Asian J. Math.
Vol. 25, No. 2, pp. 243–276, April 2021

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POISSON WAVE TRACE FORMULA FOR DIRAC RESONANCES AT SPECTRUM EDGES AND APPLICATIONS*

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Abstract. We study the self-adjoint Dirac operators $\mathbb{D} = \mathbb{D}_0 + V(x)$, where $\mathbb{D}_0$ is the free three-dimensional Dirac operator and $V(x)$ is a smooth compactly supported Hermitian matrix potential. We define resonances of $\mathbb{D}$ as poles of the meromorphic continuation of its cut-off resolvent. By analyzing the resolvent behaviour at the spectrum edges $\pm m$, we establish a generalized Birman-Krein formula, taking into account possible resonances at $\pm m$. As an application of the new Birman-Krein formula we establish the Poisson wave trace formula in its full generality. The Poisson wave trace formula links the resonances with the trace of the difference of the wave groups. The Poisson wave trace formula, in conjunction with asymptotics of the scattering phase, allows us to prove that, under certain natural assumptions on V, the perturbed Dirac operator has infinitely many resonances; a result similar in nature to Melrose's classic 1995 result for Schrödinger operators.

Key words. Scattering resonances, Dirac operators, Birman-Krein formula, Poisson wave trace formula, threshold resolvent behaviour.

Mathematics Subject Classification. 35Q41 (35P25, 47A40, 81U20,81U24).

1. Introduction. The first Poisson wave trace formula (in the sense of distributions) of the form

$$2\,\mathrm{Tr}\,\left(\cos t\sqrt{L} - \cos t\sqrt{-\Delta}\right) = \sum e^{it\lambda_j}, \tag{1.1}$$

where L is a (suitable) perturbed operator of the Laplacian $-\Delta$, on $L^2(\mathbb{R}^n)$ with n odd, and the sum extends over all resonances λ_j of L, is attributed to Lax and Phillips [17]. They proved the formula for obstacle scattering, but only for $t > 4R$ where the obstacle is located within a ball of radius R. Subsequently, Bardos, Guillot and Ralston [3] studied the exact distribution of scattering poles associated with perturbations of the wave equation in odd dimensions and one of their main ingredients is an extension of the trace formula (1.1), still based on Lax-Phillips theory [17], but again only valid for comparatively large values of t, namely $t > 2R$. Melrose [18] proved the Poisson formula (1.1) for *all* $t \neq 0$ in the case of compactly supported potentials and, subsequently, Sjöstrand and Zworski [24] carried it over to more general operators L. Later, Zworski [26] gave a new proof of the trace formula that includes $t = 0$ and avoids the use of Lax-Phillips theory and instead is based on an estimate of the scattering determinant. The Poisson trace formula has been applied to obtain lower bounds for the number of resonances in certain regions and to prove the existence of infinitely many resonances, see e.g. [19], [23] and [24]. An overview of key results regarding trace formulas for Schrödinger operators H and resonances in general can be found in [12]. Therein resonances of the Schrödinger operator is studied via the so-called cut-off resolvent $\rho(H-\lambda^2)^{-1}\rho$. Given ρ is a compactly supported bump function, one considers the meromorphic extension of the cut-off resolvent from the *physical* $\lambda = \sqrt{z}$ upper half-plane, across the real line, to $\mathbb{C}$. In the lower *unphysical* plane, the first bump function enables one to consider the resolvent as a mapping from $L^2_{\mathrm{comp}}(\mathbb{R}^3)$ to $L^2_{\mathrm{loc}}(\mathbb{R}^3)$. The second bump function therefore ensures the resultant

*Received October 29, 2019; accepted for publication July 31, 2020.
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mapping is back to $L^2_{\text{comp}}(\mathbb{R}^3)$. The isolated poles of $\rho(H-\lambda^2)^{-1}\rho$ in the unphysical plane are in turn defined as resonances.

Within the relativistic theory, a Poisson wave trace formula was first established by Kungsman and Melgaard [16] for the perturbed Dirac operator $\mathbb{D} := \mathbb{D}_0 + V(x)$ where $\mathbb{D}_0$ is the free Dirac operator in three dimensions (see below) and V is a smooth compactly supported matrix potential. In the present paper we go beyond [16]. As above, resonances are defined as poles of the meromorphic continuation of the cut-off resolvent. If $\mathcal{R}$ denotes the set of resonances of $\mathbb{D}$ and $m(k_j)$ is the multiplicity of a resonance k_j (see Section 2.3 for precise definitions of $\mathcal{R} = \mathcal{R}_- \cup \mathcal{R}_+$ and $m(k_j)$), we prove that

$$\begin{aligned} &2t^4 \operatorname{Tr}\left[\cos\left(t\sqrt{\mathbb{D}^2 - m^2}\right) - \cos\left(t\sqrt{\mathbb{D}_0^2 - m^2}\right)\right] \\ &= t^4 \sum_{\pm} \sum_{k_j \in \mathcal{R}_\pm} \pm m(k_j) e^{-i|t|k_j} + 2t^4 \sum_{E_j} m_j \cosh\left(|t|\sqrt{m^2 - E_j^2}\right) \end{aligned} \tag{1.2}$$

in the sense of distributions for all $t \in \mathbb{R}$, where E_j denote the eigenvalues of $\mathbb{D}$ with multiplicity m_j. The significance of the present work is that we allow that the spectrum edges, denoted $\pm m$, of the (absolutely) continuous spectrum of $\mathbb{D}$ belong to $\mathcal{R}$; this difficult case was not treated in [16]. This requires a substantial analysis of the resolvent behaviour at the threshold energies $\pm m$ (see Section 3), not previously carried out for the Dirac operator (and not found in [16]). The latter allows us to prove a generalized version of the Birman-Krein formula, Theorem 4.2 below, which has interest in its own right. Its proof applies the Dynkin-Helffer-Sjöstrand formula and involves Gohberg-Sigal theory. By means of the Birman-Krein formula, Weierstrass's factorization theorem, an upper bound for the resonance counting function [16], and an estimate on the scattering determinant, we establish the Poisson wave trace formula (1.2) in Theorem 5.1.

Prior to the present work and [16], the only trace formula for Dirac operators involving resonances known to us is that of Khochman [14], where a local trace formula for resonances in the spirit of Sjöstrand [22] is established.

As an application of the Poisson formula, in conjunction with asymptotics of the scattering phase, we prove that, under certain natural assumptions on V, the perturbed Dirac operator has infinitely many resonances (see Section 6); a very significant result which should be compared to a similar result for Schrödinger operators established by Melrose in 1995 [19] (see also [7]).

2. Preliminaries.

2.1. The Dirac operator. To discuss perturbed Dirac operators we begin by considering the free, or unperturbed, Dirac operator. The free Dirac operator, describing the motion of a relativistic electron or positron without external forces, is the unique self-adjoint extension of the symmetric operator

$$\mathbb{D}_0 = -i\sum_{j=1}^{3} \alpha_j \partial_j + m\beta = -i\boldsymbol{\alpha}\cdot\nabla + m\beta, \qquad \partial_j := \frac{\partial}{\partial x_j},\ \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3),$$

defined on $C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$ in the Hilbert space $L^2(\mathbb{R}^3;\mathbb{C}^4) \simeq L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$ or, with a slight abuse of notation, $L^2(\mathbb{R}^3)^4$. Here the α_j are symmetric 4×4 matrices satisfying the anti-commutation relations

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I_4, \quad j,k = 1,2,3, \tag{2.1}$$

$$\alpha_j\beta + \beta\alpha_j = 0, \qquad j = 1,2,3 \tag{2.2}$$

and $\beta^2 = I_4$; the 4×4 identity matrix which we henceforth denote by I. The extension, which we also denote by $\mathbb{D}_0$, acts on the Hilbert space $L^2(\mathbb{R}^3;\mathbb{C}^4)$ equipped with the inner product

$$\langle u,v\rangle_{L^2(\mathbb{R}^3;\mathbb{C}^4)} = \sum_{j=1}^{4}\int\limits_{\mathbb{R}^3} u_j(\boldsymbol{x})\overline{v_j(\boldsymbol{x})}\,d\boldsymbol{x} \quad \text{where } u = (u_j)_{1\le j\le 4},\ v = (v_j)_{1\le j\le 4}$$

and it has domain $H^1(\mathbb{R}^3)^4$; the Sobolev space of order one. When there is no risk of confusion we sometimes just write L^2 and H^1, respectively. It is well-known (see, e.g., Thaller [25]) that the spectrum of $\mathbb{D}_0$ is purely absolutely continuous, viz.

$$\operatorname{spec}(\mathbb{D}_0) = \operatorname{spec}_{\mathrm{ac}}(\mathbb{D}_0) = (-\infty,-m]\cup[m,\infty).$$

On the resolvent set $\rho(\mathbb{D}_0) = \mathbb{C}\setminus\operatorname{spec}(\mathbb{D}_0)$ we denote the free resolvent $(\mathbb{D}_0 - z)^{-1}$ by $R_0(z)$. As usual the Fourier transform is defined by

$$(\mathscr{F}u)(\boldsymbol{\xi}) = (2\pi)^{-3/2}\int\limits_{\mathbb{R}^3} e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}}u(\boldsymbol{x})\,d\boldsymbol{x}.$$

In the momentum space $(\mathscr{F}L^2(\mathbb{R}^3,\,\mathrm{d}\boldsymbol{x})^4) = L^2(\mathbb{R}^3,\,\mathrm{d}\boldsymbol{\xi})^4$, the free Dirac operator $\mathbb{D}_0$ acts as a multiplication matrix in the form

$$(\mathscr{F}\mathbb{D}_0\mathscr{F}^{-1})(\boldsymbol{\xi}) = \boldsymbol{\alpha}\cdot\boldsymbol{\xi} + m\beta. \tag{2.3}$$

For each $\boldsymbol{\xi} = (\xi_j)_{1\le j\le 3}$, this is a 4×4 Hermitian matrix with eigenvalues given by

$$\lambda_{1,2} = -\lambda_{3,4} = \sqrt{|\boldsymbol{\xi}|^2 + m^2} =: \lambda(\boldsymbol{\xi}),$$

where $\sqrt{\ }$ is the holomorphic square-root on $\mathbb{C}\setminus[0,\infty)$, and the projections onto the corresponding eigenspaces are given by

$$\Pi_{\pm}(\boldsymbol{\xi}) = \frac{1}{2}\left(I \pm \frac{\boldsymbol{\alpha}\cdot\boldsymbol{\xi} + m\beta}{\sqrt{|\boldsymbol{\xi}|^2 + m^2}}\right). \tag{2.4}$$

Introducing

$$U = \frac{(m+\lambda(\boldsymbol{\xi}))I_4 + m\beta\boldsymbol{\alpha}\cdot\boldsymbol{\xi}}{\sqrt{2\lambda(\boldsymbol{\xi})(m+\lambda(\boldsymbol{\xi}))}}$$

it can be shown that $\mathbb{D}_0$ and $\beta\lambda(\boldsymbol{\xi})$ are unitarily equivalent under the unitary transformation $U\mathscr{F}$. In the standard representation, this diagonalizes the free Dirac operator,

$$U\mathscr{F}\mathbb{D}_0(U\mathscr{F})^{-1}(\boldsymbol{\xi}) = \beta\lambda(\boldsymbol{\xi}).$$

We are going to consider perturbations of $\mathbb{D}_0$ by smooth compactly supported Hermitian 4×4 matrix potentials $V \in C_0^\infty(\mathbb{R}^3)\otimes M_4(\mathbb{C})$; $M_4(\mathbb{C})$ being the set of

4×4 matrices over $\mathbb{C}$, equipped with the operator norm, designated by $\| \cdot \|_{4\times 4}$. The resulting self-adjoint operator $\mathbb{D} = \mathbb{D}_0 + V$ (defined via Kato-Rellich's theorem) has domain $H^1(\mathbb{R}^3)^4$ and, according to Weyl's theorem, $\mathrm{spec}_{\mathrm{ess}}(\mathbb{D}) = (-\infty, -m] \cup [m, \infty)$. In addition, $\mathbb{D}$ can have finitely many eigenvalues of finite multiplicity in $(-m, m)$ (see, e.g., [25, Theorem 4.23]). It is well-known that under our assumptions on V there are no eigenvalues λ with $|\lambda| > m$ embedded in the continuous spectrum (see, e.g., [2]). For our main results we restrict ourselves to real-valued potentials:

ASSUMPTION 2.1. Let $V : \mathbb{R}^3 \to M_4(\mathbb{R})$ be a smooth, compactly supported Hermitian 4×4 matrix potential.

2.2. Resolvent of free Dirac operator. The resolvent of the free Dirac operator

$$R_0(z) := (\mathbb{D}_0 - z)^{-1} : L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4$$

is defined for $z \in \rho(\mathbb{D}_0)$. From the identity $\mathbb{D}_0^2 = (-\Delta + m^2)I$ it follows that

$$R_0(z) = (\mathbb{D}_0 + z) R_{00}(\sqrt{z^2 - m^2}), \tag{2.5}$$

where $R_{00}(z) = (-\Delta - z^2)^{-1}$. It is well-known that $R_{00}(z)$ is a convolution operator (see e.g. [19]) and that its kernel is given by $(4\pi)^{-1}|x|^{-1} e^{iz|x|}$ where Im $z > 0$. One can easily show that, for $z \in \rho(\mathbb{D}_0)$, $(R_0(z)u)(\boldsymbol{x}) = \int_{\mathbb{R}^3} G_0(\boldsymbol{x} - \boldsymbol{y}; z) u(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}$, $u \in \mathscr{S}(\mathbb{R}^3)$ (the Schwartz space of rapidly decreasing functions), where the kernel $G_0(\boldsymbol{x}; z)$ is given explicitly by

$$G_0(\boldsymbol{x}; z) = \left(i \frac{\boldsymbol{\alpha} \cdot \boldsymbol{x}}{|\boldsymbol{x}|^2} + k(z) \frac{\boldsymbol{\alpha} \cdot \boldsymbol{x}}{|\boldsymbol{x}|} + \beta m + z \right) \frac{e^{ik(z)|\boldsymbol{x}|}}{4\pi |\boldsymbol{x}|} \tag{2.6}$$

with $k(z) := \sqrt{z^2 - m^2}$, Im $k(z) > 0$ and, moreover, it extends to a bounded operator on $L^2(\mathbb{R}^3)$. This change of variable has the effect of mapping $\rho(\mathbb{D}_0)$ to a pair of half-planes in the k variable. Since we choose the branch of the square root such that Im $k > 0$, then $\rho(\mathbb{D}_0)$ maps to the upper (or *physical*) half-plane. The lower half-plane will be named *unphysical*. We note here that the mapping $k := k(z)$ cancels any negative sign of the spectral parameter z. Therefore for the inverse map $k \mapsto z(k)$, we use the negative prefactor to restore this negativity and write $z = \pm\sqrt{k^2 + m^2}$. We define

$$R_0^{\pm}(k) = R_0(\pm z(k)) = \begin{cases} (\mathbb{D}_0 - \sqrt{k^2 + m^2})^{-1}, & + \\ (\mathbb{D}_0 + \sqrt{k^2 + m^2})^{-1}, & - \end{cases} \tag{2.7}$$

where $\pm$ refers to the prefactor of z above and we let the corresponding kernels be denoted by $G_0^{\pm}(\bullet; k)$. Then it is easy to verify that

$$G_0^{\pm}(\boldsymbol{x} - \boldsymbol{y}; k) = \left[\frac{i\boldsymbol{\alpha} \cdot (\boldsymbol{x} - \boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^2} + k \frac{\boldsymbol{\alpha} \cdot (\boldsymbol{x} - \boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|} + \beta m \pm \sqrt{k^2 + m^2} \right] \frac{e^{ik|\boldsymbol{x} - \boldsymbol{y}|}}{4\pi |\boldsymbol{x} - \boldsymbol{y}|} \tag{2.8}$$

with the relations

$$G_0^{\pm}(\boldsymbol{x}-\boldsymbol{y};0) = \left[\frac{i\boldsymbol{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + m(\beta \pm I)\right]\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}, \tag{2.9}$$

$$\partial_k G_0^{\pm}(\boldsymbol{x}-\boldsymbol{y};k) = \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi}\left\{ im\beta \pm i\sqrt{k^2+m^2} + \right. \tag{2.10}$$

$$\left. + \frac{k}{|\boldsymbol{x}-\boldsymbol{y}|}\left(i\boldsymbol{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y}) \pm \frac{1}{\sqrt{k^2+m^2}}\right)\right\},$$

$$\partial_k G_0^{\pm}(\boldsymbol{x}-\boldsymbol{y};0) = (\beta \pm I)\frac{im}{4\pi}. \tag{2.11}$$

By standard arguments, one verifies that the free Dirac resolvents, $R_0^{\pm}(k)$, defined for $\text{Im } k > 0$, continues holomorphically to $\text{Im } k < 0$, to the family of operators

$$R_0^{\pm}(k) \,:\, L^2_{\text{comp}}(\mathbb{R}^3)^4 \to L^2_{\text{loc}}(\mathbb{R}^3)^4.$$

Here L^2_{comp} denotes the space of L^2-functions which equals zero outside a compact set and, for two normed linear vector spaces $\mathcal{K}_1$ and $\mathcal{K}_2$, a holomorphic function $A(z)$, defined on an open set $\Omega \subset \mathbb{C}$, with values in $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$, is a function with values in the space of linear operator from $\mathcal{K}_1$ into $\mathcal{K}_2$ such that $\chi_1 A(z)\chi_2$ is holomorphic for all $\chi_j \in C_0^{\infty}(\mathbb{R}^3)$.

For any $\rho \in \mathscr{D}(\mathbb{R}^3)$, the space of test functions, and $\text{diam}\,(\text{supp}\,\rho) < L$ finite, we have the estimates

$$\|\rho R_0^{\pm}(k)\rho\|_{L^2\to H^j} \le C\langle k\rangle^j e^{L(\text{Im } k)_-}, \qquad j = 0, 1, \tag{2.12}$$

where $(y)_- = \max(-y, 0)$ and $\langle y\rangle = \sqrt{1+|y|^2}$. This follows from similar cut-off estimates for the Schrödinger operator (see, e.g., [7, Section 3.1]) and (2.5).

We clarify our notation here when $k \in \mathbb{R}\setminus\{0\}$. In this case, the following limits, taken in the sense of operators $C_0^{\infty} \to C^{\infty}$ (with improved versions possible), rewritten in the k-plane, exist [1]:

$$\begin{aligned} R_0^{\pm}(k) &= \lim_{\epsilon\to 0^+}(\mathbb{D}_0 \mp \sqrt{k^2+m^2} - i\epsilon)^{-1}, \\ R_0^{\pm}(-k) &= \lim_{\epsilon\to 0^+}(\mathbb{D}_0 \mp \sqrt{k^2+m^2} + i\epsilon)^{-1}, \end{aligned} \tag{2.13}$$

where $\epsilon > 0$ This is analogous to the limiting absorption principle for free Laplacians,

$$R_{00}(\lambda) = \lim_{\epsilon\to 0^+}(-\Delta - \lambda^2 - i\epsilon)^{-1}, \quad R_{00}(-\lambda) = \lim_{\epsilon\to 0^+}(-\Delta - \lambda^2 + i\epsilon)^{-1},$$

again in the sense of operators $C_0^{\infty} \to C^{\infty}$. From (2.5) we obtain

$$R_0^{\pm}(k) = (\mathbb{D}_0 \pm \sqrt{k^2+m^2})R_{00}(k)$$

and then $\mathscr{F}\mathbb{D}_0\mathscr{F}^{-1} = k\boldsymbol{\alpha}\cdot\boldsymbol{\omega} + m\beta$ gives us that

$$
\begin{aligned}
&\left[\rho(R_0^{\pm}(k) - R_0^{\pm}(-k))\rho f\right](\boldsymbol{x}) \\
&= \rho(\boldsymbol{x})\left[(\mathbb{D}_0 \pm \sqrt{k^2+m^2})\left(R_{00}(k) - R_{00}(-k)\right)\rho f\right](\boldsymbol{x}) \\
&= \frac{ik}{8\pi^2}\rho(\boldsymbol{x})(\mathbb{D}_0 \pm \sqrt{k^2+m^2})\int_{\mathbb{S}^2} e^{ik\boldsymbol{\omega}\cdot\boldsymbol{x}}\int_{\mathbb{R}^3} e^{-ik\boldsymbol{\omega}\cdot\boldsymbol{y}}\rho(\boldsymbol{y})f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{\omega} \\
&= \frac{ik\sqrt{k^2+m^2}}{8\pi^2}\rho(\boldsymbol{x})\int_{\mathbb{S}^2} e^{ik\boldsymbol{\omega}\cdot\boldsymbol{x}}\left(\frac{k\alpha\cdot\boldsymbol{\omega}+m\beta}{\sqrt{k^2+m^2}} \pm I\right)\int_{\mathbb{R}^3} e^{-ik\boldsymbol{\omega}\cdot\boldsymbol{y}}\rho(\boldsymbol{y})f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{\omega} \\
&= \pm\frac{ik\sqrt{k^2+m^2}}{4\pi^2}\int_{\mathbb{S}^2} e^{ik\boldsymbol{\omega}\cdot\boldsymbol{x}}\rho(\boldsymbol{x})\Pi_{\pm}(k\boldsymbol{\omega})\int_{\mathbb{R}^3} e^{-ik\boldsymbol{\omega}\cdot\boldsymbol{y}}\rho(\boldsymbol{y})f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{\omega} \\
&= \pm\frac{ik\sqrt{k^2+m^2}}{4\pi^2}\left(E_{\pm}(k)^*E_{\pm}(k)f\right)(\boldsymbol{x}), \qquad (2.14)
\end{aligned}
$$

where $E_{\pm}(k)$ is defined in Section 2.4.

2.3. Resonances. Let Assumption 2.1 hold. Consider the perturbed resolvent

$$R_V(z) := (\mathbb{D} - z)^{-1} = (\mathbb{D}_0 + V - z)^{-1}$$

defined for $z \in \rho(\mathbb{D})$. Analogously to $R_0^{\pm}(k)$, we define $R_V^{\pm}(k) := R_V(\pm z(k))$. Then the perturbed resolvents

$$R_V^{\pm}(k) := (\mathbb{D} \mp \sqrt{k^2+m^2})^{-1} \;:\; L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4 \qquad (2.15)$$

defined for $\operatorname{Im} k > 0$ is a meromorphic family of operators with a finite number of poles, associated with the eigenvalues of $\mathbb{D}$. It extends to a meromorphic family of operators, for $k \in \mathbb{C}$,

$$R_V^{\pm}(k) \;:\; L^2_{\mathrm{comp}}(\mathbb{R}^3)^4 \to L^2_{\mathrm{loc}}(\mathbb{R}^3)^4.$$

Recall that for an open $\Omega \subset \mathbb{C}$ and two normed linear vector spaces $\mathcal{K}_1$ and $\mathcal{K}_2$, a meromorphic function $A(k)$ is one which is holomorphic on $\Omega \setminus S$, where $S \subset \Omega$ is a discrete set, and such that if $k_0 \in S$ then near k_0 we have

$$A(k) = \sum_{j=1}^{N} \frac{A_j}{(k-k_0)^j} + B(k)$$

with $A_j : \mathcal{K}_1 \to \mathcal{K}_2$ (continuous in the sense that $\chi_1 A_j \chi_2$ is bounded for all $\chi_j \in C_0^\infty(\mathbb{R}^3)$) of finite rank, and $B(k)$ holomorphic with values in $\mathcal{B}(\mathcal{K}_1,\mathcal{K}_2)$ for k in a neighborhood of k_0.

The poles of the meromorphic extension of $R_V^{\pm}(k)$ satisfying

$$R_V^{\pm}(k) = R_0^{\pm}(k)(I+VR_0^{\pm}(k))^{-1} = R_0^{\pm}(k)(I+VR_0^{\pm}(k)\rho)^{-1}(I-VR_0^{\pm}(k)(I-\rho)) \quad (2.16)$$

coincides with the poles of $(I + VR_0^{\pm}(k)\rho)^{-1}$, $\rho \in \mathscr{D}(\mathbb{R}^3)$, and are referred to as (scattering) resonances of $\mathbb{D}$; they are located in the lower complex plane. So, the resonances are defined on the Riemann surface of the function $z \mapsto k(z) = \sqrt{z^2-m^2}$. This surface has two finite branch points at $z = \pm m$ and two infinite branch points at $\pm\infty$. At both of these finite points there is a simple zero, so analytic continuation around them enables one to get to the other sheet. These two sets of resonances of

$\mathbb{D}$ are denoted by $\mathcal{R}_\pm$, their union is denoted $\mathcal{R} := \mathcal{R}_- \cup \mathcal{R}_+$, and their multiplicities $m_R(k)$ are defined by

$$m_R(k) := \dim \operatorname{span} \{A_1^\pm(L^2_{\text{comp}}), \ldots, A_N^\pm(L^2_{\text{comp}})\} \tag{2.17}$$

where

$$R_V^\pm(\zeta) = \sum_{n=1}^{N} \frac{A_n^\pm}{(\zeta - k)^n} + A^\pm(\zeta, k) \tag{2.18}$$

and $\zeta \mapsto A^\pm(\zeta, k)$ is holomorphic near k. When we study $\mathcal{R}_\pm$, the eigenvalues $E_j = \pm\sqrt{m^2 - (\mu_j^\pm)^2}$ correspond to the poles $k = \pm i\mu_j^\pm$ of $R_V^\pm(k)$, with $\mu_j^\pm \in (0, m]$ (the value $\mu_j^\pm = m$ is possible because $E_j = 0$ is not excluded). In Section 3 we study the structure of (2.18) near $k = 0$ (associated with the original thresholds $\pm m$). Note that, in principle, we have two variables $k^\pm \in \mathcal{R}_\pm$ but, with a slight abuse of notation, we keep with k.

We recall the following result from [16, Prop. 2.2]; here slightly improved.

Proposition 2.2. *Let Assumption 2.1 hold and suppose $\rho \in \mathscr{D}(\mathbb{R}^3)$ such that $\rho V = V$. If we also set*

$$H^\pm(k) := \det(I - (V R_0^\pm(k)\rho)^4).$$

Then

$$|H^\pm(k)| \leq \exp(C|k|^4). \tag{2.19}$$

Moreover, the number of resonances inside the disk $D(0; r)$ (for each plane) denoted

$$N(r) := \#\{\, k \in \mathcal{R} \,:\, |k| \leq r \,\}$$

satisfies

$$N(r) \leq C r^4. \tag{2.20}$$

2.4. Scattering theory. Under Assumption 2.1, the wave operators

$$W_\pm = \operatorname*{s-lim}_{t \to \pm\infty} e^{it\mathbb{D}} e^{-it\mathbb{D}_0} \tag{2.21}$$

exist, are asymptotically complete and fulfill the intertwining relation $\mathbb{D} W_\pm = W_\pm \mathbb{D}_0$ (see e.g. [25]). The scattering operator $S = W_+^* W_-$ for the pair $(\mathbb{D}, \mathbb{D}_0)$ is unitary on $L^2(\mathbb{R}^3)$ and commutes with $\mathbb{D}_0$. This enables a reduction of the scattering operator to a family of operators defined on $L^2(\mathbb{S}^2; \mathbb{C}^4)$ called the scattering matrix which maps incoming components to outgoing ones of a solution to

$$(\mathbb{D} \mp \sqrt{k^2 + m^2}) w^\pm = 0. \tag{2.22}$$

We start from solutions of the type

$$w^\pm(\boldsymbol{x}, k, \boldsymbol{\omega}) = e^{-ik\boldsymbol{x}\cdot\boldsymbol{\omega}} + u^\pm(\boldsymbol{x}, k, \boldsymbol{\omega}) \tag{2.23}$$

where the outgoing $u^{\pm}$ is obtained from the resolvent $R_V^{\pm}(k)$, except at possible poles,

$$u^{\pm}(\boldsymbol{x},k,\boldsymbol{\omega}) := -R_V^{\pm}(k)Ve^{-ik\boldsymbol{x}\cdot\boldsymbol{\omega}}. \tag{2.24}$$

A solution $u^{\pm}$ to $(\mathbb{D}\mp\sqrt{k^2+m^2})u^{\pm}=f^{\pm}$, $k\in\mathbb{R}\setminus\{0\}$, $f^{\pm}\in L^2_{\text{comp}}(\mathbb{R}^3;\mathbb{R}^4)$, is called outgoing if there exist $g^{\pm}\in L^2_{\text{comp}}(\mathbb{R}^3;\mathbb{R}^4)$ such that $u^{\pm}=R_0^{\pm}(k)g^{\pm}$, where $R_0^{\pm}(k)$ is the free resolvent given in (2.7). A solution $u^{\pm}$ is called incoming if $u^{\pm}=R_0^{\pm}(-k)g^{\pm}$, $k\in\mathbb{R}\setminus\{0\}$, for some $g^{\pm}\in L^2_{\text{comp}}(\mathbb{R}^3;\mathbb{R}^4)$. For $f^{\pm}\in\mathscr{S}(\mathbb{R}^3)$, $k\in\mathbb{R}\setminus\{0\}$, $\boldsymbol{x}=r\boldsymbol{\theta}$, $\boldsymbol{\theta}\in\mathbb{S}^2$, we have that

$$R_0^{\pm}(k)f^{\pm}(r\boldsymbol{\theta}) = h_0^{\pm}(\boldsymbol{\theta})\frac{e^{ikr}}{4\pi r}+\mathcal{O}(r^{-2}), \tag{2.25}$$

where $h_0^{\pm}(\boldsymbol{\theta})=(\beta m\pm\sqrt{k^2+m^2}+k\boldsymbol{\alpha}\cdot\boldsymbol{\theta})(2\pi)^{3/2}\widehat{f}^{\pm}(k\boldsymbol{\theta})$. Moreover, using the method of stationary phase [13], we obtain, in the sense of distributions,

$$e^{-ik\boldsymbol{x}\cdot\boldsymbol{\omega}} \sim \frac{2\pi i}{kr}\left[e^{-ikr}\delta(\boldsymbol{\theta}-\boldsymbol{\omega})-e^{ikr}\delta(\boldsymbol{\theta}+\boldsymbol{\omega})\right], \text{ as } r\to\infty. \tag{2.26}$$

Then the leading term of $w^{\pm}$ in (2.23) can be written

$$w^{\pm}(r\boldsymbol{\theta}) \sim \frac{2\pi i}{kr}\left[e^{-ikr}\delta(\boldsymbol{\theta}-\boldsymbol{\omega})-e^{ikr}\left(\delta(\boldsymbol{\theta}+\boldsymbol{\omega})+b^{\pm}(k,\boldsymbol{\theta},\boldsymbol{\omega})\right)\right], \tag{2.27}$$

where the scattering amplitude $b^{\pm}(k,\boldsymbol{\theta},\boldsymbol{\omega})$ is the leading asymptotic term of

$$u^{\pm}(\boldsymbol{x},k,\boldsymbol{\omega}) = -\frac{2\pi i}{kr}e^{ikr}b^{\pm}(k,\boldsymbol{\theta},\boldsymbol{\omega})+\mathcal{O}\left(r^{-2}\right). \tag{2.28}$$

The e^{ikr} prefactor is due to $R_V^{\pm}(k)=R_0^{\pm}(k)(I+VR_0^{\pm}(k)\rho)^{-1}$ and (2.25). Then the so-called absolute scattering matrix maps incoming (e^{-ikr}) to the outgoing (e^{ikr}) terms above:

$$S^{\pm}_{\text{abs}}(k) \,:\, \delta(\boldsymbol{\theta}-\boldsymbol{\omega})\mapsto -(\delta(\boldsymbol{\theta}+\boldsymbol{\omega})+b^{\pm}(k,\boldsymbol{\theta},\boldsymbol{\omega})). \tag{2.29}$$

By setting $V=0$, we note that $S^{\pm}_{\text{abs}}(k)f(\boldsymbol{\theta})=-f(\boldsymbol{\theta})$. By normalizing we define the scattering matrix

$$S^{\pm}(k) \,:\, \delta(\boldsymbol{\theta}-\boldsymbol{\omega})\mapsto \delta(\boldsymbol{\theta}-\boldsymbol{\omega})+b^{\pm}(k,\boldsymbol{\theta},-\boldsymbol{\omega}). \tag{2.30}$$

Here

$$S^{\pm}(k) = -S^{\pm}_{\text{abs}}(k)J \text{ with } Jf(\boldsymbol{\theta})=f(-\boldsymbol{\theta}). \tag{2.31}$$

By standard arguments [21], the scattering matrix $S^{\pm}(k)\,:\,L^2(\mathbb{S}^2;\mathbb{C}^4)\to L^2(\mathbb{S}^2;\mathbb{C}^4)$ has the representation $S^{\pm}(k)=I-A^{\pm}(k)$ where $A^{\pm}(k)$ is given by

$$A^{\pm}(k) = \pm\frac{ik\sqrt{k^2+m^2}}{4\pi^2}E_{\pm}(k)(I+VR_0^{\pm}(k)\rho)^{-1}VE_{\pm}(k)^*, \tag{2.32}$$

where $E_{\pm}(k)\,:\,L^2(\mathbb{R}^3)^4\to L^2(\mathbb{S}^2)^4$ has integral kernel

$$E_{\pm}(\boldsymbol{x},\boldsymbol{\omega}) = \Pi_{\pm}(k\boldsymbol{\theta})\rho(\boldsymbol{x})e^{-ik\langle\boldsymbol{x},\boldsymbol{\omega}\rangle};$$

here $\Pi_{\pm}(k\boldsymbol{\theta})$ is defined in (2.4).

If $V : \mathbb{R}^3 \to M_4(\mathbb{R})$ is bounded, $k \in \mathbb{R} \setminus \{0\}$ and $g^\pm \in C^\infty(\mathbb{S}^2)^4$, then there exists $f^\pm \in C^\infty(\mathbb{S}^2)^4$ and $v^\pm \in H^1_{\mathrm{loc}}(\mathbb{R}^3)^4$ such that

$$(\mathbb{D} \mp \sqrt{k^2 + m^2}) v^\pm = 0 \tag{2.33}$$

and

$$v^\pm(r\boldsymbol{\theta}) = \frac{C}{r}\left(e^{-ikr} g^\pm(\boldsymbol{\theta}) + e^{ikr} f^\pm(\boldsymbol{\theta})\right) + \mathcal{O}\left(r^{-2}\right). \tag{2.34}$$

Furthermore, $g^\pm$ and $f^\pm$ are related by the scattering matrices (2.29) and (2.30), viz.

$$S^\pm_{\mathrm{abs}}(k) : g^\pm(\boldsymbol{\theta}) \mapsto f^\pm(\boldsymbol{\theta}), \tag{2.35}$$

$$S^\pm(k) : -g^\pm(-\boldsymbol{\theta}) \mapsto f^\pm(\boldsymbol{\theta}). \tag{2.36}$$

If, moreover, Assumption 2.1 holds and we define the scattering determinant

$$s^\pm(k) := \det S^\pm(k) \tag{2.37}$$

then, with J defined in (2.31),

$$S^\pm(k)^{-1} = J S^\pm(-k) J \text{ and } s^\pm(k)^{-1} = s^\pm(-k). \tag{2.38}$$

From (2.16) and (2.32) we see that the resonances will appear as poles of the scattering matrix $S^\pm(k)$. It also follows from (2.32) that resonances appear as poles of the scattering determinant $s^\pm(k)$ in (2.37). Furthermore, if $\rho \in \mathscr{D}(\mathbb{R}^3)$ satisfies $\rho V = V$, then (2.32), the Fredholm determinant, trace cyclicity and the Jacobi determinant formula, in conjunction with the spectral theorem and (2.14), yields

$$\begin{aligned}
\mathrm{Tr}\,\left[S^\pm(k)^{-1}\partial_k S^\pm(k)\right] &= \mathrm{Tr}\,F^\pm(k) + \mathrm{Tr}\,F^\pm(-k),\\
F^\pm(k) &:= \mp\frac{k}{\sqrt{k^2+m^2}} R_0^\pm(k)(I + V R_0^\pm(k)\rho)^{-1} V R_0^\pm(k),\\
F^\pm(-k) &:= \pm\frac{k}{\sqrt{k^2+m^2}} R_0^\pm(-k)(I + V R_0^\pm(-k)\rho)^{-1} V R_0^\pm(-k).
\end{aligned} \tag{2.39}$$

2.5. Dynkin-Helffer-Sjöstrand type formula. We give a variant of the Cauchy-Riemann-Green-Stokes formula, which was used by Dynkin [8] and has been widely used since the work by Helffer and Sjöstrand [11].

Lemma 2.3. *Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Suppose $f \in \mathscr{S}(\mathbb{R})$ such that $f(z) = (z - z_0)^{-N} g(z)$ where $\mathrm{Im}\, z_0 > 0$ and $g \in \mathscr{S}(\mathbb{R})$ with an almost-analytic extension $\tilde{g}$ satisfying*

$$\tilde{g} \in \mathscr{S}(\mathbb{C}), \qquad \operatorname{supp} g \subset \{\,|\,\mathrm{Im}\, z| < 1\}.$$

Then

$$f(T) = \frac{1}{\pi i}\int_{\mathbb{C}} (T - z)^{-1}(T - z_0)^{-N} \overline{\partial}_z \tilde{g}(z)\, \mathrm{dm}(z) \tag{2.40}$$

where m denotes the Lebesgue measure on $\mathbb{C}$ and $\bar{\partial}_z$ is the z-bar derivative.

3. Resolvent near energies plus/minus m. We analyze the threshold behaviour of the resolvent of the perturbed Dirac operator; bear in mind the definition of $R_0^{\pm}(k)$ in (2.7) and $R_V^{\pm}(k)$ defined just above (2.15). Throughout this section, we let Assumption 2.1 hold.

We introduce the following subspaces

$$\begin{aligned} V_{\pm} &:= \{\, v_{\pm} \in H^1(\mathbb{R}^3)^4 \;:\; (\mathbb{D} \mp m)v_{\pm} = \mathbf{0}\,\}, \\ U_{\pm} &:= \{\, u_{\pm} \in H^1_{\mathrm{loc}}(\mathbb{R}^3)^4 \;:\; (\mathbb{D} \mp m)u_{\pm} = \mathbf{0}, \quad u_{\pm} = R_0^{\pm}(0)(\mathbb{D}_0 \pm m)u_{\pm}\,\}, \end{aligned} \tag{3.1}$$

and the orthogonal projection $\Pi_{\pm} : L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4$ that maps $L^2(\mathbb{R}^3)^4$ functions into $V_{\pm}$.

When we change variables from z to k, the resonances at $z = \pm m$ both map to the origin of the k plane. The (non-imaginary) eigenvalues of $\mathbb{D}_0$, denoted $\{E_j\}$, residing within $(-m, m)$ in the z plane map to the (entirely) imaginary set $\{i\mu_j\}$, where $0 < \mu_j < m$ in the k plane.

Lemma 3.1. *The full resolvent $R_V^{\pm}(k) : L^2_{\mathrm{comp}}(\mathbb{R}^3)^4 \to L^2_{\mathrm{loc}}(\mathbb{R}^3)^4$ near $k = 0$ can be expressed*

$$R_V^{\pm}(k) = \mp \frac{\Pi_{\pm}}{k^2}(\sqrt{k^2+m^2}+m) + \frac{iA_{\pm}}{k}\sqrt{\sqrt{k^2+m^2}+m} + B_{\pm}(k), \tag{3.2}$$

where $k \mapsto B_{\pm}(k)$ is holomorphic near 0*, and the operators* $\Pi_{\pm} : L^2(\mathbb{R}^3)^4 \to V_{\pm}$, $A_{\pm} : L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$ *satisfying* $(\mathbb{D} \mp m)\Pi_{\pm} = (\mathbb{D} \mp m)A_{\pm} = 0$ *are symmetric.*

Proof. We divide the proof into four steps.
1. For Im $k > 0$ and assuming $|k| \ll \min\{\mu_j\}$ such that we avoid all eigenvalues, we first note that

$$\sup_{s\in(-\infty,-m]\cup[m,\infty)} (s \mp \sqrt{k^2+m^2})^{-1} = (\pm m \mp \sqrt{k^2+m^2})^{-1} = \mp\frac{(\sqrt{k^2+m^2}+m)}{k^2}.$$

Hence using the spectral theorem we find that the resolvent satisfies the bound

$$\|R_V^{\pm}(k)\| \le \frac{|\sqrt{k^2+m^2}+m|}{|k|^2}, \qquad \mathrm{Im}\, k > 0,\ |k| \ll \mu_j. \tag{3.3}$$

This implies that, for k near 0, the decomposition

$$R_V^{\pm}(k) = \mp \frac{\tilde{A}_{\pm}}{k^2}(\sqrt{k^2+m^2}+m) + i\frac{A_{\pm}}{k}\sqrt{\sqrt{k^2+m^2}+m} + B_{\pm}(k), \tag{3.4}$$

holds, where $A_{\pm}, \tilde{A}_{\pm} : L^2_{\mathrm{comp}}(\mathbb{R}^3)^4 \to L^2_{\mathrm{loc}}(\mathbb{R}^3)^4$ are finite rank operators and $B_{\pm}(k)$ is holomorphic near $k = 0$. The $\mp$ coefficients will be apparent in Step 4 when we show that $\tilde{A}_{\pm} = \Pi_{\pm}$.
2. The property $(\mathbb{D}\mp m)\widetilde{A}_{\pm} = (\mathbb{D}\mp m)A_{\pm} = 0$ follows from $I = (\mathbb{D}\mp\sqrt{k^2+m^2})R_V^{\pm}(k)$, using the decomposition (3.4) and equating the coefficients of $k^{-\alpha}$ where $\alpha = \{1, 2\}$.
3. To show symmetry of $\widetilde{A}_{\pm}$ and $A_{\pm}$, set $\psi, \phi \in L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$ and then we have by self-adjointness of $R_V^{\pm}(it)$, $0 < t \ll \mu_j$,

$$\begin{aligned} \mp 2m\langle \tilde{A}_{\pm}\psi\,,\phi\rangle &= \lim_{t\to 0}\left\langle -t^2\left(R_V^{\pm}(it) - \frac{A_{\pm}}{t}\sqrt{\sqrt{m^2-t^2}+m} - B_{\pm}(it)\right)\psi\,,\phi\right\rangle \\ &= \lim_{t\to 0}\langle -t^2 R_V^{\pm}(it)\psi\,,\phi\rangle = \lim_{t\to 0}\langle \psi\,, -t^2 R_V^{\pm}(it)\phi\rangle \\ &= \lim_{t\to 0}\left\langle \psi\,, -t^2\left(R_V^{\pm}(it) - \frac{A_{\pm}}{t}\sqrt{\sqrt{m^2-t^2}+m} - B_{\pm}(it)\right)\right\rangle = \mp 2m\langle \psi, \tilde{A}_{\pm}\phi\rangle \end{aligned} \tag{3.5}$$

and, by similar arguments, we show $i\sqrt{2m}\langle A_\pm\psi\,,\phi\rangle = i\sqrt{2m}\langle\psi\,,A_\pm\phi\rangle$.
4. Finally we explore the properties of $\tilde{A}_\pm$. First $\tilde{A}_\pm : L^2 \to L^2$ is a bounded operator by (3.3) and so (3.5) holds for all ψ, $\phi \in L^2$. As we have also seen, $(\mathbb{D} \mp m)\tilde{A}_\pm = 0$ and so the range of $\tilde{A}_\pm$ is contained in $V_\pm$ as per the definition in (3.1). To show that indeed $\tilde{A}_\pm = \Pi_\pm$ then for any $v_\pm \in V_\pm$, $\phi \in L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$ and $t \ll \mu_j$:

$$\begin{aligned}
&\langle v_\pm, \phi\rangle = \langle R_V^\pm(it)(\mathbb{D} \mp \sqrt{m^2-t^2})v_\pm, \phi\rangle \\
&= \mp(\sqrt{m^2-t^2}-m)\left\langle \left[\pm\frac{\tilde{A}_\pm}{t^2}(\sqrt{m^2-t^2}+m) + \frac{A_\pm}{t}\sqrt{\sqrt{m^2-t^2}+m} + B_\pm(it)\right]v_\pm, \phi\right\rangle \\
&\to \langle \tilde{A}_\pm v_\pm, \phi\rangle \text{ as } t \to 0,
\end{aligned}$$

where we have used $(\mathbb{D} \mp m)v_\pm = 0$. □

Lemma 3.2. *Let $A_\pm$ be defined as Lemma 3.1. Then the image of $A_\pm$ is contained in $U_\pm$; not in $L^2(\mathbb{R}^3)^4$.*

Proof. We divide the proof into two steps.
1. Injectivity of $R_0^\pm(k)$ on $L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$ (the left inverse being $\mathbb{D}_0 \mp \sqrt{k^2+m^2}$) implies, for k near 0,

$$R_V^\pm(k) = R_0^\pm(k)\left(\frac{C_\pm}{k^2}(\sqrt{k^2+m^2}+m) + \frac{D_\pm}{k}\sqrt{\sqrt{k^2+m^2}+m} + E_\pm(k)\right), \quad (3.6)$$

where $C_\pm$, $D_\pm$, $E_\pm(k) : L^2_{\mathrm{comp}}(\mathbb{R}^3)^4 \to L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$ and $E_\pm(k)$ is holomorphic near $k = 0$. This form resembles the identity $R_V^\pm(k) = R_0^\pm(k)(I + VR_0^\pm(k))^{-1}$. If $\psi \in L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$, then

$$\begin{aligned}
i\sqrt{2m}A_\pm\psi &= \left[\left(kR_V^\pm(k) \pm \frac{\Pi_\pm}{k}(\sqrt{k^2+m^2}+m) - kB_\pm(k)\right)\psi\right]_{k=0} \\
&\quad + \sum_{j=1}^\infty \frac{k^j}{j!}\partial_k^j\left[\left(kR_V^\pm(k) \pm \frac{\Pi_\pm}{k}(\sqrt{k^2+m^2}+m) - kB_\pm(k)\right)\psi\right]_{k=0}.
\end{aligned} \quad (3.7)$$

If we consider only the coefficients of k^0 and use (3.6) then the first term on the right-hand side of (3.7) becomes

$$\begin{aligned}
&\left[kR_V^\pm(k)\psi\right]_{k=0} \\
&= \left[\int_{\mathbb{R}^3}\sqrt{\sqrt{k^2+m^2}+m}\left(\frac{i\boldsymbol{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + m(\beta \pm I)\right)\frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}(D_\pm\psi)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right]_{k=0} \\
&\quad + \left[\int_{\mathbb{R}^3}(\sqrt{k^2+m^2}+m)\frac{\boldsymbol{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}\frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi}(C_\pm\psi)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right]_{k=0}.
\end{aligned}$$

By taking the limit, we find that

$$\begin{aligned}
i\sqrt{2m}A_\pm\psi &= \sqrt{2m}\int_{\mathbb{R}^3}\left(\frac{i\boldsymbol{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + m(\beta \pm I)\right)\frac{(D_\pm\psi)(\boldsymbol{y})}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y} \\
&\quad + 2m\int_{\mathbb{R}^3}\frac{\boldsymbol{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}\frac{(C_\pm\psi)(\boldsymbol{y})}{4\pi}\,\mathrm{d}\boldsymbol{y} + \sum_{j=1}^\infty \frac{k^j}{j!}\partial_k^j\left[kR_V^\pm(k)\psi\right]_{k=0},
\end{aligned}$$

where $\int_{\mathbb{R}^3}(D_\pm\psi)(\boldsymbol{y})/|\boldsymbol{x}-\boldsymbol{y}|\,\mathrm{d}\boldsymbol{y}$ is not in $L^2(\mathbb{R}^3)^4$. We now show that this term does not cancel from the remaining derivative terms in the Taylor expansion. Hence for any $m \in \mathbb{N}$

$$\begin{aligned}
&\frac{k^m}{m!}\partial_k^m \left[kR_V^{\pm}(k)\psi\right]_{k=0} \\
&= \frac{k^m}{m!}\partial_k^m \left[kR_0^{\pm}(k)\left(\frac{C_\pm}{k^2}(\sqrt{k^2+m^2}+m)+\frac{D_\pm}{k}\sqrt{\sqrt{k^2+m^2}+m}+E_\pm(k)\right)\psi\right]_{k=0} \\
&= \frac{k^m}{m!}\partial_k^m \left[R_0^{\pm}(k)\left(\frac{C_\pm}{k}(\sqrt{k^2+m^2}+m)+D_\pm\sqrt{\sqrt{k^2+m^2}+m}\right)\psi\right]_{k=0} + \mathcal{O}(k)\,.
\end{aligned}$$

Equating terms of order k^0, we see no term containing $D_\pm$ as $k \to 0$ since we see from (2.8) that $\partial_k^j R_0^{\pm}(k) = \mathcal{O}(k)$ for $j = 0, \ldots, m$.
2. We now show that the range of $A_\pm$ lies in $U_\pm$. For all $k \in \mathbb{C}$, $(\mathbb{D}_0 \mp \sqrt{k^2+m^2})R_V^{\pm}(k) : L^2_{\text{comp}} \to L^2_{\text{comp}}$ we have

$$R_V^{\pm}(k) = R_0^{\pm}(k)(\mathbb{D}_0 \mp \sqrt{k^2+m^2})R_V^{\pm}(k) \tag{3.8}$$

and again expand as in (3.7) for $\psi \in L^2_{\text{comp}}(\mathbb{R}^3)^4$ such that by using (3.8) and collecting k^0 terms, we have

$$i\sqrt{2m}A_\pm\psi = \left[kR_0^{\pm}(k)(\mathbb{D}_0 \mp \sqrt{k^2+m^2})R_V^{\pm}(k)\psi\right]_{k=0} + \sum_{j=1}^{\infty}\frac{k^j}{j!}\partial_k^j\left[kR_V^{\pm}(k)\psi\right]_{k=0}. \tag{3.9}$$

Taking the 'non-derivative' term in (3.9), collecting k^0 terms and taking the limit we simply have

$$\left[kR_0^{\pm}(k)(\mathbb{D}_0 \mp \sqrt{k^2+m^2})R_V^{\pm}(k)\psi\right]_{k=0} = i\sqrt{2m}R_0^{\pm}(0)(\mathbb{D}_0 \mp m)A_\pm\psi,$$

while, for each j^{th}-derivative ($j \geq 1$), we have by collecting k^0 terms:

$$\begin{aligned}
\frac{k^j}{j!}\partial_k^j\left[kR_V^{\pm}(k)\psi\right]_{k=0} &= \frac{k^j}{j!}\partial_k^j\left[\mp\frac{\Pi_\pm}{k}(\sqrt{k^2+m^2}+m)\psi\right]_{k=0} + \mathcal{O}(k) \\
&= \mp\frac{k^j}{j!}\left[(\partial_k^{j-1}k^{-1})\partial_k(\sqrt{k^2+m^2}+m)\binom{j}{j-1}\Pi_\pm\psi\right]_{k=0} \\
&= \mp\frac{k^j}{j!}\left[\frac{(-1)^j(j-1)!}{k^j}\frac{k}{(\sqrt{k^2+m^2}+m)}\frac{j!}{(j-1)!}\Pi_\pm\psi\right]_{k=0} \\
&= \mp(-1)^j\left[\frac{k}{(\sqrt{k^2+m^2}+m)}\Pi_\pm\psi\right]_{k=0} = 0.
\end{aligned}$$

Hence we are left with

$$A_\pm\psi = R_0^{\pm}(0)(\mathbb{D}_0 \mp m)A_\pm\psi. \tag{3.10}$$

This together with the property $(\mathbb{D} \mp m)A_\pm = 0$ from Lemma 3.1 is enough to satisfy the definition in (3.1) whence the range of $A_\pm$ lies in $U_\pm$. □

Lemma 3.3. *Suppose $v_\pm \in V_\pm$ and let Assumption 2.1 hold. Then*
1. $v_\pm = R_0^{\pm}(0)f_\pm$ where $f_\pm = (\mathbb{D}_0 \mp m)v_\pm = -Vv_\pm \in L^2_{\text{comp}}(\mathbb{R}^3;\mathbb{R}^4)$ and $\int_{\mathbb{R}^3} f_\pm = 0$.

2. Uniformly in $\boldsymbol{\omega} \in \mathbb{S}^2$ *and locally uniformly in* $\boldsymbol{y} \in \mathbb{R}^3$,

$$\begin{aligned} v_\pm(\boldsymbol{y} + r\boldsymbol{\omega}) = \frac{i}{4\pi r^3} &\left[3 \sum_{i,j=1}^{3} \alpha_i \omega_i \omega_j b_j^\pm(v_\pm) - \sum_{j=1}^{3} \alpha_j b_j^\pm(v_\pm) \right] \\ &+ \frac{m(\beta \pm I)}{4\pi r^2} \sum_j b_j^\pm(v_\pm)\omega_j + \frac{3m(\beta \pm I)}{8\pi r^3} \sum_{j,k} (B_{jk}^\pm - 2b_j^\pm(v_\pm) y_k)\omega_j\omega_k \\ &+ \frac{m(\beta \pm I)}{8\pi r^3} \left(- \operatorname{Tr} B_{jk}^\pm + 2 \sum_\ell b_\ell^\pm(v_\pm) y_\ell \right) + \mathcal{O}\left(r^{-4}\right), \qquad \text{as } r \to +\infty, \end{aligned} \tag{3.11}$$

where

$$b_j^\pm(v_\pm) = \int_{\mathbb{R}^3} x_j(\mathbb{D}_0 \mp m) v_\pm(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}, \quad B_{jk}^\pm(v_\pm) = \int_{\mathbb{R}^3} x_j x_k (\mathbb{D}_0 \mp m) v_\pm(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}. \tag{3.12}$$

3. For $r > 0$ *and locally uniformly in* $\boldsymbol{y} \in \mathbb{R}^3$,

$$\begin{aligned} I_v^\pm(r, \boldsymbol{y}) &:= \int_{\mathbb{S}^2} v_\pm(\boldsymbol{y} + r\boldsymbol{\omega}) \, \mathrm{d}\boldsymbol{\omega} = \mathcal{O}\left(r^{-4}\right), \qquad \text{as } r \to +\infty, \\ \tilde{I}_v^\pm(r, \boldsymbol{y}) &:= \int_{\mathbb{S}^2} (\overline{\boldsymbol{\alpha}} \cdot \boldsymbol{\omega}) v_\pm(\boldsymbol{y} + r\boldsymbol{\omega}) \, \mathrm{d}\boldsymbol{\omega} = \mathcal{O}\left(r^{-2}\right), \qquad \text{as } r \to +\infty. \end{aligned} \tag{3.13}$$

Proof. We divide the proof into four steps.
1. Using (3.8) we have, by equating k^{-2} factors and setting $k = 0$,

$$\Pi_\pm = R_0^\pm(0)(\mathbb{D}_0 \mp m)\Pi_\pm, \quad (\mathbb{D}_0 \mp m)\Pi_\pm : L^2_{\text{comp}} \to L^2_{\text{comp}}. \tag{3.14}$$

The range of this operator lies in $V_\pm$ by Lemma 3.1. If we set $f_\pm = (\mathbb{D}_0 \mp m)v_\pm \in L^2_{\text{comp}}(\mathbb{R}^3)^4$ then $v_\pm = R_0^\pm(0) f_\pm \in \operatorname{Ran} \Pi_\pm$. Again by Lemma 3.1 we have $(\mathbb{D} \mp m)\Pi_\pm = 0$ and so we see that $f_\pm = (\mathbb{D}_0 \mp m)v_\pm = -V v_\pm$.
2. Let $r > 0$ and $\boldsymbol{\omega} \in \mathbb{S}^2$ so $r\boldsymbol{\omega} \in \mathbb{R}^3$. Using the free resolvent kernel in (2.9) we then write for $v_\pm = R_0^\pm(0) f_\pm \in V_\pm$

$$\begin{aligned} v_\pm(\boldsymbol{y} + r\boldsymbol{\omega}) &= R_0^\pm(0) f_\pm(\boldsymbol{y} + r\boldsymbol{\omega}) \\ &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \left(\frac{-i\boldsymbol{\alpha} \cdot (\boldsymbol{x} - (\boldsymbol{y} + r\boldsymbol{\omega}))}{|\boldsymbol{x} - (\boldsymbol{y} + r\boldsymbol{\omega})|^2} + m(\beta \pm I) \right) \frac{f_\pm(\boldsymbol{x})}{|\boldsymbol{x} - (\boldsymbol{y} + r\boldsymbol{\omega})|} \, \mathrm{d}\boldsymbol{x} \end{aligned}$$

where the -1 factor occurs due to the fact that the resolvent integral in (2.9) is with respect to $\boldsymbol{y}$ and not $\boldsymbol{x}$ which is the case here. Writing $\boldsymbol{x} - (\boldsymbol{y} + r\boldsymbol{\omega}) = -r[\boldsymbol{\omega} - (\boldsymbol{x} - \boldsymbol{y})/r]$ we have

$$\begin{aligned} v_\pm(\boldsymbol{y} + r\boldsymbol{\omega}) = &\frac{i}{4\pi r^2} \int_{\mathbb{R}^3} \frac{\boldsymbol{\alpha} \cdot [\boldsymbol{\omega} - (\boldsymbol{x} - \boldsymbol{y})/r]}{|\boldsymbol{\omega} - (\boldsymbol{x} - \boldsymbol{y})/r|^3} f_\pm(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \\ &+ \frac{m(\beta \pm I)}{4\pi r} \int_{\mathbb{R}^3} \frac{f_\pm(\boldsymbol{x})}{|\boldsymbol{\omega} - (\boldsymbol{x} - \boldsymbol{y})/r|} \, \mathrm{d}\boldsymbol{x}. \end{aligned} \tag{3.15}$$

It is useful to consider the Taylor expansion of the denominators in (3.15). First

$$\frac{1}{|\boldsymbol{\omega} - (\boldsymbol{x} - \boldsymbol{y})/r|} = \frac{1}{\sqrt{\langle \boldsymbol{\omega}, \boldsymbol{\omega} \rangle - 2\langle \boldsymbol{\omega}, \boldsymbol{x} - \boldsymbol{y} \rangle / r + |\boldsymbol{x} - \boldsymbol{y}|^2 / r^2}},$$

where $\langle \boldsymbol{\omega}\,,\boldsymbol{\omega}\rangle = 1$. Therefore we can use the expansion $(1+s)^{-1/2} = 1-s/2+3s^2/8+\mathcal{O}\left(s^3\right)$ and setting $s = -2\langle \boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle/r + |\boldsymbol{x}-\boldsymbol{y}|^2/r^2 = \mathcal{O}\left(r^{-1}\right)$ we therefore have

$$\frac{1}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|} = 1 + \frac{\langle \boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle}{r} - \frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{2r^2} + \frac{3\langle \boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle^2}{2r^2} + \mathcal{O}\left(r^{-3}\right).$$

Hence the second term on the right-hand side of (3.15) is

$$\begin{aligned}&\frac{1}{r}\int_{\mathbb{R}^3} \frac{f_\pm(\boldsymbol{x})}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|}\,\mathrm{d}\boldsymbol{x}\\ =&\frac{1}{r}\int_{\mathbb{R}^3} f_\pm(\boldsymbol{x})\left[1+\frac{\langle \boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle}{r} - \frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{2r^2} + \frac{3\langle \boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle^2}{2r^2}\right]\mathrm{d}\boldsymbol{x} + \mathcal{O}\left(r^{-4}\right).\end{aligned} \tag{3.16}$$

We use the previous expansion to write

$$\begin{aligned}\frac{1}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|^3} &= \left[\frac{1}{\sqrt{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|^2}}\right]^3\\ &= \left[1+\frac{\langle \boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle}{r} + \mathcal{O}\left(r^{-2}\right)\right]^3 = 1 + 3\frac{\langle \boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle}{r} + \mathcal{O}\left(r^{-2}\right).\end{aligned}$$

For the first term on the right-hand side of (3.15) we therefore have

$$\begin{aligned}&\frac{1}{r^2}\int_{\mathbb{R}^3} \frac{\boldsymbol{\alpha}\cdot[\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r]}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|^3} f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\ =&\frac{1}{r^2}\int_{\mathbb{R}^3} \boldsymbol{\alpha}\cdot[\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r]\left[1+\frac{3\langle \boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle}{r}\right] f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \mathcal{O}\left(r^{-4}\right) = \mathcal{O}\left(r^{-2}\right).\end{aligned} \tag{3.17}$$

Bringing together (3.16) and (3.17) with (3.15) we see that

$$v_\pm(\boldsymbol{y}+r\boldsymbol{\omega}) = \frac{m(\beta \pm I)}{4\pi r}\int_{\mathbb{R}^3} f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \mathcal{O}\left(r^{-2}\right). \tag{3.18}$$

We know that $v_\pm \in L^2(\mathbb{R}^3)^4$ but due to the prefactor r^{-1} we have that $\int_{\mathbb{R}^3} f_\pm(\boldsymbol{x})/r\,\mathrm{d}\boldsymbol{x} \notin L^2(\mathbb{R}^3)^4$. Therefore, we require

$$\int_{\mathbb{R}^3} f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0. \tag{3.19}$$

3. Equation (3.19) greatly simplifies our expansion of $v_\pm(\boldsymbol{y}+r\boldsymbol{\omega})$. Consider first (3.16):

$$\begin{aligned}&\frac{1}{r}\int_{\mathbb{R}^3} \frac{f_\pm(\boldsymbol{x})}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|}\,\mathrm{d}\boldsymbol{x}\\ &= \frac{1}{r^2}\int_{\mathbb{R}^3} f_\pm(\boldsymbol{x})\langle \boldsymbol{\omega},\,\boldsymbol{x}\rangle\,\mathrm{d}\boldsymbol{x} - \frac{1}{2r^3}\int_{\mathbb{R}^3} f_\pm(\boldsymbol{x})\left[\langle \boldsymbol{x},\,\boldsymbol{x}\rangle - 2\langle \boldsymbol{x},\,\boldsymbol{y}\rangle\right]\mathrm{d}\boldsymbol{x}\\ &\quad + \frac{3}{2r^3}\int_{\mathbb{R}^3} f_\pm(\boldsymbol{x})\left[\langle \boldsymbol{\omega},\,\boldsymbol{x}\rangle^2 - 2\langle \boldsymbol{\omega},\,\boldsymbol{x}\rangle\langle \boldsymbol{\omega},\,\boldsymbol{y}\rangle\right]\mathrm{d}\boldsymbol{x} + \mathcal{O}\left(r^{-4}\right).\end{aligned}$$

In explicit coordinates we have

$$
\begin{aligned}
&\frac{1}{r}\int_{\mathbb{R}^3}\frac{f_\pm(\boldsymbol{x})}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|}\,\mathrm{d}\boldsymbol{x}\\
&\quad=\frac{1}{r^2}\int_{\mathbb{R}^3}f_\pm(\boldsymbol{x})\sum_i\omega_i x_i\,\mathrm{d}\boldsymbol{x}-\frac{1}{2r^3}\int_{\mathbb{R}^3}f_\pm(\boldsymbol{x})\sum_i\left[x_i^2-2x_iy_i\right]\,\mathrm{d}\boldsymbol{x}\\
&\qquad+\frac{3}{2r^3}\int_{\mathbb{R}^3}f_\pm(\boldsymbol{x})\sum_{i,j}\left[\omega_i x_i\omega_j x_j-2\omega_i x_i\omega_j y_j\right]\,\mathrm{d}\boldsymbol{x}+\mathcal{O}\left(r^{-4}\right)
\end{aligned}
$$

and, by (3.12),

$$
\begin{aligned}
&\frac{m(\beta\pm I)}{4\pi r}\int_{\mathbb{R}^3}\frac{f_\pm(\boldsymbol{x})}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|}\,\mathrm{d}\boldsymbol{x}\\
&\qquad=\frac{m(\beta\pm I)}{4\pi r^2}\sum_i b_i^\pm(v_\pm)\omega_i-\frac{m(\beta\pm I)}{8\pi r^3}\left[\mathrm{Tr}\,B^\pm-2\sum_i y_i b_i^\pm(v_\pm)\right]\\
&\qquad\quad+\frac{3m(\beta\pm I)}{8\pi r^3}\sum_{i,j}\left[B_{ij}^\pm-2y_i b_j^\pm(v_\pm)\right]\omega_i\omega_j+\mathcal{O}(r^{-4}).
\end{aligned}
$$

Consider next the term corresponding to (3.17):

$$
\begin{aligned}
&\frac{1}{r^2}\int_{\mathbb{R}^3}\frac{\boldsymbol{\alpha}\cdot[\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r]}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|^3}f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\
&\quad=\frac{1}{r^2}\int_{\mathbb{R}^3}\boldsymbol{\alpha}\cdot[\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r]\left[1+\frac{3\langle\boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle}{r}\right]f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}+\mathcal{O}\left(r^{-4}\right)\\
&\quad=\frac{1}{r^2}\int_{\mathbb{R}^3}[\sum_i\alpha_i\omega_i]\left[1+3\sum_j\frac{\omega_j x_j}{r}\right]f_\pm(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\\
&\qquad-\frac{1}{r^3}\int_{\mathbb{R}^3}\sum_i[\alpha_i(x_i-y_i)]f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}+\mathcal{O}\left(r^{-4}\right)\\
&\quad=\frac{3}{r^3}\int_{\mathbb{R}^3}\sum_{i,j}\alpha_i\omega_i\omega_j x_j f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}-\frac{1}{r^3}\int_{\mathbb{R}^3}\sum_i\alpha_i x_i f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}+\mathcal{O}\left(r^{-4}\right).
\end{aligned}
$$

Hence by (3.12)

$$
\begin{aligned}
&\frac{i}{4\pi r^2}\int_{\mathbb{R}^3}\frac{\boldsymbol{\alpha}\cdot[\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r]}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|^3}f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\
&\qquad=\frac{i}{4\pi r^3}\left[3\sum_{i,j}\alpha_i\omega_i\omega_j b_i^\pm(v_\pm)-\sum_i\alpha_i b_i^\pm(v_\pm)\right]+\mathcal{O}(r^{-4}).
\end{aligned}
\tag{3.20}
$$

4. For (3.13) we note that

$$
\int_{\mathbb{S}^2}\omega_j\,\mathrm{d}\boldsymbol{\omega}=0,\qquad\int_{\mathbb{S}^2}\omega_j\omega_k\,\mathrm{d}\boldsymbol{\omega}=\frac{4\pi}{3}\delta_{jk},\qquad\int_{\mathbb{S}^2}\omega_j\omega_k\omega_\ell\,\mathrm{d}\boldsymbol{\omega}=0,\tag{3.21}
$$

where δ_{jk} is the Kronecker delta. For $I_\pm(r,\boldsymbol{y})$ the spherical integrals of all terms on the right-hand side of (3.11) either cancel or are equal to zero up to the power r^{-4}.

Explicitly we have

$$\sum_j b_j^{\pm}(v_{\pm}) \int_{\mathbb{S}^2} \omega_j \, \mathrm{d}\boldsymbol{\omega} = 0,$$

whilst we have the cancellation amongst the r^{-3} terms

$$\begin{aligned}
&\frac{3m(\beta \pm I)}{8\pi r^3} \sum_{i,j} \left[B_{ij}^{\pm} - 2b_i^{\pm}(v_{\pm})y_j\right] \int_{\mathbb{S}^2} \omega_i\omega_j \, \mathrm{d}\boldsymbol{\omega} \\
&\quad = \frac{3m(\beta \pm I)}{8\pi r^3} \sum_{i,j} \left[B_{ij}^{\pm} - 2b_i^{\pm}(v_{\pm})y_j\right] \left(\frac{4\pi}{3}\delta_{ij}\right) \\
&\quad = \frac{m(\beta \pm I)}{2\pi r^3} \left[\mathrm{Tr}\ B^{\pm} - 2\sum_i y_i b_i^{\pm}(v_{\pm})\right].
\end{aligned}$$

For $\tilde{I}_{\pm}(r, \boldsymbol{y})$ we retain only one term:

$$\begin{aligned}
\tilde{I}_v^{\pm}(r, \boldsymbol{y}) &= \int_{\mathbb{S}^2} \sum_i \overline{\alpha_i}\omega_i v(\boldsymbol{y} + r\boldsymbol{\omega}) \, \mathrm{d}\boldsymbol{\omega} \\
&= \sum_{i,j} \overline{\alpha_i} \int_{\mathbb{S}^2} \omega_i \left[\frac{m(\beta \pm I)}{4\pi r^2} b_j^{\pm}(v_{\pm})\omega_j\right] \mathrm{d}\boldsymbol{\omega} + \mathcal{O}\left(r^{-4}\right) \\
&= \frac{m(\beta \pm I)}{3r^2} \sum_i \overline{\alpha_i} b_i^{\pm}(v_{\pm}) + \mathcal{O}\left(r^{-4}\right).
\end{aligned}$$

Therefore $\tilde{I}_{\pm}(r, \boldsymbol{y}) = \mathcal{O}\left(r^{-2}\right)$. □

LEMMA 3.4. *Let $v \in V_{\pm}$, $\phi \in L^2_{\mathrm{comp}}(\mathbb{R}^3; \mathbb{R}^4)$ and set $u_{\pm} := R_0^{\pm}(0)\phi \in L^2_{\mathrm{loc}}(\mathbb{R}^3)^4$. Then the following limit, independent of $\boldsymbol{y} \in \mathbb{R}^3$ exists:*

$$\langle v_{\pm}, u_{\pm}\rangle_0 := \lim_{R\to\infty} \int_{B(\boldsymbol{y};R)} v_{\pm}(\boldsymbol{x})\overline{u_{\pm}(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x} = -i\langle H_v^{\pm}, \phi\rangle + m(\overline{\beta} \pm I)\langle K_v^{\pm}, \phi\rangle, \quad (3.22)$$

where

$$H_v^{\pm}(\boldsymbol{y}) := \frac{1}{4\pi} \int_0^{\infty} \tilde{I}_v^{\pm}(r, \boldsymbol{y}) \, \mathrm{d}r, \qquad K_v^{\pm}(\boldsymbol{y}) := \frac{1}{4\pi} \int_0^{\infty} r I_v^{\pm}(r, \boldsymbol{y}) \, \mathrm{d}r. \qquad (3.23)$$

Proof. We divide the proof into two steps.
1. By an application of the fundamental theorem of calculus, together with $\partial_{y_j} r = \omega_j$ and $v(\boldsymbol{x}) = \mathcal{O}\left(|\boldsymbol{x}|^{-2}\right)$ we deduce, for $\boldsymbol{y}, \boldsymbol{y'} \in \mathbb{R}^3$ and large R,

$$\begin{aligned}
&\partial_{y_j'} \int_{B(\boldsymbol{y'};R)} \frac{v_{\pm}(\boldsymbol{x})}{|\boldsymbol{x} - \boldsymbol{y}|} \left[\frac{-i\overline{\boldsymbol{\alpha}} \cdot (\boldsymbol{x} - \boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^2} + m(\overline{\beta} \pm I)\right] \mathrm{d}\boldsymbol{x} \\
&= R^2 \int_{\mathbb{S}^2} \omega_j v_{\pm}(\boldsymbol{y'} + R\boldsymbol{\omega}) \frac{-i\overline{\boldsymbol{\alpha}} \cdot (\boldsymbol{y'} - \boldsymbol{y} + R\boldsymbol{\omega})}{|\boldsymbol{y'} - \boldsymbol{y} + R\boldsymbol{\omega}|^3} \, \mathrm{d}\boldsymbol{\omega} \\
&\quad + m(\overline{\beta} \pm I)R^2 \int_{\mathbb{S}^2} \omega_j \frac{v_{\pm}(\boldsymbol{y'} + R\boldsymbol{\omega})}{|\boldsymbol{y'} - \boldsymbol{y} + R\boldsymbol{\omega}|} \, \mathrm{d}\boldsymbol{\omega}.
\end{aligned}$$

locally uniformly and, as a consequence,

$$\int_{B(\boldsymbol{y};R)} \left[\frac{-i\overline{\boldsymbol{\alpha}}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + m(\overline{\beta}\pm I)\right] \frac{v_\pm(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{x}$$
$$= \int_{B(\boldsymbol{y}';R)} \left[\frac{-i\overline{\boldsymbol{\alpha}}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + m(\overline{\beta}\pm I)\right] \frac{v_\pm(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{x} + \mathcal{O}\left(R^{-1}\right).$$

2. For each fixed $\boldsymbol{y}' \in \mathbb{R}^3$, where $\boldsymbol{x}-\boldsymbol{y} = r\boldsymbol{\omega}$ and $\phi \in L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$, we find by using (3.13),

$$-i\langle H_v^\pm, \phi\rangle + m(\overline{\beta}\pm I)\langle K_v^\pm, \phi\rangle$$
$$= \lim_{R\to\infty} \frac{1}{4\pi}\int_{B(\boldsymbol{y}';R)} v_\pm(\boldsymbol{x}) \left[\int_{\mathbb{R}^3}\left(\frac{\overline{i\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + m(\overline{\beta}\pm I)\right)\frac{\overline{\phi(\boldsymbol{y})}}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\right]\mathrm{d}\boldsymbol{x}$$
$$= \lim_{R\to\infty}\int_{B(\boldsymbol{y}';R)} v_\pm(\boldsymbol{x})\overline{u_\pm(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x} = \langle v_\pm, u_\pm\rangle_0.$$

□

Lemma 3.5.

1. The spaces $V_\pm$ and $U_\pm$ are related by

$$V_\pm = \left\{ v_\pm \in U_\pm \;:\; \int_{\mathbb{R}^3} (\mathbb{D}_0 \mp m)v_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0\right\} \tag{3.24}$$

and we define the multiplicity

$$\widetilde{m}_R(\pm m) := \dim(U_\pm/V_\pm) = \{0,1\}. \tag{3.25}$$

2. If $u_\pm \in U_\pm$ then

$$u_\pm + \frac{m\sqrt{2m}}{4\pi}(\beta\pm I)A_\pm V\int_{R^3}(\mathbb{D}_0\mp m)u_\pm \in V_\pm. \tag{3.26}$$

Proof. We divide the proof into two steps.

1. Clearly $V_\pm \subset U_\pm$. Lemma 3.3 indicates that $0 = \int_{\mathbb{R}^3} f_\pm = \int_{\mathbb{R}^3}(\mathbb{D}_0 \mp m)v_\pm$ which gives rise to (3.24). However the method leading to the expansion (3.18) can also be applied to $u_\pm \in U_\pm$ in which case the $1/r$ term is non-zero. This leads to (3.25).

2. Extending the notation in Lemma 3.4 and analogous to that in Lemma 3.3, set $u_\pm = R_0^\pm(0)g_\pm$ such that $g_\pm = (\mathbb{D}_0 \mp m)u_\pm = -Vu_\pm \in L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$. If $\rho \in C_0^\infty(\mathbb{R}^3)$ such that $\rho = 1$ on $\operatorname{supp} V$ then we consider the Taylor expansion of $u_\pm$:

$$u_\pm = R_V^\pm(k)\rho(\mathbb{D}\mp\sqrt{k^2+m^2})\sum_{j=0}^\infty \frac{k^j}{j!}[\partial_k^j R_0^\pm(k)g_\pm]_{k=0}. \tag{3.27}$$

Justification for inserting the bump function ρ follows from

$$(\mathbb{D}\mp\sqrt{k^2+m^2})R_0^\pm(k) = I + VR_0^\pm(k).$$

Since (3.27) is holomorphic on the left-hand side, then the poles on the right-hand side must cancel. Then collecting k^0 terms in the non-derivative term (3.27) becomes

$$R_V^\pm(k)\rho(\mathbb{D}\mp\sqrt{k^2+m^2})R_0^\pm(k)g_\pm \xrightarrow[k\to 0]{} 0,$$

because $(\mathbb{D} \mp m)u_\pm = 0$. Next consider the first derivative term on the right-hand side of (3.27). Collecting k^0 terms again and using (2.11) we have

$$kR_V^\pm(k)(\mathbb{D} \mp \sqrt{k^2+m^2})\left[\partial_k R_0^\pm(k)g_\pm\right] \xrightarrow[k\to 0]{} iA_\pm(\mathbb{D} \mp m)\sqrt{2m}\int \frac{im}{4\pi}(\beta \pm I)g_\pm.$$

This remaining term lies in the range of $U_\pm$. From (3.27) we note that any further k^0 terms will be mapped into $\Pi_\pm$. Hence

$$u_\pm + \frac{m\sqrt{2m}}{4\pi}(\beta \pm I)A_\pm(\mathbb{D} \mp m)\int g_\pm \in V_\pm.$$

To obtain (3.26) we show that

$$(\beta \pm I)(\mathbb{D}_0 \mp m)\int_{\mathbb{R}^3} g_\pm(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = (\beta \pm I)(-i\alpha\cdot\nabla + \beta m \mp m)\int_{\mathbb{R}^3} g_\pm(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = 0,$$

which follows from $\alpha \cdot \nabla_{\boldsymbol{x}} \int_{\mathbb{R}^3} g_\pm(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = 0$ and $(\beta \pm I)(\beta \mp I) = \beta^2 - I = 0$ by the condition mentioned after (2.2). ☐

LEMMA 3.6. *The operator $A_\pm$ has the explicit form*

$$A_\pm = \widetilde{m}_R(\pm m)(w_\pm \otimes \overline{w}_\pm), \tag{3.28}$$

where $w_\pm$ is the unique element of $U_\pm$ satisfying

$$w_\pm(\boldsymbol{x}) = -\frac{1}{\sqrt{2m}}\frac{h}{|\boldsymbol{x}|} + \mathcal{O}\left(|\boldsymbol{x}|^{-2}\right), \qquad h = (1,1,1,1)^\top. \tag{3.29}$$

Proof. We divide the proof into two steps.
1. Let $v_\pm \in V_\pm$, $\psi \in L^2_{\text{comp}}(\mathbb{R}^3)^4$ and $\rho \in \mathscr{D}(\mathbb{R}^3)$ such that $\rho V = V$ and $\rho\psi = \psi$. If $t \ll \mu_j$ then we write

$$\begin{aligned}\langle v_\pm\,, \psi\rangle &= \langle R_V^\pm(it)(\mathbb{D} \mp \sqrt{m^2-t^2})v_\pm\,, \psi\rangle = \langle[(\mathbb{D}\mp m) \mp (\sqrt{m^2-t^2}-m)]v_\pm\,, R_V^\pm(it)\psi\rangle \\ &= \mp(\sqrt{m^2-t^2}-m)\langle v_\pm\,, R_V^\pm(it)\psi\rangle \\ &= \mp(\sqrt{m^2-t^2}-m)\langle v_\pm\,, R_0^\pm(it)(\mathbb{D}_0 \mp \sqrt{m^2-t^2})R_V^\pm(it)\psi\rangle,\end{aligned}$$

where we have used $(\mathbb{D} \mp m)A_\pm = 0$ from Lemma 3.1. Explicitly we write

$$\begin{aligned}&\langle v_\pm\,, \psi\rangle \\ &= \mp(\sqrt{m^2-t^2}-m)\int_{\mathbb{R}^3} v_\pm(\boldsymbol{x})\overline{(R_0^\pm(it)(\mathbb{D}_0 \mp \sqrt{m^2-t^2})R_V^\pm(it)\psi)(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x} \\ &= \mp(\sqrt{m^2-t^2}-m)\int_{\mathbb{R}^3} v_\pm(\boldsymbol{x})\overline{\int_{\mathbb{R}^3}\left[\frac{i\boldsymbol{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + it\frac{\boldsymbol{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|} + \beta m \pm \sqrt{m^2-t^2}\right]} \\ &\quad \overline{\times\frac{e^{-t|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}}(\mathbb{D}_0 \mp \sqrt{m^2-t^2})R_V^\pm(it)\psi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Let $r\boldsymbol{\omega} = \boldsymbol{x} - \boldsymbol{y}$ where $r > 0$, $\boldsymbol{\omega} \in \mathbb{S}^2$. Hence

$$\begin{aligned}
&\langle v_\pm\,, \psi\rangle \\
&= \mp(\sqrt{m^2-t^2}-m) \\
&\quad \times \int_{\mathbb{R}^3} \lim_{R\to\infty} \int_0^R \int_{\mathbb{S}^2} v_\pm(\boldsymbol{y}+r\boldsymbol{\omega}) \left[-i\overline{\boldsymbol{\alpha}}\cdot\boldsymbol{\omega} - itr\overline{\boldsymbol{\alpha}}\cdot\boldsymbol{\omega} + r(\overline{\beta}m \pm \sqrt{m^2-t^2})\right] \\
&\quad \times \frac{e^{-tr}}{4\pi} \overline{(\mathbb{D}_0 \mp \sqrt{m^2-t^2})R_V^\pm(it)\psi(\boldsymbol{y})}\,\mathrm{d}\boldsymbol{\omega}\,\mathrm{d}r\,\mathrm{d}\boldsymbol{y} \\
&= \mp(\sqrt{m^2-t^2}-m) \int_{\mathbb{R}^3} \lim_{R\to\infty} \int_0^R \left[-i\tilde{I}_v^\pm(r,\boldsymbol{y}) - itr\tilde{I}_v^\pm(r,\boldsymbol{y}) + r(\overline{\beta}m \pm \sqrt{m^2-t^2}) I_v^\pm(r,\boldsymbol{y})\right] \\
&\quad \times \frac{e^{-tr}}{4\pi} \overline{(\mathbb{D}_0 \mp \sqrt{m^2-t^2})R_V^\pm(it)\psi(\boldsymbol{y})}\,\mathrm{d}r\,\mathrm{d}\boldsymbol{y}.
\end{aligned}$$

We proceed to use the Taylor expansion of e^x at $x = 0$. For convergence we require for

$$\int_0^\infty r^\alpha I_v^\pm(r,\boldsymbol{y})\,\mathrm{d}r, \qquad \int_0^\infty r^\beta \tilde{I}_v^\pm(r,\boldsymbol{y})\,\mathrm{d}r$$

that $\alpha < 3$ and $\beta < 1$ by (3.13). If we introduce

$$J_v^\pm(\boldsymbol{y}) := -\frac{1}{4\pi}\int_0^\infty r^2 I_v^\pm(r,\boldsymbol{y})\,\mathrm{d}r \tag{3.30}$$

and use (3.23) we write

$$\begin{aligned}
\langle v_\pm\,, \psi\rangle = \mp(\sqrt{m^2-t^2}-m) &\int_{\mathbb{R}^3} \left[-iH_v^\pm(\boldsymbol{y}) + (\overline{\beta}m \pm \sqrt{m^2-t^2})\left(K_v^\pm(\boldsymbol{y}) + tJ_v^\pm(\boldsymbol{y})\right)\right] \\
&\times \overline{(\mathbb{D}_0 \mp \sqrt{m^2-t^2})R_V^\pm(it)\psi(\boldsymbol{y})}\,\mathrm{d}\boldsymbol{y} + \mathcal{O}\left(t^{3/2}\right).
\end{aligned} \tag{3.31}$$

Expand using (3.2) the product

$$\mp(\sqrt{m^2-t^2}-m)R_V^\pm(it) = \Pi_\pm \mp iA_\pm\sqrt{\sqrt{m^2-t^2}-m} + \mathcal{O}\left(\sqrt{m^2-t^2}-m\right).$$

Back to (3.31) we therefore write

$$\begin{aligned}
\langle v_\pm\,, \psi\rangle = &-i\langle H_v^\pm\,, (\mathbb{D}_0 \mp \sqrt{m^2-t^2})\Pi_\pm\psi\rangle \\
&- i\langle H_v^\pm\,, (\mp i)\sqrt{\sqrt{m^2-t^2}-m}(\mathbb{D}_0 \mp \sqrt{m^2-t^2})A_\pm\psi\rangle \\
&+ (\overline{\beta}m \pm \sqrt{m^2-t^2})\langle K_v^\pm\,, (\mathbb{D}_0 \mp \sqrt{m^2-t^2})\Pi_\pm\psi\rangle \\
&+ (\overline{\beta}m \pm \sqrt{m^2-t^2})\langle K_v^\pm\,, (\mp i)\sqrt{\sqrt{m^2-t^2}-m}(\mathbb{D}_0 \mp \sqrt{m^2-t^2})A_\pm\psi\rangle \\
&+ t(\overline{\beta}m \pm \sqrt{m^2-t^2})\langle J_v^\pm\,, (\mathbb{D}_0 \mp \sqrt{m^2-t^2})\Pi_\pm\psi\rangle \\
&+ t(\overline{\beta}m \pm \sqrt{m^2-t^2})\langle J_v^\pm\,, (\mp i)\sqrt{\sqrt{m^2-t^2}-m}(\mathbb{D}_0 \mp \sqrt{m^2-t^2})A_\pm\psi\rangle \\
&+ \mathcal{O}\left(\sqrt{m^2-t^2}-m\right).
\end{aligned}$$

Collect $\sqrt{\sqrt{m^2-t^2}-m}$ terms, let $t \to 0$ and use Lemma 3.4 so that we have

$$\begin{aligned} 0 &= -i\langle H_v^{\pm}, (\mathbb{D}_0 \mp m)A_{\pm}\psi\rangle + m(\overline{\beta} \pm I)\langle K_v^{\pm}, (\mathbb{D}_0 \mp m)A_{\pm}\psi\rangle \\ &= \langle v_{\pm}, R_0^{\pm}(0)(\mathbb{D}_0 \mp m)A_{\pm}\psi\rangle_0 = \langle v_{\pm}, A_{\pm}\psi\rangle_0. \end{aligned} \tag{3.32}$$

2. We now concentrate on (3.32) which suggests that we have some form of orthogonality between $v_{\pm} \in V_{\pm}$ and $A_{\pm}\psi \in U_{\pm}$ on the inner product defined by Lemma 3.4. Trivially if $\widetilde{m}(\pm m) = 0$ then the sets $U_{\pm}$ and $V_{\pm}$ coincide exactly and we infer that $A_{\pm} = 0$.

Now consider when $\widetilde{m}(\pm m) = 1$. Existence and uniqueness of the element $w_{\pm}$ defined in (3.29) follow by the argument in Step 1 of the proof of Lemma 3.5. The range of $A_{\pm}$ is contained in the span of $w_{\pm}$. As $A_{\pm}$ is symmetric (see Lemma 3.1) we have for some $c \in \mathbb{C}$

$$A_{\pm} = c(w_{\pm} \otimes \overline{w}_{\pm}). \tag{3.33}$$

The constant c can be determined by inserting $w_{\pm}$ into Lemma 3.5:

$$V_{\pm} \ni w_{\pm} + \sqrt{2m}\frac{m}{4\pi}(\beta \pm I)A_{\pm}V \int_{\mathbb{R}^3} \tilde{g}_{\pm}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{3.34}$$

where we have set $\tilde{g}_{\pm} = (\mathbb{D}_0 \mp m)w_{\pm} \in L^2_{\text{comp}}(\mathbb{R}^3)^4$ and $w_{\pm} = R_0^{\pm}(0)\tilde{g}_{\pm} \in L^2_{\text{loc}}(\mathbb{R}^3)^4$. To determine $\int_{\mathbb{R}^3} \tilde{g}_{\pm}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ we expand $w_{\pm}$ as in Step 2 of the proof of Lemma 3.3. Analogous to (3.18) we obtain

$$w_{\pm}(\boldsymbol{x}) = \frac{m(\beta \pm I)}{4\pi r}\int_{\mathbb{R}^3} \tilde{g}_{\pm}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \mathcal{O}\left(r^{-2}\right). \tag{3.35}$$

Unlike the case for $v_{\pm} \in V_{\pm}$, the first term on the right-hand side is non-zero since we have shown that we require a r^{-1} term for $w_{\pm}$. Comparing the r^{-1} terms in (3.29) and (3.35) we have

$$-\frac{h}{\sqrt{2m}} = \frac{m(\beta \pm I)}{4\pi}\int_{\mathbb{R}^3} \tilde{g}_{\pm}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \text{ or } \int_{\mathbb{R}^3} \tilde{g}_{\pm}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = -\frac{4\pi}{m\sqrt{2m}}(\beta \pm I)^{-1}h.$$

Insert the latter into (3.34) and using $(A_{\pm}Vh)(\boldsymbol{x}) = cw_{\pm}(\boldsymbol{x})$ we find $w_{\pm} - cw_{\pm} \in V_{\pm}$, or $c = 1$. □

We bring all the auxiliary results above together for the following theorem:

THEOREM 3.7. *Let Assumption 2.1 hold and suppose* $m_R(\pm m) > 0$. *Then, near* $k = 0$ *the following decomposition for the full resolvent*

$$\begin{aligned} R_V^{\pm}(k) = &\mp \frac{\Pi_{\pm}}{k^2}(\sqrt{k^2+m^2}+m) \\ &+ \frac{\widetilde{m}_R(\pm m)}{k}(w_{\pm} \otimes \overline{w}_{\pm})\sqrt{\sqrt{k^2+m^2}+m} + B_{\pm}(k), \end{aligned} \tag{3.36}$$

holds, where $k \to B_{\pm}(k)$ *is holomorphic near* $k = 0$, $\Pi_{\pm}$ *is the orthogonal and symmetric projection defined in the beginning of Section 3, and the multiplicity* $\widetilde{m}_R(\pm m)$ *and* $w_{\pm}$ *are defined, respectively, by* (3.25) *and* (3.29).

REMARK 3.8. The multiplicity of a resonance k_j can also be defined as follows: suppose $k_j \in \mathcal{R} \setminus \{0\}$ and let γ_{k_j} denote a circular line integral around k_j : $k_j + \epsilon_j e^{i\theta}$,

$\theta \in [0, 2\pi)$ where $\epsilon_j > 0$ is sufficiently small such that γ_j contains no other resonances. Then the multiplicity of k_j is defined

$$m_R(k_j) = \text{rank} \oint_{\gamma_j} \frac{k}{\sqrt{k^2+m^2}} R_V^{\pm}(k)\, \mathrm{d}k. \tag{3.37}$$

This definition coincides with (2.17) provided $k_j \in \mathcal{R} \setminus \{0\}$.

4. The Birman-Krein formula for the perturbed Dirac operator. We begin with the following fact. The space of trace operators is designated by $\mathcal{B}_1$.

LEMMA 4.1. *Let Assumption 2.1 hold and let $f \in \mathscr{S}(\mathbb{R})$. Then $f(\mathbb{D}) - f(\mathbb{D}_0)$ is a trace class operator.*

Proof. First we observe that, for $z \in \rho(\mathbb{D}) \cap \{|\operatorname{Im} z| < 1\}$,

$$\begin{aligned} &\left\| R_V(z) R_V(i)^4 - R_0(z) R_0(i)^4 \right\|_{\mathcal{B}_1} \\ &\leq \begin{cases} \max\limits_{E \in \operatorname{spec}(\mathbb{D})} \dfrac{C}{|\operatorname{Re} z - E|^2 + |\operatorname{Im} z|^2}, & |\operatorname{Re} z| < m, \\ \dfrac{C}{|\operatorname{Im} z|^2}, & |\operatorname{Re} z| \geq m. \end{cases} \end{aligned} \tag{4.1}$$

The inequality follows from the second resolvent formula, together with the following estimates for singular values, valid for $\alpha, \beta \in \mathbb{N}$, $\rho \in C_0^\infty(\mathbb{R}^3)$ satisfying $V = \rho V$, and established for the free resolvent using arguments as in [16] which, by induction, also leads to the ones for the full resolvent,

$$\begin{aligned} s_{j/\alpha}\left(\rho R_0^\beta(i)\right),\ s_{j/\alpha}\left(R_0^\beta(i)\rho\right) &\leq C j^{-\beta/3}, \\ s_{j/\alpha}\left(\rho R_V^\beta(i)\right),\ s_{j/\alpha}\left(R_V^\beta(i)\rho\right) &\leq C j^{-\beta/3}. \end{aligned} \tag{4.2}$$

With these preparations, we are ready to establish the claim in the Lemma. Since $f \in \mathscr{S}(\mathbb{R})$ we can write $f(z) = (z-i)^{-4} g(z)$ where $g \in \mathscr{S}(\mathbb{R})$. Introduce also $\tilde{g}$ as an almost-analytic extension of g. An application of the Dynkin-Helffer-Sjöstrand type formula in Lemma 2.3, in conjunction with (4.1) and $\bar{\partial}_z \tilde{g} \leq C_N |\operatorname{Im} z|^N$, yields

$$\begin{aligned} &\|f(\mathbb{D}) - f(\mathbb{D}_0)\|_{\mathcal{B}_1} \\ &= \left\| \frac{1}{\pi i} \int_{\mathbb{C}} \left[(\mathbb{D} - z)^{-1} (\mathbb{D} - i)^{-4} - (\mathbb{D}_0 - z)^{-1} (\mathbb{D}_0 - i)^{-4} \right] \overline{\partial}_z \tilde{g}(z)\, \mathrm{dm}(z) \right\|_{\mathcal{B}_1} \\ &= \frac{1}{\pi i} \int_{\{|\operatorname{Re} z| \geq m\}} \|R_V(z) R_V^4(i) - R_0(z) R_0^4(i)\|_{\mathcal{B}_1} \overline{\partial}_z \tilde{g}(z)\, \mathrm{dm}(z) \\ &\quad + \frac{1}{\pi i} \int_{\{|\operatorname{Re} z| < m\}} \|R_V(z) R_V^4(i) - R_0(z) R_0^4(i)\|_{\mathcal{B}_1} \overline{\partial}_z \tilde{g}(z)\, \mathrm{dm}(z). \end{aligned}$$

It remains to use (4.1) and note that for $N \geq 4$, the integral above is finite. □

The Birman-Krein formula goes back to the classical paper [4].

THEOREM 4.2 (The Dirac Birman-Krein formula). *Let Assumption 2.1 hold. Suppose, moreover, that $f \in \mathscr{S}(\mathbb{R})$, and let $S^{\pm}(k)$ be the scattering matrix; see (2.30).*

Then

$$\begin{aligned}\operatorname{Tr}\,(f(\mathbb{D})-f(\mathbb{D}_0)) &= \frac{1}{2\pi i}\int_0^\infty f(\sqrt{k^2+m^2})\operatorname{Tr}\left[S^+(k)^{-1}\partial_k S^+(k)\right]\,\mathrm{d}k\\ &\quad-\frac{1}{2\pi i}\int_0^\infty f(-\sqrt{k^2+m^2})\operatorname{Tr}\left[S^-(k)^{-1}\partial_k S^-(k)\right]\,\mathrm{d}k \qquad (4.3)\\ &\quad+\sum_{E_j} m_j f(E_j)+\frac{1}{2}\sum_{\pm}\pm\widetilde{m}_R(\pm m)f(\pm m),\end{aligned}$$

where E_j are the eigenvalues of $\mathbb{D}$ with associated multiplicity m_j, and the resonance multiplicity $\widetilde{m}_R(\pm m)$ is defined in (3.25).

Proof. We divide the proof into six steps.

1. By the spectral theorem of self-adjoint operators and Stone's formula (see, e.g., [20]) we have

$$\begin{aligned}f(\mathbb{D}) &= \int_{-\infty}^{-m} f(z)\,\mathrm{d}E(z)+\int_m^\infty f(z)\,\mathrm{d}E(z)+\sum_{E_j\in\mathrm{spec_d}(\mathbb{D})} m_j f(E_j)u_j\otimes\bar{u}_j\\ &= \frac{1}{2\pi i}\int_{-\infty}^{-m} f(z)\left[R_V(z+i0)-R_V(z-i0)\right]\mathrm{d}z\\ &\quad+\frac{1}{2\pi i}\int_m^\infty f(z)\left[R_V(z+i0)-R_V(z-i0)\right]\mathrm{d}z\\ &\quad+\sum_{E_j\in\mathrm{spec_d}(\mathbb{D})} m_j f(E_j)u_j\otimes\bar{u}_j, \qquad (4.4)\end{aligned}$$

where E_j are the eigenvalues of $\mathbb{D}$ with corresponding eigenfunctions u_j. We concentrate first on the continuous part of the spectrum. In view of Lemma 4.1 we take the trace difference between functions of the full and free Dirac operator and rewrite in the k variable,

$$\begin{aligned}&2\pi i\operatorname{Tr}\,[f(\mathbb{D})-f(\mathbb{D}_0)]\\ &=-\operatorname{Tr}\int_\infty^{-\infty} f(-\sqrt{k^2+m^2})\left[R_V^-(k)-R_V^-(-k)-R_0^-(k)+R_0^-(-k)\right]\frac{k}{\sqrt{k^2+m^2}}\,\mathrm{d}k \qquad (4.5)\\ &\quad+\operatorname{Tr}\int_{-\infty}^\infty f(\sqrt{k^2+m^2})\left[R_V^+(k)-R_V^+(-k)-R_0^+(k)+R_0^+(-k)\right]\frac{k}{\sqrt{k^2+m^2}}\,\mathrm{d}k.\end{aligned}$$

Introduce

$$\begin{aligned}B^\pm(k) &= \pm\frac{k}{\sqrt{k^2+m^2}}\left(R_V^\pm(k)-R_0^\pm(k)\right)\\ &= \pm\frac{k}{\sqrt{k^2+m^2}}\left(-R_V^\pm(k)VR_0^\pm(k)\right)\\ &= \mp\frac{k}{\sqrt{k^2+m^2}}\left(R_0^\pm(k)(I+VR_0^\pm(k)\rho)^{-1}VR_0^\pm(k)\right)+\frac{\Pi_\pm}{k}\frac{(\sqrt{k^2+m^2}+m)}{\sqrt{k^2+m^2}},\end{aligned}$$

where we have used the second resolvent identity, (2.16) and added the singularities of the full resolvent from Theorem 3.7. Note that the simple pole $1/k$ is cancelled by the k prefactor, and that we do not include any contribution from the opposite resonance. Similarly for $B^{\pm}(-k)$ we have

$$\begin{aligned}
B^{\pm}(k) &= F^{\pm}(k) + \frac{\Pi_{\pm}}{k} \frac{(\sqrt{k^2+m^2}+m)}{\sqrt{k^2+m^2}}, \\
B^{\pm}(-k) &= F^{\pm}(-k) - \frac{\Pi_{\pm}}{k} \frac{(\sqrt{k^2+m^2}+m)}{\sqrt{k^2+m^2}}, \\
F^{\pm}(k) &= \mp \frac{k}{\sqrt{k^2+m^2}} \left(R_0^{\pm}(k)(I+VR_0^{\pm}(k)\rho)^{-1} V R_0^{\pm}(k) \right), \\
F^{\pm}(-k) &= \pm \frac{k}{\sqrt{k^2+m^2}} \left(R_0^{\pm}(-k)(I+VR_0^{\pm}(-k)\rho)^{-1} V R_0^{\pm}(-k) \right).
\end{aligned} \tag{4.6}$$

Note that $F^{\pm}(k)$ and $F^{\pm}(-k)$ above matches the definition in (2.39). We introduce $\tilde{f} \in \mathscr{S}(\mathbb{C})$ as an almost-analytic extension of f satisfying

$$\begin{gathered}
\tilde{f}\big|_{\mathbb{R}} = f, \qquad \operatorname{supp} \tilde{f} \subset \{z : |\operatorname{Im} z| \le 1\}, \\
\overline{\partial}_z \tilde{f} \le C_N \frac{|\operatorname{Im} z|^N}{\langle x \rangle^{2(N+1)}}, \ \forall N \in \mathbb{N}.
\end{gathered} \tag{4.7}$$

Hence, assuming $0 < \epsilon \ll m$, an application of Green's theorem yields

$$\begin{aligned}
&4\pi i \operatorname{Tr}\left[f(\mathbb{D}_V) - f(\mathbb{D}_0)\right] \\
&\quad = \sum_{\pm} \pm \lim_{\epsilon \to 0} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} f(\pm\sqrt{k^2+m^2}) \left(\operatorname{Tr} B^{\pm}(k) + \operatorname{Tr} B^{\pm}(-k) \right) \mathrm{d}k \\
&\qquad + \sum_{\pm} \pm \lim_{\epsilon \to 0} \int_{\gamma_{\pm}(\epsilon)} \tilde{f}(-\sqrt{k^2+m^2}) \operatorname{Tr} B^{-}(\pm k)\, \mathrm{d}k \\
&\qquad + \sum_{\pm} \mp \lim_{\epsilon \to 0} \int_{\gamma_{\pm}(\epsilon)} \tilde{f}(\sqrt{k^2+m^2}) \operatorname{Tr} B^{+}(\pm k)\, \mathrm{d}k \\
&\qquad + \sum_{\pm} \pm \lim_{\epsilon \to 0} \int_{\partial\Gamma_{\mp}(\epsilon)} \tilde{f}(-\sqrt{k^2+m^2}) \operatorname{Tr} B^{-}(\mp k)\, \mathrm{d}k \\
&\qquad + \sum_{\pm} \pm \lim_{\epsilon \to 0} \int_{\partial\Gamma_{\pm}(\epsilon)} \tilde{f}(\sqrt{k^2+m^2}) \operatorname{Tr} B^{+}(\pm k)\, \mathrm{d}k,
\end{aligned} \tag{4.8}$$

using the notation (see also Figure 1)

$$\begin{gathered}
\Gamma_{\pm}(\epsilon) = D(0;\epsilon) \cap \mathbb{C}_{\pm}, \qquad \mathbb{C}_{\pm} = \{k \in \mathbb{C} : \pm \operatorname{Im} k > 0\}, \\
\gamma_{\pm}(\epsilon) = \{\partial\Gamma_{\pm}(\epsilon) : \pm \operatorname{Im} k > 0\},
\end{gathered}$$

where the positively orientated contours $\partial\Gamma_{\pm}(\epsilon)$ enclose the open regions $\Gamma_{\pm}(\epsilon)$, and $D(0;\epsilon)$ is the open disk of radius ϵ centred at the origin. We number the terms on the right-hand side of (4.8) from 1 to 5 and evaluate each in the remaining steps.

2. In this step we examine term 1 on the right-hand side of (4.8). The explicit simple poles at the origin in (4.6) cancel and so we can take the limit $\epsilon \to 0$. By employing

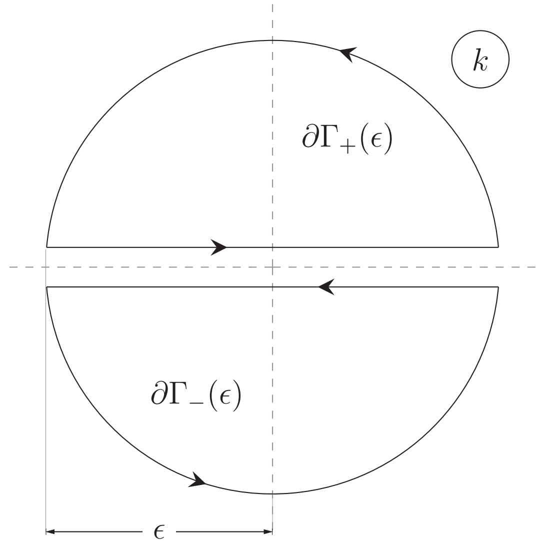

FIGURE 1. *The contours $\partial\Gamma_+(\epsilon)$ and $\partial\Gamma_-(\epsilon)$ in the k plane. We have assumed that $\epsilon \ll m$.*

(2.39) we show for term 1

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} f(\pm\sqrt{k^2+m^2})\left(\mathrm{Tr}\ B^{\pm}(k)+\mathrm{Tr}\ B^{\pm}(-k)\right)\,\mathrm{d}k\\
&\qquad= \lim_{\epsilon\to 0}\int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} f(\pm\sqrt{k^2+m^2})\left(\mathrm{Tr}\ F^{\pm}(k)+\mathrm{Tr}\ F^{\pm}(-k)\right)\,\mathrm{d}k\\
&\qquad= \int_{\mathbb{R}} f(\pm\sqrt{k^2+m^2})\,\mathrm{Tr}\ \left[S^{\pm}(k)^{-1}\partial_k S^{\pm}(k)\right]\,\mathrm{d}k.
\end{aligned}$$

By (2.39), the integrand is even in k and we take the integral on $[0,\infty)$.
3. To examine the remaining integrals in (4.8) located near the origin, we use Gohberg-Sigal theory (see [10]) to study the structure of $F^{\pm}(k)$ near this point. To this end we have

$$I+VR_0^{\pm}(k)\rho = U_1^{\pm}(k)\left(Q_2^{\pm}k^2+Q_1^{\pm}k+Q_0^{\pm}\right)U_2^{\pm}(k),$$

where $U_j^{\pm}$ are holomorphic and invertible, plus the projection operators $Q_j^{\pm}$ satisfy

$$\begin{gathered}
Q_i^{\pm}Q_j^{\pm}=\delta_{ij}Q_i^{\pm}, \qquad \mathrm{rank}(I-Q_0^{\pm})<\infty,\\
\mathrm{rank}\,Q_2^{\pm}=\mathrm{Tr}\ \Pi_{\pm}=m_R(\pm m)-\widetilde{m}_R(\pm m), \qquad \mathrm{rank}\,Q_1^{\pm}=\widetilde{m}_R(\pm m).
\end{gathered} \tag{4.9}$$

Since the free and perturbed resolvents meromorphically extend from the upper k-plane to all $\mathbb{C}$, , then we can apply the generalized argument principle [10, Chapter XI, Section 9, Theorem 9.1] with

$$N_0(I+VR_0^{\pm}(k)\rho)=2\,\mathrm{Tr}\ \Pi_{\pm}+\widetilde{m}_R(\pm m), \quad N_0((I+VR_0^{\pm}(k)\rho)^{-1})=0,$$

which count, with multiplicity, the number of zeros and poles of $I+VR_0^{\pm}(k)\rho$ respectively. Using the definitions in (4.6) we therefore have

$$\begin{aligned}\oint \mathrm{Tr}\ F^{\pm}(k)\,\mathrm{d}k &= \mp\,\mathrm{Tr}\oint\left(\frac{k}{\sqrt{k^2+m^2}}\right)R_0^{\pm}(k)(I+VR_0^{\pm}(k)\rho)^{-1}VR_0^{\pm}(k)\,\mathrm{d}k\\ &= -\,\mathrm{Tr}\oint \partial_k(I+VR_0^{\pm}(k)\rho)(I+VR_0^{\pm}(k)\rho)^{-1}\,\mathrm{d}k\\ &= -2\pi i\,(2\,\mathrm{Tr}\ \Pi_{\pm}+\widetilde{m}_R(\pm m))\\ &= -\oint\frac{(2\,\mathrm{Tr}\ \Pi_{\pm}+\widetilde{m}_R(\pm m))}{k}\,\mathrm{d}k,\end{aligned}$$

where we have used the spectral theorem. A similar argument holds for Tr $F^{\pm}(-k)$. Hence near $k=0$ we have

$$\mathrm{Tr}\ F^{\pm}(k) = -\frac{1}{k}\left[2\,\mathrm{Tr}\ \Pi_{\pm}+\widetilde{m}_R(\pm m)\right]+\varphi_{\pm}(k), \tag{4.10}$$

where $\varphi_{\pm}(k)$ is holomorphic for Im $k\geq 0$.

4. In this step we consider how terms 4 and 5 on the right-hand side of (4.8) behave as we take the limit $\epsilon\to 0$. In fact we take one particular case below with the method applicable to the remaining integrals. First write using Green's formula again

$$\begin{aligned}\left|\int_{\partial\Gamma_+(\epsilon)}\tilde{f}(-\sqrt{k^2+m^2})\,\mathrm{Tr}\ B^-(k)\,\mathrm{d}k\right| &= \left|2\int_{\Gamma_+(\epsilon)}\overline{\partial}_k\left[\tilde{f}(-\sqrt{k^2+m^2})\,\mathrm{Tr}\ B^-(k)\right]\,\mathrm{d}m\right|\\ &= \left|2\int_{\Gamma_+(\epsilon)}\left(\overline{\partial}_k\tilde{f}(-\sqrt{k^2+m^2})\right)\,\mathrm{Tr}\ B^-(k)\,\mathrm{d}m\right|,\end{aligned}$$

where m denotes the Lebesgue measure on $\mathbb{C}$ and Tr $B^-(k)$ defined by (4.6) and (4.10) is analytic. Given

$$\overline{\partial}_k\tilde{f}(-\sqrt{k^2+m^2})\leq C_N|\,\mathrm{Im}\ k|^N,\quad \forall N\in\mathbb{N},$$

and Tr $B^-(k)=\mathcal{O}\left(k^{-1}\right)$, then

$$\left|\int_{\partial\Gamma_+(\epsilon)}\tilde{f}(-\sqrt{k^2+m^2})\,\mathrm{Tr}\ B^-(k)\,\mathrm{d}k\right|\leq C_N\epsilon^{N-1}\left|\int_{\Gamma_+(\epsilon)}\mathrm{d}m\right| = C_N\epsilon^{N+1},\quad \forall N\in\mathbb{N}.$$

We conclude that this and indeed all contributions from terms 4 and 5 in the right-hand side of (4.8) tend to 0 as $\epsilon\to 0$.

5. For terms 2 and 3 in the right-hand side of (4.8), we use the indentation lemma to compute the integrals along circular arcs. Using (4.6) and (4.10) we have for term 2

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{\gamma_+(\epsilon)}\tilde{f}(-\sqrt{k^2+m^2})\,\mathrm{Tr}\ B^-(k)\,\mathrm{d}k-\lim_{\epsilon\to 0}\int_{\gamma_-(\epsilon)}\tilde{f}(-\sqrt{k^2+m^2})\,\mathrm{Tr}\ B^-(-k)\,\mathrm{d}k\\ &=\lim_{\epsilon\to 0}\int_{\gamma_+(\epsilon)}\frac{\tilde{f}(-\sqrt{k^2+m^2})}{k}\left[-(2\,\mathrm{Tr}\ \Pi_-+\widetilde{m}_R(-m))+\mathrm{Tr}\ \Pi_-\frac{\sqrt{k^2+m^2}+m}{\sqrt{k^2+m^2}}\right]\mathrm{d}k\\ &\quad-\lim_{\epsilon\to 0}\int_{\gamma_-(\epsilon)}\frac{\tilde{f}(-\sqrt{k^2+m^2})}{k}\left[(2\,\mathrm{Tr}\ \Pi_-+\widetilde{m}_R(-m))-\mathrm{Tr}\ \Pi_-\frac{\sqrt{k^2+m^2}+m}{\sqrt{k^2+m^2}}\right]\mathrm{d}k\\ &=-2\pi i\widetilde{m}_R(-m)f(-m).\end{aligned}$$

This completes the analysis on term 2 on the right-hand side of (4.8). Term 3 similarly follows

$$\lim_{\epsilon\to 0}\int_{\gamma_-(\epsilon)} \tilde{f}(\sqrt{k^2+m^2})\,\mathrm{Tr}\,B^+(k)\,\mathrm{d}k - \lim_{\epsilon\to 0}\int_{\gamma_+(\epsilon)} \tilde{f}(\sqrt{k^2+m^2})\,\mathrm{Tr}\,B^+(-k)\,\mathrm{d}k$$
$$= 2\pi i \widetilde{m}_R(m) f(m).$$

6. Bringing together all the previous steps with (4.5) produces the first, second and fourth terms on the right-hand side of (4.3). For the third term, it remains to bring back the contribution from the discrete eigenvalues. This follows immediately from the form in (4.4) where we have Tr $u_j \otimes \overline{u}_j = 1$. □

5. The Poisson wave trace formula. From (2.16) and (2.32) we see that resonances appear as poles of the scattering matrix $S^{\pm}(k)$ (see Section 2.4). It also follows that resonances appear as poles of the scattering determinant

$$s^{\pm}(k) = \det(S^{\pm}(k)) = \det(I - A^{\pm}(k)),$$

where $A^{\pm}(k)$ is given in (2.32). For $k \in \mathbb{R}\setminus\{0\}$, we have from (2.35) that $S^{\pm}_{\mathrm{abs}}(k)^{-1} = S^{\pm}_{\mathrm{abs}}(-k)$. Meromorphic extension of both sides in k, extends to all $k \in \mathbb{C}$. Since $S^{\pm}(k) = -S^{\pm}_{\mathrm{abs}}(k)J$ from (2.31) then $-S^{\pm}_{\mathrm{abs}}(k)^{-1} = JS^{\pm}(k)^{-1}$ and so

$$S^{\pm}(-k) = -S^{\pm}_{\mathrm{abs}}(-k)J = -S^{\pm}_{\mathrm{abs}}(k)^{-1}J = JS^{\pm}(k)^{-1}J.$$

Using $\mathrm{Tr}\,(JA^{\pm}(-k)J)^j = \mathrm{Tr}\,A(-k)^j$ we obtain

$$s^{\pm}(k)^{-1} = s^{\pm}(-k).$$

Hence we have that $k_j \in \mathcal{R}$ if and only if $s^{\pm}(k_j) = 0$. Moreover, $s^{\pm}(k)$ is an entire function and by the Weierstrass factorization theorem [15] we have

$$s^{\pm}(k) = (-1)^{\widetilde{m}_R(\pm m)} e^{g(k)} \frac{P_{\pm}(-k)}{P_{\pm}(k)}, \tag{5.1}$$

where $P_{\pm}$ is the canonical product

$$P_{\pm}(k) := \prod_{k_j \in \mathcal{R}_{\pm}\setminus\{0\}} E_4(k/k_j)^{m_R(k_j)}, \quad E_p(k) := (1-k)\exp\left(\sum_{n=1}^{p} k^n/n\right), \tag{5.2}$$

and g an entire function; we argue below that g is an odd polynomial for degree at most 3. We summarize the arguments. The genus of E_p equals 4 in view of (2.20) so the infinite product converges (see e.g. [6]). It follows from Proposition 2.2 that

$$e^{-C|k|^{4+\epsilon}} \le |P_{\pm}(k)| \le e^{C|k|^{4+\epsilon}}.$$

The latter, in conjunction with (following from [9, Theorem 5.1] and Cartan's minimum modulus principle for entire functions)

$$|s^{\pm}(k)| \le Ce^{C|k|^{12+\epsilon}}, \qquad k \notin \bigcup_{k_j \in \mathcal{R}_{\pm}} D(k_j; \langle k_j \rangle^{-4-\delta}), \tag{5.3}$$

implies that $|e^{g(k)}| \le Ce^{C|k|^{12+\epsilon}}$. Then the maximum modulus principle and the Borel-Carathéodory theorem allow us to deduce that g is a polynomial of degree at most 12. To show that g is a polynomial of degree no greater than 3 we use Hadamard's factorization theorem [6, Chapter XI, Lemma 3.1] with the holomorphic function $f(k) = e^{g(k)} P_\pm(k)$.

With these preparations we are ready to state and prove the main result.

THEOREM 5.1. *Let Assumption 2.1 hold. Let $\mathcal{R} = \cup_\pm \mathcal{R}_\pm$ denote the set of resonances of $\mathbb{D}$ having multiplicity $m_R(k)$, satisfying* (2.17)*. Then, in distributional sense for all $t \in \mathbb{R}$,*

$$2t^4 \operatorname{Tr}\left[\cos\left(t\sqrt{\mathbb{D}^2 - m^2}\right) - \cos\left(t\sqrt{\mathbb{D}_0^2 - m^2}\right)\right]$$
$$= t^4 \sum_\pm \sum_{k_j \in \mathcal{R}_\pm} \pm m(k_j) e^{-i|t|k_j} + 2t^4 \sum_{E_j} m_j \cosh\left(|t|\sqrt{m^2 - E_j^2}\right), \tag{5.4}$$

where, in accordance with (2.17) and (3.25),

$$m(k) = \begin{cases} m_R(k), & k \neq 0, \\ \widetilde{m}_R(\pm m), & k = 0, \end{cases}$$

and E_j are the eigenvalues of $\mathbb{D}$ with multiplicities m_j.

REMARK 5.2. Note that we have combined the contributions from the threshold resonances with the other resonances.

Proof. We divide the proof into seven steps.
1. In view of (5.1) and the facts summarized below it, we first establish

$$\partial_k^5 \log s^\pm(k) = \sum_{k_j \in \mathcal{R}_\pm \setminus \{0\}} m_R(k_j) \partial_k^4 \left(\frac{1}{k + k_j} - \frac{1}{k - k_j}\right). \tag{5.5}$$

For this purpose we recall that $g(k)$ is an odd polynomial of degree $N \leq 3$ (see above). Hence

$$\begin{aligned} &\partial_k^5 \log s^\pm(k) \\ &= \partial_k^5 \left[\log\left(e^{g(k)} \frac{P_\pm(-k)}{P_\pm(k)}\right)\right] = \partial_k^5 \left[g(k) + \log P_\pm(-k) - \log P_\pm(k)\right] \\ &= \sum_{k_j \in \mathcal{R}_\pm \setminus \{0\}} m_R(k_j) \partial_k^5 \left[\log\left(1 + \frac{k}{k_j}\right) + \sum_{\ell=1}^4 \frac{1}{\ell}\left(-\frac{k}{k_j}\right)^\ell - \log\left(1 - \frac{k}{k_j}\right) - \sum_{\ell=1}^4 \frac{1}{\ell}\left(\frac{k}{k_j}\right)^\ell\right] \\ &= \sum_{k_j \in \mathcal{R}_\pm \setminus \{0\}} m_R(k_j) \partial_k^4 \left[\frac{1}{k + k_j} - \frac{1}{k - k_j}\right]. \end{aligned}$$

2. Let $u(t) = 2t^4 \operatorname{Tr}\left[\cos\left(t\left(\mathbb{D}^2 - m^2\right)^{1/2}\right) - \cos\left(t\left(\mathbb{D}_0^2 - m^2\right)^{1/2}\right)\right] \in \mathscr{D}'(\mathbb{R})$, the

space of distributions, and let $\phi \in \mathscr{D}(\mathbb{R})$. Then

$$\begin{aligned}
&\langle u, \phi\rangle \\
&= \int 2t^4 \operatorname{Tr}\left[\cos(t\sqrt{\mathbb{D}^2 - m^2}) - \cos(t\sqrt{\mathbb{D}_0^2 - m^2})\right]\phi(t)\,\mathrm{d}t \\
&= \int t^4 \operatorname{Tr}\left[e^{it\sqrt{\mathbb{D}^2-m^2}} + e^{-it\sqrt{\mathbb{D}^2-m^2}} - e^{it\sqrt{\mathbb{D}_0^2-m^2}} - e^{-it\sqrt{\mathbb{D}_0^2-m^2}}\right]\phi(t)\,\mathrm{d}t \\
&= \sqrt{2\pi}\operatorname{Tr}\left[\sum_{\pm}\widehat{t^4\phi}\left(\pm\sqrt{\mathbb{D}^2 - m^2}\right) - \sum_{\pm}\widehat{t^4\phi}\left(\pm\sqrt{\mathbb{D}_0^2 - m^2}\right)\right] \\
&= \sqrt{2\pi}\operatorname{Tr}\left[f(\mathbb{D}) - f(\mathbb{D}_0)\right],
\end{aligned}$$

where

$$f(z) = \widehat{t^4\phi}(\sqrt{z^2 - m^2}) + \widehat{t^4\phi}(-\sqrt{z^2 - m^2}). \tag{5.6}$$

An application of Theorem 4.2 yields

$$\frac{1}{\sqrt{2\pi}}\langle u\,,\phi\rangle = A + B + C + D, \tag{5.7}$$

where

$$\begin{aligned}
A &:= \tfrac{1}{2\pi i}\textstyle\int_0^\infty f(\sqrt{k^2+m^2})\operatorname{Tr}\left(S^+(k)^{-1}\partial_k S^+(k)\right)\mathrm{d}k, \\
B &:= -\tfrac{1}{2\pi i}\textstyle\int_0^\infty f(-\sqrt{k^2+m^2})\operatorname{Tr}\left(S^-(k)^{-1}\partial_k S^-(k)\right)\mathrm{d}k, \\
C &:= \textstyle\sum_{E_j} m_j f(E_j), \\
D &:= \tfrac{1}{2}\textstyle\sum_{\pm} \pm\widetilde{m}_R(\pm m) f(\pm m).
\end{aligned} \tag{5.8}$$

3. We first study the continuous part of the spectrum. Define $h(k) := \widehat{\phi}(k)$ obeying $\partial_k^4 h(k) = \widehat{t^4\phi}(k)$. Thus we have from Jacobi's determinant formula that

$$\begin{aligned}
2\pi i A &= \int_0^\infty f(\sqrt{k^2+m^2})\operatorname{Tr}\left[S^+(k)^{-1}\partial_k S^+(k)\right]\mathrm{d}k \\
&= \frac{1}{2}\int_{\mathbb{R}} f(\sqrt{k^2+m^2})\partial_k \log s^+(k)\,\mathrm{d}k \\
&= \frac{1}{2}\int_{\mathbb{R}} \left[\widehat{t^4\phi}(k) + \widehat{t^4\phi}(-k)\right]\partial_k \log s^+(k)\,\mathrm{d}k \\
&= \frac{1}{2}\int_{\mathbb{R}} \left[\partial_k^4 h(k) + \partial_k^4 h(-k)\right]\partial_k \log s^+(k)\,\mathrm{d}k \\
&= \frac{1}{2}\int_{\mathbb{R}} \left[h(k) + h(-k)\right]\partial_k^5 \log s^+(k)\,\mathrm{d}k,
\end{aligned}$$

where (2.38) was used, together with the fact that the 'boundary' terms vanish because $h \in \mathscr{S}(\mathbb{R})$. Using (5.5) we find

$$\begin{aligned}
2\pi i A &= \frac{1}{2}\int_{\mathbb{R}}[h(k) + h(-k)]\sum_{k_j\in\mathcal{R}_+\setminus\{0\}} m_R(k_j)\partial_k^4\left[\frac{1}{k+k_j} - \frac{1}{k-k_j}\right]\mathrm{d}k \\
&= \sum_{k_j\in\mathcal{R}_+\setminus\{0\}} \frac{m_R(k_j)}{2}\int_{\mathbb{R}} \partial_k^4[h(k) + h(-k)]\left[\frac{1}{k+k_j} - \frac{1}{k-k_j}\right]\mathrm{d}k \\
&= \sum_{k_j\in\mathcal{R}_+\setminus\{0\}} \frac{m_R(k_j)}{2}\int_{\mathbb{R}} \left[\widehat{t^4\phi}(k) + \widehat{t^4\phi}(-k)\right]\left[\frac{1}{k+k_j} - \frac{1}{k-k_j}\right]\mathrm{d}k.
\end{aligned}$$

Explicitly we have

$$\begin{aligned}2\pi i A = &\sum_{k_j\in\mathcal{R}_+\setminus\{0\}} \frac{m_R(k_j)}{2\sqrt{2\pi}}\Bigg[\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-itk}\phi(t)t^4\left(\frac{1}{k+k_j}-\frac{1}{k-k_j}\right)\mathrm{d}t\,\mathrm{d}k \\ &+\int_{\mathbb{R}}\int_{\mathbb{R}} e^{itk}\phi(t)t^4\left(\frac{1}{k+k_j}-\frac{1}{k-k_j}\right)\mathrm{d}t\,\mathrm{d}k\Bigg] \\ = &\sum_{k_j\in\mathcal{R}_+\setminus\{0\}} \frac{m_R(k_j)}{2\sqrt{2\pi}}\Bigg[\int_{\mathbb{R}}\left(\int_0^\infty e^{-i|t|k}t^4\phi(t)\,\mathrm{d}t+\int_{-\infty}^0 e^{i|t|k}t^4\phi(t)\,\mathrm{d}t\right. \\ &\left.+\int_0^\infty e^{i|t|k}t^4\phi(t)\,\mathrm{d}t+\int_{-\infty}^0 e^{-i|t|k}t^4\phi(t)\,\mathrm{d}t\right)\left(\frac{1}{k+k_j}-\frac{1}{k-k_j}\right)\mathrm{d}k\Bigg].\end{aligned}$$

Hence we have

$$\begin{aligned}2\pi i A = &\sum_{k_j\in\mathcal{R}_+\setminus\{0\}} \frac{m_R(k_j)}{2\sqrt{2\pi}}\Bigg[\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i|t|k}t^4\phi(t)\left(\frac{1}{k+k_j}-\frac{1}{k-k_j}\right)\mathrm{d}t\,\mathrm{d}k \\ &+\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i|t|k}t^4\phi(t)\left(\frac{1}{k+k_j}-\frac{1}{k-k_j}\right)\mathrm{d}t\,\mathrm{d}k\Bigg].\end{aligned} \tag{5.9}$$

To utilise Jordan's lemma and obtain semi-circular arcs that produce closed loops, we note that we require Im $k<0$ and Im $k>0$ for the first and second terms on the right-hand side of (5.9). (That is, $e^{\pm i|t|k}=e^{\pm i|t|(\mathrm{Re}\ k+i\,\mathrm{Im}\ k)}\le e^{\mp|t|\,\mathrm{Im}\ k}$ and then we use the sign of Im k such that $e^{i|t|k}\to 0$ as $|k|\to\infty$). Since we defined the resonances to lie in the lower k plane, then we remove half of the terms in (5.9). Define

$$C_1(R)=[-R,R]\cup\{Re^{i\theta}\,:\,\theta\in(2\pi,\pi)\},\qquad C_2(R)=[-R,R]\cup\{Re^{i\theta}\,:\,\theta\in(0,\pi)\}$$

so that

$$\begin{aligned}2\pi i A = &\sum_{k_j\in\mathcal{R}_+\setminus\{0\}} \frac{m_R(k_j)}{2\sqrt{2\pi}}\Bigg[-\lim_{R\to\infty}\int_{\mathbb{R}}\int_{C_1(R)}\frac{e^{-i|t|k}}{k-k_j}t^4\phi(t)\,\mathrm{d}k\,\mathrm{d}t \\ &\qquad+\lim_{R\to\infty}\int_{\mathbb{R}}\int_{C_2(R)}\frac{e^{i|t|k}}{k+k_j}t^4\phi(t)\,\mathrm{d}k\,\mathrm{d}t\Bigg] \\ &=2\pi i\sum_{k_j\in\mathcal{R}_+\setminus\{0\}}\frac{m_R(k_j)}{2\sqrt{2\pi}}\int_{\mathbb{R}}\left[e^{-i|t|k_j}+e^{i|t|(-k_j)}\right]t^4\phi(t)\,\mathrm{d}t \\ &=\frac{2\pi i}{\sqrt{2\pi}}\left\langle\sum_{k_j\in\mathcal{R}_+\setminus\{0\}} m_R(k_j)t^4e^{-i|t|k_j}\,,\phi\right\rangle.\end{aligned}$$

4. We follow the same method as in the previous step to find

$$B=-\frac{1}{\sqrt{2\pi}}\left\langle\sum_{k_j\in\mathcal{R}_-\setminus\{0\}} m_R(k_j)t^4e^{-i|t|k_j}\,,\phi\right\rangle.$$

5. For the discrete spectra, invoking (5.6), we write

$$C = \sum_{E_j} m_j f(E_j) = \sum_{E_j} m_j \left[\widehat{t^4\phi}\left(\pm i\sqrt{m^2 - E_j^2}\right) + \widehat{t^4\phi}\left(\mp i\sqrt{m^2 - E_j^2}\right)\right]$$
$$= \sum_{E_j} \frac{m_j}{\sqrt{2\pi}}\left[\int_{\mathbb{R}} t^4 \left(e^{\mp t\sqrt{m^2-E_j^2}} + e^{\pm t\sqrt{m^2-E_j^2}}\right)\phi(t)\,\mathrm{d}t\right]$$
$$= \frac{1}{\sqrt{2\pi}}\left\langle \sum_{E_j} 2m_j t^4 \cosh\left(|t|\sqrt{m^2 - E_j^2}\right), \phi\right\rangle.$$

6. For the possible resonances at $z = \pm m$ we similarly use (5.6) to write

$$D = \frac{1}{2}\sum_{\pm} \pm\widetilde{m}_R(\pm m) f(\pm m) = \frac{1}{2}\sum_{\pm} \pm\widetilde{m}_R(\pm m)[\widehat{t^4\phi}(0) + \widehat{t^4\phi}(0)]$$
$$= \sum_{\pm} \pm\frac{\widetilde{m}_R(\pm m)}{\sqrt{2\pi}}\int_{\mathbb{R}} t^4\phi(t)\,\mathrm{d}t = \frac{1}{\sqrt{2\pi}}\left\langle \sum_{\pm} \pm\widetilde{m}_R(\pm m)t^4, \phi\right\rangle.$$

7. Finally, we insert the results of Steps 3, 4, 5 and 6 into (5.7) to obtain the desired result. □

We point out that a minor modification of the proof above shows that, for all $t \in \mathbb{R}\setminus\{0\}$,

$$2\,\mathrm{Tr}\left[\cos\left(t\sqrt{\mathbb{D}^2 - m^2}\right) - \cos\left(t\sqrt{\mathbb{D}_0^2 - m^2}\right)\right]$$
$$= \sum_{k_j\in\mathcal{R}_+} m_R(k_j)e^{-i|t|k_j} - \sum_{k_j\in\mathcal{R}_-} m_R(k_j)e^{-i|t|k_j} + 2\sum_{E_j} m_j\cos(|t|E_j), \tag{5.10}$$

in the sense of distributions.

6. Application. Existence of infinitely many resonances. As an application of the results above, we prove that the perturbed Dirac operator has infinitely many resonances. For this purpose we need the asymptotics of the scattering phase established in [5] which requires that we restrict ourselves to real-valued scalar potentials of the form

$$V = \begin{pmatrix} V_+I_2 & 0 \\ 0 & V_-I_2 \end{pmatrix}, \tag{6.1}$$

where I_2 is the 2×2 identity matrix. Furthermore, it is assumed that, $V_\pm \in C_0^\infty(\mathbb{R}^3)$. First we establish the following result:

THEOREM 6.1. *Define the scattering phase, $\sigma_\pm(k)$, by*

$$\sigma'_\pm(k) := \frac{1}{2\pi i}\partial_k \log\det S^\pm(k). \tag{6.2}$$

Then there exists a sequence a_j such that

$$\sigma'_+(k) - \sigma'_-(k) \sim \sum_{j}^{\infty} a_j(V)k^{2(1-j)} \text{ as } k\to\infty. \tag{6.3}$$

Moreover, in the vicinity of $t = 0$, we have the expansion

$$\widehat{\sigma}'_+(k) - \widehat{\sigma}'_-(k) = C_1\delta(t) + \sum_{j=2}^{\infty} C_j|t|^{2j-3}, \tag{6.4}$$

where

$$\begin{gathered} C_1 = -2\sqrt{2\pi}\gamma_2(V), \qquad C_j = (-1)^j\sqrt{2\pi}\frac{\gamma_{2j}(V)}{(2j-3)!}, \quad j \geq 2, \\ \gamma_2(V) = \frac{1}{2\pi^2}\int_{\mathbb{R}^3}\left[\left(\frac{V_+ - V_-}{2}\right)^2 + (V_+ - V_-) - 2\left(\frac{V_+ + V_-}{2}\right)^2\right](\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \end{gathered} \tag{6.5}$$

Proof. We divide the proofs into two steps.
1. From [5, Theorem 2.1] we have

$$-\frac{1}{2\pi i}\log\det S(z) \sim \sum_{j=1}^{\infty}\gamma_j(V)z^{2-j}, \qquad z \to \pm\infty. \tag{6.6}$$

Switching to $k = \sqrt{z^2 - m^2}$ we have that $\dfrac{\mathrm{d}k}{\mathrm{d}z} = \pm\sqrt{k^2 - m^2}/k$ and so splitting the scattering phase along the two semi-infinite axes we have

$$-\frac{1}{2\pi i}\partial_z \log\det S(z) = -\frac{1}{2\pi i}\left(\frac{\mathrm{d}k}{\mathrm{d}z}\right)\partial_k \log\det S^{\pm}(k) = \mp\frac{\sqrt{k^2 + m^2}}{k}\sigma'_{\pm}(k).$$

As $z \to \pm\infty$ corresponds to $k \to \infty$ then, in the limit $k \to \infty$,

$$\begin{aligned} \sigma'_{\pm}(k) &\sim \mp\frac{k}{\sqrt{k^2 + m^2}}\sum_{j=1}^{\infty}\gamma_j(V)(\pm\sqrt{k^2 + m^2})^{2-j} \\ &= \mp\frac{k}{\sqrt{k^2 + m^2}}\left[\pm\gamma_1(V)\sqrt{k^2 + m^2} + \gamma_2(V) \pm \frac{\gamma_3(V)}{\sqrt{k^2 + m^2}} + \frac{\gamma_4(V)}{k^2 + m^2} + \cdots\right] \\ &= -\gamma_1(V)k \mp \gamma_2(V)\frac{k}{\sqrt{k^2 + m^2}} - \gamma_3(V)\frac{k}{k^2 + m^2} \mp \gamma_4(V)\frac{k}{(k^2 + m^2)^{3/2}} + \cdots \\ &\sim -\gamma_1(V)k \mp \gamma_2(V) - \frac{\gamma_3(V)}{k} \mp \frac{\gamma_4(V)}{k^2} + \cdots, \qquad k \to \infty. \end{aligned}$$

Hence

$$\sigma'_+(k) - \sigma'_-(k) \sim -2\gamma_2(V) - \frac{2\gamma_4(V)}{k^2} + \cdots = -2\sum_{j=1}^{\infty}\frac{\gamma_{2j}(V)}{k^{2(j-1)}}$$

or

$$\sigma'_+(k) - \sigma'_-(k) \sim \sum_{j=1}^{\infty}\frac{a_j(V)}{k^{2(j-1)}}, \qquad k \to \infty, \tag{6.7}$$

where $a_j(V) = -2\gamma_{2j}(V)$ as required.
2. Write $k = \alpha\kappa$, $t = \beta\tau$ and $\kappa\tau = 1$ so that as $\kappa \to \infty$, then $\tau \to 0$. Then we take

the Fourier transform of (6.7) so that, as $\tau \to 0$, we have

$$\begin{aligned}
&\widehat{\sigma}'_+(\beta\tau) - \widehat{\sigma}'_-(\beta\tau) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\sigma'_+(\alpha k) - \sigma'_-(\alpha k)\right) e^{-i\alpha\kappa\beta\tau} \,\mathrm{d}(\alpha\kappa) \\
&\sim \sum_{j=1}^{\infty} a_j(V) \int_{\mathbb{R}} \frac{e^{-i\alpha\kappa\beta\tau}}{(\alpha\kappa)^{2(j-1)}} \,\mathrm{d}(\alpha\kappa) \\
&= \sqrt{2\pi} a_1(V)\delta(\beta\tau) + \sum_{j=2}^{\infty} a_j(V)(-i)\sqrt{\frac{\pi}{2}} \frac{(-i)^{2(j-1)-1}(\beta\tau)^{2(j-1)-1}}{(2(j-1)-1)!} \,\mathrm{sgn}(\beta\tau) \\
&= \sqrt{2\pi} a_1(V)\delta(\beta\tau) + \sqrt{\frac{\pi}{2}} \sum_{j=2}^{\infty} (-1)^{j+1} a_j(V) \frac{(\beta\tau)^{2j-3}}{(2j-3)!} \,\mathrm{sgn}(\beta\tau), \qquad \tau \to 0.
\end{aligned}$$

Let $\beta = 1$ and use $a_j(V) = -2\gamma_{2j}(V)$. Then

$$\widehat{\sigma}'_+(\tau) - \widehat{\sigma}'_-(\tau) \sim -2\sqrt{2\pi}\gamma_2(V)\delta(t) + \sqrt{2\pi} \sum_{j=2}^{\infty} (-1)^j \frac{\gamma_{2j}(V)}{(2j-3)!} |t|^{2j-3},$$

as required. □

It is useful to record the following variant of Theorem 5.1.

LEMMA 6.2. *Near $t = 0$, the trace formula in Theorem 5.1 satisfies, in distributional sense,*

$$\begin{aligned}
&2\,\mathrm{Tr}\,[\cos(t\sqrt{\mathbb{D}^2 - m^2}) - \cos(t\sqrt{\mathbb{D}_0^2 - m^2})] - T(t) \\
&\qquad = -4\pi\gamma_2(V)\delta(t) + 2\pi \sum_{j=2}^{\infty} (-1)^j \frac{\gamma_{2j}(V)}{(2j-3)!} |t|^{2j-3},
\end{aligned} \tag{6.8}$$

where $T(t) = 2\sum_{E_j} m_j \cos(|t|E_j) + \sum_{\pm} \pm\widetilde{m}_R(\pm m)$

Proof. As in the proof of Theorem 5.1, we use

$$u(t) = 2\,\mathrm{Tr}\,[\cos(t\sqrt{\mathbb{D}^2 - m^2}) - \cos(t\sqrt{\mathbb{D}_0^2 - m^2})] - T(t),$$

$\phi \in C_0^\infty(\mathbb{R})$ and $f(z) = \widehat{\phi}(\sqrt{z^2 - m^2}) + \widehat{\phi}(-\sqrt{z^2 - m^2})$ to write

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \langle u, \phi \rangle &= \frac{1}{2\pi i} \int_0^\infty f(\sqrt{k^2 + m^2}) \partial_k \log\det S^+(k) \,\mathrm{d}k \\
&\quad - \frac{1}{2\pi i} \int_0^\infty f(-\sqrt{k^2 + m^2}) \partial_k \log\det S^-(k) \,\mathrm{d}k \\
&= \int_0^\infty f(\sqrt{k^2 + m^2}) \sigma'_+(k) \,\mathrm{d}k - \int_0^\infty f(\sqrt{k^2 + m^2}) \sigma'_-(k) \,\mathrm{d}k \\
&= \int_0^\infty [\widehat{\phi}(k) - \widehat{\phi}(-k)] \sigma'_+(k) \,\mathrm{d}k - \int_0^\infty [\widehat{\phi}(k) - \widehat{\phi}(-k)] \sigma'_-(k) \,\mathrm{d}k \\
&= \langle (\widehat{\sigma}'_+ - \widehat{\sigma}'_-), \phi \rangle,
\end{aligned}$$

where we have used the fact that $\sigma'_{\pm}(k)$ is even; see (2.39). Then, by invoking (6.4) we have, as $t \to 0$,

$$u(t) = -4\pi\gamma_2(V)\delta(t) + 2\pi \sum_{j=2}^{\infty} (-1)^j \frac{\gamma_{2j}(V)}{(2j-3)!} |t|^{2j-3},$$

as claimed. □

THEOREM 6.3. *Let Assumption 2.1 hold and, moreover, let V satisfy* (6.1). *Suppose, in addition, that $\gamma_2(V) \neq 0$ and $\gamma_{2j}(V) \neq 0$ for some $j \geq 2$; see* (6.5). *Then the perturbed Dirac operator $\mathbb{D}$ has infinitely many resonances.*

The proof is a variation of Melrose's contradiction arguments for existence of infinitely many resonances for the perturbed Schrödinger operator [19]; see also [12].

Proof. We divide the proof into two steps.
1. First we prove that there exists at least one resonance of $\mathbb{D}$ away from $\pm m$. By contradiction we assume that there are a finite number of eigenvalues and that the only resonances are at $z = \pm m$. Then

$$\sum_{k_j \in \mathcal{R}_+} m_R(k_j) e^{-i|t|k_j} - \sum_{k_j \in \mathcal{R}_-} m_R(k_j) e^{-i|t|k_j} = 0. \tag{6.9}$$

Incidentally (6.9) is also zero if resonances in $\mathcal{R}_+$ and $\mathcal{R}_-$ at k_j with equal multiplicities cancel out but this automatically implies that there exists at least two resonances. Inserting (6.9) into the Poisson wave trace formula in Theorem 5.1 we therefore have

$$2\,\mathrm{Tr}\,[\cos(t\sqrt{\mathbb{D}^2 - m^2}) - \cos(t\sqrt{\mathbb{D}_0^2 - m^2})] - T(t) = 0.$$

By Lemma 6.2 this implies that

$$0 = -4\pi\gamma_2(V)\delta(t) + 2\pi \sum_{j=2}^{\infty} (-1)^j \frac{\gamma_{2j}(V)}{(2j-3)!} |t|^{2j-3}.$$

For any small $t > 0$ the delta distribution is equal to zero but for any $\gamma_{2j}(V) \neq 0$, $j \geq 2$ the right-hand side is not zero and hence gives a contradiction. Therefore there exists at least one resonance not at $z = \pm m$.
2. Next assume that there are only a finite number of resonances. Then rearranging the Poisson wave trace formula in Theorem 5.1, the right-hand side of

$$\begin{aligned} 2\,\mathrm{Tr}\left[\cos(t\sqrt{\mathbb{D}^2 - m^2}) - \cos(t\sqrt{\mathbb{D}_0^2 - m^2})\right] - T(t) \\ = \sum_{k_j \in \mathcal{R}_+} m_R(k_j) e^{-i|t|k_j} - \sum_{k_j \in \mathcal{R}_-} m_R(k_j) e^{-i|t|k_j}, \end{aligned} \tag{6.10}$$

is finite (possibly zero). Then we may continuously extend (6.10) to $t = 0$ such that the right-hand side is equal to or between $-\sum_{k_j \in \mathcal{R}_-} m_R(k_j)$ and $\sum_{k_j \in \mathcal{R}_+} m_R(k_j)$. However this contradicts (5.4) which is not continuous at $t = 0$ due to the delta distribution. We therefore conclude that there are infinitely many resonances. □

Acknowledgement. This work is sponsored by an EPSRC Doctoral Scholarship in Mathematics.

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