Asian J. Math.
Vol. 25, No. 2, pp. 195–228, April 2021

THE L^q-SPECTRUM FOR A CLASS OF SELF-SIMILAR MEASURES WITH OVERLAP*

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Abstract. It is known that the heuristic principle, referred to as the multifractal formalism, need not hold for self-similar measures with overlap, such as the 3-fold convolution of the Cantor measure and certain Bernoulli convolutions. In this paper we study an important function in the multifractal theory, the L^q-spectrum, $\tau(q)$, for measures of finite type, a class of self-similar measures that includes these examples. Corresponding to each measure, we introduce finitely many variants on the L^q-spectrum which arise naturally from the finite type structure and are often easier to understand than τ. We show that τ is always bounded by the minimum of these variants and is equal to the minimum variant for $q \geq 0$. This particular variant coincides with the L^q-spectrum of the measure μ restricted to appropriate subsets of its support. If the IFS satisfies particular structural properties, which do hold for the above examples, then τ is shown to be the minimum of these variants for all q. Under certain assumptions on the local dimensions of μ, we prove that the minimum variant for $q \ll 0$ coincides with the straight line having slope equal to the maximum local dimension of μ. Again, this is the case with the examples above. More generally, bounds are given for τ and its variants in terms of notions closely related to the local dimensions of μ.

Key words. L^q-spectrum, multifractal formalism, self-similar measure, finite type.

Mathematics Subject Classification. Primary 28A80; Secondary 28A78.

1. Introduction. By the local dimension of a probability measure μ at a point x in its support we mean the quantity

$$\dim_{\text{loc}} \mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

It is natural to ask about the values that are attained as local dimensions of the measure and the size of the sets $E_\alpha = \{x : \dim_{\text{loc}} \mu(x) = \alpha\}$. For self-similar measures that satisfy the open set condition, it is well known that the set of attainable local dimensions is a closed interval and there are simple formulas for the endpoints of this interval. The Hausdorff dimension of E_α is equal to the Legendre transform of the L^q-spectrum of μ, $\tau(q)$, at α (see Definition 4.1) meaning, $\dim E_\alpha = \inf_{q \in \mathbb{R}}(q\alpha - \tau(q))$. In this case τ is a differentiable function on all of $\mathbb{R}$. This is known as the multifractal formalism. We refer the reader to [2] for proofs of these facts.

The local behaviour of 'overlapping' self-similar measures is not as well understood and the multifractal formalism need not hold. For instance, in [11] Hu and Lau discovered that the set of local dimensions of the 3-fold convolution of the classical middle-third Cantor measure consists of a closed interval and an isolated point which is its maximum local dimension. Specifically, this maximum local dimension occurs at the two endpoints of the support of the measure. This unexpected property was later found to be true for more general overlapping regular Cantor-like measures; see [1, 9, 16]. Lau and Ngai in [12] discovered that if the self-similar measure arises from an IFS that satisfies only the weak separation property and its L^q-spectrum is differentiable at $q_0 > 0$, then $\dim E_\alpha$ is the Legendre transform of τ at α if $\alpha = \tau'(q_0)$.

*Received September 23, 2019; accepted for publication July 8, 2020. This research was supported in part by NSERC grants RGPIN 2016-03719 and 2019-03930. This paper is in final form and no version of it will be submitted for publication elsewhere.

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But there is no guarantee in their theorem that τ is differentiable and no information about $\tau(q)$ for $q < 0$. In [13], Lau and Wang showed that the L^q-spectrum of the 3-fold convolution of the Cantor measure has one point of non-differentiability at $q_0 < 0$. The L^q-spectrum is equal to the line with slope equal to its maximum local dimension for $q < q_0$ and coincides with the L^q-spectrum of the measure restricted to a closed subinterval of the interior of its support for $q \geq q_0$. A similar result was found for more general regular, Cantor-like measures in [16]. Another much studied class of overlapping measures are the uniform Bernoulli convolutions with contraction factor the inverse of a Pisot number ϱ. In the case of $\varrho =$ golden mean, Feng [4] found that even though the set of local dimensions of this measure is a closed interval, its L^q-spectrum is not differentiable at some $q_0 < 0$ and coincides with the line with slope equal to the maximum local dimension for $q < q_0$.

These overlapping measures are all examples of self-similar measures of finite type. The notion of finite type was introduced by Ngai and Wang in [14], and is essentially weaker than the open set condition and stronger than the weak separation property. In [3], Feng proved that the L^q-spectrum of any finite type measure is differentiable for all $q > 0$. In [4, 5], he extensively studied the local dimension theory of uniform Bernoulli convolutions with simple Pisot inverses as the contraction factor. The local dimension theory was extended to general self-similar measures of finite type in a series of papers by two of the authors with various coauthors, [8, 9, 10].

In this note, we continue the study of the L^q-spectrum of measures of finite type. Important structural building blocks for measures of finite type are combinatorial objects known as loop classes. Our general strategy is to 'decompose' the support of the measure into these loop classes, study the restriction of its L^q-spectrum to each such a loop class L (denoted τ_L) and then try to recover information about τ from the data collected for each loop class. As seen in [8, 9, 10], the set of local dimensions corresponding to a given loop class is often a closed interval and hence according to the multifractal philosophy, it is reasonable to expect τ_L to be better behaved than τ.

We have that $\tau \leq \min \tau_L$ where the minimum is taken over all maximal loop classes L. We prove that equality holds under certain structural assumptions. In particular, we give criteria which ensures there is a some $q_0 < 0$ such that $\tau(q) = qd$ for $d = \max\{\dim_{\text{loc}} \mu(x) : x\}$ and all $q < q_0$. The line $y = qd$ arises because it is the function $\tau_L(q)$ for a (suitable) singleton maximal loop class L. In Example 5.13 we give an example showing that if these criteria are not met, it is possible for τ not to have this property. In particular, this example has the property that $\tau(q)/q \not\to d$ as $q \to -\infty$.

In general, for $q \geq q_0$, the L^q-spectrum coincides with τ_L for L the special maximal loop class known as the essential class. It also coincides with the L^q-spectrum of the measure restricted to various proper subsets of its support which are easily described in terms of the finite type data (and which in many examples can be any proper closed subset of the interior of its support). A lower bound for q_0 is given in terms of the finite type data. These structural assumptions are satisfied in many examples, including regular Cantor-like measures and biased Bernoulli convolutions with contraction factor the inverse of a simple Pisot number.

The proofs of most of these results, including a detailed discussion of what they imply about finite type Cantor-like measures and Bernoulli convolutions, can be found in Section 5.

In Section 3 we discuss the local dimensional behaviour of the measure on loop classes and compare this with the (usual) local dimension of the measure. In Section 4 we introduce the notion of the loop class L^q-spectrum, τ_L, and establish basic properties of these variants on the L^q-spectrum. In particular, we find bounds on the functions τ_L, determine their asymptotic behaviour and prove that the L^q-spectrum τ is dominated by the minimum of these τ_L. In Section 2, we recall the definitions, notation and basic facts about finite type measures that are needed in the paper.

In [6] (see also [5]), Feng and Lau showed that suitable restrictions of self-similar measures having only the weak separation property satisfy the multifractal formalism. However, the local dimensions of the original measure and its restrictions will not, in general, coincide at all points in the support of the restricted measure. Moreover, in many examples, the L^q-spectrum of the original measure and its restrictions only agree for large q.

2. Basic Definitions and Terminology. We begin by reviewing the notion of finite type and the related concepts and terminology that will be used throughout the paper. These notes are basically summarized from [3, 8, 10] where the facts which are stated here are either proved or references given.

2.1. Iterated function systems and finite type. By an iterated function system (IFS) we will mean a finite set of contractions,

$$S_j(x) = \rho_j x + d_j : \mathbb{R} \to \mathbb{R} \text{ for } j = 0, 1, \ldots, m, \tag{2.1}$$

where $m \geq 1$ and $0 < |\rho_j| < 1$. When all ρ_j are equal and positive, the IFS is referred to as **equicontractive**. Each IFS generates a unique invariant, non-empty, compact set K, known as its associated self-similar set, satisfying $K = \bigcup_{j=0}^m S_j(K)$. By rescaling the d_j, if necessary, we can assume the convex hull of K is $[0, 1]$.

Assume we are given probabilities $p_j > 0$ satisfying $\sum_{j=0}^m p_j = 1$. There is a unique self-similar probability measure μ associated with the IFS $\{S_j\}_{j=0}^m$ and probabilities $\{p_j\}_{j=0}^m$, supported on the self-similar set associated with the IFS and satisfying the rule

$$\mu = \sum_{j=0}^m p_j \mu \circ S_j^{-1}.$$

The measure is said to be equicontractive if the IFS is equicontractive.

Given a finite word $\omega = (\omega_1, \ldots, \omega_j)$, on the alphabet $\{0, 1, \ldots, m\}$, we will let $\omega^- = (\omega_1, \ldots, \omega_{j-1})$, $S_\omega = S_{\omega_1} \circ S_{\omega_2} \circ \cdots \circ S_{\omega_j}$ and $\rho_\omega = \prod_{i=1}^j \rho_{\omega_i}$. Let

$$\rho_{\min} = \min_j |\rho_j|$$

and put

$$\Lambda_n = \{\text{finite words } \omega : |\rho_\omega| \leq \rho_{\min}^n \text{ and } |\rho_{\omega^-}| > \rho_{\min}^n\}.$$

The notion of finite type was introduced by Ngai and Wang in [14]. The definition we will use is slightly less general, but is simpler and seen to be equivalent to the finite type definition given for equicontractive IFS in [3]. It includes all the examples of finite type measures in $\mathbb{R}$ of which we are aware.

Definition 2.1. Assume $\{S_j\}$ is an IFS as in equation (2.1). The words $\omega, \tau \in \Lambda_n$ are said to be neighbours if $S_\omega(0,1) \cap S_\tau(0,1) \neq \emptyset$. Denote by $\mathcal{N}(\omega)$ the set of all neighbours of ω. We say that $\omega \in \Lambda_n$ and $\tau \in \Lambda_m$ have the same neighbourhood type if there is a map $f(x) = \pm\rho_{\min}^{n-m}x + c$ such that

$$\{f \circ S_\eta : \eta \in \mathcal{N}(\omega)\} = \{S_\nu : \nu \in \mathcal{N}(\tau)\} \quad \text{and} \quad f \circ S_\omega = S_\tau.$$

The IFS is said to be of **finite type** if there are only finitely many neighbourhood types. Any associated self-similar measure is also said to be of finite type.

It was shown in [15] that an IFS of finite type satisfies the weak separation property, but not necessarily the open set condition.

Here are two interesting and much studied classes of measures of finite type that fail to satisfy the open set condition, [14]. We will refer to these often in the paper.

EXAMPLE 2.2 (Bernoulli convolutions). Consider the IFS: $S_0(x) = \varrho x, S_1(x) = \varrho x + 1 - \varrho$, where $1 < \varrho < 2$ is the inverse of a Pisot number[1] such as the golden mean. If the two probabilities are equal ($p_0 = p_1 = 1/2$) we call the associated self-similar measure a uniform Bernoulli convolution and otherwise it is said to be biased. The self-similar set is $[0, 1]$. These measures are all of finite type.

EXAMPLE 2.3 ((m, d)-Cantor measures). Consider the IFS:

$$S_j(x) = \frac{x}{d} + \frac{j(d-1)}{dm} \text{ for } j = 0, \dots, m$$

with integers $m \geq d \geq 2$ and probabilities $\{p_j\}$. The associated self-similar measures are called (m, d)-Cantor measures and they have support $[0, 1]$. If we take $p_j = \binom{m}{j}2^{-m}$, the resulting measure is the m-fold convolution of the uniform Cantor measure on the Cantor set with ratio of dissection $1/d$, rescaled to have support $[0, 1]$. For example, if $d = 3 = m$ and the probabilities are $1/8, 3/8, 3/8, 1/8$, the self-similar measure is the rescaled 3-fold convolution of the classical middle-third Cantor measure. These measures are all of finite type.

2.2. Net intervals and characteristic vectors.

DEFINITION 2.4. For each positive integer n, let $h_1, \dots, h_{s_n}$ be the collection of elements of the set $\{S_\omega(0), S_\omega(1) : \omega \in \Lambda_n\}$, listed in increasing order. Put

$$\mathcal{F}_n = \{[h_j, h_{j+1}] : 1 \leq j \leq s_n - 1 \text{ and } (h_j, h_{j+1}) \cap K \neq \emptyset\}.$$

Elements of $\mathcal{F}_n$ are called **net intervals of level** n. The interval $[0, 1]$ is understood to be the (only) net interval of level 0.

It is known that for each IFS of finite type there is some $c > 0$ such that

$$c\rho_{\min}^n \leq \ell(\Delta) \leq \rho_{\min}^n$$

for all net intervals Δ of level n, where $\ell(\Delta)$ is the length of the interval Δ. We denote by

$$\ell_n(\Delta) = \rho_{\min}^{-n}\ell(\Delta),$$

the **normalized length** of Δ.

[1] Recall that a Pisot number is an algebraic integer greater than 1, all of whose Galois conjugates are < 1 in absolute value.

Given $\Delta = [a,b] \in \mathcal{F}_n$, suppose that $S_{\sigma_i}(K) \cap \text{int}\Delta \neq \emptyset$ for $\sigma_i \in \Lambda_n$, $i = 1, ..., J$, where the σ_i are distinct and J is maximal. The definition of a net interval ensures that then $S_{\sigma_i}(K) \supseteq \Delta \bigcap K$. We put

$$L_i = \rho_{\min}^{-n} \rho_{\sigma_i} \text{ and } a_i = \rho_{\min}^{-n}(a - S_{\sigma_i}(0))$$

and we say that σ_i **is associated with** (a_i, L_i) **and** Δ. We order the pairs (a_i, L_i) so that $a_i \leq a_{i+1}$ and if $a_i = a_{i+1}$, then $L_i < L_{i+1}$. (Note that some L_i could be negative if there are negative contraction factors.) By the **neighbour set** of Δ we mean the ordered tuple

$$V_n(\Delta) = ((a_1, L_1), (a_2, L_2), \dots, (a_J, L_J)). \tag{2.2}$$

For each $\Delta \in \mathcal{F}_n$, $n \geq 1$, there is a unique element $\widehat{\Delta} \in \mathcal{F}_{n-1}$ which contains Δ, called the **parent** of **child** Δ. Suppose $\Delta \in \mathcal{F}_n$ has parent $\widehat{\Delta}$. If $\widehat{\Delta}$ has k children with the same normalized length and neighbour set as Δ, we order these from left to right as $\Delta^{(1)}, \Delta^{(2)}, \dots, \Delta^{(k)}$ and denote by $t_n(\Delta)$ the integer t such that $\Delta^{(t)} = \Delta$.

Definition 2.5. The **characteristic vector of** $\Delta \in \mathcal{F}_n$ is defined to be the triple

$$\mathcal{C}_n(\Delta) = (\ell_n(\Delta), V_n(\Delta), t_n(\Delta)).$$

We denote by γ_0 the characteristic vector of $[0, 1]$.

Given a level n net interval Δ_n, we inductively define Δ_{j-1} to be the parent of Δ_j for $j = 1, ..., n$, where $\Delta_0 = [0, 1]$. Each $\Delta_n \in \mathcal{F}_n$ is uniquely identified by the $(n+1)$-tuple $(\gamma_0, \gamma_1, \dots, \gamma_n)$ where $\gamma_j = \mathcal{C}_j(\Delta_j)$. We call $(\gamma_0, \gamma_1, \dots, \gamma_n)$ the **symbolic representation of** Δ.

Similarly, for each $x \in [0, 1]$, the **symbolic representation of** x will be the (infinite) sequence of characteristic vectors $(\mathcal{C}_0(\Delta_0), \mathcal{C}_1(\Delta_1), \dots)$ where $x \in \Delta_n \in \mathcal{F}_n$ for each n and Δ_{j-1} is the parent of Δ_j. The symbolic representation uniquely determines x and is unique unless x is the endpoint of some net interval, in which case there can be two different symbolic representations (and two net intervals of level n containing x). We will write $\Delta_n(x)$ for any net interval of level n containing x. An important fact is that the characteristic vector of a child is uniquely determined by the characteristic vector of the parent, thus we can also speak of the parent/child of characteristic vectors.

By an **admissible path** $(\chi_1, \chi_2, \dots)$ (or path, for short) we mean a (finite or infinite) sequence of characteristic vectors where each χ_{j+1} is the characteristic vector of a child of χ_j. Note that a path need not start with γ_0. We write $|\eta|$ for the length of the finite path η. We remark that if we write $(\chi_1, \chi_2, \dots)$ (finite or infinite) for some characteristic vectors χ_i, it is implied that this is an admissible path. When we write $\sigma|n$ we mean the restriction of the path σ to its first $n + 1$ letters.

Since the characteristic vectors of the children of Δ depend only on the characteristic vector of Δ, we can construct a finite directed graph of characteristic vectors, called the **transition graph**, where we have a directed edge from γ to β if there is a net interval Δ with characteristic vector γ and a child of Δ with characteristic vector β. The paths in the graph are the admissible paths.

Example 2.6 (Bernoulli convolution with contraction factor the inverse of the golden mean). The structure of this IFS was investigated by Feng in [4] and we refer

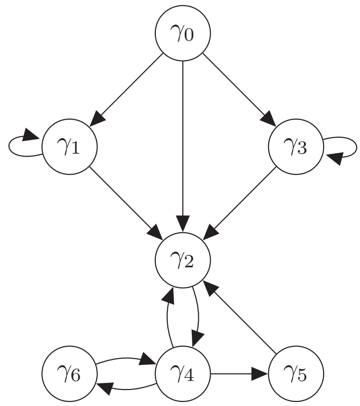

FIG. 2.1. *Bernoulli convolution*

the reader to its transition graph given in Figure 2.1. All paths in the transition graph are admissible paths. Two examples are: $(\gamma_0, \gamma_3, \gamma_2, \gamma_4, \gamma_6, \gamma_4)$ or $(\gamma_4, \gamma_2, \gamma_4, \gamma_5, \gamma_2, \gamma_4)$. The net interval $[0, 1]$ of level 0 has characteristic vector $\gamma_0 = (1, ((0, 1)), 1)$. The set Λ_n consists of the words of length n, thus

$$\{S_\omega(0), S_\omega(1) : \omega \in \Lambda_1\} = \{0, 1 - \varrho, \varrho, 1\},$$

where these are listed in increasing order. Hence there are three net intervals at level 1, the three children of $[0, 1]$, namely

$$[0, 1 - \varrho], [1 - \varrho, \varrho] \text{ and } [\varrho, 1],$$

Consider the level 1 net interval $\Delta = [1 - \varrho, \varrho]$. Its normalized length is $(2\varrho - 1)/\varrho = 1 - \varrho$. We have $S_\sigma(K) \bigcap \mathrm{int}\Delta \neq \emptyset$ for both the words σ of length one. Since $\varrho^{-1}(1-\varrho-S_0(0)) = \varrho$ and $\varrho^{-1}(1-\varrho-S_1(0)) = 0$, $\sigma = 0$ is associated with $(\varrho, 1)$ and Δ, and $\sigma = 1$ is associated with $(0, 1)$ and Δ. Thus the neighbour set of Δ is $((0, 1), (\varrho, 1))$ and its characteristic vector is $\gamma_2 = (1 - \varrho, ((0, 1), (\varrho, 1)), 1)$. The two outer net intervals have characteristic vectors $\gamma_1 = (\varrho, ((0, 1)), 1)$ and $\gamma_3 = (\varrho, ((1 - \varrho, 1)), 1)$.

There are three other characteristic vectors which arise at deeper levels, $\gamma_4 = (\varrho, ((0, 1), (1 - \varrho, 1)), 1)$, (the characteristic vector of the only child of any net interval with characteristic vector γ_2), $\gamma_5 = (2\varrho - 1, ((1 - \varrho, 1)), 1)$ and $\gamma_6 = (1 - \varrho, ((0, 1), (\varrho, 1)), 2)$, the characteristic vectors of the two children of any net interval with characteristic vector γ_4. The net intervals with characteristic vectors γ_1, γ_3 are the left and right-most net intervals respectively, at each level $n \geq 1$, thus the symbolic representation of 0 is the sequence $(\gamma_0, \gamma_1, \gamma_1, ...)$, while $(\gamma_0, \gamma_3, \gamma_3, ...)$ is the symbolic representation of 1.

EXAMPLE 2.7 ($(3, 3)$-Cantor measure). The finite type structure of this IFS was detailed in [9]. Its transition graph is given in Figure 2.2. There are eight characteristic vectors, γ_0 the characteristic vector of $[0, 1]$, and the characteristic vectors of its children, the seven net intervals of level one. The left and right-most intervals at each level have characteristic vectors γ_1 and γ_7 respectively, so the symbolic representations of 0 and 1 are the sequences $(\gamma_0, \gamma_1, \gamma_1, ...)$ and $(\gamma_0, \gamma_7, \gamma_7, ...)$ respectively.

A very important fact is that every IFS of finite type has only finitely many (distinct) characteristic vectors, [10]. We will denote this finite set of characteristic vectors by Ω.

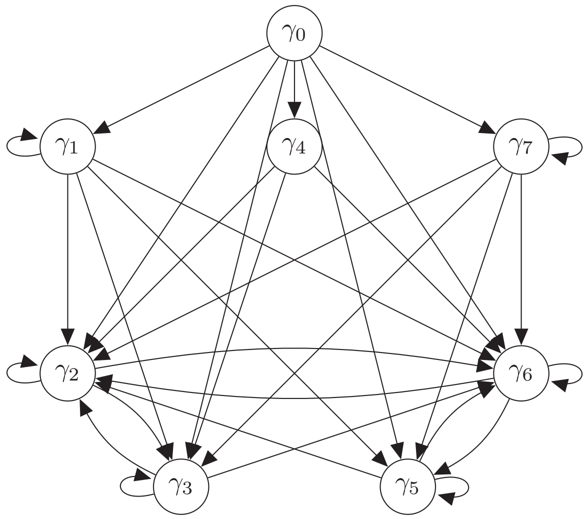

FIG. 2.2. $(3,3)$-*Cantor measure*

Suppose Δ and Δ' are any net intervals at levels n and m respectively, with the same characteristic vector. Assume their (common) neighbour set is $V = ((a_j, L_j))_{j=1}^J$. It is a consequence of the definitions that there is a constant d such that if $\sigma_i \in \Lambda_n$ is associated with (a_i, L_i) and Δ, and $\sigma_i' \in \Lambda_m$ is associated with (a_i, L_i), but with respect to Δ', then $S_{\sigma_i'} = \rho_{\min}^{m-n} S_{\sigma_i} + d$.

As $S_{\sigma_i}(K) \bigcap \operatorname{int}\Delta$ is non-empty, there is some word $\lambda_i \in \Lambda_{N_i}$ such that $S_{\sigma_i \lambda_i}[0,1] \subseteq \Delta$. But then, also, $S_{\sigma_i' \lambda_i}[0,1] \subseteq \Delta'$. Since there are only finitely many characteristic vectors and only finitely many paths of bounded length, this proves Part (i) below. Part (ii) is proved similarly.

LEMMA 2.8. *There is a finite set of words $\mathcal{W}$ and $N \in \mathbb{N}$ with the following properties:*

(i) If $\Delta_n \in \mathcal{F}_n$ and $S_\sigma[0,1] \supseteq \Delta_n$ for some $\sigma \in \Lambda_n$, then there is some $\nu \in \mathcal{W}$ such that $\sigma\nu \in \Lambda_{n+N}$ and $S_{\sigma\nu}[0,1] \subseteq \Delta_n$.

(ii) If $\Delta_{n+N} \subseteq \Delta_n$ belong to $\mathcal{F}_{n+N}$ and $\mathcal{F}_n$ respectively, and $S_\sigma[0,1] \supseteq \Delta_n$ for $\sigma \in \Lambda_n$, then there is some $v \in \mathcal{W}$ such that $\sigma v \in \Lambda_{n+N}$ and $S_{\sigma v}[0,1] \bigcap \Delta_{n+N}$ is empty.

2.3. Local dimensions and transition matrices.

DEFINITION 2.9. Given a probability measure μ on $\mathbb{R}$, by the **lower local dimension** of μ at $x \in \operatorname{supp}\mu$, we mean the number

$$\underline{\dim}_{\mathrm{loc}}\mu(x) = \liminf_{r \to 0^+} \frac{\log \mu(x-r, x+r)}{\log r}.$$

Replacing the lim inf by lim sup gives the upper local dimension, $\overline{\dim}_{\mathrm{loc}}\mu(x)$, and if these two are equal, the common value is the local dimension of μ at x, denoted $\dim_{\mathrm{loc}} \mu(x)$.

If μ is a measure of finite type, the local dimensions of μ can be expressed in terms of measures of net intervals. Indeed, as all net intervals of level n have lengths comparable to $\rho_{\min}^n$, it follows that

$$\begin{aligned}\dim_{\mathrm{loc}} \mu(x) &= \lim_{n\to\infty} \frac{\log(\mu(\Delta_n(x)) + \mu(\Delta_n^+(x)) + \mu(\Delta_n^-(x)))}{n \log \rho_{\min}} \qquad (2.3)\\ &\le \lim_{n\to\infty} \frac{\log(\mu(\Delta_n(x))}{n\log\rho_{\min}},\end{aligned}$$

where $\Delta_n^+(x), \Delta_n^-(x)$ are the adjacent, n'th level net intervals on each side of $\Delta_n(x)$. A similar statement holds for the upper and lower local dimensions.

Definition 2.10. Let μ be a measure of finite type with the notation as in the previous subsection. Let $\Delta = [a, b] \in \mathcal{F}_n$ and $\widehat{\Delta} = [c, d] \in \mathcal{F}_{n-1}$ be its parent. Assume $V_n(\Delta) = ((a_j, L_j))_{j=1}^J$ and $V_{n-1}(\widehat{\Delta}) = ((c_i, M_i))_{i=1}^I$. The **primitive transition matrix**, $T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta))$, is the $I \times J$ matrix (T_{ij}) which encapuslates information about the relationship between the $(c_i, M_i) \in V_{n-1}(\widehat{\Delta})$ and $(a_j, L_j) \in V_n(\Delta)$. To be precise, let $\sigma_i \in \Lambda_{n-1}$ be such that $\rho_{\min}^{-n+1}(c - S_{\sigma_i}(0)) = c_i$ and $\rho_{\min}^{-n+1}\rho_{\sigma_i} = M_i$. Let $\mathcal{T}_{i,j}$ be the set all ω such that $\sigma_i\omega \in \Lambda_n$, $\rho_{\min}^{-n}(a - S_{\sigma_i\omega}(0)) = a_j$ and $\rho_{\min}^{-n}\rho_{\sigma_i\omega} = L_j$. Notice that $\mathcal{T}_{i,j}$ depends only S_{σ_i} (equivalently, on c_i and M_i) and j, and not on the choice of σ_i. We define

$$T_{i,j} = \sum_{\omega\in\mathcal{T}_{i,j}} p_\omega$$

where the empty sum is taken to be 0. Given a path $(\gamma_1, \dots, \gamma_n)$, we let

$$T(\gamma_1, \dots, \gamma_n) = T(\gamma_1, \gamma_2)T(\gamma_2, \gamma_3)\cdots T(\gamma_{n-1}, \gamma_n).$$

We call any such product a **transition matrix**.

As explained in [3, 10], there are positive constants c_1, c_2 such that whenever $\Delta_n \in \mathcal{F}_n$ has symbolic representation $(\gamma_0, \gamma_1, \dots, \gamma_n)$, then

$$c_1\mu(\Delta_n) \le \|T(\gamma_0, \gamma_1, \dots, \gamma_n)\| \le c_2\mu(\Delta_n).$$

We say that $\mu(\Delta_n)$ and $\|T(\gamma_0, \gamma_1, \dots, \gamma_n)\|$ are **comparable**.

An important fact about transition matrices is that each column of any transition matrix contains a non-zero entry. Here are two useful consequences of this. The proofs are left as an exercise and follow from the definition of the matrix norm. Note that we say a matrix is **positive** if all its entries are strictly positive.

Lemma 2.11. *Let A, B, C be transition matrices with B positive.*

(i) There are positive constants a_1, a_2, depending only on A, such that

$$a_1 \|C\| \le \|AC\| \le \|A\| \|C\| \le a_2 \|C\|.$$

(ii) There is a constant $b = b(B) > 0$ (independent of A, C) such that

$$\|ABC\| \ge b \|A\| \|C\|.$$

3. Local dimensional behaviour on Loop classes. From here on, unless we say otherwise μ will be a self-similar measure arising from an IFS of finite type, with the notation as in the previous section.

3.1. Loop classes and the Essential class. A non-empty subset L of the set of characteristic vectors Ω is called a **loop class** if whenever $\alpha, \beta \in L$, then there is an admissible path $(\gamma_1, \dots, \gamma_J)$ of characteristic vectors $\gamma_i \in L$ such that $\alpha = \gamma_1$ and $\beta = \gamma_J$. The loop classes are the cycles in the transition graph.

A loop class L is called an **essential class** if, in addition, whenever $\alpha \in L$ and $\beta \in \Omega$ is a child of α, then $\beta \in L$. It was shown in [5, 10] that there is always a unique essential class which we will denote by E. A loop class is called **maximal** if it is not properly contained in any other loop class.

Given a set of characteristic vectors L containing a loop class, we will say that the infinite path σ belongs to L_e, and write $\sigma \in L_e$, if $\sigma = (\sigma_1, \sigma_2, \dots)$ with $\sigma_j \in L$ eventually, i.e., there exists an index j_0 such that $\sigma_j \in L$ for all $j \geq j_0$. Of course, every infinite path will belong to some maximal loop class L eventually. We will say that the (finite or infinite) path $\sigma \in L_a$ if all the letters of σ belong to L. If the path σ begins with γ_0, the characteristic vector of $[0, 1]$, we will write $\sigma \in L_e^0$ or L_a^0, appropriately.

If x has a symbolic representation $\sigma \in L_e^0$, we will say that $x \in K_L$. Of course, x can belong to both K_{L_1} and K_{L_2} for different maximal loop classes L_1, L_2 only if x is a boundary point of a net interval with symbolic representations coming from both L_1 and L_2. When $x \in K_E$ we will say x is an **essential point**.

Example 3.1 (Bernoulli convolution with contraction factor the inverse of the golden mean). It can be seen from the transition graph, Figure 2.1, that the maximal loop classes are the singletons $\{\gamma_1\}$ and $\{\gamma_3\}$ and the essential class $E = \{\gamma_2, \gamma_4, \gamma_5, \gamma_6\}$.

An example of an infinite path in E_e^0 is given by $(\gamma_0, \gamma_2, \gamma_4, \gamma_6, \gamma_4, \gamma_6, ...)$. The path $(\gamma_2, \gamma_4, \gamma_5, \gamma_2, \gamma_4, \gamma_5, ...)$ is in E_a and hence in E_e, but not in E_e^0. Note also that L_a^0 is the empty set for all loop classes L, as γ_0 does not belong to any loop class.

Since the only net intervals at level n whose characteristic vectors are not in the essential class are the intervals $[0, \varrho^{n-1}(1-\varrho)]$ and $[1-\varrho^{n-1}(1-\varrho), 1]$, it follows that $(0, 1)$ is the set of essential points. The two singleton loop classes correspond to the endpoints 0 and 1.

Example 3.2 ($(3, 3)$-Cantor measure). One can see from the transition graph, Figure 2.2, that the maximal loop classes are the singletons $\{\gamma_1\}$, $\{\gamma_7\}$, corresponding to the points $0, 1$ respectively, and the essential class $\{\gamma_2, \gamma_3, \gamma_5, \gamma_6\}$, with the set of essential points again being $(0, 1)$.

3.2. Local dimensional behaviour of paths. Motivated by the notion of the local dimension at a point, we introduce a related notion for paths.

Notation 3.3. *Given any infinite path σ, put*

$$\underline{d}(\sigma) = \liminf_{n \to \infty} \frac{\log \|T(\sigma|n)\|}{n \log \rho_{\min}} \text{ and } \overline{d}(\sigma) = \limsup_{n \to \infty} \frac{\log \|T(\sigma|n)\|}{n \log \rho_{\min}}.$$

We write $d(\sigma)$ if $\underline{d}(\sigma) = \overline{d}(\sigma)$.

Let L be any non-empty subset of Ω containing a loop class and set

$$d_{\min}^L = \inf_{\sigma \in L_e^0} \underline{d}(\sigma) \text{ and } d_{\max}^L = \sup_{\sigma \in L_e^0} \overline{d}(\sigma).$$

Since $\|T(\eta,\sigma)\|$ and $\|T(\sigma)\|$ are comparable whenever η, σ are finite paths, with constants of comparability depending only on η (see Lemma 2.11), $\underline{d}(\sigma) = \underline{d}(\sigma')$ where σ' omits the initial segment of σ that contains the letters not in L. A similar statement holds for $\overline{d}$. Thus

$$d_{\min}^L = \inf_{\sigma \in L_a} \underline{d}(\sigma) \text{ and } d_{\max}^L = \sup_{\sigma \in L_a} \overline{d}(\sigma).$$

Lemma 3.4. *We have*

$$0 < d_{\min}^{\Omega} = \min_L d_{\min}^L \text{ and } d_{\max}^{\Omega} = \max_L d_{\max}^L < \infty$$

where the minimum and maximum are over all maximal loop classes L.

Proof. Lemma 2.8 (ii) implies that there is an index N and a finite set of words $\mathcal{W}$ such that if ν is associated with an element of the neighbour set of $\Delta_n(x)$, then there is some $\omega \in \mathcal{W}$ such that $\nu\omega$ is not associated with any element of the neighbour set of $\Delta_{n+N}(x)$. Thus, if $\sigma = (\gamma_1, \dots, \gamma_{N+1})$ is any path of length $N+1$, then the sum of each row of the transition matrix $T(\sigma)$ is at most $1-\varepsilon$, where $\varepsilon = \min\{p_\omega : \omega \in \mathcal{W}\}$. (Think of γ_1 as the characteristic vector of some $\Delta_n(x)$ and γ_{N+1} as the characteristic vector of its descendent $\Delta_{n+N}(x)$).

If we let $\|T\|_r = \max_i \sum_j |T_{ij}|$ be the maximum row sum norm, then one can easily verify that $\|T_1 T_2\|_r \leq \|T_1\|_r \|T_2\|_r$. Furthermore, $\|T\| \leq C \|T\|_r$ where C is a bound on the number of rows of matrix T. If $\sigma = (\gamma_1, \gamma_2, \dots)$ is any infinite path, then we can factor $T(\sigma|n)$ as $T(\eta_1) \cdots T(\eta_J) T(\lambda)$ where $\eta_i = (\gamma_{(i-1)N+1}, \dots, \gamma_{iN+1})$ are paths of length $N+1$, $J = \left[\frac{n-1}{N}\right]$ and $\lambda = (\gamma_{JN+1}, \dots, \gamma_n)$ is a path of length $\leq N$. As each $\|T(\eta_i)\|_r \leq 1-\varepsilon$, and there are only finitely many paths of length at most N, the submultiplicativity of the r-norm implies

$$\|T(\sigma|n)\| \leq C \|T(\sigma|n)\|_r \leq C(1-\varepsilon)^J \max_{|\lambda| \leq N} \|T(\lambda)\| \leq C'(1-\varepsilon)^{\left[\frac{n-1}{N}\right]}$$

for a suitable constant C'. Hence

$$\underline{d}(\sigma) = \liminf_{n\to\infty} \frac{\log \|T(\sigma|n)\|}{n \log \rho_{\min}} \geq \frac{\log(1-\varepsilon)}{N \log \rho_{\min}} > 0 \text{ for all } \sigma. \tag{3.1}$$

On the other hand, if N is the integer of Lemma 2.8 (i), then any $\Delta \in \mathcal{F}_n$ contains $S_\nu[0,1]$ for some $\nu \in \Lambda_{n+N}$. Hence

$$\mu(\Delta) \geq (\min p_j)^{s(n+N)}, \tag{3.2}$$

where s is chosen such that any word in Λ_k is of length at most sk.

Equivalently, there is a constant $C > 0$ such that $\|T(\sigma|n)\| \geq C(\min p_j)^{s(n+N)}$ for all infinite paths σ. Thus

$$\overline{d}(\sigma) = \limsup_{n\to\infty} \frac{\log \|T(\sigma|n)\|}{n \log \rho_{\min}} \leq \frac{s \log(\min p_j)}{\log \rho_{\min}} < \infty. \tag{3.3}$$

The bounds (3.1) and (3.3) obviously imply $d_{\min}^L$ and $d_{\max}^L$ are bounded above and below from 0. Since every infinite path σ belongs to L eventually for a unique choice of maximal loop class L the proof is complete. □

3.3. Relationships between $d_{\min}^L, d_{\max}^L$ and local dimensions. If $x \in K_L$ has symbolic representation $\sigma \in L_e^0$ and $\Delta_n(x)$ has symbolic representation $\sigma|n$, then the comparability of $\mu(\Delta_n)$ and $\|T(\sigma|n)\|$ when $\sigma|n$ is the symbolic representation of Δ_n, together with (2.3), shows

$$\underline{d}(\sigma) = \liminf_{n\to\infty} \frac{\log \mu(\Delta_n(x))}{n \log \rho_{\min}} \geq \underline{\dim}_{\mathrm{loc}}\mu(x). \tag{3.4}$$

Similarly,

$$\overline{\dim}_{\mathrm{loc}}\mu(x) \leq \overline{d}(\sigma).$$

In particular, if L contains a loop class, then

$$\overline{\dim}_{\mathrm{loc}}\mu(x) \leq d_{\max}^L \text{ for all } x \in K_L. \tag{3.5}$$

Definition 3.5. We will say that an infinite path σ is a **periodic path with period** θ if $\sigma = (\eta, \theta^-, \theta^-, \dots)$ for some initial finite path η and **cycle** $\theta = (\theta_1, \dots, \theta_k, \theta_1)$. We call x a **periodic point** if it has a periodic symbolic representation.

An example of a periodic point is the boundary point of a net interval.

Notation 3.6. *Denote by* $\mathrm{sp}(M)$ *the spectral radius of the square matrix* M.

Example 3.7. If σ is a periodic path with period θ, then

$$d(\sigma) = \underline{d}(\sigma) = \overline{d}(\sigma) = \lim_{k\to\infty} \frac{\log \left\|(T(\theta))^k\right\|}{|\theta^-|\, k \log \rho_{\min}} = \frac{\log \mathrm{sp}(T(\theta))}{|\theta^-| \log \rho_{\min}}. \tag{3.6}$$

Similarly, if x is a periodic point with a unique periodic symbolic representation σ, then $\dim_{\mathrm{loc}} \mu(x) = d(\sigma)$. If x has two different symbolic representations, σ, τ, then these are both necessarily periodic and

$$\dim_{\mathrm{loc}} \mu(x) = \min(d(\sigma), d(\tau)).$$

See [9, Prop. 2.7] for details. In Example 3.8 we show that it is possible to have $d(\sigma) \neq d(\tau)$.

An equicontractive self-similar measure is said to be **regular** if $p_0 = p_m = \min p_j$ where the S_j are ordered so that $d_0 < d_1 < \cdots < d_m$. It was shown in [4, Thm. 3.2] (see also [8, Cor. 3.7]) that if μ is a regular, finite type measure, then the μ-measures of adjacent net intervals are comparable and consequently (2.3) implies

$$\underline{\dim}_{\mathrm{loc}}\mu(x) = \liminf_{n\to\infty} \frac{\log(\mu(\Delta_n(x))}{n \log \rho_{\min}} = \liminf_{n\to\infty} \frac{\log \|T(\sigma|n)\|}{n \log \rho_{\min}} = \underline{d}(\sigma) \tag{3.7}$$

when x has symbolic representation σ, and similarly for the (upper) local dimension. Consequently, under the regularity assumption

$$d_{\min}^L = \inf\{\underline{\dim}_{\mathrm{loc}}\mu(x) : x \in K_L\}, \tag{3.8}$$

and similarly for $d_{\max}^L$ and the upper local dimensions. But without this assumption, these statements need not be true. Here is one example.

EXAMPLE 3.8. Consider the IFS with contractions $S_j(x) = x/3 + d_j$ for $d_j = 0, 1/9, 1/3, 1/2, 2/3$ and probabilities $p_j = 4/17$ for $j = 0, 1, 3, 4$ and $p_2 = 1/17$. This IFS is of finite type and has 19 characteristic vectors.

In particular, $L = \{\gamma_4\}$ is a singleton maximal loop class with $K_L = \{1/2\}$. In this case, the only infinite word in L_e^0 is $\sigma := (\gamma_0, \gamma_4, \gamma_4, \gamma_4, \dots)$. It can be checked that

$$d(\sigma) = \frac{\log \operatorname{sp}(T(\gamma_4, \gamma_4))}{\log 3} = \frac{\log 17}{\log 3}.$$

But $1/2$ is a boundary point of a net interval and has a second symbolic representation, $\tau = (\gamma_0, \gamma_5, \gamma_{12}, \gamma_{12}, \gamma_{12} \dots)$, with $\tau \notin L_e^0$. As $d(\tau) = \log(17/4)/\log 3$, we have

$$\dim_{\mathrm{loc}} \mu(1/2) = \min(d(\tau), d(\sigma)) = d(\tau),$$

so

$$\inf_{x \in K_L} \{\underline{\dim}_{\mathrm{loc}} \mu(x)\} = \sup_{x \in K_L} \{\overline{\dim}_{\mathrm{loc}} \mu(x)\} = \dim_{\mathrm{loc}} \mu(1/2) < d(\sigma) = d_{\min}^L = d_{\max}^L.$$

We refer the reader to Example 5.13 for more details.

More can be said about the relationship between $d_{\min}^L, d_{\max}^L$ and local dimensions, but first it is useful to establish that the convergence to the limiting local behaviour is 'uniform' over $\sigma \in L_a$.

LEMMA 3.9. *Let L be any set of characteristic vectors containing a loop class. For each $\varepsilon > 0$ there is an integer k_0 such that if $\sigma \in L_a$ and $|\sigma| \geq k \geq k_0$, then*

$$\frac{\log \|T(\sigma|k)\|}{k \log \rho_{\min}} \geq d_{\min}^L - \varepsilon,$$

equivalently,

$$\sup_{\sigma \in L_a} \|T(\sigma|k)\|^{1/k} \leq \rho_{\min}^{d_{\min}^L - \varepsilon}.$$

Proof. Fix $s < d_{\min}^L$ and let $\mathcal{T}$ be the following set of transition matrices:

$$\mathcal{T} = \{T(\sigma) : \sigma = (\sigma_1, \dots, \sigma_n) \in L_a, \|T(\sigma_1, \dots, \sigma_j)\|^{1/j} > \rho_{\min}^s \text{ for } j = 1, \dots, n-1$$

$$\text{and } \|T(\sigma_1, \dots, \sigma_n)\|^{1/n} \leq \rho_{\min}^s\}.$$

If $\mathcal{T}$ is an infinite set, as there are only finitely many characteristic vectors, there must be infinitely many $T(\sigma) \in \mathcal{T}$ with all σ having the same first letter, say σ_1. Among these infinitely many $T(\sigma)$, there must be infinitely many σ all having the same second letter as well, say σ_2. Repeating this process, we create an infinite path $\sigma = (\sigma_1, \sigma_2, \dots) \in L_a$ with $\|T(\sigma_1, \dots, \sigma_k)\|^{1/k} > \rho_{\min}^s$ for every k. But then $\underline{d}(\sigma) \leq s < d_{\min}^L$ and that is a contradiction. Consequently, $\mathcal{T}$ is finite.

Let N be the maximal length of any α with $T(\alpha) \in \mathcal{T}$. Given any finite path $\sigma \in L_a$, we can factor $T(\sigma)$ as a product $\left(\prod_{j=1}^{J} T(\eta_j)\right) T(\alpha)$ where $T(\eta_j) \in \mathcal{T}$ and

$|\alpha| \leq N$. There are only finitely many possible choices for $T(\alpha)$, hence there is a constant C, independent of σ, such that

$$\|T(\sigma)\| \leq C \prod_{j=1}^{J} \|T(\eta_j)\|.$$

As $\|T(\eta_j)\| \leq \rho_{\min}^{s|\eta_j|}$ and $|\alpha| \leq N$, taking $C_1 = C\rho_{\min}^{-sN}$ gives

$$\|T(\sigma)\| \leq C\rho_{\min}^{s\sum|\eta_j|} = C\rho_{\min}^{s(|\sigma|-|\alpha|)} \leq C_1\rho_{\min}^{s|\sigma|}.$$

Hence for any $\varepsilon > 0$,

$$\|T(\sigma|k)\| \leq C_1\rho_{\min}^{sk} \leq \rho_{\min}^{k(s-\varepsilon)}$$

for sufficiently large k. As $s < d_{\min}^L$ and $\varepsilon > 0$ are arbitrary choices, this completes the proof. □

Next, we see that a similar result holds for $d_{\max}^L$ if we assume L is a loop class.

LEMMA 3.10. *Suppose L is a loop class. For each $\varepsilon > 0$ there is an integer k_0 such that if $\sigma \in L_a$ and $|\sigma| \geq k \geq k_0$, then*

$$\frac{\log \|T(\sigma|k)\|}{k \log \rho_{\min}} \leq d_{\max}^L + \varepsilon,$$

equivalently,

$$\sup_{\sigma \in L_a} \|T(\sigma|k)\|^{1/k} \geq \rho_{\min}^{(d_{\max}^L + \varepsilon)}.$$

Proof. The proof is quite different. We will use the fact that as L is a loop class, there is a finite set of paths $\mathcal{S}$ in L_a with the property that given any finite path $\sigma \in L_a$, there is some path $\beta \in \mathcal{S}$ so that the path (σ^-, β) is a cycle. Let $C = \max_{\beta \in \mathcal{S}} \|T(\beta)\|$.

Given $\sigma \in L_a$, put $\sigma_k = (\sigma|k)^-$. Pick $\beta_k \in \mathcal{S}$ so that $(\sigma_k, \beta_k) = \theta_k$ is a cycle. For all positive integers n we have

$$\|(T(\sigma_k, \beta_k))^n\|^{1/n} \leq \|T(\sigma_k, \beta_k)\| \leq \|T(\sigma|k)\| \, \|T(\beta_k)\| \leq C \, \|T(\sigma|k)\|$$

and hence

$$\frac{\log \|T(\sigma|k)\|}{k \log \rho_{\min}} \leq \frac{\log 1/C + \frac{1}{n} \log \|(T(\sigma_k, \beta_k))^n\|}{k \log \rho_{\min}}.$$

Let $\omega_k \in L_e^0$ be an infinite periodic path with period θ_k. Then

$$d_{\max}^L \geq d(\omega_k) = \frac{\log \operatorname{sp}(T(\theta_k))}{(k + |\beta_k|) \log \rho_{\min}} = \lim_n \frac{\log \|(T(\sigma_k, \beta_k))^n\|}{n(k + |\beta_k|) \log \rho_{\min}}.$$

Since $\max_{\beta \in \mathcal{S}} |\beta| < \infty$, given $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon)$ such that for all $k \geq k_0$,

$$\begin{aligned} \frac{\log \|T(\sigma|k)\|}{k \log \rho_{\min}} &\leq \frac{\log 1/C + \log \operatorname{sp}(T(\theta_k))}{k \log \rho_{\min}} \\ &\leq \frac{\log 1/C}{k \log \rho_{\min}} + d_{\max}^L \left(\frac{k + |\beta_k|}{k} \right) \leq d_{\max}^L + \varepsilon. \quad \square \end{aligned}$$

We remind the reader that in (3.4) we observed that $\underline{\dim}_{\text{loc}}\mu(x)(x) \le \underline{d}(\sigma)$ whenever σ is a symbolic representation of x and thus $\inf_{x\in K_L}\{\underline{\dim}_{\text{loc}}\mu(x)\} \le d^L_{\min}$. More can be said, particularly when K_L is relatively open.

PROPOSITION 3.11. *Suppose L is a set of vectors containing a loop class. If x belongs to the relative interior of K_L, then $\underline{\dim}_{\text{loc}}\mu(x) \ge d^L_{\min}$. Moreover, if K_L is open (in the relative topology on K), then*

$$d^L_{\min} = \inf\{\underline{\dim}_{\text{loc}}\mu(x) : x \in K_L\}.$$

REMARK 3.12. Note that if K_L is open, then we necessarily have that $E \subset L$.

Proof of Prop. 3.11. First, assume $x \in \text{int}K_L$. Choose N so large that $B(x, 2\rho^N_{\min}) \cap K \subseteq K_L$ and let $n \ge N$. Then, for any $n \ge N$, we have $\Delta_n(x)$ and the adjacent level n net intervals, $\Delta^+_n(x)$ and $\Delta^-_n(x)$, are contained in $B(x, 2\rho^N_{\min})$ and thus have symbolic representations of the form $\sigma|n = (\eta^-, \lambda)$ where $|\eta| \le N$ and $\lambda \in L_a$.

Let $C_1 = \max \|T(\eta)\|$ over the finitely many paths η of length at most N. Lemma 3.9 guarantees that for $n \ge n(\varepsilon)$,

$$\|T(\sigma|n)\| \le \|T(\eta)\| \, \|T(\lambda)\| \le C_1 \rho_{\min}^{n(d^L_{\min}-\varepsilon)}$$

for all σ of this form. Combined with (2.3), this gives

$$\begin{aligned}\underline{\dim}_{\text{loc}}\mu(x) &= \liminf_{r\to 0} \frac{\log\left(\mu(\Delta_n(x)) + \mu(\Delta^+_n(x)) + \mu(\Delta^-_n(x))\right)}{n\log\rho_{\min}} \\ &\ge \liminf_{n\to\infty} \frac{\log \rho_{\min}^{n(d^L_{\min}-\varepsilon)}}{n\log\rho_{\min}} \ge d^L_{\min} - \varepsilon.\end{aligned}$$

Now assume that K_L is open. Given any $\varepsilon > 0$, choose $\sigma \in L^0_e$ with $\underline{d}(\sigma) \le d^L_{\min}+\varepsilon$ and take $x \in K_L$ with symbolic representation σ. Then

$$d^L_{\min} + \varepsilon \ge \underline{d}(\sigma) \ge \underline{\dim}_{\text{loc}}\mu(x) \ge d^L_{\min}$$

and that proves $d^L_{\min} = \inf_{x\in K_L}\{\underline{\dim}_{\text{loc}}\mu(x)\}$. □

COROLLARY 3.13. $d^\Omega_{\min} = \inf\{\underline{\dim}_{\text{loc}}\mu(x) : x \in K\}$.

Proof. This is the special case of $L = \Omega$. □

Let L be a set of characteristic vectors containing a loop class. We will call a finite path $\eta = (\gamma_1, \ldots, \gamma_n) \in L_a$ (with $n > 1$) a boundary path if all γ_i, for $i > 1$, are left-most children of γ_{i-1}, or all are right-most children. We call η an **interior path** if it is not a boundary path. If L is a loop class and each characteristic vector in L has a unique child in L, then all paths in L will be simple cycles. We will call such a loop class **simple**. If L does not admit an interior path, then it is necessarily simple (although the converse is not necessarily true).

PROPOSITION 3.14. *Suppose L is a loop class.*
(i) If L admits an interior path, then

$$d^L_{\max} = \sup\{\overline{\dim}_{\text{loc}}\mu(x) : x \in K_L\} = \sup\{\dim_{\text{loc}}\mu(x) : x \in K_L\}. \tag{3.9}$$

(ii) Otherwise, L is simple and in this case

$$d_{\max}^{L} = \frac{\log \operatorname{sp}(T(\theta(L)))}{|L| \log \rho_{\min}}$$

(with the notation $\theta(L)$ introduced above).

Proof. Part (i). Any loop class which admits an interior path has interior paths (in the loop class), which join any two members of the class. For each pair $\alpha, \beta \in L$, pick one such interior path in L and call this finite set of interior paths $\mathcal{P}$.

Fix $\varepsilon > 0$ and choose $\sigma \in L_a$ such that $\overline{d}(\sigma) \geq d_{\max}^{L} - \varepsilon/2$. Then select a subsequence (n_k) such that $\|T(\sigma|n_k)\| \leq \rho_{\min}^{n_k(d-\varepsilon)}$ where $d = d_{\max}^{L}$. Let $\sigma_k = \sigma|n_k$ and choose a path $\lambda_k \in \mathcal{P}$ such that $\theta_k = (\sigma_k^-, \lambda_k)$ is a cycle. Let x be a periodic point in K_L with symbolic representation having period θ_k.

As λ_k is an interior path, this symbolic representation of x is unique. As per Example 3.7, the local dimension at x exists and is given by

$$\dim_{\mathrm{loc}} \mu(x) = \frac{\log \operatorname{sp}(T(\theta_k))}{|\theta_k^-| \log \rho_{\min}}.$$

For any n, the submultiplicativity of the norm implies

$$\|(T(\theta_k))^n\|^{1/n} \leq \|T(\sigma|n_k)\| \, \|T(\lambda_k)\| \leq C\rho_{\min}^{n_k(d-\varepsilon)}$$

where $C = \max_{\lambda \in \mathcal{P}} \|T(\lambda)\|$. Thus if n_k is sufficiently large, then

$$\frac{\log \operatorname{sp}(T(\theta_k))}{|\theta_k^-| \log \rho_{\min}} = \lim_{n\to\infty} \frac{\log \|(T(\theta_k))^n\|^{1/n}}{(n_k + |\lambda_k|) \log \rho_{\min}} \geq \frac{\log C\rho_{\min}^{n_k(d-\varepsilon)}}{n_k \log \rho_{\min}} \geq d - 2\varepsilon.$$

As $\varepsilon > 0$ was arbitrary, it follows that $\sup_{x\in K_L}\{\dim_{\mathrm{loc}} \mu(x)\} \geq d$. Since we previously saw that $\sup_{x\in K_L}\{\overline{\dim}_{\mathrm{loc}}\mu(x)\} \leq d_{\max}^{L}$ when L is a loop class, this is enough to establish both equalities in (3.9).

Part (ii). We have already noted that if L does not admit any interior paths, then the transition graph is the simple cycle $\theta(L)$. Thus any $\sigma \in L_e^0$ is periodic with period $\theta(L)$ and $d(\sigma) = \log spT(\theta(L))/\,|L| \log \rho_{\min} = d_{\max}^{L}$. □

Here are some immediate corollaries.

COROLLARY 3.15. *Let E be the essential class. Then $d_{\max}^{E} = \sup\{\dim_{\mathrm{loc}} \mu(x) : x \in K_E\}$.*

REMARK 3.16. We note that if L is a loop class and K_L is relatively open in K then $L = E$.

Proof of Cor. 3.15. The essential class admits an interior path since every child of a characteristic vector in E is again in E. □

Another consequence of Proposition 3.14 is that

$$d_{\max}^{\Omega} = \max\left\{\sup_{x\in K}\{\dim_{\mathrm{loc}} \mu(x)\}, \max_{L \text{ simple, max loop class}} \left\{\frac{\log \operatorname{sp}(T(\theta(L)))}{|L| \log \rho_{\min}}\right\}\right\}.$$

However, $d_{\max}^{\Omega} > \sup_{x\in K}\{\dim_{\mathrm{loc}} \mu(x)\}$ is possible, as the next example illustrates.

EXAMPLE 3.17. Consider, again, the IFS of Example 3.8. We observed there that $d^L_{\max} = \log 17/\log 3$ for $L = \{\gamma_4\}$. For each of the other singleton maximal loop classes L, one can compute that $d^L_{\underline{\max}} = \log(17/4)/\log 3$, so $d^\Omega_{\max} = \log 17/\log 3$. With more work it can be shown that $\overline{\dim}_{\text{loc}}\mu(x) \leq \log(17/2)/\log 3$ for all $x \in [0,1]$. See Example 5.13 for more details.

4. Loop class spectrum.

The L^q-spectrum can be defined for any measure.

DEFINITION 4.1. The L^q-**spectrum** of the measure μ is defined to be the function $\tau(\mu, q)$ defined on $\mathbb{R}$ by

$$\tau(\mu, q) = \liminf_{\delta\to 0} \frac{\log \sup \sum_i (\mu(B(x_i, \delta)))^q}{\log \delta},$$

where the supremum is over all countable collections of disjoint open balls, $B(x_i, \delta)$, with centres $x_i \in \text{supp}\mu$. We write $\tau(q)$ if the measure μ is clear.

REMARK 4.2. This is also known in the literature as the lower L^q-spectrum. The upper L^q-spectrum (and L^q-spectrum) could be similarly defined, but as these will not be of interest to us, we will refer to τ as the L^q-spectrum, for short.

As proven in [12, Section 3], for any measure μ, we have $\tau(\mu, \cdot)$ is an increasing concave function. Further it does not take on either of the values $\pm\infty$ if and only if

$$\liminf_{\delta\to 0} \frac{\log(\inf_{x\in\text{supp}\mu} \mu(B(x, \delta)))}{\log \delta} < \infty. \tag{4.1}$$

We show in Prop. 4.3 that this is indeed the case for measures of finite type. For finite type measures μ, balls centred in the support of μ can be replaced by net intervals in the definition of the L^q-spectrum. This was shown by Feng for equicontractive finite type measures, [3, Prop. 5.6]. The proof for the general case is similar and is included here for completeness.

PROPOSITION 4.3. *Suppose μ is a measure of finite type. The L^q-spectrum of μ can be computed as*

$$\tau(\mu, q) = \liminf_{n\to\infty} \frac{\log \sum_{\Delta\in\mathcal{F}_n} (\mu(\Delta))^q}{n \log \rho_{\min}} = \liminf_{n\to\infty} \frac{\log \sum_{|\sigma|=n, \sigma\in\Omega^0_a} \|T(\sigma)\|^q}{n \log \rho_{\min}}. \tag{4.2}$$

Moreover, $\tau(\mu, q)$ is real-valued for all q.

Proof. We first remark that as $\mu(\Delta)$ is comparable to $\|T(\sigma)\|$ when Δ has symbolic representation σ, the second equality of the display is clear.

Given $0 < \delta < \rho_{\min}$, choose the integer n such that $\rho^n_{\min} \leq \delta < \rho^{n-1}_{\min}$. Then, for any $x \in \text{supp}\mu$, $B(x, \delta)$ contains $\Delta_n(x)$. Applying (3.2), it follows that there are positive constants C, s such that $\mu(B(x, \delta)) \geq \mu(\Delta_n) \geq C (\min p_j)^{sn}$. That certainly implies the left side of (4.1) is finite and hence $\tau(q)$ is real valued.

Furthermore, if $q < 0$, then

$$(\mu(B(x, \delta)))^q \leq \mu(\Delta_n)^q,$$

while if $q \geq 0$, then

$$(\mu(B(x, \delta)))^q \leq \left(\sum_{\substack{\text{level } n \text{ net intervals } \Delta_n \\ \text{intersecting } B(x,\delta)}} \mu(\Delta_n) \right)^q.$$

Moreover, the fact that $\delta < \rho_{\min}^{n-1}$ ensures that $B(x,\delta)$ is contained in the union of the at most $2/(c\rho_{\min}) = C_0$ net intervals of level n that intersect it. Since each net interval of level n can intersect at most two balls of radius δ, we have

$$\sum_i (\mu(B(x_i,\delta)))^q \leq \begin{cases} 2C_0^q \sum_{\Delta\in\mathcal{F}_n} \mu(\Delta)^q & \text{if } q \geq 0 \\ \sum_{\Delta\in\mathcal{F}_n} \mu(\Delta)^q & \text{if } q < 0 \end{cases}. \tag{4.3}$$

It is easy to see from (4.3) that for either choice of q,

$$\tau(q) \geq \liminf_{n\to\infty} \frac{\log \sum_{\Delta\in\mathcal{F}_n} (\mu(\Delta))^q}{n \log \rho_{\min}}.$$

For the other inequality, we argue as follows. Choose N_0 such that $\rho_{\min}^{N_0} \leq c/3$. Consider any level n net interval $\Delta = [a,b]$ and its neighbour set $((a_j, L_j))_{j=1}^J$. For each j, choose $\sigma_j \in \Lambda_n$ associated with (a_j, L_j) and Δ. Select the index $i \in \{1,\ldots,J\}$ so that $\sum_{\alpha:S_\alpha=S_{\sigma_j}} p_\alpha$ is maximal for $j = i$.

Obtain an integer $N \geq N_0$ and finite set $\mathcal{W}$, as in Lemma 2.8 (i), with the property that there is some $\nu \in \mathcal{W}$ with $S_{\sigma_i\nu}[0,1] \subseteq \Delta$ and $\sigma_i\nu \in \Lambda_{n+N}$. As the length of this interval satisfies

$$\rho_{\min}^{n+N+1} \leq \ell(S_{\sigma_i\nu}[0,1]) \leq \operatorname{length}(\Delta)/3,$$

one of the two endpoints of the interval $S_{\sigma_i\nu}[0,1]$, say x_Δ, has distance at least $\ell(S_{\sigma_i\nu}[0,1])$ to both a and b. Of course, x_Δ belongs to K. Therefore

$$S_{\sigma_i\nu}[0,1] \subseteq B(x_\Delta, \ell(S_{\sigma_i\nu}[0,1])) \subseteq \Delta.$$

If we let M be the maximum number of elements in any neighbour set, then the maximality property in the choice of index i means that

$$\mu(\Delta) \geq \mu(B(x_\Delta, \ell(S_{\sigma_i\nu}[0,1]))) \geq \sum_{\omega:S_\omega=S_{\sigma_i v}} p_\omega \geq p_\nu \sum_{\alpha:S_\alpha=S_{\sigma_i}} p_\alpha \geq p_\nu \frac{\mu(\Delta)}{M}.$$

As there are only finitely many such words v, there is some $\varepsilon > 0$ such that $p_v > \varepsilon$ for all such v. Hence $\mu(B(x_\Delta, \ell(S_{\sigma_i\nu}[0,1]))) \geq \mu(\Delta)\varepsilon/M$ for all Δ.

Recall $\sigma_i\nu$ depends on Δ. Since the balls $B(x_\Delta, \ell(S_{\sigma_i\nu}[0,1]))$ are disjoint over the set of $\Delta \in \mathcal{F}_n$, centred in K and have radius comparable to $\rho_{\min}^n$, we can now conclude that for all q,

$$\tau(q) \leq \liminf_{n\to\infty} \frac{\log \sum_{\Delta\in\mathcal{F}_n} (\mu(\Delta))^q}{n \log \rho_{\min}}.$$

□

Motivated by this, we make the following definition for the loop class L^q-spectrum of a measure μ of finite type.

Definition 4.4. Let μ be a measure of finite type and let L be a set of characteristic vectors containing a loop class. We define the **L^q-spectrum of μ on L** as the function $\tau_L(\mu,q)$ defined at $q \in \mathbb{R}$ by

$$\tau_L(\mu,q) = \liminf_{k\to\infty} \frac{\log \sum_{\sigma\in L_a, |\sigma|=k} \|T(\sigma)\|^q}{k \log \rho_{\min}}.$$

Again, we suppress μ in the notation if the measure is clear.

We will first prove that $\tau_\Omega(\mu,q)=\tau(\mu,q)$ and τ_Ω is minimal over all τ_L.

THEOREM 4.5. *Let $L\subseteq\Omega$ be any set of characteristic vectors containing a loop class. Then $\tau(\mu,q)=\tau_\Omega(\mu,q)\le\tau_L(\mu,q)$ for all q and hence*

$$\tau(\mu,q)\le\min\{\tau_L(\mu,q): L \text{ maximal loop class}\} \text{ for all } q\in\mathbb{R}.$$

Proof. We first check that $\tau(\mu,q)=\tau_\Omega(\mu,q)$. Since

$$\sum_{\sigma\in\Omega_a^0,|\sigma|=k}\|T(\sigma)\|^q\le\sum_{\sigma\in\Omega_a,|\sigma|=k}\|T(\sigma)\|^q,$$

it is immediate that $\tau(\mu,q)\ge\tau_\Omega(\mu,q)$.

To prove the other inequality, we start by partitioning the right hand sum according to the first letter of σ. Let $\Omega=\{\gamma_0,\gamma_1,\dots,\gamma_{|\Omega|-1}\}$ be the complete list of characteristic vectors and for each i, choose η_i such that $(\gamma_0,\eta_i,\gamma_i)$ is an admissible path where $|\eta_i|=n_i$ (and (η_0,γ_0) should be understood to be the empty word).

As $\|T(\gamma_i,\chi)\|$ is comparable to $\|T(\gamma_0,\eta_i,\gamma_i,\chi)\|$, there is a constant $C_i(q)$ such that

$$\begin{aligned}\sum_{\sigma\in\Omega_a,|\sigma|=k}\|T(\sigma)\|^q&=\sum_{i=0}^{|\Omega|-1}\sum_{|\chi|=k-1}\|T(\gamma_i,\chi)\|^q\\&\le\sum_{i=0}^{|\Omega|-1}C_i\sum_{|\chi|=k-1}\|T(\gamma_0,\eta_i,\gamma_i,\chi)\|^q\\&\le\sum_{i=0}^{|\Omega|-1}C_i\sum_{|\sigma|=k+n_i+1,\sigma\in\Omega_a^0}\|T(\sigma)\|^q.\end{aligned}$$

The definition of $\tau(\mu,q)$ implies that given any $\varepsilon>0$ there is some $k_0=k_0(\varepsilon,q)$ such that for all $k\ge k_0$ we have

$$\frac{\log\sum_{|\sigma|=k,\sigma\in\Omega_a^0}\|T(\sigma)\|^q}{k\log\rho_{\min}}\ge\tau(\mu,q)-\varepsilon.$$

Thus for k sufficiently large

$$\sum_{|\sigma|=k+n_i+1,\sigma\in\Omega_a^0}\|T(\sigma)\|^q\le\rho_{\min}^{(k+n_i+1)(\tau(q)-\varepsilon)}.$$

We deduce that

$$\sum_{\sigma\in\Omega_a,|\sigma|=k}\|T(\sigma)\|^q\le\sum_{i=0}^{|\Omega|-1}C_i'\rho_{\min}^{k(\tau(q)-\varepsilon)}\le C\rho_{\min}^{k(\tau(q)-\varepsilon)}$$

for suitable constants C_i',C depending on ε and q. It is immediate from this that

$$\frac{\log\sum_{\sigma\in\Omega_a,|\sigma|=k}\|T(\sigma)\|^q}{k\log\rho_{\min}}\ge\frac{\log C}{k\log\rho_{\min}}+\tau(\mu,q)-\varepsilon$$

and letting $k \to \infty$, we see that $\tau_\Omega(\mu, q) \geq \tau(\mu, q) - \varepsilon$ for all $\varepsilon > 0$. Thus $\tau_\Omega(\mu, q) = \tau(\mu, q)$ for all q.

For any $L \subseteq \Omega$,

$$\sum_{\sigma \in L_a, |\sigma|=k} \|T(\sigma)\|^q \leq \sum_{\sigma \in \Omega_a, |\sigma|=k} \|T(\sigma)\|^q ,$$

hence $\tau_L(\mu, q) \geq \tau_\Omega(\mu, q)$ for all q, completing the proof. □

We next obtain pointwise bounds for the functions τ_L. By the incidence matrix of a set of characteristic vectors $L = \{\chi_1, \ldots, \chi_{|L|}\}$ we mean the $|L| \times |L|$ matrix whose (i, j) entry is 1 if χ_i has χ_j as a child and equals 0 otherwise.

Proposition 4.6. *Let L be a set of characteristic vectors containing a loop class and let I_L denote its incidence matrix. Then*

$$qd^L_{\min} \geq \tau_L(\mu, q) \geq qd^L_{\min} - \frac{\log \mathrm{sp}(I_L)}{|\log \rho_{\min}|} \text{ if } q \geq 0.$$

If L is a loop class, then

$$qd^L_{\max} \geq \tau_L(\mu, q) \geq qd^L_{\max} - \frac{\log \mathrm{sp}(I_L)}{|\log \rho_{\min}|} \text{ if } q < 0.$$

Proof. Let $\varepsilon > 0$ and assume $L = \{\chi_1, \ldots, \chi_{|L|}\}$ is a loop class. By Lemma 3.10, for sufficiently large k and $\sigma \in L_a, |\sigma| = k$, we have

$$\|T(\sigma)\| \geq \rho_{\min}^{(d^L_{\max}+\varepsilon)k}.$$

Furthermore, we can choose an infinite path $\alpha \in L_a$ such that $\overline{d}(\alpha) \geq d^L_{\max} - \varepsilon/2$, hence for infinitely many k_j we have

$$\|T(\alpha|k_j)\| \leq \rho_{\min}^{k_j(d^L_{\max}-\varepsilon)}. \tag{4.4}$$

Assume χ_i is the characteristic vector of a net interval Δ_i of level n_i. Every $\sigma \in L_a$ with $|\sigma| = k+1$ and beginning with letter χ_i, determines a unique descendent of Δ_i at level $n_i + k$ that has its characteristic vector in L. The number of such σ is the sum of the entries on row i of the matrix $(I_L)^k$. This is dominated by $\left\|(I_L)^k\right\|$ and hence by $(\mathrm{sp}(I_L) + \varepsilon)^k$ for large enough k depending on $\varepsilon > 0$.

Thus for $q < 0$ and large k,

$$\sum_{\sigma \in L_a, |\sigma|=k} \|T(\sigma)\|^q = \sum_{i=1}^{|L|} \sum_{|\sigma|=k-1} \|T(\chi_i, \sigma)\|^q \leq \rho_{\min}^{qk(d^L_{\max}+\varepsilon)} (\mathrm{sp}(I_L) + \varepsilon)^k,$$

while for α as in (4.4) and infinitely many k, we have

$$\sum_{\sigma \in L_a, |\sigma|=k} \|T(\sigma)\|^q \geq \|T(\alpha|k)\|^q \geq \rho_{\min}^{kq(d^L_{\max}-\varepsilon)}.$$

Taking logarithms, dividing by $k \log \rho_{\min}$ and letting $k \to \infty$ gives

$$\tau_L(q) = \liminf_{k\to\infty} \frac{\log \sum_{\sigma \in L_a, |\sigma|=k} \|T(\sigma)\|^q}{k \log \rho_{\min}} \left\{ \begin{array}{cc} \geq & q(d^L_{\max} + \varepsilon) - \frac{\log(\mathrm{sp}(I_L)+\varepsilon)}{|\log \rho_{\min}|} \\ \leq & q(d^L_{\max} - \varepsilon) \end{array} \right. ,$$

which proves the claim for $q < 0$.

The argument for $q \geq 0$ is similar using Lemma 3.9, instead of Lemma 3.10, and only requires L to contain a loop class. □

It was proven by Ngai and Wang in [14] that $\dim_B K = \log(\mathrm{sp}(I_\Omega))/|\log \rho_{\min}|$. More generally, the following is true.

Lemma 4.7. *Suppose L is a set of characteristic vectors containing a loop class and let J_L be the incidence matrix for the set of characteristic vectors that have a descendent in L. The box dimension of K_L exists and is equal to*

$$\dim_B K_L = \frac{\log \mathrm{sp}(J_L)}{|\log \rho_{\min}|}.$$

Remark 4.8. Note that J_L above is not the incidence matrix of L. It may contain characterstic vectors outside of L that have descendents in L. It is possible that $\mathrm{sp}(J_L) > \mathrm{sp}(I_L)$ as we show in Example 4.11.

Proof. Given a matrix J, let $\|J\|_i = \sum_j |J_{ij}|$, the sum of the moduli of the entries of row i of J.

Without loss of generality, we can assume the entries of the first row of J_L are determined by the children of γ_0 (with descendents in L). Then $\left\|(J_L)^k\right\|_1$ is the number of net intervals at level k that contain an element of K_L. This is an increasing function of k and is comparable to the number of disjoint balls of radius $c\rho_{\min}^k$ centred at points in K_L.

Furthermore, $\left\|J_L^k\right\|_i \leq \left\|J_L^{k+m_i}\right\|_1$ if the entries of row i of J are determined by the children of γ_i where γ_i is a descendent of γ_0 at level m_i which has a descendent in L. Thus if C is the number of rows of J_L and $M = \max m_i$, we have

$$\left\|J_L^k\right\|_1 \leq \left\|J_L^k\right\| = \sum_i \left\|J_L^k\right\|_i \leq C \left\|J_L^{k+M}\right\|_1.$$

This shows $\lim_{k\to\infty} \left\|J_L^k\right\|_1^{1/k} = \lim_k \left\|J_L^k\right\|^{1/k} = \mathrm{sp}(J_L)$ and therefore

$$\frac{\log \mathrm{sp}(J_L)}{|\log \rho_{\min}|} = \lim_{k\to\infty} \frac{\log \left\|J_L^k\right\|_1}{k\,|\log \rho_{\min}|} = \dim_B K_L.$$

□

Corollary 4.9. *Let L be a set of characteristic vectors containing a loop class.*
(i) Then

$$qd_{\min}^L \geq \tau_L(\mu, q) \geq qd_{\min}^L - \dim_B K_L \geq qd_{\min}^L - 1 \text{ if } q \geq 0.$$

(ii) Further, if L is a loop class, then

$$qd_{\max}^L \geq \tau_L(\mu, q) \geq qd_{\max}^L - \dim_B K_L \geq qd_{\max}^L - 1 \text{ if } q < 0.$$

(iii) For any $q \geq 0$,

$$qd_{\min}^\Omega \geq \tau(\mu, q) \geq qd_{\min}^\Omega - \dim_B K \geq qd_{\min}^\Omega - 1.$$

(iv) If $|L| = 1$ *then*

$$qd^L_{\min} = qd^L_{\max} = \tau_L(\mu, q) \text{ for all } q.$$

Proof. Part (i) and (ii). Let J_L be as in the Lemma. Then the incidence matrix of L, I_L, is a submatrix of J_L and $\|I^k_L\| \le \|J^k_L\|$ for all k, so $\mathrm{sp}(I_L) \le \mathrm{sp}(J_L)$.

Part (iii) is the special case of $L = \Omega$ in Part (i).

Part (iv). Assume $L = \{\gamma\}$. We have

$$d^L_{\min} = d^L_{\max} = \frac{\log \mathrm{sp}(T(\gamma, \gamma))}{\log \rho_{\min}}$$

and $I_L = [1]$, so $sp(I_L) = 1$. □

Remark 4.10. A more general result than (iii) can be found in [12, Proposition 3.7].

We note that it is possible to have $\mathrm{sp}(I_L) < \mathrm{sp}(J_L)$. Here is an example.

Example 4.11. Consider the finite type IFS with maps $S_j(x) = x/3 + d_j$ where $d_j = 0, 4/9, 5/9, 2/3$, discussed in Example 3.10 of [9]. There are seven characteristic vectors: γ_0 with children $\gamma_i, i = 0, 1, \dots, 5$; γ_1 with child γ_0; γ_2 with children $\gamma_1, \gamma_2, \gamma_3$; γ_5 with children $\gamma_3, \gamma_4, \gamma_5$; and $\gamma_3, \gamma_4, \gamma_6, \gamma_7$ all with children $\gamma_3, \gamma_6, \gamma_7$.

The singleton $L = \{\gamma_5\}$ is a maximal loop class. Here $I_L = [1]$ with $\mathrm{sp}(I_L) = 1$, while

$$J_L = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with $\mathrm{sp}(J_L) = 2$.

In [6, Thm. 1.2] it is shown that for any self-similar measure satisfying the weak separation property, $\tau(q)/q \to \inf\{\underline{\dim}_{\mathrm{loc}}\mu(x) : x\}$ as $q \to \infty$. More generally, we can also immediately deduce the following by combining Propositions 3.11, 3.14 and 4.6.

Corollary 4.12. *Let L be a set of characteristic vectors containing a loop class.*

(i) We have $\tau_L(q)/q \to d^L_{\min}$ *as* $q \to \infty$. *If* K_L *is relatively open, then*

$$\tau_L(q)/q \to \inf\{\underline{\dim}_{\mathrm{loc}}\mu(x) : x \in K_L\} \text{as } q \to \infty.$$

(ii) Further, if L is a loop class, then $\tau_L(q)/q \to d^L_{\max}$ *as* $q \to -\infty$. *If, in addition, L admits an interior path, then*

$$\tau_L(q)/q \to \sup\{\dim_{\mathrm{loc}} \mu(x) : x \in K_L\} \text{ as } q \to -\infty.$$

Here are some other facts about loop class L^q-spectra which will be useful later in the paper.

Proposition 4.13. *Let E be the essential class and suppose* $\Delta^{(j)}, j = 1, \dots, J$, *are distinct net intervals of level N with characteristic vectors in E. Let* $\mu_j = \mu|_{\Delta^{(j)}}$. *For any real number q,*

$$\tau_E(\mu, q) = \tau(\mu_1 + \cdots + \mu_J, q).$$

Moreover,

$$\tau(\mu, q) = \tau_E(\mu, q) \text{ if } q \geq 0.$$

Proof. We first note that since the net intervals $\Delta^{(j)}$ are disjoint, similar arguments to the proof of Proposition 4.3 show that

$$\tau(\mu_1 + \cdots + \mu_J, q) = \liminf_{k\to\infty} \frac{\log \sum_{j=1}^J \sum_{\Delta \subseteq \Delta^{(j)}, \Delta \in \mathcal{F}_k} (\mu_j(\Delta))^q}{k \log \rho_{\min}}.$$

Suppose $\Delta^{(j)}$ has symbolic representation $\delta^{(j)}$, with last letter $\gamma_j \in E$. The usual comparability arguments give

$$\begin{aligned} \tau(\mu_1 + \cdots + \mu_J, q) &= \liminf_{k\to\infty} \frac{\log \sum_{j=1}^J \sum_{|\sigma|=k} \left\| T(\delta^{(j)}, \sigma) \right\|^q}{k \log \rho_{\min}} \qquad (4.5) \\ &= \liminf_{k\to\infty} \frac{\log \sum_{j=1}^J \sum_{|\sigma|=k} \left\| T(\gamma_j, \sigma) \right\|^q}{k \log \rho_{\min}}. \end{aligned}$$

As $\gamma_j \in E$, the admissible paths σ appearing in the sum belong to E_a and therefore

$$\sum_{|\sigma|=k} \| T(\gamma_j, \sigma)\|^q \leq \sum_{|\sigma|=k+1, \sigma \in E_a} \|T(\sigma)\|^q.$$

It follows that

$$\tau(\mu_1 + \cdots + \mu_J, q) \geq \liminf_{k\to\infty} \frac{\log \sum_{|\sigma|=k, \sigma\in E_a} \|T(\sigma)\|^q}{k \log \rho_{\min}} = \tau_E(\mu, q).$$

The proof that $\tau_E(\mu, q) \geq \tau(\mu_1 + \cdots + \mu_J, q)$ uses arguments similar to those used in the proof of Theorem 4.5. Assume $E = \{\chi_1, \ldots, \chi_{|E|}\}$. As every $\sigma \in E_a$ will begin with one of the χ_i, we can write

$$\sum_{|\sigma|=k, \sigma \in E_a} \|T(\sigma)\|^q = \sum_{i=1}^{|E|} \sum_{|\sigma|=k-1, \sigma\in E_a} \|T(\chi_i, \sigma)\|^q.$$

We continue to assume $\Delta^{(1)}$ has symbolic representation $\delta^{(1)}$ (with final letter in E). For each $i = 1, \ldots, |E|$, choose a path $\lambda_i \in E_a$ linking $\delta^{(1)}$ with χ_i. Assume $|\lambda_i| = n_i$. The definition of $\tau(\mu_1)$ established in (4.5) implies that for each $\varepsilon > 0$ and all $k \geq k_0(\varepsilon)$ we have

$$\begin{aligned} \sum_{|\sigma|=k-1, \sigma\in E_a} \|T(\chi_i, \sigma)\|^q &\leq C_i \sum_{|\sigma|=k-1, \sigma \in E_a} \left\| T(\delta^{(1)}, \lambda_i, \chi_i, \sigma) \right\|^q \\ &\leq C_i \sum_{|\sigma|=k+n_i, \sigma\in E_a} \left\| T(\delta^{(1)}, \sigma) \right\|^q \leq C_i' \rho_{\min}^{k(\tau(\mu_1,q)-\varepsilon)}. \end{aligned}$$

Consequently, $\tau_E(\mu, q) \geq \tau(\mu_1, q)$.

It is easy to see that $\tau(\mu_1, q) \geq \tau(\mu_1 + \cdots + \mu_J, q)$ and hence $\tau_E(\mu, q) = \tau(\mu_1 + \cdots + \mu_J, q)$ for all q.

Finally, we note that in [3, Lemma 5.3], Feng shows (in the equicontractive case, but the same argument works in general) that for $q \geq 0$, $\tau(\mu, q) = \tau(\mu_1, q)$. Hence $\tau(\mu, q) = \tau_E(\mu, q)$ for all $q \geq 0$. □

COROLLARY 4.14. *For $q \geq 0$, the function $\tau_L(\mu, q)$ is minimized over all maximal loop classes L at L equal to the essential class E. Likewise, the minimum value of $d^L_{\min}$ is attained at $L = E$.*

Proof. The previous work shows that for all $q \geq 0$, $\tau_L(\mu, q) \geq \tau(\mu, q) = \tau_E(\mu, q)$ and that $\tau_L(\mu, q)/q \to d^L_{\min}$ as $q \to \infty$ for any loop class L. □

5. The L^q-spectrum of finite type measures.

5.1. Main Theorem. Throughout this section we continue to assume μ is a self-similar measure of finite type. We conjecture that for all real q, that $\tau(\mu, q) = \min_L \tau_L(\mu, q)$, where the minimum is taken over all the maximal loop classes L. Indeed, we have already seen in Theorem 4.5 that $\tau(q) \leq \min_L \tau_L(q)$.

In this section, we will prove the conjecture under additional assumptions, which we will see later are satisfied by many examples, including the (uniform or biased) Bernoulli convolutions with contraction factor the inverse of a simple Pisot number and the (m, d)-Cantor measures with $m \geq d$.

It is convenient to introduce further terminology: We will say that two different maximal loop classes, L_1 and L_2, are adjacent if there is a path from L_1 to L_2 that does not pass through any other maximal loop class. To be more precise, we mean there is a path $\chi_1, \ldots, \chi_s$ with $s > 1$, $\chi_1 \in L_1, \chi_s \in L_2$ and χ_i not in a maximal loop class for any $i \neq 1, s$. Any such path will be called a **transition path** from L_1 to L_2. Since $\chi_2, ..., \chi_{s-1}$ do not belong to any of the loop classes, they must all be distinct. The definition of a maximal loop class ensures that once a path exits a maximal loop class, it can never return to it. It follows that any finite type IFS has only finitely many transition paths.

EXAMPLE 5.1. For the Bernoulli convolution with contraction factor the inverse of the golden mean, one can see from Figure 2.1 that the only transition paths are (γ_1, γ_2) and (γ_3, γ_2). For the $(3, 3)$-Cantor measure, (see Figure 2.2), the transition paths are (γ_i, γ_j) for $i = 1, 7$ and $j = 2, 3, 5, 6$.

Recall that we write E for the essential class.

THEOREM 5.2. *Let μ be any self-similar measure of finite type. We have*

$$\tau(\mu, q) = \min_L \tau_L(\mu, q) = \tau_E(\mu, q) \text{ for } q \geq 0,$$

where the minimum is over all maximal loop classes L (including E) and E denotes the essential class.

If the transition matrices of all transition paths are positive matrices, then

$$\tau(\mu, q) = \min_L \tau_L(\mu, q) \text{ for } q < 0.$$

Proof. In Theorem 4.5 and Corollary 4.14, we have already seen that $\tau(q) = \min_L \tau_L(q) = \tau_E(q)$ for $q \geq 0$ and that $\tau(q) \leq \min_L \tau_L(q)$ for all q. Thus we need only prove the reverse inequality for $q < 0$ under the additional assumption, and that requires finding upper bounds on $\sum_{|\sigma|=k, \sigma\in\Omega^0} \|T(\sigma)\|^q$. To do this, we observe that a consequence of the definition of maximal loop classes and transition paths ensures that each such σ can be factored as $(\beta_1^-, \lambda_1^-, \beta_2^-, \lambda_2^-, \ldots, \lambda_\ell^-, \beta_{\ell+1})$ where

- $\beta_1 = \beta_1^{(1)} \dots \beta_1^{(j_1)}$ is the path from γ_0 to the first maximal loop class. Here $\beta_1^{(j_1)}$ is in the (first) maximal loop class associated with λ_1. This may be a singleton if γ_0 is within the loop class associated with λ_1.
- $\lambda_i = \lambda_i^{(1)} \dots \lambda_i^{(k_i)}$ is path contained within a single maximal loop class and $\lambda_i \neq \lambda_j$ for $i \neq j$.
- For $i = 2, ..., \ell$, $\beta_i = \beta_i^{(1)} \dots \beta_i^{(j_i)}$ is a transition path from the loop class given by λ_{i-1} to the loop class given by λ_i. Here $\beta_i^{(1)}$ is in the loop class associated with λ_{i-1} and $\beta_i^{(j_i)}$ is in the loop class associated with λ_i.
- $\beta_{\ell+1} = \beta_{\ell+1}^{(1)} \dots \beta_{\ell_1}^{(j_{\ell+1})}$ is a prefix of a transition path from the loop class given by λ_ℓ. This may be a singleton if the last characteristic vector in σ is in a loop class.
- We have $\beta_i^{(j_i)} = \lambda_i^{(1)}$ and $\lambda_i^{(k_i)} = \beta_{i+1}^{(1)}$.

Notice that the number ℓ is bounded above by the number of maximal loop classes. With this notation we have

$$T(\sigma) = T(\beta_1)T(\lambda_1) \dots T(\beta_\ell)T(\lambda_\ell)T(\beta_{\ell+1}).$$

Here we understand $T(\beta_i)$ or $T(\lambda_i)$ to be 1 if they contained only one characteristic vector.

We also observe that there are only a finite number of initial and transition paths, and hence a finite number of β_i. By assumption, $T(\beta_i)$ is a positive matrix for $i = 2, \dots, \ell$ and we always have that $T(\beta_1)$ is a positive matrix with row dimension 1. Hence by Lemma 2.11(ii) we get

$$\begin{aligned} \|T(\sigma)\| &\geq \|T(\beta_1)T(\lambda_1) \dots T(\beta_\ell)T(\lambda_\ell)T(\beta_{\ell+1})\| \\ &\geq C\|T(\lambda_1)\|\|T(\lambda_2)\| \dots \|T(\lambda_{\ell-1})\|\|T(\lambda_\ell)T(\beta_{\ell+1})\|. \end{aligned} \tag{5.1}$$

Recall $\beta_{\ell+1} = \beta_{\ell+1}^{(1)} \dots \beta_{\ell+1}^{(j_{\ell+1})}$ is a prefix of a transition path. If $\beta_{\ell+1}$ is a singleton, then we obtain directly that $\|T(\lambda_\ell)T(\beta_{\ell+1})\| = \|T(\lambda_\ell)\|$. Assume that it is not a singleton (and hence the path σ has exited the loop class associated with λ_ℓ). Let $\beta_{\ell+1}^{+} = \beta_{\ell+1}^{(1)} \dots \beta_{\ell_1}^{(j_{\ell+1})} \dots \beta_{\ell_1}^{(J_{\ell+1})}$ be a completion of $\beta_{\ell+1}$, meaning a transition path for which $\beta_{\ell+1}$ is a prefix. We see that

$$\begin{aligned} \|T(\lambda_\ell)T(\beta_{\ell+1})\|\|T(\beta_{\ell+1}^{(j_{\ell+1})} \dots \beta_{\ell+1}^{(J_{\ell+1})})\| &\geq \|T(\lambda_\ell)T(\beta_{\ell+1})T(\beta_{\ell+1}^{(j_{\ell+1})} \dots \beta_{\ell+1}^{(J_{\ell+1})})\| \\ &= \|T(\lambda_\ell)T(\beta_{\ell+1}^{+})\| \\ &\geq b\|T(\lambda_\ell)\|. \end{aligned}$$

The first equality follows from the definition of $\beta_{\ell+1}^{+}$. The inequality on the third line comes from Lemma 2.11(ii) (taking $C = Id$ there), which applies since $T(\beta_{\ell+1}^{+})$ is a positive matrix by assumption. There are only finitely many such completions, hence $\|T(\beta_{\ell+1}^{(j_{\ell+1})} \dots \beta_{\ell+1}^{(J_{\ell+1})}\|$ is bounded from below. Combining this observation with equation (5.1) gives us that $\|T(\sigma)\| \geq C \prod_{i=1}^{\ell} \|T(\lambda_i)\|$ for some (new) constant C, or equivalently

$$\|T(\sigma)\|^q \leq C^q \prod_{i=1}^{\ell} \|T(\lambda_i)\|^q. \tag{5.2}$$

Let κ be the number of possible tuples $(\beta_1, \beta_2, \dots, \beta_{\ell+1})$. We see that κ is finite as there are a finite number of transition paths and ℓ is bounded above. Let C be a renamed constant, the minimum value of C^q taken over all such tuples. Let $L_1, \dots, L_N$ be the complete set of loop classes (including the essential class).

With this notation we have

$$\begin{aligned}
\sum_{|\sigma|=n} \|T(\sigma)\|^q &\leq \sum_{(\beta_1,\dots,\beta_{\ell+1})} \left(\sum_{|\lambda_1|+\cdots+|\lambda_\ell|\leq n} \|T(\beta_1)T(\lambda_1)\dots T(\lambda_\ell)T(\beta_{\ell+1})\|^q \right) \\
&\leq C \sum_{(\beta_1,\dots,\beta_{\ell+1})} \left(\sum_{i_1+\cdots+i_N\leq n} \sum_{\substack{|\lambda_1|=i_1 \\ \lambda_1\in L_1}} \cdots \sum_{\substack{|\lambda_N|=i_N \\ \lambda_N\in L_N}} \|T(\lambda_1)\|^q \dots \|T(\lambda_N)\|^q \right) \\
&\leq C\kappa \sum_{i_1+\cdots+i_N\leq n} \left(\sum_{\substack{|\lambda_1|=i_1 \\ \lambda_1\in L_1}} \|T(\lambda_1)\|^q \right) \cdots \left(\sum_{\substack{|\lambda_N|=i_N \\ \lambda_N\in L_N}} \|T(\lambda_N)\|^q \right).
\end{aligned}$$

Fix $\varepsilon > 0$ and let

$$\theta(q) := \min_L \tau_L(q).$$

By the definition of τ_L, there is some n_L such that for all paths in L with length at least $n \geq n_L$,

$$\sum_{\lambda\in L_a, |\lambda|=n} \|T(\lambda)\|^q \leq \rho_{\min}^{n(\tau_L(q)-\varepsilon)} \leq \rho_{\min}^{n(\theta(q)-\varepsilon)}.$$

As there are only finitely many paths of length less than n_L and only finitely many maximal loop classes, there is some constant C_ε such that

$$\sum_{\lambda\in L_a, |\lambda|=n} \|T(\lambda)\|^q \leq C_\varepsilon \rho_{\min}^{n(\theta(q)-\varepsilon)}$$

for all n and for all maximal loop classes L. Thus

$$\begin{aligned}
\sum_{|\sigma|=n} \|T(\sigma)\|^q &\leq C\kappa \sum_{i_1+\cdots+i_N\leq n} \left(\sum_{\substack{|\lambda_1|=i_1 \\ \lambda_1\in L_1}} \|T(\lambda_1)\|^q \right) \cdots \left(\sum_{\substack{|\lambda_N|=i_N \\ \lambda_N\in L_N}} \|T(\lambda_N)\|^q \right) \\
&\leq C\kappa \sum_{i_1+\cdots+i_N\leq n} \left(C_\epsilon \rho_{\min}^{i_1(\theta(q)-\epsilon)} \right) \left(C_\epsilon \rho_{\min}^{i_2(\theta(q)-\epsilon)} \right) \cdots \left(C_\epsilon \rho_{\min}^{i_N(\theta(q)-\epsilon)} \right) \\
&\leq C\kappa C_\epsilon^N \sum_{i_1+\cdots+i_N\leq n} \rho_{\min}^{n(\theta(q)-\epsilon)} \\
&\leq C\kappa C_\epsilon^N n^N \rho_{\min}^{n(\theta(q)-\epsilon)} \\
&\leq C' \rho_{\min}^{n(\theta(q)-2\epsilon)}.
\end{aligned}$$

Recall here that $\theta(q) < 0$ when $q < 0$.

Taking the logarithm, dividing by $n \log \rho_{\min}$ and letting $n \to \infty$, we deduce that

$$\tau(q) = \liminf_n \frac{\log \sum_{|\sigma|=n} \|T(\sigma)\|^q}{n \log \rho_{\min}} \geq \theta(q) - \varepsilon.$$

As this holds for all $\varepsilon > 0$, we have $\tau(q) \geq \theta(q) = \min_L \tau_L(q)$, as we desired to show. □

REMARK 5.3. We remark that while the values of the entries of the transition matrices associated with transition paths are a function of the probabilities associated with the self-similar measure, the property of being positive matrices depends only on the finite type structure of the IFS. Thus Theorem 5.2 will apply to every self-similar measure associated with an IFS which satisfies the hypotheses.

5.2. Consequences of the Theorem. For this subsection, we will let

$$d = \max\{d_{\max}^L : L \text{ maximal loop class, } L \neq E\}.$$

PROPOSITION 5.4. *Let μ be any self-similar measure of finite type. Suppose that every maximal loop class other than the essential class is a singleton and that the primitive transition matrices associated with these singleton loop classes are scalars. Then $\tau(\mu, q) = \min_L \tau_L(\mu, q)$ for all q and consequently,*

$$\tau(\mu, q) = \begin{cases} \tau_E(\mu, q) & \text{if } q \geq 0 \\ \min\{qd, \tau_E(\mu, q)\} & \text{if } q < 0 \end{cases}.$$

Proof. If L is a singleton maximal loop class with primitive transition matrix a scalar, then the transition matrix of any transition path beginning at L is a row vector. As every column of a transition matrix has a non-zero entry, it follows that such a transition matrix is positive.

There are no transition paths starting from the essential class. As all other maximal loop classes are singletons with scalar transition matrices, we have that all transition paths are positive. Thus the hypotheses of Theorem 5.2 are satisfied and $\tau(\mu, q) = \min_L \tau_L(\mu, q)$ for all q.

The remaining claims follow upon recalling that if L is a singleton loop class, then $\tau_L(\mu, q) = qd_{\max}^L$ (and $= qd_{\min}^L$) for all q; see Corollary 4.9 (iv). □

REMARK 5.5. As with Theorem 5.2, the assumptions of this Proposition are structural properties of the underlying IFS and hence every self-similar measure associated with such an IFS will satisfy the Proposition.

For both the Bernoulli convolutions with simple Pisot inverses as contractions and the (m, d)-Cantor measures, there are three maximal loop classes, the essential class and two singletons (corresponding to the endpoints $0, 1$) and the latter have scalars as their primitive transition matrices. Hence the Proposition applies to these examples. We will discuss these classes of finite type measures in more detail later in this subsection; see Examples 5.9 and 5.10.

In fact, these Bernoulli convolutions and Cantor-like measures are special examples of the following more general situation.

LEMMA 5.6. *Suppose the IFS is equicontractive and of finite type and that the set of essential points is $(0, 1)$. Then there are (precisely) two other maximal loop classes. These are both singletons, corresponding to the points $0, 1$ respectively, with primitive transition matrices that are scalars.*

Proof. Assume the IFS consists of similarities $\{S_j\}_{j=0}^m$, ordered in the natural way. The left-most net interval at level 1 is either $[0, S_0(1)]$ or $[0, S_1(0)]$, depending on which is smaller, $S_0(1)$ or $S_1(0)$. At level 2, it is the interval $[0, a]$ where $a =$

$\min\{S_iS_j(0), S_iS_j(1) : i, j\}$. It is straight forward to check that if $S_0(1) \leq S_1(0)$, then the minimum value is $S_0S_0(1)$ and then the net interval $[0, a]$ has normalized length 1, the same as the level 1 net interval $[0, S_0(1)]$. Otherwise, the minimum value is $S_0S_1(0)$ and the net interval $[0, a]$ has the same normalized length as the level 1 net interval, $[0, S_1(0)]$. In both cases, these level one and level two net intervals have the same neighbour set, the singleton $\{(0, 0)\}$. Hence if χ_0 is the characteristic vector of the left-most interval at level one, χ_0 is also its left-most child, so $\{\chi_0\}$ is a loop class and 0 has symbolic representation $(\gamma_0, \chi_0, \chi_0, \dots)$. As there is only one neighbour set, the primitive transition matrix is a scalar. As 0 is not an essential point, χ_0 is not in the essential class.

By symmetry the same is true for the right-most net intervals with 1 having symbolic representation $(\gamma_0, \chi_1, \chi_1, \dots)$.

The loop classes $L_0 = \{\chi_0\}$ and $L_1 = \{\chi_1\}$ are maximal because otherwise there would be infinitely many points that are not essential points. There are no other maximal loop classes (other than the essential class) for the same reason. □

Corollary 5.7. *Suppose μ is an equicontractive, finite type measure and that the set of essential points is $(0, 1)$. Let Δ be any finite union of net intervals of a fixed level, with characteristic vectors in the essential class E.*

(i) Then

$$\tau(\mu, q) = \begin{cases} \tau_E(\mu, q) = \tau(\mu|_\Delta, q) & \text{if } q \geq 0 \\ \min\{qd, \tau_E(\mu, q)\} & \text{if } q < 0 \end{cases}.$$

(ii) If $d^E_{\max} < d$, then there is a choice of $q_0 < 0$ such that

$$\tau(\mu, q) = \begin{cases} \tau_E(\mu, q) = \tau(\mu|_\Delta, q) & \text{if } q \geq q_0 \\ qd & \text{if } q < q_0 \end{cases}. \tag{5.3}$$

We can take $q_0 \geq -1/(d - d^E_{\max})$ to be the unique solution to $\tau_E(q) = qd$.

(iii) If $d^E_{\max} \geq d$, then $\tau(\mu, q) = \tau_E(\mu, q) = \tau(\mu|_\Delta, q)$ for all q.

Proof. Part (i). This follows directly from the previous work.

For Part (ii), recall that we saw in Proposition 4.6 that if $q < 0$, then

$$qd^E_{\max} \geq \tau_E(q) \geq qd^E_{\max} - 1.$$

Thus if $q_1 = -1/(d - d^E_{\max})$, then $\tau_E(q_1) \geq dq_1$. It is easy to see that $\tau_E(0) < 0$. As τ_E is concave, and hence continuous, there is some $q_0 \in [q_1, 0)$ such that $\tau_E(q_0) = dq_0$. Using the fact that if f is a concave function and $z \in (x_1, x_2)$ we have

$$\frac{f(x_1) - f(z)}{z - x_1} \leq \frac{f(z) - f(x_2)}{x_2 - z},$$

it can be checked that there can only be one choice of q_0 with $\tau_E(q_0) = dq_0$. The statements of (5.3) are clearly satisfied for this choice of $q_0 \geq q_1 = -1/(d - d_0)$.

Part (iii). We simply note that if $d^E_{\max} \geq d$ and $q < 0$, then $\tau_E(q) \leq qd^E_{\max} \leq qd$ and hence $\min\{qd, \tau_E(\mu, q)\} = \tau_E(\mu, q)$. □

Corollary 5.8. *Assume μ is associated with an equicontractive, finite type IFS. Suppose $K_E = (0, 1)$ and that*

$$\sup\{\dim_{\mathrm{loc}} \mu(x) : x \in K_E\} < \max\{\dim_{\mathrm{loc}} \mu(0), \dim_{\mathrm{loc}} \mu(1)\}.$$

Then

$$d = \max_{j=0,1} \dim_{\text{loc}} \mu(j) > d^E_{\max}$$

If Δ *is any non-empty, closed subinterval of* $(0,1)$*, then*

$$\tau(\mu, q) = \begin{cases} \tau(\mu|_\Delta, q) & \text{if } q \geq q_0 \\ qd & \text{if } q < q_0 \end{cases}$$

where $q_0 < 0$ *is the unique solution to* $\tau(\mu|_\Delta, q) = qd$ *.*

Proof. As $0, 1$ are the endpoints of the support of μ, they each have a unique periodic symbolic representation. Hence $\dim_{\text{loc}} \mu(j) = d^{L_j}_{\max}$ where L_j is the singleton maximal loop class associated with $j = 0, 1$. Thus our hypothesis, together with Corollary 3.15, implies $d^E_{\max} < d$ and therefore Corollary 5.7 (ii) applies.

It is easy to see from the definition of the L^q-spectrum that if $Y \subseteq X \subseteq \text{supp}\mu$, then $\tau(\mu|_X, q) \leq \tau(\mu|_Y, q)$ for all q. Since any Δ described in the statement of the Corollary will contain a net interval with a characteristic vector in the essential class and is contained in a finite union of such net intervals, it follows that $\tau_E(\mu, q) = \tau(\mu|_\Delta, q)$ for $q \geq q_0$. □

Example 5.9 ((m, d)-Cantor measures). First consider any $(3, 3)$-Cantor measure arising from the equicontractive, finite type IFS with contractions $S_j(x) = x/3 + 2j/9$ for $j = 0, 1, 2, 3$. As previously noted, there are two singleton maximal loop classes, $L_0 = \{\gamma_1\}$ and $L_1 = \{\gamma_7\}$, corresponding to the endpoints $0, 1$. The characteristic vectors $\{\gamma_2, \gamma_3, \gamma_5, \gamma_6\}$ comprise the essential class E, with the open interval $(0, 1)$ being the set of essential points. See Figure 2.2 for the transition graph. Thus Corollary 5.7 (i) applies for any choice of probabilities.

If we take, for example, the probabilities $1/8, 3/8, 3/8, 1/8$, the associated measure is the rescaled 3-fold convolution of the classical Cantor measure. We have

$$d = \dim_{\text{loc}} \mu(j) = \frac{\log 8}{\log 3} = d^{L_j}_{\max} \text{ for } j = 0, 1.$$

It is known that $\sup_{x \in K_E} \{\dim_{\text{loc}} \mu(x)\} < \log 8/\log 3 = d$ and that the union of the net intervals of level N with characteristic vectors in E is the closed interval $\Delta(N) = [2 \cdot 3^{-N}, 1 - 2 \cdot 3^{-N}]$.

Similar statements are true for any choice of regular probabilities, hence we can appeal to Corollary 5.7 (ii) in the regular case to conclude that for any N,

$$\tau(\mu, q) = \begin{cases} \tau(\mu|_{\Delta(N)}, q) & \text{if } q \geq q_0 \\ qd & \text{if } q < q_0 \end{cases}$$

where $q_0 < 0$ is the unique solution to $\tau(\mu|_{\Delta(1)}, q) = qd$.

More generally, similar statements hold for any (m, d)-Cantor measure with $m \geq d \geq 2$, with, again, the stronger conclusions if regular probabilities are chosen. In this case, the union of the net intervals of level N with characteristic vectors in E is the closed interval

$$\Delta(N) = [(d-1)/(md^N), 1 - (d-1)/(md^N)].$$

Most of these results were previously found for the special cases of the 3-fold convolution of the Cantor measure in [7, Thm. 1.1, 1.3] and for $m < 2d - 2$ in [16, Thm. 1.5]. The reader can refer to [8, Section 7] and [9, Section 5] for proofs of the finite type structural properties of the (m, d)-Cantor measures and facts about their local dimensions.

EXAMPLE 5.10 (Bernoulli convolutions). Consider the IFS $S_0(x) = \varrho x, S_1(x) = \varrho x + 1 - \varrho$ which generates the Bernoulli convolution with contraction factor ϱ the inverse of the golden mean. Recall, there are two singleton maximal loop classes, $L_0 = \{\gamma_1\}$ and $L_1 = \{\gamma_3\}$, corresponding to the endpoints $0, 1$. The remaining maximal loop class given by $\{\gamma_2, \gamma_4, \gamma_5, \gamma_6\}$ form the essential class E, so again the open interval $(0, 1)$ is the set of essential points. See Figure 2.1.

If μ is the Bernoulli convolution arising from the probabilities $p_0, p_1 = 1 - p_0$, then

$$d_{\max}^{L_j} = \dim_{\text{loc}} \mu(j) = \frac{\log p_j}{\log \varrho} \text{ for } j = 0, 1.$$

In the case that $p_0 < p_1$ (the biased case), it was shown in [9, Thm. 4.3] that $d_{\max}^E < d = d_{\max}^{L_0}$. In contrast, it was shown in [4, Thm. 1.5] that in the uniform case $d = d_{\max}^{L_j} = d_{\max}^E$ for $j = 0, 1$.

According to Corollary 5.7(i), in either the uniform or biased case we have

$$\tau(\mu, q) = \begin{cases} \tau_E(\mu, q) = \tau(\mu|_\Delta, q) & \text{if } q \geq 0 \\ \min\{qd,\ \tau_E(\mu, q)\} & \text{if } q < 0 \end{cases}.$$

where Δ is any finite union of net intervals of a fixed level with characteristic vectors in the essential class E. We can take as Δ any interval of the form $[\varrho^N(1 - \varrho), 1 - \varrho^N(1-\varrho)]$, for example. In the biased case, $\tau(\mu, q)$ is the straight line qd for $q < q_0$, as per Corollary 5.7 (ii), while in the uniform case, Corollary 5.7 (iii) implies $\tau_E(\mu, q) = \tau(\mu, q)$ for all q.

Similar statements hold for the Bernoulli convolutions with contraction factor the inverse of a simple Pisot number other than the golden mean. (A real number is called a simple Pisot number if it is the (unique) positive root of a polynomial $x^k - x^{k-1} \cdots -x - 1 = 0$ for some integer $k \geq 2$.) In particular, in the biased case, it was shown in [9, Section 4] that $d > d_{\max}^E$, so, again from Corollary 5.7, we have $\tau(\mu, q) = qd$ for $q < q_0$.

REMARK 5.11. In [4], Feng studied uniform Bernoulli convolutions with contraction factor the inverse of a simple Pisot number. He showed that in the case of the golden mean, $\tau(\mu, q)$ is the line qd for $q \leq q_1$ for a suitable choice of $q_1 < 0$ and is not differentiable at that point. Hence τ_E is not differentiable at q_1. In the non-golden mean case, he proved that $\tau(\mu, q) = qd - \log x(q)/\log \varrho$ for an infinitely differentiable function x and that always $x(q) < 1$. Thus, in contrast to the golden mean case, we have $\tau(\mu, q) = \tau_E(\mu, q) < qd$ for all q and τ_E is differentiable everywhere.

Here is another example which satisfies the conditions of Proposition 5.4, but the set of essential points is not the full open interval $(0, 1)$.

EXAMPLE 5.12. Consider the equicontractive IFS with $S_j(x) = x/3 + d_j$ for $d_j = 0, 1/5, 2/5, 2/3$. One can verify that there are three maximal loop classes other than the essential class. These are all singletons corresponding to the points $x = 0, 1$ and $x_0 \in (0, 1)$ respectively, (so $K_E \neq (0, 1)$) and all have scalar primitive transition matrices. Consequently, Theorem 5.2 applies and thus $\tau(\mu,q)=\min \tau_L(\mu,q)$ for all q and all associated self-similar measures.

As a particular example, take $p_3=1/156, p_0=5/256, p_2=25/156$ and $p_1=125/156$. Although we have not been able to prove this, computational work suggests that $d^E_{\max}\leq 2.25$ and $d\geq 4.5$. If this is true, then the L^q-spectrum would be the line $y=qd$ for large negative q.

For our final example, we study the L^q-spectrum of the self-similar measure from Example 3.8.

Example 5.13. Take the IFS with $S_j(x)=x/3+d_j$ for $d_j=0,1/9,1/3,1/2,2/3$ and probabilities $p_j=4/17$ for $j=0,1,3,4$ and $p_2=1/17$, introduced in Example 3.8. We refer the reader to Figure 5.1 for the transition graph. There are 19 characteristic vectors and four singleton maximal loop classes, $L_i=\{\gamma_i\}$ for $i=1,4,7,8$, in addition to the essential class E with 9 characteristic vectors (whose transitions are not detailed in the figure). The maximal loop classes L_1 and L_7 correspond to the points at 0 and 1. The loop class at L_4 corresponds to a right most path, whose corresponding left most path is in the essential class. The loop class at L_8 also corresponds to a right most path, whose corresponding left most path is in the essential class. The singleton maximal loop classes L_1, L_4 (corresponding to the point 1/2) and L_7 have scalar primitive transition matrices, so the transition matrices of transition paths beginning with these loop classes are all positive. However, the maximal loop class L_8 has a 2×2 lower triangular, primitive transition matrix. Furthermore, there are 10 transition paths between L_8 and the essential class, namely

$\gamma_8\gamma_3\gamma_{12}$	$\gamma_8\gamma_3\gamma_{13}$	$\gamma_8\gamma_3\gamma_{14}\gamma_5\gamma_9$	$\gamma_8\gamma_3\gamma_{14}\gamma_5\gamma_{10}$
$\gamma_8\gamma_3\gamma_{14}\gamma_5\gamma_{11}$	$\gamma_8\gamma_3\gamma_{14}\gamma_5\gamma_{12}$	$\gamma_8\gamma_3\gamma_{14}\gamma_5\gamma_{13}$	$\gamma_8\gamma_3\gamma_{14}\gamma_9$
$\gamma_8\gamma_3\gamma_{14}\gamma_{10}$	$\gamma_8\gamma_3\gamma_{14}\gamma_{11}$		

Some of the transition matrices for these transition paths (such as $\gamma_8\gamma_3\gamma_{12}$) are not positive. So Theorem 5.2 does not apply directly.

However, the transition matrix $T(\gamma_8\gamma_8\gamma_3)$ is positive and one can only exit L_8 through γ_3. Furthermore, one can only enter L_8 from γ_2. With this in mind, we can modify our definitions slightly so that the proof of the Theorem will still apply.

(i) Include in the list of initial paths the paths starting with $\gamma_0\gamma_2\gamma_8\gamma_3\ldots$.
(ii) Include in the list of transition paths from L_1 to E the paths starting with $\gamma_1\gamma_2\gamma_8\gamma_3\ldots$.
(iii) Require transition paths from L_8 to E to start with $\gamma_8\gamma_8\gamma_3\ldots$.
(iv) Decompose σ using these modified transition paths.

With these modifications, there are still only finitely many initial and transition paths and these are now all positive. There are also still only finitely many choices for loop class components λ_i. From here the Theorem follows as before and we again deduce that $\tau(\mu,q)=\min_L \tau_L(\mu,q)$ for all q.

We have $\tau_{L_j}(q)=qd^{L_j}_{\max}$ with $d^{L_j}_{\max}=\log(17/4)/\log 3$ for $j=1,4,8$ and $d^{L_j}_{\max}=\log 17/\log 3$ for $j=7$. Computational work, as explained in Appendix A, shows that

$$\max\{\dim_{\mathrm{loc}}\mu(x):x\in K\}\leq \log(17/2)/\log 3.$$

In particular, $d^E_{\max}<d^{\Omega}_{\max}=\log 17/\log 3=d$ and thus for a suitable choice of q_0, $\tau(\mu,q)$ is the straight line $\tau(q)=qd^{\Omega}_{\max}$ for $q\leq q_0$. But $\{x:\dim_{\mathrm{loc}}\mu(x)=d^{\Omega}_{\max}\}$ is empty, contrary to the spirit of the multifractal formalism.

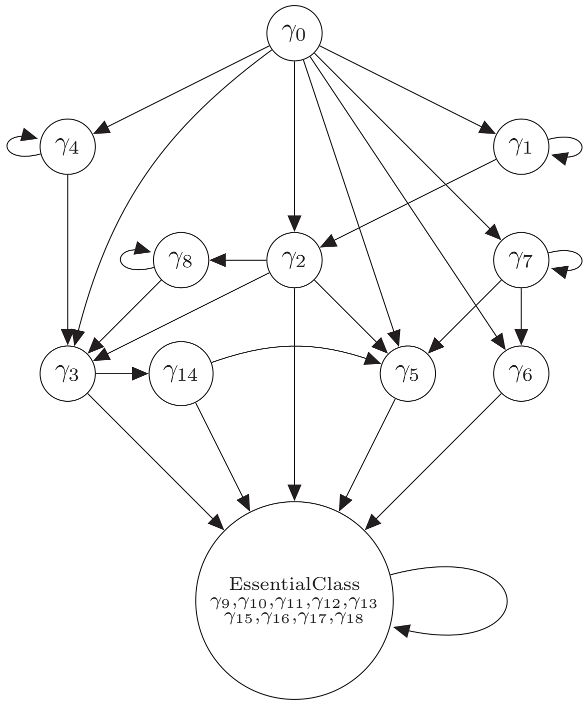

FIG. 5.1. *Example 5.13*

5.3. Open questions. We conclude with a short list of questions we have not been able to answer.

(i) Does the equality $\tau(\mu, q) = \min\{\tau_L(\mu, q) : L \text{ maximal loop class}\}$ hold for all finite type measures μ? If not, in what generality does it hold?

(ii) In Example 5.10 we saw that the function $\tau_E(\mu, q)$ can have a point of non-differentiability. Under what assumptions are the functions $\tau_L(\mu, q)$ differentiable for all q? real analytic?

(iii) Can $d^{\Omega}_{\min}$ be computed?

(iv) What more can be learned about the multifractal analysis of μ from the functions τ_L and their Legendre transforms?

(v) The finite type model can be thought of as an analog of the Markov chain model where the probabilities are replaced by matrices with non-negative entries. Can one develop an analogous theory? For instance, could one define an analogue of the notion of an invariant measure and use this measure to obtain information about the self-similar measure?

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Appendix A. Details of Example 5.13. As was noted in the example, there are four loop classes outside of the essential class and the points associated with these loop classes have local dimension bounded above from $\frac{\log 17/4}{\log 3}$. Hence it suffices to bound the local dimension of points in the essential class. One can check that the transition matrices within the essential class are given by

$$T(\gamma_9,\gamma_{15}) = \frac{1}{17}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 4 \\ 4 & 0 & 1 \end{bmatrix} \qquad T(\gamma_9,\gamma_{16}) = \frac{1}{17}\begin{bmatrix} 4 & 4 & 0 & 0 \\ 0 & 0 & 4 & 4 \\ 0 & 4 & 0 & 1 \end{bmatrix}$$

$$T(\gamma_9,\gamma_{17}) = \frac{1}{17}\begin{bmatrix} 0 & 4 & 4 & 0 \\ 1 & 0 & 0 & 4 \\ 4 & 0 & 4 & 0 \end{bmatrix} \qquad T(\gamma_9,\gamma_{18}) = \frac{1}{17}\begin{bmatrix} 0 & 4 & 4 \\ 1 & 0 & 0 \\ 4 & 0 & 4 \end{bmatrix}$$

$$T(\gamma_9,\gamma_{10}) = \frac{1}{17}\begin{bmatrix} 1 & 0 & 4 \\ 4 & 1 & 0 \\ 0 & 4 & 0 \end{bmatrix} \qquad T(\gamma_9,\gamma_{11}) = \frac{1}{17}\begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 0 & 4 \end{bmatrix}$$

$$T(\gamma_{10},\gamma_{12}) = \frac{1}{17}\begin{bmatrix} 4 & 0 \\ 4 & 1 \\ 4 & 4 \end{bmatrix} \qquad T(\gamma_{10},\gamma_{13}) = \frac{1}{17}\begin{bmatrix} 4 & 4 & 0 \\ 0 & 4 & 1 \\ 0 & 4 & 4 \end{bmatrix}$$

$$T(\gamma_{10},\gamma_9) = \frac{1}{17}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 0 & 4 \\ 0 & 0 & 4 \end{bmatrix} \qquad T(\gamma_{11},\gamma_{10}) = \frac{1}{17}\begin{bmatrix} 1 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

$$T(\gamma_{11},\gamma_{11}) = \frac{1}{17}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \qquad T(\gamma_{12},\gamma_{12}) = \frac{1}{17}\begin{bmatrix} 4 & 0 \\ 4 & 1 \end{bmatrix}$$

$$T(\gamma_{12},\gamma_{13}) = \frac{1}{17}\begin{bmatrix} 4 & 4 & 0 \\ 0 & 4 & 1 \end{bmatrix} \qquad T(\gamma_{12},\gamma_9) = \frac{1}{17}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 0 & 4 \end{bmatrix}$$

$$T(\gamma_{13},\gamma_{10}) = \frac{1}{17}\begin{bmatrix} 4 & 0 & 0 \\ 1 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix} \qquad T(\gamma_{13},\gamma_{13}) = \frac{1}{17}\begin{bmatrix} 4 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$T(\gamma_{15},\gamma_{15}) = \frac{1}{17}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 4 \\ 4 & 0 & 1 \end{bmatrix} \qquad T(\gamma_{15},\gamma_{16}) = \frac{1}{17}\begin{bmatrix} 4 & 4 & 0 & 0 \\ 0 & 0 & 4 & 4 \\ 0 & 4 & 0 & 1 \end{bmatrix}$$

$$T(\gamma_{15},\gamma_{17}) = \frac{1}{17}\begin{bmatrix} 0 & 4 & 4 & 0 \\ 1 & 0 & 0 & 4 \\ 4 & 0 & 4 & 0 \end{bmatrix} \qquad T(\gamma_{15},\gamma_{18}) = \frac{1}{17}\begin{bmatrix} 0 & 4 & 4 \\ 1 & 0 & 0 \\ 4 & 0 & 4 \end{bmatrix}$$

$$T(\gamma_{16},\gamma_{10}) = \frac{1}{17}\begin{bmatrix} 4 & 0 & 0 \\ 1 & 0 & 4 \\ 4 & 1 & 0 \\ 0 & 4 & 0 \end{bmatrix} \qquad T(\gamma_{16},\gamma_{13}) = \frac{1}{17}\begin{bmatrix} 4 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$T(\gamma_{17},\gamma_{15}) = \frac{1}{17}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 4 \\ 4 & 0 & 1 \\ 4 & 0 & 4 \end{bmatrix} \qquad T(\gamma_{17},\gamma_{16}) = \frac{1}{17}\begin{bmatrix} 4 & 4 & 0 & 0 \\ 0 & 0 & 4 & 4 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 0 & 4 \end{bmatrix}$$

$$T(\gamma_{17},\gamma_{17}) = \frac{1}{17}\begin{bmatrix} 0 & 4 & 4 & 0 \\ 1 & 0 & 0 & 4 \\ 4 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \qquad T(\gamma_{17},\gamma_{18}) = \frac{1}{17}\begin{bmatrix} 0 & 4 & 4 \\ 1 & 0 & 0 \\ 4 & 0 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$T(\gamma_{18},\gamma_{10}) = \frac{1}{17}\begin{bmatrix} 1 & 0 & 4 \\ 4 & 1 & 0 \\ 0 & 4 & 0 \end{bmatrix} \qquad T(\gamma_{18},\gamma_{11}) = \frac{1}{17}\begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 0 & 4 \end{bmatrix}$$

Consider the following following $K_i := Cone(V_i) = \{\sum a_j v_j : v_j \in V_i, \sum a_j \geq 1, a_j \geq 0\}$ where the V_i are given by

$$\begin{aligned}
V_9 &= \{[1,1,1],[4,0,0],[4,4,0],[8,0,8]\} \\
V_{10} &= \{[0,0,4],[0,4,0],[1,1,1],[2,0,2],[2,2,1/2],[16,4,4]\} \\
V_{11} &= \{[1,1],[2,0]\} \\
V_{12} &= \{[0,2],[1,1]\} \\
V_{13} &= \{[0,0,2],[0,4,3],[2,0,0],[2,8,0]\} \\
V_{15} &= \{[0,0,2],[1,1,1],[10,8,8],[20,0,52],[9/2,2,6]\} \\
V_{16} &= \{[0,0,4,4],[1,1,1,1],[2,0,8,0],[10,8,8,0],[20,0,52,20],[9/2,2,6,2]\} \\
V_{17} &= \{[0,4,4,0],[0,8,0,8],[0,52,20,16],[1,1,1,1],[2,6,2,5/2],[8,8,0,10]\} \\
V_{18} &= \{[1,1,1],[2,0,2],[4,4,0],[6,2,5/2],[8,0,10],[52,20,16]\}
\end{aligned}$$

One can check that $T(\gamma_i,\gamma_j)V_i \subset \frac{2}{17}V_j$ for all valid combinations of γ_i and γ_j. This implies that a product $\|T(\gamma_{i_0}\gamma_{i_1}\dots\gamma_{i_n})\| \geq \left(\frac{2}{17}\right)^n$ which proves the desired result.