ASIAN J. MATH.
Vol. 25, No. 3, pp. 341–368, June 2021

STRONGLY HOMOTOPY LIE ALGEBRAS AND DEFORMATIONS OF CALIBRATED SUBMANIFOLDS*

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Abstract. For an element Ψ in the graded vector space $\Omega^*(M, TM)$ of tangent bundle valued forms on a smooth manifold M, a Ψ-submanifold is defined as a submanifold N of M such that $\Psi_{|N} \in \Omega^*(N, TN)$. The class of Ψ-submanifolds encompasses calibrated submanifolds, complex submanifolds and all Lie subgroups in compact Lie groups. The graded vector space $\Omega^*(M, TM)$ carries a natural graded Lie algebra structure, given by the Frölicher-Nijenhuis bracket $[-,-]^{FN}$. When Ψ is an odd degree element with $[\Psi, \Psi]^{FN} = 0$, we associate to a Ψ-submanifold N a strongly homotopy Lie algebra, which governs the formal and (under certain assumptions) smooth deformations of N as a Ψ-submanifold, and we show that under certain assumptions these deformations form an analytic variety. As an application we revisit formal and smooth deformation theory of complex closed submanifolds and of φ-calibrated closed submanifolds, where φ is a parallel form in a real analytic Riemannian manifold.

Key words. Ψ-submanifold, calibrated submanifold, Frölicher-Nijenhuis bracket, strongly homotopy Lie algebra, derived bracket, formal deformation, smooth deformation, complex submanifold.

Mathematics Subject Classification. Primary 32G10, 53C38; Secondary 17B55, 53C29, 58D27.

1. Introduction.

Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection. A differential form φ is called *parallel* if $\nabla\varphi = 0$. In this case we shall write (M, g, φ). When we want to stress that φ has degree p we shall write φ^p for φ. If φ is parallel, φ is closed and its comass is constant. Normalizing the comass of φ, we regard φ as a calibration. All important calibrated submanifolds are φ-calibrated submanifolds for some parallel differential form φ [Dao1977, HL1982, Le1990, McLean1998, Joyce2007][1]. On the other hand, φ-calibrated submanifolds play an important rôle in the geometry of manifolds with special holonomy, in higher dimensional gauge theory and in string theory as "supersymmetric cycles" or "branes" [SYZ1996, DT1998, Tian2000, GYZ2003, AW2003, Joyce2007, DS2011, Walpuski2012, Walpuski2014]. Note that manifolds with special holonomy always admit parallel forms, see Subsection 2.1 below.

Deformation theory of closed calibrated submanifolds has been initiated by McLean [McLean1998] inspired by similarities between calibrated submanifolds and complex submanifolds. McLean considered deformations of special Lagrangian, associative, coassociative and Cayley submanifolds. In [LV2017] Lê-Vanžura observed that any φ-calibrated submanifold L in a Riemannian manifold (M, g) considered by

*Received March 6, 2019; accepted for publication September 2, 2020.

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[1] In [Dao1977] Dao, based on previous work by Federer and Lawson, proposed to use parallel differential forms to study area-minimizing real currents, but he did not invent the word "calibration".

McLean (as well as any Kähler submanifold) satisfies the following Harvey-Lawson identity [LV2017, Definition 1.1]

$$|\varphi(\xi)|^2 + |\Psi_E(\xi)|^2 = |\xi|^2 \text{ for all } x \in M, \tag{1.1}$$

with ξ an element in the Grassmannian of unit decomposable k-vectors in T_xM, for some E-valued form $\Psi_E \in \Omega^*(M, E)$, where E is a Riemannian vector bundle over M. In this case the defining equation of φ-calibrated submanifolds L is equivalent to $(\Psi_E)_{|L} = 0$. McLean showed that, in the reformulation of [LV2017] using the Harvey-Lawson identity, the equation $(\Psi_E)_{|L} = 0$ is essentially elliptic for special Lagrangian and coassociative submanifolds, and using the standard elliptic theory he proved that deformations of those submanifolds are unobstructed. Additionally, he proved that the equation $(\Psi_E)_{|L} = 0$ is elliptic for associative and Cayley submanifolds L, but deformation of those calibrated submanifolds may be obstructed.

Further works on deformations of calibrated submanifolds are devoted to the smoothness and the Zariski tangent space to the moduli space of closed calibrated submanifolds that are special Lagrangian, associative, coassociative and Cayley in (tamed) almost/nearly Calabi-Yau, G_2 and Spin(7)-manifolds [AS2008, AS2008b, GIP2003, Gayet2014, Kawai2017, Ohst2014], or to similar questions concerning calibrated submanifolds with elliptic boundary condition [Butscher2003, KL2009, GW2011, Ohst2014] and non-compact calibrated submanifolds of certain type [JS2005, KL2012, Lotay2009].

In the present paper we propose a new approach to deformations of calibrated submanifolds. Firstly, we do not look for a Harvey-Lawson type identity. Instead, using the first cousin principle we characterize φ-calibrated submanifolds up to first order via the vector-valued form $\hat{\varphi} \in \Omega^*(M, TM)$ that is obtained from φ by contraction with the metric (Lemma 3.1, see also Remark 3.2). Motivated by Lemma 3.1, we introduce the notion of a Ψ-submanifold (Definition 3.3) and develop a general deformation theory for closed Ψ-submanifolds for any square-zero element Ψ of odd degree in the graded Lie algebra $\Omega^*(M, TM)$, using strongly homotopy Lie algebras (Proposition 5.3). This generalizes the assignment of a strongly homotopy Lie algebra to a complex submanifold (Remark 6.7). In particular, we show that under some natural assumptions, the deformation space of Ψ-submanifolds is a finite dimensional analytic space (Theorem 5.10).

Applying this to a parallel calibration, we prove that the moduli space of $\hat{\varphi}$-submanifolds within a given φ-transversal homology class is an analytic space and hence, both the formal and the smooth deformation problem for closed φ-calibrated submanifolds in (M, g, φ) are encoded in its associated L_∞-algebra (Theorem 6.4).

This paper is organized as follows. In Section 2, we collect known results concerning parallel differential forms and the Frölicher-Nijenhuis bracket that are important for the main part of the paper. In Section 3, we introduce the notion of a Ψ-submanifold (Definition 3.3) which seems a good notion to understand deformations of calibrated submanifolds (Corollary 3.6). In Section 4, we assign to each Ψ-submanifold a canonical strongly homotopy Lie algebra, if Ψ is a square-zero element of odd degree in the graded Lie algebra $(\Omega^*(M, TM), [-,-]^{FN})$ (Theorem 4.1). In Section 5 we define the deformation problem for Ψ-submanifolds and study formal deformations using this strongly homotopy Lie algebra (Proposition 5.3). Moreover, we show that under certain conditions the deformation space is an analytic variety (Theorem 5.10). In Section 6, we apply these results to study infinitesimal, smooth and formal deformations of calibrated submanifolds in detail (Proposition 6.1, Theorem 6.4) and revisit the deformation theory of complex submanifolds (Theorem 6.6, Remark 6.7).

Notations and conventions.

• In this paper, manifolds and their submanifolds are denoted by capital Latin letters M, L, etc. When we want to emphasize the dimension of a manifold M (resp. a submanifold L) we write M^m (resp. L^l). The tangent map to a smooth map $f : M \to N$ is denoted by $Tf : TM \to TN$, and its value at the point $x \in M$ by $T_x f : T_x M \to T_{f(x)} N$.

• Small Greek letters usually denote scalar valued forms and capital Greek letters denote vector valued forms. Also, we use the Einstein summation convention summing over repeated indices whenever convenient.

• For a scalar valued form φ on M we denote by $\hat{\varphi}$ the associated TM-valued form on M obtained from φ by contraction with the metric (see (2.6) and the sentence that follows for explanation).

• For a finite dimensional (resp. infinite dimensional) vector space V we denote by $0 \in V$ (resp. $\mathbf{0} \in V$) the origin of V.

• We adopt Getzler's conventions about L_∞-algebras [Getzler2009].

2. Preliminaries.

2.1. Parallel differential forms on a Riemannian manifold. In this section, we recall the classification of parallel differential forms on a Riemannian manifold (M, g), described in Tables 1, 2, 3, 4 from [Besse1987, Chapter 10].

Let φ be a parallel form on (M, g) such that φ is not a multiple of the volume form. Then the restricted holonomy group $Hol^0(M, g)$ is contained in the stabilizer $Stab(\varphi)$ and therefore is strictly smaller than the group $O(m)$. Since locally a Riemannian manifold (M, g) is a product of Riemannian manifolds whose holonomy group action on the tangent bundle is irreducible, the classification of parallel forms on (M, g) is reduced to the case of irreducible Riemannian manifolds (M, g). Symmetric Riemannian spaces are examples of manifolds admitting parallel forms.

• The algebra of parallel forms on an irreducible symmetric space $M = G/H$ is isomorphic to the algebra of Ad_H-invariant forms on $T_e G/H$. In particular, if $M = G/H$ is compact, then the algebra of parallel forms is isomorphic to the de Rham cohomology algebra $H^*(M, \mathbb{R})$. A list of the Poincaré polynomials of all the simply connected compact irreducible symmetric spaces has been compiled by Takeuchi in [Takeuchi1962].

In 1955, Marcel Berger proved that if (M, g) is a simply-connected Riemannian manifold with irreducible holonomy group and nonsymmetric, then $Hol^0(M, g)$ must be one of $SO(n)$, $U(m)$ (Kähler manifolds) , $SU(m)$ (Ricci flat Kähler manifolds, in particular Calabi-Yau manifolds), $Sp(m)$ (hyper-Kähler manifolds), $Sp(m) \times Sp(1)$ (quaternionic Kähler manifolds), G_2 (G_2-manifolds) or Spin(7) (Spin(7)-manifolds).

• The algebra of parallel forms on a Kähler manifold is generated by the Kähler 2-form ω.

• The algebra of parallel forms on a Ricci flat Kähler manifold is generated by the Kähler 2-form ω and the real and imaginary parts $\mathrm{Re}\,\mathrm{vol}_{\mathbb{C}}$, $\mathrm{Im}\,\mathrm{vol}_{\mathbb{C}}$ of the complex volume form. The latter are called special Lagrangian forms, abbreviated as SL-forms.

• The algebra of parallel forms on a quaternionic Kähler manifold is generated by the quarternionic 4-form ψ.

• The algebra of parallel forms on a hyper-Kähler manifold is generated by the three Kähler 2-forms.

• The algebra of parallel forms on a G_2-manifold is generated by the associative 3-form φ and its dual coassociative 4-form $*\varphi$.

• The algebra of parallel forms on a Spin(7)-manifold is generated by the self-dual Cayley 4-form κ.

We also refer the reader to [Bryant1987, Salamon1989] for the geometry of parallel forms on manifolds with special holonomy.

2.2. Frölicher-Nijenhuis bracket. Let us recall the definition of the Frölicher-Nijenhuis bracket on $\Omega^*(M, TM)$ following [KMS1993, §8], see also [KLS2017a, §2.1] for a short account.

The space $Der(\Omega^*(M))$ of graded derivations of the graded commutative algebra $\Omega^*(M)$ is a graded Lie algebra. First we recall the definition of *algebraic graded derivations* in $Der(\Omega^*(M))$. They are defined by insertions ι_K for $K \in \Omega^*(M, TM)$. For $K = \alpha^l \otimes X$ we define $\iota_K \in Der(\Omega^*(M))$ as follows

$$\iota_{\alpha^l \otimes X} \beta^r := \alpha^l \wedge (\iota_X \beta^r) \in \Omega^{l+r-1}(M).$$

Next we define the linear map

$$\mathcal{L} : \Omega^*(M, TM) \to Der(\Omega^*(M)),\ K \mapsto \mathcal{L}_K,$$

$$\mathcal{L}_K := [\iota_K, d] \in Der(\Omega^*(M)). \tag{2.1}$$

Proposition 2.1 ([KMS1993, Theorem 8.3, p. 69]). *For any graded derivation $D \in Der(\Omega^*(M))$ there are unique $K \in \Omega^*(M, TM)$ and $K' \in \Omega^*(M, TM)$ such that*

$$D = \mathcal{L}_K + \iota_{K'}.$$

We have $K' = 0$ if and only if $[D, d] = 0$ and D is algebraic if and only if $K = 0$.

It follows from Proposition 2.1 that the map $\mathcal{L}$ is injective and its image $\mathcal{L}(\Omega^*(M, TM))$ is the centralizer of d in $Der(\Omega^*(M))$:

$$\mathcal{L}(\Omega^*(M, TM)) = \{D \in Der(\Omega^*(M)) |\ [D, d] = 0\}. \tag{2.2}$$

Hence, $\mathcal{L}(\Omega^*(M, TM))$ is closed under the graded Lie bracket $[-,-]$ on $Der(\Omega^*(M))$. Then we define *the Frölicher-Nijenhuis bracket* $[-,-]^{FN}$ on $\Omega^*(M, TM)$ as the pull-back of the graded Lie bracket on $Der(\Omega^*(M))$ via the linear embedding $\mathcal{L}$, i.e.,

$$\mathcal{L}_{[K,L]^{FN}} := [\mathcal{L}_K, \mathcal{L}_L]. \tag{2.3}$$

Thus, the Frölicher-Nijenhuis bracket provides $\Omega^*(M, TM)$ with the structure of a $\mathbb{Z}$-graded (hence $\mathbb{Z}_2$-graded) Lie algebra.

Furthermore the Frölicher-Nijenhuis bracket enjoys the following functoriality with respect to local diffeomorphisms. First of all, for a local diffeomorphism $f : M \to N$ and any $K \in \Omega^*(N, TN)$, the pull-back of K by f is defined as follows:

$$(f^*K)_x(X_1, \cdots, X_l) := (T_x f)^{-1} K_{f(x)}(T_x f \cdot X_1, \cdots, T_x f \cdot X_l). \tag{2.4}$$

Then we have [KMS1993, 8.16, p. 74]

$$f^*[K, L]^{FN} = [f^*K, f^*L]^{FN}. \tag{2.5}$$

Let (M, g) be a Riemannian manifold. Recall that the contraction $\partial_g : \Lambda^l T^*M \longrightarrow \Lambda^{l-1}T^*M \otimes TM$ is defined pointwise as follows [KLS2017a, (2.5)]

$$\partial_g(\varphi^l) := (\iota_{e_i}\varphi^l) \otimes (e_i), \tag{2.6}$$

where the sum is taken pointwise over some orthonormal basis (e_i) of T_xM.

We also abbreviate $\partial_g(\varphi)$ by $\hat{\varphi}$.

Remark 2.2. To express $\hat{\varphi}$, we shall also use the more convenient notion of a *$(l-1)$-fold alternating vector product $\rho_\varphi \in \Omega^{(l-1)}(M, TM)$ defined by a differential form $\varphi \in \Omega^l(M)$*, [Robles2012, (3.1)] such that

$$\varphi^l(u, v_2, \cdots, v_l) = g(u, \rho_\varphi(v_2, \cdots, v_l)). \tag{2.7}$$

Then we have $\hat{\varphi} = \rho_\varphi$. The notion of an alternating vector product, introduced by Robles, is a generalization of the notion of a *multi-linear vector cross product*, introduced by Gray [Gray1969], where Gray imposed a further compatibility between a vector cross product, which is an alternating vector product, on a pseudo-Riemannian manifold (M, g) and the pseudo-Riemannian metric g. Vector cross products have been used intensively by Harvey-Lawson in their study of calibrated geometry [HL1982].

A straightforward computation via geodesic normal coordinates yields the following

Proposition 2.3 (cf. [KLS2017a, Proposition 2.2]). *For any parallel differential form φ on a Riemannian manifold (M, g) we have $[\hat{\varphi}, \hat{\varphi}]^{FN} = 0$.*

Definition 2.4. We say that an element $\Psi \in \Omega^{2l+1}(M, TM)$ is *of square-zero*, if $[\Psi, \Psi]^{FN} = 0$.

Observe that $[\Psi, \Psi]^{FN} = 0$ for any $\Psi \in \Omega^{2l}(M, TM)$ as $[-, -]^{FN}$ is graded skew-symmetric.

3. φ-calibrated submanifolds and Ψ-submanifolds. In this section, motivated by the geometry of calibrated submanifolds (Lemma 3.1), we introduce the notion of a Ψ-submanifold for any $\Psi \in \Omega^*(M, TM)$ (Definition 3.3). We relate Lemma 3.1 and the notion of a Ψ-submanifold with previous work on calibrated submanifolds, their further extensions and related results (Remarks 3.2, 3.4). We provide examples of Ψ-submanifolds that are not calibrated submanifolds (Example 3.7), including all complex submanifolds as well as all Lie subgroups in compact Lie groups. Finally we give a shorter proof of Robles' result that if φ^l is a parallel l-form on a Riemannian manifold (M, g) then $\hat{\varphi}^l$-submanifolds L are minimal submanifolds if the restriction of φ^l to L does not vanish (Theorem 3.5), which shall be needed in a later section.

For a submanifold L in a manifold M we denote by $NL = TM_{|L}/TN$ the normal bundle of L and by $\mathrm{pr} : TM_{|L} \to NL$ the canonical projection. If M is endowed with a Riemannian metric g then we also identify NL with the (Riemannian) normal bundle of L that is the orthogonal complement to the tangent bundle TL.

Lemma 3.1. *Let φ^l be a calibration on a Riemannian manifold (M, g) and L a φ^l-calibrated submanifold. Then* $\mathrm{pr} \circ \hat{\varphi}^l_{|L} = 0 \in \Omega^*(L, NL)$.

Proof. Let L be a φ^l-calibrated submanifold. Let (e_i) and (f_j) be orthonormal bases of T_xL and N_xL, respectively. Then for any $i \in [1,k]$ we have

$$\mathrm{pr} \circ \hat{\varphi}(e_1 \wedge \cdots \widehat{e_i} \cdots \wedge e_l) = \sum_{j=1}^{n-l} \varphi(f_j \wedge e_1 \wedge \cdots \widehat{e_i} \cdots \wedge e_l) \otimes f_j, \tag{3.1}$$

where $\widehat{e_i}$ stands for omission. By the first cousin principle for calibrated submanifolds [HL1982, p. 78] the right hand side of (3.1) vanishes. This completes the proof. □

REMARK 3.2. (1) Let us denote by $G_l(T_xM)$ the Grassmannian of all unit decomposable l-vectors in T_xM and by $\vec{T_xL}$ the unit l-vector associated to the oriented tangent space T_xL whose orientation is defined by the volume form $\varphi^l_{|L}$. The Grassmanian $G_l(T_xM)$ has the natural Riemannian metric induced from the Riemannian metric on T_xM. Note that the tangent space $T_{\vec{T_xL}}G_l(T_xM)$ has an orthogonal basis consisting of l-vectors of the form $f_j \wedge e_1 \wedge \cdots \widehat{e_i} \cdots \wedge e_l$. Let $\tilde{\varphi}^l(x)$ denote the restriction of $\varphi^l(x)$ to $G_l(T_xM)$. Then we have

$$\langle \mathrm{pr} \circ \varphi^l(e_1 \wedge \cdots \widehat{e_i} \cdots \wedge e_l), f_j \rangle = \langle d_{e_1 \wedge \cdots \wedge e_l} \tilde{\varphi}^l(x), f_j \wedge e_1 \wedge \cdots \widehat{e_i} \cdots \wedge e_l \rangle \tag{3.2}$$

where the pairing in the LHS of (3.2) is defined via the Riemannian metric and where again, $\widehat{e_i}$ stands for omission. Thus the condition $\mathrm{pr} \circ \hat{\varphi}^l_{|L} = 0 \in \Omega^*(L, NL)$ is equivalent to the condition that L is a φ-critical manifold, i.e., $\vec{T_xL}$ is a critical point of φ_x for all $x \in L$; see also [Le1990], [HL2009], [Robles2012] for a study of φ-critical submanifolds.

(2) In [Robles2012, Proposition 3.4] Robles gives a nice characterization of φ-critical submanifolds L in terms of the alternating vector product ρ_φ, namely TL is ρ_φ-closed, i.e., $\rho_{\varphi|L} \in \Omega^*(L, TL)$.

(3) Although the equality $\mathrm{pr} \circ \hat{\varphi}_{|L} = 0$ is equivalent to the condition that L is ρ_φ-closed, we prefer the expression $\mathrm{pr} \circ \hat{\varphi}_{|L} = 0$, since it says that L is the zero set of an NL-valued differential form, what we shall use in our deformation theory in later sections. Moreover, this expression is similar to that appearing in the deformation theory for coisotropic submanifolds in Poisson and Jacobi manifolds (see, e.g., [LOTV2014], and references therein), which led us to our search for L_∞-algebras governing deformations of calibrated submanifolds.

Lemma 3.1 motivates the following

DEFINITION 3.3. Let M be a smooth manifold and $\Psi \in \Omega^l(M, TM)$. A submanifold $L^r \subset M$, where $r \geq l$, will be called *a Ψ-submanifold*, if $\mathrm{pr} \circ \Psi_{|L} = 0 \in \Omega^l(L^r, NL^r)$ or, equivalently, $\Psi_{|L} \in \Omega^l(L^r, TL^r)$.

REMARK 3.4. The class of Ψ-submanifolds of a manifold M is larger than the class of φ-critical submanifolds, since this definition does not require a metric on M. For instance, any almost complex submanifold in an almost complex manifold is a Ψ-submanifold. In contrast, the notion of a φ-critical submanifold in M implicitly requires a Riemannian metric g on M, which allows us to associate a tangent space T_xL of an oriented submanifold $L \subset M$ with the unit l-vector $\vec{T_xL}$.

Let us recall the following result of Robles, which we shall need later.

THEOREM 3.5 ([Robles2012, Theorem 1.2]). *Assume that φ^l is a parallel form on a Riemannian manifold (M, g). Then a $\hat{\varphi}^l$-submanifold L is a minimal submanifold if $\varphi^l_{|L} \neq 0$.*

We provide below a short proof of Theorem 3.5, using the argument in the proof of Lemma 1.1 in [Le1990].

Proof. Let L be a $\hat{\varphi}^l$-submanifold in (M, g). We shall compute the mean curvature H of L. Let $V_1, \cdots, V_l$ be local vector fields on L such that $|V_1 \wedge \cdots \wedge V_l| = 1$. By Remark 3.2(1), for each $x \in L$ the unit l-vector $\vec{T_x L}$ is a critical point of the function $\xi \mapsto \varphi^l(\xi)$. Hence we have

$$\varphi^l(\vec{T_x L}) = c \tag{3.3}$$

for some constant c (this follows from the classification of parallel forms on (M, g), see e.g. [Besse1987, Theorem 10.108, Corollary 10.110] and Subsection 2.1). Recall that $c \neq 0$ by the assumption of Theorem 3.5.

Let X be a normal vector on L. Using the argument in the proof of Lemma 1.1 in [Le1990] we compute

$$\begin{aligned} 0 =& (\iota_X d\varphi^l)(V_1, \cdots, V_l) = \sum_{i=1}^{l} (-1)^i V_i\big(\varphi^l(X, V_1, \cdots, \hat{V_i}, \cdots, V_l)\big) \\ & - X\big(\varphi^l(V_1, \cdots, V_l)\big) + \sum_{1 \le i < j \le l} (-1)^{i+j} \varphi^l([V_i, V_j], X, \cdots, \hat{V_i}, \cdots, \hat{V_j}, \cdots, V_l) \\ & + \sum_{i=1}^{l} (-1)^i \varphi^l([X, V_i], \cdots \hat{V_i}, \cdots V_l), \end{aligned} \tag{3.4}$$

where $\hat{V_i}$ stands for omission of this entry. The first and third terms in (3.4) are zero, since L is a $\hat{\varphi}^l$-submanifold. The second term is zero by (3.3). Hence we get from (3.4)

$$0 = \sum_{i=1}^{l} (-1)^i \varphi^l([X, V_i], \cdots, \hat{V_i}, \cdots, V_l) = c \sum_{i=1}^{l} \langle [X, V_i], V_i \rangle = c\langle -H, X \rangle.$$

Since $c \neq 0$ we obtain $H = 0$. This proves Theorem 3.5. □

COROLLARY 3.6. *Let φ be a parallel calibration. A deformation of a closed φ-calibrated submanifold inside the class of $\hat{\varphi}$-submanifolds remains in the subclass of φ-calibrated submanifolds.*

Proof. Let φ be a calibration and let L_t, $t \in [0, 1]$, be a continuous family of closed $\hat{\varphi}$-submanifolds such that L_0 is a φ-calibrated submanifold. Then $\varphi_{|L_0} \neq 0$ and therefore $\varphi_{|L_t} \neq 0$ for sufficiently small t. By Theorem 3.5, L_t is also a minimal submanifold for such small t, in particular the volume of L_t is constant around $t = 0$. Since L_0 is a calibrated submanifold, it follows that the L_t are calibrated submanifolds for all sufficiently small t. Then the set of all values t such that L_t is a φ-calibrated submanifold is an open subset in the interval $[0, 1]$. On the other hand, since the volume function is continuous, the set of all values t such that L_t is a φ-calibrated submanifold is also closed. This completes the proof of Corollary 3.6. □

EXAMPLE 3.7. 1. By Remark 3.2 (1), each φ-calibrated submanifold is a $\hat{\varphi}$-submanifold. In particular, every associative submanifold L^3 in a G_2-manifold M^7 is a $\hat{\varphi}$-submanifold, where φ is the associative 3-form on M^7. We claim that every 3-dimensional $\hat{\varphi}$-submanifold is an associative submanifold. To prove this assertion we regard $\hat{\varphi} \in \Omega^2(M^7, TM^7)$ as the 2-fold cross product $TM^7 \times TM^7 \to TM^7$: $\varphi(X, Y, Z) = \langle X \times Y, Z \rangle$ where $\times$ denotes the cross product [HL1982, KLS2017a]. Then our assertion follows from the first cousin principle for $\hat{\varphi}$-submanifolds and the observation that a 3-plane is associative if and only if it is invariant under the 2-fold cross product [HL1982].

2. Let us consider a complex manifold (M, g, J). We regard J as an element in $\Omega^1(M, TM)$. Clearly a submanifold L in M is a J-submanifold if and only if it is a complex submanifold.

3. Let $*\varphi$ be the coassociative 4-form on a G_2-manifold M^7. The associated form $\widehat{*\varphi} \in \Omega^3(M^7, TM^7)$ is often denoted by χ and called the 3-fold cross product [HL1982, KLS2017a].

(a) It is shown in the proof of Lemma 5.6 in [KLS2018] that a 3-submanifold $L^3 \subset M^7$ is a χ-submanifold if and only if it is an associative submanifold.

(b) By Remark 3.2 (1), every coassociative submanifold L^4 is a χ-submanifold. We claim that a 4-dimensional χ-submanifold is a coassociative submanifold. To prove this it suffices to show that the coassociative planes (up to orientation) are the only critical points of the function $\widetilde{*\varphi}$ defined in Remark 3.2 (1). This assertion is equivalent to the statement that the associative planes (up to orientation) are the only critical points of the function $\tilde{\varphi}$, which has been proved in Example 3.7 1.

It is not hard to conclude from (a) and (b) that a χ-submanifold in a G_2-manifold is either an associative or a coassociative submanifold. Thus, we regard χ as an analogue of the complex form $J \in \Omega^1(M, TM)$ in complex geometry. In [KLS2017a] Kawai-Lê-Schwachhöfer gave another interpretation of this fact, proving that a G_2-structure is torsion-free if and only if $[\chi, \chi]^{FN}$ vanishes.

4. Let $\alpha := Re(vol_{\mathbb{C}})$ be the SL-calibration on a Calabi-Yau manifold $(M, g, \omega, vol_{\mathbb{C}})$. Remark 3.2 (1) implies that every special Lagrangian submanifold $L \subset M$ is a $\hat{\alpha}$-submanifold.

5. Let M^7 be a G_2-manifold and φ the defining associative 3-form. In [LV2017] Lê-Vanžura define a form $\tau \in \Omega^4(M^7, TM^7)$ as follows. For $x, y, z, w \in TM^7$ we set ([HL1982, (1.17), Theorem 1.18, p. 117], see also [LV2017, Remark 4.2])

$$\tau(x, y, z, w) := -(\varphi(y, z, w)x + \varphi(z, x, w)y + \varphi(x, y, w)z + \varphi(y, x, z)w). \tag{3.5}$$

Then any 4-submanifold in M^7 is a τ-submanifold.

6. Let M^8 be a Spin(7)-manifold and ψ^4 its defining Cayley form. Recall that $\psi^4(X, Y, Z, W) = \langle P(X, Y, Z), W \rangle$ where P is the 3-fold vector cross product, see e.g. [Fernandez1986]. By Remark 3.2(1), every Cayley submanifold is a $\hat{\psi}^4$-submanifold. Since any $\hat{\psi}^4$-submanifold L is invariant under the triple product P, L must be a Cayley submanifold.

7. Let G be a compact Lie group provided with the Killing metric. Denote by ω^3 the Cartan 3-form on G. The calibration ω^3 has been first considered by Dao in [Dao1977] and later by Tasaki [Tasaki1985]. By Theorem 3.1 in [Le1990] any 3-dimensional Lie subgroup in G is a $\hat{\omega}$-submanifold. Since the tangent space T_eG is invariant under the Lie bracket, any Lie subgroup in G is a $\hat{\omega}$-submanifold. In [Le1990, Section 3] Lê classified stably minimal 3-dimensional subgroups in compact semi-simple Lie groups of classical type, see also [Le1990b] for the classification of all stably minimal simple Lie subgroups in classical Lie groups. Clearly, non-stably minimal Lie subgroups cannot be calibrated submanifolds, since calibrated submanifolds are area-minimizing, and hence stably minimal.

8. Let $\theta^3, \theta^5, \cdots, \theta^{2m-1}$ be bi-invariant forms on $SU(m)$. By Theorem 3.4 in [Le1990] for any $n < m$ the standard subgroup $SU(n) \subset SU(m)$ is a $\hat{\phi}$-submanifold for $\phi = \theta^1 \wedge \cdots \wedge \theta^{2n-1}$.

4. The L_∞-algebra associated to a Ψ-submanifold. We use the notation in the previous sections, in particular, NL denotes the normal bundle of a submanifold L in a manifold M. In this section, using Voronov's derived bracket construction [Voronov2005], we prove the following.

THEOREM 4.1. *Let $\Psi \in \Omega^*(M, TM)$ be an odd degree element which is square-zero, i.e., such that $[\Psi, \Psi]^{FN} = 0$, and let L be a Ψ-submanifold. Then the cochain complex $\Omega^*(L, NL)[-1]$ carries a canonical $\mathbb{Z}_2$-graded L_∞-algebra structure. If* $\deg \Psi = 1$ *then this $\mathbb{Z}_2$-graded L_∞-algebra is also a $\mathbb{Z}$-graded L_∞-algebra.*

As a corollary, taking into account Example 3.7 (1),(2), we obtain the following

COROLLARY 4.2. *1. Assume that φ^l is a parallel l-form on a Riemannian manifold (M, g) and L is a closed φ^l-calibrated submanifold. If l is even, then there is a canonical $\mathbb{Z}_2$-graded L_∞-algebra structure on $\Omega^*(L, NL)[-1]$. If l is odd, then there is a canonical $\mathbb{Z}_2$-graded L_∞-algebra structure on $\Omega^*(L \times S^1, N(L \times S^1))[-1]$.*

2. (cf. [Manetti2007]) For any closed complex submanifold L in a complex manifold M there is a canonical $\mathbb{Z}$-graded L_∞-algebra structure on $\Omega^*(L, NL)[-1]$.

3. For every closed associative submanifold L^3 in a G_2-manifold (M^7, φ^3) there are canonical $\mathbb{Z}_2$-graded L_∞-algebra structures both on $\Omega^*(L^3, NL^3)[-1]$ and on $\Omega^*(L^3 \times S^1, N(L^3 \times S^1))[-1]$.

Proof. Let L be a closed φ^l-calibrated submanifold of the Riemannian manifold (M, g). If l is even, then L is a $\hat{\varphi}^l$-submanifold of M. If l is odd, then $L \times S^1$ is a $\widehat{\varphi^l \wedge dt}$-submanifold of $M \times S^1$. This proves statement *1*. Statement *2* is immediate, as any complex submanifold is a J-submanifold. Finally, any associative submanifold L^3 of a G_2-manifold (M^7, φ^3) is a φ^3-calibrated submanifold, so we have an L_∞-structure on $\Omega^*(L^3 \times S^1, N(L^3 \times S^1))[-1]$ by statement *1*. On the other hand, it has been showed in [KLS2018] that L^3 is $\widehat{*\varphi^3}$-submanifold (see Example 3.7.3). This proves statement *3*. □

The remainder of this section is devoted to the proof of Theorem 4.1. First, let us recall Voronov's construction of a $\mathbb{Z}_2$-graded L_∞-algebra from a set of V-data. A set of *V-data* is a quintuple $(\mathfrak{g}, \mathfrak{a}, \mathrm{j}, P, \triangle)$, where

- $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}_2$-graded Lie algebra (with Lie bracket $[-,-]$),
- $\mathfrak{a}$ is an abelian Lie algebra;
- $\mathrm{j} : \mathfrak{a} \to \mathfrak{g}$ is a Lie algebra inclusion;
- $P : \mathfrak{g} \to \mathfrak{a}$ is a (not necessarily bracket preserving) projection, inverting j from the left and such that $\ker P \subseteq \mathfrak{g}$ is a Lie subalgebra,
- $\triangle \in (\ker P) \cap \mathfrak{g}_1$ is an element such that $[\triangle, \triangle] = 0$.

PROPOSITION 4.3 ([Voronov2005, Theorem 1, Corollary 1]). *Let $(L, \mathfrak{a}, \mathrm{j}, P, \triangle)$ be a set of V-data. Then $\mathfrak{a}[-1]$ is a $\mathbb{Z}_2$-graded L_∞-algebra with multibrackets*

$$\mathfrak{l}_n(a_1, \cdots, a_n) = (-1)^\star P[\cdots[[\triangle, \mathrm{j}(a_1)], \mathrm{j}(a_2)], \cdots, \mathrm{j}(a_n)]. \tag{4.1}$$

where

$$\star = (n-1)|a_1| + (n-2)|a_2| + \cdots + |a_{n-1}| + \frac{n(n+1)}{2},$$

and the vertical bars $|-|$ *denote the degree in* $\mathfrak{a}[-1]$.

Replacing $\mathbb{Z}_2$ by $\mathbb{Z}$ in the definition of V-data, Formula (4.1) gives a $\mathbb{Z}$-graded L_∞-algebra. A homotopy Lie theoretic interpretation of Voronov's L_∞-algebra structure on $\mathfrak{a}[-1]$ can be found in [Bandiera2015].

The proof of Theorem 4.1 will now go through several steps. The first step consists in associating V-data to a Ψ-submanifold L equipped with a tubular neighborhood τ.

If $j : L \hookrightarrow M$ is a submanifold, a *tubular neighborhood of* L *in* M is defined to be a diffeomorphism $\tau : N_\epsilon L \to U \subset M$ from an open neighborhood $N_\epsilon L \subset NL$ of $\mathbf{0}$ onto an open neighborhood of L in M such that $\tau \circ \mathbf{0} = j$, where $\mathbf{0}\colon L \to NL$ is the zero section. Clearly, such maps exist, e.g. using the normal exponential w.r.t. some Riemannian metric on M. Furthermore, since we may assume that $N_\epsilon L \subset NL$ is a disc bundle and hence bundle equivalent to all of NL, it follows that we may w.l.o.g. replace $N_\epsilon L$ by all of NL. That is, from now on we shall assume that a tubular neighborhood is a diffeomorphism $\tau : NL \xrightarrow{\sim} U \subset M$.

Definition 4.4. Let $\Psi \in \Omega^*(M, TM)$ be an odd degree element with $[\Psi, \Psi]^{FN} = 0$, let $j\colon L \hookrightarrow M$ be a Ψ-submanifold and $\tau : NL \to U \subset M$ be a tubular neighborhood of L in M. Denote by $\pi\colon NL \to L$ the projection. The 5-tuple $(\mathfrak{g}_L, \mathfrak{a}_L, \mathrm{j}_L, P_L, \triangle_{L,\tau})$ is defined as follows:

- The graded Lie algebra $\mathfrak{g}_L$ is $\Omega^*(NL, TNL)$ with the FN bracket;
- the abelian graded Lie algebra $\mathfrak{a}_L$ is the graded vector space $\Omega^*(L, NL)$ endowed with the zero bracket;
- the graded vector space morphism $\mathrm{j}_L\colon \mathfrak{a}_L \to \mathfrak{g}_L$ is defined on decomposable elements as $\mathrm{j}_L(\omega \otimes X) = \pi^*(\omega) \otimes \hat{X}$, where $\hat{X}$ is the canonical vertical lift of X given by the natural identification $N_{\pi(x)}L \cong \ker(\pi_*\colon T_x NL \to T_{\pi(x)}L)$;
- the graded vector space morphism $P_L\colon \mathfrak{g}_L \to \mathfrak{a}_L$ is the composition

$$\Omega^*(NL, TNL) \xrightarrow{|_L} \Omega^*(L, TNL\big|_L) \xrightarrow{\mathrm{pr}} \Omega^*(L, NL),$$

  where the rightmost arrow pr is the natural projection induced by the projection $TNL_{|L} \to NL$, also denoted by pr in Section 3, by identifying L with a submanifold of NL via the zero section $L \hookrightarrow NL$ (equivalently, pr is induced by the canonical splitting $TNL|_L = TL \oplus NL$);
- the element $\Delta_{L,\tau}$ in $\mathfrak{g}_{L,\tau}$ is $\Delta_{L,\tau} = \tau^*\Psi$, where

$$\tau^*\colon \Omega^*(M, TM) \to \Omega^*(NL, TNL)$$

  is the pullback of tensors along the local diffeomorphism τ.

Remark 4.5. Notice that, as the notation suggests, $\Delta_{L,\tau}$ is the only component of the 5-tuple $(\mathfrak{g}_L, \mathfrak{a}_L, \mathrm{j}_L, P_L, \triangle_{L,\tau})$ which actually depends on the tubular neighborhood τ.

Proposition 4.6. *The 5-tuple* $(\mathfrak{g}_L, \mathfrak{a}_L, \mathrm{j}_L, P_L, \triangle_{L,\tau})$ *associated with a* Ψ-*manifold is a 5-tuple of V-data. As a consequence the graded vector space* $\mathfrak{a}_L[-1] = \Omega^*(L, NL)[-1]$ *carries a* $\mathbb{Z}_2$-*graded* L_∞-*algebra structure induced by this data. When* Ψ *has degree 1, this is actually a* $\mathbb{Z}$-*graded* L_∞-*algebra structure.*

Proof. The map j_L is injective, the map P_L is surjective, and one manifestly has $P_L \circ \mathrm{j}_L = \mathrm{id}_{\mathfrak{a}_L}$ so we are left with showing $[\mathrm{j}_L\mathfrak{a}_L, \mathrm{j}_L\mathfrak{a}_L] = 0$, that $\ker P_L$ is a Lie subalgebra of $\mathfrak{g}_L$, that $\triangle_{L,\tau} \in \ker P_L$ and $[\triangle_{L,\tau}, \triangle_{L,\tau}] = 0$. To this aim, consider the composition

$$\tilde{P}_L = \mathrm{j}_L \circ P_L \colon \Omega^*(NL, TNL) \to \Omega^*(NL, TNL).$$

It is shown in [KLS2018] that the image of $\tilde{P}_L$ is an abelian subalgebra of the graded Lie algebra $(\Omega^*(NL, TNL), [-,-]^{FN})$ and that $\ker \tilde{P}_L$ is closed under the Frölicher-Nijenhuis bracket. As P_L is surjective, the image of $\tilde{P}_L$ coincides with the image of j_L, so that $\mathrm{j}_L(\mathfrak{a}_L)$ is an abelian subalgebra of $\mathfrak{g}_L$. As j_L is injective, we have $\ker \tilde{P}_L = \ker P_L$, and so $\ker P_L$ is a Lie subalgebra of $\mathfrak{g}_L$. By the naturality of the Frölicher-Nijenhuis bracket, we have

$$[\triangle_{L,\tau}, \triangle_{L,\tau}] = [\tau^*\Psi, \tau^*\Psi]^{FN} = \tau^*[\Psi, \Psi]^{FN} = 0. \tag{4.2}$$

Finally, as L is a Ψ-submanifold of M and τ is a diffeomorphism relative to L in a neighborhood of L (identified with the zero section in NL), we have that L is a $\triangle_{L,\tau}$-manifold in NL. Therefore, $P_L \triangle_{L,\tau} = 0$ by definition of $\triangle_{L,\tau}$-manifold. □

The underlying graded vector space of the L_∞-algebra structure induced on $\Omega^*(L, NL)[-1]$ by Proposition 4.6 is independent of τ. Our next step will consist in showing that also the L_∞-algebra structure is actually independent of τ, up to isomorphism. To begin with, let us show that a reparameterization of the tubular neighborhood leaves the L_∞-algebra structure unchanged up to isomorphism.

Lemma 4.7. *Let τ_0 and τ_1 be two tubular neighborhoods of L in M such that $\tau_1 = \tau_0 \circ \psi$ for some diffeomorphism ψ of NL relative to L. Then ψ induces an isomorphism of V-data between $(\mathfrak{g}_L, \mathfrak{a}_L, \mathrm{j}_L, P_L, \Delta_{L,\tau_0})$ and $(\mathfrak{g}_L, \mathfrak{a}_L, \mathrm{j}_L, P_L, \Delta_{L,\tau_1})$. In particular $(\mathfrak{g}_L, \mathfrak{a}_L, \mathrm{j}_L, P_L, \Delta_{L,\tau_0})$ and $(\mathfrak{g}_L, \mathfrak{a}_L, \mathrm{j}_L, P_L, \Delta_{L,\tau_1})$ induce isomorphic L_∞-algebra structures on $\Omega^*(L, NL)[-1]$.*

Proof. As ψ is a diffeomorphism of NL relative to L the pullback along ψ induces commutative diagrams

$$\begin{array}{ccc} \Omega^*(L,NL) & \xrightarrow{\mathrm{j}_L} & \Omega^*(NL,TNL) \\ \downarrow{\scriptstyle \psi^*} & & \downarrow{\scriptstyle \psi^*} \\ \Omega^*(L,NL) & \xrightarrow{\mathrm{j}_L} & \Omega^*(NL,TNL) \end{array} \quad ; \quad \begin{array}{ccc} \Omega^*(NL,TNL) & \xrightarrow{P_L} & \Omega^*(L,NL) \\ \downarrow{\scriptstyle \psi^*} & & \downarrow{\scriptstyle \psi^*} \\ \Omega^*(NL,TNL) & \xrightarrow{P_L} & \Omega^*(L,NL) \end{array}$$

Finally, we have

$$\psi^*\Delta_{L,\tau_0} = (\tau_0^{-1} \circ \tau_1)^*(\tau_0^*\Psi) = \tau_1^*\Psi = \Delta_{L,\tau_1}.$$

□

In order to prove that the L_∞-algebra structure $\Omega^*(L, NL)[-1]$ is generally independent of τ, up to isomorphism, as we can not directly compare two distinct tubular neighborhoods of L in M, it is convenient to pass to formal neigborhoods.

Definition 4.8. Let $\Psi \in \Omega^*(M, TM)$ be an odd degree element with $[\Psi, \Psi]^{FN} = 0$, let $L \subset M$ be a Ψ-submanifold and $\tau : NL \to U \subset M$ be a tubular neighborhood of L in M. Finally, let $NL_{\mathrm{for}} \hookrightarrow NL$ be the formal neighborhood of L in NL via the zero section embedding $s_0 \colon L \hookrightarrow NL$. We recall that working in the formal neighborhood of L means working only with ∞-*jets* (of functions, sections, etc.) transverse to

L (see, e.g., [CS2008, Section 4.1], were a similar situation is discussed in details). In the same notation as Proposition 4.6, the 5-tuple $(\mathfrak{g}_L^{\text{for}}, \mathfrak{a}_L^{\text{for}}, P_L^{\text{for}}, \mathrm{j}_L^{\text{for}}, \triangle_{L,\tau}^{\text{for}})$ is the restriction to NL_{for} of the 5-tuple $(\mathfrak{g}_L, \mathfrak{a}_L, P_L, \mathrm{j}_L, \triangle_{L,\tau})$.

Remark 4.9. Notice that the graded abelian Lie algebras $\mathfrak{a}_L$ and $\mathfrak{a}_L^{\text{for}}$ actually coincide: they are both the graded vector space $\Omega^*(L, NL)$ endowed with the zero bracket. In particular the restriction to NL_{for} is the identity morphism on $\Omega^*(L, NL)$.

Proposition 4.10. *The 5-tuple $(\mathfrak{g}_L^{\text{for}}, \mathfrak{a}_L^{\text{for}}, \mathrm{j}_L^{\text{for}}, P_L^{\text{for}}, \triangle_{L,\tau}^{\text{for}})$ is a set of V-data and so induces a $\mathbb{Z}_2$-graded L_∞-algebra structure on $\mathfrak{a}_L^{\text{for}}[-1] = \Omega^*(L, NL)[-1]$. Moreover this $\mathbb{Z}_2$-graded L_∞-algebra structure coincides with that induced on $\Omega^*(L, NL)$ by the V-data $(\mathfrak{g}_L, \mathfrak{a}_L, P_L, \mathrm{j}_L, \triangle_{L,\tau})$.*

Proof. The proof follows analogous lines as those of [CS2008, Sections 4.1], and we leave the obvious translation to the reader. □

Lemma 4.11. *Let τ_0 and τ_1 be two isotopic tubular neighborhoods of L in M. Then $\triangle_{L,\tau_0}^{\text{for}}$ and $\triangle_{L,\tau_1}^{\text{for}}$ are gauge equivalent square-zero elements in $\mathfrak{g}_L^{\text{for}}$. In particular the V-data $(\mathfrak{g}_L^{\text{for}}, \mathfrak{a}_L^{\text{for}}, \mathrm{j}_L^{\text{for}}, P_L^{\text{for}}, \triangle_{L,\tau_0}^{\text{for}})$ and $(\mathfrak{g}_L^{\text{for}}, \mathfrak{a}_L^{\text{for}}, \mathrm{j}_L^{\text{for}}, P_L^{\text{for}}, \triangle_{L,\tau_1}^{\text{for}})$ induce isomorphic L_∞-algebra structures on $\mathfrak{a}_L^{\text{for}}[-1] = \Omega^*(L, NL)[-1]$.*

Proof. By definition of isotopic tubular neighborhoods, there exists a smooth family Φ_t of maps $\Phi_t \colon NL \to M$, with $t \in [0,1]$, which are diffeomorphisms on their images and such that $\Phi_t \circ s_0 = j$ for every $t \in [0,1]$, such that $\Phi_0 = \tau_0$ and $\Phi_1 = \tau_1$. Let $\hat{\Phi}_t$ be the composition of Φ_t with the embedding $NL_{\text{for}} \hookrightarrow NL$ of the formal neighborhood NL_{for} of L into NL. Then $\hat{\Phi}_t$ is a formal diffeomorphism between NL_{for} and the formal neighborhood $\hat{L}_M$ of L inside M. Let $\Delta_t^{\text{for}} = \hat{\Phi}_t^*(\Psi\big|_{\hat{L}_M})$. Then $\Delta_0^{\text{for}} = \Delta_{L,\tau_0}^{\text{for}}$ and $\Delta_1^{\text{for}} = \Delta_{L,\tau_1}$. Moreover, writing $\hat{\Xi}_t$ for the formal diffeomorphism of NL_{for} relative to L given by $\hat{\Xi}_t = \hat{\Phi}_0^{-1} \circ \hat{\Phi}_t$ we have

$$\Delta_t^{\text{for}} = \hat{\Phi}_t^*(\hat{\Phi}_0^{-1})^* \hat{\Phi}_0^*(\Psi\big|_{\hat{L}_M}) = \hat{\Xi}_t^* \Delta_0^{\text{for}}$$

As $\hat{\Xi}_0 = \mathrm{id}_{NL_{\text{for}}}$, differentiating the above equation with respect to t we find

$$\frac{d}{dt}\Delta_t^{\text{for}} = \mathcal{L}_{\hat{\xi}_t} \Delta_t^{\text{for}},$$

where $\mathcal{L}_{\hat{\xi}_t} \Delta_t^{\text{for}}$ is the Lie derivative of the tensor field Δ_t^{for} with respect to the vector field $\hat{\xi}_t = \frac{d}{dt}\hat{\Xi}_t$. For every t, the vector field $\hat{\xi}_t$ is an element in $\Omega^0(NL_{\text{for}}, TNL_{\text{for}}) = (\mathfrak{g}_L^{\text{for}})_0$. Moreover, $\mathcal{L}_{\hat{\xi}_t} \Delta_t = [\hat{\xi}_t, \Delta_t^{\text{for}}]^{FN}$. Thus, the family of elements Δ_t^{for} satisfies

$$\begin{cases} \frac{d}{dt}\Delta_t^{\text{for}} = [\hat{\xi}_t, \Delta_t^{\text{for}}]^{FN} \\ \\ \Delta_0^{\text{for}} = \Delta_{L,\tau_0}^{\text{for}} \\ \\ \Delta_1^{\text{for}} = \Delta_{L,\tau_1}^{\text{for}} \end{cases}$$

and it is therefore a gauge equivalence between $\Delta_{\tau_0}^{L,\text{for}}$ and $\Delta_{L,\tau_1}^{\text{for}}$ in $\mathfrak{g}_L^{\text{for}}$. The final part of the statement follows from the following

Proposition 4.12 (Cattaneo & Schätz, cf. [CS2008, Theorem 3.2]). *Let* $(\mathfrak{g}, \mathfrak{a}, \mathrm{j}, P, \Delta_0)$ *and* $(\mathfrak{g}, \mathfrak{a}, \mathrm{j}, P, \Delta_1)$ *be V-data, and let* $\mathfrak{a}[-1]_0$ *and* $\mathfrak{a}[-1]_1$ *be the associated* L_∞*-algebras. If* Δ_0 *and* Δ_1 *are gauge equivalent and they are intertwined by a gauge transformation preserving* $\ker P$*, then* $\mathfrak{a}[-1]_0$ *and* $\mathfrak{a}[-1]_1$ *are* L_∞*-isomorphic.*

□

Corollary 4.13. *Let* τ_0 *and* τ_1 *be two isotopic tubular neighborhoods of* L *in* M*. Then the V-data* $(\mathfrak{g}_L, \mathfrak{a}_L, \mathrm{j}_L, P_L, \triangle_{L,\tau_0})$ *and* $(\mathfrak{g}_L, \mathfrak{a}_L, P_L, \mathrm{j}_L, \triangle_{L,\tau_1})$ *induce isomorphic* L_∞*-algebra structures on* $\Omega^*(L, NL)[-1]$*.*

Proof. Immediate from Proposition 4.10 and Lemma 4.11. □

Putting Proposition 4.10 and Corollary 4.13 together, we obtain the following statement, which is a rephrasing of Theorem 4.1.

Proposition 4.14. *Let* $\Psi \in \Omega^*(M, TM)$ *be an odd square-zero element, and* L *a* Ψ*-submanifold of* M*. Then the* $\mathbb{Z}_2$*-graded* L_∞*-algebra structure on* $\Omega^*(L, NL)[-1]$ *induced by the V-data* $(\mathfrak{g}_L, \mathfrak{a}_L, \mathrm{j}_L, P_L, \triangle_{L,\tau})$ *is independent of the tubular neighborhood* τ*, up to isomorphism.*

Proof. Given two tubular neighborhoods τ_0 and τ_1 of L in M, there always exists a third tubular neighborhood $\tilde{\tau}_1$ such that τ_0 and $\tilde{\tau}_1$ are isotopic relative to L and $\tilde{\tau}_1 = \tau_1 \circ \psi$ for a suitable diffeomorphism of NL relative to L, see, e.g., [H1997, Theorem 5.3]. □

5. Deformations of Ψ-submanifolds. Let $\Psi \in \Omega^{2l-1}(M, TM)$ be an odd degree, square zero element and L a closed Ψ-submanifold in M. As in the proof of Theorem 4.1, we use a tubular neighborhood $\tau : NL \to U \subset M$ to identify the normal bundle NL with an open neighborhood U of L in M, and we thus may replace M by NL. In particular, we may regard Ψ as a square zero element in $\Omega^*(NL, TNL)$.

A smooth small deformation of L in NL can be identified with a (smooth) section $L \to NL$, i.e., with an element in $\Omega^0(L, NL)$. In other words, when thinking of small deformations we implicitly identify L with the image of the zero section $\mathbf{0} : L \to NL$. We say that a section $s\colon L \to NL$ is *a* Ψ*-section*, if its image $s(L)$ is a Ψ-submanifold in NL. These have an elegant characterization in terms of the maps $\mathrm{j}_L : \Omega^*(L, NL) \to \Omega^*(NL, TNL)$ and $P_L : \Omega^*(NL, TNL) \to \Omega^*(L, NL)$ from Definition 4.4.

Proposition 5.1. *Let* $F_\Psi : \Gamma(NL) \to \Omega^*(L, NL)$ *be the map defined by*

$$F_\Psi(s) := P_L(\exp \mathrm{j}_L(-s)^*\Psi). \tag{5.1}$$

Then a section $s : L \to NL$ *is a* Ψ*-section if and only if* $F_\Psi(s) = 0 \in \Omega^*(L, NL)$*.*

Proof. Let $x \in L$. We begin with two simple remarks. First of all, for $v \in T_xL$ we have

$$\exp \mathrm{j}_L(s)_* v = T_x s \cdot v. \tag{5.2}$$

Second, let $w \in T_{s(x)}NL$. Then w can be uniquely written as $w = w_s + w_N$ where w_s is tangent to $s(L)$ and w_N is a tangent vector vertical with respect to projection $NL \to L$. In particular w_N is the vertical lift of a, necessarily unique, vector in N_xL that we denote $w_N^\downarrow$. Finally, we have

$$\pi_{NL} \exp \mathrm{j}_L(-s)_* w = w_N^\downarrow, \tag{5.3}$$

where $\pi_{NL} : TNL \to NL$ is the projection to the base. Both (5.2) and (5.3) can be easily checked, e.g. in local coordinates. Now, we compute $F_\Psi(s)$ explicitly. So, let $v_1, \dots, v_{2l-1} \in T_xL$. Then

$$\begin{aligned}
&F_\Psi(s)_x(v_1, \dots, v_{2l-1}) \\
&= P_L(\exp \mathrm{j}_L(-s)^*\Psi)_x(v_1, \dots, v_{2l-1}) \\
&= \pi_{NL} \exp \mathrm{j}_L(-s)_* \left(\Psi_{s(x)}(\exp \mathrm{j}_L(s)_* v_1, \dots, \exp \mathrm{j}_L(s)_* v_{2l-1})\right) \\
&= \pi_{NL} \exp \mathrm{j}_L(-s)_* \left(\Psi_{s(x)}(T_xs \cdot v_1, \dots, T_xs \cdot v_{2l-1})\right) && \text{by (5.2)} \\
&= \Psi_{s(x)}(T_xs \cdot v_1, \dots, T_xs \cdot v_{2l-1})^{\downarrow}_N. && \text{by (5.3)}
\end{aligned}$$

This shows that $F_\Psi(s) = 0$ if and only if $\Psi(w_1, \dots, w_{2l-1})$ is tangent to $s(L)$ for all $w_1, \dots, w_{2l-1}$ tangent to $s(L)$, i.e. $s(L)$ is a Ψ-submanifold. □

As a step towards our investigation of smooth deformations of a Ψ-submanifold L, in this section we first study formal Ψ-deformations of L (Definition 5.2) and show how they are governed by the Maurer-Cartan equation of the L_∞-algebra attached to L (Proposition 5.3). Then we study infinitesimal and smooth deformations of Ψ-submanifolds and prove the main theorem of this section on the local structure of the pre-moduli space of analytic Ψ-submanifolds in an analytic manifold M (Theorem 5.10) under the condition that Ψ is multi-symplectic (Definition 5.8).

5.1. Formal deformations of Ψ-submanifolds. Let ε be a formal parameter. Let us recall that a formal series $s(\varepsilon) = \sum_{i=0}^\infty \varepsilon^i s_i \in \Gamma(NL)[[\varepsilon]], s_i \in \Gamma(NS)$ such that $s_0 = 0$ is called *a formal deformation of* L, and $s_1 \in \Gamma(NL)$ is called its *initial velocity*.

Denote by $\mathfrak{X}(NL)$ and $\mathcal{T}^{(r,s)}(NL)$ the space of smooth vector fields and (r, s)-tensor fields on NL, where $\mathfrak{X}(NL)$ is interpreted as the derivations of the (commutative) algebra of smooth functions $C^\infty(NL)$. The Lie derivative of tensor fields naturally extends to formal power series; for a *formal vector field* $\xi(\varepsilon) := \sum_0^\infty \varepsilon^i \xi_i \in \mathfrak{X}(NL)[[\varepsilon]]$ and a *formal* (r, s)*-tensor field* $T(\varepsilon) := \sum_0^\infty \varepsilon^i T_i \in \mathcal{T}^{(r,s)}(NL)[[\varepsilon]]$ we define the *formal Lie derivative*

$$\mathcal{L}_{\xi(\varepsilon)} T(\varepsilon) := \sum_{i=0}^\infty \varepsilon^k \sum_{i+j=k} \mathcal{L}_{\xi_i} T_j \tag{5.4}$$

and the formal exponential acting on $\mathcal{T}^{(r,s)}(NL)[[\varepsilon]]$ as

$$\exp \mathcal{L}_{\xi(\varepsilon)} := \sum_{n=0}^\infty \frac{1}{n!} \mathcal{L}^n_{\xi(\varepsilon)}. \tag{5.5}$$

Any section $s\colon L \to NL$ defines the constant vector field $\mathrm{j}_L(s)$ on NL: the flow on NL generated by the vector field $\mathrm{j}_L(s)$ on NL is given by $\Phi^t_{\mathrm{j}_L(s)} y_x = y_x + ts(x)$ for all $y_x \in N_xL$. The same applies to formal series $s(\varepsilon) = \sum_{i=0}^\infty \varepsilon^i s_i \in \Gamma(NL)[[\varepsilon]]$.

Proposition (5.1) motivates the following

Definition 5.2. A formal deformation $s(\varepsilon)$ of L is called *a Ψ-formal deformation*, if $F_\Psi(s(\varepsilon)) := P_L(\exp \mathcal{L}_{\mathrm{j}_L(-s(\varepsilon))} \Psi) = 0 \in \Omega^*(L, NL)[[\varepsilon]]$.

An *infinitesimal Ψ-deformation of* L is a smooth section $s : L \to NL$ for which $F_\Psi(\varepsilon s(L)) = O(\varepsilon^2)$.

If $s(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i s_i$ is a formal Ψ-deformation of L, then its initial velocity s_1 is evidently an infinitesimal Ψ-deformation. Conversely, given an infinitesimal Ψ-deformation s_1, we say that s_1 is *unobstructed*, if there exists a formal Ψ-deformation with initial velocity s_1. If all infinitesimal deformations are unobstructed, then we say that *the formal deformation problem is unobstructed*. Otherwise it is *obstructed* (cf. [LO2016, §10], [LOTV2014, Remark 4.8], [LS2014, Definition 4.8]).

Recall the multibracket $\mathfrak{l}_n$ of the L_∞-algebra associated to a Ψ-submanifold (Theorem 4.1) has been defined in (4.1) in Proposition 4.3.

PROPOSITION 5.3. *The formal Ψ-deformations of L are governed by the L_∞-algebra $\Omega^*(L, NL)$. Namely, a formal deformation $s(\varepsilon)$ of L is a formal Ψ-deformation if and only if $s(\varepsilon)$ is a solution of the (formal) Maurer-Cartan equation*

$$MC(s(\varepsilon)) := \sum_{n=1}^{\infty} \frac{1}{n!} \mathfrak{l}_n(s(\varepsilon), \cdots, s(\varepsilon)) = 0. \tag{5.6}$$

Proof. From the definition of $\mathfrak{l}_n$ we get

$$P_L(\mathcal{L}^n_{\mathrm{j}_L(-s)}\Psi) = \mathfrak{l}_n(s, \cdots, s), \quad n \geq 1, \tag{5.7}$$

for $s \in \Gamma(NL)$. This implies the identity of formal power series

$$P_L(\mathcal{L}^n_{\mathrm{j}_L(-s(\varepsilon))}\Psi) = \mathfrak{l}_n(s(\varepsilon), \cdots, s(\varepsilon)), \quad n \geq 1,$$

for any $s(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i s_i \in \Gamma(NL)[[\varepsilon]]$ and so

$$MC(s(\varepsilon)) = \sum_{n=1}^{\infty} \frac{1}{n!} P_L(\mathcal{L}^n_{\mathrm{j}_L(-s(\varepsilon))}\Psi) = P_L(\exp \mathcal{L}_{\mathrm{j}_L(-s(\varepsilon))}\Psi)$$

for any formal series $s(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i s_i \in \Gamma(NL)[[\varepsilon]]$. $\square$

COROLLARY 5.4. *Let $s : L \to NL$ be a smooth section. Then εs is an infinitesimal Ψ-deformation of L if and only if $\mathfrak{l}_1(s) = 0$, i.e., if and only if $s \in \ker d_{\mathbf{0}}F_\Psi$, where $d_{\mathbf{0}}F_\Psi$ is the differential of F_Ψ at the point $\mathbf{0}$ of $\Gamma(NL)$.*

We shall denote the space of infinitesimal Ψ-deformations as

$$J_\Psi(L) := \ker \mathfrak{l}_1 = \ker d_{\mathbf{0}}F_\Psi. \tag{5.8}$$

5.2. Smooth and infinitesimal deformations of Ψ-submanifolds.

DEFINITION 5.5. A *smooth Ψ-deformation of L* is a smooth one-parameter deformation $\{s_t\}$ of the zero section of the vector bundle $NL \to L$ such that each section in the family is a *Ψ-section*.

Clearly if $\{s_t\}$ is a smooth Ψ-deformation, then the section $\frac{ds_t}{dt}_{|t=0} : L \to NL$ is an infinitesimal Ψ-deformation. More generally, the Taylor expansion

$$\sum_{n=0}^{\infty} \left(\frac{d^n}{dt^n}\bigg|_{t=0} s_t \right) \varepsilon^n$$

of a smooth Ψ-deformation $\{s_t\}$ is a formal Ψ-deformation. Smooth obstructedness/unobstructedness are defined in a similar way as formal obstructedness/unobstructedness.

For $\Psi \in \Omega(M, TM)$ denote by $\mathrm{Diff}_\Psi(M)$ the subgroup of the diffeomorphism group $\mathrm{Diff}(M)$ whose elements preserve Ψ.

Definition 5.6. Given $\Psi \in \Omega^*(M, TM)$ and a homology class $\alpha \in H_l(M, \mathbb{Z})$, we denote by $\mathcal{M}_\Psi(\alpha)$ the set of all closed l-dimensional Ψ-submanifolds representing the homology class α and call it *the pre-moduli space of* α. The quotient $\mathcal{M}_\Psi(\alpha)/\mathrm{Diff}_\Psi(M)$ is called *the moduli space of* Ψ*-submanifolds of homology class* α.

Furthermore, for a Ψ-submanifold $L \subset M$, we denote by $\mathcal{M}_\Psi(L)$ the pre-moduli space of closed Ψ-submanifolds in M that are obtained from L by smooth Ψ-deformations, so that $\mathcal{M}_\Psi(L) \subset \mathcal{M}_\Psi([L])$.

Here we will work with the pre-moduli spaces only and will not discuss the *moduli problem*. But note that in most applications, $\mathrm{Diff}_\Psi(M)$ is a (finite dimensional) Lie group. Under suitable analiticity and nondegeneracy conditions on Ψ this will imply that the moduli space of Ψ-submanifolds in the connected component of L is a finite dimensional analytic space, see Theorem 5.10 and Remark 6.5.

Since locally, $\mathcal{M}_\Psi(L)$ is the set of C^1-small solutions of the equation $F_\Psi(s) = 0$, we shall use the tools provided in the proof of [LS2014, Theorem 4.9], see also the pioneering paper by Koiso [Koiso1983] for a similar idea.

Being a differential operator, F_Ψ extends for each $k \geq 1$ to a map denoted by the same symbol

$$F_\Psi : L^2_k\Omega^0(L, NL) \longrightarrow L^2_{k-1}\Omega^{l-1}(L, NL) \tag{5.9}$$

where $L^2_k\Omega^l(L, NL)$ denotes the Sobolev space of L^2_k l-forms on L with values in NL, i.e., the completion of $\Omega^l(L, NL)$ in the L^2_k norm.

Proposition 5.7. *Let M be a real analytic manifold, $\Psi \in \Omega^{l-1}(M, TM)$ analytic and $L \subset M$ a closed analytic Ψ-submanifold. Then for each k, the map F_Ψ from (5.9) is analytic in a neighborhood of the zero-form $\mathbf{0} \in L^2_k\Omega^0(L, NL)$,*

$$F_\Psi(s) = P_L(\exp \mathcal{L}_{\mathrm{j}_L(-s)}\Psi) = \sum_{n=1}^{\infty} \frac{1}{n!} P_L(\mathcal{L}^n_{\mathrm{j}_L(-s)}\Psi) \overset{(5.7)}{=} \sum_{n=1}^{\infty} \frac{1}{n!} \mathfrak{l}_n(s, \cdots, s). \tag{5.10}$$

Proof. We follow an approach similar to [LS2014]. First we consider the restriction of the map F_Ψ defined in (5.9) to the space $C^k\Omega^0(L, NL) \subset L^2_k\Omega^0(L, NL)$. Abusing notation, the restriction is also denoted by F_Ψ. Note that the image $F_\Psi(C^k\Omega^0(L, NL))$ belongs to $C^{k-1}\Omega^l(L, NL)$. Now we consider spaces $C^k\Omega^0(L, NL)$ and $C^{k-1}\Omega^l(L, NL)$ as Banach spaces with C^k-norm and C^{k-1}-norm respectively.

Choose a real analytic local trivialization (x^i, y^r) of NL where (x^i) are the coordinates on L. For any C^k-function $f(x^i)$ in this neighbourhood we define its *C^k-norm at x* as

$$\|f\|_{C^k;x} := \sum_{|I| \leq k} \|(D_I s)_x\|,$$

and likewise, for a section $s \in C^k(L, NL)$ we define its C^k-norm (at x) as

$$\|s\|_{C^k;x} := \sum_{|I| \leq k} \|(D_I s)_x\|, \text{ and } \|s\|_{C^k} := \sup_{p \in L} \|s\|_{C^k;p} \tag{5.11}$$

where the sum is taken over all multi-indices I, and D_I denotes multiple partial derivatives with respect to the given coordinates. Observe that there is a constant C_k such that for all C^k-functions f and g

$$\|fg\|_{C^k;x} \leq C_k \|f\|_{C^k;x} \, \|g\|_{C^k;x}, \tag{5.12}$$

In these coordinates, $\Psi \in \Omega^{2k-1}(NL, TNL)$ takes the form

$$\Psi = \sum_{|I|+|R|=2k-1} dx^I \wedge dy^R \otimes \left(f^r_{I;R}(x,y)\frac{\partial}{\partial y^r} + f^i_{I;R}(x,y)\frac{\partial}{\partial x^i} \right) \tag{5.13}$$

where I, R are skew-symmetric multi-indices, and where we use the Einstein convention of summation over repeated indices. A section $s \in \Gamma(L, NL)$ and its graph $s(L) \subset NL$ are given as

$$s(x) = s^r(x)\frac{\partial}{\partial y^r} \qquad \text{and} \qquad y^r = s^r(x), \tag{5.14}$$

respectively, and now a straightforward calculation yields

$$F_\Psi(ts) = \sum_{|I|+|R|=2k-1} (-t)^{|R|} \left(f^r_{I;R}(x, -ts(x)) + t\, f^i_{I;R}(x, -ts(x))\frac{\partial s^r}{\partial x^i}(x) \right) dx^I \wedge ds^R \otimes \frac{\partial}{\partial y^r} \tag{5.15}$$

where, for $R = (r^1, \cdots, r^p)$ we set

$$ds^R = ds^{r_1} \wedge \cdots \wedge ds^{r_p} = \frac{\partial s^{r_1}}{\partial x^{i_1}} \cdots \frac{\partial s^{r_p}}{\partial x^{i_p}} dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{5.16}$$

In particular,

$$\left.\frac{d^n}{dt^n}\right|_{t=0} F_\Psi(ts) = \sum_{|I|=l-1} p^r_I \left(x; s^r, \frac{\partial s^r}{\partial x^i} \right) dx^I \otimes \frac{\partial}{\partial y^r},$$

where each p^r_I is a homogeneous polynomial of degree n in the variables $\left(s^r, \frac{\partial s^r}{\partial x^i}\right)$ whose coefficients are linear combinations of functions of the form

$$D_S f^r_{I;R}, D_S f^i_{I;R}, \qquad |S| \leq n,$$

where $S = (r^1, \cdots r^n)$ is a multi-index in the y^r-variables only. Since $f^r_{I;R}, f^i_{I;R}$ are real analytic, it follows (cf. [KP2002, Proposition 2.2.10]) that – after possibly shrinking the coordinate neighborhood – there are positive constants A, K such that for any multi-index $I = (i_1, \cdots i_l)$ and for all x, $|D_I D_S f^r_{I;R}|, |D_I D_S f^i_{I;R}| \leq n! l! A K^n K^l$ and hence,

$$\|D_S f^r_{I;R}\|_{C^k;x}, \|D_S f^i_{I;R}\|_{C^k;x} \leq n! \tilde{A} K^n \tag{5.17}$$

for all x and $|S| \leq n$, with fixed $\tilde{A}, K > 0$. Finally,

$$\|s^r\|_{C^k;x} \leq C_0 \|s\|_{C^k;x} \leq C_0 \|s\|_{C^{k+1};x}, \qquad \left\| \frac{\partial s^r}{\partial x^i}(x) \right\| \leq C_0 \|s\|_{C^{k+1};x} \tag{5.18}$$

for some constant C_0. Therefore, as p^r_I is a homogeneous polynomial, it follows from (5.17), (5.12) and (5.18) that for all x,

$$\left\| p^r_I\left(x; s^r(x), \frac{\partial s^r}{\partial x^i}(x)\right)\right\|_{C^k;x} \le n!\tilde{A}(C_0C_kK)^n\|s\|^n_{C^{k+1};x},$$

whence in this coordinate neighborhood

$$\left\| \frac{d^n}{dt^n}\bigg|_{t=0} F_\Psi(ts)\right\|_{C^k;x} \le n!A_0K_0^n\|s\|^n_{C^{k+1};x} \tag{5.19}$$

for all x, and since by compactness L may be covered by finitely many such neighborhoods, we may assume that (5.19) holds for all $x \in L$ for fixed constants $A_0, K_0 > 0$. That is,

$$\left\| \frac{d^n}{dt^n}\bigg|_{t=0} F_\Psi(ts)\right\|_{C^k} \le n!A_0K_0^n\|s\|^n_{C^{k+1}}. \tag{5.20}$$

By [LS2014, Lemma 6.2], the estimate (5.20) implies that the map $F_\Psi : C^k\Omega^0(L, NL) \to C^{k-1}\Omega^{l-1}(L, NL)$ is an analytic map between Banach spaces. Since L is compact, as in [LS2014], this implies that the map $F_\Psi : L^2_k\Omega^0(L, NL) \to L^2_{k-1}\Omega^{l-1}(L, NL)$ is also an analytic map between Banach spaces. □

In order to utilize this analyticity, we shall need some regularity on the linearization $d_{\mathbf{0}}F_\Psi$. For $\xi \in T^*_xM$, we define the linear map

$$\sigma_\xi : \Lambda^{l-1}T^*_xM \otimes T_xM \longrightarrow \Lambda^l T^*_xM, \quad \alpha^{l-1} \otimes v \longmapsto \xi(v)\xi \wedge \alpha. \tag{5.21}$$

Definition 5.8. We call $\Psi \in \Omega^{l-1}(M, TM)$ *multi-symplectic* if $\sigma_\xi\Psi \neq 0$ for all $\xi \neq 0$. We say that Ψ is *multi-symplectic on* L for a Ψ-submanifold L, if $\Psi_{|L}$ is multi-symplectic in $\Omega^{l-1}(L, TL)$.

This terminology generalizes the notion of multi-symplecticity of differential forms, as it follows from (2.6) that $\Psi = \hat{\varphi}$ is multi-symplectic (on L) iff φ ($\varphi_{|L}$, respectively) is multi-symplectic, meaning that $\imath_\xi\varphi = 0$ only if $\xi = 0$.

Proposition 5.9. *Let Ψ be multi-symplectic on the Ψ-submanifold $L \subset M$. Then $d_{\mathbf{0}}F_\Psi$ is an overdetermined elliptic differential operator. In particular, this is the case if $\Psi = \hat{\varphi}$ and $\varphi_{|L}$ is multi-symplectic.*

Proof. We pick coordinates (x^i, y^r) as in the proof of Proposition 5.7. Then (5.15) yields

$$\begin{aligned} d_0F_\Psi(s) = \mathfrak{l}_1(s) = \frac{d}{dt}\bigg|_{t=0} F_\Psi(ts) \\ &= \sum_{|I|=2k-1}\left(f^i_{I;\emptyset}(x,0)\frac{\partial s^r}{\partial x^i}(x) - s^u\frac{\partial}{\partial y^u}f^r_{I,\emptyset}(x,0)\right)dx^I \otimes \frac{\partial}{\partial y^r} \\ &\quad - \sum_{|J|=2k-2} f^r_{J;u}(x,0)\frac{\partial s^u}{\partial x^i}dx^J \wedge dx^i \otimes \frac{\partial}{\partial y^r}. \end{aligned}$$

Thus, for $\xi = \xi_i dx^i \in T^*_{x_0}L$, the symbol of $d_0 F_\Psi$ is

$$\sigma_\xi d_0 F_\Psi(s) =$$
$$= \sum_{|I|=2k-1} f^i_{I;\emptyset}(x_0,0)\xi_i s^r dx^I \otimes \frac{\partial}{\partial y^r} - \sum_{|J|=2k-2} f^r_{J;u}(x_0,0)\xi_i s^u dx^J \wedge dx^i \otimes \frac{\partial}{\partial y^r}$$
$$\overset{(5.14)}{=} \sum_{|I|=2k-1} f^i_{I;\emptyset}(x_0,0)\xi_i dx^I \otimes s - \xi \wedge \sum_{|J|=2k-2} f^r_{J;u}(x_0,0) s^u dx^J \otimes \frac{\partial}{\partial y^r}$$

which implies

$$\xi \wedge \sigma_\xi d_0 F_\Psi(s) = \xi \wedge \sum_{|I|=2k-1} f^i_{I;\emptyset}(x_0,0)\xi_i dx^I \otimes s \overset{(5.21),(5.13)}{=} \iota_\xi \Psi_{|L} \otimes s.$$

Since $\Psi_{|L}$ is multi-symplectic, $\iota_\xi \Psi_{|L} \neq 0$ for all $\xi \neq 0$, whence the symbol $\sigma_\xi d_0 F_\Psi$ is injective for all $\xi \neq 0$, showing the assertion. □

We are almost in the position now to construct a local analytic chart on $\mathcal{M}_\Psi(L)$ using the *Inverse Function Theorem* (IFT) for analytic mappings between real analytic Banach manifolds [Douady1966], see [LS2014, Appendix] for a short account. However, the main difference to the situation handled there is that we do not know a priori if the space $J_\Psi(L)$ of infinitesimal Ψ-deformations is finite dimensional, whence we need to impose this as an additional condition.

THEOREM 5.10. *Let M be an analytic manifold with an analytic section $\Psi \in \Omega^{l-1}(M, TM)$.*

(1) *If $L \subset M$ is an analytic Ψ-submanifold such that Ψ is multi-symplectic on L and $J_\Psi(L)$ is finite dimensional, then the pre-moduli space $\mathcal{M}_\Psi(L)$ of all Ψ-submanifolds C^1-close to L forms a finite dimensional analytic variety.*
(2) *If any Ψ-submanifold in $\mathcal{M}_\Psi(L)$ shares the properties given in (1), then $\mathcal{M}_\Psi(L)$ is a finite dimensional analytic space.*

Proof. As Ψ is fixed, we shall simply write F instead of F_Ψ. Since Ψ is multi-symplectic on L and hence $d_{\mathbf{0}}F$ is overdetermined elliptic by Proposition 5.9, we have the following L^2-orthogonal decomposition (see e.g. [Besse1987, Corollary 32, p. 464])

$$L^2\Omega^{l-1}(L, NL) = d_{\mathbf{0}}F(L^2_1\Gamma(NL)) \oplus \big(\ker(d_{\mathbf{0}}F)^* \cap L^2\Omega^{l-1}(L, NL)\big). \tag{5.22}$$

- Let $\Pi_1 : L^2\Omega^{l-1}(L, NL) \to d_{\mathbf{0}}F(L^2_1\Gamma(NL))$ be the orthogonal projection with respect to the decomposition in (5.22). Being bounded linear, Π_1 is an analytic map between Banach spaces.
- Let $U(\mathbf{0})$ denote an open neighborhood of $\mathbf{0}$ in $L^2_1(\Gamma(NL))$ such that the restriction of the map F to $U(\mathbf{0})$ is analytic. The existence of $U(\mathbf{0})$ is ensured by Proposition 5.7.
- Denote by $\pi : L^2_1\Gamma(NL) \to J_\Psi(L)$ the orthogonal projection. Since $J_\Psi(L)$ is assumed to be finite dimensional, π is bounded linear and hence analytic.

Then we set

$$\hat{F} := \pi \oplus (-\Pi_1 \circ F) : L^2_1(\Gamma(NL)) \supset U(\mathbf{0}) \to J_\Psi(L) \oplus d_{\mathbf{0}}F(L^2_1\Gamma(NL)). \tag{5.23}$$

By Proposition 5.7, the map $\hat{F}$ is analytic in $U(\mathbf{0})$ and its differential at $\mathbf{0}$ is an isomorphism. Therefore the IFT for analytic mappings of Banach spaces implies that there is an analytic inverse of $\hat{F}$

$$G : V(0, \mathbf{0}) \to U(\mathbf{0})$$

where $V(0,\mathbf{0})$ is an open neighborhood of $(0,\mathbf{0}) \in J_\Psi(L) \oplus d_{\mathbf{0}}F(L^2_1\Gamma(NL))$.

Let $V^{J_\Psi}(0,\mathbf{0}) := V(0,\mathbf{0}) \cap (J_\Psi(L),\mathbf{0})$. Next we define the map

$$\tau : V^{J_\Psi}(0,\mathbf{0}) \to J_\Psi(L),\ s \mapsto \pi \circ G(s) - i(s) \tag{5.24}$$

where $i : (J_\Psi(L),\mathbf{0}) \to J_\Psi(L)$ is the natural identification map. We now assert that

(1) The map τ is analytic.
(2) The restriction of the projection π to $F^{-1}(\mathbf{0}) \cap U(\mathbf{0})$ is injective.
(3) An element $y \in V^{J_\Psi}(0,\mathbf{0})$ belongs to $\tau^{-1}(0)$ if and only if $y = i^{-1} \circ \pi(z)$ for some $z \in (\Pi_1 \circ F)^{-1}(\mathbf{0})$.

The first statement holds as both π and G are analytic maps, whereas the second holds since $\hat{F}$ is locally invertible at the origin.

To see the last assertion, let us first proof the "if"-part. Assume that $y = i^{-1}\circ\pi(z)$ and $\Pi_1 \circ F(z) = \mathbf{0}$. Then $\hat{F}(z) = i^{-1} \circ \pi(z) = y$. It follows $z = G(y)$ and $\tau(y) = \pi \circ G(y) - i(y) = \pi(z) - \pi(z) = 0$, which proves the "if"-assertion.

Now assume that $\tau(y) = 0$. Then $\pi \circ G(y) = i(y)$. Set $z = G(y)$. Then $\hat{F}(z) = y = \pi(z)$ and therefore, $\Pi_1 \circ F(z) = \mathbf{0}$, so that the third assertion is shown as well.

We now consider the restriction $F : \tau^{-1}(0) \to \ker \Pi_1$. Then $F^{-1}(\mathbf{0})$ consists of the common zeroes of the analytic functions $F_\xi(\cdot) := \langle \xi, F(\cdot)\rangle$ on $\tau^{-1}(0)$ for all $\xi \in \ker \Pi_1$ and where $\langle\cdot,\cdot\rangle$ refers to the scalar product on $L^2\Omega^{l-1}(L,NL)$.

Since $\tau^{-1}(0) \subset V^{J_\Psi}(0,\mathbf{0})$ is a finite dimensional real analytic variety, the ring of germs of analytic functions at $\mathbf{0}$ is Noetherian [Frisch1967, Theorem I.9]. Therefore, $F^{-1}(\mathbf{0})$ is given as the zero set of *finitely many* analytic functions $F_{\xi_1},\dots,F_{\xi_N}$. In other words, there is an analytic function

$$\hat{\tau} := (\tau, F_{\xi_1},\dots,F_{\xi_N}) : V^J(0,\mathbf{0}) \longrightarrow J_\Psi(L) \oplus \mathbb{R}^N$$

for some finite number N such that $F^{-1}(y) = 0$ iff $y = i^{-1}\circ\pi(z)$ for some $z \in F^{-1}(\mathbf{0})$. This allows to identify the C^1-neighborhood $F^{-1}(\mathbf{0}) \cap U(\mathbf{0})$ of L in the pre-moduli space $\mathcal{M}_\Psi(L)$ with the pre-image $\hat{\tau}^{-1}(0)$ in the neighborhood of $0 \in J_\Psi(L)$ via the map $i^{-1} \circ \pi : F^{-1}(\mathbf{0}) \cap U(\mathbf{0}) \to \hat{\tau}^{-1}(0)$, where $\hat{\tau}$ is an analytic map between open neighborhoods of finite dimensional vector spaces. Since $F^{-1}(\mathbf{0}) \cap U(\mathbf{0})$ models a C^1-neighborhood $U(L)$ of L in $\mathcal{M}_\Psi(L)$, this completes the proof of the first statement of the theorem.

To show the second statement, assume that $L_1 \in \mathcal{M}_\Psi(L)$ lies in a C^1-neighborhood of L, so that $L_1 = s(L)$ for some analytic section s of NL. Then s induces, via $\exp \mathrm{j}_L(s)$, an invertible analytic map between the Sobolev spaces $L^2_1\Gamma(NL)$ and $L^2_1\Gamma(NL_1)$ as well as an invertible analytic map between $L^2_1\Omega^{l-1}(L,NL)$ and $L^2_1\Omega^{l-1}(L,NL)$.

This means that the charts on $U_1(L_1)$ constructed via maps π^{L_1}, τ^{L_1} are equivalent to the analytic structure induced from the one on $U(L)$. In other words, any two analytic charts are compatible, which completes the proof. □

6. Deformations of φ-calibrated submanifolds. In this section we consider smooth $\hat{\varphi}$-deformation of a closed φ-calibrated submanifold L, where φ^l is a parallel calibration on the Riemannian manifold (M,g). First, we prove that the space of all infinitesimal $\hat{\varphi}$-deformations of L coincides with the space of Jacobi vector fields on L, regarding L as a minimal submanifold (Proposition 6.1). Then we prove our main theorem stating that the formal and smooth deformations of a closed φ-calibrated submanifold are encoded in its cananically associated $\mathbb{Z}_2$-graded strongly homotopy Lie algebra (Theorem 6.4). In Remark 6.5 we discuss some related results. Then we revisit the deformation theory of complex submanifolds using the methods developed in the present paper (Theorem 6.6). Finally, in the last subsection we summarize the achievements of the present paper.

6.1. Infinitesimal deformations of $\hat{\varphi}$-submanifolds. Given a *parallel* calibration φ and a closed φ-calibrated submanifold L, the premoduli space $\mathcal{M}_{\hat{\varphi}}(L)$ consists of all closed minimal submanifolds that are obtained from L by smooth deformation. The following Proposition, in which we do not require the calibration φ to be parallel, is an infinitesimal analogue of Corollary 3.6.

PROPOSITION 6.1. *Let $\varphi \in \Omega^l(M)$ be a calibration and $L \subset M$ closed such that either*

(1) *L is φ-calibrated, or*

(2) *L is a $\hat{\varphi}$-submanifold such that $\varphi_{|L} \neq 0$, and φ is parallel.*

Then L is minimal, and

$$J_{\hat{\varphi}}(L) \subset J(L), \tag{6.1}$$

where $J_{\hat{\varphi}}(L)$ is given in (5.8) and $J(L)$ is the space of Jacobi fields on L. Moreover, under condition (1), equality holds in (6.1).

Proof. The minimality of L is evident if L is φ-calibrated, and it follows from Theorem 3.5 under condition (2). Assume that L is a closed $\hat{\varphi}$-submanifold and $s \in \Gamma(NL)$ is an infinitesimal $\hat{\varphi}$-deformation. Let us recall that $\psi_t = \exp\big(\mathrm{j}_L(ts)\big)$. Since L is compact, there exist a positive number A and a positive number ε_0 such that, for any $x \in L$

$$|\mathrm{pr}\big((\psi_t)^*\hat{\varphi}\big)_{|L}(x)| \leq A \cdot t^2 \tag{6.2}$$

for any $t \leq \varepsilon_0$. Denote by $G_\varphi(x)$ the space of unit decomposable l-vectors w in $G_l(T_xM)$ such that $\varphi(w) = 1$. Denote by ρ the distance on the Grassmannian $G_l(T_xM)$ induced by the Riemannian metric on T_xM.

We shall abbreviate $L_t := \psi_t(L)$, $x_t := \psi_t(x)$ and $\hat{\varphi}_t := (\psi_t)^*\hat{\varphi}$.

LEMMA 6.2. *The inequality (6.2) is equivalent to the existence of a positive number B and a positive number ε_1 such that*

$$\rho\Big(\vec{T_{x_t}L_t}, G_\varphi(x_t)\Big) \leq B \cdot t^2 \tag{6.3}$$

for all $t \in (0, \varepsilon_1)$ (Recall that $\vec{T_{x_t}L_t}$ is the unit l-vector associated to the oriented tangent space T_xL).

Proof. Since $\psi_0 = Id$ we observe that $|\mathrm{pr}\hat{\varphi}_t| = O(t^2)$ if and only if $|\mathrm{pr}\hat{\varphi}_{|L_t}| = O(t^2)$. Recall that we denoted by $\tilde{\varphi}(x)$ the form φ (at the point x) regarded as a function on the Grassmannian $G_l(T_xM)$ of unit decomposable l-vectors. Then the function $|d_w\tilde{\varphi}(x)|$ is smooth in the variable $w \in G_l(T_xM)$. Since $d_w\varphi(x) = 0$ if $w \in G_\varphi(x)$, this implies that there exist positive constants C_1, C_2 such that

$$C_1 \cdot |d_w\tilde{\varphi}(x)| \leq \rho(w, G_\varphi(x)) \leq C_2 \cdot |d_w\tilde{\varphi}(x)|. \tag{6.4}$$

Now, Lemma 6.2 follows from (3.2), which implies

$$|\mathrm{pr}\hat{\varphi}_{|T_{x_t}L_t}| = |d_{\vec{T_{x_t}L_t}}\tilde{\varphi}(x_t)|. \qquad \square \tag{6.5}$$

Observe that $c_0 := \varphi^k(\vec{T_x L})$ is constant on L. Indeed, if φ is parallel, then this follows from (3.3), and if L is φ-calibrated, this holds for $c_0 := 1$ by definition. Thus, Lemma 6.2 implies that there exist a constant C_3 and $0 < \varepsilon_2 < \varepsilon_1$ such that for all $x \in L$ and all $t \in (0, \varepsilon_2)$ we have

$$c_0 - C_3 t^4 \leq \langle \varphi, \vec{T_{x_t} L_t} \rangle \leq c_0 + C_3 t^4. \tag{6.6}$$

Since $\psi_0 = Id$, there exist a constant C_4 and a positive number $\varepsilon_3 < \varepsilon_2$ such that, for all $x \in L$ and all $t \in (0, \varepsilon_3)$, we have

$$1 - C_4 t \leq |(\psi_t)_* \vec{T_x L}| \leq 1 + C_4 t. \tag{6.7}$$

It follows from (6.6) and (6.7) that there exists a constant C_5 such that for all $x \in L$ and all $t \in (0, \varepsilon_3)$, we have

$$(c_0 - C_5 t^3)\langle \varphi, (\psi_t)_*(\vec{T_x L})\rangle \leq |(\psi_t)_* \vec{T_x L}| \leq (c_0 + C_5 t^3)\langle \varphi, (\psi_t)_*(\vec{T_x L})\rangle. \tag{6.8}$$

Finally, it follows from (6.8) and $\langle \varphi, (\psi_t)_*(\vec{T_x L})\rangle = \langle \psi_t^*(\varphi)\vec{T_x L}\rangle$, that

$$\frac{d^2}{dt^2}_{|t=0} vol(\psi_t(L)) = \int_L \frac{d^2}{dt^2}_{|t=0} |(\psi_t)_*(\vec{T_x L})| \, dvol_L = \int_L \frac{d^2}{dt^2}_{|t=0} (\psi_t)^* \varphi = 0.$$

Hence s is a Jacobi vector field. This proves (6.1).

For the last statement, assume that s is a Jacobi vector field on the φ-calibrated submanifold L. By Remark 2.3 in [LV2017], s is an infinitesimal deformation of L as a φ-calibrated submanifold. This is the same to say that (6.3) holds for some B and ε_0. By Lemma 6.2, this implies that $s \in J_{\hat{\varphi}}(L)$. □

Note that (6.1) in case of hypothesis (2) in Proposition 6.1 can also be seen by an argument along the lines of the proof of Theorem 3.5.

Recall that $J(L)$ is finite dimensional and its dimension is called the *nullity of* L [Simons1968].

From Proposition 6.1 we obtain Corollary 6.3 below, which has been first proved by Simons in [Simons1968, Theorem 3.5.1] by computing the Jacobi operator on a compact Kähler submanifold L. Simons' computation has been generalized by McLean [McLean1998] for calibrated submanifolds, and simplified by Lê-Vanžura [LV2017], using different methods.

Corollary 6.3. *Let L be a compact and closed Kähler submanifold in a Kähler manifold (M, g, ω^2). Then the nullity of L is equal to the dimension of the space of globally defined holomorphic sections in NL.*

We are now ready to show the main result of this section.

Theorem 6.4 (Main Theorem). *Let $\varphi \in \Omega^l(M)$ be a parallel calibration on a real analytic Riemannian manifold (M, g), let $\alpha \in H_l(M, \mathbb{Z})$ be a homology class such that $\langle [\varphi], \alpha \rangle \neq 0$, and let $L \in \mathcal{M}_{\hat{\varphi}}(\alpha)$.*

(1) *The pre-moduli space $\mathcal{M}_{\hat{\varphi}}(\alpha)$ is a finite dimensional analytic space, and so is, in particular, $\mathcal{M}_{\hat{\varphi}}(L) \subset \mathcal{M}_{\hat{\varphi}}(\alpha)$.*
(2) *If L is φ-calibrated, then a formally unobstructed Jacobi field $s \in J(L)$ is smoothly unobstructed.*
(3) *If L is φ-calibrated, then there is a canonical $\mathbb{Z}_2$-graded strongly homotopy Lie algebra that governs formal and smooth deformations of L in the class of φ-calibrated submanifolds.*

Proof. Let $\mathcal{N}_\varphi := \{v \in TM \mid \imath(v)\varphi = 0\}$ be the annihilator of φ which induces the parallel orthogonal decomposition $TM = \mathcal{N}_\varphi \oplus \mathcal{N}_\varphi^\perp$, and clearly, the restriction of $\hat{\varphi}$ to $\mathcal{N}_\varphi^\perp$ is multi-symplectic. As L is φ-calibrated, it follows that $TL \subset \mathcal{N}_\varphi^\perp$, so that L is contained in a maximal leaf $M_0 \subset M$ of the (parallel and hence integrable) distribution $\mathcal{N}_\varphi^\perp$. The normal bundle of L decomposes orthogonally as

$$NL = NL_0 \oplus (\mathcal{N}_\varphi)_{|L}, \tag{6.9}$$

where NL_0 is the normal bundle of the inclusion $L \hookrightarrow M_0$. As L is closed, there is an $\varepsilon > 0$ such that the normal exponential $\exp : NL_\varepsilon \to M$ is a fiberwise local diffeomorphism, where $NL_\varepsilon \subset NL$ denotes the ε-disc bundle. Let $g_N := \exp^*(g)$ be the induced metric on NL_ε. As g is a local product metric, the g_N-orthogonal complement of the fibers of $((\mathcal{N}_\varphi)_{|L})_\varepsilon \to L$ induces a flat connection on this disc bundle, and decomposing a small section $s = s_0 + s_1 \in \Omega^0(L, NL_\varepsilon) = \Omega^0(L, (NL_0)_\varepsilon) \oplus \Omega^0(L, ((\mathcal{N}_\varphi)_{|L})_\varepsilon)$ according to (6.9), the definition of $F_{\hat{\varphi}}$ implies that $F_{\hat{\varphi}}(s) = 0$ iff $F_{\hat{\varphi}}(s_0) = 0$ and s_1 is parallel. Thus,

$$\mathcal{M}_{\hat{\varphi}}(L) \cong \mathcal{M}^0_{\hat{\varphi}}(L) \times \mathbb{R}^k,$$

where $\mathcal{M}^0_{\hat{\varphi}}(L)$ is the premoduli space of $L \subset M_0$, and k is the dimension of the space of parallel sections in $\Omega^0(L, (\mathcal{N}_\varphi)_{|L})$.

With this, it suffices to show the theorem for $L \subset (M_0, \varphi_{|M_0})$, and, after replacing M_0 by M, we may therefore assume w.l.o.g. that φ is multi-symplectic.

Being parallel, φ is harmonic and hence analytic. If $L \in \mathcal{M}_{\hat{\varphi}}(\alpha)$, then $\langle [\varphi], \alpha \rangle \neq 0$ implies that $\varphi_{|L} \neq 0$ and moreover, by Theorem 3.5, (3.3) is satisfied, and $L \subset M$ is a minimal submanifold. In particular, L is analytic by the Morrey regularity theorem [Morrey1954, Morrey1958, Morrey2008].

Therefore, Proposition 6.1 implies that $J_{\hat{\varphi}}(L) \subset J(L)$ is finite dimensional, so that $\mathcal{M}_{\hat{\varphi}}(L)$ is an analytic space by Theorem 5.10. Since this is the case for any $L \in \mathcal{M}_{\hat{\varphi}}(\alpha)$, it follows that $\mathcal{M}_{\hat{\varphi}}(\alpha)$ is an analytic space as well.

The second assertion of Theorem 6.4 is a corollary of the first assertion, Proposition 6.1, and the Artin's approximation theorem [Artin1968, Theorem 1.2], which implies that, in a finite dimensional analytic space, smooth and formal obstructedness are equivalent.

For the last statement, assume that L is a φ^{2l}-calibrated submanifold. Then the last assertion of Theorem 6.4 for L follows from the second assertion and Proposition 5.3.

Now assume that L is a φ^{2l-1}-calibrated submanifold. Then $L \times S^1$ is a $\varphi^{2l-1} \wedge dt$-calibrated submanifold in $(M \times S^1, g + dt^2, \varphi \wedge dt)$. It is not hard to see that, if $\tilde{L}_t$ is a smooth deformation of $L \times S^1$ in the class of minimal submanifolds in $M \times S^1$, then $\tilde{L}_t = L_t \times S^1$ for some family of φ-calibrated submanifolds L_t. Hence, the formal and smooth deformations of φ^{2l-1}-calibrated submanifold are governed by the $\mathbb{Z}_2$-graded strongly homotopy Lie algebra associated to $L \times S^1$. This completes the proof of Theorem 6.4. □

Remark 6.5. 1. Theorem 6.4 is also valid for open φ-calibrated submanifolds with compactly supported variation fields.

2. Assume that L is simultaneously a φ-calibrated submanifold and a φ'-calibrated submanifold, where φ and φ' are calibrations on (M, g). Then any φ-calibrated closed submanifold L' that is homologous to L is also a φ'-calibrated submanifold. This implies that deformations of such calibrated submanifolds are easier

to control. For example, let $(M^6, g, \omega^2, \alpha = \mathrm{Re}\, vol_{\mathbb{C}})$ be a Calabi-Yau 6-manifold and $C \subset (M^6, g, \omega^2, \alpha)$ a complex curve. Clearly, the product $L := S^1 \times C$ is simultaneously calibrated with respect to both the associative calibration $\varphi := dt \wedge \omega^2 + \alpha$ and the calibration $dt \wedge \omega^2$. Hence, any deformation L' of L in the class of associative submanifolds is also calibrated by $dt \wedge \omega$. In particular, L' is invariant under the flow generated by the vector field ∂_t. This flow preserves the Calabi-Yau structure on each slice $\{t\} \times M^6$. We conclude that all the slices $L' \cap \{t = \text{constant}\}$ are isomorphic as complex curves in M^6. It follows that $L' = S^1 \times C'$, where C' is a complex deformation of C. In particular, if C is isolated then L is isolated. The last assertion has been obtained in [CHNP2012, Lemma 5.11] by computing the kernel of the corresponding linearized operators that control the corresponding deformations. In [Leung2002] Leung studies deformations of simultaneously calibrated submanifolds using integral estimates. We refer the interested reader to [Le1993] for the relation between the calibration method and the integral estimate method in the theory of minimal submanifolds.

3. As we noted in Corollary 4.2, there are two natural $\mathbb{Z}_2$-graded strongly homotopy Lie algebras associated to an associative submanifold L in a G_2-manifold (M^7, g, φ). It is known that the smooth and infinitesimal χ-deformations of L coincide with the smooth and infinitesimal deformations of L as a minimal submanifold [McLean1998, LV2017]. Thus, the strongly homotopy Lie algebra attached to L via χ also governs smooth and formal deformations of L as a φ-calibrated submanifold.

4. The action of $\mathrm{Diff}_\Psi(M)$ preserves the analytic structure on $\mathcal{M}_\Psi(L)$, whence the moduli space $\mathcal{M}_\Psi(L)/\mathrm{Diff}_\Psi(M)$ is an analytic space as well. In particular, generic points of $\mathcal{M}_\Psi(L)$ or $\mathcal{M}_\Psi(L)/\mathrm{Diff}_\Psi(M)$, respectively, are smooth, and hence (formally) unobstructed.

6.2. Deformations of complex submanifolds revisited.

THEOREM 6.6. *Assume that L is a closed complex submanifold in a complex manifold (M, J).*

(1) *The pre-moduli space $\mathcal{M}_J(L)$ with the C^1-topology has the structure of a finite dimensional analytic space.*
(2) *A formally unobstructed holomorphic normal field is smoothly unobstructed.*
(3) *There is a canonical $\mathbb{Z}$-graded strongly homotopy Lie algebra that governs formal and smooth complex deformations of L.*

Proof of Theorem 6.6. Theorem 6.6 is proved in the same way as Theorem 6.4 and we omit its proof. □

REMARK 6.7. Deformations of complex submanifolds have been examined by Ji in [Ji2014], using his general theory of deformations of Lie subalgebroids. Since the Frölicher-Nijenhuis bracket of $J \in \Omega^*(M, TM)$ is $-i/2$-times the Dolbeault operator $\bar{\partial}$, Ji's strongly homotopy Lie algebra is the same as ours up to an uninfluential global factor (see also [Manetti2007] for an equivalent formulation).

6.3. Conclusion. In our paper, inspired by the principle that every deformation problem over a field of characteristic zero is governed by a differential graded Lie algebra (or, equivalently, by an L_∞-algebra), we found new connections between seemingly unrelated subjects: coisotropic submanifolds in Jacobi manifolds, calibrated and more generally φ-critical submanifolds, complex submanifolds in complex manifolds. As a consequence of this deformation theory approach, we and other authors were then led to discover new structures defined by the Frölicher-Nijenhuis bracket on G_2 and Spin(7)-manifolds [KLS2017a, KLS2017b, KLS2018], [CKT2018].

In the present paper, we also give a new treatment of Jacobi vector fields on calibrated submanifolds (Proposition 6.1), generalizing results by Simons, McLean and Lê-Vanžura. We prove a result on the structure of the pre-moduli space of φ-calibrated submanifolds, where φ is a parallel calibration in a real analytic Riemannian manifold (Theorem 6.4), and a similar theorem for Ψ-submanifolds under a certain condition (Theorem 5.10), which generalizes a known result on deformations of complex submanifolds in complex manifolds.

Acknowledgement. LV and LS warmly thank HVL and the Institute of Mathematics of the Czech Academy of Sciences for their hospitality during their respective stays in Prague, where part of this project was developed. LV is member of the GNSAGA of INdAM. We also thank the referees for their careful reading of this manuscript, which helped us to correct some inaccuracies and to significantly improve its exposition.

REFERENCES

[AS2008] S. Akbulut and S. Salur, *Calibrated manifolds and gauge theory*, J. reine angew. Math., 625 (2008), pp. 187–214.

[AS2008b] S. Akbulut and S. Salur, *Deformations in G_2-manifolds*, Adv. Math., 217 (2008), pp. 2130–2140.

[Artin1968] M. Artin, *On the solutions of analytic equations*, Inven. Math., 5 (1968), pp. 277–291.

[AW2003] M. Atiyah and E. Witten, *M-Theory dynamics on a manifold of G_2-holonomy*, Adv. Theor. Math. Phys., 6 (2003), pp. 1–106.

[Bandiera2015] R. Bandiera, *Nonabelian higher derived brackets*, J. Pure Appl. Algebra, 219 (2015), pp. 3292–3313.

[Besse1987] A. Besse, *Einstein manifolds*, Springer-Verlag, 1987.

[Bryant1987] R. L. Bryant, *Metrics with exceptional holonomy*, Ann. Math., 126 (1987), pp. 525–576.

[Butscher2003] A. Butscher, *Deformations of minimal Lagrangian submanifolds with boundary*, Proc. Amer. Math. Soc., 131 (2003), pp. 1953–1964.

[CS2008] A. S. Cattaneo and F. Schätz, *Equivalences of higher derived brackets*, J. Pure Appl. Algebra, 212 (2008), pp. 2450–2460.

[CHNP2012] A. Corti, M. Haskins, J. Nordström and T. Pacini, *G_2-manifolds and associative submanifolds via semi-fano 3-folds*, Duke Math. J., 164 (2015), pp. 1971–2092.

[CKT2018] K. F. Chan, S. Karigiannis and C. C. Tsang, *The $\mathcal{L}_B$-cohomology on compact torsion-free* G_2 *manifolds and an application to 'almost' formality*, Annals of Global Analysis and Geometry, 55 (2019), pp. 325-369, arXiv:1801.06410.

[Dao1977] D. C. Thi, *Minimal real currents on compact Riemannian manifolds*, Izv. Akad. Nauk, USSR, Ser. Math., 41 (1977), pp. 853–867.

[Douady1966] A. Douady, *Le probléme des modules pour les sous-espaces analytiques compacts d'un espace analytique donné*, Ann. Inst. Fourier, 16 (1966), pp. 1–95.

[DK1990] S. K. Donaldson and P. B. Kronheimer, *The Geometry of four-manifolds*, Clarendon Press, Oxford, 1990.

[DS2011] S. K. Donaldson and E. Segal, *Gauge theory in higher dimensions, II.* Surveys in differential geometry. Volume XVI. Geometry of special holonomy and related topics, Surv. Differ. Geom. 16, Int. Press, Somerville, MA, 2011, pp. 1–41.

[DT1998] S. K. Donaldson and R. P. Thomas, *Gauge Theory in higher dimensions*, in: The Geometric Universe, Huggett et al. Eds., Oxford Univ. Press, 1998.

[Fernandez1986] M. Fernandez, *A classification of Riemannian manifolds with structure group Spin(7)*, Ann. Mat. Pur. Appl., 143 (1986), pp. 101–122.

[Frisch1967] J. Frisch, *Points de platitude d'un morphisme d'espaces analytiques complexes*, Inv. Math. 4 (1967), pp. 118–138.

[Gayet2014] D. Gayet, *Smooth moduli spaces of associative submanifolds*, Q. J. Math., 65 (2014), pp. 1213–1240.

[Gray1969] A. Gray, *Vector cross products on manifolds*, Trans. Amer. Math. Soc., 141 (1969), pp. 465–504.

[GW2011] D. Gayet and F. Witt, *Deformations of associative submanifolds with boundary*, Adv. Math., 226 (2011), pp. 2351–2370.

[Getzler2009] E. Getzler, *Lie theory for nilpotent L_∞-algebras*, Ann. Math., 170 (2009), pp. 271–301.

[GIP2003] J. Gutowski, S. Ivanov and G. Papadopoulos, *Deformations of generalized calibrations and compact non-Kähler manifolds with vanishing first Chern class*, Asian J. Math., 7 (2003), pp. 39–79.

[GYZ2003] S. Gukov, S.-T. Yau and E. Zaslow, *Duality and fibrations on G_2-manifolds*, Turk. J. Math., 27 (2003), pp. 61–97.

[HL1982] R. Harvey and H. B. Lawson, *Calibrated geometry*, Acta Math., 148 (1982), pp. 47–157.

[HL2009] R. Harvey and H. B. Lawson, *An introduction to potential theory in calibrated geometry*, Amer. J. Math., 131:4 (2009), pp. 839–944.

[HM1986] R. Harvey and F. Morgan, *The faces of the Grassmannian of three-planes in $\mathbb{R}^7$*, Inv. Math., 83 (1986), pp. 191–228.

[H1997] M. W. Hirsch, *Differential Topology*, Graduate Text in Mathematics, Springer, New York, 1997.

[Ji2014] X. Ji, *Simultaneous deformations of a Lie algebroid and its Lie subalgebroid*, J. Geom. Phys., 84 (2014), pp. 8–29.

[Joyce2007] D. D. Joyce, *Riemannian holonomy groups and calibrated geometry*, Oxford Univ. Press, 2007.

[JS2005] D. D. Joyce and S. Salur, *Deformations of asymptotically cylindrical coassociative submanifolds with fixed boundary*, Geom. Topol., 9 (2005), pp. 1115–1146.

[Kawai2017] K. Kawai, *Deformations of homogeneous associative submanifolds in nearly parallel G_2-manifolds*, Asian J. Math., 21 (2017), pp. 429–462.

[KLS2017a] K. Kawai, H. V. Lê and L. Schwachhöfer, *The Frölicher-Nijenhuis bracket and the geometry of G_2-and Spin(7)-manifolds*, Ann. Math. Pur. Appl., 197 (2018), pp. 411–432.

[KLS2017b] K. Kawai, H. V. Lê and L. Schwachhöfer, *Frölicher-Nijenhuis cohomology on G_2 and Spin (7)-manifolds*, International Journal of Mathematics, 29:12 (2018), 1850075 (36 pages), arXiv:1703.05133.

[KLS2018] K. Kawai, H. V. Lê and L. Schwachhöfer, *Frölicher-Nijenhuis bracket on manifolds with special holonomy*, in: Lectures and Surveys on G_2-Manifolds and Related Topics, S. Karigiannis, N. Leung, J. Lotay (Eds.), Fields Institute Communications, pp. 201–215, arXiv:1810.12714.

[Koiso1983] N. Koiso, *Einstein metric and complex structures*, Inv. Math., 73 (1983), pp. 71–106.

[KMS1993] I. Kolar, P. W. Michor and J. Slovak, *Natural operators in differential geometry*, Springer, 1993.

[KL2009] A. Kovalev and J. Lotay, *Deformations of compact coassociative 4-folds with boundary*, J. Geom. Phys., 59 (2009), pp. 63–73.

[KL2012] S. Karigiannis and N. C. Leung, *Deformations of calibrated subbundles of Euclidean spaces via twisting by special sections*, Ann. Glob. Anal. Geom., 42 (2012), pp. 371–389.

[KP2002] S. G. Krantz and H. R. Park, *A prime of real analytic functions*, Birkhäuse, 2002.

[Le1990] H. V. Lê, *Relative calibration and the problem of stability of minimal surfaces*, Lect. Notes in Math. 1453, Springer-Verlag, 1990, pp. 245–262.

[Le1990b] H. V. Lê, *Jacobi equations on minimal homogenous submanifolds in homogenous Riemannian spaces*, (Russian) Funkt. Anal. i Prilozhen., 24 (1990), pp. 50–62; translation in Funct. Anal. Appl., 24 (1990), pp. 125–135.

[Le1993] H. V. Lê, *Application of integral geometry to minimal surfaces*, Int. J. Math., 4 (1993), pp. 89–111.

[Le2013] H. V. Lê, *Geometric structures associated with a simple Cartan 3-fom*, J. Geom. Phys., 70 (2013), pp. 205–223.

[LO2016] H. V. Lê and Y.-G. Oh, *Deformations of coisotropic submanifolds in locally conformal symplectic manifolds*, Asian J. Math., 20 (2016), pp. 555–598.

[Lotay2009] J. Lotay, *Deformation theory of asymptotically conical coassociative 4-folds*, Proc. Lond. Math. Soc., 99 (2009), pp. 386–424.

[LOTV2014] H. V. Lê, Y-G. Oh, A. G. Tortorella and L. Vitagliano, *Deformations of coisotropic submanifolds in Jacobi manifolds*, J. Sympl. Geom., 16 (2018), pp. 1051–1116.

[LS2014] H. V. Lê and L. Schwachhöfer, *Lagrangian submanifolds in strict nearly Kähler 6-manifolds*, Osaka J. of Math., 56 (2019), pp. 601–629, arXiv:1408.6433.

[LV2017] H. V. Lê and J. Vanžura, *McLean's second variation formula revisited*, J. Geom. Phys., 113 (2017), pp. 188–196.

[Leung2002] N. C. Leung, *Topological quantum field theory for Calabi-Yau threefolds and G_2-manifolds*, Adv. Theor. Math. Phys., 6 (2002), pp. 575–591.

[Manetti2007] M. Manetti, *Lie description of higher obstructions to deforming submanifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci., 6 (2007), pp. 631–659.

[McLean1998] R. C. McLean, *Deformations of calibrated submanifolds*, Comm. Anal. Geom., 6 (1998), pp. 705–747.

[Morrey1954] C. B. Morrey, *Second order elliptic system of partial differential equations*, pp. 101–160, in Contribution to the theory of Partial differential equations, Ann. of Math. Study, 33, Princeton Univ. Press, Princeton, 1954.

[Morrey1958] C. B. Morrey, *On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations, I , II*, AJM, 80 (1958), pp. 198–218, pp. 219–237.

[Morrey2008] C. B. Morrey, *Multiple Integrals in the Calculus of the variations*, Springer, 2008.

[Ohst2014] M. Ohst, *Deformations of Compact Cayley Submanifolds with Boundary*, arXiv:1405.7886.

[Robles2012] C. Robles, *Parallel calibrations and minimal submanifolds*, Illinois J. Math., 56 (2012), pp. 383–395.

[Salamon1989] S. M. Salamon, *Riemannian geometry and holonomy groups*, Pitman Res. Notes in Math., 201, Longman, Harlow, 1989.

[Simons1968] J. Simons, *Minimal subvarieties in Riemannian manifolds*, Ann. Math., 88 (1968), pp. 62–105.

[SYZ1996] A. Strominger, S.-T. Yau and E. Zaslow, *Mirror symmetry is T-duality*, Nucl. Phys. B, 479 (1996), pp. 243–259.

[Takeuchi1962] M. Takeuchi, *On Pontrjagin classes of compact symmetric spaces*, J. Fac. Sci. Univ. Tokyo Sect., 9 (1962), pp. 313–328.

[Tasaki1985] H. Tasaki, *Certain minimal or homologically volume minimizing submanifolds in compact symmetric spaces*, Tsukuba J. Math., 9 (1985), pp. 117–131.

[Tian2000] G. Tian, *Gauge theory and calibrated geometry. I*, Ann. Math., 151 (2000), pp. 193–268.

[Voronov2005] T. Voronov, *Higher derived brackets and homotopy algebras*, J. Pure Appl. Algebra, 202 (2005), pp. 133–153.

[Walpuski2012] T. Walpuski, *G_2-instantons, associative submanifolds and Fueter sections*, Comm. Anal. Geom., 25 (2017), pp. 847–893.

[Walpuski2014] T. Walpuski, *Spin(7)-instantons, Cayley submanifolds and Fueter sections*, Comm. Math. Phys., 352 (2017), pp. 1–36.

[ZJ2005] J. Zhou, *Morse functions on Grassmann manifolds*, Proc. Roy. Soc. Edinburgh Sect A 135:1 (2005), pp. 209–221.