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HARISH-CHANDRA MODULES OVER INVARIANT SUBALGEBRAS IN A SKEW-GROUP RING*

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Abstract. We construct a new class of algebras resembling enveloping algebras and generalizing orthogonal Gelfand-Zeitlin algebras and rational Galois algebras studied by [EMV, FGRZ, RZ, Har]. The algebras are defined via a geometric realization in terms of sheaves of functions invariant under an action of a finite group. A natural class of modules over these algebra can be constructed via a similar geometric realization. In the special case of a local reflection group, these modules are shown to have an explicit basis, generalizing similar results for orthogonal Gelfand-Zeitlin algebras from [EMV] and for rational Galois algebras from [FGRZ]. We also construct a family of canonical simple Harish-Chandra modules and give sufficient conditions for simplicity of some modules.

Key words. Gelfand-Zeitlin modules, invariant polynomial, Gelfand-Zeitlin algebras, rational Galois algebras, Harish-Chandra modules.

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1. Introduction. In the last decade there was a significant progress in understanding infinite dimensional simple modules over the Lie algebra $\mathfrak{gl}_n$, see e.g. [FGR, Ni1, Ni2, EMV] and references therein. An essential part of this progress is related to the study of so-called *Gelfand-Zeitlin modules* which originate from [DOF] based on [GZ] (see [EMV] for a detailed literature overview on Gelfand-Zeitlin modules). Various approaches to the study of Gelfand-Zeitlin modules rely on different realizations of the universal enveloping algebras which led to a number of generalizations of such algebras. These include *orthogonal Gelfand-Zeitlin algebras* introduced in [Ma] and *Galois algebras* introduced in [FO]. These generalizations include also finite W-algebras of type A, see [Ar, Har], and were studied in, in particular, [EMV, Har, FGRZ, RZ]. The recent preprint [KTWWY] establishes a relation between orthogonal Gelfand-Zeitlin algebras and Khovanov-Lauda-Rouquier algebras from [KL, Ro] and, in particular, leads to a (not very explicit) classification of simple Gelfand-Zeitlin modules over orthogonal Gelfand-Zeitlin algebras.

In the present paper we define and study a simultaneous geometric generalization of orthogonal Gelfand-Zeitlin algebras and Galois algebras. Both our construction and methods of study are inspired by the geometric approach of [Vi1, Vi2] to singular Gelfand-Zeitlin modules and is formulated in elementary sheaf-theoretic terms. To any semidirect product $G \ltimes V$ of a finite group G and a complex-analytic or linear algebraic group V, we associate the corresponding skew-group ring $\mathcal{S}$, see Subsection 2.1 for a precise definition. We denote by $\mathcal{O}$ the sheaf of holomorphic or polynomial functions on V. There is a natural action of G on $\mathcal{O}$ and it is natural to consider the sheaf $\mathcal{O}^G$ of G-invariants in $\mathcal{O}$. Main protagonists of the present paper are subalgebras in $\mathcal{S}$ that preserve the sheaf $\mathcal{O}^G$. Orthogonal Gelfand-Zeitlin algebras, Galois algebras, finite W-algebras and Galois orders, studied in [EMV, FGRZ, RZ, Har], are all special cases of our construction. In a special case which we call *standard algebras of type A*,

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we give an explicit description of our algebras as subalgebras in the *universal ring* as introduced in [Vi2]. Our geometric approach also naturally provides a construction of a large family of (simple) modules over our algebras, generalizing [Vi1, Vi2, EMV]. We note that, the general case of our construction seems to be outside the scope of *Harish-Chandra subalgebras and Gelfand-Zeitlin modules* as defined in [DFO]. However, it still fits into the general Harish-Chandra setup which studies modules over some algebra on which a certain subalgebra acts locally finitely. In particular, our results significantly generalize and simplify many results from [FGRZ].

The paper is organized as follows: Section 2 contains a description of our setup and preliminaries. Section 3 defines and provides basic structure results for our algebras. Sections 4, 5 and 7 study in detail the spacial case of rational Galois orders. Sections 6 describes applications of our approach to the study of Gelfand-Zeitlin modules. Finally, in Sections 8 we construct canonical simple Harish-Chandra modules over our algebras and give a sufficient condition for simplicity of these modules.

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2. Preliminaries.

2.1. Skew-group ring. Throughout the paper we work over the field $\mathbb{C}$ of complex numbers. Let G and V be two complex-analytic or linear algebraic Lie groups such that G acts on V. Let $G \ltimes V$ be the corresponding semidirect product. To simplify notations we will write G and V for subgroups $G \times \{e\}$ and $\{e\} \times V$ in $G \ltimes V$, respectively. On V we have a free transitive action of V by left translations ϕ_ξ, where $\xi \in V$, and an action of G given by $v \mapsto g \cdot v = gvg^{-1}$. Both actions are assumed to be holomorphic or algebraic. Note that $e \in V$ is a fixed point of the action of G. Let $\daleth$ be a subgroup of V, which we keep fixed throughout the paper.

Denote by $\mathbb{C}\daleth$ the group algebra of $\daleth$. Consider the vector space of global meromorphic (or rational) sections of the trivial vector bundle $V \times \mathbb{C}\daleth \to V$. We will denote this vector space by $\mathcal{S}(\daleth)$ or simply by $\mathcal{S}$, if $\daleth$ is clear from the context. We assume that any section of $\mathcal{S}(\daleth)$ has the form $f = \sum\limits_{i=1}^{s} f_i \phi_{\xi_i}$, where f_i are meromorphic (or rational) functions on V, $\xi_i \in \daleth$ and $s < \infty$. The vector space $\mathcal{S}(\daleth)$ has the natural structure of a skew-group ring defined in the following way:

$$\sum_i f_i \phi_{\xi_i} \circ \sum_j f'_j \phi_{\xi'_j} = \sum_{i,j} f_i \phi_{\xi_i}(f'_j) \phi_{\xi_i \circ \xi'_j}.$$

Here, by definition, $\phi_{\xi_i}(f'_j)(x) := f'_j(\xi_i^{-1}(x))$ for any $x \in V$. Clearly, for any subgroup $\daleth' \subset \daleth$ the ring $\mathcal{S}(\daleth')$ can be viewed as a subring in $\mathcal{S}(\daleth)$ in the obvious way.

The action of G on V induces an action of G on $\mathcal{S}(V)$ and also an action of G on $S(\daleth)$ provided that $\daleth$ is G-invariant. More precisely, $g \cdot f\phi_\xi = (g \cdot f)\phi_{g\xi g^{-1}}$ and $g \cdot f$ is a function on V defined as follows $g \cdot f(v) = f(g^{-1} \cdot v)$ for $v \in V$. Let $\daleth$ be G-invariant. Then we have the subring $\mathcal{S}(\daleth)^G$ of G-invariant sections of $\mathcal{S}(\daleth)$. Denote by $\mathcal{M}$ and by $\mathcal{O}$ the sheaves of meromorphic (or rational) and holomorphic (or polynomial) functions on V, respectively. For any $v \in V$, we denote by $\mathcal{M}_v$ and

$\mathcal{O}_v$ the corresponding algebras of germs of meromorphic and holomorphic functions at v. We put

$$\mathfrak{M} := \bigoplus_{x\in V} \mathcal{M}_x, \quad \mathfrak{O} := \bigoplus_{x\in V} \mathcal{O}_x.$$

If $W \subset V$ is a subset, we set $\mathfrak{M}|_W := \bigoplus_{x\in W} \mathcal{M}_x$ and $\mathfrak{O}|_W := \bigoplus_{x\in W} \mathcal{O}_x$.

The ring $\mathcal{S}(V)$ acts on the vector space $\mathfrak{M}$ in the following way:

$$f\phi_\xi : \mathcal{M}_v \to \mathcal{M}_{\xi(v)}, \quad F_v \mapsto (f\phi_\xi(F_v))_{\xi(v)}.$$

Consequently, the ring $\mathcal{S}(\beth)$ acts on the vector space $\mathfrak{M}|_{\beth\cdot v}$, where $v \in V$ and $\beth \cdot v$ is the $\beth$-orbit of v. Note that, in general, we do not have any action of $\mathcal{S}$ on $\mathfrak{O}$, since sections of $\mathcal{S}$ are assumed to be meromorphic (resp. rational) and not holomorphic (resp. polynomial).

In case we need to distinguish complex-analytic and algebraic categories, we will use the subscripts C and A, respectively. For example, we will write $\mathcal{S}_{\mathsf{C}}$ and $\mathcal{O}_{\mathsf{C}}$ to specify that we are working with meromorphic sections of our skew-ring and with holomorphic functions on V.

2.2. Example: the classical Gelfand-Zeitlin operators. For $n \geq 2$, denote by V the vector space

$$V = \mathbb{C}^{n(n+1)/2} = \{(v_{ki}) \mid 1 \leq i \leq k \leq n\}.$$

An element of V is called a *Gelfand-Zeitlin tableaux.* Let $\beth \simeq \mathbb{Z}^{n(n-1)/2}$ be the subgroup of V generated by Kronecker vectors $\delta^{st} = (\delta^{st}_{ki})$, where k and i are as above, $1 \leq t \leq s \leq n-1$ and $\delta^{st}_{ki} = 1$, if $k = s$ and $i = t$, and $\delta^{st}_{ki} = 0$ otherwise. The product $G = S_1 \times S_2 \times \cdots \times S_n$ of symmetric groups acts on V in the following way: the element $s = (s_1, \ldots, s_n) \in G$ acts on $v = (v_{ki})$ via $(s(v))_{ki} = v_{ks_k(i)}$. For $a \in \mathbb{C}$, set $\xi^a_k = a\delta^{k1}$. Consider the following elements in $\mathcal{S}(\beth)^G$:

$$E_{k,k+1} = \sum_{g\in G} g \cdot \frac{\prod_{j=1}^{k+1} (v_{k1} - v_{k+1,j})}{\prod_{j=2}^{k} (v_{k1} - v_{kj})} \phi_{\xi^1_k}; \quad E_{k+1,k} = \sum_{g\in G} g \cdot \frac{\prod_{j=1}^{k-1} (v_{k1} - v_{k-1,j})}{\prod_{j=2}^{k} (v_{k1} - v_{kj})} \phi^{-1}_{\xi^1_k};$$

$$E_{kk} = \sum_{i=1}^{k} (v_{ki} + i - 1) - \sum_{i=1}^{k-1} (v_{k-1,i} + i - 1).$$

The subalgebra $U \subset \mathcal{S}(\beth)^G$ generated by E_{st} is isomorphic to universal enveloping algebra $\mathcal{U}(\mathfrak{gl}_n(\mathbb{C}))$, see e.g. [DFO, Ma] for details.

2.3. Orthogonal Gelfand-Zeitlin algebras. Orthogonal Gelfand-Zeitlin algebras are generalizations of $\mathcal{U}(\mathfrak{gl}_n(\mathbb{C}))$ introduced in [Ma]. Fix a positive integer $n \geq 2$ and let n_k, where $k = 1, \ldots, n$, be positive integers. Denote by V the following vector space

$$V = \mathbb{C}^{\sum_k n_k} = \{v = (v_{ki}) \mid 1 \leq i \leq n_k,\ 1 \leq k \leq n\}.$$

Let $\beth \simeq \mathbb{Z}^{\sum_k n_k}$ be the subgroup of V generated by $\delta^{st} = (\delta^{st}_{ki})$, where $1 \leq t \leq n_k$, $1 \leq s \leq n$, as in Subsection 2.2. The group $G = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_n}$ acts on V as in Subsection 2.2 which defines the ring $\mathcal{S}$ and its subring $\mathcal{S}^G$.

With ξ_k^1 defined as in Subsection 2.2, an *orthogonal Gelfand-Zeitlin algebra* is a subalgebra in $\mathcal{S}^G$ generated by all G-invariant polynomials on V and by the elements

$$E_k = \sum_{g\in G} g \cdot \frac{\prod_{j=1}^{n_{k+1}} (v_{k1} - v_{k+1,j})}{\prod_{j=2}^{n_k} (v_{k1} - v_{kj})} \phi_{\xi_k^1}; \quad F_k = \sum_{g\in G} g \cdot \frac{\prod_{j=1}^{n_{k-1}} (v_{k1} - v_{k-1,j})}{\prod_{j=2}^{n_k} (v_{k1} - v_{kj})} \phi_{\xi_k^1}^{-1}.$$

The algebra $\mathcal{U}(\mathfrak{gl}_n(\mathbb{C}))$ from Subsection 2.2 is just a special case of this construction, for $n_k = k$.

Note that the generators E_k and F_k of the orthogonal Gelfand-Zeitlin algebra are rational (and not polynomial), however, it was shown in [EMV, Proposition 1] that the operators E_k and F_k preserve the vector space $H^0(V, \mathcal{O})^G$. By [W], orthogonal Gelfand-Zeitlin algebras are related to shifted Yangians and generalized W-algebras in type A, see also [KTWWY].

2.4. Standard algebras of type $\mathbb{A}$. Let V and G be as in Section 2.3. An element A in $\mathcal{S}^G$ is called *standard* if $A = \sum_{g\in G} g \cdot (f\phi_{\xi_k^a})$, where $a \in \mathbb{C}$.

Definition 1. *A subalgebra $\mathcal{A} \subset \mathcal{S}(V)^G$ is called standard of type $\mathbb{A}$ if $\mathcal{A}$ is generated by linear combinations of standard elements.*

Orthogonal Gelfand-Zeitlin algebras are examples of standard algebras of type $\mathbb{A}$. Other examples of such algebras are: finite W-algebras of type A and, more general, standard Galois orders of type A, see [FGRZ, Section 8] or [Har] for definition. In Section 5 we will show that standard algebras of type $\mathbb{A}$ that preserve the vector space $\mathfrak{O}^G$ are exactly standard Galois orders of type A.

2.5. Harish-Chandra modules. In this paper we will study modules which fit into the general philosophy of *Harish-Chandra modules*. Let $\mathcal{A} \subset \mathcal{S}^G$ be a subalgebra containing, as a subalgebra, the algebra $\mathcal{B}$ of all global G-invariant functions on V.

Definition 2. *We say that an $\mathcal{A}$-module M is a Harish-Chandra module provided that the action of $\mathcal{B}$ on M is locally finite.*

Gelfand-Zeitlin modules for orthogonal Gelfand-Zeitlin algebras and Galois orders, studied in [EMV, FGRZ, Vi1, Vi2] are examples of Harish-Chandra modules.

3. Algebras preserving the vector space $\mathfrak{O}^G$ and their modules.

3.1. A fibration corresponding to the sheaf of invariant functions. Consider a semidirect product $G \ltimes V$, where G is a finite group and V is a complex-analytic or linear algebraic group. As above we denote by $\mathcal{O}$ the structure sheaf of the complex-analytic (or algebraic) variety V. In other words, we assume that all sections of $\mathcal{O}$ are holomorphic or polynomial functions on V, respectively. We now define the sheaf $\mathcal{O}^G$ of G-invariant holomorphic (or polynomial) functions on V. For a G-invariant open set U in V, we let $\mathcal{O}^G(U)$ be the algebra of G-invariant holomorphic (or polynomial) functions on U. Below we will consider the algebra $\mathcal{O}_v^G$ of germs of G-invariant functions at a point $v \in V$. By definition, $\mathcal{O}_v^G$ is the algebra of germs $f \in \mathcal{O}_v$ such that there exists a G-invariant function $F \in \mathcal{O}_{G\cdot v}$ that has the germ f at the point v. We put

$$\mathfrak{M}^G := \bigoplus_{\bar{x}\in V/G} \mathcal{M}_{\bar{x}}^G, \quad \mathfrak{O}^G := \bigoplus_{\bar{x}\in V/G} \mathcal{O}_{\bar{x}}^G.$$

If $W \subset V$ is a G-invariant subset, we set $\mathfrak{M}^G|_W := \bigoplus_{\bar{x}\in W/G} \mathcal{M}_{\bar{x}}$ and $\mathfrak{O}^G|_W := \bigoplus_{\bar{x}\in W/G} \mathcal{O}_{\bar{x}}$.

If v is a fixed point of the action of G, then the algebra $\mathcal{O}_v^G$ is invariant with respect to the action of G. The group G has at least one fixed point, namely, the identity $e \in V$. Consider the algebra $\mathcal{O}_e$ of germs of functions at the point e and its G-invariant subalgebra $\mathcal{O}_e^G \subset \mathcal{O}_e$. Denote by $\mathcal{J}_e$ the ideal in $\mathcal{O}_e$ generated by functions from $\mathcal{O}_e^G$ that are equal to 0 at e. As above, we denote by $\phi_\xi : \mathcal{O}_x \to \mathcal{O}_{\xi x}$, $f \mapsto \phi_\xi(f) = f \circ \xi^{-1}$, the left translation by $\xi \in V$.

Lemma 3. *Let G be a finite group, $\xi \in V$, G_ξ the stabilizer of ξ and $\xi G \xi^{-1} \subset G \ltimes V$ the group obtained from G by conjugation with ξ. We have*

$$\phi_\xi(\mathcal{O}_e^{G_\xi}) = \mathcal{O}_\xi^G \quad \textit{and} \quad \phi_\xi(\mathcal{O}_e^G) = \mathcal{O}_\xi^{\xi G \xi^{-1}}.$$

In particular, we have

$$\phi_\xi(\mathcal{O}_e^{G_\xi}/(\mathcal{O}_e^{G_\xi} \cap \mathcal{J}_e)) = \mathcal{O}_\xi^G/\langle \mathcal{O}_\xi^G \cap (\mathcal{O}_\xi^{\xi G \xi^{-1}})^+\rangle,$$

where the superscript $+$ means that we consider all functions from $\mathcal{O}_\xi^{\xi G\xi^{-1}}$ that are equal to 0 at ξ and $\langle \mathcal{O}_\xi^G \cap (\mathcal{O}_\xi^{\xi G \xi^{-1}})^+\rangle$ denotes the ideal in $\mathcal{O}_\xi^G$ generated by $\mathcal{O}_\xi^G \cap (\mathcal{O}_\xi^{\xi G \xi^{-1}})^+$.

Proof. First of all, we note that $\mathcal{O}_\xi^G = \mathcal{O}_\xi^{G_\xi}$. Indeed, if $f \in \mathcal{O}_\xi^G$, then, clearly, $f \in \mathcal{O}_\xi^{G_\xi}$. Further, if $f \in \mathcal{O}_\xi^{G_\xi}$, then the sum of germs $\sum_{g\in G} g(f)$ is an element of $\bigoplus_{g\in G} \mathcal{O}^G_{g\cdot\xi}$. Therefore $f \in \mathcal{O}_\xi^G$. Furthermore, the sheaf isomorphism $\phi_\xi : \mathcal{O}_e \to \mathcal{O}_\xi$ is G_ξ-equivariant. Therefore, $\phi_\xi(\mathcal{O}_e^{G_\xi}) = \mathcal{O}_\xi^{G_\xi}$. The second and the third statements are clear, details are left to the reader. □

The above defines the vector space $\mathcal{O}_e/\mathcal{J}_e$ and its subspaces $\mathcal{O}_e^{G_\xi}/(\mathcal{O}_e^{G_\xi} \cap \mathcal{J}_e)$, for any $\xi \in V$. Consider the following correspondence:

$$V \ni \xi \longmapsto \mathbb{E}_\xi := \mathcal{O}_\xi^G/\langle \mathcal{O}_\xi^G \cap (\mathcal{O}_\xi^{\xi G \xi^{-1}})^+\rangle = \phi_\xi(\mathcal{O}_e^{G_\xi}/(\mathcal{O}_e^{G_\xi} \cap \mathcal{J}_e)).$$

This correspondence defines a fibration $\mathbb{E} = (\mathbb{E}_\xi)_{\xi\in V}$ of vector spaces over V. The group G acts naturally on the fibration $\mathbb{E}$. Indeed, if $f \in \mathcal{O}_\xi^G$, then $g \cdot f \in \mathcal{O}^G_{g\cdot\xi}$, and if $f \in (\mathcal{O}_\xi^{\xi G\xi^{-1}})^+$, then $g \cdot f \in (\mathcal{O}_{g\cdot\xi}^{g\cdot\xi G(g\cdot\xi)^{-1}})^+$. Now we can define the fibration $\mathbb{E}^G = (\mathbb{E}^G_{\bar{\xi}})_{\bar{\xi}\in V/G}$ on V/G in the following way: $\mathbb{E}^G_{\bar{\xi}}$ is the vector space of all G-invariant elements from $\bigoplus_{\xi'\in\bar{\xi}} \mathbb{E}_{\xi'}$. We set

$$\mathfrak{E} := \bigoplus_{x\in V} \mathbb{E}_x, \quad \mathfrak{E}|_{W'} := \bigoplus_{x\in W'} \mathbb{E}_x, \quad \mathfrak{E}^G := \bigoplus_{\bar{x}\in V/G} \mathbb{E}^G_{\bar{x}}, \quad \mathfrak{E}^G|_W := \bigoplus_{\bar{x}\in W/G} \mathbb{E}^G_{\bar{x}}$$

for a subset $W' \subset V$ and for a G-invariant subset $W \subset V$.

3.2. Action of elements in $\mathcal{S}$ on $\mathfrak{E}$. Let $\daleth$ be a G-invariant subgroup in V and $v \in V$. Consider the G-invariant subset $(G \ltimes \daleth) \cdot v$. Then $\mathfrak{E}^G|_{(G\ltimes\daleth)\cdot v/G}$ is defined. Take an element A in $\mathcal{S}$ preserving the vector space $\mathfrak{D}^G|_{(G\ltimes\daleth)\cdot v/G}$. Any such A has the following form on $\mathfrak{D}^G|_{(G\ltimes\daleth)\cdot v/G}$:

$$A|_{\mathfrak{D}^G|_{(G\ltimes\daleth)\cdot v/G}} = \sum_i \sum_{h\in G} h \cdot (f_i \phi_{\xi_i}). \tag{1}$$

Note that A may be meromorphic. Also, we do not assume that $A(\mathfrak{D}) \subset \mathfrak{D}$.

THEOREM 4. *Assume that G is a finite group and A sends $\mathfrak{D}^G|_{(G\ltimes\daleth)\cdot v/G}$ to itself. Then the action of A on $\mathfrak{D}^G|_{(G\ltimes\daleth)\cdot v/G}$ induces an action of A on $\mathfrak{E}^G|_{(G\ltimes\daleth)\cdot v/G}$.*

Proof. The element $A \in \mathcal{S}$ acts on $\mathfrak{D}^G|_{(G\ltimes\daleth)\cdot v/G}$. We need to show that this action induces an action on $\mathfrak{E}^G|_{(G\ltimes\daleth)\cdot v/G}$, or, equivalently, that it induces an action on the vector space

$$\bigoplus_{\bar{\xi}\in(G\ltimes\daleth)\cdot v/G} \Big[\bigoplus_{\xi'\in\bar{\xi}} \mathcal{O}^G_{\xi'}/\phi_{\xi'}(\mathcal{O}_e^{G_{\xi'}} \cap \mathcal{J}_e)\Big]^G.$$

In other words, we need to show that $A(F)$, where $F \in [\bigoplus_{g\in G} \phi_{g\cdot\xi}(\mathcal{O}_e^{G_{g\cdot\xi}} \cap \mathcal{J}_e)]^G$, is a sum of elements from $[\bigoplus_{g\in G} \phi_{g\cdot\xi'}(\mathcal{O}_e^{G_{g\cdot\xi'}} \cap \mathcal{J}_e)]^G$ for various ξ'. Let us take F such that there exists $F' \in \mathcal{O}^G_e$ and $X \in \mathcal{O}_e^{G_{g\cdot\xi}}$ with

$$F = \sum_{g\in G} (F' \circ g\cdot\xi^{-1})[g\cdot(X\circ\xi^{-1})].$$

Note that F' is either in the ideal $\mathcal{J}_e$ or is an invertible G-invariant element. We have

$$A(F) = \sum_i \sum_{h,g\in G} (h\cdot f_i) F' \circ (g\cdot\xi^{-1}\circ h\cdot\xi_i^{-1})[g\cdot(X)\circ g\cdot\xi^{-1}\circ h\cdot\xi_i^{-1}].$$

This is a sum of G-invariant germs supported at the points $h\cdot\xi_i\circ g\cdot\xi$. Consider, for example, the germ of $A(F)$ at the point $\eta := h_0\cdot\xi_{i_0}\circ\xi$:

$$\begin{aligned} A(F)_\eta = \sum_{(g,h,i)\in\Lambda} (h\cdot f_i) F' \circ (g\cdot\xi^{-1}\circ h\cdot\xi_i^{-1})[g\cdot(X)\circ\eta^{-1}] = \\ F'\circ\eta^{-1} \sum_{(g,h,i)\in\Lambda} (h\cdot f_i)[g\cdot(X)\circ\eta^{-1}], \end{aligned} \tag{2}$$

where $\Lambda = \{(g,h,i) \mid (h\cdot\xi_i)\circ(g\cdot\xi) = \eta\}$. We see that the product of a meromorphic function

$$H := \sum_{(g,h,i)\in\Lambda} (h\cdot f_i) g\cdot(X)\circ\eta^{-1}$$

and a holomorphic function $F'\circ\eta^{-1}$ is holomorphic, since $A(F)_\eta$ is holomorphic. This holds for any $F' \in \mathcal{O}^G_e$, in particular, for constant F'. The latter implies that H is holomorphic at η. Similarly, we conclude that H is in $\mathcal{O}^G_\eta$. Summing up, we have $F'\circ\eta^{-1} \in \mathcal{O}_\eta^{\eta G\eta^{-1}}$ and $H \in \mathcal{O}^G_\eta$. Note that, from $F' \in \mathcal{J}_e$, it follows that $F'\circ\eta^{-1} \in (\mathcal{O}_\eta^{\eta G\eta^{-1}})^+$. Now the assertion of the theorem follows from Lemma 3. □

3.3. $\mathcal{A}$-modules corresponding to $\mathfrak{E}$. For convenience we put

$$M(G,(G\ltimes\gimel)\cdot v)=\mathfrak{E}|_{\bar{\xi}\in(G\ltimes\gimel)\cdot v/G}. \tag{3}$$

Denote by $M^*(G,(G\ltimes\gimel)\cdot v)$ the vector space

$$M^*(G,(G\ltimes\gimel)\cdot v):=\bigoplus_{\bar{\xi}\in(G\ltimes\gimel)\cdot v/G}(\mathbb{E}^G_{\bar{\xi}})^*. \tag{4}$$

Note that, in general, $M^*(G,(G\ltimes\gimel)\cdot v)\subsetneq\big(\mathfrak{E}|_{\bar{\xi}\in(G\ltimes\gimel)\cdot v/G}\big)^*$. We will need the following lemma.

LEMMA 5. *Assume that A sends the vector space $\mathfrak{D}^G|_{(G\ltimes\gimel)\cdot v/G}$ to itself. Then the action of A on $\mathfrak{D}^G|_{(G\ltimes\gimel)\cdot v/G}$ induces an action of A on $M^*(G,(G\ltimes\gimel)\cdot v)$.*

Proof. Let us take $A=\sum\limits_i\sum\limits_{h\in G}h\cdot(f_i\phi_{\xi_i})$, $\alpha\in(\mathbb{E}^G_{\bar{\eta}})^*$ and $\sum\limits_{g\in G}g\cdot F\in\mathcal{O}_{\bar{\xi}}$. We have

$$[A(\alpha)]\big(\sum_{g\in G}g\cdot F\big)=\alpha\big(\sum_i\sum_{h,g\in G}h\cdot(f_i)(g\cdot F)\circ h\cdot\xi_i^{-1}\big).$$

Inside the brackets on the right hand side we have a sum of G-invariant germs supported at the points from the finite set $\{h\cdot\xi_i\circ g\cdot\xi\mid g,h\in G\}$. Therefore, $[A(\alpha)]\big(\sum\limits_{g\in G}g\cdot F\big)=0$, if $\bar{\eta}\notin\{h\cdot\xi_i\circ g\cdot\xi\mid g,h\in G\}/G$. In other words,

$$A(\alpha)\subset\bigoplus_{\bar{\xi}'\in\Lambda/G}(\mathbb{E}^G_{\bar{\xi}})^*,\quad\text{where}\quad\Lambda=\{h\cdot\xi_i^{-1}\circ g\cdot\eta\mid g,h\in G\}$$

and the proof is complete. □

As a consequence of Theorem 4 and Lemma 5, we have the following statement.

COROLLARY 6. *Let $\mathcal{A}$ be a subalgebra in $\mathcal{S}(\gimel)$ that preserves the vector space $\mathfrak{D}^G|_{(G\ltimes\gimel)\cdot v}$. Then both $M(G,(G\ltimes\gimel)\cdot v)$ and $M^*(G,(G\ltimes\gimel)\cdot v)$ are $\mathcal{A}$-modules.*

In the next sections we will consider the case when G acts locally as a reflection group. In this case all vector spaces $\mathbb{E}^G_{\bar{\xi}}$ are finite dimensional of dimension $|G|$ by Chevalley-Shephard-Todd Theorem.

3.4. Construction of new $\mathcal{A}$-modules. Recall that $\gimel$ is a G-invariant subgroup in V. Let $\mathcal{A}$ be a subalgebra in $\mathcal{S}(\gimel)$ that preserves the vector space $\mathfrak{D}^G|_{(G\ltimes\gimel)\cdot v/G}$, where $v\in V$ is a fixed point. Denote by $G_{\gimel\cdot v}$ the stabilizer in G of the orbit $\gimel\cdot v$. Let $W:=(G\ltimes\gimel)\cdot v\setminus\gimel\cdot v$. In other words, $W\subset V$ is the union of all orbits of $\gimel$ in $(G\ltimes\gimel)\cdot v$ except for $\gimel\cdot v$. By definition, the group $G_{\gimel\cdot v}$ acts on $\gimel\cdot v$. Therefore, $G_{\gimel\cdot v}$ acts on W too.

Further, we have a natural projection $\pi_G:\mathfrak{D}^G|_{(G\ltimes\gimel)\cdot v/G}\to\mathfrak{D}^{G_{\gimel\cdot v}}|_{\gimel\cdot v/G_{\gimel\cdot v}}$ defined by the following formula:

$$\mathfrak{D}^G|_{(G\ltimes\gimel)\cdot v/G}\ni F=\sum_{g\in G_{\gimel\cdot v}}g\cdot f_\xi+\sum_{g\in L}g\cdot f_\xi\longmapsto\sum_{g\in G_{\gimel\cdot v}}g\cdot f_\xi\in\mathfrak{D}^{G_{\gimel\cdot v}}|_{\gimel\cdot v/G_{\gimel\cdot v}}, \tag{5}$$

where $\xi\in\gimel\cdot v$, $f_\xi\in\mathcal{O}_\xi^{G_\xi}$ is a G_ξ-invariant germ and

$$L:=G\setminus G_{\gimel\cdot v}=\{g\in G\mid g\cdot\xi\notin\gimel\cdot v\}.$$

Note that, for any such F, there exists f_ξ with $\xi \in \daleth\cdot v$ and the map (5) is independent of the choice of $\xi \in \daleth \cdot v$.

Lemma 7. *The map π_G is a bijection.*

Proof. Assume that $\pi_G(F) = 0$. Then $f_\xi = 0$ and hence $F' := \sum_{g\in G_{\daleth\cdot v}} g \cdot f_\xi = 0$. Further, let us take $F' = \sum_{g\in G_{\daleth\cdot v}} g \cdot f_\xi \in \mathcal{O}^{G_{\daleth\cdot v}}|_{\daleth\cdot v/G_{\daleth\cdot v}}$. Then

$$F' = \pi_G(\sum_{g\in G_{\daleth\cdot v}} g \cdot f_\xi + \sum_{g\in L} g \cdot f_\xi).$$

Explicitly, the map π_G^{-1} is given by

$$\pi_G^{-1}(\sum_{g\in G_{\daleth\cdot v}} g \cdot f_\xi) = \frac{1}{|G_{\daleth\cdot v}|} \sum_{g'\in G} g' \cdot (\sum_{g\in G_{\daleth\cdot v}} g \cdot f_\xi).$$

□

We will need the following proposition.

Proposition 8. *Let $\mathcal{A}$ be a subalgebra in $\mathcal{S}(\daleth)$ that preserves the vector space $\mathfrak{O}^G|_{(G\ltimes\daleth)\cdot v/G}$. Then $\mathcal{A}$ also preserves $\mathfrak{O}^{G_{\daleth\cdot v}}|_{\daleth\cdot v/G_{\daleth\cdot v}}$ and the map π is an isomorphism of $\mathcal{A}$-modules.*

Proof. Let $A \in \mathcal{A}$ be as in (1). We apply A to a germ $F = \sum_{g\in G} g \cdot f_\xi \in \mathcal{O}^G_{\bar\xi}$, where $f_\xi \in \mathcal{O}^{G_\xi}_\xi$, $\bar\xi = G \cdot \xi$ and $\xi \in \daleth \cdot v$. We get

$$A(F) = \sum_i \sum_{h,g\in G} (h \cdot f_i)[(g \cdot f_\xi) \circ h \cdot \xi_i^{-1}].$$

Note that, if $(h \cdot \xi_i) \circ (g \cdot \xi) \in \daleth \cdot v$, then $g \cdot \xi \in \daleth \cdot v$ and hence $g \in G_{\daleth\cdot v}$.

Now we compute the germ of $A(F)$ at the point $\eta := (h_0 \cdot \xi_{i_0}) \circ (g_0 \cdot \xi) \in \daleth \cdot v$:

$$A(F)_\eta = \sum_{(g,h,i)\in\Lambda} (h \cdot f_i)[(g \cdot f_\xi) \circ h \cdot \xi_i^{-1}],$$

where $\Lambda = \{(g, h, i) \mid (h \cdot \xi_i) \circ (g \cdot \xi) = \eta\}$. If $(h \cdot \xi_i) \circ (g \cdot \xi) = \eta$, then $(g, h, i) \in \Lambda$ and we have $g \cdot \xi = \phi_{h\cdot\xi_i^{-1}}(\eta)$ implying $g \in G_{\daleth\cdot v}$. As a consequence of these observations, we obtain

$$\pi(A(F)) = A(\pi(F)).$$

In particular, this equality implies that $A(\pi(F))$ is holomorphic and therefore A preserves $\mathfrak{O}^{G_{\daleth\cdot v}}|_{\daleth\cdot v/G_{\daleth\cdot v}}$. It also implies that π is a homomorphism of $\mathcal{A}$-modules. The proof is complete. □

Here comes yet another construction of $\mathcal{A}$-modules. Let $\mathcal{A}$ be as above and H a subgroup of G such that $\mathcal{A}$ preserves the vector space $\mathfrak{O}^H|_{(H\ltimes\daleth)\cdot v/H}$. For the pair $H \subset G$, we have the obvious inclusion $P^G_H : \mathcal{O}^G|_{(G\ltimes\daleth)\cdot v/G} \hookrightarrow \mathcal{O}^H|_{(G\ltimes\daleth)\cdot v/H}$.

LEMMA 9. *Assume that $\mathcal{A}$ preserves the vector spaces $\mathfrak{D}^G|_{(G\ltimes\daleth)\cdot v/G}$ and $\mathfrak{D}^H|_{(H\ltimes\daleth)\cdot v/H}$. Then the diagram*

$$\begin{array}{ccc} \mathfrak{D}^G|_{(G\ltimes\daleth)\cdot v/G} & \xrightarrow{\pi_G} & \mathfrak{D}^{G_{\daleth\cdot v}}|_{\daleth\cdot v/G_{\daleth\cdot v}} \\ {\scriptstyle P_H^G}\downarrow & & \downarrow{\scriptstyle P_{H_{\daleth\cdot v}}^{G_{\daleth\cdot v}}} \\ \mathfrak{D}^H|_{(G\ltimes\daleth)\cdot v/G} & \xrightarrow{\pi_H} & \mathfrak{D}^{H_{\daleth\cdot v}}|_{\daleth\cdot v/H_{\daleth\cdot v}}, \end{array}$$

in which all maps are homomorphisms of $\mathcal{A}$-modules, commutes.

Proof. This follows directly from the definitions. □

The above leads us to the following theorem.

THEOREM 10. *Assume that $\mathcal{A}$ preserves the vector spaces $\mathfrak{D}^G|_{(G\ltimes\daleth)\cdot v/G}$ and $\mathfrak{D}^H|_{(H\ltimes\daleth)\cdot v/H}$. Then we have the following commutative diagram of $\mathcal{A}$-modules:*

$$\begin{array}{ccc} M(G,(G\ltimes\daleth)\cdot v) & \xrightarrow{\tilde{\pi}_G} & M(G_{\daleth\cdot v},\daleth\cdot v) \\ {\scriptstyle \mathbf{P}_H^G}\downarrow & & \downarrow{\scriptstyle \mathbf{P}_{H_{\daleth\cdot v}}^{G_{\daleth\cdot v}}} \\ M(H,(G\ltimes\daleth)\cdot v) & \xrightarrow{\tilde{\pi}_H} & M(H_{\daleth\cdot v},\daleth\cdot v), \end{array}$$

where $\tilde{\pi}_G$ and $\tilde{\pi}_H$ are induced by π_G and π_H from Proposition 8, respectively. Moreover, the map

$$\Upsilon = \mathbf{P}_H^G \circ \pi_G^{-1} : \mathfrak{D}^{G_{\daleth\cdot v}}|_{\daleth\cdot v/G_{\daleth\cdot v}} \longrightarrow M(H,(G\ltimes\daleth)\cdot v)$$

is also a homomorphism of $\mathcal{A}$-modules.

Proof. Theorem 4 defines all involved $\mathcal{A}$-module structures. Let us argue, for example, that the morphism $\tilde{\pi}_G$ of $\mathcal{A}$-modules induced by π_G is well-defined. This follows from the fact that, to obtain the module $M(G,(G\ltimes\daleth)\cdot v)$, we factor out by the ideal generated by G-invariants and, to obtain the module $M(G_{\daleth\cdot v},\daleth\cdot v)$, we factor out by the ideal generated by $G_{\daleth\cdot v}$-invariants. As we obviously have $G_{\daleth\cdot v}\subset G$, the necessary statement is obtained by the standard factorization argument. The commutativity of the diagram follows from Lemma 9. □

3.5. The vector space $(\mathcal{O}_\mathsf{C}/\mathcal{J}_\mathsf{C})_e$ is finite dimensional. In this section we show that the vector space $(\mathcal{O}_\mathsf{C}/\mathcal{J}_\mathsf{C})_e$ is finite dimensional. In particular, this implies that the fibration $\mathbb{E}$ has finite dimensional fibers. Several observations of this section were pointed out to us by D. Timashev.

Let V be a complex-analytic or linear algebraic Lie group. Any linear algebraic group is a complex-analytic Lie group, see [Hum]. Recall that we emphasize by the subscripts C and A objects in the complex-analytic and the algebraic category, respectively. For example, we denote by $\mathcal{O}_\mathsf{C}$ and by $\mathcal{O}_\mathsf{A}$ the sheaves of complex-analytic (holomorphic) and algebraic (polynomial) functions, respectively.

Let V be a linear algebraic group. Note that we can choose coordinates (x_i) in a neighborhood U of the identity $e\in V$ such that e is the origin and the vector space $W=\langle x_1,\ldots,x_n\rangle$ is G-invariant. Indeed, denote by $\mathfrak{m}_e$ the maximal ideal in $(\mathcal{O}_\mathsf{A})_e$. Then $\mathfrak{m}_e^2$ is a G-invariant subspace in $\mathfrak{m}_e$. We choose any coordinates $\{y_1,\ldots,y_n\}$ in U. Let W' be the $\mathbb{C}$-span of $\{g\cdot y_i \mid i=1,\ldots,n, g\in G\}$. Then W' and $W'\cap\mathfrak{m}_e^2$

are G-invariant. Since G is finite, there exists G-invariant subspace W such that $W' = W \oplus (W' \cap \mathfrak{m}_e^2)$. Let $x_1, \dots, x_n$ be a basis in W. If $f \in (\mathcal{O}_{\mathsf{C}}^G)_e$, then there exists a decomposition $f = \sum\limits_{k=0}^{\infty} f_k$, where f_k are G-invariant homogeneous polynomials in (x_i) of degree k. If V is complex analytic but not algebraic, we mean by $(\mathcal{O}_{\mathsf{A}})_e$ the algebra of germs of polynomial functions in (x_i).

A classical fact from the invariant theory is that the extension $(\mathcal{O}_{\mathsf{A}}^G)_e \subset (\mathcal{O}_{\mathsf{A}})_e$ of rings is integral. Indeed, any polynomial $f \in (\mathcal{O}_{\mathsf{A}})_e$ is integral over $(\mathcal{O}_{\mathsf{A}}^G)_e$ since it is a root of the polynomial $\prod\limits_{g \in G} (t - g \cdot f)$. In particular, $f^{|G|}$ is a linear combination of f^p, where $p < |G|$, with coefficients from $(\mathcal{O}_{\mathsf{A}}^G)_e$.

Lemma 11. *We have that $(\mathcal{O}_{\mathsf{A}})_e$ is a finitely generated $(\mathcal{O}_{\mathsf{A}}^G)_e$-module and the minimal number of generators is less than or equal to $|G|^{\dim V}$.*

Proof. The proof follows from the fact that $x_i^{|G|}$ is a linear combination of x_i^p, where $p < |G|$, with coefficients from $(\mathcal{O}_{\mathsf{A}}^G)_e$. □

Corollary 12. *The vector space $(\mathcal{O}_{\mathsf{A}}/\mathcal{J}_{\mathsf{A}})_e$ is finite dimensional and its dimension is less than or equal to $|G|^{\dim V}$.*

Theorem 13. *Let V be a complex analytic or linear algebraic group and G a finite group acting on V. Then*

$$(\mathcal{O}_{\mathsf{A}}/\mathcal{J}_{\mathsf{A}})_e \simeq (\mathcal{O}_{\mathsf{C}}/\mathcal{J}_{\mathsf{C}})_e.$$

In particular, $(\mathcal{O}_{\mathsf{C}}/\mathcal{J}_{\mathsf{C}})_e$ is finite dimensional and its dimension is less than or equal to $|G|^{\dim V}$.

Proof. We have the obvious map

$$(\mathcal{O}_{\mathsf{A}}/\mathcal{J}_{\mathsf{A}})_e \longrightarrow (\mathcal{O}_{\mathsf{C}}/\mathcal{J}_{\mathsf{C}})_e, \quad f \mapsto f + (\mathcal{J}_{\mathsf{C}})_e. \tag{6}$$

Let us show that this map is a bijection.

Step 1. Let us first show that the map (6) is injective. To start with, assume that $f \in (\mathcal{O}_{\mathsf{A}})_e \cap (\mathcal{J}_{\mathsf{C}})_e$. Then $f = \sum\limits_{j=1}^{s} f_{1j} f_{2j}$, where $f_{1j} = \sum\limits_{k=0}^{\infty} f_k^{j1} \in (\mathcal{O}_{\mathsf{C}})_e$, $f_{2j} = \sum\limits_{p=1}^{\infty} f_p^{j2} \in (\mathcal{O}_{\mathsf{C}}^G)_e$, f_k^{j1} are homogeneous polynomials in (x_i) of degree k and f_p^{j2} are homogeneous G-invariant polynomials in (x_i) of degree p. We see that the polynomial $f = \sum\limits_{j=1}^{s} \sum\limits_{k=0}^{\infty} \sum\limits_{p=1}^{\infty} f_k^{j1} f_p^{j2}$ is an element in $(\mathcal{J}_{\mathsf{A}})_e$.

Step 2. Let us now show that the map (6) is surjective. Denote by $z_1, \dots, z_p$ a system of generators for the $(\mathcal{O}_{\mathsf{A}}^G)_e$-module $(\mathcal{O}_{\mathsf{A}})_e$ and set $N = \max\limits_s \{\deg z_s\}$. Let us take $f \in \mathfrak{m}_e^{N+1}$, where $\mathfrak{m}_e$ is the maximal ideal in $(\mathcal{O}_{\mathsf{C}})_e$.

Assume first that $f = \sum\limits_{i=N+1}^{t} f_i$, where f_i is a homogeneous polynomial of degree i, is a polynomial. The polynomial $\prod\limits_{g \in G} (t - g \cdot f)$, considered above, is homogeneous. Hence we can assume that z_j are homogeneous and $f_i = \sum\limits_j f_{ij} z_j$ is a decomposition with homogeneous G-invariant coefficients. Since $\deg f_i > N$, we conclude that $f \in (\mathcal{J}_{\mathsf{C}})_e$.

Further, let us take $f = \sum_{i=N+1}^{\infty} f_i \in (\mathcal{O}_{\mathsf{C}})_e$, where f_i are homogeneous polynomials in (x_i) of degree i. Assume that f is not identically equal to zero on the x_n-axis (we may ensure this by a linear change of coordinates). By the Weierstrass preparation theorem, we have $f = Pf_1$, where $P = x_n^r + a_{r-1}x_n^{r-1} + \cdots + a_1 x_n + a_0$ is a Weierstrass polynomial and f_1 is a unit. Here a_i is a holomorphic function in $x_1, \ldots, x_{n-1}$, for any i. Since f_1 is a unit, $P = ff_1^{-1} \in \mathfrak{m}_e^{N+1}$. Note that in the Taylor expansions of $a_\alpha x_n^\alpha$ and $a_\beta x_n^\beta$ in a neighborhood of e, where $\alpha \neq \beta$, we do not have equal summands. Therefore, $a_\alpha x_n^\alpha \in \mathfrak{m}_e^{N+1}$ for any α. Similarly, we apply the Weierstrass preparation theorem to a_α and proceed inductively. We obtain a polynomial in $\mathfrak{m}_e^{N+1}$ that, by the above, belongs to $(\mathcal{J}_{\mathsf{C}})_e$. Now, assume, by induction, that $a_\alpha x_n^\alpha \in (\mathcal{J}_{\mathsf{C}})_e$. Hence $P \in (\mathcal{J}_{\mathsf{C}})_e$ and therefore $f = Pf_1 \in (\mathcal{J}_{\mathsf{C}})_e$.

Now we can show that the map (6) is surjective. Indeed, by the above, any element $F \in (\mathcal{O}_{\mathsf{C}}/\mathcal{J}_{\mathsf{C}})_e$ has a polynomial representative. This completes the proof. □

Let $\mathcal{A}$ be as in Theorem 10 and $\mathcal{B} \subset \mathcal{S}$ be the algebra of G-invariant functions. Assume, in addition, that $\mathcal{B} \subset \mathcal{A}$

PROPOSITION 14. *The $\mathcal{A}$-modules constructed in Theorem 10 are Harish-Chandra modules.*

Proof. This follows from Theorem 13. □

4. Rational Galois orders and their modules.

4.1. Reflection groups and divided difference operators. Let $V_{\mathbb{R}}$ be a vector space over $\mathbb{R}$ equipped with a non degenerate symmetric bilinear form $(\,,\,)$. Set $V = \mathbb{C} \otimes_{\mathbb{R}} V_{\mathbb{R}}$ and denote the corresponding to $(\,,\,)$ inner product on V by the same symbol. For $v \in V$, the *reflection σ_v with respect to v* is the linear transformation of V that fixes the hyperplane $\{w \in V \mid (w, v) = 0\}$ and maps v to $-v$. It is given by the formula $\sigma_v(x) = x - \frac{2(x,v)}{(v,v)}v$. A *root system* Φ is a finite subset in $V_{\mathbb{R}} \setminus \{0\}$ that satisfies the following properties:

(I) If $x, y \in \Phi$, then $\sigma_x(y) \in \Phi$.

(II) If x and kx in Φ, for some $k \in \mathbb{R}$, then $k = \pm 1$.

For a root system Φ, the corresponding *reflection group* $G \subset GL(V)$ is the group generated by all reflections α_v, where $v \in \Phi$. A *system of simple roots* or a *basis* of Φ is a linearly independent subset in Φ such that every $x \in \Phi$ can be written as a linear combination of elements from Ψ with all non-negative or all non-positive coefficients. Any root system Φ has a basis. If a basis $\Psi \in \Phi$ is fixed, we get a partition $\Phi = \Phi^+ \cup \Phi^-$, where Φ^+ is the system of positive roots and Φ^- is the system of negative ones. Here a root x is called *positive* (resp. *negative*) with respect to Ψ, if it is a linear combination of vectors from Ψ with all non-negative (resp. non-positive) coefficients. We denote by Θ the set of *simple reflections*, that is reflections corresponding to elements in Ψ.

Let G be a reflection group, Ψ be a system of simple roots and Θ be the corresponding system of simple reflections. For any $x \in V$, we have a unique $\gamma_x \in V^*$ such that $\gamma_x(y) = (x, y)$, for all $y \in V$. Further, for any simple reflection $\sigma_x \in \Theta$, we define the corresponding *divided difference operator* ∂_{σ_x} on the set of holomorphic (or meromorphic, or rational or polynomial) functions on V via

$$\partial_{\sigma_x} \cdot f := \frac{f - \sigma_x \cdot f}{\gamma_x}.$$

For any $w \in G$, we set $\partial_w = \partial_{\sigma_1} \circ \cdots \circ \partial_{\sigma_p}$, where $w = \sigma_1 \circ \cdots \circ \sigma_p$ is a reduced expression. By [BGG, Page 5], we have $\partial_w = 0$, if the expression $w = \sigma_1 \circ \cdots \circ \sigma_p$ is not reduced. Moreover, the operator ∂_w is independent of the choice of a reduced expression.

4.2. Rational Galois orders. Rational Galois orders is a large class of algebras introduced in [Har, Section 4]. This class includes, for instance, orthogonal Gelfand-Zeitlin algebras, finite W-algebras of type A and, as we will see in Section 5, standard algebras of type $\mathbb{A}$ that preserve the vector space $\mathfrak{O}^G$. Note that a particular case of rational Galois orders was considered earlier in [Vi1, Vi2]. In the terminology of [Vi1, Vi2], these are finitely generated over $H^0(V,\mathcal{O})^G$ subalgebras in the so-called *universal ring*.

Let G be a reflection group in V as in Subsection 4.1 (note that the definition of a rational Galois order was given in [Har] for a more general case of a pseudo-reflection group or a complex reflection group G). Let $\chi : G \to \mathbb{C}^\times$ be a character. The space of relative invariants

$$H^0(V,\mathcal{O})^G_\chi := \{f \in H^0(V,\mathcal{O}) \mid g\cdot f = \chi(g)f \text{ for all } g \in G\}$$

is, naturally, an $H^0(V,\mathcal{O})^G$-module. This module is free of rank 1 and is generated by

$$d_\chi = \prod_{H\in A(G)} (\gamma_H)^{a_H},$$

where $A(G)$ is the set of all hyperplanes H that are fixed by a certain element σ_H in G, $\gamma_H \in V^*$ with $\ker \gamma_H = H$ and a_H is the minimal non-negative integer such that $\chi(\sigma_H) = \det(\sigma_H^*)^{a_H}$. If G is a reflection group, then $a_H = 0$ or 1, see [Ter, Section 2] for details.

Definition 15 ([Har, Definition 4.3]). *A* rational Galois order *is a subalgebra $\mathcal{R}$ in $\mathcal{S}(V)^G$ that contains $H^0(V,\mathcal{O})^G$ and that is generated by a finite number of elements $X \in \mathcal{S}(V)^G$ such that, for any such X, there exists a character χ of G such that $d_\chi X$ is holomorphic in V.*

In [Har, Theorem 4.2] it was shown that a rational Galois order preserves $H^0(V,\mathcal{O})^G$. In the following Lemma we prove a more general result: a rational Galois order preserves the vector space $\mathfrak{O}^G$.

Lemma 16. *Let X be a generator of a rational Galois order. Then $X(\mathfrak{O}^G) \subset \mathfrak{O}^G$.*

Proof. Let χ be a character of G such that $d_\chi X$ is holomorphic in V. We take $F_{G\cdot\xi} \in \mathcal{O}^G_{G\cdot\xi}$ and consider a germ P_η of $P = X(F_{G\cdot\xi})$ at a point $\eta \in V$. Denote by d_η the product of all divisors γ_H of d_χ such that $\gamma_H(\eta) = 0$. The corresponding reflections σ_H generate the group G_η. Then $P_\eta = P'_\eta/\chi_\eta$, where P'_η is a holomorphic function at η. We see that P'_η is a relative invariant for the character χ_η, where $\chi_\eta(h) = (h\cdot d_\eta)/d_\eta$, $h \in G_\eta$. By [Ter, Section 2], we have $P'_\eta = d_\eta P''_\eta$, where P''_η is holomorphic at η. Therefore, P_η is also holomorphic at η. □

Here is an example.

Example 17. Assume that we are in the setup of Subsection 2.2. Let $n \geq 4$ and consider for example the classical Gelfand-Zeitlin operator E_{34}. We will now show explicitly that $E_{34}(F)$ is holomorphic, where $F = \sum_{g\in G} g\cdot(f_{\xi^1_3}) \in \bigoplus_{g\in G} \mathcal{O}^G_{g\cdot\xi^1_3}$. We compute, for example, the germ of $E_{34}(F)$ at the point $\eta := \xi_3^1 + \xi'$, where $\xi' = (\delta_{32}^{ki})$. We have

$$E_{34}(F)_\eta = \frac{\prod_{j=1}^{4}(v_{31}-v_{4j})}{(v_{31}-v_{32})(v_{31}-v_{33})} f_{\xi_3^1}\circ(\xi')^{-1} + \frac{\prod_{j=1}^{4}(v_{32}-v_{4j})}{(v_{32}-v_{31})(v_{32}-v_{33})} f_{\xi'}\circ(\xi_3^1)^{-1} =$$

$$\frac{(v_{31}-v_{33})\prod_{j=1}^{4}(v_{32}-v_{4j}) f_{\xi'}\circ(\xi_3^1)^{-1} - (v_{32}-v_{33})\prod_{j=1}^{4}(v_{31}-v_{4j}) f_{\xi_3^1}\circ(\xi')^{-1}}{(v_{32}-v_{33})(v_{32}-v_{31})(v_{31}-v_{33})}.$$

We see that the polynomial in the numerator changes the sign, if we permute v_{32} and v_{31}. Therefore the factor $v_{32} - v_{31}$ cancels and the fraction is a holomorphic function at η. Another important observation here is that we have to consider the holomorphic category instead of the algebraic one. Indeed, the rational operator E_{34} sends a polynomial germ F to the holomorphic germ $E_{34}(F)_\eta$ plus other holomorphic summands.

Representation theory of rational Galois orders was developed in [FGRZ]. In this paper, we generalize some of the constructions from [FGRZ] for any finite group, see Section 6.

4.3. Bases in some modules over rational Galois orders. Assume that there is a G-invariant neighborhood U of $e \in V$ such that G acts as a reflection group in U. In this case, we will call G a *local reflection group*. An example of this situation is $G = S_n$ and $V \simeq \mathbb{C}^n$, where S_n acts via its permutation representation. Another example is $G = S_n$ and $V = \mathbb{C}^n/\mathbb{Z}^n$. More generally, G is a generalized Weyl group acting on $\mathbb{C}^n$ and $V = \mathbb{C}^n/\daleth'$, where $\daleth'$ is a G-invariant discrete lattice in $\mathbb{C}^n$.

In this subsection we will describe the finite dimensional vector spaces $\mathbb{E}_\xi^*$ using divided difference operators. If G is a local reflection group, by Chevalley-Shephard-Todd Theorem, the factor space $\mathcal{O}_e/\mathcal{J}_e$ is finite dimensional and has dimension $|G|$. Denote by $\Delta(\Psi)$ the product of all α_x, where $x \in \Phi^+$. For any $g \in G$, we put $\mathcal{P}_g := \partial_{g^{-1}w_0}\Delta(\Psi)$. The obtained polynomials are called *Schubert polynomials* and their images in $\mathcal{O}_e/\mathcal{J}_e$ form there a basis. Note that $\mathcal{P}_w(e) = 0$ if $w \neq e$ and $\mathcal{P}_e$ is a non-zero constant. Now we can easily construct the dual basis. Consider

$$B(\Theta) := \langle ev_e \circ \partial_w \mid w \in G\rangle, \tag{7}$$

where ev_e is the evaluation at $e \in V$. To show that $B(\Theta)$ is a basis of $(\mathcal{O}_e/\mathcal{J}_e)^*$, we note that $ev_e \circ \partial_w(\mathcal{P}_g)$ is 0, if and only if $g \neq w$. If Θ' is another system of simple reflections in G and $\rho(\Theta) = \Theta'$, then

$$B(\Theta') = \langle ev_e \circ \rho \circ \partial_w \circ \rho^{-1} \mid w \in G\rangle$$

is another basis of $(\mathcal{O}_e/\mathcal{J}_e)^*$. We note also that a basis of

$$(\mathcal{O}_e^{G_\xi}/\mathcal{O}_e^{G_\xi} \cap \mathcal{J}_e)^* \subset (\mathcal{O}_e/\mathcal{J}_e)^*$$

is given by $\langle ev_e \circ \partial_w \mid w \in (G/G_\xi)^{short}\rangle$, where $(G/G_\xi)^{short}$ denotes the set of shortest coset representatives.

Assume that Θ is fixed. In any class $\bar{\xi} \in V/G$, we can choose a representative $\tilde{\xi}$ such that $G_{\tilde{\xi}}$ is parabolic with respect to Θ. A description of the basis in $(\mathbb{E}_{\tilde{\xi}}^G)^*$ corresponding to $B(\Theta)$ is given in the following straightforward statement:

LEMMA 18. *Let Θ be a system of simple roots, $\tilde{\xi}$ be as above and $B(\Theta)$ be the corresponding basis of $(\mathcal{O}_e/\mathcal{J}_e)^*$. Then $\{ev_e \circ \partial_w \circ \phi_{\tilde{\xi}},\ w \in (G/G_{\tilde{\xi}})^{short}\}$ is a basis of $(\mathbb{E}^G_{\tilde{\xi}})^*$.*

We summarize the above results in the following theorem.

THEOREM 19. *Let G be a local reflection group, Θ be a system of simple reflections and $\mathcal{A}$ be a subalgebra in the skew-ring $\mathcal{S}$ that preserves the vector space $\mathfrak{O}^G|_{(G\ltimes \daleth)\cdot v}$, for a subgroup $\daleth \subset V$. Then*

$$\bigcup_{\tilde{\xi}\in \daleth/G} \{ev_e \circ \partial_w \circ \phi_{\tilde{\xi}},\ w \in (G/G_{\tilde{\xi}})^{short}\},$$

is basis of the $\mathcal{A}$-module $M^*(G,(G \ltimes \daleth)\cdot v)$.

Proof. The statement follows from Corollary 6 and Lemmata 5 and 18. □

For instance, we have Theorem 19 for all rational Galois orders.

5. Characterization of rational Galois orders. Let V and G be as in Subsections 2.3 and 2.4. In this case we can describe all elements $A \in \mathcal{S}(V)^G$ such that A preserves the vector space $\mathfrak{O}^G$. More precisely we prove that all such elements satisfy Definition 15 for $d_\chi = \Delta$, where Δ is the product of all $x_{ki} - x_{kj}$, where $i \neq j$. Denote by (x_{ki}) the standard dual basis in V^*, that is, $x_{ki}(v) = v_{ki}$, where $v = (v_{st}) \in V$.

THEOREM 20. *Let $A = \sum\limits_{s=1}^{p} \sum\limits_{g\in G} g \cdot (f_s \phi_{\xi^{a_s}_{i_s}})$ be any element in $\mathcal{S}(V)^G$. Assume that A preserves the vector space $\mathfrak{O}^G$. Then ΔA is holomorphic.*

Proof. Step 1. We start by reducing the statement to the case $p = 1$. For this, we show that $B_s := \sum\limits_{g\in G} g \cdot (f_s \phi_{\xi^{a_s}_{i_s}})$ also preserves the vector space $\mathfrak{O}^G$, for any $s = 1,\dots,p$. Denote by S_t the G-invariant polynomial

$$\sum_{g\in G} g \cdot x_{i_t,1} = \frac{|G|}{n_{i_t}} \sum_{j=1}^{n_{i_t}} x_{i_t,j},$$

where $t \in \{1,\dots,p\}$. Consider the operator $S_t id \in \mathcal{S}(V)^G$ and the following composition of operators

$$A \circ S_t id = S_t \sum_{s=1}^{p} \sum_{g\in G} g \cdot (f_s \phi_{\xi^{a_s}_{i_s}}) - \frac{a_{i_t}|G|}{n_{i_t}} \sum_{g\in G} g \cdot (f_t \phi_{\xi^{a_t}_{i_t}}) = S_t A - \frac{a_{i_t}|G|}{n_{i_t}} B_t.$$

The operators $A \circ S_t id$, $S_t id$ and $S_t A$ all preserve $\mathfrak{O}^G$. Hence the element B_t also preserves $\mathfrak{O}^G$, in case $a_t \neq 0$.

Consider now the case $a_t = 0$. Let us rewrite the operator A:

$$A = \sum_{a_s\neq 0} \sum_{g\in G} g \cdot (f_s \phi_{\xi^{a_s}_{i_s}}) + \sum_{g\in G} g \cdot (f_s \phi_{\xi^{0}_{i_s}}) = \sum_{a_s\neq 0} \sum_{g\in G} g \cdot (f_s \phi_{\xi^{a_s}_{i_s}}) + H id,$$

where H is G-invariant. Since A and the first summand preserve $\mathfrak{O}^G$, we deduce that $H id$ also preserves $\mathfrak{O}^G$.

Therefore to prove our theorem it is enough to show that, if $C := \sum_{g\in G} g\cdot(f\phi_{\xi_i^a})$ preserves the vector space $\mathfrak{O}^G$, then $C \in \mathcal{D}$.

Step 2. Assume that $C = \sum_{g\in G} g\cdot(f\phi_{\xi_i^a})$ preserves the vector space $\mathfrak{O}^G$. Let us show that every function $g\cdot f$ is holomorphic in any Weyl chamber. In other words, we want to show that the function $g\cdot f$ is holomorphic at any point $w\in V$ such that $w=(w_{ki})$, where $w_{ki}\neq w_{kj}$, for any k and $i\neq j$.

First of all we note that, if $a=0$, then the operator C is holomorphic at any point $v\in V$. Indeed, in this case $C = Hid$, where H is a G-invariant meromorphic function. Let us take $\sum_{h\in G} h\cdot c\in \mathcal{O}^G_{\bar v}$, where $c\in\mathbb{C}\setminus\{0\}$. Then

$$C(\sum_{h\in G} h\cdot c) = H\sum_{h\in G} h\cdot c\in\mathcal{O}^G_{\bar v},$$

where $\bar v = G\cdot v$. Therefore, cH is holomorphic at any $h\cdot v$. Hence H is holomorphic on V.

Assume now that $a\neq 0$. Let us take $\sum_{h\in G} h\cdot F\in\mathcal{O}^G_{\bar v}$, where $F = e\cdot F\in\mathcal{O}^G_v$. Then $C(\sum_{h\in G} h\cdot F)\in\mathfrak{O}^G$ is a sum of G-invariant germs supported at the points from the set

$$T=\{h\cdot v+g\cdot\xi_i^a \mid g,h\in G\}.$$

Let us show that, from the fact that $h\cdot v+g\cdot\xi_i^a = h'\cdot v+g'\cdot\xi_i^a$ is a point in a Weyl chamber, it follows that $h\cdot v = h'\cdot v$ and $g\cdot\xi_i^a = g'\cdot\xi_i^a$.

Take $w=(w_{kj}) = h\cdot v+g\cdot\xi_i^a\in T$, a point from a Weyl chamber. Assume that there is $w'=(w'_{kj}) = h'\cdot v+g'\cdot\xi_i^a\in T$ such that $w'=w$. First of all, from $w=w'$, it follows that $w_{kj}=w'_{kj}$, for any $k\neq i$ and for any j. Further, we have two possibilities: $v_{ij}+a = v_{ip}+a$ or $v_{ij}+a=v_{ip}$, for some p. In the first case, we have $v_{ij}=v_{ip}$. Using that w is in a Weyl chamber, we conclude that $h=id$ or h is the transposition that sends v_{ij} to v_{ip}. In particular, $h\cdot v = h'\cdot v$. Consider the case $v_{ij}+a=v_{ip}$, where $p\neq j$. In this case we have a contradiction with the assumption that w is in a Weyl chamber. Summing up, we have $h\cdot v=h'\cdot v$, and hence $g\cdot\xi_i^a = g'\cdot\xi_i^a$.

Now consider the summand

$$\sum_{h_1\in G_v}(hh_1\cdot F)\circ(g\cdot\xi_i^a)^{-1}\sum_{g_1\in G_{\xi_i^a}}(gg_1\cdot f) = \alpha[(h\cdot F)\circ(g\cdot\xi_i^a)^{-1}](g\cdot f)\in\mathcal{O}^G_w, \quad (8)$$

where $\alpha\in\mathbb{C}\setminus\{0\}$, from $C(\sum_{h\in G} g\cdot F)$, supported at the point $w=h\cdot v+g\cdot\xi_i^a$ from a Weyl chamber. Note that, to obtain (8), we use the fact that $G_F=G_v$ and $G_f = G_{\xi_i^a}$. Further, putting $F = \mathit{const}\neq 0$, we see that $g\cdot f$ is holomorphic at w.

Step 3. Our goal now is to show that $C\in\mathcal{D}$. Take $w=(w_{kj}) = h\cdot v+g\cdot\xi_i^a\in T$ such that the stabilizer of w has order 2. We have two possibilities:

(1) $v_{ks}=v_{kt}$, for some $s\neq t$, $G_w=\{id,\sigma\}$, where σ is the transposition that swaps the point v_{ks} and v_{kt};
(2) $v_{ij}+a=v_{ip}$, for some $j\neq p$, $G_w=\{id,\tau\}$, where τ is the transposition that swaps the point $v_{ij}+a$ and v_{ip}.

In the first case, as in Step 2, we get that $h \cdot f$ is holomorphic at w. Consider the second possibility. The summand from $C(\sum_{h\in G} h \cdot F)$ supported at the point w is

$$\begin{aligned} &\sum_{h_1\in G_x} (hh_1 \cdot F) \circ (g \cdot \xi_i^a)^{-1} \sum_{g_1 \in G_{\xi_i^a}} (gg_1 \cdot f) + \\ \tau[&\sum_{h_1\in G_v} (hh_1 \cdot F) \circ (g \cdot \xi_i^a)^{-1} \sum_{g_1 \in G_{\xi_i^a}} (gg_1 \cdot f)] \in \mathcal{O}_w^G. \end{aligned} \tag{9}$$

Let $F = c \in \mathbb{C}\setminus\{0\}$. From (9), we get that $g \cdot f + \tau(g \cdot f) \in \mathcal{O}_w^G$. We put $z_1 := x_{ij} - x_{ip}$ and $z_2 := x_{ij} + x_{ip}$. Then (z_1, z_2, x_{kt}), where $(kt) \neq (ij), (ip)$, form a new coordinate system. Moreover, z_2 and x_{kt} are τ-invariant and $\tau(z_1) = -z_1$.

From Step 2 it follows that $g \cdot f$ is a holomorphic function in a neighborhood of w, except for points y with $z_1(y) = 0$. Any such function possesses a Hartogs-Laurent series, see [Sh, Section 8]. Let this series be $g \cdot f = \sum_{s=q}^{\infty} H_s z_1^s$, where H_s are holomorphic functions in z_2 and all x_{kt}. We have

$$g \cdot f + \tau(g \cdot f) = \sum_{s=q}^{\infty} (1 + (-1)^s) H_s z_1^s \in \mathcal{O}_w^G.$$

We obtain that $H_s = 0$, for all $s = 2r < 0$.

Further, we note that $G_{h\cdot v} = \{id\}$ or $G_{h\cdot v} = \{id, \theta\}$, where θ is an involution that swaps v_{ij} with some $v_{ij'}$, where $j' \neq p$. In the first case, set $h \cdot F = z_1 \in \mathcal{O}_{h\cdot v}^{G_{h\cdot v}}$. In the second case, set $h \cdot F = z_1 + \theta(z_1) \in \mathcal{O}_{h\cdot v}^{G_{h\cdot v}}$. In both cases, using (9), we obtain

$$z_1 (\sum_{s=q}^{\infty} H_s z_1^s - \sum_{s=q}^{\infty} (-1)^s H_s z_1^s) \in \mathcal{O}_w^G.$$

This is possible only if $H_s = 0$, for $s < 1$. Therefore $g \cdot f$ has only a simple pole at w.

Summing up, above we proved that f is holomorphic in any Weyl chamber and it has a simple pole or it is holomorphic at all points with the stabilizer of order 2. This implies that $f\Delta$ is holomorphic at all point with the stabilizer of order 1 or 2. By the Riemann extension theorem, see e.g. [Dem, Corollary 6.4], singularities of codimension at least 2 are removable. It follows that $f\Delta = H$ is homomorphic in V. The proof is complete. □

COROLLARY 21. *Let $\mathcal{A} \subset \mathcal{S}(V)^G$ be a finitely generated over $H^0(V, \mathcal{O})^G$ standard algebra of type $\mathbb{A}$ that preserves the vector space $\mathfrak{O}^G$. Then $\mathcal{A}$ is a rational Galois order.*

This description of standard algebras of type $\mathbb{A}$ that preserve $\mathfrak{O}^G$ is surprising. It would be interesting to prove an analog of this result (or to find a counter-example) for other reflection groups.

6. Applications of Theorem 4 to Gelfand-Zeitlin modules. Let $\mathcal{A}$ be a subalgebra in $\mathcal{S}(V)$ that preserves the vector space $\mathfrak{O}^G|_{(G\ltimes \daleth)\cdot v}$, for some $v \in V$, and $\mathcal{B}$ be the algebra of global G-invariant functions on V. Then, by Corollary 6 and Proposition 14, $M(G, (G \ltimes \daleth) \cdot v)$ and $M^*(G, (G \ltimes \daleth) \cdot v)$ are $\mathcal{A}$-modules. These $\mathcal{A}$-modules and their submodules were studied, for some special cases, simultaneously and independently in [RZ] (the case of $\mathcal{A} = U(\mathfrak{gl}_n(\mathbb{C}))$) and in [EMV] (the case of $\mathcal{A}$

being an orthogonal Gelfand-Zeitlin algebra). The case when $\mathcal{A}$ is a rational Galois orders corresponding to any reflection group was later considered in [FGRZ]. In this section, we show how to obtain [RZ, Section 5.6, Theorem], [EMV, Theorem 10] and [FGRZ, Theorem 7.4] using Corollary 6, Theorem 10 and Proposition 14.

6.1. The case of orthogonal Gelfand-Zeitlin algebras. Let V, $\gimel$ and G be as in Subsection 2.3. The classical Gelfand-Zeitlin operators E_{st} and the generators of the orthogonal Gelfand-Zeitlin algebra E_k and F_k are rational, however as it was shown in Lemma 16, we have $E_k(\mathfrak{O}^G) \subset \mathfrak{O}^G$ and $F_k(\mathcal{O}^G) \subset \mathcal{O}^G$. Clearly, the same holds for E_{st}. Further let us take $v' \in V$. It is easy to see that there exists $v \in \gimel \cdot v'$ such that G_v includes all stabilizers G_w, where $w \in \gimel \cdot v'$.

LEMMA 22. *We have* $G_v = G_{\gimel \cdot v}$.

Proof. For $v = (v_{ki})$, the following holds: if $v_{ki} - v_{kj} \in \mathbb{Z}$, then $v_{ki} = v_{kj}$. Further, it is clear that $G_v \subset G_{\gimel \cdot v}$. If $g \in G_{\gimel \cdot v}$, then $g \cdot v \in \gimel \cdot v$ or, equivalently, $g \cdot v - v \in \gimel$. Hence $(g \cdot v)_{ki} - v_{ki} \in \mathbb{Z}$ and thus $(g \cdot v)_{ki} = v_{ki}$, implying $g \in G_v$. □

By Lemma 22 and Proposition 8, we get that $M(G_v, \gimel \cdot v)$ and $M^*(G_v, \gimel \cdot v)$ are $\mathcal{A}$-modules. From Proposition 14 it follows that these modules are Harish-Chandra modules and therefore Gelfand-Zeitlin modules. This recovers the corresponding results from [RZ] and [EMV].

6.2. The case of rational Galois orders. Let V, $\gimel$ and G be as in Subsection 4.2. Take $v \in V$ and let H be a subgroup in G that contains all stabilizers G_w, where $w \in \gimel \cdot v$. Then it is easy to check (we refer to Theorem 26 for details) that a rational Galois order $\mathcal{A}$ preserves the vector space $\mathfrak{O}^H|_{(H \ltimes \gimel) \cdot v}$. By Lemma 16, the algebra $\mathcal{A}$ preserves also the vector space $\mathfrak{O}^G|_{(G \ltimes \gimel) \cdot v}$. Therefore we may apply Theorem 10 to obtain a family of the corresponding modules. In the case when H is a reflection group and satisfies some other conditions (it has to be parabolic with respect to a fixed system of simple roots), the $\mathcal{A}$-module $Im(\Upsilon^*)$, cf Theorem 10, was constructed in [FGRZ, Theorem 7.4]. This recovers the corresponding result of [FGRZ].

7. Structure theorem for rational Galois order.

7.1. Further examples of algebras that preserve the vector space $\mathfrak{O}^G$. In this section we assume that G is a reflection group on $V \simeq \mathbb{C}^n$. Let us fix a system Ψ of simple roots and let Θ be the set of the corresponding simple reflections. Our goal now is to define two classes of algebras preserving the vector space $\mathfrak{O}^G$. As above, we denote by $\circ$ composition of operators or the product in $G \ltimes V$ and we use $\cdot$ to denote the action of G. For example, if $g \in G$ and $\xi \in V$, then $g \cdot \xi = g \circ \xi \circ g^{-1}$ and $g \cdot \phi_\xi = g \circ \phi_\xi \circ g^{-1}$.

Algebras of type I. These are subalgebras of $\mathcal{S}$ generated by elements of the form $\sum_i \partial_{w_i} \circ p_i \phi_{v_i}$, where, for each i, the stabilizer G_{v_i} of $v_i \in V$ is parabolic with respect to Θ, the function p_i is G_{v_i}-invariant and holomorphic (or meromorphic, or rational or polynomial) and w_i is the longest element in $(G/G_{v_i})^{short}$.

Algebras of type II. These are subalgebras of $\mathcal{S}$ generated by elements in the form $\sum_i \partial_{w_i} \cdot p_i \phi_{v_i}$, where v_i, p_i and w_i are as in type I.

REMARK 23. *Note the difference of using $\cdot$ in type II instead of $\circ$ in type I.*

Let $\mathbf{A}$ be an algebra of type I. Denote by $\gimel$ the subgroup of V generated by all possible $g \cdot v_i$, where $g \in G$ and v_i appears in a generator of $\mathbf{A}$, see above.

Proposition 24. *Let $E = \sum_i \partial_{w_i} \circ p_i \phi_{v_i}$ be a generator of the algebra* $\mathbf{A}$*. If all p_i are holomorphic in V, then*

$$E(\mathfrak{O}^G) \subset \mathfrak{O}^G.$$

Proof. Take a simple reflection $\tau \in \Theta$. Then

$$(id - \tau) \circ \partial_{w_i} \circ p_i \phi_{v_i} = \gamma_\tau \partial_\tau \circ \partial_{w_i} \circ p_i \phi_{v_i}. \tag{10}$$

Since w_i is the longest element in $(G/G_{v_i})^{short}$, the operator $\partial_\tau \circ \partial_{w_i}$ is either zero or can be written as $\partial_u \circ \partial_s$, where $\partial_s \in G_{v_i}$. Therefore the right hand side of (10) is identically zero on $\mathfrak{M}^G$. Hence, for any $F \in \mathfrak{M}^G$ and $g \in G$, we have $g \circ \partial_{w_i} \circ p_i \phi_{v_i}(F) = \partial_{w_i} \circ p_i \phi_{v_i}(F)$ implying $\partial_{w_i} \circ p_i \phi_{v_i}(\mathfrak{M}^G) \subset \mathfrak{M}^G$.

Further, we have $\partial_\tau(\mathfrak{O}) \subset \mathfrak{O}$. Indeed, let us take $f_x \in \mathcal{O}_x$ and consider $\partial_\tau(f_x)$. If $\tau(x) = x$, then $\gamma_\tau(x) = 0$. In this case γ_τ is a divisor of $f_x - \tau(f_x) \in \mathcal{O}_x$. Therefore, $\partial_\tau(f_x) \in \mathcal{O}_x$. If $\tau(x) \neq x$, then $\gamma_\tau(x) \neq 0$. Hence $f_x/\gamma_\tau \in \mathcal{O}_x$ and $\tau(f_x)/\gamma_\tau \in \mathcal{O}_{\tau(x)}$. □

Let $\mathbf{A}$ be an algebra of type I and $\mathbf{B}$ be an algebra of type II. Assume that for each generator $E = \sum_i \partial_{w_i} \circ p_i \phi_{v_i}$ of $\mathbf{A}$ there is a generator $E' = \sum_i \partial_{w_i} \cdot p_i \phi_{v_i}$ of $\mathbf{B}$ and vice versa. The next lemma describes when the actions of E and E' coincide.

Lemma 25. *Assume that all p_i are holomorphic. We have the equality of operators*

$$E|_{\mathfrak{O}^G} = E'|_{\mathfrak{O}^G}.$$

Therefore, the actions of algebras $\mathbf{A}$ *and* $\mathbf{B}$ *as above on $\mathfrak{O}^G$ coincide.*

Proof. Consider first the operators $\partial_\rho \circ f\phi_x$ and $\partial_\rho \cdot f\phi_x$, where $f \in \mathcal{M}$ is any meromorphic function and $\rho \in G$ is any (not necessary longest) element with reduced expression $\rho = \tau_1\tau_2\cdots\tau_k$. Let us prove, by induction on k, that

$$\partial_\rho \circ f\phi_x|_{\mathfrak{M}^G} = \partial_\rho \cdot f\phi_x|_{\mathfrak{M}^G}.$$

For $k = 1$, the claim is obvious. To establish the induction step, we have

$$\begin{aligned}\partial_{\tau_1} \circ \cdots \circ \partial_{\tau_k} \circ f\phi_x|_{\mathfrak{M}^G} = \partial_{\tau_1} \circ \cdots \circ \partial_{\tau_{k-1}} \circ (\partial_{\tau_k} \cdot f\phi_x)|_{\mathfrak{M}^G} = \\ \partial_{\tau_1} \circ \cdots \circ \partial_{\tau_{k-1}} \circ (f/\gamma_{\tau_k}\phi_x - (\tau_k \cdot f)/\gamma_{\tau_k}\tau_k \cdot \phi_x)|_{\mathfrak{M}^G} = \\ \partial_{\tau_1} \circ \cdots \circ \partial_{\tau_{k-1}} \cdot (f/\gamma_{\tau_k}\phi_x - (\tau_k \cdot f)/\gamma_{\tau_k}\tau_k \cdot \phi_x)|_{\mathfrak{M}^G} = \\ \partial_{\tau_1} \cdots \partial_{\tau_k} \cdot f\phi_x|_{\mathfrak{M}^G}.\end{aligned}$$

The result now follows from Proposition 24. □

7.2. Structure theorem for rational Galois order. In this section we assume that G is a reflection group on $V \simeq \mathbb{C}^n$, where Φ is a root system with basis Ψ and Θ is the set of corresponding simple reflections (cf. Subsection 4.1). We have the decomposition $\Phi = \Phi^+ \cup \Phi^-$ corresponding to Ψ. Consider the following product of linear functions on V:

$$\Delta := \prod_{x \in \Phi^+} \gamma_x,$$

where $\gamma_x(v) = (x, v)$, for any $v \in V$, see Section 4.1. We have $\sigma_x \cdot \Delta = -\Delta$, for any simple reflection $\sigma_x \in \Theta$. If $G = S_n$, then Δ may be identified with the classical Vandermonde determinant.

Let us take $v \in V$ such that the stabilizer G_v is parabolic in G with respect to Θ. Denote by Δ' the product of γ_x, where σ_x is a reflection in G_v. Let us take also a polynomial (or a holomorphic function) p' and let w be the longest element in $(G/G_v)^{short}$.

Consider an element of the form $\sum_{\tau \in G} \tau \cdot (\frac{p'}{\Delta}\phi_v)$ from a rational Galois order $\mathcal{A}$, see Subsection 4.2. We always can choose p' such that it satisfies

$$\tau \cdot p' = \chi(\tau)p', \quad \text{where} \quad \chi(\tau) := \frac{\tau \cdot \Delta}{\Delta}, \quad \text{for} \quad \tau \in G_v.$$

Therefore we have $p' = \Delta' p$, where p is a G_v-invariant polynomial or a holomorphic function (cf. Subsection 4.2). In other words, if $\Phi' \subset \Phi$ is the root subsystem corresponding to G_v, then

$$\Delta' := \prod_{x \in \Phi'^+} \gamma_x,$$

where Φ'^+ is the subsystem of positive roots generated by $\Psi \cap \Phi'$. If w_0 is the longest element in G, then we have the following equality on global rational functions

$$\partial_{w_0} = \sum_{\tau \in G} \tau \cdot \frac{1}{\Delta}, \tag{11}$$

see [Hill, Section IV, Proposition 1.6]. From Dedekind's Theorem it follows that the operators (11) are equal as elements in $\mathcal{S}$. Therefore we have

$$\partial_{w_0}|_{\mathfrak{D}} = \sum_{\tau \in G} \tau \cdot \frac{1}{\Delta}|_{\mathfrak{D}}, \tag{12}$$

The following theorem generalizes [EMV, Proposition 7].

Theorem 26 (Structure Theorem).
(*a*) *We have*

$$\sum_{\tau \in G} \tau \cdot \frac{\Delta'}{\Delta} p\phi_v\big|_{\mathfrak{D}^G} = a\partial_w \circ p\phi_v\big|_{\mathfrak{D}^G}, \tag{13}$$

where $a \neq 0$ is a scalar.
(*b*) *Let G' be any subgroup in G which is parabolic with respect to Θ. Then*

$$\sum_{\tau \in G} \tau \cdot \frac{\Delta'}{\Delta} p\phi_v\big|_{\mathfrak{D}^G} = \sum_{s=1}^{k} \partial_{w_s} \circ t_s \phi_{v_s}|_{\mathfrak{D}^G}, \tag{14}$$

where $w_s \in G'/G'_{v_s}$ is the longest reduced element and t_s are rational functions defined in Weyl chambers and at $\ker \gamma_x$, *where $x \in \Phi^+$ and $\sigma_x \in G'$.*

Proof. Note that we always can choose v such that G_v is parabolic with respect to Θ. Using (12), we have

$$\sum_{\tau \in G} \tau \cdot \frac{\Delta'}{\Delta} p\phi_v\big|_{\mathfrak{D}^G} = \sum_{\tau \in G} \tau \cdot \frac{1}{\Delta} \Delta' p\phi_v|_{\mathfrak{D}^G} = \partial_{w_0} \circ \Delta' p\phi_v|_{\mathfrak{D}^G} =$$

$$\partial_w \circ \partial_{w_0'} \circ \Delta' p\phi_v|_{\mathfrak{D}^G} = \partial_w \circ p\phi_v \partial_{w_0'} \circ (\Delta')|_{\mathfrak{D}^G} = a\partial_w \circ p\phi_v|_{\mathfrak{D}^G},$$

where w_0' is the longest element in G_v. This implies claim (a).

To prove claim (b), let G' be a subgroup in G which is parabolic with respect to Θ. We have

$$\sum_{\tau\in G}\tau\cdot\frac{\Delta'}{\Delta}p\phi_v=\sum_{\tau\in G'}\tau\cdot\sum_{s=1}^{k}\tau_s\cdot\frac{\Delta'}{\Delta}p\phi_v.$$

Here $\tau_s\in G'\backslash G$ is a coset representative and $k=|G'\backslash G|$. Note that we can choose the representatives τ_s such that $\phi_{v_s}:=\tau_s\cdot\phi_v$ has a parabolic stabilizer G'_{v_s} with respect to Θ.

Denote by $\tilde{\Delta}$ the product of γ_x, where $x\in\Phi^+$ and $\sigma_x\in G'$, and by $\tilde{\Delta}_s$ the product of γ_x, where $x\in\Phi^+$ and $\sigma_x\in G'_{v_s}$. Clearly, $\tilde{\Delta}$ is a divisor of Δ and $\tilde{\Delta}_s$ is a divisor of $\tau_s\cdot\Delta'$. Denote by $l(\tau_s)$ the length of τ_s. We have

$$\tau_s\cdot\frac{\Delta'}{\Delta}p\phi_v=\frac{(-1)^{l(\tau_s)}\tau_s\cdot\Delta'}{\Delta}p_s\phi_{v_s}=\frac{\tilde{\Delta}_s}{\tilde{\Delta}}\frac{(-1)^{l(\tau_s)}(\tau_s\cdot\Delta')/\tilde{\Delta}_s}{\Delta/\tilde{\Delta}}p_s\phi_{v_s},$$

where $p_s:=\tau_s\cdot p$. We put

$$t_s:=\frac{(-1)^{l(\tau_s)}(\tau_s\cdot\Delta')/\tilde{\Delta}_s}{\Delta/\tilde{\Delta}}p_s.$$

We see that t_s are rational functions defined in Weyl chambers and at $\ker\gamma_x$, where $x\in\Phi^+$ and $\sigma_x\in G'$.

Using (a), we obtain

$$\sum_{s=1}^{k}\sum_{\tau\in G'}\tau\cdot\frac{\tilde{\Delta}_s}{\tilde{\Delta}}t_s\phi_{v_s}|_{\mathfrak{O}^G}=\sum_{s=1}^{k}a_s\partial_{w_s}\circ t_s\phi_{v_s}|_{\mathfrak{O}^G},$$

where $a_s\neq 0$ are scalars and $w_s\in G'/G'_{v_s}$ are longest element in the set of shortest coset representatives. □

In the case $G=S_n$, formula (13) was conjectured by the second author in [Vi3] and later independently proved in [RZ, EMV]. It was extended to an arbitrary reflection group in [FGRZ] where it was also shown that it plays a crucial role in construction and study of simple Gelfand-Zeitlin modules for rational Galois orders.

Consider a rational Galois order $\mathcal{A}$ as above. Fix $v\in V$ and denote by H the subgroup in G generated by all stabilizers G_v, where $v\in\beth\cdot v$. In the proof of Theorem 26 we obtained the following expression

$$A=\sum_{\tau\in G}\tau\cdot\frac{\Delta'}{\Delta}p\phi_v=\sum_{s=1}^{k}\sum_{\tau\in H}\tau\cdot\frac{\tilde{\Delta}_s}{\tilde{\Delta}}t_s\phi_{v_s},$$

where $\tilde{\Delta}$ is the product of γ_x, for $x\in\Phi^+$ and $\sigma_x\in H$, and $\tilde{\Delta}_s$ the product of γ_x, for $x\in\Phi^+$ and $\sigma_x\in H_{v_s}$. By Lemma 16, the operator A preserves the vector spaces $\mathfrak{O}^G$ and $\mathfrak{O}^H$. Therefore we have the families of modules given by Theorem 10. In particular, we have the $\mathcal{A}$-modules $M^*(G,(G\ltimes\beth)\cdot v)$ and $M^*(H,\beth\cdot v)$. A basis of these modules is constructed in Theorem 19. Using Theorem 26, we get the following fairly explicit result.

Corollary 27. *With respect to the basis of Theorem 19, the action of* $\mathcal{A}$ *on the modules* $M^*(G,(G\ltimes \daleth)\cdot v)$ *or* $M^*(H,\daleth\cdot v)$ *can be computed using the following formula:*

$$(ev_0\circ\partial_w\circ\phi_\xi)\circ A|_{\mathfrak{O}^G}=(ev_0\circ\partial_w\circ\phi_\xi)\circ(\partial_w\circ p\phi_v)|_{\mathfrak{O}^G}=\sum_{s=1}^{k}a_s ev_0\circ\partial_w\circ\partial_{w_s}\circ(\phi_\xi\cdot t_s)\phi_{\xi\circ v_s}|_{\mathfrak{O}^G},\tag{15}$$

where $a_s\in\mathbb{C}\setminus\{0\}$*, cf. Theorem 26. Here* t_s *and* v_s *correspond to* $G'=G_\xi$*.*

8. A construction of simple modules and sufficient conditions for simplicity .

8.1. Canonical simple Harish-Chandra modules. In this section we constract a family of simple modules which we will call *canonical Harish-Chandra modules.* This construction generalizes the corresponding constructions from [EMV] and [Har]. Assume that V is a complex-analytic Lie group, G is a finite group, $\daleth\subset V$ is a subgroup and $v\in V$. Let $\mathcal{A}\subset S(V)^G$ be a subalgebra containing $H^0(V,\mathcal{O})^G$, which preserves the vector space $\mathfrak{O}^G|_{(G\ltimes\daleth)\cdot v}$. Consider the $\mathcal{A}$-module $M(G,(G\ltimes\daleth)\cdot v)$. Denote by $N_{\bar w}$, where $\bar w=G\cdot w$ for some $w\in(G\ltimes\daleth)\cdot v$, the submodule in $M(G,(G\ltimes\daleth)\cdot v)$ generated by $\tilde 1_{\bar w}\in\mathbb{E}^G_{\bar w}$, where $\tilde 1_{\bar w}$ is the class generated by the constant function 1.

Proposition 28. *Assume that* $H^0(V,\mathcal{O})^G$ *separates orbits in* $(G\ltimes\daleth)\cdot v$ *and that the* $H^0(V,\mathcal{O})^G$*-module* $\mathbb{E}^G_{\bar w}$ *is generated by* $\tilde 1_{\bar w}$*. Then* $N_{\bar w}$ *has a unique maximal submodule.*

Proof. The unique maximal submodule is the sum of all submodules N' in $N_{\bar w}$ such that $N'\cap\mathbb{E}^G_{\bar w}\subset\mathfrak{n}_{\bar w}\cdot\tilde 1_{\bar w}$, where $\mathfrak{n}_{\bar w}\subset H^0(V,\mathcal{O})^G$ is the ideal of all G-invariant functions that are equal to 0 at $\bar w$. □

The quotient of $N_{\bar w}$ by its unique maximal submodule is denoted $L_{\bar w}$ and is called the *canonical simple Harish-Chandra module associated to* $\bar w$.

8.2. Standard algebras of type $\mathbb{A}$. Let V and G be as in Subsection 2.3 and 2.4. As we have seen in Corollary 21, a finitely generated over $H^0(V,\mathcal{O})^G$ standard algebra of type $\mathbb{A}$ that preserves the vector space $\mathfrak{O}^G$ is a rational Galois order. Consider a special case of such algebras, the algebra $\mathcal{A}$ that is generated by $H^0(V,\mathcal{O})^G$ and by the following elements:

$$E_i=\sum_{g\in G}g\cdot(\frac{\Delta' H_i^E}{\Delta}\phi_{\xi_i^a}),\quad F_i=\sum_{g\in G}g\cdot(\frac{\Delta' H_i^F}{\Delta}\phi_{\xi_i^{-a}}),\quad i=1,\dots,n,$$

where $a\in\mathbb{C}\setminus\{0\}$, H_i^E,H_i^F are holomorphic functions in V such that we have $G_{H_i^E}=G_{H_i^F}=G_{\xi_i^a}$, for $i=1,\dots,n$, and Δ and Δ' are as in Subsection 7.2.

Let $\daleth$ be a subgroup in V generated by ξ_i^a, where $i=1,\dots,n$, and $v'\in V$ be any point. In this case, for $G_{\daleth\cdot v'}$ we have an analogue of Lemma 22. That is, there exists $v\in\daleth\cdot v'$ such that $G_{\daleth\cdot v'}=G_v$. The module $M^*(G_{\daleth\cdot v},\daleth\cdot v)=M^*(G_v,\daleth\cdot v)$ was studied in [EMV, Theorem 11]. More precisely, in [EMV] the following theorem was proved.

Theorem 29. *[EMV, Theorem 11] Assume that* H_i^E,H_i^F*, where* $i=1,\dots,n$*, have no zeros on* $\daleth\cdot v$*. Then the* $\mathcal{A}$*-module* $M^*(G_v,\daleth\cdot v)$ *is irreducible.*

In [EMV], this theorem was proved only for a special choice of functions H_i^E, H_i^F. However exactly the same proof as in [EMV] works for any functions H_i^E, H_i^F. This fact was noticed in [FGRZ, Theorem 8.5], where the result [EMV, Theorem 11] was discussed in detail.

8.3. Regular modules. Assume that V is a complex-analytic Lie group, $\beth \subset V$ is a subgroup and $v \in V$. Let $\mathcal{A} \subset S(V)$ be a finitely generated, over $H^0(V, \mathcal{O})$, subalgebra which preserves the vector space $\mathfrak{O}|_{\beth \cdot v} = \mathfrak{O}^e|_{\beth \cdot v}$. We denote by Γ an oriented graph that is defined in the following way. The vertices of Γ are all points from $\beth \cdot v$ and we connect x and y with an arrow $x \to y$ if there exists $A = \sum f_i \phi_{\xi_i} \in \mathcal{A}$ and i_0 such that $\phi_{\xi_{i_0}}(x) = y$ and $f_{i_0}(y) \neq 0$. Note that, in this case, all f_i are holomorphic in $\beth \cdot v$.

Proposition 30. *Assume that $\mathcal{A} \subset S(V)$ is a finitely generated over $H^0(V, \mathcal{O})$ subalgebra that preserves the vector space $\mathfrak{O}|_{\beth \cdot v} = \mathfrak{O}^e|_{\beth \cdot v}$, $H^0(V, \mathcal{O})$ separates points of $\beth \cdot v$ and Γ is connected as an oriented graph. Then the $\mathcal{A}$-module $M(\{e\}, \beth \cdot v)$ is irreducible.*

Proof. First of all, we note that the $\mathcal{A}$-module $M(\{e\}, \beth \cdot v)$ is a direct sum of $\mathbb{E}_\xi = \phi_\xi(\mathcal{O}_e^{\{e\}}/(\mathcal{O}_e^{\{e\}} \cap \mathcal{J}_e)) \simeq \mathbb{C}$. In other words, $M(\{e\}, \beth \cdot v)$ is a vector space of all finite linear combinations of points $v_s \in \beth \cdot v$.

Let $\sum a_s v_s$ be an element in a submodule N. Since $H^0(V, \mathcal{O})$ separates points of $\beth \cdot v$, we see that $v_s \in N$ for any s. Let us take a submodule $N' \subset M(\{e\}, \beth \cdot v)$ that contains a point $x \in \beth \cdot v$. Further let $y \in \beth \cdot v$. Since Γ is connected as an oriented graph, there exists a sequence

$$x_0 = x, x_1, \dots, x_{n-1}, x_n = y$$

such that the path $x_0 \to x_1 \to \cdots \to x_n$ connects x and y. Assume, by induction, that we proved that $x_{s-1} \in N$. From our assumptions, there exists $A = \sum f_i \phi_{\xi_i} \in \mathcal{A}$ and i_0 such that $\phi_{\xi_{i_0}}(x_{s-1}) = x_s$ and $f_{i_0}(x_s) \neq 0$. We have $A(x_{s-1}) = \sum f_i(x_s) \phi_{\xi_i}(x_{s-1}) \in N$. Therefore $x_s \in N'$. □

Assume that $\mathcal{A}$ is generated by $H^0(\beth \cdot v, \mathcal{O})$ and, additionally, by a finite set of elements $E_i = \sum f_{ij} \phi_{\xi_{ij}}$. Let $\beth$ be the group generated by all ξ_{ij}. Denote by $Q(\xi_{ij})$ the monoid generated by all ξ_{ij}.

Proposition 31. *Assume that*
(*i*) *$H^0(V, \mathcal{O})$ separates points of $\beth \cdot v$;*
(*ii*) *$Q(\xi_{ij}) = \beth$;*
(*iii*) *every f_{ij} has no zeros at $\beth \cdot v$.*
Then the $\mathcal{A}$-module $M(\{e\}, \beth \cdot v)$ is irreducible.

Proof. Due to assumptions (i) and (iii), to be able to use Proposition 30, we only need to show that Γ is connected. The latter, however, follows directly from assumption (ii). Therefore the claim follows from Proposition 30. □

8.4. Singular modules. Assume that V is a complex-analytic Lie group, $\beth \subset V$ is a subgroup and $v \in V$. Let $\mathcal{A} \subset S(V)^{G_{\beth \cdot v}}$ be a finitely generated over $H^0(V, \mathcal{O})^{G_{\beth \cdot v}}$ subalgebra which preserves the vector space $\mathfrak{O}^{G_{\beth \cdot v}}|_{\beth \cdot v}$. Assume that $H^0(V, \mathcal{O})^{G_{\beth \cdot v}}$ separates $G_{\beth \cdot v}$-orbits in $\beth \cdot v$ and that the $H^0(V, \mathcal{O})^{G_{\beth \cdot v}}$-module $\mathbb{E}_{\bar{\xi}}^{G_{\beth \cdot v}}$, see (3), is generated by a non-trivial constant $c \in \mathbb{C} \setminus \{0\}$, for any $\xi \in \beth \cdot v$.

We denote by Γ the oriented graph defined as follows:

- the vertices of Γ are all $G_{\daleth\cdot v}$-orbits in $\daleth \cdot v$;
- for two orbits $\bar{\xi}$ to $\bar{\eta}$, there is an oriented arrow from $\bar{\xi}$ to $\bar{\eta}$, if there exists $A = \sum f_i \phi_{\xi_i} \in \mathcal{A}$ such that the function $H := \sum_{(g,h,i)\in\Lambda} h \cdot f_i$, cf. (2) for $X = 1$, exists and is not equal to 0 at η. (Note that the function H depends on the orbits $\bar{\xi}$ and $\bar{\eta}$ and on the element A.)

Theorem 32. *In the above situation, we have:*

(*i*) *For every $\bar{\xi}$, the module $M(G_{\daleth\cdot v}, \daleth \cdot v)$ has a unique submodule $N(\bar{\xi})$ which is maximal, with respect to inclusions, among all submodules of $M(G_{\daleth\cdot v}, \daleth \cdot v)$ that do not contain $\mathbb{E}_{\bar{\xi}}^{G_{\daleth\cdot v}}$.*

(*ii*) *If Γ is connected as an oriented graph, then $M(G_{\daleth\cdot v}, \daleth\cdot v)$ is generated by the class of a non-trivial constant function and also has a unique maximal submodule.*

Proof. Claim (i) follows from Proposition 30. Further we have

$$M(G_{\daleth\cdot v}, \daleth \cdot v) = \bigoplus_{\bar{\xi}\in\daleth\cdot v/G_{\daleth\cdot v}} \mathbb{E}_{\bar{\xi}}^{G_{\daleth\cdot v}},$$

see (3). Denote by N the $\mathcal{A}$-submodule generated by the class $c_{\bar{\xi}} \in \mathbb{E}_{\bar{\xi}}^{G_{\daleth\cdot v}}$ of a non-trivial constant function c. Let $\bar{y} \subset \daleth \cdot v$. Since Γ is connected as an oriented graph, there exists a sequence

$$\bar{x}_0 = \bar{x}, \bar{x}_1, \ldots, \bar{x}_{n-1}, \bar{x}_n = \bar{y}$$

such that the path $\bar{x}_0 \to \bar{x}_1 \to \cdots \to \bar{x}_n$ connects $\bar{x}$ and $\bar{y}$. Assume, by induction, that we proved that $1_{\bar{x}_{s-1}} \in N$. From our assumptions, there is $A = \sum f_i \phi_{\xi_i} \in \mathcal{A}$ that sends $1_{\bar{x}_{s-1}}$ to $a_{\bar{x}_{s-1}}$ with a constant non-trivial representative $a \neq 0$, see (2). This implies the first part of claim (ii) and the second part of claim (ii) follows from the first part of claim (ii) and claim (i). □

Let $M(\bar{\xi})$ denote the $\mathcal{A}$-submodule of $M(G_{\daleth\cdot v}, \daleth \cdot v)$ generated by $\mathbb{E}_{\bar{\xi}}^{G_{\daleth\cdot v}}$. The simple quotient $M(\bar{\xi})/N(\bar{\xi})$, whose existence is guaranteed by Theorem 32(i), is the *canonical* Harish-Chandra $\mathcal{A}$-module associated to $\bar{\xi}$.

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