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ON THE 2-ADIC LOGARITHM OF UNITS OF CERTAIN TOTALLY IMAGINARY QUARTIC FIELDS*

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Abstract. In this paper, we prove a result on the 2-adic logarithm of the fundamental unit of the field $\mathbb{Q}(\sqrt[4]{-q})$, where $q \equiv 3 \bmod 4$ is a prime. When $q \equiv 15 \bmod 16$, this result confirms a speculation of Coates-Li and has consequences for certain Iwasawa modules arising in their work.

Key words. 2-adic logarithm, units, class groups, pure quartic fields.

Mathematics Subject Classification. 11R29, 11R27, 11R29.

1. Introduction. Let q be any prime $\equiv 3 \bmod 4$, and define

$$K = \mathbb{Q}(\sqrt{-q}), \;\; F = K(\sqrt[4]{-q}).$$

Then there is a unique prime $\mathfrak{P}$ of F lying above 2 which is ramified in the extension $F/\mathbb{Q}$ (see Lemma 3 below), and we write $\mathrm{ord}_{\mathfrak{P}}$ for the usual order valuation at $\mathfrak{P}$. Moreover, K has odd class number, and it is not difficult to show that F also has odd class number (see Lemma 4 below). The unit group of F has rank 1, and we write η for a fundamental unit of F. We have $\eta \equiv 1 \bmod \mathfrak{P}$ when $q > 3$, so that the usual logarithmic series $\log_{\mathfrak{P}}(\eta)$ will converge in the completion $F_{\mathfrak{P}}$ of F at $\mathfrak{P}$ (see Lemma 4 below, where we also point out how to deal with the slightly exceptional case of $q = 3$). We shall use elementary arguments to prove the following result.

THEOREM 1. *Let q be any prime $\equiv 3 \bmod 4$. Let η be a fundamental unit of F, and let $\mathfrak{P}$ be the unique ramified prime of F above 2. Then (1) If $q \equiv 3 \bmod 8$, we have $\mathrm{ord}_{\mathfrak{P}}(\log_{\mathfrak{P}}(\eta)) = 0$; (2) If $q \equiv 7 \bmod 16$, we have $\mathrm{ord}_{\mathfrak{P}}(\log_{\mathfrak{P}}(\eta)) = 2$; and (3) If $q \equiv 15 \bmod 16$, we have $\mathrm{ord}_{\mathfrak{P}}(\log_{\mathfrak{P}}(\eta)) \geq 4$.*

We first remark that assertions (1) and (2) can be viewed as an exact $\mathfrak{P}$-adic form of the Brauer-Siegel theorem as q varies. Secondly, our motivation for proving the above theorem came from a recent paper of J. Coates and Y. Li [1], which uses 2-adic arguments from Iwasawa theory to prove various non-vanishing theorems for the values at $s = 1$ of the complex L-series of certain elliptic curves with complex multiplication. In fact, the results in [1] are concerned with the field $F^* = \mathbb{Q}(\sqrt{-\sqrt{-q}})$, but we note that the fields F and F^* are isomorphic extensions of $\mathbb{Q}$, and so Theorem 1 remains valid with F^* replacing F. Assume first that $q \equiv 7 \bmod 8$, so that 2 splits in K, and let $\mathfrak{p}$ be the unique prime of K lying below $\mathfrak{P}$. By class field theory, there is a unique extension K_∞/K with Galois group $\mathrm{Gal}(K_\infty/K) \xrightarrow{\sim} \mathbb{Z}_2$, which is unramified outside the prime $\mathfrak{p}$. Define $F^*_\infty = F^* K_\infty$, and let $\Gamma = \mathrm{Gal}(F^*_\infty/F)$. Let $M(F^*_\infty)$ (resp. $M(F^*)$) denote the maximal abelian 2-extension of F^*_∞ (resp. F^*) which is unramified outside the primes of F^*_∞ (resp. F^*) lying above $\mathfrak{p}$. Let $X(F^*_\infty) = \mathrm{Gal}(M(F^*_\infty)/F^*_\infty)$. Now $M(F^*_\infty)$ is clearly a Galois extension of F^*, and hence, as always in Iwasawa theory [4], Γ will act on $X(F^*_\infty)$ by lifting inner automorphisms. Writing $X(F^*_\infty)_\Gamma$ for the Γ-coinvariants of $X(F^*_\infty)$, we see immediately that $X(F^*_\infty)_\Gamma = \mathrm{Gal}(M(F^*)/F^*_\infty)$. Moreover we have $X(F^*_\infty) = 0$ if and only if $X(F^*_\infty)_\Gamma = 0$. By global class field theory,

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the Galois group $\mathrm{Gal}(M(F^*)/F^*_\infty)$ is a finite group, and a classical theorem of Coates and Wiles (see [1, Theorem 8.2]) shows that

$$[M(F^*) : F^*_\infty] = 2^{(\mathrm{ord}_{\mathfrak{P}}(\log_{\mathfrak{P}}(\eta))-2)/2}, \tag{1.1}$$

where η now denotes a fundamental unit of the field F^*. Now when $q \equiv 7 \bmod 16$, Coates and Li show in [1] by a simple Iwasawa theoretic argument based on Nakayama's lemma that $X(F^*_\infty) = 0$, whence it follows from (1.1) that $\mathrm{ord}_{\mathfrak{P}}(\log_{\mathfrak{P}}(\eta)) = 2$. Based on numerical computations carried out by Zhibin Liang, they also conjecture in [1] that $\mathrm{ord}_{\mathfrak{P}}(\log_{\mathfrak{P}}(\eta)) \geq 4$ when $q \equiv 15 \bmod 16$, but say that they cannot prove this conjecture by the arguments of Iwasawa theory. Thus our theorem above confirms their conjecture, as well as giving a new and simple proof of their result when $q \equiv 7 \bmod 16$. In fact, when combined with the arguments from Iwasawa theory given in [1], our result shows that $X(F^*_\infty)$ is a free finitely generated $\mathbb{Z}_2$-module of strictly positive rank when $q \equiv 15 \bmod 16$. Let B be the abelian variety defined over K, which is the restriction of scalars from the Hilbert class field of K to K of the elliptic curve A, with complex multiplication by the ring of integers of K, which was first defined by Gross (an equation for this elliptic curve is recalled in [1], p. 1). Then in fact, when $q \equiv 15 \bmod 16$, our result shows that either $B(F^*_\infty)$ contains a point of infinite order, or the Tate-Shafarevich group of B/F^*_∞ contains a copy of $\mathbb{Q}_2/\mathbb{Z}_2$. When $q \equiv 3 \bmod 8$, none of the above Iwasawa theoretic arguments remain literally valid, because 2 now remains prime in K. Nevertheless, we cannot help speculating whether assertion (1) of Theorem 1 for F^* could somehow be used to attack the non-vanishing Conjecture 1.8 of [1]. However, our theorem has the following consequence for primes $q \equiv 3 \bmod 8$.

Corollary 2. *Suppose $q \equiv 3 \bmod 8$. Let F_∞ be the compositum of all $\mathbb{Z}_2$-extensions of F. Let $M(F)$ denote the maximal abelian 2-extension of F which is unramified outside $\mathfrak{P}$. Then $M(F) = F_\infty$ and $\mathrm{Gal}(M(F)/F) \cong \mathbb{Z}_2^3$.*

We end this Introduction with two unrelated remarks. Firstly, the arguments used to prove Theorem 1 break down completely for primes $q \equiv 1 \bmod 4$, because then both K and F have even class numbers. Secondly, the elementary arguments given in the next section hinge on the following simple observations. Firstly, we use repeatedly the identity

$$\eta^2 \pm 1 = \eta(\eta \pm \eta^{-1}).$$

Secondly, since the prime $\mathfrak{P}$ has ramification index 2, we have $\mathrm{ord}_{\mathfrak{P}}(\log_{\mathfrak{P}}(w)) = \mathrm{ord}_{\mathfrak{P}}(w-1)$ for any element of w of F with $\mathrm{ord}_{\mathfrak{P}}(w-1) > 2$.

Note added in proof. By considerably more elaborate arguments, the author can now prove that $\mathrm{ord}_{\mathfrak{P}}(\log_{\mathfrak{P}}(\eta)) \geq 6$ when $q \equiv 15 \bmod 16$. This stronger assertion was suggested by the original numerical calculations carried out by Zhibin Liang for Coates and Li [1]. Also, in his paper [3] (see Remark 3.2), Gras presents some numerical data in support of our results.

2. Proofs.

In this section, we present our elementary proof for Theorem 1. Next we prove Corollary 2 by using a standard result of class field theory. Finally, we give another very simple proof for Theorem 1(3) by the Coates-Wiles formula (1.1).

Lemma 3. *There exists a unique ramified prime ideal $\mathfrak{P}$ of F above 2 which has ramification index 2 in the extension $F/\mathbb{Q}$.*

Proof. A number field is ramified at a rational prime if and only if its Galois closure is ramified at that prime. It follows that $F/\mathbb{Q}$ is ramified at 2 since its Galois closure $F(\sqrt{-1})$ is clearly ramified at 2. If $q \equiv 3 \bmod 8$, then 2 is inert in K. Hence $\mathfrak{p} = 2\mathcal{O}_K$ must be ramified in F/K, with ramification index 2. Assume next that $q \equiv 7 \bmod 8$. Then 2 splits in K, say $2\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$. The prime ideal $\mathfrak{p}$ induces an embedding from K to $\mathbb{Q}_2$. We fix the choice of $\sqrt{-q}$ such that $\sqrt{-q} \equiv 3 \bmod 8\mathbb{Z}_2$ when $q \equiv 7 \bmod 16$ and that $\sqrt{-q} \equiv 7 \bmod 8\mathbb{Z}_2$ when $q \equiv 15 \bmod 16$. Then $\mathfrak{p}$ is ramified in F. Note that $\bar{\mathfrak{p}}$ is inert in F when $q \equiv 7 \bmod 16$ and that $\bar{\mathfrak{p}}$ splits in F when $q \equiv 15 \bmod 16$. This proves the lemma. □

Lemma 4. (1) *Assume $q > 3$. Then the norm $N(\eta)$ of η from F to K is* 1 *and η is congruent to* 1 *modulo $\mathfrak{P}$.*

(2) *The class number h of F is odd.*

Proof. Note that $N(\eta)$ is a unit of K and hence $N(\eta) = \pm 1$. Since $q \equiv 3 \bmod 4$, the quadratic Hilbert symbol in the local field $\mathbb{Q}_q(\sqrt{-q})$

$$\left(\frac{-1, \sqrt{-q}}{\mathbb{Q}_q(\sqrt{-q})}\right) = \left(\frac{-1, q}{\mathbb{Q}_q}\right) = -1.$$

It follows that $-1 \notin N(F^\times)$. In particular, $N(\eta) = 1$.

If $q \equiv 7 \bmod 8$, then $\mathcal{O}_F/\mathfrak{P} \cong \mathbb{F}_2$ by the above lemma. Hence $\eta \equiv 1 \bmod \mathfrak{P}$ clearly. Suppose next that $q \equiv 3 \bmod 8$. Note that the polynomial $(x+1)^2 - \sqrt{-q}$ is Eisenstein in $K_{\mathfrak{p}}[x]$ where $K_{\mathfrak{p}} = \mathbb{Q}_2(\sqrt{3})$ is the completion of K at $\mathfrak{p} = 2\mathcal{O}_K$. It follows that the ring of integers of F is $\mathcal{O}_K[\sqrt[4]{-q}]$. Write $\eta = a + b\sqrt[4]{-q}$ with $a, b \in \mathcal{O}_K$. By (1), the conjugate of η is η^{-1} and hence $\eta + \eta^{-1} = 2a \equiv 0 \bmod \mathfrak{P}$. Thus $\eta \equiv 1 \bmod \mathfrak{P}$ by the structure of the finite field $\mathcal{O}_F/\mathfrak{P} = \mathbb{F}_4$. This proves (1).

For (2), we first note that K has odd class number by genus theory. The ambiguous class number formula [5, Chapter 13, Lemma 4.1] states that for a cyclic extension F/K of number fields, the order of the $\mathrm{Gal}(F/K)$-invariant subgroup of the ideal class group Cl_F of F is given by:

$$|\mathrm{Cl}_F^{\mathrm{Gal}(F/K)}| = |\mathrm{Cl}_K| \frac{\prod_v e_v}{[F:K][\mathcal{O}_K^\times : \mathcal{O}_K^\times \cap N(F^\times)]}.$$

Here Cl_K is the ideal class group of K, the product runs over all the places of K and e_v is the ramification index of v in F/K. In our case, the ramified places are $\sqrt{-q}\mathcal{O}_K$ and $\mathfrak{p}$. Recall that $\mathfrak{p}$ is the prime of K lying below $\mathfrak{P}$. By (1), we know that $-1 \notin N(F^\times)$. Applying the above formula gives $2 \nmid |\mathrm{Cl}_F^{\mathrm{Gal}(F/K)}|$. Hence $2 \nmid h = |\mathrm{Cl}_F|$ by Nakayama's lemma. □

We remark that for $q = 3$, multiplying η by a third root of unity if needed, we can also assume that $\eta \equiv 1 \bmod \mathfrak{P}$.

Lemma 5. (1) *If $q \equiv 3 \bmod 8$, then $\mathrm{ord}_{\mathfrak{P}}(\eta + \eta^{-1}) = \mathrm{ord}_{\mathfrak{P}}(\eta - \eta^{-1}) = 2$;*

(2) *If $q \equiv 7 \bmod 16$, then $\mathrm{ord}_{\mathfrak{P}}(\eta + \eta^{-1}) = 4$.*

(3) *If $q \equiv 15 \bmod 16$, then $\mathrm{ord}_{\mathfrak{P}}(\eta + \eta^{-1}) \geq 6$.*

Proof of Lemma 5. The ideas of the proofs are the same for all cases. We first consider the case $q \equiv 3 \bmod 8$ which is slightly easier to handle. If $q = 3$, then $\eta = \frac{\sqrt{-3}+1}{2} - \sqrt[4]{-3}$, and it is readily verified that (1) holds. Assume now that $q > 3$. We have $\mathfrak{p} = 2\mathcal{O}_K = \mathfrak{P}^2$. Then $\mathfrak{P} = \gamma\mathcal{O}_F$ for some $\gamma \in \mathcal{O}_F$ since the class number h of F is odd. It follows that $\frac{\gamma^2}{2}$ is a unit of $\mathcal{O}_F$. Thus $\frac{\gamma^2}{2} = \pm\eta^k$ for some integer k. We claim that k is odd. Indeed, if k is even, we would have that $(\gamma\eta^{-k/2})^2 = \pm 2$, whence $F = K(\sqrt{\pm 2})$, which is a contradiction. This proves the claim. By replacing γ by $\gamma\eta^{-\frac{k-1}{2}}$, we may assume that $\frac{\gamma^2}{2}$ is the fundamental unit η. In the proof of part (2) of Lemma 4, we have shown that $\mathcal{O}_F = \mathcal{O}_K[\sqrt[4]{-q}]$. Thus we can write $\gamma = a + b\sqrt[4]{-q}$ with $a, b \in \mathcal{O}_K$, whence

$$\eta = \frac{a^2 + b^2\sqrt{-q}}{2} + ab\sqrt[4]{-q} \quad \text{and} \quad N(\gamma) = a^2 - b^2\sqrt{-q} = \pm 2.$$

In fact, one can show that $N(\gamma) = -2$ by computing the Hilbert symbols of -2 and $\sqrt{-q}$, but we will not need this finer result. We need to calculate $a \bmod 2 \in \mathcal{O}_K/2\mathcal{O}_K \cong \mathbb{F}_4$. It is easy to see that $a \not\equiv 0 \bmod 2\mathcal{O}_K$. We claim that $a \not\equiv 1 \bmod 2\mathcal{O}_K$. Note that $\sqrt{-q} \equiv 1 \bmod 2\mathcal{O}_K$. It follows that $a^2 \equiv b^2 \bmod 2\mathcal{O}_K$. Suppose $a \equiv 1 \bmod 2\mathcal{O}_K$. Then $a^2 \equiv b^2 \equiv 1 \bmod 4\mathcal{O}_K$. This contradicts to the equality $N(\gamma) = \pm 2$ and this proves the claim. Since $a \not\equiv 1 \bmod 2\mathcal{O}_K$, we have $a^2 + 1 \not\equiv 0 \bmod 2\mathcal{O}_K$ by the structure of the finite field $\mathbb{F}_4$. Since $N(\eta) = 1$, the conjugate of η is η^{-1}. We then have $\mathrm{ord}_{\mathfrak{P}}(\eta + \eta^{-1}) = \mathrm{ord}_{\mathfrak{P}}(a^2 + b^2\sqrt{-q}) = \mathrm{ord}_{\mathfrak{P}}(2(a^2+1)) = 2$ and $\mathrm{ord}_{\mathfrak{P}}(\eta - \eta^{-1}) = \mathrm{ord}_{\mathfrak{P}}(2ab\sqrt[4]{-q}) = 2$. This completes the proof for $q \equiv 3 \bmod 8$.

Now we assume $q \equiv 7 \bmod 8$ in the rest of the proof. We have $\mathfrak{P}^h = \gamma\mathcal{O}_F$ for some $\gamma \in \mathcal{O}_F$. Put $\pi = N(\gamma) \in \mathcal{O}_K$. The equalities of ideals $\mathfrak{p}^h\mathcal{O}_F = \mathfrak{P}^{2h} = \pi\mathcal{O}_F = \gamma^2\mathcal{O}_F$ gives a unit $\frac{\gamma^2}{\pi}$ of F. We have $\frac{\gamma^2}{\pi} = \pm\eta^k$ for some odd integer k, for the same reason as in the case $q \equiv 3 \bmod 8$. As $\eta \equiv 1 \bmod \mathfrak{P}$, we have $\mathrm{ord}_{\mathfrak{P}}(\pm\eta^k \pm \eta^{-k}) = \mathrm{ord}_{\mathfrak{P}}(\eta + \eta^{-1})$. We may assume that $\frac{\gamma^2}{\pi}$ is the fundamental unit η. Write $\gamma = a + b\sqrt[4]{-q}$ with $a, b \in K$. Then

$$\eta = \frac{a^2 + \sqrt{-q}b^2}{\pi} + \frac{2ab\sqrt[4]{-q}}{\pi} \quad \text{and} \quad a^2 - \sqrt{-q}b^2 = \pi.$$

From now on, we work in $F_{\mathfrak{P}}$, which is a quadratic extension of $K_{\mathfrak{p}} = \mathbb{Q}_2$. Recall that as in the proof of Lemma 3, the embedding induced by $\mathfrak{p}$ is chosen so that $\sqrt{-q} \equiv 3 \bmod 8$ when $q \equiv 7 \bmod 16$ and that $\sqrt{-q} \equiv 7 \bmod 8$ when $q \equiv 15 \bmod 16$. Note that the ring of integers of $F_{\mathfrak{P}}$ is $\mathbb{Z}_2[\sqrt[4]{-q}]$. Since γ is integral in $F_{\mathfrak{P}}$, we have $a, b \in \mathbb{Z}_2$. Since $\mathrm{ord}_{\mathfrak{p}}(\pi) = h$, we can write $\pi = 2^h u$ with $u \in \mathbb{Z}_2^\times$. Note that one must have $\mathrm{ord}_2(a) = \mathrm{ord}_2(b)$. Otherwise, the valuation of $\pi = N_{F_{\mathfrak{P}}/K_{\mathfrak{p}}}(a + b\sqrt[4]{-q})$ at 2 is even which contradicts to the fact that h is odd. Also note that if $c, d \in \mathbb{Z}_2^\times$, then $N_{F_{\mathfrak{P}}/K_{\mathfrak{p}}}(c + d\sqrt[4]{-q}) \equiv 2 \bmod 4\mathbb{Z}_2$. It follows that $\mathrm{ord}_2(a) = \mathrm{ord}_2(b) = (h-1)/2$. Because $\pi = N_{F_{\mathfrak{P}}/K_{\mathfrak{p}}}(\gamma)$ is a norm, we conclude the following values of the Hilbert symbols

$$\Big(\frac{2^h u, \sqrt{-q}}{K_{\mathfrak{p}}}\Big) = \Big(\frac{2u, 3}{\mathbb{Q}_2}\Big) = 1 \text{ if } q \equiv 7 \bmod 16$$

and

$$\Big(\frac{2^h u, \sqrt{-q}}{K_{\mathfrak{p}}}\Big) = \Big(\frac{2u, 7}{\mathbb{Q}_2}\Big) = 1 \text{ if } q \equiv 15 \bmod 16.$$

This implies that $u \equiv 3 \bmod 4$ if $q \equiv 7 \bmod 16$ and that $u \equiv 1 \bmod 4$ if $q \equiv 15 \bmod 16$. Thus

$$\begin{aligned}\frac{\eta + \eta^{-1}}{2} &= \frac{a^2 + \sqrt{-q}b^2}{\pi} = \frac{2a^2 - \pi}{\pi} \\ &= \Big(\frac{a}{2^{\frac{h-1}{2}}}\Big)^2 u^{-1} - 1 \equiv u^{-1} - 1 \equiv \begin{cases} 2 \bmod 4 & \text{if } q \equiv 7 \bmod 16, \\ 0 \bmod 4 & \text{if } q \equiv 15 \bmod 16. \end{cases}\end{aligned}$$

This finishes the proof of Lemma 5 by the fact $\mathrm{ord}_{\mathfrak{P}}(2) = 2$. $\square$

Proof of Theorem 1. As we mentioned in the end of the introduction, the basic fact that $\mathrm{ord}_{\mathfrak{P}}(\log_{\mathfrak{P}}(x)) = \mathrm{ord}_{\mathfrak{P}}(x-1)$ if $\mathrm{ord}_{\mathfrak{P}}(x-1) > 2$ will be used. For a proof, see [6, Lemma 5.5]. Assume $q \equiv 3 \bmod 8$. Then $\mathrm{ord}_{\mathfrak{P}}(\eta^2+1) = \mathrm{ord}_{\mathfrak{P}}(\eta^2+\eta\eta^{-1}) = \mathrm{ord}_{\mathfrak{P}}(\eta+\eta^{-1}) = 2$ and $\mathrm{ord}_{\mathfrak{P}}(\eta^2-1) = \mathrm{ord}_{\mathfrak{P}}(\eta^2-\eta\eta^{-1}) = \mathrm{ord}_{\mathfrak{P}}(\eta-\eta^{-1}) = 2$. Hence $\mathrm{ord}_{\mathfrak{P}}(\eta^4-1) = 4$. This gives $\mathrm{ord}_{\mathfrak{P}}\log_{\mathfrak{P}}(\eta^4) = 4$. Thus $\mathrm{ord}_{\mathfrak{P}}(\log_{\mathfrak{P}}(\eta)) = \mathrm{ord}_{\mathfrak{P}}\log_{\mathfrak{P}}(\eta^4) - \mathrm{ord}_{\mathfrak{P}}(4) = 0$. This proves (1).

Assume $q \equiv 7 \bmod 16$. We have $\mathrm{ord}_{\mathfrak{P}}(\eta^2+1) = \mathrm{ord}_{\mathfrak{P}}(\eta^2+\eta\eta^{-1}) = \mathrm{ord}_{\mathfrak{P}}(\eta+\eta^{-1}) = 4$. Then $\mathrm{ord}_{\mathfrak{P}}(\eta^2-1) = \mathrm{ord}_{\mathfrak{P}}(\eta^2+1-2) = \mathrm{ord}_{\mathfrak{P}}(2) = 2$. This gives $\mathrm{ord}_{\mathfrak{P}}(\eta^4-1) = 6$. Thus $\mathrm{ord}_{\mathfrak{P}}(\log_{\mathfrak{P}}(\eta^4)) = \mathrm{ord}_{\mathfrak{P}}(\eta^4-1) = 6$. Hence $\mathrm{ord}_{\mathfrak{P}}(\log_{\mathfrak{P}}(\eta)) = 6 - \mathrm{ord}_{\mathfrak{P}}(4) = 2$. This proves (2).

Assume $q \equiv 15 \bmod 16$. Then $\mathrm{ord}_{\mathfrak{P}}(\eta^4-1) = \mathrm{ord}_{\mathfrak{P}}(\eta^2+1) + \mathrm{ord}_{\mathfrak{P}}(\eta^2-1) \geq 6+2 = 8$. Then $\mathrm{ord}_{\mathfrak{P}}(\log_{\mathfrak{P}}(\eta^4)) = \mathrm{ord}_{\mathfrak{P}}(\eta^4-1) \geq 8$. Thus $\mathrm{ord}_{\mathfrak{P}}(\log_{\mathfrak{P}}(\eta)) \geq 4$. This completes the proof of Theorem 1. $\square$

Now, we prove Corollary 2, and we begin by recalling a classical result from global class field theory. Let L be any number field, and p be a prime number. For a prime ideal v of L, let $U_{1,v}$ denote the principal units in the completion L_v of L, and put $U_1 = \prod_{v|p} U_{1,v}$. Let ϕ be the canonical embedding $L \hookrightarrow \prod_{v|p} L_v$. Denote by $\mathcal{E}_1$ the group of global units of L whose images lie in U_1, and let $\overline{\phi(\mathcal{E}_1)}$ denote the closure of $\phi(\mathcal{E}_1)$ in U_1 under the p-adic topology. Let H be the p-Hilbert class field of L. Finally let $M(L)$ be the maximal abelian p-extension of L, which is unramified outside the primes of L lying above p. Then the Artin map induces an isomorphism

$$U_1/\overline{\phi(\mathcal{E}_1)} \cong \mathrm{Gal}(M(L)/H).$$

This is a standard consequence of global class field theory (see, for example, [6, Theorem 13.4]). Note that U_1 is a finitely generated $\mathbb{Z}_p$-module of rank $[L:\mathbb{Q}]$. Moreover, the $\mathbb{Z}_p$-module $\overline{\phi(\mathcal{E}_1)}$ has rank $\leq r_1+r_2-1$, and Leopoldt's conjecture asserts that this rank is always equal to r_1+r_2-1; here r_1 and r_2 are the number of real and complex places of L, respectively.

Proof of Corollary 2. We apply the above isomorphism to the field F with $q \equiv 3 \bmod 8$ and the prime 2. In this case, $U_1 = 1+\mathfrak{P}\mathcal{O}_{F_{\mathfrak{P}}}$ has $\mathbb{Z}_2$-rank $[F:\mathbb{Q}] = 4$, and $\overline{\phi(\mathcal{E}_1)} = \overline{\langle \eta, -1\rangle}$ clearly has $\mathbb{Z}_2$-rank 1. Moreover, the 2-Hilbert class field of F is F itself since F has odd class number by Lemma 4. Thus we obtain an isomorphism of $\mathbb{Z}_2$-modules

$$(1+\mathfrak{P}\mathcal{O}_{F_{\mathfrak{P}}})/\overline{\langle \eta, -1\rangle} \cong \mathrm{Gal}(M(F)/F).$$

In order to prove $M(F) = F_\infty$, it suffices to show that there is no nontrivial torsion element in the group on the left. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \{\pm 1\} & \longrightarrow & \overline{\phi(\mathcal{E}_1)} & \xrightarrow{\log_{\mathfrak{P}}} & \mathbb{Z}_2 \log_{\mathfrak{P}}(\eta) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mu(1+\mathfrak{P}\mathcal{O}_{F_{\mathfrak{P}}}) & \longrightarrow & 1+\mathfrak{P}\mathcal{O}_{F_{\mathfrak{P}}} & \xrightarrow{\log_{\mathfrak{P}}} & \log_{\mathfrak{P}}(1+\mathfrak{P}\mathcal{O}_{F_{\mathfrak{P}}}) & \longrightarrow & 0. \end{array}$$

Here $\mu(1+\mathfrak{P}\mathcal{O}_{F_{\mathfrak{P}}})$ is the group of roots of unity in $1+\mathfrak{P}\mathcal{O}_{F_{\mathfrak{P}}}$ which equals $\{\pm 1\}$ as one can check that $\sqrt{-1} \notin F_{\mathfrak{P}}$. Thus the logarithm induces an isomorphism

$$(1+\mathfrak{P}\mathcal{O}_{F_{\mathfrak{P}}})/\overline{\langle \eta, -1\rangle} \cong \log_{\mathfrak{P}}(1+\mathfrak{P}\mathcal{O}_{F_{\mathfrak{P}}})/\mathbb{Z}_2 \log_{\mathfrak{P}}(\eta).$$

Since $\mathrm{ord}_{\mathfrak{P}}(2) = 2$, it is clear from the logarithmic series that $\log_{\mathfrak{P}}(1+\mathfrak{P}\mathcal{O}_{F_{\mathfrak{P}}}) \subset \mathcal{O}_{F_{\mathfrak{P}}}$. We claim that the $\mathbb{Z}_2$-module $\log_{\mathfrak{P}}(1+\mathfrak{P}\mathcal{O}_{F_{\mathfrak{P}}})/\mathbb{Z}_2 \log_{\mathfrak{P}}(\eta)$ is free. Suppose not. Then there exists an element a in $\log_{\mathfrak{P}}(1+\mathfrak{P}\mathcal{O}_{F_{\mathfrak{P}}}) \subset \mathcal{O}_{F_{\mathfrak{P}}}$ but not in $\mathbb{Z}_2 \log_{\mathfrak{P}}(\eta)$ such that $2a \in \mathbb{Z}_2 \log_{\mathfrak{P}}(\eta)$. Write $2a = r \log_{\mathfrak{P}}(\eta)$ with $r \in \mathbb{Z}_2$. Note that r must be in $\mathbb{Z}_2^{\times}$. This would give $\mathrm{ord}_{\mathfrak{P}}(\log_{\mathfrak{P}}(\eta)) = \mathrm{ord}_{\mathfrak{P}}(2a) > 0$ which contradicts to Theorem 1. Thus we have that $\mathrm{Gal}(M(F)/F) \cong \log_{\mathfrak{P}}(1+\mathfrak{P}\mathcal{O}_{F_{\mathfrak{P}}})/\mathbb{Z}_2 \log_{\mathfrak{P}}(\eta)$ is a free $\mathbb{Z}_2$-module of rank 3 and hence $M(F) = F_\infty$. This completes the proof. □

We end this paper by noting a second and very simple proof of Theorem 1(3). Suppose $q \equiv 7 \bmod 8$, so that 2 splits in K, and recall that $\mathfrak{p}$ is the restriction of $\mathfrak{P}$ to K. As before, let $M(F)$ be the maximal abelian 2-extension which is unramified outside $\mathfrak{P}$. By class field theory and the fact that F has odd class number [2, Theorem 11], we have

$$(1+\mathfrak{P}\mathcal{O}_{F_{\mathfrak{P}}})/\overline{\langle \eta, -1 \rangle} \cong \mathrm{Gal}(M(F)/F).$$

Suppose now $q \equiv 15 \bmod 16$. The embedding $K \hookrightarrow K_{\mathfrak{p}} = \mathbb{Q}_2$ induced by $\mathfrak{p}$ makes that $\sqrt{-q} \equiv -1 \bmod 8$ whence $F_{\mathfrak{P}} = \mathbb{Q}_2(\sqrt{-1})$. Clearly $\sqrt{-1}$ is in $1+\mathfrak{P}\mathcal{O}_{F_{\mathfrak{P}}}$ but not in $\overline{\langle \eta, -1 \rangle}$. Thus $\mathrm{Gal}(M(F)/F)$ has an element of order 2. Now let $F_\infty = FK_\infty$, where K_∞ is the unique $\mathbb{Z}_2$-extension of K unramified outside $\mathfrak{p}$. Since $\mathrm{Gal}(F_\infty/F)$ is a free $\mathbb{Z}_2$-module of rank 1, it follows that $\mathrm{Gal}(M(F)/F_\infty)$ must contain the element of order 2, and so $\mathrm{Gal}(M(F)/F_\infty) \neq 0$. By the formula (1.1) of Coates-Wiles, it follows that we must have $\mathrm{ord}_{\mathfrak{P}}(\log_{\mathfrak{P}}(\eta)) \geq 4$, as required.

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