may consult [Ko], [GS4] and the discussion in the appendix of [GS2]. If A is very ample, the existence of the lift implies that there is a naturally induced unitary representation of G on $H^0(M, A)$, and that M is an invariant projectively embedded submanifold of $\mathbb{P}\left(H^0(M, A)^\vee\right)$. At any rate, in the present article we shall focus on the particular case $G = SU(2)$, so that the existence of the lift is granted *a priori*.

Under these assumptions, there is a naturally induced unitary representation of G on the Hardy space $H(X) \subset L^2(X)$; hence $H(X)$ can be equivariantly decomposed over the irreducible representations of G:

$$H(X) = \bigoplus_{\boldsymbol{\nu} \in \widehat{G}} H(X)_{\boldsymbol{\nu}}, \tag{2}$$

where $\widehat{G}$ is the collection of all irreducible representations of G. As is well-known, if $\Phi(m) \neq 0$ for every $m \in M$, then each isotypical component $H(X)_{\boldsymbol{\nu}}$ is finite dimensional (see e.g. §2 of [P4]).

For example, suppose that $G = S^1$ and μ is trivial, with moment map $\Phi = 1$. By (1), the lifted action on X is fiber rotation $x \mapsto e^{-\imath\vartheta}\, x$ $(x \in X)$. The irreducible representations of S^1 are indexed by the integers $k \in \mathbb{Z}$, and the isotypical component $H(X)_k$ may be naturally and unitarily identified with the space $H^0\left(M, A^{\otimes k}\right)$ of global holomorphic sections of $A^{\otimes k}$. Hence, from this perspective, the spaces $H(X)_{\boldsymbol{\nu}}$ in (2) may be viewed as a broad representation-theoretic extension of the usual algebro-geometric notion of linear series, although they may *not*, in general, be interpreted as spaces of sections of some power of A. It is then natural to study their global and local properties and their geometric consequences, in analogy with the classical case. Here the paradigm is offered by the TYZ asymptotic expansion and its near-diagonal extension; we shall work in the general conceptual framework of [BG] and [GS1], and adopt more specifically the approach developed in [Z], [BSZ] and [SZ], which is based on the description of Π as an FIO in [BS].

Hence, as discussed in the introductions of [P4] and [GP], the present perspective departs form the setting of Berezin-Toeplitz quantization; rather, it may be considered a variant of it, where the structure S^1-action on X is replaced by the contact lift of a general Hamiltonian action of a compact Lie group on M (see for instance [Ch], [MM], [MZ], [Sch] and references therein for a discussion of Berezin-Toeplitz quantization and different approaches to near-diagonal kernel asymptotics).

Let $\Pi_{\boldsymbol{\nu}} : L^2(X) \to H(X)_{\boldsymbol{\nu}}$ be the orthogonal projector (the $\boldsymbol{\nu}$-equivariant Szegö projector); if $H(X)_{\boldsymbol{\nu}}$ is finite dimensional, the corresponding distributional kernel is smooth, $\Pi_{\boldsymbol{\nu}}(\cdot,\cdot) \in \mathcal{C}^\infty(X \times X)$ (the $\boldsymbol{\nu}$-equivariant Szegö kernel). If $\mathbf{0} \notin \Phi_G(M)$, then $H(X)_{\boldsymbol{\nu}}$ is finite dimensional for any $\boldsymbol{\nu}$. Under this hypothesis, we are generally interested in the asymptotic properties of the kernels $\Pi_{\boldsymbol{\nu}}(\cdot,\cdot)$ when $\boldsymbol{\nu}$ tends to infinity in weight space, and in finding analogues of the TYZ-asymptotic expansion and of the near-diagonal scaling asymptotics which have been the object of considerable attention in the standard case $G = S^1$, $\Phi = 1$. The case where G is a torus has been considered in [P4], [P5], [Cm]; in [GP], we have focused on the case $G = U(2)$.

Here, we shall consider the case $G = SU(2)$. We shall henceforth write $G = SU(2)$ and $\mathfrak{g} = \mathfrak{su}(2)$ (the Lie algebra of 2×2 traceless skew-Hermitian matrices).

The irreducible representations of G are indexed by the integers $\nu > 0$. To emphasize the difference with the case of S^1 without burdening the notation, we shall label the representations by the pairs $\boldsymbol{\nu} = (\nu, 0)$, and denote them by $V_{\boldsymbol{\nu}}$. As is well-known (see for instance [V]), $V_{\boldsymbol{\nu}} = \mathrm{Sym}^{\nu-1}\left(\mathbb{C}^2\right)$, and the restriction to the standard torus $T \leqslant G$ of the corresponding character $\chi_{\boldsymbol{\nu}}$ is

$$\chi_{\boldsymbol{\nu}}\left(\begin{pmatrix} e^{\imath\,\vartheta} & 0 \\ 0 & e^{-\imath\,\vartheta}\end{pmatrix}\right) = \frac{e^{\imath\,\nu\,\vartheta} - e^{-\imath\,\nu\,\vartheta}}{e^{\imath\,\vartheta} - e^{-\imath\,\vartheta}} = e^{\imath\,(\nu-1)\,\vartheta} + e^{\imath\,(\nu-3)\,\vartheta} + \ldots + e^{-\imath\,(\nu-1)\,\vartheta}. \tag{3}$$

We shall fix $\boldsymbol{\nu}$, and consider the pointwise asymptotics of $\Pi_{k\boldsymbol{\nu}}(\cdot,\cdot)$ when $k \to +\infty$. To begin with, we shall show that $\Pi_{k\boldsymbol{\nu}}(x, y)$ is rapidly decreasing, unless $y \to G \cdot x$ (the orbit of x) at a sufficiently fast pace. Let dist_X be, as above, the Riemannian distance function on X.

Theorem 1.1. *Let us fix C, $\epsilon > 0$. Then, uniformly for $(x, y) \in X \times X$ satisfying*

$$\mathrm{dist}_X\left(x, G \cdot y\right) \geq C\, k^{\epsilon - 1/2},$$

we have $\Pi_{k\boldsymbol{\nu}}(x, y) = O\left(k^{-\infty}\right)$.

Let us consider the asymptotics of $\Pi_{k\boldsymbol{\nu}}$ near a point $x \in X$. To build-up to our next Theorems, we need to introduce some more terminology.

If $x \in X$, we shall set $m_x := p(x)$.

Definition 1.1. For $m \in M$, $\Phi_G(m) \in \mathfrak{g}$ is a traceless skew-Hermitian 2×2 matrix. Suppose $\Phi_G(m) \neq 0$. Then

1. $\lambda(m) > 0$ will denote the (unique) positive eigenvalue of $-\imath\,\Phi_G(m)$;
2. $h_m\, T \in G/T$ will denote the unique coset such that

$$\Phi_G(m) = \imath\, h_m \begin{pmatrix} \lambda(m) & 0 \\ 0 & -\lambda(m)\end{pmatrix} h_m^{-1}; \tag{4}$$

3. we shall set, for $\nu \in \mathbb{N}$,

$$u_0(\nu, m) := \frac{\nu}{2\,\lambda(m)}.$$

If $x \in X$, we shall generally write $u_0(\nu, x)$ for $u_0(\nu, m_x)$.

The assignments $\lambda : M \to (0, +\infty)$ and $m \in M \mapsto h_m\, T \in G/T$ are $\mathcal{C}^\infty$, provided of course that $\Phi_G(m) \neq 0$ for every $m \in M$.

Remark 1.1. The positive eigenvalue $\lambda(m)$ has a symplectic interpretation, being closely related to the moment map for the action restricted to a suitable torus $T_m \leqslant G$. Let us set

$$\beta := \begin{pmatrix} \imath & 0 \\ 0 & -\imath\end{pmatrix}; \tag{5}$$

thus β is the infinitesimal generator of the standard torus T, and therefore $\mathrm{Ad}_{h_m}(\beta)$ is the infinitesimal generator of the torus $T_m := C_{h_m}(T)$ (here $C_g(h) := g\,h\,g^{-1}$, for all $g,\,h \in G$). Then for any $m \in M$ we have

$$2\,\lambda(m) = \big\langle h_m^{-1}\,\Phi_G(m)\,h_m, \beta\big\rangle = \big\langle \Phi_G(m), \mathrm{Ad}_{h_m}(\beta)\big\rangle. \tag{6}$$

In §3, we shall prove that if $x \in X$ and $\Phi_G(m_x) \neq \mathbf{0}$, then $\widetilde{\mu}$ is locally free at x. Hence the stabilizer subgroup $G_x \leqslant G$ is finite. In addition, by equivariance of Φ_G, G_x stabilizes $\Phi_G(m)$. By (4), $G_x \subset h_{m_x}\,T\,h_{m_x}^{-1}$. Thus G_x is finite and Abelian. Therefore, there exist $e^{\imath\vartheta_j} \in S^1$, $j = 1, \ldots, N_x$, such that

$$G_x = \left\{ h_{m_x} \begin{pmatrix} e^{\imath\vartheta_j} & 0 \\ 0 & e^{-\imath\vartheta_j} \end{pmatrix} h_{m_x}^{-1} \,:\, j = 1, \ldots, N_x \right\}. \tag{7}$$

We shall set, for every $j = 1, \ldots, N_x$,

$$t_j := \begin{pmatrix} e^{\imath\vartheta_j} & 0 \\ 0 & e^{-\imath\vartheta_j} \end{pmatrix}, \quad g_j := h_{m_x}\,t_j\,h_{m_x}^{-1}. \tag{8}$$

Definition 1.2. Let $Z := \{\pm I_2\} \leqslant G$ be the center of G, and set $Z_x := G_x \cap Z$.

We shall see that there is a contribution to the asymptotics of $\Pi_{k\boldsymbol{\nu}}$ near x associated to each $g \in G_x$, and that the shape of the contribution is different depending on whether $g \in Z_x$ or $g \in G_x \setminus Z_x$.

If $h \in G_x \setminus Z_x$, then $h \neq h^{-1}$. Hence $G_x \setminus Z_x$ has even cardinality $b_x = 2\,a_x$, and G_x has cardinality $b_x + h$, where $h = 1$ or 2. Perhaps after renumbering, we can assume that

$$G_x \setminus Z_x = \big\{g_1, \ldots, g_{a_x}, g_{a_x+1} = g_1^{-1}, \ldots, g_{b_x} = g_{a_x}^{-1}\big\} \tag{9}$$

(it may well be that $a_x = 0$).

Definition 1.3. If $\ell \in \mathbb{Z}$, let us define $f_\ell : T \to \mathbb{C}$ by letting

$$f_\ell : e^{\vartheta\,\beta} \in T \mapsto e^{\imath\,\ell\,\vartheta} \in \mathbb{C}^*.$$

Let us first consider the on-diagonal asymptotics of $\Pi_{k\boldsymbol{\nu}}\,(x, x)$, assuming only that $\widetilde{\mu}$ is locally free at x.

Definition 1.4. If $z \in \mathbb{C}$, let us set

$$A(z) := \imath \begin{pmatrix} 0 & z \\ \overline{z} & 0 \end{pmatrix} \in \mathfrak{g}.$$

Suppose $g_j \in G_x \setminus Z_x$. Then, with notation as in (8), the $\mathbb{R}$-linear map

$$\eta_j : z \in \mathbb{C} \mapsto \big(\mathrm{Ad}_{t_j^{-1}} - \mathrm{id}_{\mathfrak{g}}\big)\big(A(z)\big) \in \mathfrak{g}$$

is injective. Therefore, since $\widetilde{\mu}$ is locally free at x, there is a positive definite 2×2 matrix $C(x; j)$ such that

$$\big\|\mathrm{Ad}_{h_{m_x}}\big(\eta_j(z)\big)_X(x)\big\|^2 = \frac{1}{2} \cdot Z^t\,C(x; j)\,Z \quad (z \in \mathbb{C})$$

where $Z := \begin{pmatrix} a & b \end{pmatrix}^t \in \mathbb{R}^2$ if $z = a + \imath\, b$. Let us define

$$B(x; j) := C(x; j) + 4\,\imath\, \sin(2\vartheta_j) \cdot \lambda(m_x)\, I_2. \tag{10}$$

Finally, let us denote by V_3 the area of the unit sphere $S^3 \subset \mathbb{R}^4$, and set

$$D_{G/T} := 2\pi/V_3. \tag{11}$$

The geometric meaning of $D_{G/T}$ will be elucidated by Lemma 4.6.

We shall prove the following.

THEOREM 1.2. *Assume that $\Phi_G(m_x) \neq \mathbf{0}$, and that*

$$G_x \setminus Z_x = \left\{g_1, g_1^{-1}, \ldots, g_{a_x}, g_{a_x}^{-1}\right\}. \tag{12}$$

Then as $k \to +\infty$ there is an asymptotic expansion

$$\Pi_{k\,\boldsymbol{\nu}}(x, x) \sim \Pi_{k\,\boldsymbol{\nu}}(x, x)_{Z_x} + \Pi_{k\,\boldsymbol{\nu}}(x, x)_{G_x \setminus Z_x},$$

where

$$\Pi_{k\,\boldsymbol{\nu}}(x, x)_{Z_x} \sim \frac{1}{2\,\lambda(m_x)} \cdot \left(\frac{\nu\, k}{2\,\pi\,\lambda(m_x)}\right)^d \cdot \sum_{g \in G_x} f_{1-k\cdot\nu}(g) \cdot \left[1 + \sum_{j\geq 1}^{+\infty} k^{-j}\, B_{gj}(m_x)\right], \tag{13}$$

and

$$\Pi_{k\,\boldsymbol{\nu}}(x, x)_{G_x \setminus Z_x} \sim 4\,\pi \cdot D_{G/T} \cdot \left(\frac{\nu\, k}{2\,\pi \cdot \lambda(m_x)}\right)^d \cdot \sum_{j=1}^{a_x} \left[\Re\left(\frac{\imath\, \sin(\vartheta_j) \cdot e^{-\imath k \nu \cdot \vartheta_j}}{\sqrt{\det\left(B(x; j)\right)}}\right) + \sum_{l \geq 1} k^{-l}\, P_{jl}(m_x)\right], \tag{14}$$

where $B(x; j)$ is as in (10), and B_{gj}, P_{jl} are appropriate $\mathcal{C}^\infty$ real functions on the image in M of the locus of those $x' \in X$ such that $|G_{x'}| = |G_x|$.

Let us elucidate (14). As will emerge from the proof, $\Pi_{k\,\boldsymbol{\nu}}(x, x)_{G_x \setminus Z_x}$ asymptotically splits as a sum of local contributions, each coming from a small neighborhood of an element of $G_x \setminus Z_x$. Each such contribution, say associated to $g_j \in G_x \setminus Z_x$, may be estimated by a stationary phase computation, and the resulting expansion, given by (120), has the familiar shape divisible by $\sqrt{\det\left(B(x; j)\right)}$; here we are numbering the elements of $G_x \setminus Z_x$ as in (9), say. If on the other hand we list the elements of $G_x \setminus Z_x$ as in (12), and pair the local contributions of g_j and g_j^{-1} ($j = 1, \ldots, a_x$), we are left with an expansion of the form (14); in other words, the j-th summand in (14) is the joint contribution of g_j and g_j^{-1}.

Let us next consider the near-diagonal asymptotics of $\Pi_{k\boldsymbol{\nu}}$. As usual, these are conveniently expressed in Heisenberg local coordinates (HLC's) on X (the reader is referred to [SZ] for a precise definition and a general discussion thereof), and involve an invariant ψ_2 that we shall recall below. To simplify our treatment, we shall assume in this case that $G_x = Z_x$.

Thus, given $x \in X$, let us choose a system of Heisenberg local coordinates (HLC's) on X centered at x. Following [SZ], we shall denote this by the additive expression $x + \upsilon \in X$, where $\upsilon = (\theta, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^{2n}$, with $\theta \in (-\pi, \pi)$ and $\mathbf{v}$ sufficiently small. Translation in the 'angular' coordinate θ corresponds to rotation in a fixed fiber of the circle bundle $X \to M$: whenever both sides are defined,

$$x + (\theta + \vartheta, \mathbf{v}) = r_\theta\big(x + (\vartheta, \mathbf{v})\big),$$

where $r_\theta(y)$ is the standard action of $e^{\imath\,\theta} \in S^1$ on $y \in X$. On the other hand, the curve $\gamma : \tau \mapsto x + (\theta, \tau\,\mathbf{v})$ is 'horizontal' for $\tau = 0$, that is, $\dot\gamma(0) \in \mathcal{H}_{r_\theta(x)}(X/M)$. When $\theta = 0$, we shall abridge $x + (0, \mathbf{v})$ to $x + \mathbf{v}$.

HLC's come with built-in unitary isomorphisms $T_{m_x}M \cong \mathbb{C}^d$ and $T_xX \cong \mathbb{R} \times T_{m_x}X$; with this in mind, we shall use the expression $x + (\theta, \mathbf{v})$ for $(\theta, \mathbf{v}) \in T_xX$ or $x + \mathbf{v}$ for $\mathbf{v} \in T_{m_x}M$, where $m_x = \pi(x)$.

If $g \in G_x$, then $\mathrm{d}_{m_x}\mu_g : T_{m_x}M \to T_{m_x}M$ is a unitary isomorphism. For any $\mathbf{v} \in T_{m_x}M$, we shall set

$$\mathbf{v}^{(g)} := \mathrm{d}_{m_x}\mu_{g^{-1}}(\mathbf{v}).$$

The following invariant plays an ubiquitous role in the study of rescaled local asymptotics of equivariant Szegö kernels.

Definition 1.5. If $m \in M$ and $\mathbf{v}_1,\, \mathbf{v}_2 \in T_mM$, following [SZ] let us set

$$\psi_2(\mathbf{v}_1,\, \mathbf{v}_2) := -\imath\,\omega_m(\mathbf{v}_1, \mathbf{v}_2) - \frac{1}{2}\,\big\|\mathbf{v}_1 - \mathbf{v}_2\big\|_m^2,$$

where $\omega_m : T_mM \times T_mM \to \mathbb{R}$ is the symplectic form, and $\|\cdot\|_m : T_mM \to \mathbb{R}$ is the norm function.

Theorem 1.3. *Let us assume that $x \in X$ and that $\widetilde{\mu}$ is locally free on X in a neighborhood of x. Let $G_x \leqslant G$ be the stabilizer subgroup of x, and suppose that $G_x \leqslant Z$. Let us a choose system of Heisenberg local coordinates on X centered at x. Let us fix $C > 0$ and $\epsilon \in (0, 1/6)$. Then, uniformly for $\mathbf{v}_1,\, \mathbf{v}_2 \in T_{m_x}M$ satisfying $\|\mathbf{v}_j\| \le C\,k^\epsilon$ $(j = 1, 2)$, and belonging to a subspace of $T_{m_x}M$ transverse to the G-orbit through m_x, we have for $k \to +\infty$ an asymptotic expansion of the form*

$$\begin{aligned}
&\Pi_{k\boldsymbol{\nu}}\left(x + \frac{1}{\sqrt{k}}\,\mathbf{v}_1, x + \frac{1}{\sqrt{k}}\,\mathbf{v}_2\right)\\
&\sim \frac{1}{2\,\lambda(m_x)} \cdot \left(\frac{\nu\,k}{2\,\pi\,\lambda(m_x)}\right)^d \cdot \sum_{g \in G_x} f_{1-k\cdot\nu}(g) \cdot e^{u_0(\nu, m_x)\cdot\psi_2\left(\mathbf{v}_1^{(g)}, \mathbf{v}_2\right)}\\
&\quad \cdot \left[1 + \sum_{j\ge1}^{+\infty} k^{-j/2} A_{gj}(x; \mathbf{v}_1^{(g)}, \mathbf{v}_2)\right],
\end{aligned}$$

where $A_{gj}(x; \cdot, \cdot)$ is a polynomial of degree $\le 3\,j$ and parity $(-1)^j$.

We can apply Theorems 1.2 and 1.3 to estimate the dimension of $H(X)_{k\boldsymbol{\nu}}$ when $k \to +\infty$. Let us make this explicit in the case where $\widetilde{\mu}$ is generically free, leaving the possible variants to the interested reader.

Corollary 1.1. *Assume that $\widetilde{\mu}$ is generically free (that is, G_x is trivial for the general $x \in X$). Then*

$$\lim_{k\to+\infty} \left[\left(\frac{\pi}{k\,\nu}\right)^d \cdot \dim\big(H(X)_{k\boldsymbol{\nu}}\big)\right] = \int_M \mathrm{d}V_M(m) \left[\left(\frac{1}{2\,\lambda(m)}\right)^{d+1}\right]. \qquad (15)$$

In closing this introduction, it is in order to illustrate some natural possible developments, given by variants of the current problem to which the present approach can be applied but won't be treated here. Since the G-action on X, $\tilde{\mu}$, and the standard S^1 action given by fiber rotation commute, they combine to yield an action $\hat{\rho}$ of $S^1 \times G$ on X. Correspondingly, there is an induced unitary representation of $S^1 \times G$ on $H(X)$. The irreducible representations of $S^1 \times G$ are indexed by the pairs $(j, \boldsymbol{\nu})$, where $j \in \mathbb{Z}$ and $\boldsymbol{\nu} = (\nu, 0)$, $\nu > 0$; hence we have an equivariant unitary decomposition

$$H(X) = \bigoplus_{j=0}^{+\infty} \bigoplus_{\boldsymbol{\nu}} H(X)_{j,\boldsymbol{\nu}}, \quad \text{where} \quad H(X)_{j,\boldsymbol{\nu}} := H(X)_j \cap H(X)_{\boldsymbol{\nu}}.$$

Let $\Pi_{j,\boldsymbol{\nu}} : H(X) \to H(X)_{j,\boldsymbol{\nu}}$ denote the corresponding equivariant Szegö projection; it is smoothing, and its distributional kernel $\Pi_{j,\boldsymbol{\nu}} \in \mathcal{C}^\infty(X \times X)$ is simply the j-th Fourier component of $\Pi_{\boldsymbol{\nu}}$. It is natural to investigate the local asymptotics of $\Pi_{j,\boldsymbol{\nu}}$ when (j, ν) drifts to infinity along various rays in $\mathbb{Z} \times \mathbb{Z}$.

For instance, if we fix j_0 and let $\nu \to +\infty$, then $H(X)_{j_0,\boldsymbol{\nu}} = (0)$ for $\nu \gg 0$; this is so because, by the theory of [GS3], $H(X)_{j_0,\boldsymbol{\nu}} = (0)$ unless $\boldsymbol{\nu} \in j_0\, \Phi_G(M)$, and this condition obviously fails when $\nu \gg 0$, since $j_0\, \Phi_G(M)$ is a fixed compact subset of $\mathfrak{g}^\vee$. At the opposite extreme, we can fix $\boldsymbol{\nu}_0 = (\nu_0, 0)$, and let $j \to +\infty$. If, as we are presently assuming, $\mathbf{0} \notin \Phi_G(M)$, then $\boldsymbol{\nu}_0 \notin j\, \Phi_G(M)$ for $j \gg 0$, whence $\Pi_{j,\boldsymbol{\nu}_0} = 0$ for $j \gg 0$. On the other hand, if $\mathbf{0} \in \Phi_G(M)$, and $\mathbf{0}$ is a regular value of Φ_G, then the local asymptotics of $\Pi_{j,\boldsymbol{\nu}_0}$ for $j \to +\infty$ are non-trivial and concentrate along the submanifold $\Phi_G^{-1}(\mathbf{0})$; in fact, such asymptotics were already studied by Szegö kernel techniques and for arbitrary G in [P1], [P2] and [P3]. On the other hand, we can fix a pair $(j_0, \boldsymbol{\nu}_0)$ and consider the local asymptotics of $\Pi_{(k\,j_0,k\,\boldsymbol{\nu}_0)}$; these ladder asymptotics are precisely of the kind described at the beginning of this introduction, with G replaced by $S^1 \times G$, and they can be studied by a combination of the methods of the present paper and those in [Z] and [SZ]. Differently from the present setting, the concentration will be along the inverse image under Φ_G of a sphere of radius ν_0/j_0 [P1]. Still more generally, one might consider commuting Hamiltonian actions of G and of an r-dimensional torus T, and combine the methods of the present paper, [P1] and [P4] to study the local asymptotics of the corresponding equivariant Szegö kernels (thus with G now replaced by $T \times G$).

Another direction for future research motivated by the present results lies in the quest for a geometric interpretation of the local invariants associated to the pointwise asymptotic expansions of the equivariant Szegö kernels, in terms of the local properties of appropriate symplectic manifolds/orbifolds.

The present paper covers part of the PhD thesis of the first author at the University of Milano Bicocca.

Acknowledgments. We gratefully thank the referee for suggesting many useful expository improvements and proposing some motivating remarks.

2. Examples. Given $Z \in \mathbb{C}^{d+1} \setminus \{0\}$, we shall denote by $[Z] \in \mathbb{P}^d$ its image in projective space. If $Z = (z_0, \cdots, z_d)$, then $[Z] = [z_0 : \cdots : z_d]$.

To begin with, let us test our normalizations against the simplest case of the standard action of G on $\mathbb{P}^1$.

EXAMPLE 2.1. Let ω_{FS} denote the Fubini-Study form on $\mathbb{P}^1$. The standard action of G on $\mathbb{P}^1$, given by $\mu_A([Z]) := [AZ]$, is Hamiltonian with respect to $2\,\omega_{FS}$, with nowhere vanishing moment map

$$\Psi\big([z_0 : z_1]\big) := \frac{\imath}{|z_0|^2 + |z_1|^2} \begin{pmatrix} \frac{1}{2}\,\big(|z_0|^2 - |z_1|^2\big) & z_0 \cdot \overline{z}_1 \\ \overline{z}_0 \cdot z_1 & \frac{1}{2}\,\big(|z_1|^2 - |z_0|^2\big) \end{pmatrix}. \tag{16}$$

Then $\lambda([z_0 : z_1]) = 1/2$ for any $[z_0 : z_1] \in \mathbb{P}^1$, and the contact action $\widetilde{\mu}$ on S^3 is free, since it may be identified with action of $SU(2)$ on itself by left translations. Furthermore, $H_{k\,\boldsymbol{\nu}}(X) = H_{k\nu-1}(X)$, where the right hand side is the $(k \cdot \nu - 1)$-th isotype for the S^1-action. With $\nu = 1$ the leading order term of the expansion of Theorem 1.3 is

$$\left(\frac{k}{\pi}\right)^d \cdot e^{\psi_2(\mathbf{v}_1, \mathbf{v}_2)},$$

in agreement with the standard off-diagonal scaling asymptotics for Szegö kernels on $\mathbb{P}^1$ ([BSZ], [SZ]).

EXAMPLE 2.2. Let us consider the diagonal action of G on $\mathbb{P}^1 \times \mathbb{P}^1$,

$$\mu_A\big([Z], [W]\big) = \big([AZ], [AW]\big).$$

For $r = 1, 2, \ldots$, consider the symplectic structure $\Omega_r := \omega_{FS} \boxplus (r\,\omega_{FS})$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Then μ is Hamiltonian with respect to $2\,\Omega_r$, with moment map

$$\Phi_r : \big([Z], [W]\big) \mapsto \Psi\big([Z]\big) + r\,\Psi\big([W]\big).$$

If $r \geq 2$, then Φ_r is nowhere vanishing.

On the other hand, Ω_r is the normalized curvature of the positive line bundle $A_r := \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(r)$. The unit circle bundle X_r associated to A_r is the image of $S^3 \times S^3$ under the map

$$(Z, W) \in S^3 \times S^3 \subset \mathbb{C}^2 \times \mathbb{C}^2 \mapsto Z \otimes W^{\otimes r} \in \mathbb{C}^{2^{r+1}},$$

and the contact lift of μ is given by

$$\widetilde{\mu}_A\left(Z \otimes W^{\otimes r}\right) = (AZ) \otimes (AW)^{\otimes r}.$$

Let us consider the stabilizer subgroup of $Z \otimes W^{\otimes r}$. We have

$$\widetilde{\mu}_A\left(Z \otimes W^{\otimes r}\right) = Z \otimes W^{\otimes r} \quad \Leftrightarrow \quad AZ = \lambda_1\,Z, \quad AW = \lambda_2\,W$$

for certain $\lambda_1, \lambda_2 \in S^1$ with $\lambda_1 \cdot \lambda_2^r = 1$.

If Z and W are linearly dependent, then $\lambda_1 = \lambda_2$ and $\lambda_1^{r+1} = 1$. The stabilizer subgroup of $Z \otimes W^{\otimes r}$ is therefore cyclic of order $r + 1$. Otherwise, (Z, W) is an eigenbasis of A and $\lambda_2 = \lambda_1^{-1}$, $\lambda_1^{r-1} = 1$. Hence, assuming that $Z \wedge W \neq 0$, the stabilizer subgroup of $Z \otimes W^{\otimes r}$ is cyclic of order $r - 1$ when (Z, W) is an orthonormal basis of $\mathbb{C}^2$, and otherwise it is trivial when r is even and $\{\pm I_2\}$ when r is odd. Thus $\widetilde{\mu}$ is locally free for $r \geq 2$. Furthermore, the action is generically free when r is even, and the stabilizer is generically of order two when r is odd.

Let us now consider how $V_{k\boldsymbol{\nu}}$ appears in

$$H(X_r) = \bigoplus_{l=0}^{+\infty} H_l(X_r), \quad H_l(X_r) \cong H^0\left(\mathbb{P}^1 \times \mathbb{P}^1, A_r^{\otimes l}\right).$$

Since $A_r^{\otimes l} = \mathcal{O}_{\mathbb{P}^1}(l) \boxtimes \mathcal{O}_{\mathbb{P}^1}(l\,r)$, by the Künneth formula we have

$$\begin{aligned} H^0\left(\mathbb{P}^1 \times \mathbb{P}^1, A_r^{\otimes l}\right) &\cong H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(l)\right) \otimes H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(l\,r)\right) \\ &\cong V_{(l+1,0)} \otimes V_{(lr+1,0)}. \end{aligned}$$

Thus the character of $H^0\left(\mathbb{P}^1 \times \mathbb{P}^1, A_r^{\otimes l}\right)$ as a G-representation is $\chi_{l+1} \cdot \chi_{l\,r+1}$. Using (3), we see by a few computations that

$$\begin{aligned} &(\chi_{l+1} \cdot \chi_{l\,r+1})\left(\begin{pmatrix} e^{\imath\,\vartheta} & 0 \\ 0 & e^{-\imath\,\vartheta} \end{pmatrix}\right) \\ &= \left(e^{\imath\,l\,\theta} + e^{\imath\,(l-2)\,\theta} + \cdots + e^{-\imath\,l\,\theta}\right) \cdot \frac{e^{\imath\,(l\,r+1)\,\theta} - e^{-\imath\,(l\,r+1)\,\theta}}{e^{\imath\,\theta} - e^{-\imath\,\theta}} \\ &= \sum_{j=0}^{l} \frac{e^{\imath\,(l+l\,r+1-2j)\,\theta} - e^{-\imath\,(l+l\,r+1-2j)\,\theta}}{e^{\imath\,\theta} - e^{-\imath\,\theta}}. \end{aligned} \tag{17}$$

Therefore,

$$H^0\left(\mathbb{P}^1 \times \mathbb{P}^1, A_r^{\otimes l}\right) \cong \bigoplus_{j=0}^{l} V_{(l+l\,r+1-2j,0)}.$$

We conclude that $V_{k\boldsymbol{\nu}}$ appears at most once in each $H_l(X_r)$; it does appear once, in fact, if and only if $k\,\nu$ and $l\,(r+1)+1$ have the same parity, and

$$\frac{k\,\nu - 1}{r-1} \geq l \geq \frac{k\,\nu - 1}{r+1}. \tag{18}$$

Suppose, for example, that $k\,\nu$ and $r+1$ are both even. Then $l\,(r+1)+1$ is odd for any choice of l and we conclude that $H_{k\boldsymbol{\nu}}(X)$ vanishes. Notice that at the general $x \in X_r$ we have $G_x = \{\pm I_2\}$, and $\sum_{g\in G_x} f_{1-k\,\nu}(g) = 0$. If, on the other hand, $r+1$ is even and $k\,\nu$ is odd, then there is a copy of $V_{k\boldsymbol{\nu}}$ in $H_l(X_r)$ for every integer l satisfying (18). Hence the number of copies of $V_{k\boldsymbol{\nu}}$ in $H(X_r)$ is $\sim 2\,k\,\nu/\left(r^2-1\right)$, so that the dimension of $H_{k\boldsymbol{\nu}}(X)$ is $\sim 2\,(k\,\nu)^2/\left(r^2-1\right)$. For the general $x \in X_r$, we have in this case $\sum_{g\in G_x} f_{1-k\,\nu}(g) = 2$.

When $r+1$ is odd, on the other hand, the generic stabilizer is trivial. For the general $x \in X_r$, therefore, $\sum_{g\in G_x} f_{1-k\,\nu}(g) = 1$ irrespective of $k\,\nu$. If $k\,\nu$ is even (respectively, odd) then there is a copy of $V_{k\boldsymbol{\nu}}$ in $H_l(X_r)$ if and only if l is odd (respectively, even) and satisfies (18). Thus the number of copies of $V_{k\boldsymbol{\nu}}$ in $H(X_r)$ is $\sim k\,\nu/\left(r^2-1\right)$, so that the dimension of $H_{k\boldsymbol{\nu}}(X)$ is $\sim (k\,\nu)^2/\left(r^2-1\right)$.

3. Preliminaries. In this Section, we shall collect various basic concepts and foundational results that will be invoked in the following proofs; in addition, in §3.5 we shall establish a technical preamble to the proofs in §4.

For any $x \in X$ and $g \in G$,

$$\Pi_{k\boldsymbol{\nu}}\left(\widetilde{\mu}_g(x), \widetilde{\mu}_g(x)\right) = \Pi_{k\boldsymbol{\nu}}(x, x);$$

furthermore, transplanting a system of HLC's centered at x by $g \in G$ (in an obvious sense) yields a system of HLC's centered at $\widetilde{\mu}_g(x)$. Therefore, with no loss of generality we might replace x by $\widetilde{\mu}_{h_{m_x}}(x)$ (recall (4)), and assume that

$$\Phi_G(m_x) = \imath \begin{pmatrix} \lambda(m_x) & 0 \\ 0 & -\lambda(m_x) \end{pmatrix}. \tag{19}$$

Since however it is convenient to keep explicit track of h_{m_x}, we shall make the generic assumption that $\Phi_G(m_x)$ is not anti-diagonal.

The following Lemma is specific to the case $G = SU(2)$.

LEMMA 3.1. *Suppose $x \in X$ and $\Phi_G(m_x) \neq \mathbf{0}$. Then $\widetilde{\mu}$ is locally free at x.*

Proof. Given that $\Phi_G(m_x) \neq \mathbf{0}$, $\widetilde{\mu}$ is locally free at x if and only if Φ_G is transverse at m_x to the ray $\mathbb{R}_+ \cdot \Phi_G(m_x)$ (see the discussion in §2 of [P4]). By the equivariance of Φ_G, this is equivalent to the condition that Φ_G be transverse at m_x to the cone over the coadjoint orbit through $\Phi_G(m_x)$. However $\mathfrak{g} \cong \mathbb{R}^3$ and the non-trivial orbits in $\mathfrak{g} \cong \mathfrak{g}^\vee$ are spheres centered at the origin, hence the previous condition is trivially satisfied.

Perhaps more explicitly, recall from Remark 1.1 that $G_x \subseteq h_{m_x} T\, h_{m_x}^{-1} \cong S^1$, and that the latter torus is generated by $\mathrm{Ad}_{h_m}(\beta)$. On the other hand, by (1) and (6),

$$\mathrm{Ad}_{h_m}(\beta)_X(m_x) = \mathrm{Ad}_{h_m}(\beta)_M(m_x)^\sharp - 2\,\lambda(m)\ \partial_\theta|_m\,,$$

which is certainly non vanishing if $\lambda(m) > 0$. □

3.1. Recalls on Szegö kernels. Let $\Pi : L^2(X) \to H(X)$ be the Szegö projector, $\Pi(\cdot,\cdot) \in \mathcal{D}'(X \times X)$ the Szegö kernel (that is, the distributional kernel of X). After [BS] (see also the discussions in [Z], [BSZ], [SZ]), Π is a FIO with complex phase, of the form

$$\Pi(x, y) = \int_0^{+\infty} e^{iu\psi(x,y)}\, s(x, y, u)\, \mathrm{d}u, \tag{20}$$

where $\Im(\psi) \geq 0$ and

$$s(x, y, u) \sim \sum_{j \geq 0} u^{d-j}\, s_j(x, y).$$

We shall rely on the rather explicit description of ψ in Heisenberg local coordinates in §3 of [SZ].

3.2. The Weyl Integration Formula. By composing Π with the equivariant projector associated to $\boldsymbol{\mu} = (\mu > 0)$ (see the discussion in [GS1]), we have

$$\Pi_{\boldsymbol{\mu}}(x', x'') = \mu \cdot \int_G \mathrm{d}V_G(g)\left[\overline{\chi_{\boldsymbol{\mu}}(g)}\, \Pi\left(\widetilde{\mu}_{g^{-1}}(x'), x''\right)\right]. \tag{21}$$

We can remanage (21) as follows. Let us define $F : T \to \mathcal{D}'(X \times X)$ by setting

$$F(t; x', x'') := \int_{G/T} \mathrm{d}V_{G/T}(g\,T)\ \left[\Pi\left(\widetilde{\mu}_{g\,t^{-1}\,g^{-1}}(x'), x''\right)\right] \quad (t \in T).$$

Since t and t^{-1} are conjugate in G, $F(x', x''; t) = F\left(x', x''; t^{-1}\right)$. Let t_1 and $t_2 = t_1^{-1}$ denote the diagonal entries of $t \in T$. Then by the Weyl Integration and character formulae [V]

$$\begin{aligned} &\Pi_{\boldsymbol{\mu}}\left(x', x''\right) \\ &= \frac{\mu}{2} \cdot \int_T \mathrm{d}V_T(t)\ \left(t_1^{-\nu} - t_1^{\nu}\right)\left(t_1 - t_1^{-1}\right)\ F(t; x', x'') \\ &= I_+(\mu; x', x'') - I_-(\mu; x', x''), \end{aligned} \tag{22}$$

where

$$I_\pm(\mu; x', x'') := \frac{\mu}{2} \cdot \int_T \mathrm{d}V_T(t)\ \left[t_1^{\mp\nu} \cdot \left(t_1 - t_1^{-1}\right) \cdot F(t; x', x'')\right]. \tag{23}$$

In (23), the change of variable $t \mapsto t^{-1}$ shows that $I_-(\mu; x', x'') = -I_+(\mu; x', x'')$. Hence,

$$\begin{aligned} \Pi_{\boldsymbol{\mu}}\left(x', x''\right) &= 2\, I_+(\mu; x', x'') \\ &= \mu \cdot \int_T \mathrm{d}V_T(t)\ \left[t_1^{-\nu} \cdot \left(t_1 - t_1^{-1}\right) \cdot F(t; x', x'')\right]. \end{aligned} \tag{24}$$

3.3. The Haar measure on G/T. As is well-known, G is diffeomorphic to the unit sphere $S^3 \subset \mathbb{C}^2$ by the map

$$\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \in G \xrightarrow{\gamma} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in S^3. \tag{25}$$

Furthermore, γ intertwines the right action of $T \cong S^1$ on G with the standard circle action on S^3. Therefore, the projection $G \to G/T$ may be identified with the Hopf map $S^3 \to \mathbb{P}^1 \cong S^2$. It follows that the Haar measure on G/T is a positive multiple of the pull-back of the standard measure on S^2. Explicitly, using the local coordinates $(\theta, \delta) \in (0, \pi/2) \times (-\pi, \pi)$ for the coset in G/T of the matrix (25) with $\alpha = \cos(\theta)\, e^{\imath\delta}$, $\beta = \sin(\theta)$, then the Haar volume element on G/T is

$$\mathrm{d}V_{G/T}(g\,T) = \frac{1}{2\pi}\, \sin(2\,\theta)\, \mathrm{d}\theta\, \mathrm{d}\delta.$$

3.4. G_x-equivariant Heisenberg Local Coordinates. Although inessential, it will slightly simplify our exposition to make a convenient choice of HLC's centered at $x \in X$. These depend on a system of *preferred* adapted local coordinates at m_x, and of a *preferred* local frame for A at m_x [SZ]. As to the former (which needn't be holomorphic), we may use the exponential map at m_x, and for the latter we may assume without loss that it is G_x-invariant (by an argument as in §3 of [P3]). With this choice, we have the convenient equality

$$\widetilde{\mu}_g\big(x + (\theta, \mathbf{v})\big) = x + \big(\theta, \mathrm{d}_{m_x}\widetilde{\mu}_g(\mathbf{v})\big) \quad (g \in G_x). \tag{26}$$

3.5. Reduction to compactly supported integrals. For an arbitrary pair $(x_1, x_2) \in X \times X$, we consider the asymptotics of $\Pi_{k\boldsymbol{\nu}}(x_1, x_2)$. Since

$$\Pi_{k\boldsymbol{\nu}}(x_1, x_2) = \Pi_{k\boldsymbol{\nu}}\left(\widetilde{\mu}_g(x_1), \widetilde{\mu}_g(x_2)\right) \quad \forall\, g \in G, \tag{27}$$

we may assume without loss, by choosing $g \in G$ general, that $\Phi_G \circ \pi(x_2)$ is not anti-diagonal; choosing $g = h_{m_{x_2}}$ (Definition 1.1), we may even reduce to the case where $\Phi_G(m_{x_2})$ is diagonal.

By (21) with $\mu = k\,\nu$,

$$\Pi_{k\boldsymbol{\nu}}(x_1, x_2) = k\nu \int_G \mathrm{d}V_G(g) \left[\overline{\chi_{k\boldsymbol{\nu}}(g)}\, \Pi\left(\widetilde{\mu}_{g^{-1}}(x_1), x_2\right)\right]. \tag{28}$$

For some suitably small $\delta > 0$, let us define

$$\begin{aligned} G_{<\delta}(x_1, x_2) &:= \left\{g \in G \,:\, \mathrm{dist}_X\left(\widetilde{\mu}_{g^{-1}}(x_1), x_2\right) < \delta\right\}, \\ G_{>\delta}(x_1, x_2) &:= \left\{g \in G \,:\, \mathrm{dist}_X\left(\widetilde{\mu}_{g^{-1}}(x_1), x_2\right) > \delta\right\}. \end{aligned} \tag{29}$$

Then $\mathcal{U} := \{G_{<2\,\delta}(x, y),\, G_{>\delta}(x, y)\}$ is an open cover of G, and we may consider a $\mathcal{C}^\infty$ partition of unity $\{\varrho, 1-\varrho\}$ of G subordinate to $\mathcal{U}$. One can see that $\varrho = \varrho_{x_1,x_2}$ may be chosen to depend smoothly on $(x_1, x_2) \in X \times X$; we shall omit the dependence on (x_1, x_2).

When $\varrho(g) \neq 1$, we have $\mathrm{dist}_X\left(\widetilde{\mu}_{g^{-1}}(x_1), x_2\right) \geq \delta > 0$. Because Π is smoothing away from the diagonal, the function

$$g \mapsto \big(1 - \varrho(g)\big) \cdot \Pi\left(\widetilde{\mu}_{g^{-1}}(x_1), x_2\right)$$

is $\mathcal{C}^\infty$ on G. Therefore, taking Fourier transforms and arguing as in §3.2, we obtain the following Proposition.

Proposition 3.1. *Only a rapidly decreasing contribution to the asymptotics is lost, if the integrand of (28) is multiplied by $\varrho(g)$.*

On the support of ϱ, $\left(\widetilde{\mu}_{g^{-1}}(x_1), x_2\right)$ lies in a small neighborhood of the diagonal; since any smoothing term will contribute negligibly to the asymptotics, we may replace Π by its representation as an FIO (§3.1). If we insert (20) in (28) (with the factor $\varrho(g)$ included), and apply the rescaling $u \mapsto k\,u$ we obtain

$$\begin{aligned} \Pi_{k\boldsymbol{\nu}}(x_1, x_2) \sim k^2\nu \int_G \mathrm{d}V_G(g) \int_0^{+\infty} \mathrm{d}u \\ \left[\varrho(g) \cdot \overline{\chi_{k\boldsymbol{\nu}}(g)}\, e^{\imath\, k\, u\, \psi\left(\widetilde{\mu}_{g^{-1}}(x_1), x_2\right)} \cdot s\left(\widetilde{\mu}_{g^{-1}}(x_1), x_2, k\,u\right)\right]. \end{aligned} \tag{30}$$

Integration in (30) can be reduced to a suitable compact domain without altering the asymptotics.

Proposition 3.2. *Let $D \gg 0$ and let $\rho \in \mathcal{C}^\infty_c(\mathbb{R})$ be ≥ 0, supported in $\big(1/D, D\big)$, and $\equiv 1$ on $(2/D, D/2)$. Then only a rapidly decreasing contribution to the asymptotics is lost, if the integrand on the last line of (30) is multiplied by $\rho(u)$.*

Proof of Proposition 3.2. Let us deal with the cases $u \gg 0$ and $0 < u \ll 1$ separately.

Case 1: $u \gg 0$**.** To begin with, let $\rho_1' : (0,+\infty) \to [0,+\infty)$ be $\mathcal{C}^\infty$, $\equiv 1$ on $(0, D/2)$ and $\equiv 0$ on $(D,+\infty)$ [1]. Let us set $\rho_2'(u) := 1 - \rho_1'(u)$. By (30),

$$\Pi_{k\boldsymbol{\nu}}(x_1,x_2) \sim \Pi_{k\boldsymbol{\nu}}(x_1,x_2)_1 + \Pi_{k\boldsymbol{\nu}}(x_1,x_2)_2, \tag{31}$$

where

$$\Pi_{k\boldsymbol{\nu}}(x_1,x_2)_j \sim k^2\nu \int_G \mathrm{d}V_G(g) \int_0^{+\infty} \mathrm{d}u \left[e^{\imath\, k\, u\, \psi(\widetilde{\mu}_{g^{-1}}(x_1),x_2)} \right.$$
$$\left. \varrho(g)\cdot \rho_j'(u) \cdot \overline{\chi_{k\boldsymbol{\nu}}(g)} \cdot s\left(\widetilde{\mu}_{g^{-1}}(x_1), x_2, k\, u\right)\right]. \tag{32}$$

We need to show that $\Pi_{k\boldsymbol{\nu}}(x_1,x_2)_2 = O\left(k^{-\infty}\right)$.

Let us set

$$G' := \left\{ g \in G \,:\, \mathrm{dist}_G\big(g, \{\pm I_2\}\big) < 2\,\delta \right\}, \tag{33}$$
$$G'' := \left\{ g \in G \,:\, \mathrm{dist}_G\big(g, \{\pm I_2\}\big) > \delta \right\}.$$

Then $\{G', G''\}$ is also an open cover of G, and we may consider a $\mathcal{C}^\infty$ partition of unity $\beta' + \beta'' = 1$ on G subordinate to it. Let us set

$$\varrho' := \varrho \cdot \beta', \quad \varrho'' := \varrho\cdot\beta''.$$

Then $\varrho = \varrho' + \varrho''$, where ϱ' is supported in a small neighborhood of $\{\pm I_2\}$, and ϱ'' is supported away from $\{\pm I_2\}$.

Accordingly, we have

$$\Pi_{k\boldsymbol{\nu}}(x_1,x_2)_2 = \Pi_{k\boldsymbol{\nu}}(x_1,x_2)_2' + \Pi_{k\boldsymbol{\nu}}(x_1,x_2)_2'', \tag{34}$$

where in the former (respectively, latter) summand $\varrho(g)$ has been replaced by $\varrho'(g)$ (respectively, $\varrho''(g)$).

We shall deal with the two summands in (34) separately.

LEMMA 3.2. *$\Pi_{k\boldsymbol{\nu}}(x_1,x_2)_2' = O\left(k^{-\infty}\right)$ as $k \to +\infty$.*

REMARK 3.1. In the integration defining $\Pi_{k\boldsymbol{\nu}}(x_1,x_2)_2'$, $\widetilde{\mu}_{g^{-1}}(x_1)$ is close to x_2, g is close to $\{\pm I_2\}$, and u is very large.

Proof of Lemma 3.2. Let us assume to fix ideas that $k\,\nu = 2\ell + 1$ is odd. Then $V_{k\boldsymbol{\nu}}$ may be identified with the vector space $\mathbb{C}^{(2\ell)}[z_1,z_2]$ of complex homogeneous polynomials of degree $2\,\ell$ in two variables. A natural basis of the latter is given by the monomials $P_\mu(z_1,z_2) := z_1^{\ell-\mu}\, z_2^{\ell+\mu}$, where $\mu \in \{-\ell,\ldots,0,\ldots,\ell\}$. We shall accordingly denote the matrix elements of the representation $V_{k\boldsymbol{\nu}}$ by $\mathcal{M}^{(k\boldsymbol{\nu})}_{a,b}(g)$, where $g \in G$ and $a, b \in \{-\ell,\ldots,0,\ldots,\ell\}$. Thus

$$\Pi_{k\boldsymbol{\nu}}(x_1,x_2)_2' \sim \sum_{a=-\ell}^{\ell} \Pi_{k\boldsymbol{\nu}}(x_1,x_2)_{2,a}', \tag{35}$$

where

$$\Pi_{k\boldsymbol{\nu}}(x_1,x_2)_{2,a}' := k^2\nu \int_G \mathrm{d}V_G(g) \int_{D/2}^{+\infty} \mathrm{d}u \left[e^{\imath\, k\, u\, \psi(\widetilde{\mu}_{g^{-1}}(x_1),x_2)} \right. \tag{36}$$
$$\left. \cdot\overline{\mathcal{M}^{(k\boldsymbol{\nu})}_{a,a}(g)} \cdot \varrho'(g) \cdot \rho_2'(u) \cdot s\left(\widetilde{\mu}_{g^{-1}}(x_1), x_2, k\, u\right)\right].$$

[1] Throughout this proof, the primes do not stand for derivatives.

On the support of ϱ', either $g \sim I_2$ or $g \sim -I_2$; hence we can write

$$g = \begin{pmatrix} A(g)\, e^{\imath\theta_G(g)} & -\overline{\gamma(g)} \\ \gamma(g) & A(g)\, e^{-\imath\theta_G(g)} \end{pmatrix}, \tag{37}$$

where $A(g) > 0$, and either $\theta_G(g) \sim 0$ or $\theta_G(g) \sim \pi$. Furthermore, A, γ, θ_G are $\mathcal{C}^\infty$ functions of $g \in G$ on a neighborhood of the support of ρ'.

By the discussion in §2.6.3 of [A] and in §11 of [RT], with g as in (37) we have

$$\mathcal{M}^{(k\boldsymbol{\nu})}_{a,a}(g) = e^{-2\imath a\theta_G(g)} \cdot R_{\ell,a}\left(A(g)^2\right), \tag{38}$$

where $R_{\ell,a}$ may be expressed in terms of suitable Jacobi polynomials, and is itself a real polynomial. Since the left hand side of (38) is an entry of a unitary matrix, we have at any rate $\left|R_{\ell,a}\left(A(g)^2\right)\right| \leq 1$.

Inserting (38) in (36), we obtain:

$$\begin{aligned} \Pi_{k\boldsymbol{\nu}}\left(x_1,x_2\right)'_{2,a} := k^2\nu \int_G \mathrm{d}V_G(g) \int_{D/2}^{+\infty} \mathrm{d}u\, \Big[& e^{\imath\, k\, \Psi_{a/k}(x_1,x_2;u,g)} \\ & \cdot R_{\ell,a}\left(A(g)^2\right) \cdot \varrho'(g) \cdot \rho_2'(u) \cdot s\left(\widetilde{\mu}_{g^{-1}}(x_1), x_2, k\,u\right)\Big]. \end{aligned} \tag{39}$$

where for every $b \in \mathbb{R}$ we set

$$\Psi_b(x_1,x_2;u,g) := u \cdot \psi\left(\widetilde{\mu}_{g^{-1}}(x_1), x_2\right) + 2\,b \cdot \theta_G(g). \tag{40}$$

Since $|a/k| \leq \nu/2$, the phases $\Psi_{a/k}$ form a bounded family.

Let $E : \eta \in \mathfrak{g} \mapsto E(\eta) := e^\eta \in G$ be the exponential map, and let β be as in Remark 1.1. Then every element of the standard torus $T \leqslant G$ may be written

$$t = E(\theta\,\beta) = e^{\theta\beta} = \begin{pmatrix} e^{\imath\theta} & 0 \\ 0 & e^{-\imath\theta} \end{pmatrix}. \tag{41}$$

Let us view β as a left-invariant vector field on G; the corresponding 1-parameter group of diffeomorphisms is $\varphi_\tau(g) := g\, e^{\tau\,\beta}$. Thus

$$\varphi_\tau : \begin{pmatrix} u & -\overline{v} \\ v & \overline{u} \end{pmatrix} \mapsto \begin{pmatrix} u \cdot e^{\imath\tau} & -\overline{v} \cdot e^{-\imath\tau} \\ v \cdot e^{\imath\tau} & \overline{u} \cdot e^{-\imath\tau} \end{pmatrix}. \tag{42}$$

Therefore, if L_β is the same vector field viewed as a differential operator on G, then $L_\beta(\theta_G) = 1$ on the support of ϱ'. Also, L_β is a skew-hermitian operator on $L^2(G)$, since ϕ_τ induces a 1-parameter group of unitary automorphisms of $L^2(G)$; hence, $L_\beta^t = -\overline{L}_\beta$. Furthermore, the function $g \mapsto A(g)^2$ is in $\mathcal{C}^\infty(G)$, real and φ_τ-invariant for every $\tau \in \mathbb{R}$; therefore, $L_\beta\left(A(g)^2\right) = \overline{L}_\beta\left(A(g)^2\right) = 0$.

On the other hand, by (1) we have

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{\mu}_{\varphi_\tau(g)^{-1}}(x_1)\right|_{\tau=0} &= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{\mu}_{e^{-\tau\,\beta}}\left(\widetilde{\mu}_{g^{-1}}(x_1)\right)\right|_{\tau=0} \\ &= -\beta_X\left(\widetilde{\mu}_{g^{-1}}(x_1)\right) \\ &= -\beta_M\left(\widetilde{\mu}_{g^{-1}}(m_{x_1})\right)^\sharp + \left\langle \Phi_G\left(\mu_{g^{-1}}(m_{x_1})\right), \beta\right\rangle \partial_\theta. \end{aligned} \tag{43}$$

For $\varrho(g) \neq 0$ we have $\mathrm{dist}_X\left(\widetilde{\mu}_{g^{-1}}(x_1), x_2\right) \leq 2\,\delta$; therefore,

$$\left\langle \Phi_G\left(\mu_{g^{-1}}(m_{x_1})\right), \beta\right\rangle = \left\langle \Phi_G\left(m_{x_2}\right), \beta\right\rangle + O(\delta). \tag{44}$$

If δ is sufficiently small, (44) is non-zero, since we are assuming that $\Phi_G(m_{x_2})$ is not anti-diagonal; assuming to fix ideas that $\Phi_G(m_{x_2})$ is diagonal, then the right hand side of (44) is $\langle \Phi_G(m_{x_2}), \beta\rangle = 2\,\lambda(m_x) + O(\delta)$.

In addition, by the discussion in §3.1, where $\varrho(g) \neq 0$

$$\mathrm{d}_{(\widetilde{\mu}_{g^{-1}}(x_1),x_2)}\psi = \Big(\alpha_{\widetilde{\mu}_{g^{-1}}}(x_1), -\alpha_{x_2}\Big) + O(\delta). \tag{45}$$

Therefore,

$$L_\beta\big(\Psi_b(x_{1k}; u, g)\big) = 2\left[u \cdot \lambda(m_x) + b\right] + O(\delta). \tag{46}$$

For $u \gg 0$, we conclude that $L_\beta\big(\Psi_{a/k}(u,g)\big) \geq C' \cdot u + 1$ for some $C' > 0$, which can be chosen uniformly for all $a \in \{-\ell, \ldots, 0, \ldots, \ell\}$; by iteratively 'integrating by parts' in (39) by the transpose operator $L_\beta^t = -\overline{L}_\beta$, we conclude that $\Pi_{k\boldsymbol{\nu}}\big(x_1,x_2\big)'_{2,a} = O\left(k^{-\infty}\right)$ uniformly for $a \in \{-\ell, \ldots, \ell\}$. Since this holds uniformly for each of the $k\,\nu$ summands in (35), the statement of Lemma 3.2 is established in the case where $k\,\nu$ is odd.

The case where $k\,\nu$ is even is only slightly different - one takes ℓ to be half-integer (see Theorem 11.7.1 of [RT]). □

Lemma 3.3. *$\Pi_{k\boldsymbol{\nu}}\big(x_1,x_2\big)''_2 = O\left(k^{-\infty}\right)$ as $k \to +\infty$.*

Remark 3.2. In the integration defining $\Pi_{k\boldsymbol{\nu}}\big(x_1,x_2\big)''_2$, $\widetilde{\mu}_{g^{-1}}(x_1)$ is close to x_2, g is at a positive distance from $\{\pm I_2\}$, and u is very large.

Proof of Lemma 3.3. The proof is similar to the one of Lemma 3.2, except that we shall use eigenvalues rather than matrix elements, so we'll be somewhat sketchy.

If $g \in G \setminus \{\pm I_2\}$, then there is a unique $\vartheta_G(g) = \cos^{-1}\big(\mathrm{trace}(g)/2\big) \in (0,\pi)$ such that the eigenvalues of g are $e^{\pm\imath\,\vartheta_G(g)}$. The map $g \in G \setminus \{\pm I_2\} \mapsto \vartheta_G(g) \in (0,\pi)$ is $\mathcal{C}^\infty$. On the same domain, the character of $V_{k\nu}$ is thus given by

$$\chi_{k\boldsymbol{\nu}}(g) = \sum_{j=0}^{k\,\nu-1} e^{\imath\,(k\nu-1-2j)\,\vartheta_G(g)}. \tag{47}$$

In place of (35), we shall now write

$$\Pi_{k\boldsymbol{\nu}}\big(x_1,x_2\big)''_2 \sim \sum_{j=0}^{k\,\nu-1} \Pi_{k\boldsymbol{\nu}}\big(x_1,x_2\big)''_{2,j}, \tag{48}$$

where

$$\begin{aligned}\Pi_{k\boldsymbol{\nu}}\big(x_1,x_2\big)''_{2,j} &:= k^2\nu \int_G \mathrm{d}V_G(g) \int_{D/2}^{+\infty} \mathrm{d}u\,\Big[e^{\imath\,k\,u\cdot\psi\left(\widetilde{\mu}_{g^{-1}}(x_1),x_2\right)} \\ &\qquad \cdot e^{-\imath\,(k\nu-1-2j)\cdot\vartheta_G(g)} \cdot \varrho''(g)\cdot\rho_2'(u)\cdot s\left(\widetilde{\mu}_{g^{-1}}(x_1), x_2, k\,u\right)\Big] \\ &= k^2\nu \int_G \mathrm{d}V_G(g) \int_{D/2}^{+\infty} \mathrm{d}u\,\Big[e^{\imath\,k\,\Gamma_{(1+2j)/k}(x_1,x_2;u,g)} \\ &\qquad \cdot\varrho''(g)\cdot\rho_2'(u)\cdot s\left(\widetilde{\mu}_{g^{-1}}(x_1), x_2, k\,u\right)\Big],\end{aligned} \tag{49}$$

where we have set for $b \in \mathbb{R}$

$$\Gamma_b(x_1,x_2;u,g) := u\,\psi\left(\widetilde{\mu}_{g^{-1}}(x_1), x_2\right) - (\nu + b)\cdot\vartheta_G(g). \tag{50}$$

Again, the phases $\Gamma_{(1+2j)/k}(x_{jk};\cdot,\cdot)$ vary in a bounded family.

Furthermore, since δ is small but fixed, ϑ_G is bounded in $\mathcal{C}^r$ norm for every $r \geq 0$ on the support of ϱ''. We can then complete the proof by arguing as in the final part of the proof of Lemma 3.2. □

Given (34), Lemmata 3.2 and 3.3 imply that for $\Pi_{k\boldsymbol{\nu}}\big(x_1,x_2\big)_2 = O\left(k^{-\infty}\right)$ for $k \to +\infty$.

Case 2: $0 < u \ll 1$. Let $\rho_1'' : (0,+\infty) \to \mathbb{R}$ be $\mathcal{C}^\infty$, ≥ 0, $\equiv 0$ on $(0,1/D)$ and $\equiv 1$ on $(2/D,+\infty)$. Let us set $\rho_2'' := 1 - \rho_1''$. By the above, we can replace (34) by

$$\Pi_{k\boldsymbol{\nu}}\big(x_1,x_2\big) \sim \Pi_{k\boldsymbol{\nu}}\big(x_1,x_2\big)_1 = \Pi_{k\boldsymbol{\nu}}\big(x_1,x_2\big)_{11} + \Pi_{k\boldsymbol{\nu}}\big(x_1,x_2\big)_{12}, \tag{51}$$

where $\Pi_{k\boldsymbol{\nu}}\big(x_1,x_2\big)_{1j}$ is defined as in (32), except that in the integrand $\rho_j'(u)$ is replaced by $\rho_1'(u)\cdot\rho_j''(u)$.

Lemma 3.4. *$\Pi_{k\boldsymbol{\nu}}\big(x_1,x_2\big)_{12} = O\left(k^{-\infty}\right)$ as $k \to +\infty$.*

Proof of Lemma 3.4. Let us rewrite $\Pi_{k\boldsymbol{\nu}}\big(x_{1k},x_{2k}\big)_{12}$ by means of the Weyl integration and character formulae, as in §3.2. Introducing coordinates on T in (24), we obtain

$$\begin{aligned}
&\Pi_{k\boldsymbol{\nu}}\big(x_1,x_2\big)_{12} \\
&= \frac{k^2\cdot\nu}{2\,\pi}\cdot\int_{G/T}\mathrm{d}V_{G/T}(gT)\int_{-\pi}^{\pi}\mathrm{d}\vartheta\int_0^{+\infty}\mathrm{d}u\,\Big[e^{\imath\,k\,\Psi(x_1,x_2;u,gT,\vartheta)}\cdot\rho_1'(u)\,\rho_2''(u) \\
&\quad\cdot\varrho''\left(g\,e^{-\vartheta\,\beta}\,g^{-1}\right)\cdot\left(e^{\imath\,\vartheta}-e^{-\imath\,\vartheta}\right)\cdot s\left(\widetilde{\mu}_{g\,e^{-\vartheta\,\beta}\,g^{-1}}(x_1),x_2,k\,u\right)\Big],
\end{aligned} \tag{52}$$

where we have set

$$\Psi(x_1,x_2;u,gT,\vartheta) := u\cdot\psi\left(\widetilde{\mu}_{g\,e^{-\vartheta\,\beta}\,g^{-1}}(x_{1k}),x_{2k}\right) - \nu\cdot\vartheta. \tag{53}$$

Thus if $D \gg 0$ on the support of $\rho_2'(u)$ we have

$$\partial_\vartheta\Psi(x_1,x_2;u,gT,\vartheta) \leq -\frac{\nu}{2};$$

the claim then follows in a standard manner by iteratively integrating by parts in $\mathrm{d}\vartheta$. □

We conclude that $\Pi_{k\boldsymbol{\nu}}\big(x_1,x_2\big) \sim \Pi_{k\boldsymbol{\nu}}\big(x_1,x_2\big)_{11}$. To complete the proof of Proposition 3.2, we need only factor $\rho(u) = \rho_1'(u)\cdot\rho_1''(u)$. □

Remark 3.3. Let $\mathbf{v}_j \in T_{m_x}M$ be as in the statement of Theorem 1.3, and set $x_{jk} := x + k^{-1/2}\,\mathbf{v}_j$. As a variant of Proposition 3.2, we can replace (x_1,x_2) by (x_{1k},x_{2k}). The same arguments apply with minor modifications; in particular, one will replace $G_{<\delta}(x_1,x_2)$ in (29) by a δ-neighborhood of the stabilizer subgroup G_x.

4. The proofs.

We collect in this Section the proofs of Theorems 1.1, 1.2, 1.3.

4.1. Theorem 1.1.

Proof of Theorem 1.1. We can replace x and y by any other points in their respective orbits, and therefore we may assume without loss that $\Phi_G\circ\pi(y)$ is diagonal, in form (4). By Proposition 3.2 (with $x_1 = x$ and $x_2 = y$), we have

$$\begin{aligned}
\Pi_{k\boldsymbol{\nu}}\big(x,y\big) &\sim k^2\nu\int_G\mathrm{d}V_G(g)\int_0^{+\infty}\mathrm{d}u \\
&\quad\Big[\rho(u)\cdot\varrho(g)\cdot\overline{\chi_{k\boldsymbol{\nu}}(g)}\,e^{\imath\,k\,u\,\psi\left(\widetilde{\mu}_{g^{-1}}(x),y\right)}\cdot s\left(\widetilde{\mu}_{g^{-1}}(x),y,k\,u\right)\Big],
\end{aligned} \tag{54}$$

where $\varrho \in \mathcal{C}^\infty(G)$ is a bump function supported where $\mathrm{dist}_X\left(\widetilde{\mu}_{g^{-1}}(x), y\right) \leq 2\,\delta$, and identically $\equiv 1$ where $\mathrm{dist}_X\left(\widetilde{\mu}_{g^{-1}}(x), y\right) \leq \delta$, while ρ is as in Proposition 3.2.

Where $\mathrm{dist}_X(G \cdot x, y) \geq C\, k^{\epsilon - 1/2}$, by Corollary 1.3 of [BS] we have

$$\left|\psi\left(\widetilde{\mu}_{g^{-1}}(x), y\right)\right| \geq \Im\left(\psi\left(\widetilde{\mu}_{g^{-1}}(x), y\right)\right| \geq C_1\, k^{2\,\epsilon-1} \tag{55}$$

for some constant $C_1 > 0$.

The statement of Theorem 1.1 then follows by iteratively integrating by parts in $\mathrm{d}u$, since at each step we introduce a factor $O\left(k^{-2\epsilon}\right)$. □

For expository reasons, we shall give the proof of Theorem 1.3 before the one of Theorem 1.2.

4.2. Theorem 1.3.

Proof of Theorem 1.3. By (27), we may assume without loss that $\Phi_G(m_x)$ is not antidiagonal. Let x_{jk} be as in Remark 3.3. By the discussion in §3.5, with β as in (5), we have

$$\begin{aligned} &\Pi_{k\boldsymbol{\nu}}\left(x_{1k}, x_{2k}\right) \\ &\sim \frac{k^2\nu}{2\pi}\int_{1/D}^{D}\mathrm{d}u\int_{-\pi/2}^{3\pi/2}\mathrm{d}\vartheta\int_{G/T}\mathrm{d}V_{G/T}(gT)\,\left[e^{\imath\, k\,\left[u\,\psi\left(\widetilde{\mu}_{ge^{-\imath\vartheta B}g^{-1}}(x_{1k}), x_{2k}\right) - \nu\,\vartheta\right]}\right. \\ &\quad \left.\cdot\rho(u)\cdot\varrho\left(ge^{-\vartheta\beta}g^{-1}\right)\cdot\left(e^{\imath\vartheta} - e^{-\imath\,\vartheta}\right)\cdot s\left(\widetilde{\mu}_{ge^{-\vartheta\beta}g^{-1}}(x_{1k}), x_{2k}, k\,u\right)\right]. \end{aligned} \tag{56}$$

The bump function ϱ is supported near $G_x = Z_x \leqslant \{\pm I_2\}$ (Remark 3.3). Hence, $\varrho = \varrho_+ + \varrho_-$, where ϱ_+ is supported in a small neighborhood of I_2, while ϱ_- is supported in a small neighborhood of $-I_2$, and vanishes identically if G_x is trivial. Writing $\varrho = \varrho_+ + \varrho_-$ in (56), we obtain (with an obvious interpretation)

$$\Pi_{k\boldsymbol{\nu}}\left(x_{1k}, x_{2k}\right) \sim \Pi_{k\boldsymbol{\nu}}\left(x_{1k}, x_{2k}\right)_+ + \Pi_{k\boldsymbol{\nu}}\left(x_{1k}, x_{2k}\right)_-; \tag{57}$$

let us examine the two summands in (57) separately.

4.2.1. The asymptotics of $\Pi_{k\boldsymbol{\nu}}\left(x_{1k}, x_{2k}\right)_+$. Integration in $\mathrm{d}V_G(g)$ is supported in a small neighborhood of I_2.

Proposition 4.1. *Under the assumptions of Theorem 1.3, as $k \to +\infty$ we have*

$$\begin{aligned} \Pi_{k\boldsymbol{\nu}}\left(x_{1k}, x_{2k}\right)_+ &\sim \frac{1}{2\,\lambda(m_x)}\cdot\left(\frac{\nu\,k}{2\,\pi\,\lambda(m_x)}\right)^d\cdot e^{u_0(\nu, m_x)\cdot\psi_2(\mathbf{v}_1, \mathbf{v}_2)} \\ &\quad\cdot\left[1 + \sum_{j\geq 1}^{+\infty} k^{-j/2} A_j^+(x; \mathbf{v}_1, \mathbf{v}_2)\right], \end{aligned}$$

where $A_j^+(x; \cdot, \cdot)$ is a polynomial of degree $\leq 3\,j$ and parity $(-1)^j$.

Proof of Proposition 4.1. $\Pi_{k\boldsymbol{\nu}}\left(x_{1k}, x_{2k}\right)_+$ is given by (56), with the cut-off $\varrho\left(ge^{-\imath\vartheta B}g^{-1}\right)$ replaced by $\varrho_+\left(ge^{-\imath\vartheta B}g^{-1}\right)$; therefore, integration in $\mathrm{d}\vartheta$ is restricted to $(-2\,\delta, 2\,\delta)$; we may assume that $\varrho_+\left(ge^{-\imath\vartheta B}g^{-1}\right)$ is identically equal to one for $\vartheta \in (-\delta, \delta)$.

Let us fix constants $C_1 > 0$, $\epsilon_1 \in (0, 1/6)$. Iteratively integrating by parts in $\mathrm{d}u$, similarly to the proof of Theorem 1.1, we conclude that the locus where $|\vartheta| >$

$C_1\, k^{\epsilon_1 - 1/2}$ contributes negligibly to the asymptotics of $\Pi_{k\boldsymbol{\nu}}\big(x_{1k}, x_{2k}\big)_0$. Hence we conclude the following.

LEMMA 4.1. *Suppose that $\varrho_1 \in \mathcal{C}_c(\mathbb{R})$ is ≥ 0, supported in $\big(-2, 2\big)$, and $\equiv 1$ on $(-1, 1)$. Then the asymptotics of $\Pi_{k\boldsymbol{\nu}}\big(x_{1k}, x_{2k}\big)_+$ are unchanged, if the integrand is multiplied by $\varrho_1\left(k^{1/2-\epsilon_1}\,\vartheta\right)$.*

Applying the rescaling $\vartheta \mapsto \vartheta/\sqrt{k}$, we recover

$$\begin{aligned} &\Pi_{k\boldsymbol{\nu}}\big(x_{1k}, x_{2k}\big)_+ \\ &\sim \frac{k^{3/2}\,\nu}{2\pi} \int_0^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \int_{G/T} \mathrm{d}V_{G/T}(gT) \left[e^{\imath\, k\, \Psi_k^+} \cdot \varrho_1\left(k^{-\epsilon_1}\,\vartheta\right) \cdot \rho(u) \right. \\ &\quad \left. \cdot \left(e^{\imath \vartheta/\sqrt{k}} - e^{-\imath\,\vartheta/\sqrt{k}} \right) \cdot s\left(\widetilde{\mu}_{ge^{-\vartheta\beta/\sqrt{k}}g^{-1}}(x_{1k}), x_{2k}, k\,u \right) \right], \end{aligned} \tag{58}$$

where

$$\Psi_k^+(u, \mathbf{v}_1, \mathbf{v}_2, \vartheta, g\,T) := u\,\psi\left(\widetilde{\mu}_{ge^{-\vartheta\beta/\sqrt{k}}g^{-1}}(x_{1k}), x_{2k} \right) - \frac{\vartheta}{\sqrt{k}}\,\nu. \tag{59}$$

Integration in $\mathrm{d}\vartheta$ is over an interval of length $4\,k^{\epsilon_1}$ centered at the origin.

Let us make Ψ_k more explicit. By Corollary 2.2 of [P4], with $m_x = \pi(x)$ we have

$$\begin{aligned} &\widetilde{\mu}_{ge^{-\imath\vartheta B/\sqrt{k}}g^{-1}}(x_{1k}) = \widetilde{\mu}_{e^{-\vartheta \mathrm{Ad}_g(\beta)/\sqrt{k}}}(x_{1k}) \\ &= x + \left(\Theta_k(\mathbf{v}_1, \vartheta, g\,T), \frac{1}{\sqrt{k}}\,\mathrm{V}(\mathbf{v}_1, \vartheta, gT) + R_2\left(\frac{1}{\sqrt{k}}\,\vartheta, \frac{1}{\sqrt{k}}\,\mathbf{v}_1 \right) \right), \end{aligned} \tag{60}$$

where (for appropriate C^∞ functions R_j vanishing to j-th order at the origin and allowed to vary from line to line)

$$\begin{aligned} \Theta_k(\upsilon_1, \vartheta, g\,T) := &\frac{1}{\sqrt{k}}\,\vartheta \cdot \Big\langle \Phi_G(m_x), \mathrm{Ad}_g(\beta) \Big\rangle \\ &+ \frac{1}{k}\,\vartheta \cdot \omega_{m_x}\big(\mathrm{Ad}_g(\beta)_M(m), \mathbf{v}_1\big) + R_3\left(\frac{1}{\sqrt{k}}\,\vartheta, \frac{1}{\sqrt{k}}\,\mathbf{v}_1 \right), \end{aligned} \tag{61}$$

$$\mathrm{V}(\mathbf{v}_1, \vartheta, gT) := \mathbf{v}_1 - \vartheta\,\mathrm{Ad}_g\big(\beta\big)_M(m_x). \tag{62}$$

We shall use the abridged notation Θ_k and V.

In abridged notation, let us set

$$\widetilde{\Theta}_k(\upsilon_1, \upsilon_2, \vartheta, g\,T) := \frac{1}{\sqrt{k}}\,A + \frac{1}{k}\,B + R_3\left(\frac{1}{\sqrt{k}}\,\vartheta, \frac{1}{\sqrt{k}}\,\mathbf{v}_1 \right),$$

where

$$A = A(\mathbf{v}_1, \mathbf{v}_2, \vartheta, g\,T) := \vartheta \cdot \Big\langle \Phi_G(m_x), \mathrm{Ad}_g(\beta) \Big\rangle, \tag{63}$$

$$B = B(\mathbf{v}_1, \vartheta, g\,T) := \vartheta \cdot \omega_{m_x}\big(\mathrm{Ad}_g(\beta)_M(m), \mathbf{v}_1\big). \tag{64}$$

Then

$$\Psi_k^+(u,\mathbf{v}_1,\mathbf{v}_2,\vartheta,g\,T) = \imath\,u\,\left[1-e^{\imath\,\widetilde{\Theta}_k}\right] - \frac{\vartheta}{\sqrt{k}}\,\nu - \imath\,\frac{u}{k}\,\psi_2\Big(V,\mathbf{v}_2\Big) + R_3\left(\frac{1}{\sqrt{k}}\,(\vartheta,\mathbf{v}_1,\mathbf{v}_2)\right). \tag{65}$$

In view of §3 of [SZ] (see especially (65)), by a few computations we obtain the following.

Lemma 4.2. *We have*

$$\Psi_k^+(u,\mathbf{v}_1,\mathbf{v}_2,\vartheta,g\,T) := \frac{1}{\sqrt{k}}\,\mathcal{G}(u,\vartheta,g\,T) + \frac{1}{k}\,\mathcal{D}(u,\mathbf{v}_1,\mathbf{v}_2,\vartheta,g\,T) + R_3\left(\frac{1}{\sqrt{k}}\,(\vartheta,\mathbf{v}_1,\mathbf{v}_2)\right),$$

where

$$\mathcal{G}(u,\vartheta,g\,T) = u\,A - \vartheta\,\nu = \vartheta\cdot\left[u\,\big\langle\Phi_G(m_x),\mathrm{Ad}_g(\beta)\big\rangle - \nu\right],$$

$$\mathcal{D}(u,\mathbf{v}_1,\mathbf{v}_2,\vartheta,g\,T) = u\left[B + \imath\left(\frac{1}{2}\,A^2 - \psi_2\big(V,\mathbf{v}_2\big)\right)\right].$$

From (58) and Lemma 4.2, we conclude that

$$\begin{aligned}&\Pi_{k\boldsymbol{\nu}}\big(x_{1k},x_{2k}\big)_+ \\ &\sim \frac{k^{3/2}\,\nu}{2\pi}\int_0^{+\infty}\mathrm{d}u\int_{-\infty}^{+\infty}\mathrm{d}\vartheta\int_{G/T}\mathrm{d}V_{G/T}(gT) \\ &\quad\left[e^{\imath\,\sqrt{k}\,\mathcal{G}(u,\vartheta,g\,T)}\cdot e^{\imath\,u\,B + u\left[\psi_2\left(V,\mathbf{v}_2\right)-\frac{1}{2}\,A^2\right]}\cdot\varrho_1\left(k^{-\epsilon_1}\,\vartheta\right)\cdot e^{\imath\,k\cdot R_3\left(\frac{1}{\sqrt{k}}\,(\vartheta,\mathbf{v}_1,\mathbf{v}_2)\right)}\right. \\ &\quad\left.\cdot\rho(u)\cdot\left(e^{\imath\vartheta/\sqrt{k}} - e^{-\imath\,\vartheta/\sqrt{k}}\right)\cdot s\left(\widetilde{\mu}_{ge^{-\vartheta\beta/\sqrt{k}}g^{-1}}(x_{1k}),x_{2k},k\,u\right)\right],\end{aligned} \tag{66}$$

with A and B as in (63) and (64).

We can make (66) yet more explicit, introducing coordinates (θ,δ) on G/T as in §3.3. Furthermore, let $h_m\,T\in G/T$ be as in (4), and operate the change of variable $g\,T\mapsto h_m\,g\,T$ in G/T; we shall write g in the form (25) with $\alpha=\cos(\theta)\,e^{\imath\delta}$ and $\beta=\sin(\theta)$. Then

$$\begin{aligned}\mathcal{G}(u,\vartheta,h_m g\,T) &= \vartheta\cdot\left[u\cdot\left\langle\imath\,g^{-1}\begin{pmatrix}\lambda(m_x) & 0\\ 0 & -\lambda(m_x)\end{pmatrix}g,\beta\right\rangle-\nu\right] \\ &= \vartheta\cdot\big[u\cdot\cos(2\theta)\cdot 2\,\lambda(m_x)-\nu\big].\end{aligned} \tag{67}$$

Let $\mathcal{G}'(u,\vartheta,\theta)$ denote the expression on the last line of (67). We can rewrite (66) in the following form:

$$\begin{aligned}
&\Pi_{k\boldsymbol{\nu}}\big(x_{1k},x_{2k}\big)_+ \qquad (68)\\
&\sim \frac{k^{3/2}\,\nu}{4\pi^2}\int_0^{+\infty}\mathrm{d}u\int_{-\infty}^{+\infty}\mathrm{d}\vartheta\int_0^{\pi/2}\mathrm{d}\theta\int_{-\pi}^{\pi}\mathrm{d}\delta\\
&\Big[e^{\imath\sqrt{k}\,\mathcal{G}'(u,\vartheta,\theta)}\cdot e^{\imath\,u\,B+u\,\left[\psi_2\left(V,\mathbf{v}_2\right)-\frac{1}{2}A^2\right]}\cdot\varrho_1\left(k^{-\epsilon_1}\vartheta\right)\cdot e^{\imath\,k\cdot R_3\left(\frac{1}{\sqrt{k}}(\vartheta,\mathbf{v}_1,\mathbf{v}_2)\right)}\\
&\cdot\rho(u)\cdot\left(e^{\imath\vartheta/\sqrt{k}}-e^{-\imath\,\vartheta/\sqrt{k}}\right)\cdot s\left(\widetilde{\mu}_{ge^{-\vartheta\beta/\sqrt{k}}g^{-1}}(x_{1k}),x_{2k},k\,u\right)\,\sin(2\theta)\Big],
\end{aligned}$$

where (with abuse of notation) $g = g(\theta,\delta)$ and $A = A(\theta,\delta)$, $B = B(\theta,\delta)$ by the obvious change of variables.

Setting $t = \cos(2\theta)$, we can reformulate (68) as follows. With slight abuse, let us write $g\,T = g(t,\delta)\,T$ and define

$$\Gamma(t;u,\vartheta) := \mathcal{G}'(u,\vartheta,\theta) = \vartheta\cdot\big[2\,\lambda(m_x)\cdot u\cdot t-\nu\big]. \qquad (69)$$

Then

$$\begin{aligned}
&\Pi_{k\boldsymbol{\nu}}\big(x_{1k},x_{2k}\big)_+ \qquad (70)\\
&\sim \frac{1}{2}\cdot\frac{k^{3/2}\,\nu}{(2\,\pi)^2}\int_0^{+\infty}\mathrm{d}u\int_{-\infty}^{+\infty}\mathrm{d}\vartheta\int_{-1}^{1}\mathrm{d}t\int_{-\pi}^{\pi}\mathrm{d}\delta\\
&\Big[e^{\imath\sqrt{k}\,\Gamma(t;u,\vartheta)}\cdot e^{\imath\,u\,B_t+u\,\left[\psi_2\left(V_t,\mathbf{v}_2\right)-\frac{1}{2}A_t^2\right]}\cdot\varrho_1\left(k^{-\epsilon_1}\vartheta\right)\cdot e^{\imath\,k\cdot R_3\left(\frac{1}{\sqrt{k}}(\vartheta,\mathbf{v}_1,\mathbf{v}_2)\right)}\\
&\cdot\rho(u)\cdot\left(e^{\imath\vartheta/\sqrt{k}}-e^{-\imath\,\vartheta/\sqrt{k}}\right)\cdot s\left(\widetilde{\mu}_{ge^{-\vartheta\beta/\sqrt{k}}g^{-1}}(x_{1k}),x_{2k},k\,u\right)\Big];
\end{aligned}$$

we have denoted by A_t, B_t the functions

$$A_t\big(\mathbf{v}_1,\mathbf{v}_2,\vartheta,\delta\big) = A_t\big(\mathbf{v}_1,\mathbf{v}_2,\vartheta,g(t,\delta)\,T\big),\quad B_t(\mathbf{v}_1,\vartheta,\delta) = B(\mathbf{v}_1,\vartheta,g(t,\delta)\,T),$$

and similarly for V_t.

Let us remark that

$$\begin{aligned}
e^{\imath\vartheta/\sqrt{k}}-e^{-\imath\,\vartheta/\sqrt{k}} &= \frac{2\,\imath}{\sqrt{k}}\cdot\vartheta\cdot\sum_{j=0}^{+\infty}\frac{(-1)^j}{(2j+1)!}\cdot\frac{\vartheta^{2j}}{k^j} \qquad (71)\\
&= \frac{2\,\imath}{\sqrt{k}}\cdot\vartheta\cdot\left[1+R_2\left(\frac{\vartheta}{\sqrt{k}}\right)\right].
\end{aligned}$$

Furthermore, working in HLC's, Taylor expansion yields an asymptotic expansion

$$s\left(\widetilde{\mu}_{ge^{-\imath\vartheta B/\sqrt{k}}g^{-1}}(x_{1k}),x_{2k},k\,u\right)\sim\left(\frac{k\,u}{\pi}\right)^d\cdot\left[1+R_1\left(\frac{\vartheta}{\sqrt{k}},\frac{\mathbf{v}_j}{\sqrt{k}}\right)\right]. \qquad (72)$$

As a consequence, we have an asymptotic expansion

$$\begin{aligned}
&e^{\imath\,k\cdot R_3\left(\frac{1}{\sqrt{k}}(\vartheta,\mathbf{v}_1,\mathbf{v}_2)\right)}\cdot\left(e^{\imath\vartheta/\sqrt{k}}-e^{-\imath\,\vartheta/\sqrt{k}}\right)\cdot s\left(\widetilde{\mu}_{ge^{-\vartheta\beta/\sqrt{k}}g^{-1}}(x_{1k}),x_{2k},k\,u\right)\\
&\sim\left(\frac{k\,u}{\pi}\right)^d\cdot\frac{2\,\imath}{\sqrt{k}}\cdot\vartheta\cdot\left[1+\sum_{j\geq 1}k^{-j/2}\,P_j(x,u;\vartheta,\mathbf{v}_1,\mathbf{v}_2)\right], \qquad (73)
\end{aligned}$$

where $P_j(x, u; \vartheta, \mathbf{v}_1, \mathbf{v}_2)$ is a polynomial of degree $\leq 3j$ and parity $(-1)^j$.

Therefore, the amplitude in (70) is given by an asymptotic expansion in descending half-integer powers of k, and the asymptotic expansion for the integrand may be integrated terms by term. By a few computations, by (69) one sees that the dominant term of the resulting expansion for (70) is the dominant term of the expansion for the following oscillatory integral:

$$
\begin{aligned}
I_{\mathbf{v}_1,\mathbf{v}_2}(k) :=& \frac{\imath}{\sqrt{k}} \cdot \left(\frac{k}{\pi}\right)^d \cdot \frac{k^{3/2}\,\nu}{(2\,\pi)^2} \int_0^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \int_{-1}^{1} \mathrm{d}t \int_{-\pi}^{\pi} \mathrm{d}\delta \\
& \left[e^{\imath\sqrt{k}\,\Gamma(t;u,\vartheta)} \cdot e^{\imath\,u\,B_t + u\left[\psi_2\left(V_t,\mathbf{v}_2\right) - \frac{1}{2}\,A_t^2\right]} \right. \\
& \left. \cdot \rho(u) \cdot \varrho_1\left(k^{-\epsilon_1}\,\vartheta\right) \cdot \vartheta \cdot u^d \right] \\
=& \frac{\nu}{8\,\pi} \cdot \left(\frac{k}{\pi}\right)^{d+1} \int_0^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \int_{-1}^{1} \mathrm{d}t \int_{-\pi}^{\pi} \mathrm{d}\delta \\
& \left[e^{\imath\sqrt{k}\,\Gamma(t;u,\vartheta)} \cdot (2\imath \cdot \vartheta \cdot u) \right. \\
& \left. \cdot e^{\imath\,u\,B_t + u\left[\psi_2\left(V_t,\mathbf{v}_2\right) - \frac{1}{2}\,A_t^2\right]} \cdot \rho(u) \cdot \varrho_1\left(k^{-\epsilon_1}\,\vartheta\right) \cdot u^{d-1} \right].
\end{aligned}
$$

The latter may in turn be rewritten

$$
\begin{aligned}
I_{\mathbf{v}_1,\mathbf{v}_2}(k) =& \frac{\nu}{8\,\pi} \cdot \left(\frac{k}{\pi}\right)^{d+1} \frac{1}{\sqrt{k} \cdot \lambda(m_x)} \\
& \cdot \int_{-\pi}^{\pi} \mathrm{d}\delta \int_0^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \int_{-1}^{1} \mathrm{d}t \left[\partial_t \left(e^{\imath\sqrt{k}\,\Gamma(t;u,\vartheta)} \right) \right. \\
& \left. \cdot e^{\imath\,u\,B_t + u\left[\psi_2\left(V_t,\mathbf{v}_2\right) - \frac{1}{2}\,A_t^2\right]} \cdot \varrho_1(u) \cdot \varrho_2\left(k^{-\epsilon_1}\,\vartheta\right) \cdot u^{d-1} \right].
\end{aligned} \tag{74}
$$

Integrating by parts in dt, we obtain

$$
\begin{aligned}
I_{\mathbf{v}_1,\mathbf{v}_2}(k) =& \frac{\nu}{8\,\pi} \cdot \left(\frac{k}{\pi}\right)^{d+1} \frac{1}{\sqrt{k} \cdot \lambda(m_x)} \\
& \cdot \left[J'_{\mathbf{v}_1,\mathbf{v}_2}(k) - J''_{\mathbf{v}_1,\mathbf{v}_2}(k) - J'''_{\mathbf{v}_1,\mathbf{v}_2}(k) \right],
\end{aligned} \tag{75}
$$

where

$$
\begin{aligned}
J'_{\mathbf{v}_1,\mathbf{v}_2}(k) :=& \int_{-\pi}^{\pi} \mathrm{d}\delta \int_0^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \left[e^{\imath\sqrt{k}\,\Gamma(1;u,\vartheta)} \right. \\
& \left. \cdot e^{\imath\,u\,B_1 + u\left[\psi_2\left(V_1,\mathbf{v}_2\right) - \frac{1}{2}\,A_1^2\right]} \cdot \rho(u) \cdot \varrho_1\left(k^{-\epsilon_1}\,\vartheta\right) \cdot u^{d-1} \right],
\end{aligned} \tag{76}
$$

$$
\begin{aligned}
J''_{\mathbf{v}_1,\mathbf{v}_2}(k) :=& \int_{-\pi}^{\pi} \mathrm{d}\delta \int_0^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \left[e^{\imath\sqrt{k}\,\Gamma(-1;u,\vartheta)} \right. \\
& \left. \cdot e^{\imath\,u\,B_{-1} + u\left[\psi_2\left(V_{-1},\mathbf{v}_2\right) - \frac{1}{2}\,A_{-1}^2\right]} \cdot \rho(u) \cdot \varrho_1\left(k^{-\epsilon_1}\,\vartheta\right) \cdot u^{d-1} \right],
\end{aligned} \tag{77}
$$

$$J'''_{\mathbf{v}_1,\mathbf{v}_2}(k) := \int_{-1}^{1} \mathrm{d}t \int_{-\pi}^{\pi} \mathrm{d}\delta \int_{0}^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \left[e^{\imath \sqrt{k}\,\Gamma(t;u,\vartheta)} \right. \tag{78}$$
$$\left. \cdot \partial_t \left(e^{\imath\, u\, B_t + u\, \left[\psi_2\left(V_t, \mathbf{v}_2\right) - \frac{1}{2} A_t^2\right]} \right) \cdot \rho(u) \cdot \varrho_1\left(k^{-\epsilon_1}\vartheta\right) \cdot u^{d-1} \right].$$

Let us estimate the three terms (76), (77) and (78) separately.

Lemma 4.3. *As $k \to +\infty$, there is an asymptotic expansion of the form*

$$J'_{\mathbf{v}_1,\mathbf{v}_2}(k) \sim \frac{4\,\pi^2}{\sqrt{k}} \cdot \frac{1}{2\,\lambda(m_x)} \cdot e^{u_0(\nu,m_x)\cdot\psi_2(\mathbf{v}_1,\mathbf{v}_2)} \cdot \left(\frac{\nu}{2\,\lambda(m_x)}\right)^{d-1}$$
$$\cdot \left[1 + \sum_{l=1}^{+\infty} k^{-l/2}\, a_l(m_x; \mathbf{v}_1, \mathbf{v}_2) \right],$$

where $a_l(m_x;\cdot,\cdot)$ is a polynomial of degree $\leq 3l$ and parity $(-1)^l$, whose coefficients are $\mathcal{C}^\infty$ functions on M.

Proof of Lemma 4.3. Let us view $J'_{\mathbf{v}_1,\mathbf{v}_2}(k)$ as an oscillatory integral in the parameter $\sqrt{k}$, with real phase

$$\Gamma(1; u, \vartheta) = \vartheta \cdot \left[2\,\lambda(m_x)\cdot u - \nu\right], \tag{79}$$

and amplitude

$$e^{\imath\, u\, B_1 + u\, \left[\psi_2\left(V_1, \mathbf{v}_2\right) - \frac{1}{2} A_1^2\right]} \cdot \varrho_1(u) \cdot \varrho_2\left(k^{-\epsilon_1}\vartheta\right) \cdot u^{d-1}. \tag{80}$$

Explicitly, the exponent is

$$\begin{aligned} \mathcal{E}_1(u,\vartheta,\mathbf{v}_1,\mathbf{v}_2) &:= \imath\, u\, B_1 + u\left[\psi_2\left(V_1,\mathbf{v}_2\right) - \frac{1}{2}\,A_1^2\right] \\ &= u\, \Big[-\imath\,\omega_{m_x}(\mathbf{v}_1,\mathbf{v}_2) + \imath\,\vartheta\,\omega_{m_x}\Big(\mathrm{Ad}_{h_{m_x}}(\beta)_M(m_x), \mathbf{v}_1+\mathbf{v}_2\Big) \\ &\quad -\frac{1}{2}\left\| (\mathbf{v}_1-\mathbf{v}_2) - \vartheta\,\mathrm{Ad}_{h_{m_x}}(\beta)_X(x)\right\|^2 \Big]. \end{aligned} \tag{81}$$

Under the hypothesis of the Theorem, therefore, $\Re(\mathcal{E}_1) \leq -C'\,\vartheta^2 + C''$ for some constants C', $C'' > 0$.

Furthermore, the phase has a unique critical point, given by $(u_0(\nu,m_x),0)$ (Definition 1.1), and Hessian matrix

$$\mathrm{Hess}\big(\Gamma(1;\cdot,\cdot)\big) = \begin{pmatrix} 0 & 2\,\lambda(m_x) \\ 2\,\lambda(m_x) & 0 \end{pmatrix}.$$

Hence the Hessian determinant is $-4\,\lambda(m_x)^2$ and its signature is zero. Thus the critical point is non-degenerate, and the critical value is $\Gamma(1;u_0,0) = 0$. At the critical point, the exponent in the amplitude is

$$\mathcal{E}_1\big(u_0(\nu,m_x),0,\mathbf{v}_1,\mathbf{v}_2\big) = u_0(\nu,m_x)\cdot\psi_2(\mathbf{v}_1,\mathbf{v}_2).$$

When applying the Stationary Phase Lemma, at the l-th step we need to let the differential operator $k^{-l/2}\,R_\Gamma^l$ act on the amplitude (80), where

$$R_\Gamma := \frac{\imath}{4\cdot\lambda(m_x)}\cdot\frac{\partial^2}{\partial u\,\partial\vartheta},$$

and evaluate the result at the critical point. One sees inductively that

$$k^{-l/2}\,R_\Gamma^l\left(e^{\imath\,\mathcal{E}_1(u,\vartheta,\mathbf{v}_1,\mathbf{v}_2)}\right) = H_l\,e^{\imath\,\mathcal{E}_1(u,\vartheta,\mathbf{v}_1,\mathbf{v}_2)},$$

where H_l is a polynomial of degree $\le 3l$ in $(\vartheta,\mathbf{v}_1,\mathbf{v}_2)$ and parity $(-1)^l$.

The proof of Lemma 4.3 is complete. □

Lemma 4.4. *As $k\to+\infty$, we have $J''_{\mathbf{v}_1,\mathbf{v}_2}(k) = O\left(k^{-\infty}\right)$.*

Proof of Lemma 4.4. Let us view $J''_{\mathbf{v}_1,\mathbf{v}_2}(k)$ as an oscillatory integral in $\sqrt{k}$, with phase $\Gamma_{\theta_1,\theta_2}(-1;u,\vartheta)$. By (69),

$$\partial_\vartheta\Gamma(-1;u,\vartheta) = -2\,u\cdot\lambda(m_x) - \nu \le -\nu.$$

The claim follows by iterated integration by parts in ϑ. □

Lemma 4.5. *$\mathcal{J}'''_{\mathbf{v}_1,\mathbf{v}_2}(k;t,\delta)_2 = O\left(k^{-1}\right)$ as $k\to+\infty$; furthermore, $\mathcal{J}'''_{\mathbf{v}_1,\mathbf{v}_2}(k;t,\delta)_2$ admits an asymptotic expansion of the same kind as $\mathcal{J}'_{\mathbf{v}_1,\mathbf{v}_2}(k;t,\delta)_2$ (of lower leading order).*

Proof of Lemma 4.5. Let us choose $\epsilon' \in \big(0,\nu/\big(4\,D\cdot\lambda(m_x)\big)\big)$, and consider the open cover $\mathcal{U} := \{[-1,2\epsilon'),(\epsilon',1]\}$. Let $\gamma_1(t)+\gamma_2(t) = 1$ be a smooth partition of unity on $[-1,1]$ subordinate to $\mathcal{U}$. Thus

$$J'''_{\mathbf{v}_1,\mathbf{v}_2}(k) = J'''_{\mathbf{v}_1,\mathbf{v}_2}(k)_1 + J'''_{\mathbf{v}_1,\mathbf{v}_2}(k)_2,$$

where $J'''_{\upsilon_1,\upsilon_2}(k)_j$ is defined as in (78), with the extra factor $\gamma_j(t)$. Explicitly, let us set

$$\mathcal{E}_t(u,\vartheta,\mathbf{v}_1,\mathbf{v}_2) := \imath\,u\,B_t + u\left[\psi_2\big(V_t,\mathbf{v}_2\big) - \frac{1}{2}\,A_t^2\right]. \tag{82}$$

Then

$$J'''_{\mathbf{v}_1,\mathbf{v}_2}(k)_j = \int_{-\pi}^{\pi}\mathrm{d}\delta\int_{-1}^{1}\mathrm{d}t\,\left[\mathcal{J}'''_{\mathbf{v}_1,\mathbf{v}_2}(k;t,\delta)_j\right],$$

where

$$\begin{aligned}\mathcal{J}'''_{\mathbf{v}_1,\mathbf{v}_2}(k;t,\delta)_j = \int_0^{+\infty}\mathrm{d}u\int_{-\infty}^{+\infty}\mathrm{d}\vartheta\,\Big[e^{\imath\sqrt{k}\,\Gamma(t;u,\vartheta)}\cdot\gamma_j(t)\\ \cdot\partial_t\Big(\mathcal{E}_t(u,\vartheta,\mathbf{v}_1,\mathbf{v}_2)\Big)\cdot e^{\mathcal{E}_t(u,\vartheta,\mathbf{v}_1,\mathbf{v}_2)}\cdot\rho(u)\cdot\varrho_1\left(k^{-\epsilon_1}\,\vartheta\right)\cdot u^{d-1}\Big].\end{aligned} \tag{83}$$

Let us view $\mathcal{J}'''_{\mathbf{v}_1,\mathbf{v}_2}(k;t,\delta)_j$ as an oscillatory integral with phase $\Gamma_t(u,\vartheta) := \Gamma(t;u,\vartheta)$.

On the support of $\rho(u)\cdot\gamma_1(t)$, we have $u\le D$ and $t\le\epsilon'$; therefore,

$$\partial_\vartheta\Gamma_t(u,\vartheta) = 2u\cdot t\cdot\lambda(m_x) - \nu \le 2\,D\cdot\epsilon'\cdot\lambda(m_x) - \nu \le -\frac{\nu}{2}.$$

Therefore, integration by parts in ϑ implies that $\mathcal{J}'''_{\mathbf{v}_1,\mathbf{v}_2}(k;t,\delta)_j = O\left(k^{-\infty}\right)$, uniformly for t in the support of γ_1. Hence $J'''_{\mathbf{v}_1,\mathbf{v}_2}(k)_1 = O\left(k^{-\infty}\right)$.

On the support of γ_2, on the other hand, $\Gamma(t;\cdot,\cdot)$ has the non-degenerate critical point

$$\big(u(t),0\big) = \left(\frac{\nu}{2\,t\,\lambda(m_x)},0\right),$$

with Hessian matrix

$$\mathrm{Hess}\big(\Gamma_t\big) = \begin{pmatrix} 0 & 2\,t\cdot\lambda(m_x) \\ 2\,t\cdot\lambda(m_x) & 0 \end{pmatrix}.$$

Therefore, we can apply the Stationary Phase Lemma as in the proof of Lemma 4.3, viewing t as a parameter. The asymptotic expansion will be non trivial only for $t\in\big[\nu/(2\,\lambda(m_x)\,D),1\big]$, for otherwise ρ vanishes identically in a neighborhood of $u(t)$.

Given (62), (63), and (64) (with $g\,T$ replaced by $h_{m_x}\,g(t,\delta)\,T$), the integrand in (78) is divisible by ϑ; hence it vanishes at the critical point. Furthermore, the integrand is of class L^1 as a function of the parameter t, since the exponent getting differentiated is a smooth function of t and $\sqrt{1-t^2}$.

Hence $\mathcal{J}'''_{\mathbf{v}_1,\mathbf{v}_2}(k;t,\delta)_2$ admits an asymptotic expansion in descending half-integer powers of k, with leading power k^{-1}, and coefficients of class L^1 as functions of t. The general term of the expansion will be a scalar multiple of

$$k^{-(l+1)/2}\cdot R^l_{\Gamma_t}\left(\partial_t\Big(\mathcal{E}_t(u,\vartheta,\mathbf{v}_1,\mathbf{v}_2)\Big)\cdot e^{\mathcal{E}_t(u,\vartheta,\mathbf{v}_1,\mathbf{v}_2)}\right). \tag{84}$$

Given integers $a,b\geq 0$, let us denote by $H_{a,b}(\vartheta;\mathbf{v}_1,\mathbf{v}_2)$ a generic polynomial in $(\vartheta,\mathbf{v}_1,\mathbf{v}_2)$, which is separately homogeneous of degree a in ϑ, and of degree b in $(\mathbf{v}_1,\mathbf{v}_2)$, and by $H_a(\vartheta,\mathbf{v}_1,\mathbf{v}_2)$ a generic polynomial in $(\vartheta;\mathbf{v}_1,\mathbf{v}_2)$ homogeneous of degree a (but perhaps not polyhomogeneous); both $H_{a,b}$ and H_a are allowed to vary from line to line, and their coefficients depend smoothly on u. Thus $\mathcal{E}_t = u\cdot H_2 = u\cdot(H_{2,0}+H_{1,1}+H_{0,2})$, $\partial_t\mathcal{E}_t = u(H_{2,0}+H_{1,1})$ (here the polynomials do not depend on u). Hence we can split (84) as

$$\begin{aligned} &k^{-(l+1)/2}\cdot\Big[R^l_{\Gamma_t}\left(\rho(u)\cdot H_{2,0}\cdot e^{u\cdot(H_{2,0}+H_{1,1}+H_{0,2})}\right) \\ &\quad + R^l_{\Gamma_t}\left(\rho(u)\cdot H_{1,1}\cdot e^{u\cdot(H_{2,0}+H_{1,1}+H_{0,2})}\right)\Big]. \end{aligned} \tag{85}$$

The proof of Lemma 4.5 is completed by the following two claims, which can be proved inductively from the cases $l=0,1$:

Claim 4.1. *For $l=0,1,2,\ldots$,*

$$R^l_{\Gamma_t}\left(\rho(u)\cdot H_{1,1}\cdot e^{\mathcal{E}_t}\right)$$

is a sum of terms of the form

$$\Big[H_{0,1}\cdot H_{p_l}+H_{1,1}\cdot H_{q_l}\Big]\cdot e^{\mathcal{E}_t},$$

where $p_l+1\leq 3\,l$, $(-1)^{p_l+1}=(-1)^l$, and $q_l\leq 3l$, $(-1)^{q_l}=(-1)^l$.

Claim 4.2. *For $l=0,1,2,\ldots$,*

$$R^l_{\Gamma_t}\left(\rho(u)\cdot H_{2,0}\cdot e^{\mathcal{E}_t}\right)$$

is a sum of terms of the form $H_{a,b} \cdot e^{\mathcal{E}_t}$, *where* $b \leq 3\,l$ *and* $(-1)^{a+b} = (-1)^l$. □

Since at the critical point $\vartheta = 0$, the summands with a factor of the form $H_{a,b}$ with $a \geq 1$ all vanish at the critical point. It follows that the asymptotic expansion for (74) is as in the statement of Proposition 4.1. The contributions to the asymptotic expansion for (70) coming from the lower order terms in (73) can be dealt with by similar arguments. This completes the proof of Proposition 4.1. □

4.2.2. $\Pi_{k\boldsymbol{\nu}}\big(x_{1k}, x_{2k}\big)_-$**.** Let us now consider the asymptotics of the second summand in (57). We shall prove the following analogue of Proposition 4.1.

PROPOSITION 4.2. *Under the assumptions of Theorem 1.3, suppose in addition that* $-I_2 \in G_x$. *Let us set* $\mathbf{v}_1^{-I_2} := \mathrm{d}_{m_x}\mu_{-I_2}(\mathbf{v}_1)$. *Then as* $k \to +\infty$ *we have an asymptotic expansion*

$$\Pi_{k\boldsymbol{\nu}}\left(x_{1k}, x_{2k}\right)_- \sim \frac{e^{\imath\,\pi(1-k\cdot\nu)}}{2\,\lambda(m_x)} \cdot \left(\frac{\nu\,k}{2\,\pi\,\lambda(m_x)}\right)^d \cdot e^{u_0(\nu,m_x)\cdot\psi_2(\mathbf{v}_1^{-I_2},\mathbf{v}_2)} \cdot \left[1 + \sum_{j\geq 1}^{+\infty} k^{-j/2} A_j^-(x;\mathbf{v}_1,\mathbf{v}_2)\right],$$

where $A_j^-(x;\cdot,\cdot)$ *is a polynomial of degree* $\leq 3\,j$ *and parity* $(-1)^j$.

Proof of Proposition 4.2. The proof is a slight modification of the one for Proposition 4.1.

$\Pi_{k\boldsymbol{\nu}}\big(x_{1k}, x_{2k}\big)_-$ is given by (56), with ϱ replaced by ϱ_-; therefore, integration in $\mathrm{d}\vartheta$ is now restricted to $(\pi - 2\,\delta, \pi + 2\,\delta)$. With the change of variable $\vartheta \mapsto \vartheta + \pi$, $ge^{-\imath\vartheta B}g^{-1}$ gets replaced by $-ge^{-\imath\vartheta B}g^{-1}$ and $\vartheta \in (-\delta, \delta)$.

Lemma 4.1 still applies, so that we can again rescale in ϑ. By (26), we have

$$x_{1k}^{-I_2} := \widetilde{\mu}_{-I_2}(x_{1k}) = x + \frac{1}{\sqrt{k}}\,\mathbf{v}_1^{-I_2}. \tag{86}$$

Therefore, in place of (58), we obtain the following:

$$\begin{aligned}&\Pi_{k\boldsymbol{\nu}}\big(x_{1k}, x_{2k}\big)_- \\ &\sim \frac{k^{3/2}\,\nu}{2\pi} \int_0^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \int_{G/T} \mathrm{d}V_{G/T}(gT) \left[e^{\imath\,k\,\Gamma_k} \cdot \varrho_2\left(k^{-\epsilon_1}\,\vartheta\right)\right. \\ &\quad \left.\cdot \rho(u) \cdot \left(e^{\imath\left(\pi+\vartheta/\sqrt{k}\right)} - e^{-\imath\left(\pi+\vartheta/\sqrt{k}\right)}\right) \cdot s\left(\widetilde{\mu}_{ge^{-\imath\vartheta B/\sqrt{k}}g^{-1}}(x_{1k}^{-I_2}), x_{2k}, k\,u\right)\right],\end{aligned} \tag{87}$$

where

$$\begin{aligned}\Gamma_k(u, \mathbf{v}_1, \mathbf{v}_2, \vartheta, g\,T) &:= u\,\psi\left(\widetilde{\mu}_{ge^{-\imath\vartheta B/\sqrt{k}}g^{-1}} \circ (x_{1k}^{-I_2}), x_{2k}\right) - \frac{\vartheta}{\sqrt{k}} \cdot \nu - \pi \cdot \nu \\ &= \Psi_k(u, \mathbf{v}_1^{-I_2}, \mathbf{v}_2, \vartheta, g\,T) - \pi \cdot \nu.\end{aligned} \tag{88}$$

Let us write $\Psi_k' := \Psi_k(u, \upsilon_1', \upsilon_2, \vartheta, g\,T)$. Thus we may rewrite (58) in the following manner:

$$\begin{aligned}
&\Pi_{k\boldsymbol{\nu}}(x_{1k}, x_{2k})_- \qquad (89)\\
&\sim e^{\imath\,\pi(1-k\cdot\nu)}\cdot\frac{k^{3/2}\,\nu}{2\pi}\int_0^{+\infty}\mathrm{d}u\int_{-\infty}^{+\infty}\mathrm{d}\vartheta\int_{G/T}\mathrm{d}V_{G/T}(gT)\left[e^{\imath\,k\,\Psi'_k}\cdot\varrho_2\left(k^{-\epsilon_1}\,\vartheta\right)\right.\\
&\quad\left.\cdot\varrho_1(u)\cdot\left(e^{\imath\,\vartheta/\sqrt{k}}-e^{-\imath\,\vartheta/\sqrt{k}}\right)\cdot s\left(\widetilde{\mu}_{ge^{-\imath\vartheta B/\sqrt{k}}g^{-1}}(x'_{1k}), x_{2k}, k\,u\right)\right]\\
&\sim e^{\imath\,\pi(1-k\cdot\nu)}\cdot\Pi_{k\boldsymbol{\nu}}\left(x+\frac{1}{\sqrt{k}}\,\mathbf{v}_1^{-I_2}, x+\frac{1}{\sqrt{k}}\,\mathbf{v}_2\right)_+. \qquad (90)
\end{aligned}$$

The statement of Proposition 4.2 follows from (89) and Proposition 4.1. □

The proof of Theorem 1.3 is complete. □

4.3. Theorem 1.2.

Proof of Theorem 1.2. Let $\varrho : G \to [0, +\infty)$ be a smooth bump function supported in a small neighborhood of I_2, and identically equal to 1 on a smaller neighborhood. With g_j as in (8), $j = 1, \ldots, N_x$, let us set $\varrho_j(g) := \varrho\left(g\,g_j^{-1}\right)$. Then

$$\begin{aligned}
\Pi_{k\,\boldsymbol{\nu}}(x,x) &= k\,\nu\cdot\int_G \mathrm{d}V_G(g)\,\left[\overline{\chi_{k\,\boldsymbol{\nu}}(g)}\,\Pi\left(\widetilde{\mu}_{g^{-1}}(x), x\right)\right]\\
&\sim\sum_{j=1}^{N_x} k\,\nu\cdot\int_G \mathrm{d}V_G(g)\,\left[\varrho_j(g)\cdot\overline{\chi_{k\,\boldsymbol{\nu}}(g)}\,\Pi\left(\widetilde{\mu}_{g^{-1}}(x), x\right)\right]. \qquad (91)
\end{aligned}$$

Let us write $\Pi_{k\,\boldsymbol{\nu}}(x,x)_j$ for the j-th summand in (91).

With Z_x as in Definition 1.2, we can rewrite (91) as follows:

$$\Pi_{k\,\boldsymbol{\nu}}(x,x)\sim\Pi_{k\,\boldsymbol{\nu}}(x,x)_{Z_x}+\Pi_{k\,\boldsymbol{\nu}}(x,x)_{G_x\setminus Z_x}, \qquad (92)$$

where

$$\Pi_{k\,\boldsymbol{\nu}}(x,x)_{Z_x} := \sum_{g_j\in Z_x}\Pi_{k\,\boldsymbol{\nu}}(x,x)_j,\quad \Pi_{k\,\boldsymbol{\nu}}(x,x)_{G_x\setminus Z_x} := \sum_{g_j\notin Z_x}\Pi_{k\,\boldsymbol{\nu}}(x,x)_j. \qquad (93)$$

The asymptotic expansion (13) for the first summand in (92) follows from Theorem 1.3, with $\mathbf{v}_1 = \mathbf{v}_2 = 0$. Hence, Theorem 1.2 will be proved by establishing the following.

PROPOSITION 4.3. *Assume, as in (9), that* $G_x\setminus Z_x = \left\{g_j, g_{j+a_x} := g_j^{-1}\right\}_{j=1}^{a_x}$, *and let* $B(x;j)$ *be as in Definition 1.4. Then as* $k \to +\infty$ *there is an asymptotic expansion*

$$\begin{aligned}
\Pi_{k\,\boldsymbol{\nu}}(x,x)_{G_x\setminus Z_x} &\sim 4\,\pi\left(\frac{\nu\,k}{2\,\pi\cdot\lambda(m_x)}\right)^d\\
&\quad\cdot\left[\sum_{j=1}^{a_x}\Re\left(\frac{\imath\,\sin(\vartheta_j)\cdot e^{-\imath k\nu\cdot\vartheta_j}}{\sqrt{\det\left(B(x;j)\right)}}\right)+\sum_{l\ge 1}k^{-l/2}\,P_{jl}(\nu;m_x)\right],
\end{aligned}$$

where $P_{jl}(\nu;\cdot)$ *is* $\mathcal{C}^\infty$ *on the loci (in* M*) defined by the cardinality of* G_x, $x \in p^{-1}(m)$.

Proof of Proposition 4.3. Let $\rho : G/T \times T \to G$, $(g\,T, t) \mapsto g\,t\,g^{-1}$; each $g_j \in G_x \setminus Z_x$, being a regular element of G, is a regular value of ρ. If t_j is as in (8), then

$$\rho^{-1}(g_j) = \left\{(h_{m_x}\,T, t_j),\,(k_{m_x}\,T, t_j^{-1})\right\}, \quad k_{m_x} := h_{m_x}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{94}$$

Let $E : \xi \in \mathfrak{g} \mapsto e^{\xi} \in G$ be the exponential map. We shall write the general $t \in T$ in exponential form as

$$t = \begin{pmatrix} e^{\imath\vartheta} & 0\\ 0 & e^{-\imath\vartheta}\end{pmatrix} = E(\vartheta\,\beta),$$

where β is as in Remark 1.1.

By the Weyl integration and character formulae, we have

$$\begin{aligned}\Pi_{k\,\boldsymbol{\nu}}(x,x)_j = {} & \frac{k\,\nu}{2\pi}\cdot\int_{G/T}\mathrm{d}V_{G/T}(g\,T)\int_{-\pi}^{\pi}\mathrm{d}\vartheta\\ & \left[\varrho_j\left(g\,e^{\vartheta\beta}\,g^{-1}\right)\cdot e^{-\imath k\nu\cdot\vartheta}\,\Pi\left(\widetilde{\mu}_{ge^{-\vartheta\beta}g^{-1}}(x), x\right)\,\left(e^{\imath\,\vartheta} - e^{-\imath\,\vartheta}\right)\right].\end{aligned} \tag{95}$$

Since $\varrho_j\circ\rho : \left(g\,T, e^{\imath\,\vartheta}\right) \mapsto \varrho_j\left(g\,e^{\vartheta\beta}\,g^{-1}\right)$ is supported in a small open neighborhood of the pair (94), we can split (95) as

$$\Pi_{k\,\boldsymbol{\nu}}(x,x)_j = \Pi_{k\,\boldsymbol{\nu}}(x,x)_{j1} + \Pi_{k\,\boldsymbol{\nu}}(x,x)_{j2}, \tag{96}$$

where in $\Pi_{k\,\boldsymbol{\nu}}(x,x)_{j1}$ (respectively, $\Pi_{k\,\boldsymbol{\nu}}(x,x)_{j2}$) integration is over a small neighborhood of $(h_{m_x}\,T, t_j)$ (respectively, $(k_{m_x}\,T, t_j^{-1})$).

Let us first consider each $\Pi_{k\,\boldsymbol{\nu}}(x,x)_{j1}$. To this end, we shall introduce local coordinates on G/T and on T.

First, for some suitably small $\delta > 0$ and $z \in D(0,\delta) \subset \mathbb{C}$, we set

$$h(z) := h_{m_x}\,E\big(A(z)\big), \tag{97}$$

where $A(z)$ is as in Definition 1.4; then

$$z \in D(0,\delta) \mapsto h(z)\,T \in G/T \tag{98}$$

is a system of local coordinates on G/T centered at $h_{m_x}\,T$. The Haar measure on G/T, expressed in the z coordinates, is $\mathcal{V}_{G/T}(z)\,\mathrm{d}V_{\mathbb{C}}(z)$, for an appropriate smooth function on $\mathcal{V}_{G/T}$. The proof of the following Lemma will be omitted.

Lemma 4.6. *Let us set $D_{G/T} = \mathcal{V}_{G/T}(0)$. Then $D_{G/T} = 2\pi/V_3$, where V_3 is the surface area of S^3.*

Next, as a system of local coordinates on T centered at t_j we shall adopt $\theta \in (-\delta,\delta) \mapsto t_j\,E(\theta\,\beta) \in T$. Furthermore, since $\left(\widetilde{\mu}_{ge^{-\vartheta\beta}g^{-1}}(x), x\right)$ is in a small neighborhood of the diagonal in $X \times X$, we may replace Π by its representation as an FIO. After performing the rescaling $u \mapsto ku$, and recalling Proposition 3.2, we obtain

$$\begin{aligned}&\Pi_{k\,\boldsymbol{\nu}}(x,x)_{j1}\\ &\sim \frac{k^2\,\nu}{2\pi}\cdot e^{-\imath k\nu\cdot\vartheta_j}\cdot\int_{D(0,\delta)}\mathrm{d}V_{\mathbb{C}}(z)\int_{-\delta}^{\delta}\mathrm{d}\theta\int_0^{+\infty}\mathrm{d}u\\ &\left[e^{\imath\,k\left[u\,\psi\left(\widetilde{\mu}_{h(z)E(-\theta\beta)t_j^{-1}h(z)^{-1}}(x),x\right)-\nu\cdot\theta\right]}\cdot\mathcal{V}_{G/T}(z)\right.\\ &\left.\cdot\rho(u)\cdot s_{j1}\left(\widetilde{\mu}_{h(z)E(-\theta\beta)t_j^{-1}h(z)^{-1}}(x), x, k\,u\right)\cdot\left(e^{\imath\,(\vartheta_j+\theta)} - e^{-\imath\,(\vartheta_j+\theta)}\right)\right].\end{aligned} \tag{99}$$

Here, $dV_{\mathbb{C}}(z)$ is the Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$, and s_{j1} denotes the usual amplitude of the representation of Π as an FIO, with the above cut-offs incorporated.

In order to proceed, we need to express the phase more explicitly. We have

$$\begin{aligned} & h(z)E(-\theta\beta)t_j^{-1}h(z)^{-1} \\ & = C_{h_{m_x}}\Big(E\big(A(z)\big)\,E(-\theta\beta)E\big(-\mathrm{Ad}_{t_j^{-1}}\big(A(z)\big)\big)\Big)\,g_j^{-1} \\ & = E\Big(-\mathrm{Ad}_{h_{m_x}}\big(\gamma_j(z,\theta)\big)\Big)\,g_j^{-1}, \end{aligned} \tag{100}$$

where (by use of the Baker-Campbell-Hausdorff formula)

$$\gamma_j(z,\theta) = \gamma_{j1}(z,\theta) + \gamma_{j2}(z,\theta) + R_3(z,\theta),$$

with

$$\begin{aligned} \gamma_{j1}(z,\theta) &:= \theta\,\beta + \big(\mathrm{Ad}_{t_j^{-1}} - \mathrm{id}_{\mathfrak{g}}\big)\big(A(z)\big), \\ \gamma_{j2}(z,\theta) &:= -\frac{1}{2}\left[\theta\,\beta, A(z) + \mathrm{Ad}_{t_j^{-1}}\big(A(z)\big)\right] \\ & \quad + \frac{1}{2}\left[A(z), \mathrm{Ad}_{t_j^{-1}}\big(A(z)\big)\right], \end{aligned} \tag{101}$$

while R_j denotes a generic $\mathcal{C}^\infty$ function vanishing to j-th order at the origin. Note that γ_j is homogeneous of degree j in $\big(\Re(z), \Im(z), \theta\big)$,

By Corollary 2.2 of [P4], we obtain in HLC's

$$\begin{aligned} \widetilde{\mu}_{h(z)E(-\theta\beta)t_j^{-1}h(z)^{-1}}(x) &= \widetilde{\mu}_{E\left(-\mathrm{Ad}_{h_{m_x}}\left(\gamma_j(z,\theta)\right)\right)}(x) \\ &= x + \Big(\Theta(z,\theta), V(\theta,z)\Big), \end{aligned} \tag{102}$$

where

$$\begin{aligned} \Theta(z,\theta) &:= \Big\langle \Phi_G(m_x), \mathrm{Ad}_{h_{m_x}}\big(\gamma_j(z,\theta)\big)\Big\rangle + R_3(z,\theta) \\ V(\theta,z) &:= -\mathrm{Ad}_{h_{m_x}}\big(\gamma_j(z,\theta)\big)_M(m) + R_2(z,\theta). \end{aligned} \tag{103}$$

By the discussion in §3 of [SZ], we conclude that

$$\begin{aligned} & u\,\psi\left(\widetilde{\mu}_{h(z)E(-\vartheta\beta)t_j^{-1}h(z)^{-1}}(x), x\right) - \nu\cdot\theta \\ & = u\,\psi\left(x + \Big(\Theta(z,\theta), V(\theta,z)\Big), x\right) - \nu\cdot\theta \\ & = \imath\,u\cdot\left[1 - e^{\imath\,\Theta(z,\theta)}\right] + \frac{\imath\,u}{2}\cdot\big\|V(\theta,z)\big\|^2 + u\,R_3(z,\theta) \\ & = u\,\Theta(z,\theta) + \frac{\imath\,u}{2}\cdot\left[\Theta(z,\theta)^2 + \big\|V(\theta,z)\big\|^2\right] + u\,R_3(z,\theta). \end{aligned} \tag{104}$$

Let us choose $C>0$, $\epsilon\in(0,1/6)$. Since p is a local diffeomorphism at $(h_{m_x}T, t_j)$ and $\widetilde{\mu}$ is locally free at x, the contribution to the asymptotics of (99) of the locus where $\|(z,\theta)\| \ge C\,k^{\epsilon-1/2}$ is $O\left(k^{-\infty}\right)$. Adopting the rescaling $z \mapsto z/\sqrt{k}$, $\theta \mapsto \theta/\sqrt{k}$ we can rewrite (99) in the following form:

$$\Pi_{k\,\boldsymbol{\nu}}(x,x)_{j1} \sim \frac{k^{1/2}\,\nu}{2\pi}\cdot e^{-\imath k\nu\cdot\vartheta_j}\cdot\int_{\mathbb{C}} dV_{\mathbb{C}}(z)\,\big[I_k(x;z)\big], \tag{105}$$

where

$$I_{jk}(x;z) := \int_{-\infty}^{+\infty} \mathrm{d}\theta \int_0^{+\infty} \mathrm{d}u \left[e^{\imath\, k\, \Psi_k(x;u,\theta,z)} \cdot A_{jk}(x;u,\theta,z)\right]; \tag{106}$$

in (106) we have set

$$\begin{aligned}\imath\, k\, \Psi_k(x;u,\theta,z) &:= \imath \sqrt{k} \left[u \cdot \left\langle \Phi_G(m_x), \mathrm{Ad}_{h_{m_x}}\left(\gamma_{j1}(z,\theta)\right)\right\rangle - \theta \cdot \nu\right] \\ &+ \imath\, u \cdot \left\langle \Phi_G(m_x), \mathrm{Ad}_{h_{m_x}}\left(\gamma_{j2}(z,\theta)\right)\right\rangle - \frac{u}{2} \cdot \left\|\mathrm{Ad}_{h_{m_x}}\left(\gamma_{j1}(z,\theta)\right)_X(x)\right\|^2 \\ &+ k\, R_3\left(\frac{z}{\sqrt{k}}, \frac{\theta}{\sqrt{k}}\right),\end{aligned} \tag{107}$$

$$\begin{aligned}A_{jk}(x;u,\theta,z) := s_{j1}&\left(\widetilde{\mu}_{h(z/\sqrt{k})E(-\vartheta\beta/\sqrt{k})t_j^{-1}h(z/\sqrt{k})^{-1}}(x), x, k\, u\right) \cdot \mathcal{V}_{G/T}\left(\frac{z}{\sqrt{k}}\right) \\ &\cdot \varrho\left(k^{-\epsilon}\,(z,\theta)\right) \cdot \left(e^{\imath\,(\vartheta_j + \theta/\sqrt{k})} - e^{-\imath\,(\vartheta_j + \theta/\sqrt{k})}\right),\end{aligned} \tag{108}$$

with ϱ an appropriate bump function. Integration in (z,θ) in (105) is over a ball of radius $O\left(k^{\epsilon}\right)$ centered at the origin.

Let us derive an asymptotic expansion for (106).

Proposition 4.4. *As $k \to +\infty$, we have*

$$\begin{aligned}I_k(x;z) \sim \left(\frac{k\, u_0}{\pi}\right)^d \cdot \frac{\pi}{\sqrt{k}} \cdot \frac{2\,\imath\, \sin(\vartheta_j)}{\lambda(m_x)} \cdot e^{-\frac{u_0}{2}\, Z^t\, A(x;j)\, Z} \\ \cdot \left[1 + \sum_{l \geq 1} k^{-l/2}\, R_{jl}(m_x; Z)\right],\end{aligned} \tag{109}$$

where $R_{jl}(m_x;Z)$ is polynomial in Z of degree $\leq 3l$ and parity $(-1)^l$.

Proof of Proposition 4.4. The second summand in $\gamma_1(z,\theta)$ in (101) is anti-diagonal. Therefore,

$$\begin{aligned}&\left\langle \Phi_G(m_x), \mathrm{Ad}_{h_{m_x}}\left(\gamma_{j1}(z,\theta)\right)\right\rangle \\ &= \left\langle \mathrm{Ad}_{h_{m_x}^{-1}}\left(\Phi_G(m_x)\right), \gamma_{j1}(z,\theta)\right\rangle = \left\langle \lambda(m_x)\,\beta, \theta\,\beta\right\rangle = 2\,\lambda(m_x)\,\theta.\end{aligned} \tag{110}$$

Thus we have

$$I_{jk}(x;z) = \int_{-\infty}^{+\infty} \mathrm{d}\theta \int_0^{+\infty} \mathrm{d}u \left[e^{\imath \sqrt{k}\, \Psi_x(u,\theta)} \cdot \mathcal{A}_{jk}(x;u,\theta,z)\right], \tag{111}$$

where now

$$\begin{aligned}\Psi_x(u,\theta) &:= \theta \cdot \left[2\,\lambda(m_x) \cdot u - \nu\right] \\ \mathcal{A}_{jk}(x;u,\theta,z) &:= e^{\mathcal{E}(u,\theta,z)} \cdot e^{k\, R_3\left(\frac{z}{\sqrt{k}}, \frac{\theta}{\sqrt{k}}\right)} \cdot A_{jk}(x;u,\theta,z),\end{aligned} \tag{112}$$

with

$$\begin{aligned}\mathcal{E}(u,\theta,z) := \imath\, u \cdot &\left\langle \Phi_G(m_x), \mathrm{Ad}_{h_{m_x}}\left(\gamma_2(z,\theta)\right)\right\rangle \\ &- \frac{u}{2} \cdot \left\|\mathrm{Ad}_{h_{m_x}}\left(\gamma_1(z,\theta)\right)_X(x)\right\|^2.\end{aligned} \tag{113}$$

It follows from (101) and (113) that $\mathcal{E}(u,\theta,z)$ is homogeneous of degree 2 in $\big(\Re(z),\Im(z),\theta\big)$; furthermore, since $\widetilde{\mu}$ is locally free at x, on the support of the integrand we have

$$\Re\big(\mathcal{E}_k(z,\theta)\big)\le -D'\cdot\big(|z|^2+|\theta|^2\big) \tag{114}$$

for some positive constant $D>0$.

Noting that $\sin(\vartheta_j)\neq 0$ as $g_j\notin Z_x$, we have

$$e^{\imath\,(\vartheta_j+\theta/\sqrt{k})}-e^{-\imath\,(\vartheta_j+\theta/\sqrt{k})}=2\,\imath\,\sin(\vartheta_j)\cdot\left[1+\sum_{l\ge 1}k^{-l/2}\,a_{jl}\cdot\theta^l\right] \tag{115}$$

for certain $a_{jl}\in\mathbb{R}$. Similarly,

$$\mathcal{V}_{G/T}(z)\cdot e^{k\,R_3\left(\frac{z}{\sqrt{k}},\frac{\theta}{\sqrt{k}}\right)}=D_{G/T}+\sum_{r\ge 1}k^{-r/2}\cdot P_r(z,\theta) \tag{116}$$

where $P_r(z,\theta)$ is a polynomial in $(\Re(z),\Im(z),\theta)$ of degree $\le 3\,r$ and parity $(-1)^r$ (possibly also depending on j). Pairing (115) and (116) we conclude that

$$\begin{aligned}\mathcal{A}_{jk}(x;u,\theta,z)\sim 2\,\imath\,\sin(\vartheta_j)\cdot\left(\frac{k\,u}{\pi}\right)^d\cdot e^{\mathcal{E}(u,\theta,z)}&\\ \cdot\left[1+\sum_{r\ge 1}k^{-r/2}\,P_{jr}(z,\theta)\right],&\end{aligned} \tag{117}$$

where $P_{jr}(z,\theta)$ is again a polynomial in $(\Re(z),\Im(z),\theta)$ of degree $\le 3\,r$ and parity $(-1)^r$.

The following is straighforward.

Lemma 4.7. *Ψ_x has a unique critical point, which is non-degenerate and given by $(u_0,\theta_0)=\big(\nu/\big(2\,\lambda(m_x)\big),0\big)$; we have $\Psi_x(u_0,\theta_0)=0$. The Hessian matrix has determinant $-4\,\lambda(m_x)^2$ and vanishing signature.*

We can apply the Stationary Phase Lemma to determine the asymptotic expansion of (111). In view of (101), by a few computations we get

$$\begin{aligned}\gamma_{j1}(z,0)&=\imath\begin{pmatrix}0 & \big(e^{-2\imath\,\vartheta_j}-1\big)\cdot z\\ \big(e^{2\imath\,\vartheta_j}-1\big)\cdot\overline{z} & 0\end{pmatrix},\\ \gamma_{j2}(z,0)&=-|z|^2\cdot\sin(2\,\vartheta_j)\,\beta.\end{aligned} \tag{118}$$

If $z=a+\imath\,b$ with $a,b\in\mathbb{R}$, let $Z=\begin{pmatrix}a & b\end{pmatrix}^t\in\mathbb{R}^2$ be the corresponding vector; thus $|z|=\|Z\|$. Then

$$\left\|\mathrm{Ad}_{h_{m_x}}\big(\gamma_{j1}(z,0)\big)_X(x)\right\|^2=\frac{1}{2}\cdot Z^t\,C(x;j)\,Z$$

where $C(x;j)$ is as in Definition 1.4. From (113) and (118) we conclude

$$\mathcal{E}(u_0,0,z)=-\frac{u_0}{2}\,Z^t\,B(x;j)\,Z,$$

where $B(x;j)$ is also as in Definition 1.4.

Finally, by (117) the general term of the asymptotic expansion is the evaluation at the critical point of an expression of the form

$$C_{r,l} \cdot k^{d-s/2} \left(\frac{\partial^2}{\partial\theta\partial u} \right)^s \left(k^{-r/2} \, P_{jr}(z,\theta) \, e^{\mathcal{E}(u,\theta,z)} \right), \tag{119}$$

for some constant $C_{r,l} \in \mathbb{C}$. Writing $\mathcal{E}(u,\theta,z) = u \cdot (H_{2,0} + H_{1,1} + H_{0,2})$, where $H_{a,b}$ is bihomogeneous of bidegree (a,b) in (θ, Z), as in Claims 4.1 and 4.2 we conclude by an inductive argument that (119) has the form $k^{d-(s+r)/2} \, Q_{r,s}(\theta, Z) \, e^{\mathcal{E}(u,\theta,z)}$, where $Q_{r,s}$ is a polynomial of degree $\leq 3\,(r+s)$, and parity $(-1)^{r+s}$. The proof of Lemma 4.7 is complete. □

Let us insert (109) in (105), and remark that by parity odd polynomials do not contribute to the integral over $\mathbb{C}$. Hence after integration the half-integer powers of k drop out and we obtain

$$\begin{aligned} \Pi_{k\,\boldsymbol{\nu}}(x,x)_{j1} &\sim \nu \cdot e^{-\imath k\nu\cdot\vartheta_j} \cdot \left(\frac{k\,u_0}{\pi} \right)^d \cdot \frac{\imath\, \sin(\vartheta_j)}{\lambda(m_x)} \cdot \frac{2\pi}{u_0 \cdot \sqrt{\det\big(B(x;j)\big)}} \\ &\quad \cdot \left[D_{G/T} + \sum_{l\geq 1} k^{-l}\, P_{jl}(m_x) \right] \\ &= 4\pi \cdot \frac{\imath\, \sin(\vartheta_j) \cdot e^{-\imath k\nu\cdot\vartheta_j}}{\sqrt{\det\big(B(x;j)\big)}} \cdot \left(\frac{\nu\,k}{2\,\pi \cdot \lambda(m_x)} \right)^d \\ &\quad \cdot \left[D_{G/T} + \sum_{l\geq 1} k^{-l}\, P_{jl}(m_x) \right]. \end{aligned} \tag{120}$$

Let us remark that $g_j \neq g_j^{-1}$ since $g_j \neq \pm I_2$; summing the contributions (120) corresponding to g_j and $g_{j+a_x} = g_j^{-1}$, we obtain

$$\begin{aligned} \Pi_{k\,\boldsymbol{\nu}}(x,x)_{j1} + \Pi_{k\,\boldsymbol{\nu}}(x,x)_{j'1} &= 8\pi \cdot \Re \left(\frac{\imath\, \sin(\vartheta_j) \cdot e^{-\imath k\nu\cdot\vartheta_j}}{\sqrt{\det\big(B(x;j)\big)}} \right) \cdot \left(\frac{\nu\,k}{2\,\pi \cdot \lambda(m_x)} \right)^d \\ &\quad \cdot \left[D_{G/T} + \sum_{l\geq 1} k^{-l/2}\, R_{jl}(m_x) \right]. \end{aligned} \tag{121}$$

To deal with $\Pi_{k\,\boldsymbol{\nu}}(x,x)_{j2}$, in view of (94) we need only go over the previous computations replacing h_{m_x} with k_{m_x}, and t_j with t_j^{-1}. In the analogue of (111), in place of the phase Ψ_x in (112), we obtain

$$\Psi'_x(u,\theta) = -\theta \cdot \big[2\,\lambda(m_x) \cdot u + \nu \big],$$

so that $\partial_\vartheta \Psi'_x(u,\theta) = -\big[2\,\lambda(m_x) \cdot u + \nu \big] \leq -\nu$ on the domain of integration. Thus $\Pi_{k\,\boldsymbol{\nu}}(x,x)_{j2} = O\left(k^{-\infty}\right)$.

The proof of Proposition 4.3 is complete. □

□

Proof of Corollary 1.1. The function $x \mapsto \Pi_{k\,\boldsymbol{\nu}}(x,x)$ is S^1-invariant, hence it may be interpreted as a $\mathcal{C}^\infty$ function on M (pulled back to X). On the other hand, Theorems 1.2 and 1.3 imply that $x \mapsto (\pi/k)^d \cdot \Pi_{k\boldsymbol{\nu}}(x,x)$ is bounded, and that, with the previous interpretation, it converges almost everywhere to the integrand on the right hand side of (15). The claim follows by the dominated convergence Theorem, in view of the choice of volume form on X. □

REFERENCES

[A] D. Applebaum, *Probability on compact Lie groups*, With a foreword by Herbert Heyer. Probability Theory and Stochastic Modelling, 70. Springer, Cham, 2014.

[BSZ] P. Bleher, B. Shiffman and S. Zelditch, *Universality and scaling of correlations between zeros on complex manifolds*, Invent. Math., 142 (2000), pp. 351–395.

[BG] L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators*, Annals of Mathematics Studies, 99 (1981), Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo.

[BS] L. Boutet de Monvel and J. Sjöstrand, *Sur la singularité des noyaux de Bergman et de Szegö*, Astérisque, 34-35 (1976), pp. 123–164.

[Cm] S. Camosso, *Scaling asymptotics of Szegö kernels under commuting Hamiltonian actions*, Ann. Mat. Pura Appl. (4), 195:6 (2016), pp. 2027–2059.

[Ch] L. Charles, *Quantization of compact symplectic manifolds*, J. Geom. Anal., 26:4 (2016), pp. 2664–2710.

[GP] A. Galasso and R. Paoletti, *Equivariant Asymptotics of Szegö kernels under Hamiltonian $U(2)$ actions*, Ann. Mat. Pura Appl. (4), 198:2 (2019), pp. 639–683.

[GS1] V. Guillemin and S. Sternberg, *Homogeneous quantization and multiplicities of group representations*, J. Funct. Anal., 47:3 (1982), pp. 344–380.

[GS2] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math., 67:3 (1982), pp. 491–513.

[GS3] V. Guillemin and S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math., 67:3 (1982), pp. 515–538.

[GS4] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, Cambridge, 1984.

[Ko] B. Kostant, *Quantization and unitary representations. I. Prequantization*, Lectures in modern analysis and applications, III, pp. 87–208. Lecture Notes in Math., Vol. 170, Springer, Berlin, 1970.

[MM] X. Ma and G. Marinescu, *Berezin-Toeplitz quantization and its kernel expansion. Geometry and quantization*, pp. 125–166, Trav. Math., 19, Univ. Luxemb., Luxembourg, (2011).

[MZ] X. Ma and W. Zhang, *Bergman kernels and symplectic reductions*, Astérisque, 318, SMF (2008).

[P1] R. Paoletti, *Moment maps and equivariant Szegö kernels* J. Symplectic Geom., 2:1 (2003), pp. 133–175.

[P2] R. Paoletti, *The Szegö kernel of a symplectic quotient*, Adv. Math., 197:2 (2005), pp. 523–553.

[P3] R. Paoletti, *Scaling limits for equivariant Szegö kernels*, J. Symplectic Geom., 6:1 (2008), pp. 9-32, and *Corrigendum*, J. Symplectic Geom., 11:2 (2013), pp. 317-318.

[P4] R. Paoletti, *Asymptotics of Szegö kernels under Hamiltonian torus actions*, Israel Journal of Mathematics, 191:1 (2012), pp. 363–403.

[P5] R. Paoletti, *Lower-order asymptotics for Szegö and Toeplitz kernels under Hamiltonian circle actions*, Recent advances in algebraic geometry, pp. 321–369, London Math. Soc. Lecture Note Ser., 417, Cambridge Univ. Press, Cambridge, 2015.

[RT] M. Ruzhansky and V. Turunen, *Pseudo-differential operators and symmetries. Background analysis and advanced topics*. Pseudo-Differential Operators. Theory and Applications, 2. Birkhäuser Verlag, Basel, 2010.

[Sch] M. Schlichenmaier, *Berezin-Toeplitz quantization for compact Kähler manifolds. An introduction*, Geometry and quantization, pp. 97-124, Trav. Math., 19, Univ. Luxemb., Luxembourg, 2011.

[SZ] B. Shiffman and S. Zelditch, *Asymptotics of almost holomorphic sections of ample line bundles on symplectic manifolds*, J. Reine Angew. Math., 544 (2002), pp. 181–222.

[V] V. S. Varadarajan, *An introduction to harmonic analysis on semisimple Lie groups*, Corrected reprint of the 1989 original. Cambridge Studies in Advanced Mathematics, 16. Cambridge University Press, Cambridge, 1999. x+316 pp. ISBN: 0-521-34156-6.

[Z] S. Zelditch, *Szegö kernels and a theorem of Tian*, Int. Math. Res. Not., 6 (1998), pp. 317–331.