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AREA OF MINIMAL HYPERSURFACES IN THE UNIT SPHERE*

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Abstract. A well-known conjecture of Yau states that the area of one of Clifford minimal hypersurfaces $S^k\big(\sqrt{\frac{k}{n}}\,\big)\times S^{n-k}\big(\sqrt{\frac{n-k}{n}}\,\big)$ gives the lowest value of area among all non-totally geodesic compact minimal hypersurfaces in the unit sphere $S^{n+1}(1)$. The present paper shows that Yau conjecture is true for minimal rotational hypersurfaces, more precisely, the area $|M^n|$ of compact minimal rotational hypersurface M^n is either equal to $|S^n(1)|$, or equal to $|S^1(\sqrt{\frac{1}{n}})\times S^{n-1}(\sqrt{\frac{n-1}{n}})|$, or greater than $2(1-\frac{1}{\pi})|S^1(\sqrt{\frac{1}{n}})\times S^{n-1}(\sqrt{\frac{n-1}{n}})|$. As the application, the entropies of some special self-shrinkers are estimated.

Key words. Minimal hypersurfaces, Yau conjecture, area, self-shrinkers, entropy.

Mathematics Subject Classification. 53C42, 53A10.

1. Introduction. The study of minimal hypersurfaces in space forms (that is, $\mathbb{R}^{n+1}$, the sphere $S^{n+1}(1)$, and hyperbolic space $H^{n+1}(-1)$), is one of the most important subjects in differential geometry. There are a lot of nice results on this topic (see [3], [6], [8], [9], [14] and many others). The simplest examples of minimal hypersurfaces in $S^{n+1}(1)$ are the totally geodesic n-spheres. Another basic examples are the so-called Clifford minimal hypersurfaces $S^k\big(\sqrt{\frac{k}{n}}\,\big)\times S^{n-k}\big(\sqrt{\frac{n-k}{n}}\,\big)$.

Cheng, Li and Yau [5] proved in 1984 that if M^n is a compact minimal hypersurface in the unit sphere $S^{n+1}(1)$ and M^n is not totally geodesic, then there exists a constant $c(n)>0$, such that the area $|M^n|$ of M^n satisfies $|M^n|>(1+c(n))|S^n(1)|$, that is, the area of the totally geodesic n-sphere $S^n(1)\subset S^{n+1}(1)$ is the smallest among all compact minimal hypersurfaces in $S^{n+1}(1)$.

In 1992, S.T. Yau [21] posed the following conjecture (P288, Problem 31):

Yau Conjecture. *The area of one of Clifford minimal hypersurfaces* $S^k\big(\sqrt{\frac{k}{n}}\,\big)\times S^{n-k}\big(\sqrt{\frac{n-k}{n}}\,\big)$ *gives the lowest value of area among all non-totally geodesic compact minimal hypersurfaces in the unit sphere* $S^{n+1}(1)$.

In this paper, we consider a restricted problem of Yau conjecture for compact minimal rotational hypersurfaces M^n in $S^{n+1}(1)$. As one of the main results of this paper, we prove

Theorem 1.1. *If M^n is a compact minimal rotational hypersurface in $S^{n+1}(1)$, then the area $|M^n|$ of M^n satisfies either $|M^n|=|S^n(1)|$, or $|M^n|=|S^1(\sqrt{\frac{1}{n}})\times S^{n-1}(\sqrt{\frac{n-1}{n}})|$, or $|M^n|>2(1-\dfrac{1}{\pi})|S^1(\sqrt{\frac{1}{n}})\times S^{n-1}(\sqrt{\frac{n-1}{n}})|$.*

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REMARK 1.1. *From Theorem 1.1, we can know that Yau conjecture is true for minimal rotational hypersurfaces.*

COROLLARY 1.1. *If M^n is a compact minimal rotational hypersurface in $S^{n+1}(1)$, then the area $|M^n|$ of M^n satisfies either $|M^n| = |S^n(1)|$, or $|M^n| = |S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})|$, or $|M^n| = |M^n(3,2)|$, or $|M^n| > 3(1-\dfrac{1}{\pi})|S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})|$, where $M^n(3,2)$ is the compact minimal rotational hypersurface with 3-fold rotational symmetry and rotation number 2.*

REMARK 1.2. *From Theorem 1.1, we have that the conjecture proposed by Perdomo and the second author in [17] is true except the case $M^n(3,2)$ since $3(1-\dfrac{1}{\pi}) > 2$.*

By considering the upper bounds of some integral, we show another main result of the present paper concerning the areas of compact minimal rotational hypersurfaces M^n in $S^{n+1}(1)$, stated as follows

THEOREM 1.2. *The lowest value of area among all compact minimal rotational hypersurfaces with non-constant principal curvatures in the unit sphere $S^{n+1}(1)$ is the area of either $M^n(3,2)$ or $M^n(5,3)$, where $M^n(k,l)$ is a compact minimal rotational hypersurface in $S^{n+1}(1)$ with k-fold rotational symmetry and rotation number l, its periodic is $K = l \times \frac{2\pi}{k} = \frac{2l\pi}{k}$.*

According to the above theorems, we propose the following conjecture.

CONJECTURE. *The lowest value of area among all compact minimal rotational hypersurfaces with non-constant principal curvatures in the unit sphere $S^{n+1}(1)$ is the area of M^n with 3-fold rotational symmetry and rotation number 2.*

2. Preliminaries. Without loss of generality, we assume that the minimal rotational hypersurface is neither $S^n(1)$ nor $S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})$. From [11] and [15], we have the following description (see also [4], [17]) for each minimal rotational hypersurface in $S^{n+1}(1)$, whose principal curvatures are not constant.

For any positive number $a < a_0 = \frac{(n-1)^{n-1}}{n^n}$, let $r(t)$ be a solution of the following ordinary differential equation

$$(r'(t))^2 = 1 - r(t)^2 - ar(t)^{2-2n}. \tag{2.1}$$

Since $0 < a < a_0$, we have that the function $q(v) = 1 - v^2 - av^{2-2n}$ has two positive roots r_1 and r_2 between 0 and 1. Therefore it is not difficult to check that the solution of the differential equation (2.1) is a periodic function with period $T = 2\int_{r_1}^{r_2} \frac{1}{\sqrt{q(v)}}\, dv$ that takes values between r_1 and r_2. Moreover, since the differential equation (2.1) does not depend on t explicitly, then, for any k we have that $r(t-k)$ is a solution, provided $r(t)$ is a solution. Therefore we can assume that $r(0) = r_1$ and $r(\frac{T}{2}) = r_2$.

If we define

$$\theta(t) = \int_0^t \frac{\sqrt{a}\, r^{1-n}(\tau)}{1 - r^2(\tau)}\, d\tau,$$

then the hypersurface $\phi : S^{n-1} \times [0,T] \to S^{n+1}$ given by

$$\phi(y,t) = \big(\, r(t)\, y,\ \sqrt{1-r^2(t)}\, \cos(\theta(t)),\ \sqrt{1-r^2(t)}\, \sin(\theta(t))\,\big)$$

is called the *fundamental portion* of M_a. The curve

$$\alpha(t) = \big(\sqrt{1-r^2(t)}\,\cos(\theta(t)),\ \sqrt{1-r^2(t)}\,\sin(\theta(t))\big)$$

is called the *profile curve* of M_a. It turns out that the whole hypersurface M_a is the union of rotations of the fundamental portion. We also have that the hypersurface M_a is compact if and only if the number (see, for example, [11])

$$K(a) = \theta(T) = 2\int_0^{\frac{T}{2}} \frac{\sqrt{a}\,r^{1-n}(\tau)}{1-r^2(\tau)}\,d\tau = 2\pi\frac{p}{s}, \tag{2.2}$$

for some pair of relatively prime integers p and s. In this case, M_a is made out of exactly s copies of the fundamental portion, that is, $M^n(s,p)$ is a hypersurface with *s-fold rotational symmetry* and *rotation number* p. When $\frac{K(a)}{2\pi}$ is not a rational number, we have that the hypersurface is not compact.

The following lemma is due to Otsuki [12], [13]. A proof for the particular case $n = 2$ can also be found in [1] and in [16].

Lemma 2.1. *The function $K(a)$ given in (2.2) is strictly increasing and differentiable on $(0, a_0)$ and*

$$\lim_{a\to 0} K(a) = \pi, \qquad \lim_{a\to a_0} K(a) = \sqrt{2}\pi.$$

In [17], Perdomo and the second author proved the following lemma,

Lemma 2.2. *If M^n is a compact minimal rotational hypersurface in $S^{n+1}(1)$ with non-constant principal curvatures, then the area of M^n, denoted by $|M^n|$, is equal to*

$$w(a)p,$$

where p is the rotational number greater than 1 (see (2.2)),

$$w(a) = 2\pi\sigma_{n-1}\frac{\int_{x_1}^{x_2}\frac{x^{n-\frac{3}{2}}}{\sqrt{x^{n-1}-x^n-a}}dx}{K(a)}, \tag{2.3}$$

$a \in (0, a_0)$, $x_1 < x_2$ are the only two roots in the interval $(0,1)$ of the polynomial $z(x) = x^{n-1} - x^n - a$, σ_{n-1} denotes the area of $S^{n-1}(1)$. Moreover, we have

$$\begin{aligned}
&\lim_{a\to a_0} w(a)\\
&= \lim_{a\to a_0} 2\pi\sigma_{n-1}\frac{\int_{x_1}^{x_2}\frac{x^{n-\frac{3}{2}}}{\sqrt{x^{n-1}-x^n-a}}dx}{K(a)}\\
&= 2\pi\sigma_{n-1}\frac{\sqrt{2a_0}\pi}{\sqrt{2}\pi} = 2\pi\sigma_{n-1}\sqrt{a_0}\\
&= \left|S^1\left(\sqrt{\frac{1}{n}}\right)\times S^{n-1}\left(\sqrt{\frac{n-1}{n}}\right)\right|.
\end{aligned} \tag{2.4}$$

3. Proofs of Theorem 1.1 and Corollary 1.1.

Proof of Theorem 1.1. From Lemma 2.1 and Lemma 2.2, it is sufficient to prove Theorem 1.1 if we can get the following inequality

$$2\int_{x_1}^{x_2} \frac{x^{n-\frac{3}{2}}}{\sqrt{x^{n-1}-x^n-a}}dx > 2(1-\frac{1}{\pi})\sqrt{2a_0}\pi, \tag{3.1}$$

for a compact minimal rotational hypersurface M^n in $S^{n+1}(1)$ with non-constant principal curvatures, where $0 < x_1 < x_0 = \frac{n-1}{n} < x_2 < 1$ are two roots of $z(x) = x^{n-1} - x^n - a$, $0 < a < a_0$.

We next prove (3.1). Let

$$y = x^{n-\frac{1}{2}}, \tag{3.2}$$

then

$$2\int_{x_1}^{x_2} \frac{x^{n-\frac{3}{2}}}{\sqrt{x^{n-1}-x^n-a}}dx = \frac{4}{2n-1}\int_{y_1}^{y_2} \frac{1}{\sqrt{y^{\frac{2n-2}{2n-1}}-y^{\frac{2n}{2n-1}}-a}}dy. \tag{3.3}$$

(3.1) is equivalent to

$$\int_{y_1}^{y_2} \frac{1}{\sqrt{y^{\frac{2n-2}{2n-1}}-y^{\frac{2n}{2n-1}}-a}}dy > 2(1-\frac{1}{\pi})\frac{2n-1}{4}\sqrt{2a_0}\pi := 2(1-\frac{1}{\pi})A_0\pi, \tag{3.4}$$

where y_1, y_2 are two roots of $f(y) = y^{\frac{2n-2}{2n-1}} - y^{\frac{2n}{2n-1}} - a$, $A_0 = \frac{2n-1}{4}\sqrt{2a_0}$.

We construct a function $g_1(y)$ as follows:

$$g_1(y) = \begin{cases} c(\sqrt{y_1}-\sqrt{y_c})^2 - c(\sqrt{y}-\sqrt{y_c})^2, & y \in [y_1, y_c], \\ b(y_2-y_c)^2 - b(y-y_c)^2, & y \in (y_c, y_2], \end{cases} \tag{3.5}$$

where $c = \frac{8\sqrt{n(n-1)}}{(2n-1)^2}$, $b = \frac{2(n-1)}{(2n-1)^2}\left(\frac{n-1}{n}\right)^{-n}$ and $y_c = (\frac{n-1}{n})^{\frac{1}{2}(2n-1)}$.

Let

$$h_1(y) = g_1(y) - f(y), \quad \text{for } y \in [y_1, y_2], \tag{3.6}$$

we will prove

$$h_1(y) \geq 0 \tag{3.7}$$

and

$$\begin{aligned} &\int_{y_1}^{y_2} \frac{1}{\sqrt{y^{\frac{2n-2}{2n-1}}-y^{\frac{2n}{2n-1}}-a}}dy \\ &= \int_{y_1}^{y_2} \frac{1}{\sqrt{f(y)}}dy \geq \int_{y_1}^{y_2} \frac{1}{\sqrt{g_1(y)}}dy \\ &> \left(2-\frac{2}{\pi}+\frac{2}{\pi}\sqrt{\frac{y_1}{y_c}}\right)A_0\pi. \end{aligned} \tag{3.8}$$

We next consider two cases.

Case 1: $y \in [y_c, y_2]$**.**
By a direct calculation, we obtain

$$h_1'(y) = -2b(y - y_c) - \frac{2n-2}{2n-1} y^{\frac{-1}{2n-1}} + \frac{2n}{2n-1} y^{\frac{1}{2n-1}}, \tag{3.9}$$

$$h_1''(y) = -2b + \frac{2n-2}{2n-1}\frac{1}{2n-1} y^{-\frac{2n}{2n-1}} + \frac{2n}{2n-1}\frac{1}{2n-1} y^{-\frac{2n-2}{2n-1}}, \tag{3.10}$$

and

$$h_1''(y_c) = 0, \tag{3.11}$$

it is easy from (3.10) to see that $h_1''(y)$ is a monotonic decreasing function on an interval $[y_c, y_2]$, then

$$h_1''(y) \le h_1''(y_c) = 0, \quad \text{for } y \in [y_c, y_2], \tag{3.12}$$

that is, $h_1'(y)$ is a monotonic decreasing function on an interval $[y_c, y_2]$. Since $h_1'(y_c) = 0$, we have $h_1'(y) \le 0$ for $y \in [y_c, y_2]$, that is, $h_1(y)$ is a monotonic decreasing function on an interval $[y_c, y_2]$. Since $h_1(y_2) = 0$, we conclude that

$$h_1(y) \ge h_1(y_2) = 0, \quad \text{for } y \in [y_c, y_2], \tag{3.13}$$

it follows that

$$\frac{1}{\sqrt{f(y)}} - \frac{1}{\sqrt{g_1(y)}} \ge 0, \tag{3.14}$$

$$\int_{y_c}^{y_2} \left(\frac{1}{\sqrt{f(y)}} - \frac{1}{\sqrt{g_1(y)}} \right) dy \ge 0. \tag{3.15}$$

On the other hand,

$$\int_{y_c}^{y_2} \frac{1}{\sqrt{g_1(y)}} dy = A_0 \pi, \tag{3.16}$$

we have

$$\int_{y_c}^{y_2} \frac{1}{\sqrt{y^{\frac{2n-2}{2n-1}} - y^{\frac{2n}{2n-1}} - a}} dy = \int_{y_c}^{y_2} \frac{1}{\sqrt{f(y)}} dy \ge \int_{y_c}^{y_2} \frac{1}{\sqrt{g_1(y)}} dy = A_0 \pi. \tag{3.17}$$

Case 2: $y \in [y_1, y_c]$**.**
By a direct calculation, we have

$$h_1'(y) = g_1'(y) - f'(y) = c\left(\sqrt{\frac{y_c}{y}} - 1 \right) - \frac{2n-2}{2n-1} y^{-\frac{1}{2n-1}} + \frac{2n}{2n-1} y^{\frac{1}{2n-1}}, \tag{3.18}$$

$$h_1''(y) = -\frac{c}{2}\sqrt{\frac{y_c}{y^3}} + \frac{2n-2}{2n-1}\frac{1}{2n-1} y^{-\frac{2n}{2n-1}} + \frac{2n}{2n-1}\frac{1}{2n-1} y^{-\frac{2n-2}{2n-1}}, \tag{3.19}$$

and

$$h_1''(y_c) = 0. \tag{3.20}$$

Then, we obtain from (3.19) and (3.20) that

$$h_1''(y_c) \le 0, \tag{3.21}$$

$$h_1''(y) \le 0, \quad \text{for } y \in [y_1, y_c]. \tag{3.22}$$

From the above equations and $h_1'(y_c) = 0$, $h_1(y_1) = 0$, we get

$$h_1(y) \ge 0, \quad \text{for } y \in [y_1, y_c]. \tag{3.23}$$

Hence we have

$$\int_{y_1}^{y_c} \frac{1}{\sqrt{f(y)}} dy \ge \int_{y_1}^{y_c} \frac{1}{\sqrt{g_1(y)}} dy = \left(1 - \frac{2}{\pi} + \frac{2}{\pi}\sqrt{\frac{y_1}{y_c}}\right) A_0\pi. \tag{3.24}$$

From the above two cases, we conclude that

$$\int_{y_1}^{y_2} \frac{1}{\sqrt{f(y)}} dy \ge \left(2 - \frac{2}{\pi} + \frac{2}{\pi}\sqrt{\frac{y_1}{y_c}}\right) A_0\pi > \left(2 - \frac{2}{\pi}\right) A_0\pi. \tag{3.25}$$

This completes proof of Theorem 1.1. □

Proof of Corollary 1.1. From Lemma 2.1 and Lemma 2.2, we have the area $|M^n|$ of a compact minimal rotational hypersurface M^n in $S^{n+1}(1)$ with non-constant principal curvatures except $M^n(3,2)$ is greater than

$$3 \times 2\pi\sigma_{n-1} \frac{\inf\limits_{a\in(0,a_0)} \int_{x_1}^{x_2} \frac{x^{n-\frac{3}{2}}}{\sqrt{x^{n-1}-x^n-a}} dx}{K(a)} \tag{3.26}$$

since the rotation number of $M^n(3,2)$ is 2 and the rotation numbers of other hypersurfaces are greater than 2.

From the proof of Theorem 1.1, we get that

$$3 \times 2\pi\sigma_{n-1} \frac{\int_{x_1}^{x_2} \frac{x^{n-\frac{3}{2}}}{\sqrt{x^{n-1}-x^n-a}} dx}{K(a)} > 3(1 - \frac{1}{\pi})\left|S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})\right|. \tag{3.27}$$

This completes proof of Corollary 1.1. □

4. Estimate of an upper bound.

THEOREM 4.1. *The area of $M^n(3,2)$ satisfies*

$$|M^n(3,2)| < 3\left|S^1\left(\sqrt{\frac{1}{n}}\right)\times S^{n-1}\left(\sqrt{\frac{n-1}{n}}\right)\right|. \tag{4.1}$$

Proof. Since the rotation number p of $M^n(3,2)$ is 2, we know from Lemma 2.2 that the area of $M^n(3,2)$ is

$$4\pi\sigma_{n-1} \frac{\int_{x_1}^{x_2} \frac{x^{n-\frac{3}{2}}}{\sqrt{x^{n-1}-x^n-a}} dx}{K(a)} = 4\pi\sigma_{n-1} \frac{\int_{x_1}^{x_2} \frac{x^{n-\frac{3}{2}}}{\sqrt{x^{n-1}-x^n-a}} dx}{\frac{4\pi}{3}} \tag{4.2}$$

for some $a < a_0$ and the area of $S^1\left(\sqrt{\frac{1}{n}}\right)\times S^{n-1}\left(\sqrt{\frac{n-1}{n}}\right)$ is $2\pi\sigma_{n-1}\sqrt{a_0}$. Hence it is sufficient to prove the following inequality

$$\int_{x_1}^{x_2} \frac{x^{n-\frac{3}{2}}}{\sqrt{x^{n-1}-x^n-a}}dx < 2\pi\sqrt{a_0}, \tag{4.3}$$

that is,

$$\int_{y_1}^{y_2} \frac{1}{\sqrt{f(y)}}dy < (2n-1)\pi\sqrt{a_0} = \frac{4\pi}{\sqrt{2}}A_0. \tag{4.4}$$

First of all, we construct a function $g_2(y)$ as follows:

$$g_2(y) = \begin{cases} C(y_1-y_c)^2 - C(y-y_c)^2, & y \in [y_1, y_c], \\ B(y_2-y_c)^2 - B(y-y_c)^2, & y \in (y_c, 1], \end{cases} \tag{4.5}$$

where $B = \frac{1}{2n-1}$, $C = \frac{2(n-1)}{(2n-1)^2}\left(\frac{n-1}{n}\right)^{-n}$ and $y_c = (\frac{n-1}{n})^{\frac{1}{2}(2n-1)}$, then we claim

$$g_2(y) \le f(y), \quad y \in [y_1, y_2]. \tag{4.6}$$

Let $h_2(y) = g_2(y) - f(y)$. We have to consider two cases.

Case 1: $y \in [y_c, y_2]$**.**
By a direct calculation, we obtain

$$h_2'(y) = g_2'(y) - f'(y) = -2B(y-y_c) - \frac{2n-2}{2n-1}y^{\frac{-1}{2n-1}} + \frac{2n}{2n-1}y^{\frac{1}{2n-1}}, \tag{4.7}$$

$$h_2''(y) = -2B + \frac{2n-2}{2n-1}\frac{1}{2n-1}y^{-\frac{2n}{2n-1}} + \frac{2n}{2n-1}\frac{1}{2n-1}y^{-\frac{2n-2}{2n-1}} \tag{4.8}$$

and

$$h_2''(1) = 0, \tag{4.9}$$

we can see from (4.8) that $h_2''(y)$ is a monotonic decreasing function on an interval $[y_c, 1]$, then

$$h_2''(y) \ge h_2''(y_2) \ge h_2''(1) = 0, \quad \text{for } y \in [y_c, y_2], \tag{4.10}$$

that is, $h_2'(y)$ is a monotonic increasing function on an interval $[y_c, y_2]$. Since $h_2'(y_c) = 0$, we have $h_2'(y) \ge 0$ for $y \in [y_c, y_2]$, that is, $h_2(y)$ is a monotonic increasing function on an interval $[y_c, y_2]$. Since $h_2(y_2) = 0$, we conclude that

$$h_2(y) \le h_2(y_2) = 0, \quad \text{for } y \in [y_c, y_2], \tag{4.11}$$

it follows that

$$\frac{1}{\sqrt{f(y)}} - \frac{1}{\sqrt{g_2(y)}} \le 0, \tag{4.12}$$

$$\int_{y_c}^{y_2} \left(\frac{1}{\sqrt{f(y)}} - \frac{1}{\sqrt{g_2(y)}}\right)dy \le 0. \tag{4.13}$$

On the other hand,

$$\int_{y_c}^{y_2} \frac{1}{\sqrt{g_2(y)}} dy = \frac{\sqrt{2n-1}}{2}\pi, \tag{4.14}$$

we have

$$\int_{y_c}^{y_2} \frac{1}{\sqrt{y^{\frac{2n-2}{2n-1}} - y^{\frac{2n}{2n-1}} - a}} dy = \int_{y_c}^{y_2} \frac{1}{\sqrt{f(y)}} dy \le \int_{y_c}^{y_2} \frac{1}{\sqrt{g_2(y)}} dy = \frac{\sqrt{2n-1}}{2}\pi. \tag{4.15}$$

Case 2: $y \in [y_1, y_c]$**.**
By a direct calculation, we have

$$h_2'(y) = -2C(y - y_c) - \frac{2n-2}{2n-1} y^{\frac{-1}{2n-1}} + \frac{2n}{2n-1} y^{\frac{1}{2n-1}}, \tag{4.16}$$

$$h_2''(y) = -2C + \frac{2n-2}{2n-1}\frac{1}{2n-1} y^{-\frac{2n}{2n-1}} + \frac{2n}{2n-1}\frac{1}{2n-1} y^{-\frac{2n-2}{2n-1}} \tag{4.17}$$

and

$$h_2''(y_c) = 0, \tag{4.18}$$

then we obtain from (4.17) that $h_2''(y)$ is a monotonic decreasing function on an interval $[y_1, y_c]$, then

$$h_2''(y) \ge h_2''(y_c) = 0, \quad \text{for } y \in [y_1, y_c], \tag{4.19}$$

that is, $h_2'(y)$ is a monotonic increasing function on an interval $[y_1, y_c]$. Since $h_2'(y_c) = 0$, we have $h_2'(y) \le 0$ for $y \in [y_1, y_c]$, that is, $h_2(y)$ is a monotonic decreasing function on an interval $[y_1, y_c]$. Since $h_2(y_1) = 0$, we conclude that

$$h_2(y) \le h_2(y_1) = 0, \quad \text{for } y \in [y_1, y_c], \tag{4.20}$$

it follows that

$$\frac{1}{\sqrt{f(y)}} - \frac{1}{\sqrt{g_2(y)}} \le 0. \tag{4.21}$$

On the other hand,

$$\int_{y_1}^{y_c} \frac{1}{\sqrt{g_2(y)}} dy = \frac{1}{2}\sqrt{\frac{(2n-1)^2}{2(n-1)}\left(\frac{n}{n-1}\right)^{-n}}\pi, \tag{4.22}$$

we have

$$\begin{aligned}
&\int_{y_1}^{y_c} \frac{1}{\sqrt{y^{\frac{2n-2}{2n-1}} - y^{\frac{2n}{2n-1}} - a}} dy \\
&= \int_{y_1}^{y_c} \frac{1}{\sqrt{f(y)}} dy \\
&\le \int_{y_1}^{y_c} \frac{1}{\sqrt{g_2(y)}} dy \\
&= \frac{1}{2}\sqrt{\frac{(2n-1)^2}{2(n-1)}\left(\frac{n}{n-1}\right)^{-n}}\pi.
\end{aligned} \tag{4.23}$$

From (4.15) and (4.23), we have

$$\begin{aligned}\int_{y_1}^{y_2}\frac{1}{\sqrt{f(y)}}dy &\le \frac{\sqrt{2n-1}}{2}\pi+\frac{1}{2}\sqrt{\frac{(2n-1)^2}{2(n-1)}\left(\frac{n}{n-1}\right)^{-n}}\pi\\ &=\left(1+\sqrt{\frac{2(n-1)}{2n-1}\left(\frac{n}{n-1}\right)^{n}}\right)A_0\pi\\ &=\left(1+\sqrt{\frac{2}{(2n-1)a_0}}\right)A_0\pi\\ &<\left(1+\sqrt{\left(\frac{n}{n-1}\right)^{n}}\right)A_0\pi.\end{aligned} \tag{4.24}$$

When $n=3$, we see from (4.24)

$$\int_{y_1}^{y_2}\frac{1}{\sqrt{f(y)}}dy \le \left(1+\sqrt{\frac{2(n-1)}{2n-1}\left(\frac{n}{n-1}\right)^{n}}\right)A_0\pi=\left(1+\sqrt{\frac{27}{10}}\right)A_0\pi, \tag{4.25}$$

when $n\ge 4$, we get

$$\int_{y_1}^{y_2}\frac{1}{\sqrt{f(y)}}dy < \left(1+\sqrt{\left(\frac{n}{n-1}\right)^{n}}\right)A_0\pi \le (1+\frac{16}{9})A_0\pi=\frac{25}{9}A_0\pi. \tag{4.26}$$

Hence, we obtain from (4.25) and (4.26) that

$$\int_{y_1}^{y_2}\frac{1}{\sqrt{f(y)}}dy < \frac{25}{9}A_0\pi<\frac{4}{\sqrt{2}}A_0\pi. \tag{4.27}$$

This completes proof of Theorem 4.1. □

Proof of Theorem 1.2. From Lemma 2.1 and Lemma 2.2, we know that the area of a compact minimal rotational hypersurface M^n in $S^{n+1}(1)$ with non-constant principal curvatures except $M^n(3,2)$ and $M^n(5,3)$ is greater than

$$4\times 2\pi\sigma_{n-1}\frac{\inf\limits_{a\in(0,a_0)}\int_{x_1}^{x_2}\frac{x^{n-\frac{3}{2}}}{\sqrt{x^{n-1}-x^n-a}}dx}{K(a)} \tag{4.28}$$

since the rotation number of $M^n(3,2)$ is 2, the rotation number of $M^n(5,3)$ is 3, the rotation number of $M^n(7,4)$ is 4 and the rotation numbers of other hypersurfaces are greater than 4.

We know from Lemma 2.2 and the proof of Theorem 1.1 that

$$\begin{aligned}|M^n(7,4)| &= 4\times 2\pi\sigma_{n-1}\frac{\int_{x_1}^{x_2}\frac{x^{n-\frac{3}{2}}}{\sqrt{x^{n-1}-x^n-a}}dx}{\frac{8}{7}\pi}\\ &> 4(1-\frac{1}{\pi})\times\frac{\sqrt{2}}{\frac{8}{7}}\left|S^1(\sqrt{\frac{1}{n}})\times S^{n-1}(\sqrt{\frac{n-1}{n}})\right|\\ &> 3\left|S^1(\sqrt{\frac{1}{n}})\times S^{n-1}(\sqrt{\frac{n-1}{n}})\right|\\ &> |M^n(3,2)|\end{aligned} \tag{4.29}$$

for some $a \in (0, a_0)$ and the area of other hypersurface M^n except $M^n(3,2)$, $M^n(5,3)$ and $M^n(7,4)$ satisfies

$$\begin{aligned} |M^n| &> 5(1-\frac{1}{\pi})\left|S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})\right| \\ &> 3|S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})\Big| \\ &> |M^n(3,2)|. \end{aligned} \tag{4.30}$$

Hence, the lowest value of area among all compact minimal rotational hypersurfaces with non-constant principal curvatures in the unit sphere $S^{n+1}(1)$ is the area of either $M^n(3,2)$ or $M^n(5,3)$. This completes proof of Theorem 1.2. □

5. Entropies of some special self-shrinkers. In this section, we estimate the entropies of some special self-shrinkers as the application of the estimate of the areas.

An immersed hypersurface $X : M^n \to \mathbb{R}^{n+1}$ in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ is called a *self-shrinker* if it satisfies

$$H + \langle X, N \rangle = 0, \tag{5.1}$$

where N is the unit normal vector of $X : M^n \to \mathbb{R}^{n+1}$, H is the mean curvature. The *entropy* $\lambda(M^n)$ of self-shrinker M^n can be defined as follows:

$$\lambda(M^n) = \frac{1}{(2\pi)^{n/2}} \int_{M^n} e^{-|X|^2/2} d\mu. \tag{5.2}$$

$S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$ are standard examples of self-shrinkers, here $0 \le k \le n$. Stone [20] computed the entropy of $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$ and showed that $\lambda(S^n(\sqrt{n}))$ is decreasing in n.

From the definition of self-shrinkers, we know that if $X : M^n \to S^{n+1}(1)$ is a minimal rotational hypersurface, then $C(M^n)$, the *cone* over M^n, satisfies the self-shrinker equation (5.1) in $\mathbb{R}^{n+2}$. The entropy $\lambda(C(M^n))$ of the cone $C(M^n)$ over a compact minimal rotational hypersurface $M^n \subset S^{n+1}(1)$ in $\mathbb{R}^{n+2}$ is

$$\begin{aligned} \lambda(C(M^n)) &= \frac{1}{(2\pi)^{(n+1)/2}} |M^n| \int_0^{+\infty} t^n e^{-t^2/2} dt \\ &= \frac{1}{2}\pi^{-(n+1)/2}\Gamma(\frac{n+1}{2})|M^n| \\ &= \frac{1}{\sigma_n}|M^n|, \end{aligned} \tag{5.3}$$

where $|M^n|$ denotes the area of M^n, $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt$ is the Gamma function, σ_n denotes the n-area of $S^n(1)$.

By a computation, we have

Theorem 5.1. *If M^n is a compact minimal rotational hypersurface in $S^{n+1}(1)$, then the entropy $\lambda(C(M^n))$ of the cone $C(M^n)$ over M^n in $\mathbb{R}^{n+2}$ satisfies either $\lambda(C(M^n)) = 1$, or $\lambda(C(M^n)) = \dfrac{2\pi\sigma_{n-1}\sqrt{a_0}}{\sigma_n}$, or $\lambda(C(M^n)) > \dfrac{4(\pi-1)\sigma_{n-1}\sqrt{a_0}}{\sigma_n}$, where $a_0 = \frac{(n-1)^{n-1}}{n^n}$, σ_n denotes the n-area of $S^n(1)$.*

Proof. Combining Theorem 1.1, (5.3) and using

$$\left|S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})\right| = 2\pi\sigma_{n-1}\sqrt{a_0}, \tag{5.4}$$

we can proof Theorem 5.1. □

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