

INT-AMPLIFIED ENDOMORPHISMS OF COMPACT KÄHLER SPACES*

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Abstract. Let X be a normal compact Kähler space of dimension n . A surjective endomorphism f of such X is int-amplified if $f^*\xi - \xi = \eta$ for some Kähler classes ξ and η . First, we show that this definition generalizes the notion in the projective setting. Second, we prove that for the cases of X being smooth, a surface or a threefold with mild singularities, if X admits an int-amplified endomorphism with pseudo-effective canonical divisor, then it is a Q -torus. Finally, we consider a normal compact Kähler threefold Y with only terminal singularities and show that, replacing f by a positive power, we can run the minimal model program (MMP) f -equivariantly for such Y and reach either a Q -torus or a Fano (projective) variety of Picard number one.

Key words. compact Kähler space, int-amplified endomorphism, minimal model program.

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1. Introduction. We work over the field \mathbb{C} of complex numbers. By the fundamental work of S. Meng and D.-Q. Zhang (cf. [26] and [27]) during the past several years, we have known the building blocks and characteristic properties of polarized and int-amplified endomorphisms of normal projective varieties. Also, Meng and Zhang show that we can run the minimal model program equivariantly for mildly singular normal projective varieties and finally reach either a Q -abelian variety or a Fano variety of Picard number one. As an application, one can use the results to study the totally invariant divisors of polarized or int-amplified endomorphisms (cf. [41, Theorem 1.3] and [42, Theorem 1.1]).

Now, we pose a natural question. Does there exist such nice building blocks for compact Kähler spaces admitting non-isomorphic surjective endomorphisms? This is open since the minimal model program for higher-dimensional compact Kähler spaces is unknown.

In this article, we consider an arbitrary normal compact Kähler space X . A surjective endomorphism f of X is said to be *int-amplified*, if $f^*\xi - \xi = \eta$ for some Kähler classes ξ and η . We first prove that when X is projective, the generalized definition coincides with the previous one (cf. [26]). Then, we follow the idea of [26] to study the normal compact Kähler space admitting an int-amplified endomorphism in terms of the Kähler cone and canonical divisor. We show that most of these properties are preserved when the objects are extended to the analytic setting. Moreover, we study the periodic points or totally invariant analytic subvarieties of some special complex spaces such as Q -tori, i.e., quasi-étale finite quotients of complex tori. Finally, as a consequence of the existence of the minimal model program (MMP) for compact Kähler threefolds (cf. [20] and [21]), we prove that every int-amplified endomorphism of a compact Kähler threefold with at worst terminal singularities has an equivariant descending, when running the MMP.

Our work can be divided into two parts. First, we recall notation and terminology for the singular analytic space such as differential forms, currents and invariant cones. We spend some time showing that all of these objects possess nice properties parallel to

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normal projective cases. Second, we apply these common properties in the preliminary to generalize results in [26]. Here, we list them below for the convenience of readers.

Theorem 1.1 generalizes [26, Theorem 1.1]. The proof of the equivalent conditions is the same as [26] while the coincidence of these two definitions of int-amplified endomorphisms when X is projective is new (cf. Remark 2.12 and the proof of Theorem 1.1 in section 3).

THEOREM 1.1. *Suppose that $f : X \rightarrow X$ is a surjective endomorphism of a normal compact Kähler space X with at worst rational singularities. Let $\varphi := f^*|_{H_{\text{BC}}^{1,1}(X)}$, where $H_{\text{BC}}^{1,1}(X)$ denotes the Bott-Chern cohomology space. Then the following are equivalent.*

- (1) *The endomorphism f is int-amplified.*
- (2) *All the eigenvalues of φ are of modulus greater than 1.*
- (3) *There exists some big $(1, 1)$ -class $[\theta]$ such that $f^*[\theta] - [\theta]$ is big, i.e., it can be represented by a Kähler current T .*
- (4) *If C is a non-empty and non-zero φ -invariant convex cone in $H_{\text{BC}}^{1,1}(X)$, then $\emptyset \neq (\varphi - \text{id})^{-1}(C) \subseteq C$.*

Furthermore, if X is projective, then the two definitions of int-amplified endomorphisms coincide (cf. Definition 2.11 in the Kähler setting and [26] in the projective setting).

As an application of Theorem 1.1, we are able to show the following result, which asserts that the composition of sufficient iterations of int-amplified endomorphisms is still int-amplified. Readers may refer to [26, Theorem 1.4] for the projective version.

THEOREM 1.2. *Let f and g be two surjective endomorphisms of a normal compact Kähler space X with at worst rational singularities. Suppose that f is int-amplified. Then both $f^i \circ g$ and $g \circ f^i$ are int-amplified for sufficiently large $i \gg 1$.*

We refer the readers to [35, Sections 4 and 6] for the definitions of Cartier divisor, linear system and Kodaira dimension for a complex analytic variety. Also, the positivity of differential forms and currents will be discussed in the preliminary part.

The following theorem is important for the equivariant minimal model program of compact Kähler threefolds (cf. [26, Theorem 2.5] for the normal projective case).

THEOREM 1.3. *Let X be a normal compact Kähler space with at worst rational singularities. Suppose that X admits an int-amplified endomorphism f and the canonical divisor K_X is \mathbb{Q} -Cartier. Then, $-K_X$ is pseudo-effective. In particular, the Kodaira dimension $\kappa(X) \leq 0$.*

As an application of Theorem 1.3 and the main theorem in [16], we get Theorem 1.4 below so that when running the MMP for compact Kähler threefolds, we may reduce the case of canonical divisor K_X being pseudo-effective to the \mathbb{Q} -torus case.

THEOREM 1.4. *Let X be one of the following cases: (i) a compact Kähler manifold of any dimension, (ii) a normal compact Kähler threefold with at worst canonical singularities or (iii) a normal \mathbb{Q} -factorial compact Kähler surface. Suppose that X admits an int-amplified endomorphism $f : X \rightarrow X$ and the canonical divisor K_X is pseudo-effective. Then X is a \mathbb{Q} -torus, i.e., a quasi-étale finite quotient of a complex torus.*

We generalize [26, Theorem 1.10] to the following equivariant MMP. We refer to [27, Section 6] for the technical details. The author will highlight and compare the differences between projective varieties and compact Kähler spaces in Section 8.

THEOREM 1.5. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a normal \mathbb{Q} -factorial compact Kähler threefold with at worst terminal singularities. Then, replacing f by a positive power, there exist a \mathbb{Q} -torus Y , a morphism $X \rightarrow Y$ and an f -equivariant minimal model program of X over Y : $X = X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots \dashrightarrow X_r = Y$ (i.e., $f = f_1$ descends to f_i on each X_i), with every $X_i \dashrightarrow X_{i+1}$ a contraction of a divisorial ray, a flip or a Mori fibre contraction, of a K_{X_i} -negative extremal ray, such that:*

- (1) *If K_X is pseudo-effective, then $X = Y$ and it is a \mathbb{Q} -torus.*
- (2) *If K_X is not pseudo-effective, then for each i , f_i is int-amplified and $X_i \rightarrow Y$ is an equi-dimensional (holomorphic) morphism with every fibre irreducible and rationally connected. Also, $X_{r-1} \rightarrow X_r = Y$ is a Mori fibre contraction.*

Finally, we pose the question below. If X is assumed to be projective, the answer is affirmative by Fakhruddin's very motivating result (cf. [13, Lemma 5.1]).

QUESTION 1.6. *Suppose that $f : X \rightarrow X$ is an int-amplified endomorphism of a normal compact Kähler space. Does there exist a periodic point of f ? If so, is the set of f -periodic points $\text{Per}(f)$ open dense in X ?*

REMARK 1.7 (Differences with previous papers). In this article, we fix our attention in the complex analytic setting. Our first work is to sort out the notation and properties in compact Kähler spaces which are analogous to normal projective settings. Most of them are well-known to experts, but we were not able to find a good reference for these results. So we give the complete proofs in the preliminary part (cf. Propositions 2.2 and 2.6). Second, in the non-projective setting, we do not have ample divisors and hence an effective cycle cannot be simply written as a summand of the intersection of ample divisors. However, we shall apply the openness of Kähler cones to develop some numerical statements (cf. Lemmas 4.4~4.5). As an application, we apply these new results to show Theorem 1.4. Comparing with [26, Theorem 5.2], we cannot follow the original proof to show Theorem 1.4 due to the lack of [18, Theorem 1.1]. Throughout the article, the author will skip the same proof and highlight the differences when extending previous results.

The proofs of Theorems 1.1~1.3 are in Section 3. The proof of Theorem 1.4 is in Section 6. The proof of Theorem 1.5 is in Section 8. Now, for the organization of the paper, we begin with preliminaries in Section 2.

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2. Preliminaries. Let X be a normal compact complex space. We refer to [1], [17] and [35] for basic notation and properties for complex spaces. In this preliminary, we shall recall the differential forms (resp. currents) and their pull-backs (resp. push-forwards) in the singular setting.

Let $f : X \rightarrow Y$ be a surjective morphism between normal compact complex spaces. Here, a morphism is a holomorphic map in complex geometry. The morphism f is said to be *finite* (resp. *generically finite*) if f is proper and has discrete fibres

(resp. proper and finite outside a nowhere dense analytic closed subspace of Y). An *endomorphism* of X is a surjective holomorphic map $f : X \rightarrow X$.

In this article, we mainly deal with the singular case. Most of the properties in complex manifolds need to be extended to the singular setting. The following idea will be used more than once during the preliminary part.

Main Reduction: Let $f : X \rightarrow Y$ be a surjective morphism between normal compact complex spaces. Since Y is compact, by Hironaka desingularization theorem, there exists a modification $h : \tilde{Y} \rightarrow Y$ obtained by a finite sequence of blowing-up along smooth subspaces, such that \tilde{Y} is a complex manifold. By [1, Chapter I, Lemma 7.1], $h : \tilde{Y} \rightarrow Y$ lifts to $g' : X' \rightarrow X$ obtained by a finite sequence of blowing-up such that $f \circ g'$ factors through \tilde{Y} with the induced morphism $f' : X' \rightarrow \tilde{Y}$. Taking a resolution $\sigma : \tilde{X} \rightarrow X'$, we get $g := g' \circ \sigma : \tilde{X} \rightarrow X$, $\tilde{f} := f' \circ \sigma : \tilde{X} \rightarrow \tilde{Y}$ and the commutative diagram $f \circ g = h \circ \tilde{f}$, such that all of the morphisms are surjective. By the classical differential geometry, the morphism \tilde{f} admits very nice properties. Hence, to detect good properties on f , we need to fix our attention to the resolution h especially to a single blow-up.

REMARK 2.1. Sometimes, the choice of \tilde{X} varies. For example, we can also choose \tilde{X} to be the resolution of an irreducible component of the product $X \times_Y \tilde{Y}$ dominating X and also get a commutative diagram.

2.1. Differential forms and currents. In this subsection, we refer to [8] for the definitions and properties below.

Let X be a normal compact complex space of dimension n . Then, X is locally a closed analytic subset of an open subset of \mathbb{C}^N , i.e., for any $x \in X$, there exists an open neighbourhood U of x and an open subset $\Omega \subseteq \mathbb{C}^N$ such that $j : U \rightarrow \Omega$ is a closed embedding. A *differential form* ω of type (p, q) on X is a differential form on the smooth locus X_{reg} such that for any $x \in X$, with the closed embedding $x \in U \subseteq \Omega \subseteq \mathbb{C}^N$, there exists a differential form α on Ω such that $\omega|_{X_{\text{reg}}} = j^* \alpha|_{X_{\text{reg}}}$ locally. We denote by $\mathcal{C}^{p,q}$ the space of differential (p, q) -forms.

A *current* T of bidegree (p, q) is defined as an element in the dual space of differential (smooth) $(n-p, n-q)$ -forms. We denote by $\mathcal{D}^{p,q}(X)$ the space of (p, q) -currents with bidimension $(n-p, n-q)$ on X . It is well-defined, as well as the closedness and positivity.

Suppose $f : X \rightarrow Y$ is a morphism between two normal compact complex spaces of dimensions m and n , respectively. We refer the readers to [21] for the pullback of differential forms by pulling back forms locally. Suppose further that f is proper. Then, one can define the push-forward of currents f_* by setting $\langle f_* T, \alpha \rangle := \langle T, f^* \alpha \rangle$ for each $\alpha \in \mathcal{C}^{p,q}(X)$. For any (p, p) -current T and (q, q) -form α on X , we denote by $T \wedge \alpha$ the $(p+q, p+q)$ -current such that for each $(n-p-q, n-p-q)$ -form β on X , $\langle T \wedge \alpha, \beta \rangle = \langle T, \alpha \wedge \beta \rangle$. In addition, since each differential (p, q) -form α on X can also be naturally regarded as a (p, q) -current, we write $[\alpha]$ to represent a (p, q) -current, i.e., for each $(n-p, n-q)$ -form β , $\langle [\alpha], \beta \rangle := \int_{X_{\text{reg}}} \alpha \wedge \beta$.

We recall the following well-known projection formula which will be heavily used in this article.

PROPOSITION 2.2 (Projection formula). *Suppose $f : X \rightarrow Y$ is a proper surjective morphism between normal compact complex spaces with $\dim X = m$ and $\dim Y = n$. Then for any (l, l) -current T on X and differential (k, k) -form β on Y , we have $f_*(T \wedge f^* \beta) = f_* T \wedge \beta$ in the sense of currents. In particular, if $m = n$*

(and hence f is generically finite), then $f_*[f^*\beta] = \deg f \cdot [\beta]$ as currents.

Proof. For any differential $(n - k - l, n - k - l)$ -form γ on Y , it follows from the push-forward of currents that

$$\langle f_*(T \wedge f^*\beta), \gamma \rangle = \langle T \wedge f^*\beta, f^*\gamma \rangle = \langle T, f^*(\beta \wedge \gamma) \rangle = \langle f_*T, \beta \wedge \gamma \rangle = \langle f_*T \wedge \beta, \gamma \rangle,$$

which proves the first statement. Now we take two steps to prove the latter claim.

Step 1. From **Main Reduction**, we get a commutative diagram $h \circ \tilde{f} = f \circ g$ such that $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ is a generically finite morphism of complex manifolds and $\deg f = \deg \tilde{f}$. Then, it follows from the Gysin morphism that $\tilde{f}_*\tilde{f}^*\alpha = \deg f \cdot \alpha$ for each differential form α on \tilde{Y} (cf. [37, Remark 7.29]).

Step 2. For any $(n - k, n - k)$ -form γ on Y , we have

$$\langle f_*[f^*\beta], \gamma \rangle = \langle [f^*\beta], f^*\gamma \rangle = \int_{X_{\text{reg}}} f^*(\beta \wedge \gamma) = \int_{\tilde{X}} g^*f^*(\beta \wedge \gamma).$$

The third equality is because a proper closed subvariety of a compact Kähler manifold has Lebesgue measure zero. Then,

$$\int_{\tilde{X}} g^*f^*(\beta \wedge \gamma) = \int_{\tilde{X}} \tilde{f}^*h^*(\beta \wedge \gamma) = \deg \tilde{f} \cdot \int_{\tilde{Y}} h^*(\beta \wedge \gamma).$$

Note that the right hand side is equal to $\deg f \cdot \int_{Y_{\text{reg}}} (\beta \wedge \gamma)$ which is nothing but $\deg f \cdot \langle [\beta], \gamma \rangle$. Then, our second statement holds. \square

2.2. Classes and cones. In this subsection, we recall the notation and terminology which will be heavily used in this article. First we refer the readers to [3, Definition 4.6.2] or [21, Section 3] for the definition of Bott-Chern cohomology $H_{\text{BC}}^{1,1}(X)$ for singular spaces.

On the one hand, for any holomorphic map $f : X \rightarrow Y$ between normal compact complex spaces, the pull-back of differential forms f^* induces a well-defined morphism $f^* : H_{\text{BC}}^{1,1}(Y) \rightarrow H_{\text{BC}}^{1,1}(X)$ (cf. [21, Remark 3.2]): for any $c \in H_{\text{BC}}^{1,1}(Y)$, we can write $c = \omega + dd^c u$ where ω is a $(1, 1)$ -form with local potentials and u is a global smooth function. Then, the pull-back f^*c is written as the form $f^*\omega + dd^c(u \circ f)$ and it is easy to see that $f^*\omega$ is a $(1, 1)$ -form with local potentials since the real parts of holomorphic functions of Y pull back to real parts of holomorphic functions of X . Hence, f^*c defines an element in $H_{\text{BC}}^{1,1}(X)$.

On the other hand, if X has at worst rational singularities in the Fujiki class \mathcal{C} , then it follows from the Leray sequence (cf. [21, Remark 3.7]) that there is an injection: $H_{\text{BC}}^{1,1}(X) \hookrightarrow H^2(X, \mathbb{R})$, and hence one can define the intersection product on the Bott-Chern cohomology space $H_{\text{BC}}^{1,1}(X)$ via the cup-product for $H^2(X, \mathbb{R})$. In addition, this embedding implies that $H_{\text{BC}}^{1,1}(X)$ is a finite dimensional vector space.

REMARK 2.3. Suppose X is a compact Kähler manifold. Then by Hodge theory, there is a natural isomorphism from the Bott-Chern cohomology group to the Dolbeault group with real coefficients $H_{\text{BC}}^{1,1}(X) \cong H^{1,1}(X, \mathbb{R}) := H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$.

To make connections with the normal projective setting ([21, Definition 3.6 and 3.8]), let $N_1(X)$ be the vector space of real closed currents of bidegree $(n - 1, n - 1)$ modulo the following equivalence: $T_1 \equiv T_2$ if and only if: $T_1(\eta) = T_2(\eta)$ for all real closed $(1, 1)$ -forms η with local potentials.

REMARK 2.4. Suppose $f : X \rightarrow Y$ is a proper morphism between normal compact complex spaces and T_C (or denoted by $[C]$) is a *current of integration* (cf. [21, Section 3.B]) of an irreducible curve C on X . We claim that: if $f(C)$ is a curve, then $f_*[C] = \deg f|_C \cdot [f(C)]$; if C is contracted by f , then $f_*[C] = 0$. This coincides with the push-forward of cycles in the projective setting.

Proof. We follow the **Main Reduction**. First, we prove that for any resolution $h : \tilde{Y} \rightarrow Y$ with $E \subseteq \tilde{Y}$ a contracted subvariety, $h_*[E] = 0$. Indeed, for the local map $h' : \tilde{Y} \rightarrow Y \hookrightarrow \mathbb{C}^N$ and each differential (k, k) -form γ on Y with $\gamma = i^* \gamma_0$ and $k = \dim E$,

$$\langle h_*[E], \gamma \rangle = \langle [E], h^* \gamma \rangle = \langle [E], (h')^* \gamma_0 \rangle = 0.$$

The last equality is due to $\dim h'(E) < \dim E$ and h' is locally a holomorphic map between manifolds. Therefore, $h_*[E] = 0$. Second, suppose the curve C is contracted by f . Then, no matter $C \subseteq \text{Sing } X$ or not, any analytic curve of \tilde{X} dominating C must be contracted by either \tilde{f} or h . Hence, $f_*[C] = 0$ when C is contracted by f .

From now on, suppose $f(C)$ is a curve. Then for any differential $(n-1, n-1)$ -form γ on Y , we apply **Main Reduction**. If $C \not\subseteq \text{Sing}(X)$, set $\tilde{C} \subseteq \tilde{X}$ to be the proper transform of C ; if $C \subseteq \text{Sing}(X)$, let $\tilde{C} \subseteq \tilde{X}$ be any curve dominating C (which exists since the resolution g is projective). In both cases,

$$\langle f_*[C], \gamma \rangle = \int_{C_{\text{reg}}} f^* \gamma = \frac{1}{\deg g|_{\tilde{C}}} \cdot \int_{\tilde{C}} \tilde{f}^* h^* \gamma = \deg f|_C \cdot \int_{f(C)} \gamma = \deg f|_C \cdot \langle [f(C)], \gamma \rangle.$$

Therefore, we complete the proof of our remark. \square

Let $\overline{\text{NA}}(X) \subseteq N_1(X)$ be the closed cone generated by the classes of positive closed $(n-1, n-1)$ -currents. We define the Mori cone $\overline{\text{NE}}(X) \subseteq \overline{\text{NA}}(X)$ as the closure of the cone generated by the currents of integration T_C , where $C \subseteq X$ is an irreducible curve.

From now on, we begin to discuss the positivity and cones in complex geometry. In the beginning, we refer the readers to [11, Chapter I] and [9, Introduction and main results] for the definition and basic properties of (strictly) plurisubharmonic (psh) functions.

Suppose X is a normal compact complex space of dimension m . Then, it follows from [11, Chapter I, Theorem 5.8] that a smooth function φ is psh on X if and only if $dd^c \varphi \geq 0$ is a positive current.

A normal compact complex space X is said to be *Kähler* if there exists a Kähler form ω on X , i.e., a positive closed real $(1, 1)$ -form $\omega \in \mathcal{C}^{1,1}(X)$ such that the following holds. For every point $x \in X_{\text{sing}}$, there exist an open neighbourhood $x \in U \subseteq X$, a closed embedding $i_U : U \subseteq \Omega$ into an open set $\Omega \subseteq \mathbb{C}^N$ and a strictly psh C^∞ -function $f : \Omega \rightarrow \mathbb{C}$ such that $\omega|_{U \cap X_{\text{reg}}} = (dd^c f)|_{U \cap X_{\text{reg}}}$.

Similar to the notation of relative ample divisor related to a projective morphism, we refer the readers to [36] for the information of relative Kähler metric related to a Kähler morphism. Now, we come to discuss the invariant cones.

DEFINITION 2.5. *Let (X, ω) be a normal compact Kähler space, where ω is a fixed Kähler form. A class $[\alpha] \in H_{\text{BC}}^{1,1}(X)$ (where α is a smooth form) is said to be*

- (1) *Kähler, if it contains a representative of a smooth Kähler form, i.e., there is a smooth function φ such that $\alpha + dd^c \varphi \geq \varepsilon \omega$ for some $\varepsilon > 0$;*
- (2) *nef, if for every $\varepsilon > 0$, there is a smooth function f_ε such that $\alpha + dd^c f_\varepsilon \geq -\varepsilon \omega$;*

- (3) *pseudo-effective, if it contains a positive current (that may not be smooth), i.e., there exists an almost psh function $\varphi = \psi + h$ where ψ is psh and h is smooth on X such that $\alpha + dd^c\varphi \geq 0$ in the sense of currents.*
- (4) *big, if it contains a Kähler current, i.e., there exists an almost psh function φ such that $\alpha + dd^c\varphi \geq \varepsilon\omega$ in the sense of currents for some $\varepsilon \geq 0$.*

Let X be a normal compact Kähler space. We use the following to represent the cones in Definition 2.5 to make connections with the projective setting.

- $\mathcal{K}(X)$: the cone of Kähler classes in $H_{BC}^{1,1}(X)$;
- $\text{Nef}(X)$: the cone of nef classes in $H_{BC}^{1,1}(X)$;
- $\mathcal{E}(X)$: the cone of pseudo-effective classes in $H_{BC}^{1,1}(X)$;
- $\mathcal{E}^0(X)$: the cone of big classes in $H_{BC}^{1,1}(X)$.

Observe that $\mathcal{K}(X)$ is an open (convex) cone (cf. [16, Proposition 3.6]) and $\text{Nef}(X) \subseteq \mathcal{E}(X)$ are closed convex cones by the weak compactness of bounded sets in the space of currents (cf. [9, Section 6]). Besides, it follows from [21, Remark 3.12] that $\text{Nef}(X) = \overline{\mathcal{K}}(X)$. Furthermore, $\mathcal{E}^0(X)$ is the interior part of $\mathcal{E}(X)$.

We formulate the following important proposition which states that all of these cones mentioned above are preserved under suitable morphisms. Hence, we can apply the cone theory developed in [26] to study the Bott-Chern cohomology space.

PROPOSITION 2.6 (Invariant cones). *Suppose $f : X \rightarrow Y$ is a proper surjective morphism of two normal compact Kähler spaces. Then, for any $[\alpha] \in H_{BC}^{1,1}(Y)$, the following hold.*

- (1) *If $[\alpha]$ is pseudo-effective, then so is $f^*[\alpha]$;*
- (2) *If $[\alpha]$ is nef, then so is $f^*[\alpha]$ and the converse holds when f is further assumed to be a modification of normal compact Kähler threefolds;*
- (3) *If f is further assumed to be finite and $[\alpha]$ is Kähler, then so is $f^*[\alpha]$;*
- (4) *If f is a modification and $[\alpha]$ is big, then so is $f^*[\alpha]$;*
- (5) *If f is generically finite and $[\alpha]$ is big, then so is $f^*[\alpha]$.*

Proof. The pseudo-effective cone is invariant under any holomorphic map since the pull-back of a psh function is still psh, which gives (1). Let ω_X and ω_Y be two Kähler forms on X and Y respectively such that $f^*\omega_Y \leq a\omega_X$ for some constant $a > 0$. For any $\varepsilon > 0$, since $[\alpha]$ is nef, there exists $g_\varepsilon \in \mathcal{C}^0(Y)$ such that $\alpha + dd^c g_\varepsilon \geq -\frac{\varepsilon}{a} \cdot \omega_Y$. Then

$$f^*\alpha + dd^c(g_\varepsilon \circ f) \geq -\frac{\varepsilon}{a} f^*\omega_Y \geq -\varepsilon\omega_X,$$

which proves one direction of (2). For the converse, see [21, Lemma 3.13].

For (3), see [16, Proposition 3.5]. Now, we begin to prove (4). For each big class $[\alpha]$ on Y , there exists a Kähler form ω_Y on Y and a representative T' of $[\alpha]$ such that $T' \geq \omega_Y$. Therefore, it is enough to prove that the pull-back of each Kähler class on Y is big on X under any modification. Let $f : X \rightarrow Y$ be a modification and ω_Y a Kähler form on Y . By Hironaka’s Chow lemma (cf. [1, Theorem 7.8]), there exists a normal compact complex space Z with $p : Z \rightarrow Y$ and $q : Z \rightarrow X$ such that $p = f \circ q$, where p and q are compositions of finite sequences of blowing-up along smooth centers.

We claim that $p^*[\omega_Y]$ is a big class. Indeed, we only need to consider the case when p is a single blow-up. Note that a blow-up of Y is locally the restriction of a blow-up of \mathbb{C}^N . Denote by E the exceptional divisor of \underline{p} . Then E is locally the restriction of the exceptional divisor E_i of the blow-up $p_i : \underline{U}_i \rightarrow U_i$, where U_i is an open subset of

\mathbb{C}^N . For the smooth manifold U_i , it follows from the calculation in [37, Proposition 3.24 and Lemma 3.26] that for sufficiently large $C_i \gg 1$, $(C_i p_i^* \omega_{U_i} - E_i)|_Y$ is a positive $(1, 1)$ -form (also cf. [1, pp.116]). Since Y is compact, there exists a sufficiently large $C \gg 1$ such that $C p^* \omega_Y - E$ is a positive $(1, 1)$ -form. Hence, $p^*[\omega_Y]$ is a big class.

Moreover, $f^* \omega_Y = q_* q^* f^* \omega_Y = q_* p^* \omega_Y$ as currents by projection formula. Since $p^*[\omega_Y]$ is big, there exists a current $T \in p^*[\omega_Y] = [p^* \omega_Y]$ such that $T \geq \omega_Z$ as currents for some Kähler form ω_Z on Z . Then, we have $q_* T \in [f^* \omega_Y] = f^*[\omega_Y]$ such that $q_* T \geq q_* \omega_Z$ as currents. In addition, for any Kähler form ω_X on X , there exists $a > 0$ such that $q^* \omega_X \leq a \omega_Z$. Hence, $q_* T \geq q_* \omega_Z \geq \frac{1}{a} \omega_X$ as currents and then $f^*[\omega_Y]$ is big on Y .

Finally, for (5), if f is a generically finite surjective morphism, then taking the Stein factorization of f (cf. [35, Theorem 1.9]), we get a modification $f' : X \rightarrow Y'$ followed by a finite morphism $f'' : Y' \rightarrow Y$. For any big class $[\alpha]$ on Y , there exists a Kähler current $T \in [\alpha]$ such that $T \geq \omega_Y$ as currents for some Kähler form ω_Y on Y . Then, the pull-back $(f'')^* T \geq (f'')^* \omega_Y$ and the class of the right hand side is Kähler by (3). Hence, $(f'')^*[\alpha]$ is big. Further, it follows from the assertion (4) that $f^*[\alpha] = (f')^*(f'')^*[\alpha]$ is also big, and thus (5) has been checked. \square

Recall that if $f : X \rightarrow Y$ is a surjective morphism between normal projective varieties, then it follows from the projection formula and Lefschetz hyperplane theorem that the pull-back operation gives an injection $f^* : \text{NS}_{\mathbb{R}}(Y) \rightarrow \text{NS}_{\mathbb{R}}(X)$ on the Néron-Severi groups of X and Y . However, in the analytic case when X and Y are possibly non-projective, there is no ample divisor and then Lefschetz hyperplane theorem may not be true. Nevertheless, we give the lemma below by requiring mild singularities.

LEMMA 2.7. *Suppose $f : X \rightarrow Y$ is a surjective holomorphic map between normal compact Kähler spaces with at worst rational singularities. Then $f^* : H_{\text{BC}}^{1,1}(Y) \rightarrow H_{\text{BC}}^{1,1}(X)$ is an injection.*

Proof. We shall apply **Main Reduction** to prove our lemma. Take a resolution $h : \tilde{Y} \rightarrow Y$ and let \tilde{X} be the resolution of an irreducible component of the product $X \times_Y \tilde{Y}$ dominating X with $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ and $g : \tilde{X} \rightarrow X$. Since \tilde{f} is a surjective morphism between compact Kähler manifolds, the induced linear operation $\tilde{f}^* : H_{\text{BC}}^{1,1}(\tilde{Y}) \rightarrow H_{\text{BC}}^{1,1}(\tilde{X})$ is an injection (cf. [37, Lemma 7.28] and Remark 2.3).

Since both g and h are resolutions, $H_{\text{BC}}^{1,1}(Y)$ (resp. $H_{\text{BC}}^{1,1}(X)$) embeds into $H_{\text{BC}}^{1,1}(\tilde{Y})$ (resp. $H_{\text{BC}}^{1,1}(\tilde{X})$); see [21, Lemma 3.3]. By the commutative diagram, the induced linear operation $f^*|_{H_{\text{BC}}^{1,1}(Y)}$ is an injection. \square

In particular, we get the following proposition by setting $X = Y$ in Lemma 2.7.

PROPOSITION 2.8. *Let X be a normal compact Kähler space with at worst rational singularities and $f : X \rightarrow X$ a surjective endomorphism. Then, f induces an isomorphism of the Bott-Chern cohomology space $H_{\text{BC}}^{1,1}(X)$.*

At the end of this subsection, we state the fact of spanning cones below. Indeed, it follows immediately from the openness of Kähler cone.

LEMMA 2.9. *Suppose X is a normal compact Kähler space. Then, the pseudo-effective cone defined in Definition 2.5 spans the Bott-Chern cohomology space.*

2.3. Polarized and int-amplified endomorphisms. In this short subsection, we come to define the key notion in this article.

First, we prove the next lemma which asserts that for each holomorphic self-map of a compact Kähler space with mild singularities, the surjectivity will imply the finiteness.

LEMMA 2.10. *Let $f : X \rightarrow X$ be a surjective holomorphic map of a normal compact Kähler space X . Suppose that X has at worst rational singularities. Then f is finite.*

Proof. Since X is compact, any holomorphic self-map of X is proper. Besides, f is a surjective self-map and thus generically finite. By taking the Stein factorization $X \rightarrow Y \rightarrow X$ and applying Hironaka’s Chow lemma (cf. [1, Theorem 7.8]) for the modification $X \rightarrow Y$, we see that f is dominated by a composition of finite sequences of blow-ups followed by a finite morphism $Y \rightarrow X$, which is projective. Therefore, every positive dimensional fibre of f will be covered by curves.

Suppose $C \subseteq X$ is a curve lying in the fibre of f . We fix a Kähler class $[\omega]$ on X . On the one hand, since f is proper and surjective, it follows from Proposition 2.6 (2) that $f^*\text{Nef}(X) \subseteq \text{Nef}(X)$. On the other hand, since X has at worst rational singularities, by Proposition 2.8, $f^*|_{\mathbb{H}_{\text{BC}}^{1,1}(X)}$ is invertible; hence it maps interior part of $\text{Nef}(X)$ to the interior part of $\text{Nef}(X)$. In particular, $f^*[\omega]$ is still a Kähler class.

Considering the composite morphism $j : \tilde{C} \rightarrow C \hookrightarrow X$, where \tilde{C} is the normalization of the curve C , we see that $j^*f^*[\omega]$ is also a Kähler class by Proposition 2.6 (3). Therefore, $0 < \int_{\tilde{C}} j^*f^*\omega = \int_C f^*\omega = \langle [C], f^*\omega \rangle$ by the numerical characterization of Kähler classes (cf. [10]). However, the right hand side coincides with $\langle f_*[C], \omega \rangle$ by the projection formula and hence vanishes (cf. Proposition 2.2 and Remark 2.4), a contradiction. \square

DEFINITION 2.11. *Let $f : X \rightarrow X$ be a surjective endomorphism of a normal compact Kähler space. We say that f is*

- (1) *polarized, if there exists a Kähler class $[\alpha] \in \mathbb{H}_{\text{BC}}^{1,1}(X)$ such that $f^*[\alpha] = q[\alpha]$ for some integer $q > 1$;*
- (2) *amplified, if there exist a class $[\xi] \in \mathbb{H}_{\text{BC}}^{1,1}(X)$ and a Kähler class $[\eta] \in \mathbb{H}_{\text{BC}}^{1,1}(X)$ such that $f^*[\xi] - [\xi] = [\eta]$;*
- (3) *int-amplified, if there exist Kähler classes $[\xi]$ and $[\eta]$ such that $f^*[\xi] - [\xi] = [\eta]$.*

REMARK 2.12. We shall prove in Section 3 that, an int-amplified endomorphism in the sense of Definition 2.11 coincides with the definition in [26], when X is projective.

3. Properties of int-amplified endomorphisms: Proof of Theorem 1.1 ~ 1.3. In this section, we study the basic properties of int-amplified endomorphisms. Most of the following are generalizations of [26, Section 3] for the case when X is projective and the proofs are similar except we apply new results developed in Section 2.

To begin with this part, we prove Theorem 1.1 by applying the results in [26]. As reminded by referees, we require X to have at worst rational singularities to make sure the induced linear operation $f^*|_{\mathbb{H}_{\text{BC}}^{1,1}(X)}$ is invertible (cf. Proposition 2.8).

Proof of Theorem 1.1. It is clear that (4) \Rightarrow (1) \Rightarrow (3) by letting $C = \mathcal{K}(X)$. Suppose condition (3). Then, applying [26, Lemma 3.1] with $V = \mathbb{H}_{\text{BC}}^{1,1}(X)$ and $C = \mathcal{E}(X)$, we get (2). Finally, (2) implies (4) is exactly [26, Proposition 3.2].

Next, we show the equivalence of two kinds of definitions when X is projective. Indeed, it is obvious that if f is int-amplified in the sense of ample divisors, then f

is int-amplified in the sense of Kähler classes. For the converse direction, suppose X is projective and f is int-amplified in the sense of Kähler classes. Then by (2), each eigenvalue of the invertible linear operation $f^*|_{\mathbb{H}_{\text{BC}}^{1,1}(X)}$ has modulus greater than 1. Since the Néron Severi space $\text{NS}_{\mathbb{R}}(X)$ is a subspace of $\mathbb{H}_{\text{BC}}^{1,1}(X)$, each eigenvalue of $f^*|_{\text{NS}_{\mathbb{R}}(X)}$ also has modulus greater than 1. Hence, the converse direction holds by [26, Theorem 1.1]. \square

The next lemma shows that an endomorphism being int-amplified is equivalent to its power being int-amplified. The proof is easy by using Theorem 1.1 and Proposition 2.6.

LEMMA 3.1. *Let $f : X \rightarrow X$ be a surjective endomorphism of a normal compact Kähler space with at worst rational singularities. Then the following are equivalent.*

- (1) f is int-amplified.
- (2) f^s is int-amplified for any positive integer s .
- (3) f^s is int-amplified for some positive integer s .

In the following, we shall apply Theorem 1.1 to prove several important lemmas below.

LEMMA 3.2. *Let $\pi : X \rightarrow Y$ be a surjective morphism of normal compact Kähler spaces with at worst rational singularities. Let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be two surjective endomorphisms such that $g \circ \pi = \pi \circ f$. If f is int-amplified, then so is g .*

Proof. It follows from Lemma 2.7 that $\pi^* : \mathbb{H}_{\text{BC}}^{1,1}(Y) \rightarrow \mathbb{H}_{\text{BC}}^{1,1}(X)$ is injective. Then, each eigenvalue of $g^*|_{\mathbb{H}_{\text{BC}}^{1,1}(Y)}$ is an eigenvalue of $f^*|_{\mathbb{H}_{\text{BC}}^{1,1}(X)}$ and thus has modulus greater than one, since f is int-amplified. Therefore, our lemma follows from Theorem 1.1. \square

LEMMA 3.3. *Let $\pi : X \rightarrow Y$ be a generically finite surjective morphism between normal compact Kähler spaces with at worst rational singularities. Let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be surjective endomorphisms such that $g \circ \pi = \pi \circ f$. If g is int-amplified, then so is f .*

Proof. Suppose g is int-amplified. Then, there exist Kähler classes $[\xi]$ and $[\eta]$ on Y such that $g^*[\xi] = [\eta] + [\xi]$. Note that the Kähler cone $\mathcal{K}(Y)$ is contained in the big cone $\mathcal{E}^0(Y)$ and big classes pull back to big classes under any generically finite morphism by Proposition 2.6. Then, $\pi^*[\xi]$ and $\pi^*[\eta]$ are big classes on X and the commutative diagram implies $\pi^*[\eta] = f^*\pi^*[\xi] - \pi^*[\xi]$. Hence, our lemma follows from Theorem 1.1. \square

Next, we follow the idea of [26] to prove Theorems 1.2 and 1.3 below in our present setting. Both of them are consequences of Theorem 1.1.

Proof of Theorem 1.2. We fix a norm on the Bott-Chern cohomology space $V := \mathbb{H}_{\text{BC}}^{1,1}(X)$. Let $\phi_f := f^*|_V$ and $\phi_g := g^*|_V$. Since f is int-amplified, by Theorem 1.1, all the eigenvalues of ϕ_f are of modulus greater than 1. Recall that the norm of a linear operator is given by $\|\phi_f\| := \max_{v \in V} \frac{\|\phi_f(v)\|}{\|v\|}$, and the right hand side is no less than the spectral radius of ϕ_f . Also, by Gelfand’s formula, $\lim_{i \rightarrow \infty} \|\phi_f^{-i}\|^{\frac{1}{i}} = \rho(\phi_f^{-1}) < 1$, where $\rho(\phi_f^{-1})$ is the spectral radius of ϕ_f^{-1} . Then, there exists i_0 such that for all $i \geq i_0$, $\|\phi_f^{-i}\phi_g^{-1}\| \leq \|\phi_f^{-i}\| \cdot \|\phi_g^{-1}\| < 1$. So, for each $v \in V$, $\|\phi_g\phi_f^i(v)\| > \|v\|$. Let $h = f^i \circ g$ with $i \geq i_0$ and $\phi_h = h^*|_V$.

We claim that all the eigenvalues of ϕ_h are of modulus greater than 1. Indeed, let r be the smallest absolute value of eigenvalues of ϕ_h . Since X is Kähler, V has nonvoid interior. So, it follows from the Perron-Frobenius theorem that V contains an eigenvector v whose eigenvalue is the spectral radius of ϕ_h^{-1} , i.e., $\|\phi_h^{-1}(v)\| = \frac{1}{r}\|v\|$. Therefore, with v replaced by $\phi_h(v)$, we have $r > 1$ by the inequality: $r\|v\| = \|\phi_h(v)\| = \|\phi_g(\phi_f^i(v))\| > \|v\|$. Hence h is int-amplified. Similar argument works for $g \circ f^i$. So, Theorem 1.2 holds. \square

Proof of Theorem 1.3. By the ramification divisor formula, $f^*(-K_X) - (-K_X) = R_\phi$, where the ramification divisor R_ϕ is effective and thus pseudo-effective in the sense of currents. By Theorem 1.1 (4), $-K_X$ is also pseudo-effective.

Suppose the contrary that $\kappa(X) > 0$. Then, there exists a non-zero effective divisor D on X , such that $sK_X \sim D$ for some $s \in \mathbb{N}^+$. Note that D is not contained in the singular locus due to the normality of X . Therefore, for any Kähler form ω on X ,

$$sK_X \wedge \omega^{n-1} = [D] \wedge \omega^{n-1} = \int_{D_{\text{reg}}} (\omega|_D)^{n-1} > 0,$$

a contradiction to $-K_X$ being pseudo-effective. Then, Theorem 1.3 holds. \square

Let $f : X \rightarrow X$ be a surjective endomorphism of a normal compact Kähler space X of dimension n with at worst rational singularities (cf. [21, Remark 3.7]). Denote by

$$L^k(X) := \left\{ \sum [\alpha_1] \cup \dots \cup [\alpha_k] \mid [\alpha_i] \in H_{\text{BC}}^{1,1}(X) \right\},$$

the subspace of $H^{2k}(X, \mathbb{R})$. In particular, $L^1(X) = H_{\text{BC}}^{1,1}(X)$. Let $N^k(X) := L^k(X) / \equiv_w$, where $[\alpha] \in L^k(X)$ is weakly numerically equivalent (denoted by \equiv_w) to zero if and only if for any $[\beta_{k+1}], \dots, [\beta_n] \in H_{\text{BC}}^{1,1}(X)$, $[\alpha] \cup [\beta_{k+1}] \cup \dots \cup [\beta_n] = 0$. Moreover, for any $[\alpha], [\beta] \in L^k(X)$, $[\alpha] \equiv_w [\beta]$ if and only if $[\alpha] - [\beta] \equiv_w 0$.

Let $L_{\mathbb{C}}^k(X) = L^k(X) \otimes_{\mathbb{R}} \mathbb{C}$ and $N_{\mathbb{C}}^k(X) = N^k(X) \otimes_{\mathbb{R}} \mathbb{C}$. Note that the pull-back operation f^* on Bott-Chern cohomology space induces a linear operation on the subspace $L_{\mathbb{C}}^k(X) \subseteq H^{2k}(X, \mathbb{C})$ for each k . With the notation settled, it follows from Proposition 2.8 that f^* also gives a well-defined linear operation on the quotient space $N_{\mathbb{C}}^k(X)$.

We generalize [26, Lemma 3.6] to the following. We skip its proof and readers may refer to [26, Lemma 3.6] by applying Theorem 1.1 and Proposition 2.8. For its application, see Theorem 3.5 and Proposition 6.1.

LEMMA 3.4. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a normal compact Kähler space of dimension n . Suppose that X has at worst rational singularities. Then, for each $0 < k < n$, all the eigenvalues of $f^*|_{N_{\mathbb{C}}^k(X)}$ are of modulus less than $\deg f$. In particular, $\lim_{i \rightarrow +\infty} \frac{(f^i)^*[x]}{(\deg f)^i} \equiv_w 0$ for any $[x] \in L_{\mathbb{C}}^k(X)$.*

THEOREM 3.5. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a positive dimensional compact Kähler space with only rational singularities. Then $\deg f > 1$.*

Proof. If $\dim X = 1$, then X is normal since X has at worst rational singularities. Therefore, X is smooth and projective. By Theorem 1.1, there exist two ample divisors D and H such that $f^*D - D = H$. If $\deg f = 1$, taking the degree of both sides, we get a contradiction.

If $\dim X \geq 2$, then by Theorem 1.1, all the eigenvalues of $f^*|_{H_{\mathbb{B}\mathbb{C}}^{1,1}(X)}$ have modulus greater than 1. Note that the eigenvectors of the quotient space $N_{\mathbb{C}}^1(X)$ come from the elements in $H_{\mathbb{B}\mathbb{C}}^{1,1}(X)$. By applying Lemma 3.4 for $k = 1$, $\deg f > 1$. \square

4. Some dynamics of int-amplified endomorphisms. In this section, we discuss some dynamics for int-amplified endomorphisms in terms of fixed points and totally invariant subvarieties. Note that Lemma 4.1 follows immediately from [36, Proposition 1.3.1] and Lemma 4.2 is an extension of [26, Lemma 2.4].

LEMMA 4.1. *If f is an int-amplified endomorphism of a normal compact Kähler space X and Z is an f -invariant analytic subvariety (i.e., $f(Z) = Z$) of X , then the restriction $f|_Z$ is an int-amplified endomorphism of Z .*

LEMMA 4.2. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a compact Kähler space X . Then, the set of periodic points $\text{Per}(f)$ is countable.*

Proof. Suppose $\text{Per}(f)$ is uncountable. Then, there exists some $s > 0$, such that the set P_s of all f^s -fixed points is infinite. Let Z be an irreducible component of the closure of P_s in X with $\dim Z > 0$. Then $f^s|_Z = \text{id}_Z$, which is absurd by Lemmas 3.1 and 4.1. \square

Unlike the normal projective setting, for a compact Kähler space X admitting an amplified endomorphism f , it is still open about the density of periodic points. Nevertheless, the proposition below partially answers Question 1.6 in the case of complex tori. Further, we will show in Section 5 that the result can be extended to Q -tori case.

PROPOSITION 4.3. *Let $f : T \rightarrow T$ be an int-amplified endomorphism of a complex torus T . Then, there exists a fixed point of f .*

Proof. By Künneth formula, if $\lambda_1, \dots, \lambda_k$ (may not be distinct) are all the eigenvalues of $f^*|_{H^1(T, \mathbb{C})}$, then each eigenvalue of $f^*|_{H^i(T, \mathbb{C})}$ is of the form $\prod_j \lambda_j^{l_j}$, where $\sum l_j = i$.

Since f is int-amplified, each eigenvalue of $f^*|_{H^{1,1}(T, \mathbb{C})}$ has modulus greater than 1. By Hodge theory, all of the eigenvalues of $f^*|_{H^1(T, \mathbb{C})}$ (and hence $f^*|_{H^i(T, \mathbb{C})}$) are of modulus greater than 1. Then, it follows from the Lefschetz fixed point theorem that,

$$L(f) = \sum_i (-1)^i \text{Trace}(f^*|_{H^i(T, \mathbb{C})}) = \sum_{k_1, \dots, k_i} (-\lambda_{k_1}) \cdots (-\lambda_{k_i}) = \prod_i (1 - \lambda_i),$$

with $|\lambda_i| > 1$ for each i . Then $L(f) \neq 0$ implies that f has a fixed point. \square

In the following, we fix our attention to manifolds temporarily. We extend some early results in [26, Section 3] to the analytic case. Recall the definition of $L^k(X)$ and its quotient $N^k(X)$ in Section 3. The following lemma holds due to the openness of the cone $\mathcal{K}_k(X)$, the set of classes of strictly positive closed (k, k) -forms (cf. [11, Chapter III]).

LEMMA 4.4. *Let (X, ω) be a compact Kähler manifold of dimension n . Then, for any closed (k, k) -current T , there exists $A > 0$ such that $A\omega^k - T$ is a positive (k, k) -current.*

When X is a normal projective variety, one may use Bertini’s theorem to prove that for any closed subvariety Z , there exists an effective cycle C such that the sum

$Z + C$ is the intersection of ample divisors. However, in the analytic setting, we may not have ample divisors. Nevertheless, Lemma 4.4 says that for any closed current, there exists a positive “complement” such that the sum is (numerically) equal to a power of some Kähler form. As an application of Lemma 4.4, we show the lemma below. Note that when X is smooth, $H_{\text{BC}}^{1,1}(X) \cong H^{1,1}(X, \mathbb{R})$ and hence we may regard each $(n - k, n - k)$ -current as a function on $L^k(X)$ using the Poincaré duality and Stokes’ formula.

In what follows, we show a useful lemma, which will be applied in the proof of Proposition 4.6 and Lemma 8.2.

LEMMA 4.5. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a normal compact Kähler space with at worst rational singularities of dimension n . Let ξ be a Kähler form on X . Then for any k -dimensional analytic proper subvariety Z of X , which is not contained in the singular locus, $\lim_{i \rightarrow +\infty} \int_Z \frac{(f^i)^*(\xi^k)}{(\deg f)^i} = 0$.*

Proof. Let $x_i := \frac{(f^i)^*(\xi^k)}{(\deg f)^i} \in L^k(X)$. We take a resolution $\pi : \tilde{X} \rightarrow X$ and consider the pull-back $y_i := \pi^*x_i$, the class of which is a product of nef classes in \tilde{X} . Assume that outside a proper closed subset $E \subseteq \tilde{X}$, we have $\tilde{X} \setminus E \cong X \setminus \text{Sing}(X)$. Since $Z \not\subseteq \text{Sing}(X)$, we denote by \tilde{Z} the proper transform of Z in \tilde{X} .

For any fixed Kähler form ω on X , there exists $c \gg 1$ such that $\eta := c \cdot \pi^*\omega - E'$ is a Kähler form on \tilde{X} (cf. [37, pp.81]), where $E' \geq 0$ is a positive linear combination of the components in E . We may identify E' with E in the following. Since $[\tilde{Z}]$ is a positive $(n - k, n - k)$ -current and $[y_i]$ is a product of nef classes, by Lemma 4.4, there exists $a > 0$ such that $\langle a \cdot \eta^{n-k} - [\tilde{Z}], [y_i] \rangle \geq 0$ for any i ; hence

$$\int_Z x_i = \int_{\tilde{Z}} y_i = \langle [\tilde{Z}], [y_i] \rangle \leq [y_i] \cup a \cdot \eta^{n-k} \leq [y_i] \cup (ac)\pi^*\omega \cup \eta^{n-k-1}.$$

Note that the last inequality is due to the nefness of $[y_i]$ and $[\eta]$ (and hence $[y_i] \cup E \cup \eta^{n-k-1} \geq 0$). Repeat the above process and apply the nefness of $[\pi^*\omega]$, we have

$$\int_Z x_i \leq a \cdot c^{n-k} [y_i] \cup (\pi^*\omega)^{n-k} = a \cdot c^{n-k} [x_i] \cup \omega^{n-k}. \tag{1}$$

With \tilde{Z} replaced by $-\tilde{Z}$, the same arguments above together with Lemma 4.4 show that there exists $b > 0$ such that $\langle b \cdot \eta^{n-k} + [\tilde{Z}], [y_i] \rangle \geq 0$ for any i ; hence

$$-\int_Z x_i = -\int_{\tilde{Z}} y_i \leq [y_i] \cup b \cdot \eta^{n-k} \leq b \cdot c^{n-k} [x_i] \cup \omega^{n-k}. \tag{2}$$

By Lemma 3.4, $\lim_{i \rightarrow \infty} [x_i] \equiv_w 0$ in $L_{\mathbb{C}}^k(X)$; hence taking the upper limits of both sides for (1) and (2), we have $0 \leq \varliminf_{i \rightarrow \infty} \langle [Z], x_i \rangle \leq \overline{\lim}_{i \rightarrow \infty} \langle [Z], x_i \rangle \leq 0$. So our result follows. \square

PROPOSITION 4.6. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a normal compact Kähler space X with at worst rational singularities of dimension n . Let Y be an analytic closed subvariety of X not contained in the singular locus such that $f^{-1}(Y) = Y$ and $\deg f|_Y = \deg f$. Then $X = Y$.*

Proof. Suppose $\dim Y = m < n$ and we fix a Kähler form ξ on X . On the one hand, by Lemma 4.5, $\lim_{i \rightarrow \infty} \int_Y \frac{(f^i)^*\xi^m}{(\deg f)^i} = 0$. On the other hand, we have the

natural integration on the topological space $\langle [Y], (f^i)^*\xi^m \rangle = (\deg f|_Y)^i \langle [Y], \xi^m \rangle = (\deg f)^i \langle [Y], \xi^m \rangle$.

Therefore, we get a contradiction by the numerical characterization (cf. [10]):

$$0 < \int_{Y_{\text{reg}}} \xi^m = \int_Y \xi^m = \frac{1}{(\deg f)^i} \int_Y (f^i)^*\xi^m = \lim_{i \rightarrow \infty} \int_Y \frac{(f^i)^*\xi^m}{(\deg f)^i} = 0.$$

Then, our proposition follows from this contradiction. \square

Resulting from Proposition 4.6, we end up this section with the following lemma.

LEMMA 4.7. *Let $f : T \rightarrow T$ be an int-amplified endomorphism of a complex torus T . Let Z be a non-empty f^{-1} -periodic closed analytic subvariety of T . Then $Z = T$.*

Proof. Since f^s is int-amplified for any positive integer s (cf. Lemma 3.1), with f replaced by its power, we may assume Z is irreducible and $f^{-1}(Z) = Z$. It follows from the ramification divisor formula and the purity of branch loci that f is étale. Therefore we get $\deg f|_Z = \deg f$; and hence $Z = T$ by Proposition 4.6. \square

5. The quotients of complex tori. In this section, we deal with the case of Q -tori admitting int-amplified endomorphisms. Recall that a normal compact Kähler space X is said to be a Q -torus if there exists a complex torus (full rank) T and a finite surjective morphism $\pi : T \rightarrow X$ such that π is étale in codimension 1 (cf. [19, Notation 2.1]). Besides, we say that a finite surjective morphism $f : X \rightarrow Y$ is *quasi-étale* if f is étale in codimension 1. It follows from the ramification divisor formula that any surjective endomorphism of a Q -torus is quasi-étale.

To begin with this section, we extend the results in [33, Lemma 2.12] and [32, Proposition 4.3] to the following. Readers may refer to [2, Proposition 3] for its proof.

PROPOSITION 5.1. *Let X be a Q -torus. Then there exists a quasi-étale cover $\tau : T \rightarrow X$ satisfying the following conditions:*

- (1) *T is a complex torus;*
- (2) *τ is Galois;*
- (3) *If $\tau' : T' \rightarrow X$ is another quasi-étale cover from a torus T' , then there exists an étale morphism $\sigma : T' \rightarrow T$ such that $\tau' = \tau \circ \sigma$.*

In particular, we call the quasi-étale cover τ in Proposition 5.1 the Albanese closure of X in codimension one.

As a consequence of Proposition 5.1, we have the lifting lemma below. Then, Corollary 5.3 follows immediately from Proposition 4.3 and Lemma 5.2.

LEMMA 5.2. *Let $f : X \rightarrow X$ be a surjective endomorphism of a Q -torus X . Then there exist a complex torus T , a quasi-étale morphism $\tau : T \rightarrow X$ and an étale endomorphism $\sigma_T : T \rightarrow T$ such that $\tau \circ \sigma_T = f \circ \tau$.*

Proof. Note that f is quasi-étale by the ramification divisor formula. Let $\tau : T \rightarrow X$ be the Albanese closure as in Proposition 5.1. Since the composition $f \circ \tau$ is still quasi-étale, by Proposition 5.1 (3), such σ_T exists. \square

COROLLARY 5.3. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a Q -torus. Then, there exists a fixed point of f .*

Comparing with Lemma 4.7, we have the same argument for Q -tori. For the application of Lemma 5.4, see the proof of Theorem 1.5 in Section 8.

LEMMA 5.4. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a Q -torus X . Then, there is no f^{-1} -periodic proper subvariety of X .*

Proof. Let Z be a non-empty f^{-1} -periodic closed subset of X , i.e., $f^{-s}(Z) = Z$ for some $s > 0$. One needs to show that $Z = X$. Since X is a finite quotient of some complex torus T , it has only quotient and hence rational (cf. [25, Proposition 5.15]) singularities. Also, it follows from the ramification divisor formula that f is quasi-étale. By Lemma 5.2, there exists a quasi-étale cover $\tau : T \rightarrow X$ from a complex torus T . Since f is quasi-étale, so is the composition $f \circ \tau$. Then, one gets a surjective endomorphism $\sigma_T : T \rightarrow T$ such that $f \circ \tau = \tau \circ \sigma_T$. By Lemma 3.3, σ_T is int-amplified. Further, the commutative diagram gives that $\sigma_T^{-s}(\tau^{-1}(Z)) = \tau^{-1}(Z)$. Since σ_T^s is also int-amplified, $\tau^{-1}(Z) = T$ (cf. Lemma 4.7) and hence $Z = X$. \square

From now on, we aim to show that each dominant meromorphic map from a normal compact Kähler space with mild singularities to a Q -torus is holomorphic. We first give a generalized version of [35, Lemma 9.11] by weakening the condition. Readers are referred to [24, Lemma 8.1] for the projective case but the proofs are different. To prove Lemma 5.5, we need to recall the *Albanese torus* for a singular complex space ([15, Section 3]).

LEMMA 5.5. *Let $f : X \dashrightarrow T$ be a dominant meromorphic map from a normal compact Kähler space with only rational singularities to a torus T . Then, f is a morphism.*

Let X be a compact complex space. A holomorphic map $\alpha : X \rightarrow A$ to a complex torus A is said to be the *Albanese map* of X , and A is called the *Albanese torus* of X , if any map from X to a complex torus T factors uniquely through α . Note that the Albanese torus is unique up to isomorphism if it exists. Let $A = \text{Alb}(X)$ and $\alpha = \text{alb}_X$. If X is further assumed to have at worst rational singularities, then X admits an Albanese torus and $\text{Alb}(X) = \text{Alb}(\tilde{X})$, where $\pi : \tilde{X} \rightarrow X$ is a projective desingularization such that $\text{alb}_X \circ \pi = \text{alb}_{\tilde{X}}$ (cf. [15, Theorem 3.4]). Now, we are ready to prove Lemma 5.5.

Proof. [Proof of Lemma 5.5] Let $\pi : \tilde{X} \rightarrow X$ be a projective resolution, $\tilde{f} : \tilde{X} \dashrightarrow T$ the induced dominant meromorphic map, $\text{alb}_X : X \rightarrow \text{Alb}(X)$ and $\beta : \text{Alb}(X) \dashrightarrow T$ the induced map (cf. [15, Section 3]). Since \tilde{f} is a dominant meromorphic map from a complex manifold to a torus, by [35, Lemma 9.11], \tilde{f} is a morphism. Moreover, $\text{Alb}(X) = \text{Alb}(\tilde{X})$ and thus β is a morphism by [35, Definition 9.6 and Theorem 9.7]. Therefore, f is a morphism. \square

We end up this section with the following proposition. The proof is the same as in the projective setting (cf. [27, Lemma 5.3] or [26, Lemma 4.4]) except that we apply Lemma 5.5 instead of [27, Lemma 5.1].

PROPOSITION 5.6. *Let $\pi : X \dashrightarrow Y$ be a proper dominant meromorphic map from a normal compact Kähler space X to a Q -torus Y . Suppose that X has at worst Kawamata log terminal (klt) singularities and the normalization of the graph $\Gamma_{X/Y}$ is equi-dimensional over Y (this holds when the general fibre of π is connected, $f : X \rightarrow X$ is int-amplified and f descends to some int-amplified endomorphism $g : Y \rightarrow Y$). Then, π is a morphism.*

6. K_X pseudo-effective case: Proof of Theorem 1.4. Proceeding from this section, we will gradually fix our attention to the normal compact Kähler threefolds. In Section 8, we will discuss the equivariant MMP for threefolds and before that, some preparations are necessary. In this section, we consider the case when the canonical divisor is pseudo-effective. Now, we are ready to prove Theorem 1.4. We first divide

the theorem into three parts (cf. Propositions 6.1, 6.2 and 6.5) and then, we put them together to show Theorem 1.4 for the convenience of readers.

PROPOSITION 6.1. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a compact Kähler manifold of dimension $n \geq 1$ with pseudo-effective K_X . Then X is a Q -torus.*

Proof. Since $-K_X$ is pseudo-effective by Theorem 1.3 and K_X is pseudo-effective by assumption, $K_X \equiv 0$ in the sense of currents. Then, f is étale by the ramification divisor formula and Purity of the branch locus theorem of Grauert-Remmert (cf. [17]).

Fix a Kähler class $[\omega]$ on X . We claim that $c_2(X) \cdot [\omega]^{n-2} = 0$. Indeed, according to [16, Proposition 5.6], $(\deg f)c_2(X) \cdot [\omega]^{n-2} = c_2(X) \cdot (f^*[\omega])^{n-2} = c_2(X) \cdot f^*([\omega]^{n-2})$. Then,

$$c_2(X) \cdot [\omega]^{n-2} = c_2(X) \cdot \frac{f^*([\omega]^{n-2})}{\deg f} = c_2(X) \cdot \frac{(f^i)^*([\omega]^{n-2})}{(\deg f)^i} =: c_2(X) \cdot [x_i].$$

Here, $\lim_{i \rightarrow \infty} [x_i] \equiv_w 0$ by Lemma 3.4. Moreover, applying Lemma 4.4 for both $c_2(X)$ and $-c_2(X)$, there exist two positive closed $(2, 2)$ -currents T_1 and T_2 and a positive number A such that $c_2(X) + T_1 = A[\omega]^2$ and $-c_2(X) + T_2 = A[\omega]^2$ in the sense of currents. Then, we get the inequalities: $\lim_{i \rightarrow \infty} c_2(X) \cdot [x_i] \leq 0$ and $\lim_{i \rightarrow \infty} -c_2(X) \cdot [x_i] \leq 0$, which in turn force $c_2(X) \cdot [x_i]$ tends to 0 when $i \rightarrow \infty$. Therefore, we complete the proof of our claim.

Note that the vanishing of the first Chern class of X implies that we can find a Ricci flat metric (still denoted ω) in the Kähler class $[\omega]$ (cf. [38] and [39]). The vanishing $\int_X c_2(X) \wedge \omega^{n-2} = 0$ implies that the full curvature tensor of ω is identically zero. By the uniformisation theorem, the universal cover of X is an affine space and X is a quotient of a complex torus T by a finite group G acting freely on T . \square

PROPOSITION 6.2. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a normal compact Kähler threefold with at worst canonical singularities. Suppose that the canonical divisor K_X is pseudo-effective. Then X is a Q -torus.*

We refer the readers to [33, Theorem 3.3] and [26, Theorem 5.2] for the projective version while our present proof will be different.

Proof. To prove Proposition 6.2, we resort to [16, Theorem 1.1] by showing that $c_1(X) = 0 \in H^2(X, \mathbb{R})$ and $\tilde{c}_2(X) \cdot [\omega] = 0$ for some Kähler class $[\omega] \in H^2(X, \mathbb{R})$. Here $c_1(X)$ is the first Chern class and $\tilde{c}_2(X) \in H^2(X, \mathbb{R})^\vee$ is the orbifold second Chern class of X (cf. [16, Section 5]). Since $-K_X$ is pseudo-effective by Theorem 1.3 and K_X is pseudo-effective by assumption, K_X is numerically trivial in the sense of currents, i.e., $c_1(X)_\mathbb{R} = 0$. It follows from the ramification divisor formula $K_X = f^*K_X + R_f$ that R_f is zero and thus f is quasi-étale. Let $d := \deg f > 1$ and fix a Kähler class $[\omega]$ on X .

Now we prove that $\tilde{c}_2(X) \cdot [\omega] = 0$. Since f is quasi-étale, by [16, Proposition 5.6], we get $\tilde{c}_2(X) \cdot f^*[\omega] = d(\tilde{c}_2(X) \cdot [\omega])$. Then with the equality divided by d , we have

$$\tilde{c}_2(X) \cdot [\omega] = \tilde{c}_2(X) \cdot \frac{f^*[\omega]}{d} = \tilde{c}_2(X) \cdot \frac{(f^i)^*[\omega]}{d^i} =: \tilde{c}_2(X) \cdot [x_i],$$

where $\lim_{i \rightarrow \infty} [x_i] \equiv_w 0$ by Lemma 3.4. Note that $\tilde{c}_2(X) \in H^2(X, \mathbb{R})^\vee$ and we can regard it as an element in the dual of the Bott-Chern cohomology space and hence

$\tilde{c}_2(X) \in N_1(X)$ (cf. [21, Proposition 3.9]). Besides, for any normal compact Kähler threefold X , the nef cone $\text{Nef}(X)$ and the cone $\overline{\text{NA}}(X)$ generated by positive closed currents of bidimension $(1, 1)$ are dual (cf. [21, Proposition 3.15]). Therefore, $[\omega]^2$ is an interior point of $\overline{\text{NA}}(X)$ because as a linear function on $H_{\text{BC}}^{1,1}(X)$, $[\omega]^2$ is strictly positive on $\text{Nef}(X) \setminus \{0\}$ by Hodge-Riemann theorem (also cf. [12, Théorème 3.1]). Therefore, there exists $A > 0$ such that $A[\omega]^2 \geq \tilde{c}_2(X)$ and $A[\omega]^2 \geq -\tilde{c}_2(X)$ in the sense of currents. Hence, with the same argument as in the proof of Proposition 6.1, $\tilde{c}_2(X) \cdot [\omega] = \lim_{i \rightarrow \infty} \tilde{c}_2(X) \cdot [x_i] = 0$.

Then, applying the criterion in [16, Theorem 1.1], we see that X is a Q -torus. \square

LEMMA 6.3. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a normal compact Kähler surface with at worst klt singularities. Suppose that the canonical divisor K_X is pseudo-effective. Then X is a Q -torus.*

Proof. With the same arguments as in the previous proof (cf. Proposition 6.2), K_X is numerically trivial in the sense of currents, i.e., $c_1(X)_{\mathbb{R}} = 0$. Therefore, it follows from the ramification divisor formula that f is quasi-étale. Also, $\tilde{c}_2(X) = 0$ according to [16, Proposition 5.6]. Hence, X is a Q -torus (cf. [16, Proposition 7.2]). \square

LEMMA 6.4. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a normal compact Kähler non-uniruled surface such that K_X is \mathbb{Q} -Cartier. Then X is a Q -torus.*

Proof. By Theorem 1.3, $-K_X$ is pseudo-effective. Let $\pi : \tilde{X} \rightarrow X$ be a projective resolution. Since X and hence \tilde{X} are non-uniruled Kähler surfaces, $K_{\tilde{X}}$ is pseudo-effective by the classification of surfaces. Further, the equality $\pi_* K_{\tilde{X}} = K_X$ gives that K_X is also pseudo-effective. Therefore, $K_X \equiv 0$.

Since $\dim X = 2$, with the same proof as in [23, Lemma 2.4], X has only canonical singularities. Then it follows from Lemma 6.3 that X is a Q -torus. \square

As a consequence of Lemma 6.4, we prove the next proposition.

PROPOSITION 6.5. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a normal \mathbb{Q} -factorial compact Kähler surface X such that the canonical divisor K_X is pseudo-effective. Then X is a Q -torus.*

Proof. Since K_X is pseudo-effective, with the same reason as in previous proofs, $K_X \equiv 0$. Then, it follows from [14, Theorem 1.2] that K_X is semi-ample and $K_X \sim_{\mathbb{Q}} 0$. If X is non-uniruled, this is the case in Lemma 6.4. Therefore, we consider the case of X being uniruled. Suppose X is a uniruled surface. Then, X is bimeromorphic to a ruled surface which is algebraic. Thus, X is Moishezon by [1, Theorem 7.14]. Taking the global index-one log-canonical cover $\tilde{X} \rightarrow X$ associated with $K_X \sim_{\mathbb{Q}} 0$, we have the following results (cf. [29, Lemma 3.2.7] or [30, Lemma 4.21] and Lemma 3.3): (1) \tilde{X} is a normal Moishezon and Kähler surface; (2) $\tilde{X} \rightarrow X$ is étale in codimension one; (3) \tilde{X} has only rational double points; (4) $K_{\tilde{X}} \sim 0$; and (5) f lifts to an int-amplified endomorphism \tilde{f} of \tilde{X} . Therefore, it follows from [29, Proposition 7.3.1] or [31, Proposition 5.2] that \tilde{X} and hence X are Q -tori (cf. [16, Lemma 7.4]). \square

Proof of Theorem 1.4. Note that Theorem 1.4 is a direct consequence of Propositions 6.1, 6.2 and 6.5, which together prove the cases: manifolds, threefolds and surfaces respectively. \square

It is natural to consider whether Theorem 1.4 can be generalized to arbitrary dimensional spaces with mild singularities. For the normal projective setting, the

answer is affirmative (cf. [26, Theorem 5.2]). In our present context, it is still an open question.

We end up this section by assuming the following hypothesis when discussing the minimal model program for the case when K_X is pseudo-effective. With the assumption of **Hyp A**, we refer the readers to [21] for details.

Hyp A: X is a normal \mathbb{Q} -factorial compact Kähler threefold with at worst terminal singularities. Also, the canonical divisor K_X is pseudo-effective.

7. K_X not pseudo-effective case. In this short section, let X be a normal compact Kähler threefold with at worst canonical singularities. We refer the readers to [20, Introduction] for the arguments below.

By a theorem of Brunella [4] and [22, Theorem 4.2], the canonical divisor K_X is not pseudo-effective if and only if X is covered by rational curves. We shall deal with the uniruled case in the following. Let $\pi : \tilde{X} \rightarrow X$ be a projective resolution and $\tilde{X} \dashrightarrow Y$ the maximal rationally connected (MRC) fibration.

If $\dim Y = 0$, then \tilde{X} is algebraic (cf. [5]). We suppose that $\dim Y = 1$. Then, a general fibre F of the MRC fibration is smooth and rationally connected of dimension two in \tilde{X} . If $H^0(\tilde{X}, \Omega_{\tilde{X}}^2) \neq 0$, then there exists a global 2-form η on \tilde{X} and hence we get a non-zero holomorphic form on some rationally connected fibre F , a contradiction (cf. [7, Corollary 4.18]). So, applying Hodge theory, we have $H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ and also $H^2(\tilde{X}, \mathbb{C}) = H^{1,1}(\tilde{X}, \mathbb{C})$. By [15, Proposition 4.10], \tilde{X} is projective and thus X is Moishezon (cf. [1, Theorem 7.14]).

In both cases, X is Moishezon and also projective thanks to [34, Theorem 1.6]. In conclusion, together with [26, Theorem 1.10], we get the following lemma.

LEMMA 7.1. *Let X be a compact Kähler threefold with at worst canonical singularities. Suppose X is uniruled and the dimension of the base Y of the MRC fibration $\pi : X \dashrightarrow Y$ is at most 1. Then X is projective. In particular, if X further admits an int-amplified endomorphism, then there exists an f -equivariant MMP for X after iteration.*

From now on, when discussing the MMP for normal compact Kähler uniruled threefolds, we always assume the following hypothesis (cf. [20]).

Hyp B: X is a normal \mathbb{Q} -factorial compact Kähler threefold with at worst terminal singularities. Also, the canonical divisor K_X is not pseudo-effective and the base Y of the MRC fibration $\pi : X \dashrightarrow Y$ has dimension two.

8. Equivariant minimal model program: Proof of Theorem 1.5. In this section, we always assume **Hyp A** in Section 6 or **Hyp B** in Section 7 when discussing the MMP for Kähler threefolds. Before proving Theorem 1.5, we refer to [21, Definitions 3.19, 4.3 and 7.1] for the basic notation and respective properties of contraction morphisms, divisorial rays and small rays. Indeed, all of them are similar to those in projective settings. By [21, Remark 7.2], if the extremal ray R is small, then the corresponding Mori contraction contracts only finitely many curves. Moreover, [21] shows that if we assume **Hyp A** in Section 6 or **Hyp B** in Section 7, then after a contraction of either divisorial or small rays, the end product is still a normal \mathbb{Q} -factorial compact Kähler threefold with at worst terminal singularities. In addition, under the same assumption, [20] shows that if $\pi : X \rightarrow Y$ is a Mori fibre space, then Y is a normal \mathbb{Q} -factorial compact Kähler surface with at worst klt singularities.

In what follows, we slightly generalize [40, Lemma 2.11] to the following lemma. We shall skip the proofs since they are the same as the proofs in [40, Lemma 2.11] by applying Proposition 2.8 and [21, Proposition 3.9] to the present case.

LEMMA 8.1. *Let X be a normal compact Kähler space with at worst rational singularities and $f : X \rightarrow X$ a surjective endomorphism. Let $R_\Gamma := \mathbb{R}_{\geq 0}[\Gamma] \subseteq \overline{NA}(X)$ (which is the closed cone in $N_1(X)$ generated by the classes of positive closed currents of bidimension $(1, 1)$) be an extremal ray (not necessarily K_X -negative) with Γ a positive closed current of bidimension $(1, 1)$. Then, we have:*

- (1) $R_{f_*\Gamma}$ is an extremal ray;
- (2) If C_1 is a curve such that $[f(C_1)] \in R_\Gamma$, then $R_{[C_1]}$ is an extremal ray;
- (3) Denote by Σ_Γ the set of curves whose classes are in R_Γ . Then $f(\Sigma_\Gamma) = \Sigma_{f_*\Gamma}$.
- (4) $\Sigma_\Gamma = f^{-1}(\Sigma_{f_*\Gamma}) := \{C \text{ a curve} \mid f(C) \in \Sigma_{f_*\Gamma}\}$.

The result below is known in the projective case (cf. [26, Lemma 8.1]) and we show that it also holds in the analytic setting. We rewrite the proof in the analytic version by highlighting the differences and skipping the same parts. The main difference is that, here, we may not have integral ample divisors in the analytic setting. Nevertheless, since every analytic subvariety outside the singular locus determines an integral homology class, the idea of the previous proof is valid in the present case.

LEMMA 8.2. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a normal compact Kähler space, which is of dimension $n \geq 1$ and has at worst rational singularities. Suppose $A \subseteq X$ is a closed subvariety with $f^{-i}f^i(A) = A$ for all $i \geq 0$. Then $M(A) := \{f^i(A) \mid i \in \mathbb{Z}\}$ is a finite set.*

Proof. Let $M_{\geq 0}(A) := \{f^i(A) \mid i \geq 0\}$. With the same reason as in the first step of the proof in [26, Lemma 8.1], we only need to prove $M_{\geq 0}(A)$ is finite.

We show that $M_{\geq 0}(A)$ is a finite set by induction on the codimension of A in X . If $A = X$, there is nothing to prove. Suppose $k := \dim A < \dim X$ and $d := \deg f > 1$. Denote by Σ the union of $\text{Sing}(X)$, $f^{-1}(\text{Sing}(X))$ and the irreducible components in the ramification divisor R_f of f . Let $A_i := f^i(A)$ ($i \geq 0$). Note that $\dim A_i = \dim A = k$.

We claim that A_i is contained in Σ for infinitely many i . If not, with A replaced by some A_{i_0} , we may assume that A_i is not contained in Σ for all $i \geq 0$. Since $f^{-i-1}(A_{i+1}) = A$ and f^i is surjective, we have $f^{-1}(A_{i+1}) = A_i$. Besides, f is unramified on each A_i , and hence $\deg f|_{A_i} = \deg f$. Fix a Kähler form ξ on X . Then, by projection formula, $(\deg f)(\xi^k \cdot [A_{i+1}]) = (\deg f|_{A_i})(\xi^k \cdot [A_{i+1}]) = (f^*\xi)^k \cdot [A_i]$. Substituting this expression repeatedly, we have

$$\xi^k \cdot [A_{i+1}] = \frac{((f^i)^*\xi)^k \cdot [A_1]}{d^i}. \tag{3}$$

On the one hand, all of these A_i are integral subvarieties of dimension k and not contained in the singular locus by assumption. Also, the homology classes of these $A_i \subseteq X$ form a discrete set (denoted by $\text{NE}_k(X)$), which is contained in the integral lattice $H_{2k}(X, \mathbb{Z})/\text{tors}$. In addition, ξ^k defines a norm on the closure of $\text{NE}_k(X)$ and ξ^k is strictly positive on $\overline{\text{NE}}_k(X) \setminus \{0\}$. **Thus, there exists a real constant $\mu > 0$ such that $\xi^k \cdot [A_i] \geq \mu$ for every analytic subvariety A_i of dimension k , which is not contained in the singular locus.** As a result, the left hand side of Equation (3) is always no less than μ .

On the other hand, by Lemma 4.5, $\lim_{i \rightarrow \infty} \xi^k \cdot [A_{i+1}] = 0$. Therefore, taking the

upper limit of Equation (3), we get a contradiction and prove the claim. The remaining proof is the same as [26, Lemma 8.1] after replacing [26, Lemma 2.3] by Lemma 4.1. \square

Let $f : X \rightarrow X$ be a surjective endomorphism of a compact Kähler manifold X . Suppose $\pi : X \dashrightarrow Y$ is a meromorphic map to a complex variety. In [19, Propositions 2.9 and 2.12], Höring and Peternell gave us nice criteria for f to descend as a (meromorphic) endomorphism of Y in terms of Albanese map, Iitaka fibration, rationally connected quotient or algebraic reduction.

Now, it is natural to ask whether there exists an equivariant descending for the contraction morphism of a K_X -negative extremal ray. Actually, it has an affirmative answer under the assumption of **Hyp A** in Section 6 or **Hyp B** in Section 7 (cf. Lemma 8.3). The proof is the same as in [27, Lemma 6.2] for the projective setting after replacing [27, Lemma 2.13] by Lemma 8.1 and replacing [25, Corollary 3.17] by [21, Proposition 8.1].

LEMMA 8.3. *Suppose either **Hyp A** in Section 6 or **Hyp B** in Section 7. Let f be a surjective endomorphism of X and $\pi : X \rightarrow Y$ a contraction of a K_X -negative extremal ray $R_\Gamma := \mathbb{R}_{\geq 0}[\Gamma]$ generated by a positive closed $(2, 2)$ -current Γ . Suppose further that $E \subseteq X$ is an analytic subvariety such that $\dim(\pi(E)) < \dim E$ and $f^{-1}(E) = E$. Then, up to replacing f by its power, $f(R_\Gamma) = R_\Gamma$ (hence, for any curve C such that $[C] \in R_\Gamma$, its image $[f(C)]$ still lies in R_Γ). Therefore, the contraction π is f -equivariant, i.e., f descends to a surjective endomorphism g on Y .*

Lemma 8.3 is also valid for any Kähler surface with klt singularities. Besides, Lemma 8.3 applies when π is a Mori fibre space (with $E = X$): in that case, π is f^s -equivariant for some $s > 0$.

PROPOSITION 8.4. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a normal \mathbb{Q} -factorial compact Kähler threefold X with at worst terminal singularities. If $\pi : X \rightarrow Y$ is a divisorial contraction of a K_X -negative extremal ray $R_\Gamma := \mathbb{R}_{\geq 0}[\Gamma]$ generated by a positive closed $(2, 2)$ -current Γ . Then f^s induces an int-amplified endomorphism of Y for some $s > 0$.*

Proof. Let $S := \bigcup_{C \subseteq X, [C] \in R_\Gamma} C$. By [21, Lemma 7.5], S is an irreducible Moishezon surface and $S \cdot C < 0$ for all curves C with $[C] \in R_\Gamma$. By Lemma 8.1, $f^{-i}f^i(S) = S$ for all $i \geq 0$, since f^i is surjective. Then, it follows from Lemma 8.2 that $M_{\geq 0}(S)$ is a finite set. So, we may assume $f^{-1}(S) = S$ with f replaced by its power. Also, π is f^s -equivariant for some $s > 0$ (cf. Lemma 8.3). By Lemma 3.2, the induced endomorphism on Y is int-amplified. \square

Indeed, without Lemma 8.2, the finiteness of the number of such S in Proposition 8.4 also comes from the fact that all of these Moishezon surfaces consist of the integral part of the Zariski decomposition for K_X (cf. [21, Section 4.B. and the proof of Lemma 7.5]).

The following result is the analytic version of [40, Lemma 3.6] (cf. [28, (0.4.1)] for the existence of flips of threefolds), the proof of which is the same since each contraction morphism of compact Kähler threefolds is projective ($-K_X$ is π -ample).

LEMMA 8.5. *Let $f : X \rightarrow X$ be a surjective endomorphism of a normal \mathbb{Q} -factorial compact Kähler threefold X with at worst terminal singularities and $\sigma : X \dashrightarrow X'$ a flip with $\pi : X \rightarrow Y$ the corresponding flipping contraction of a K_X -negative extremal ray $R_\Gamma := \mathbb{R}_{\geq 0}[\Gamma]$ generated by some positive closed $(2, 2)$ -current Γ . Suppose that $R_{f_*\Gamma} = R_\Gamma$. Then the dominant meromorphic map $f^+ : X^+ \dashrightarrow X^+$*

induced from f , is holomorphic. Both f and f^+ descend to the same endomorphism of Y .

PROPOSITION 8.6. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a normal \mathbb{Q} -factorial compact Kähler threefold X with at worst terminal singularities and $\sigma : X \dashrightarrow X'$ a flip with $\pi : X \rightarrow Y$ the corresponding flipping contraction of a K_X -negative extremal ray $R_\Gamma := \mathbb{R}_{\geq 0}[\Gamma]$ generated by some positive closed current Γ of bidimension $(1, 1)$. Then there exists a K_X -flip of π , $\pi^+ : X^+ \rightarrow Y$ such that $\pi = \pi^+ \circ \sigma$ and for some $s > 0$, the commutativity is f^s -equivariant.*

Proof. The proof is the same as [27, Lemma 6.5] after replacing [27, Lemmas 2.13, 6.1, 6.2 and 6.6] by Lemmas 8.1, 8.2, 8.3 and 8.5. Also, Lemmas 3.2 and 3.3 show that the induced endomorphism f^+ in Lemma 8.5 is int-amplified. \square

The next Theorem follows from Lemmas 8.3, 8.5 and Propositions 8.4, 8.6.

THEOREM 8.7. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a \mathbb{Q} -factorial compact Kähler space X with at worst terminal singularities. Let $\pi : X \dashrightarrow Y$ be a dominant rational map which is either a divisorial contraction or a Fano contraction or a flip induced by a K_X -negative extremal ray R_Γ . Then there exists an int-amplified endomorphism $g : Y \rightarrow Y$ such that $g \circ \pi = \pi \circ f$ with f replaced by its power.*

Proof of Theorem 1.5. First, if K_X is pseudo-effective, then $X = Y$ is a Q -torus by Proposition 6.2. Next, we consider the case when K_X is not pseudo-effective. If the base of the MRC fibration $X \dashrightarrow Z$ has dimension ≤ 1 , then Theorem 1.5 follows from Lemma 7.1 and [26, Theorem 1.10], since X is projective in this case.

Now, we assume **Hyp B** in Section 7. By [20, Theorem 1.1], we may run the MMP for finitely many steps: $X = X_1 \dashrightarrow \cdots \dashrightarrow X_j$ consisting of contractions of extremal rays and flips, such that X_j admits a Mori fibre space $X_j \rightarrow S = X_{j+1}$ onto a normal \mathbb{Q} -factorial compact Kähler surface. Note that S has at worst klt singularities (cf. [20, Theorem 1.1]). Note also that S is bimeromorphic to the base of the MRC fibration of X ; hence S is non-uniruled and K_S is pseudo-effective by considering a resolution of S . By Proposition 6.5, S is a Q -torus, which is our end product. Replacing f by its power, the above sequence is f -equivariant by Theorem 8.7.

By Proposition 5.6, the composition $u_i : X_i \dashrightarrow Y$ is a morphism for each i . Also, for each small contraction $\pi_i : X_i \rightarrow Z_i$, we claim that **the meromorphic map $Z_i \dashrightarrow Y$ is also a morphism**. In the following, let $\text{Exc}(\pi_i)$ denote the exceptional locus of π_i .

If the claim does not hold, then by the rigidity lemma, there exists a curve C on X_i with $[C] \in R$ such that $u_i(C)$ is a curve on Y . Here, R is a K_{X_i} -negative small ray and π_i is the small contraction of R . Recall that, under **Hyp B**, Y is a Q -torus of dimension 2. By Lemmas 8.1 and 8.2, with f replaced by its power, $f^{-1}(\text{Exc}(\pi_i)) = \text{Exc}(\pi_i)$. Since $\dim \text{Exc}(\pi_i) = 1$, with f further replaced by its power, $u_i(C)$ is totally invariant under the induced int-amplified endomorphism $Y \rightarrow Y$ (cf. [6, Lemma 7.5]), a contradiction to Lemma 5.4. Thus, $Z_i \rightarrow Y$ is also a morphism and the MMP is over Y .

By [27, Lemmas 2.6 and 5.2], $X_i \rightarrow Y$ is equi-dimensional with every fibre being irreducible and rationally connected. Also, K_{X_i} is not pseudo-effective for any $i < r$, otherwise X_i is a Q -torus for some $i < r$ and the MMP ends. Therefore, we complete the proof of Theorem 1.5. \square

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