ASIAN J. MATH.
Vol. 25, No. 4, pp. 505–520, August 2021

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ON STRONG EXCEPTIONAL COLLECTIONS OF LINE BUNDLES OF MAXIMAL LENGTH ON FANO TORIC DELIGNE-MUMFORD STACKS*

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Abstract. We study strong exceptional collections of line bundles on Fano toric Deligne-Mumford stacks $\mathbb{P}_{\mathbf{\Sigma}}$ with rank of Picard group at most two. We prove that any strong exceptional collection of line bundles generates the derived category of $\mathbb{P}_{\mathbf{\Sigma}}$, as long as the number of elements in the collection equals the rank of the (Grothendieck) K-theory group of $\mathbb{P}_{\mathbf{\Sigma}}$.

Key words. toric Deligne-Mumford stacks, Picard groups, strong exceptional collections, line bundles, derived categories.

Mathematics Subject Classification. Primary 14M25; Secondary 14F05, 14C20.

1. Introduction. Constructing phantom and quasiphantom subcategories of the derived category of coherent sheaves on smooth projective varieties has attracted considerable interest over the years. A quasi-phantom subcategory is an admissible subcategory with trivial Hochschild homology and with a finite Grothendieck group. A phantom subcategory is an admissible subcategory with trivial Hochschild homology and a trivial Grothendieck group.

The authors of [5, 1, 12] construct some quasi-phantom subcategories as semiorthogonal complements to exceptional collections of maximal possible length on certain surfaces of general type for which $q = p_g = 0$ with the Grothendieck group of a quasiphantom isomorphic to the torsion part of the Picard group of a corresponding surface. It is natural to ask whether there exists a phantom as a semiorthogonal complement to an exceptional collection of maximal length on a simply connected surfaces of general type with $q = p_g = 0$. It was achieved by Böhning, H-Ch. Graf von Bothmer, L. Katzarkov, and P. Sosna in [6]. They show that in a small neighbourhood of the surface constructed by Barlow in the moduli space of determinantal Barlow surfaces, the generic surface has a semiorthogonal decomposition of its derived category into a length 11 exceptional sequence of line bundles and a category with trivial Grothendieck group and Hochschild homology.

Additionally, in [11], S. Gorchinskiy and D. Orlov construct geometric phantom categories by considering admissible subcategories generated by the tensor product of two quasi-phantoms for which orders of their (Grothendieck) K-theory groups are coprime. They also prove that these phantom categories have trivial K-motives and, hence, all their higher K-groups are trivial too. Under certain assumptions on the semi-orthogonal decomposition, this result has implications for the structure of the Chow motive of a variety admitting a phantom category [21].

However, [18] shows that there are no quasi-phantoms, phantoms or universal phantoms in the derived category of smooth projective curves over a field k. Furthermore, Kuznetsov has conjectured that if a triangulated category of geometric origin contains a full exceptional collection, then it can not contain a phantom subcategory.

*Received September 21, 2020; accepted for publication December 10, 2020.
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In particular, all exceptional collections of the same length must be full. In this paper we will try to address Kuznetsov's conjecture in a very special case, namely of toric DM stacks of Picard rank at most two. Moreover, we will only focus on strong exceptional collections.

Specifically, our Conjecture 1.1 given below states that it is impossible to build a phantom as a semiorthogonal complement to a strong exceptional collection of line bundles of maximal length in the derived category of a Fano toric DM stack $\mathbb{P}_{\mathbf{\Sigma}}$. Our main result in the paper shows this in the case of Picard rank less or equal to two.

The subject of exceptional collections on toric varieties and stacks has its own rich history. Kawamata constructed exceptional collections in the bounded derived categories of of coherent sheaves on complete toric Deligne-Mumford stacks in [16]. Alastair King conjectured in [17] that every smooth toric variety has a full strong exceptional collection of line bundles. Although the conjecture was proved to be false in [13], rich and varied results related to the conjecture were proved in [4, 14, 19, 9, 15, 20]. In particular, it was proved in [4] that there exist full strong exceptional collections of line bundles on smooth toric Fano DM stacks of Picard number no more than two and of any Picard number in dimension two.

The full strong exceptional collections of line bundles constructed in [4] have length equal the rank of the (Grothendieck) K-theory group, which is known to be necessary, see for example [11]. It is natural to ask whether any strong exceptional collection of line bundles of this length is a full strong exceptional collection. That is to say that the subcategory generated by all elements in the strong collection equals $\mathbf{D}^b(coh(\mathbb{P}_{\mathbf{\Sigma}}))$, and there is no orthogonal complement phantom category. We propose the following conjecture.

Conjecture 1.1. *Any strong exceptional collection of line bundles of maximal length on a Fano toric DM stack is a full strong exceptional collection.*

In this paper, we prove Conjecture 1.1 for $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 1$ (Theorem 3.9) and $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 2$ (Theorem 4.14). Our main idea is to "shrink" the strong exceptional collection by moving some specific elements successively and eventually obtain a standard full strong exceptional collection given in [4].

The paper is organized as follows. Section 2 recalls basic facts on toric DM stacks and (strong) exceptional collection of line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$. In Section 3, we prove Conjecture 1.1 for the case of $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 1$. In Section 4, Conjecture 1.1 for the case of the rank of $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ equals two is settled. Section 5 contains brief discussion of further directions.

Acknowledgements. This work was prompted by a question of Shizhuo Zhang. Lev Borisov was partially supported by NSF grant DMS-1601907. We thank the referee for useful comments, see in particular Remark 4.16.

2. (Strong) exceptional collections of line bundles on toric Deligne-Mumford stacks. In this section, we give an overview of toric DM stacks $\mathbb{P}_{\mathbf{\Sigma}}$, the corresponding Grothendieck group and (strong) exceptional collections of line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$. Since all of this is well known, we try to be brief.

Let Σ be a complete simplicial fan with m one-dimensional cones in a lattice N which is a free abelian group of finite rank. The assumption that N has no torsion allows us to refrain from the technicalities of the derived Gale duality of [2]. We pick a lattice point v in each of the one-dimensional cones of Σ and get a complete stacky fan $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^m)$, see [2]. The toric DM stack $\mathbb{P}_{\mathbf{\Sigma}}$ associated to the stacky fan $\mathbf{\Sigma}$ is constructed in [2] as a stack version of the homogeneous coordinate ring construction of a toric variety [7]. Line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ are described in [3, 4] similar to the scheme case of [8, 10]. Denote E_i to be the toric divisor for v_i.

Proposition 2.1. *The Picard group of $\mathbb{P}_{\mathbf{\Sigma}}$ is generated by $\{E_i\}_{i=1}^m$ with relations $\sum_{i=1}^m (w \cdot v_i)E_i$ for all w in the character lattice $M = N^*$.*

Proof. See [4]. □

Definition 2.2. *An object F in $\mathbf{D}^b(coh(\mathbb{P}_{\mathbf{\Sigma}}))$ is exceptional if $\mathrm{Hom}(F, F) = \mathbb{C}$ and $\mathrm{Ext}^t(F, F) = \mathrm{Hom}(F, F[t]) = 0$ for $t \neq 0$. A sequence of exceptional objects $(F_1, F_2, \ldots, F_n)$ in $\mathbf{D}^b(coh(\mathbb{P}_{\mathbf{\Sigma}}))$ is called an exceptional collection if*

$$\mathrm{Ext}^t(F_i, F_j) = \mathrm{Hom}(F_i, F_j[t]) = 0$$

for all $i > j$ and all $t \in \mathbb{Z}$. An exceptional collection is further called a strong exceptional collection if

$$\mathrm{Ext}^t(F_i, F_j) = 0$$

for all $i < j$ and all $t \in \mathbb{Z} \setminus \{0\}$.

Remark 2.3. *A subset $\mathcal{T}$ of $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ can be indexed to form a strong exceptional collection if and only if $\mathrm{Ext}^t(\mathcal{L}_1, \mathcal{L}_2) = 0$ for any $\{\mathcal{L}_1, \mathcal{L}_2\} \in \mathcal{T}$ and any $t > 0$. The reason is that the existence of nonzero $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2)$ induces a partial order on the set $\mathcal{T}$ which can be extended to a linear order.*

Definition 2.4 ([4]). *Let $\mathcal{T}$ be a finite set of line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ (which are always exceptional objects on $\mathbb{P}_{\mathbf{\Sigma}}$). We call $\mathcal{T}$ a full strong exceptional collection if*

$$\mathrm{Ext}^t(\mathcal{L}_1, \mathcal{L}_2)$$

for any $\{\mathcal{L}_1, \mathcal{L}_2\} \in \mathcal{T}$ and any $t > 0$ and the derived category of $\mathbb{P}_{\mathbf{\Sigma}}$ is generated by the line bundles in $\mathcal{T}$.

Definition 2.5. *A toric DM stack $\mathbb{P}_{\mathbf{\Sigma}}$ is called Fano if the chosen points v_i are precisely the vertices of a simplicial convex polytope in $N_{\mathbb{R}}$.*

Definition 2.6 ([3]). *Let $\mathbb{P}_{\mathbf{\Sigma}}$ be a smooth DM stack. The (Grothendieck) K-theory group $K_0(\mathbb{P}_{\mathbf{\Sigma}})$ is defined to be the quotient of the free abelian group generated by coherent sheaves $\mathcal{F}$ on $\mathbb{P}_{\mathbf{\Sigma}}$ by the relations $[\mathcal{F}_1] - [\mathcal{F}_2] + [\mathcal{F}_3]$ for all exact sequences $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$.*

Lemma 2.7 ([11]). *Let $\mathbb{P}_{\mathbf{\Sigma}}$ be a Fano toric DM stack and $(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n)$ be an exceptional collection of objects in $\mathbf{D}^b(coh(\mathbb{P}_{\mathbf{\Sigma}}))$. If $n = \mathrm{rank} K_0(\mathbb{P}_{\mathbf{\Sigma}})$, then $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ is a basis of $K_0(\mathbb{P}_{\mathbf{\Sigma}})$.*

Corollary 2.8. *Let $(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n)$ be an exceptional collection of objects in $\mathbf{D}^b(coh(\mathbb{P}_{\mathbf{\Sigma}}))$. Then $n \leq \mathrm{rk}(K_0(\mathbb{P}_{\mathbf{\Sigma}}))$.*

3. The case of $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 1$**.** In the section, we prove Conjecture 1.1 when the rank of $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ is one.

Let $\mathbb{P}_{\mathbf{\Sigma}}$ be a Fano toric DM stack such that $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ has no torsion and rank one. In this case $\mathbb{P}_{\mathbf{\Sigma}}$ is a weighted projective space which we denote by $W\mathbb{P}(w_1, \ldots, w_m)$, where $gcd(w_1, \ldots, w_m) = 1$. [1] The rank of $K_0(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}}))$ is $\sum_{i=1}^m w_i$. The Picard group $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ is $\{\mathcal{O}(d) | d \in \mathbb{Z}\}$, where $\mathcal{O}(E_i) = \mathcal{O}(w_i)$. By [4], we know that $\mathbb{P}_{\mathbf{\Sigma}}$ possesses a full strong exceptional collection of line bundles.

PROPOSITION 3.1. *[4] Let* $\mathcal{T} = \{\mathcal{O}(w) | -\mathrm{rk}(K_0(\mathbb{P}_{\mathbf{\Sigma}})) + 1 \leq w \leq 0\}$. *Then* $\mathcal{T}$ *forms a full strong exceptional collection in the derived category of* $W\mathbb{P}(w_1, \ldots, w_m)$.

Proof. See [4]. □

From [4], for any $d_1, d_2 \in \mathbb{Z}$, we know that

$$\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(d_1), \mathcal{O}(d_2)) \neq 0 \Leftrightarrow d_2 - d_1 = \sum_{i=1}^m a_i w_i, \text{ for some } a_i \in \mathbb{Z}_{<0};$$

$$\mathrm{Hom}(\mathcal{O}(d_1), \mathcal{O}(d_2)) \neq 0 \Leftrightarrow d_2 - d_1 = \sum_{i=1}^m a_i w_i, \text{ for some } a_i \in \mathbb{Z}_{\geq 0}.$$

REMARK 3.2. *In the case of* $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 1$*, any exceptional collection on* $X = \mathbb{P}_{\mathbf{\Sigma}}$ *is a strong exceptional collection. Indeed, let*

$$\mathcal{T} = (\mathcal{O}(s_1), \ldots, \mathcal{O}(s_n))$$

be an exceptional collection on $\mathbb{P}_{\mathbf{\Sigma}}$*. We have* $\mathrm{Hom}(\mathcal{O}(s_j), \mathcal{O}(s_i)) = 0$ *for* $j > i$*. Then* $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(s_i), \mathcal{O}(s_j)) = 0$ *for* $j > i$*. Otherwise, we get* $s_j - s_i = \sum_{i=1}^m a_i w_i$*, where* $a_i \in \mathbb{Z}_{<0}$*. This implies* $s_j - s_i = \sum_{i=1}^m b_i w_i$*, where* $b_i = -a_i \in \mathbb{Z}_{\geq 0}$*, which contradicts* $\mathrm{Hom}(\mathcal{O}(s_j), \mathcal{O}(s_i)) = 0$.

Main idea. Starting from an exceptional collection $\mathcal{T}$ of line bundles of maximal length, i.e., with $\sum_{i=1}^m w_i$ elements, we construct other exceptional collections of maximal length in $\mathcal{D}\langle\mathcal{T}\rangle$, the subcategory generated by elements in $\mathcal{T}$. Eventually, we will get to the exceptional collection in Proposition 3.1 given in [4]. This allows us to conclude that $\mathcal{D}\langle\mathcal{T}\rangle = \mathbf{D}^b(coh(\mathbb{P}_{\mathbf{\Sigma}}))$.

The main step is to "move" the smallest element of the exceptional collection $\mathcal{T}$ by $\sum_{i=1}^m w_i$, see Figure 1.

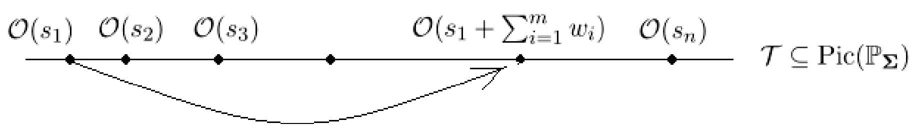

FIG. 1.

Specifically: If line bundles $\mathcal{O}(s_1), \ldots, \mathcal{O}(s_n)$, where $s_1 < s_2 < \cdots < s_n$, form a strong exceptional collection $\mathcal{T}$ of maximal length, then

(1) $\mathcal{O}(s_1 + \sum_{i=1}^m w_i)$ is not in the strong exceptional collection $\mathcal{T}$ (Lemma 3.5);

[1]This condition comes from our assumption that N has no torsion.

(2) By replacing $\mathcal{O}(s_1)$ with $\mathcal{O}(s_1 + \sum_{i=1}^m w_i)$ and reordering, we get another strong exceptional collection (Lemma 3.6);
(3) $\mathcal{O}(s_1 + \sum_{i=1}^m w_i) \in \mathcal{D}\langle\mathcal{T}\rangle$, so the new collection generates a subcategory of $\mathcal{D}\langle\mathcal{T}\rangle$ (Corollary 3.8).

Once we know these that these moves are possible, we can "shrink" the exceptional collection to make it one from Propostion 3.1 (Theorem 3.9).

Example 3.3. *We consider an exceptional collection on* $W\mathbb{P}(5,6)$

$$(\mathcal{O}(-15), \mathcal{O}(-13), \mathcal{O}(-10), \mathcal{O}(-9), \mathcal{O}(-8), \mathcal{O}(-7), \mathcal{O}(-6), \mathcal{O}(-5), \mathcal{O}(-3), \mathcal{O}(-1), \mathcal{O})$$

of maximal length 11. *We replace* $\mathcal{O}(-15)$ *by* $\mathcal{O}(-15+11) = \mathcal{O}(-4)$ *to get another strong exceptional collection*

$$(\mathcal{O}(-13), \mathcal{O}(-10), \mathcal{O}(-9), \mathcal{O}(-8), \mathcal{O}(-7), \mathcal{O}(-6), \mathcal{O}(-5), \mathcal{O}(-4), \mathcal{O}(-3), \mathcal{O}(-1), \mathcal{O}).$$

Then we replace $\mathcal{O}(-13)$ *by* $\mathcal{O}(-13+11) = \mathcal{O}(-2)$ *to get*

$$(\mathcal{O}(-10), \mathcal{O}(-9), \mathcal{O}(-8), \mathcal{O}(-7), \mathcal{O}(-6), \mathcal{O}(-5), \mathcal{O}(-4), \mathcal{O}(-3), \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O})$$

which is a full strong exceptional collection in Proposition 3.1 given in [4].

Remark 3.4. *It was pointed to us by an anonymous referee that, in general, if* $(F_1, \ldots, F_k)$ *is an exceptional collection, then so is* $(F_2, \ldots, F_k, S^{-1}(F_1)$ *where* S^{-1} *is the inverse of the Serre functor. Together with Remark 3.2, this would allow for a shorter argument in Picard one case. However, our direct argument below provides a blueprint for the far more complicated case of Picard rank two, where Serre functor on its own is not enough to achieve our goals.*

Lemma 3.5. *Let* $\mathcal{T} = \{\mathcal{O}(s_1), \ldots, \mathcal{O}(s_n)\}$ *be a strong exceptional collection. Then* $\mathcal{O}(s_1 + \sum_{i=1}^m w_i) \notin \mathcal{T}$.

Proof. If $\mathcal{O}(s_1 + \sum_{i=1}^m w_i) \in \mathcal{T}$, then $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(s_1 + \sum_{i=1}^m w_i), \mathcal{O}(s_1)) \neq 0$ since $s_1 - (s_1 + \sum_{i=1}^m w_i) = -\sum_{i=1}^n w_i$. This contradicts the assumption that $\mathcal{T}$ is a strong exceptional collection. □

Lemma 3.6. *Let* $\mathcal{T} = \{\mathcal{O}(s_1), \ldots, \mathcal{O}(s_n)\}$ *be a strong exceptional collection of maximal length on* $\mathbb{P}_{\mathbf{\Sigma}}$, *where* $s_1 < s_2 < \cdots < s_n$. *By replacing* $\mathcal{O}(s_1)$ *with* $\mathcal{O}(s_1 + \sum_{i=1}^m w_i)$ *and reordering, we get another strong exceptional collection.*

Proof. Let $\mathcal{T}^1$ be a collection obtained by replacing $\mathcal{O}(s_1)$ with $\mathcal{O}(s_1 + \sum_{i=1}^m w_i)$. For any $i \in \{2, \ldots, n\}$, we have $s_i - s_1 - \sum_{i=1}^m w_i > -\sum_{i=1}^m w_i$. Thus $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(s_1 + \sum_{i=1}^m w_i), \mathcal{O}(s_i)) = 0$. Also for any $i \in \{2, \ldots, n\}$, we have $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(s_i), \mathcal{O}(s_1 + \sum_{i=1}^m w_i)) = 0$. Otherwise, we get $s_1 + \sum_{i=1}^m w_i - s_i = \sum_{i=1}^m a_i w_i$, where $a_i \leq -1$. Thus $s_1 - s_i = \sum_{i=1}^m b_i w_i$, where $b_i < -1$. This implies $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(s_i), \mathcal{O}(s_1)) \neq 0$, which contradicts the assumption that $\mathcal{T}$ is an exceptional collection. □

Lemma 3.7. *Let* $\mathcal{T} = \{\mathcal{O}(s_1), \ldots, \mathcal{O}(s_n)\}$ *be a strong exceptional collection of maximal length on* $\mathbb{P}_{\mathbf{\Sigma}}$, *where* $s_1 < s_2 < \cdots < s_n$. *Then* $\mathcal{O}(s_1 + \sum_{j \in J} w_j)$ *is in* $\mathcal{T}$ *for any proper subset* $J \subsetneq \{1, 2, \ldots, m\}$.

Proof. Let $s = s_1 + \sum_{j \in J} w_j$. We have $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(s), \mathcal{O}(s_k)) = 0$ for all $k \in \{1, 2, \ldots, n\}$. Otherwise, we have $s_k - s \in \sum_{i=1}^m \mathbb{Z}_{<0} w_m$ for some k. However, we have $s_k - s_1 \geq 0$. So $s_k - s = s_k - s_1 - \sum_{j \in J} w_j > -\sum_{j=1}^m w_j$, which leads to contradiction.

We have $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(s_k), \mathcal{O}(s)) = 0$ for all $k \in \{1, 2, \dots, n\}$. Otherwise, we get $s_1 + \sum_{j\in J} w_j - s_k = s - s_k = \sum_{i=1}^m a_i w_i$ for some k, where $a_i \leq -1$. Thus $s_1 - s_k = \sum_{i=1}^m b_i w_i$, where $b_i \leq -1$. Therefore $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(s_k), \mathcal{O}(s_1)) \neq 0$, which contradicts that $\mathcal{T}$ is an exceptional collection.

If $\mathcal{O}(s)$ is not in $\mathcal{T}$, we can get another exceptional collection with $\sum_{i=1}^m w_i + 1$ elements by inserting $\mathcal{O}(s)$ into $\mathcal{T}$. This is impossible by Corollary 2.8. □

Corollary 3.8. *Let $\mathcal{T} = \{\mathcal{O}(s_1), \dots, \mathcal{O}(s_n)\}$ be a strong exceptional collection of maximal length on $\mathbb{P}_{\mathbf{\Sigma}}$, where $s_1 < s_2 < \cdots < s_n$. Then we have $\mathcal{O}(s_1 + \sum_{i=1}^m w_i) \in \mathcal{D}\langle\mathcal{T}\rangle$.*

Proof. We consider the Koszul complex [4]

$$0 \to \mathcal{O}(-\sum_{i=1}^m w_i) \to \cdots \to \bigoplus_{i=1}^m \mathcal{O}(-w_i) \to \mathcal{O} \to 0.$$

Then we tensor this complex by $\mathcal{O}(s_1 + \sum_{i=1}^m w_i)$ and get

$$0 \to \mathcal{O}(s_1) \to \cdots \to \bigoplus_{i=1}^m \mathcal{O}(-\sum_{j\neq i} w_j + s_1) \to \mathcal{O}(s_1 + \sum_{i=1}^m w_i) \to 0.$$

By Lemma 3.7, we have that $\mathcal{O}(s_1 + \sum_{j\in J} w_j)$ is in $\mathcal{T}$ for any proper subset $J \subsetneqq \{1, 2, \dots, m\}$. Thus $\mathcal{O}(s_1 + \sum_{i=1}^m w_i) \in \mathcal{D}\langle\mathcal{T}\rangle$. □

Theorem 3.9. *Let $X = \mathbb{P}_{\mathbf{\Sigma}}$ be a Fano toric DM stack with* $\mathrm{rank}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 1$. *Assume $\mathcal{T} = \{\mathcal{O}(s_1), \dots, \mathcal{O}(s_n)\}$ is a strong exceptional collection of maximal length. Then $\mathcal{T}$ is a full strong exceptional collection.*

Proof. Without loss of generality, we assume $s_1 < s_2 < \cdots < s_n$. If $s_1 + \sum_{i=1}^m w_i \geq s_n$, then $\sum_{i=1}^m w_i \geq s_n - s_1$. Then $(s_1, \dots, s_n) = (s_1, s_1 + 1, \dots, s_1 + \sum_{i=1}^m w_i)$. So $\mathcal{T}$ is a twist of the collection of [4] and is therefore full. If $s_1 + \sum_{i=1}^m w_i < s_n$, we get a new strong exceptional collection

$$\mathcal{T}^1 = \{\mathcal{O}(s_2), \dots, \mathcal{O}(s_1 + \sum_{i=1}^m w_i), \dots, \mathcal{O}(s_n)\}$$

in $\mathcal{D}\langle\mathcal{T}\rangle$ by Lemma 3.6 and Corollary 3.8.

This process decreases $s_n - s_1$ and therefore terminates. So eventually we will be in the situation $s_1 + \sum_{i=1}^m w_i \geq s_n$. □

Remark 3.10. *When* $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ *has torsion, the arguments go without significant changes. The details are left to the reader.*

4. The case of $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 2$**.** In this section, we consider Fano toric Deligne-Mumford stack $\mathbb{P}_{\mathbf{\Sigma}}$ associated to a stacky fan $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^m)$ in the lattice N with $\mathrm{rk}(N) = m - 2$. In this case, the rank of Picard group $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}}))$ equals 2. Our aim is to prove Conjecture 1.1 in this case. We first assume that $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ has no torsion for ease of exposition.

We recall the results of [4].

Proposition 4.1 ([4]). *There exists a unique up to scaling collection of rational numbers α_i such that $\sum_{i=1}^m \alpha_i = 0$ and $\sum_{i=1}^m \alpha_i v_i = 0$. Moreover, all α_i in this relation are nonzero.*

Proof. See Proposition 5.4 in [4]. □

We pick one such relation $\sum_{i=1}^{m} \alpha_i v_i = 0$. Let $I_+ = \{i|\alpha_i > 0\}$ and $I_- = \{i|\alpha_i < 0\}$. Then we have $\{1, \ldots, m\} = I_+ \sqcup I_-$. Let $E_+ = \sum_{i \in I_+}(E_i)$ and $E_- = \sum_{i \in I_-}(E_i)$. We consider a linear function α on $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ with $\alpha(E_i) = \alpha_i$ from Proposition 4.1. Then $\alpha(E_+) + \alpha(E_-) = 0$.

Moreover, from [4], we can pick and fix a collection of positive rational numbers r_i, $i = 1, \ldots, m$ such that $\sum_i r_i = 1$ and $\sum_i r_i v_i = 0$. This collection of positive numbers gives a linear function f on $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ with $f(E_i) = r_i > 0$.

Let P be a parallelogram in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ given by

$$|f(x)| \leq \frac{1}{2}, |\alpha(x)| \leq \frac{1}{2} \sum_{i \in I_+} \alpha_i.$$

Pick a generic point $p \in \mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ so that the lines along the sides of the parallelogram $p + P$ do not contain any points from $\mathrm{Pic}_{\mathbb{Q}}(\mathbb{P}_{\mathbf{\Sigma}})$. Then we have the following.

Proposition 4.2 ([4]). *The set S of line bundles in $p + P$ forms a full strong exceptional collection on $\mathbb{P}_{\mathbf{\Sigma}}$.*

Proof. See Theorem 5.11 in [4]. □

Notation. The following notations will be used in our arguments. Let $\mathcal{T} = \{\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)\}$ be a collection of line bundles. We will abuse the notation slightly and denote by $\max(\alpha(\mathcal{T}))$ the maximum value of $\alpha(D_i)$ for $\mathcal{O}(D_i)$ in $\mathcal{T}$ (and similarly, for min and f). We denote $\mathcal{T}_{\min(f)} = \{D_i \in \mathcal{T} | f(D_i) = \min(f(\mathcal{T}))\}$.

Main idea. The idea of the proof is similar to the case $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 1$. Starting from an exceptional collection $\mathcal{T}$ of line bundles of maximal length, we construct other exceptional collections of maximal length in $\mathcal{D}\langle\mathcal{T}\rangle$, the subcategory generated by elements in $\mathcal{T}$. Eventually, we get to the exceptional collection in Proposition 4.2.

Step 1. The first step is to "move" the largest elements in terms of the linear function α in the strong exceptional collection by $-E_+$ or E_- to construct a new strong exceptional collection in $\mathcal{D}\langle\mathcal{T}\rangle$, see Figure 2.

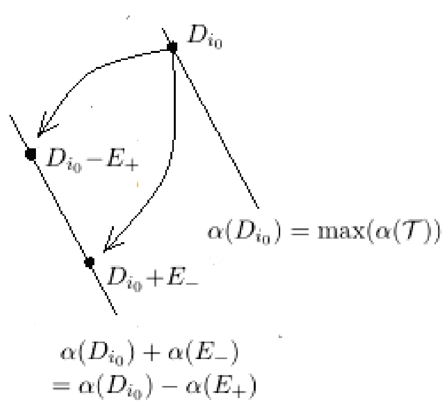

Fig. 2.

Specifically: let $\mathcal{T} = (\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n))$ be a strong exceptional collection of line bundles of maximal length. We pick $i_0 \in \{1, \ldots, n\}$ such that $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. Then

(1) Neither $\mathcal{O}(D_{i_0} - E_+)$ nor $\mathcal{O}(D_{i_0} + E_-)$ is in the strong exceptional collection $\mathcal{T}$ (Lemma 4.3);
(2) Either replacing $\mathcal{O}(D_{i_0})$ with $\mathcal{O}(D_{i_0} - E_+)$ or with $\mathcal{O}(D_{i_0} + E_-)$, we get another strong exceptional collection after reordering (Lemma 4.6, Lemma 4.9 and Lemma 4.10);
(3) The new exceptional collection in (2) is in $\mathcal{D}\langle\mathcal{T}\rangle$ (Lemma 4.7 and Lemma 4.8).

By repeating the above step (Theorem 4.11), we can reduce the problem to the strong exceptional collection $\mathcal{S}$ in $\mathcal{D}\langle\mathcal{T}\rangle$ such that all the line bundles in $\mathcal{S}$ are within a strip of width less than $\alpha(E_+)$, i.e., $\max(\alpha(\mathcal{S})) - \min(\alpha(\mathcal{S})) < \alpha(E_+) = \alpha(-E_-)$.

Step 2. From now on, we consider a strong exceptional collection $\mathcal{T} = (\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n))$ of maximal length within a strip of width less than $\alpha(E_+)$. If $\max(f(\mathcal{T})) - \min(f(\mathcal{T})) < f(E_+ + E_-) = 1$, then $\mathcal{T}$ is a full strong exceptional collection in Proposition 4.2. This allows us to conclude that $\mathcal{D}\langle\mathcal{T}\rangle = \mathbf{D}^b(coh(\mathbb{P}_{\mathbf{\Sigma}}))$.

Now, we assume $\max(f(\mathcal{T})) - \min(f(\mathcal{T})) \geq f(E_+ + E_-) = 1$. We pick $j_0 \in \{1, \ldots, n\}$ such that $\alpha(D_{j_0}) = \max(\alpha(\mathcal{T}))$. Then we can replace $\mathcal{O}(D_{i_0})$ with $\mathcal{O}(D_{i_0} - E_+)$ or $\mathcal{O}(D_{i_0} + E_-)$ to get another strong exceptional collection $\mathcal{T}'$ such that (Proposition 4.12):

(1) $\max(f(\mathcal{T}')) \leq \max(f(\mathcal{T}))$;
(2) $\min(f(\mathcal{T}')) \geq \min(f(\mathcal{T}))$;
(3) $\sharp(\mathcal{T}'_{\min(f)}) \leq \sharp(\mathcal{T}_{\min(f)})$ if $\min(f(\mathcal{T}')) = \min(f(\mathcal{T}))$;
(4) $\sharp(\{\mathcal{O}(D_i) \in \mathcal{T}' | f(\mathcal{O}(D_i)) = \min(f(\mathcal{T}))\}) < \sharp(\mathcal{T}_{\min(f)})$ if $f(\mathcal{O}(D_{i_0})) = \min(f(\mathcal{T}))$.

By repeating the above step (Theorem 4.14), we get a new strong exceptional collection $\mathcal{S}$ such that $\max(\alpha(\mathcal{S})) - \min(\alpha(\mathcal{S})) < \alpha(E_+) = \alpha(-E_-)$ and $\max(f(\mathcal{S})) - \min(f(\mathcal{S})) < f(E_+ + E_-) = 1$ which is one in Proposition 4.2. This allows us to conclude that $\mathcal{D}\langle\mathcal{T}\rangle = \mathbf{D}^b(coh(\mathbb{P}_{\mathbf{\Sigma}}))$.

Details of the proof. The nonzero Ext groups between line bundles on DM toric Fano stacks of Picard number two have been calculated in [4]. Specifically, we introduce two numbers a_+ and a_- by

$$a_\pm = |I_\mp| - 1.$$

The Fano condition implies that $a_\pm > 0$. The spaces Ext^i can only be nonzero if $i \in \{0, a_+, a_-, \mathrm{rk}(N)\}$. Specifically, for any $D_1, D_2 \in \mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$, we have

$$\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(D_1), \mathcal{O}(D_2)) \neq 0 \Leftrightarrow D_2 - D_1 \in \sum_{i \in \{1,\ldots,m\}} \mathbb{Z}_{<0} E_i;$$

$$\mathrm{Ext}^{a_+}(\mathcal{O}(D_1), \mathcal{O}(D_2)) \neq 0 \Leftrightarrow D_2 - D_1 \in \sum_{i \in I_-} \mathbb{Z}_{<0} E_i + \sum_{i \in I_+} \mathbb{Z}_{\geq 0} E_i;$$

$$\mathrm{Ext}^{a_-}(\mathcal{O}(D_1), \mathcal{O}(D_2)) \neq 0 \Leftrightarrow D_2 - D_1 \in \sum_{i \in I_+} \mathbb{Z}_{<0} E_i + \sum_{i \in I_-} \mathbb{Z}_{\geq 0} E_i;$$

$$\mathrm{Hom}(\mathcal{O}(D_1), \mathcal{O}(D_2)) \neq 0 \Leftrightarrow D_2 - D_1 \in \sum_{i \in \{1,\ldots,m\}} \mathbb{Z}_{\geq 0} E_i.$$

LEMMA 4.3. *Let* $\mathcal{T} = (\mathcal{O}(D_1), \dots, \mathcal{O}(D_n))$ *be a strong exceptional collection of line bundles on* $\mathbb{P}_{\mathbf{\Sigma}}$. *If* $i_0 \in \{1, \dots, n\}$, *then both* $\mathcal{O}(D_{i_0} - E_+)$ *and* $\mathcal{O}(D_{i_0} + E_-)$ *are not in* $\mathcal{T}$.

Proof. Immediately follows from the above description of Ext spaces between these bundles and $\mathcal{O}(D_{i_0})$ and the definition of a strong exceptional collection. ◻

For any subset $I \subseteq \{1, \dots, m\}$, we denote $E_I = \sum_{i \in I} E_i$.

LEMMA 4.4. *Let* $\mathcal{T} = \{\mathcal{O}(D_1), \dots, \mathcal{O}(D_n)\}$ *be a strong exceptional collection of line bundles on* $\mathbb{P}_{\mathbf{\Sigma}}$. *We pick* $i_0 \in \{1, \dots, n\}$ *such that* $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. *Then for any proper subset* J *of* I_+ *and any* $k \in \{1, \dots, n\}$, *we have*

$$\begin{aligned} &\mathrm{Ext}^*(\mathcal{O}(D_{i_0} - E_J), \mathcal{O}(D_k)) = 0, \ \textit{where} * = \mathrm{rk}(N), a_+, a_-; \\ &\mathrm{Ext}^*(\mathcal{O}(D_k), \mathcal{O}(D_{i_0} - E_J)) = 0, \ \textit{where} * = a_+, a_-. \end{aligned}$$

Proof. The vanishing of the spaces

$$\mathrm{Ext}^{\mathrm{rk}(N) \text{ or } a_-}(\mathcal{O}(D_{i_0} - E_J), \mathcal{O}(D_k)), \ \mathrm{Ext}^{a_+}(\mathcal{O}(D_k), \mathcal{O}(D_{i_0} - E_J))$$

follows immediately from the vanishing of the corresponding Ext spaces between $\mathcal{O}(D_{i_0})$ and $\mathcal{O}(D_k)$.

Suppose $\mathrm{Ext}^{a_+}(\mathcal{O}(D_{i_0} - E_J), \mathcal{O}(D_k)) \neq 0$. Then

$$D_k - D_{i_0} + E_J \in \sum_{i \in I_-} \mathbb{Z}_{<0} E_i + \sum_{i \in I_+} \mathbb{Z}_{\geq 0} E_i$$

and

$$D_k - D_{i_0} \in -E_- - E_J + \sum_{i \in I_-} \mathbb{Z}_{\leq 0} E_i + \sum_{i \in I_+} \mathbb{Z}_{\geq 0} E_i. \tag{4.1}$$

Therefore,

$$\alpha(D_k) - \alpha(D_{i_0}) \geq \alpha(-E_-) - \alpha(E_J) = \alpha(E_+) - \alpha(E_J) = \alpha(E_{I_+ \setminus J}) > 0 \tag{4.2}$$

in contradiction to maximality of $\alpha(D_{i_0})$.

The argument for $\mathrm{Ext}^{a_-}(\mathcal{O}(D_k), \mathcal{O}(D_{i_0} - E_J)) = 0$ similarly uses maximality of $\alpha(D_{i_0})$ and is left to the reader. ◻

LEMMA 4.5. *Let* $\mathcal{T} = \{\mathcal{O}(D_1), \dots, \mathcal{O}(D_n)\}$ *be a strong exceptional collection of line bundles on* $\mathbb{P}_{\mathbf{\Sigma}}$. *We pick* $i_0 \in \{1, \dots, n\}$ *such that* $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. *Then for any proper subset* L *of* I_- *and any* $j \in \{1, \dots, n\}$, *we have*

$$\begin{aligned} &\mathrm{Ext}^*(\mathcal{O}(D_{i_0} + E_L), \mathcal{O}(D_j)) = 0, \ \textit{where} * = a_+, a_-; \\ &\mathrm{Ext}^*(\mathcal{O}(D_j), \mathcal{O}(D_{i_0} + E_L)) = 0, \ \textit{where} * = \mathrm{rk}(N), a_+, a_-; \end{aligned}$$

Proof. The proof is analogous to the proof of Lemma 4.4 and is left to the reader. ◻

Note that Lemmas 4.4, 4.5 only cover vanishing of five out of possible six $\mathrm{Ext}^{>0}$ spaces. The next Lemma addresses the remaining space.

LEMMA 4.6. *Let* $\mathcal{T} = \{\mathcal{O}(D_1), \dots, \mathcal{O}(D_n)\}$ *be a strong exceptional collection of line bundles on* $\mathbb{P}_{\mathbf{\Sigma}}$. *We pick* $i_0 \in \{1, \dots, n\}$ *such that* $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$.

Then either $\mathrm{Ext}^{\mathrm{rk}(N)}(D_k, D_{i_0} - E_J) = 0$ *for all* $k \in \{1, \dots, n\}$ *and all* $J \subseteq I_+$ *or* $\mathrm{Ext}^{\mathrm{rk}(N)}(D_{i_0} + E_L, D_j) = 0$ *for all* $j \in \{1, \dots, n\}$ *and all* $L \subseteq I_-$*, or both.*

Proof. If $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(D_k), \mathcal{O}(D_{i_0} - E_J)) \neq 0$ for some k and some $J \subseteq I_+$, then

$$D_{i_0} - D_k - E_J \in \sum \mathbb{Z}_{<0} E_i = -E_- + \sum_{I^-} \mathbb{Z}_{\leq 0} E_i + \sum_{I^+} \mathbb{Z}_{<0} E_i.$$

If $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(D_{i_0} + E_L), \mathcal{O}(D_j)) \neq 0$ for some j and some $L \subseteq I_-$, then

$$D_j - D_{i_0} - E_L \in \sum \mathbb{Z}_{<0} E_i = -E_+ + \sum_{I^+} \mathbb{Z}_{\leq 0} E_i + \sum_{I^-} \mathbb{Z}_{<0} E_i.$$

We add the two statements to get

$$D_j - D_k - E_J - E_L \in -E_+ - E_- + \sum_{I^+} \mathbb{Z}_{<0} E_i + \sum_{I^-} \mathbb{Z}_{<0} E_i.$$

Thus

$$\begin{aligned} D_j - D_k &\in (-E_+ + E_J) + (-E_- + E_L) + \sum_{I^+} \mathbb{Z}_{<0} E_i + \sum_{I^-} \mathbb{Z}_{<0} E_i \\ &\subseteq \sum_{I^+} \mathbb{Z}_{<0} E_i + \sum_{I^-} \mathbb{Z}_{<0} E_i \end{aligned}$$

since $J \subseteq I_+$ and $L \subseteq I_-$. This implies $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(D_k), \mathcal{O}(D_j)) \neq 0$ which contradicts the assumption that $\mathcal{T}$ is a strong exceptional collection. □

LEMMA 4.7. *Let* $\mathcal{T} = \{\mathcal{O}(D_1), \dots, \mathcal{O}(D_n)\}$ *be a strong exceptional collection of maximal length on* $\mathbb{P}_{\mathbf{\Sigma}}$*. We pick* $i_0 \in \{1, \dots, n\}$ *such that* $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$*. Assume* $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(D_k), \mathcal{O}(D_{i_0} - E_J)) = 0$ *for all* $k \in \{1, \dots, n\}$ *and all proper subsets* $J \subsetneq I_+$*. Then* $\mathcal{O}(D_{i_0} - E_+) \in \mathcal{D}\langle\mathcal{T}\rangle$*.*

Proof. We have $\mathcal{O}(D_{i_0} - E_J) \in \mathcal{T}$ for all $J \subsetneq I_+$. Otherwise, there is $J \subsetneq I_+$ such that $\mathcal{O}(D_{i_0} - E_J) \notin \mathcal{T}$. By Lemma 4.4, we can add $\mathcal{O}(D_{i_0} - E_J)$ to $\mathcal{T}$ to get a strong exceptional collection with more than $\mathrm{rk}(K_0(\mathbb{P}_{\mathbf{\Sigma}}))$ elements. This is impossible by Corollary 2.8.

Now we consider the Koszul complex

$$0 \to \mathcal{O}(-E_+) \to \cdots \to \bigoplus_{i \in I_+} \mathcal{O}(-E_i) \to \mathcal{O} \to 0.$$

We tensor the complex by $\mathcal{O}(D_{i_0})$ to get

$$0 \to \mathcal{O}(D_{i_0} - E_+) \to \cdots \to \bigoplus_{i \in I_+} \mathcal{O}(-E_i + D_{i_0}) \to \mathcal{O}(D_{i_0}) \to 0.$$

Since $\mathcal{O}(D_{i_0} - E_J) \in \mathcal{T}$ for all $J \subsetneq I_+$, we get $\mathcal{O}(D_{i_0} - E_+) \in \mathcal{D}\langle\mathcal{T}\rangle$. □

LEMMA 4.8. *Let* $\mathcal{T} = \{\mathcal{O}(D_1), \dots, \mathcal{O}(D_n)\}$ *be a strong exceptional collection of maximal length on* $\mathbb{P}_{\mathbf{\Sigma}}$*. We pick* $i_0 \in \{1, \dots, n\}$ *such that* $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$*. Assume* $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(D_{i_0} + E_L), \mathcal{O}(D_j)) = 0$ *for any* $j \in \{1, \dots, n\}$ *for any subset* $L \subsetneq I_-$*. Then* $\mathcal{O}(D_{i_0} + E_-) \in \mathcal{D}\langle\mathcal{T}\rangle$*.*

Proof. Analogous to Lemma 4.7. □

LEMMA 4.9. *Let* $\mathcal{T} = \{\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)\}$ *be a strong exceptional collection of line bundles of maximal length on* $\mathbb{P}_{\mathbf{\Sigma}}$*. We pick* $i_0 \in \{1, \ldots, n\}$ *such that* $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$*. Assume* $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(D_k), \mathcal{O}(D_{i_0} - E_J)) = 0$ *for any* $k \in \{1, \ldots, n\}$ *and any subset* $J \subseteq I_+$*. Then we can get a new strong exceptional collection by replacing* $\mathcal{O}(D_{i_0})$ *with* $\mathcal{O}(D_{i_0} - E_+)$ *and reordering.*

Proof. We need to check vanishing of all six types of $\mathrm{Ext}^{>0}$ spaces with the new element of the collection. The argument is very similar to that of Lemma 4.4, with $J = I_+$ in this case. However, in (4.2) we will have $\alpha(D_k) - \alpha(D_{i_0}) \geq 0$ which is not a contradiction on its own, since we may have $\alpha(D_k) = \alpha(D_{i_0})$. However equality can only happen if in (4.1) we get

$$D_k - D_{i_0} = -E_- - E_J.$$

This implies $\mathrm{Ext}^{\mathrm{rk}(N)}(D_{i,0}, D_k) \neq 0$, in contradiction with $\mathcal{T}$ being a strong exceptional collection.

Similarly, $\mathrm{Ext}^{a_-}(\mathcal{O}(D_k), \mathcal{O}(D_{i_0} - E_+)) \neq 0$ can only happen if $D_k = D_{i_0}$, which is impossible, since the latter is no longer in the collection. Finally, by assumption, $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(D_k), \mathcal{O}(D_{i_0} - E_+)) = 0$ for all $k \in \{1, \ldots, n\}$. □

LEMMA 4.10. *Let* $\mathcal{T} = \{\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)\}$ *be a strong exceptional collection of line bundles of maximal length on* $\mathbb{P}_{\mathbf{\Sigma}}$*. We pick* $i_0 \in \{1, \ldots, n\}$ *such that* $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$*. Assume* $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(D_{i_0} + E_L), \mathcal{O}(D_j)) = 0$ *for any* $j \in \{1, \ldots, n\}$ *for any subset* $L \subseteq I_-$*. Then we can get a new strong exceptional collection in* $\mathcal{D}\langle\mathcal{T}\rangle$ *by replacing* $\mathcal{O}(D_{i_0})$ *with* $\mathcal{O}(D_{i_0} + E_-)$ *and reordering.*

Proof. Analogous to Lemma 4.9. □

PROPOSITION 4.11. *Let* $\mathcal{T} = \{\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)\}$ *be a strong exceptional collection of line bundles of maximal length on* $\mathbb{P}_{\mathbf{\Sigma}}$*. We can construct a new strong exceptional collection* $\mathcal{S}$ *in* $\mathcal{D}\langle\mathcal{T}\rangle$ *such that* $\max(\alpha(\mathcal{S})) - \min(\alpha(\mathcal{S})) < \alpha(E_+) = \alpha(-E_-)$.

Proof. The argument is similar to that of Theorem 3.9. Let $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. By Lemma 4.6, we have either $\mathrm{Ext}^{\mathrm{rk}(N)}(D_k, D_{i_0} - E_J) = 0$ for any $k \in \{1, \ldots, m\}$ and any $J \subseteq I_+$ or $\mathrm{Ext}^{\mathrm{rk}(N)}(D_{i_0} + E_L, D_j) = 0$ for any $j \in \{1, \ldots, m\}$ and any $L \subseteq I_-$. By Lemma 4.7, Lemma 4.8, Lemma 4.9 and Lemma 4.10, we get a new strong exceptional collection $\mathcal{T}'$ in $\mathcal{D}\langle\mathcal{T}\rangle$ by replacing $\mathcal{O}(D_{i_0})$ by $\mathcal{O}(D_{i_0} - E_+)$ or $\mathcal{O}(D_{i_0} + E_-)$, and reordering. See Figure 2.

We have $\max(\alpha(\mathcal{T}')) \leq \max(\alpha(\mathcal{T}))$ since $\alpha(D_{i_0} - E_+) = \alpha(D_{i_0} + E_-) < \alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. After a finite number of steps, we replace successively all $\mathcal{O}(D_i)$ such that $\alpha(D_i) = \max(\alpha(\mathcal{T}))$ by $\mathcal{O}(D_i - E_+)$ or $\mathcal{O}(D_i + E_-)$ to get a new strong exceptional collection $\mathcal{T}^1$ in $\mathcal{D}\langle\mathcal{T}\rangle$ such that $\max(\alpha(\mathcal{T}^1)) < \max(\alpha(\mathcal{T}))$.

If $\min(\alpha(\mathcal{T}^1)) < \min(\alpha(\mathcal{T}))$, there exists some D_i such that $\alpha(D_i) = \max(\alpha(\mathcal{T}))$ and $\alpha(D_i \mp E_\pm) = \min(\alpha(\mathcal{T}^1))$. Now we have

$$\max(\alpha(\mathcal{T}^1)) - \min(\alpha(\mathcal{T}^1)) < \max(\alpha(\mathcal{T})) - \min(\alpha(\mathcal{T}^1)) = \alpha(D_i) - \alpha(D_i \mp E_\pm) = \alpha(E_+).$$

If $\min(\alpha(\mathcal{T}^1)) \geq \min(\alpha(\mathcal{T}))$, then $\max(\alpha(\mathcal{T}^1)) - \min(\alpha(\mathcal{T}^1)) < \max(\alpha(\mathcal{T})) - \min(\alpha(\mathcal{T}))$.

This process decreases $\max(\alpha(\mathcal{T})) - \min(\alpha(\mathcal{T}))$. Since α is rational, these numbers have a fixed denominator and thus the process terminates. This means that eventually we will be in the situation with $\max(\alpha(\mathcal{T})) - \min(\alpha(\mathcal{T})) < \alpha(E_+) = \alpha(-E_-)$. □

We will now proceed with the second step of the process.

Proposition 4.12. *Let $\mathcal{T} = \{\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)\}$ be a strong exceptional collection of line bundles with length $n = \mathrm{rk}(K_0(\mathbb{P}_{\mathbf{\Sigma}}))$. Assume*

$$\max(f(\mathcal{T})) - \min(f(\mathcal{T})) \geq f(E_+ + E_-) = 1.$$

We pick $i_0 \in \{1, \ldots, n\}$ such that $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. Then we can replace $\mathcal{O}(D_{i_0})$ with $\mathcal{O}(D_{i_0} - E_+)$ or $\mathcal{O}(D_{i_0} + E_-)$ to get another strong exceptional collection $\mathcal{T}'$ in $\mathcal{D}\langle\mathcal{T}\rangle$ such that:

(1) $\max(f(\mathcal{T}')) \leq \max(f(\mathcal{T}))$;
(2) $\min(f(\mathcal{T}')) \geq \min(f(\mathcal{T}))$;
(3) $\sharp(\mathcal{T}'_{\min(f)}) \leq \sharp(\mathcal{T}_{\min(f)})$ *if* $\min(f(\mathcal{T}')) = \min(f(\mathcal{T}))$;
(4) $\sharp(\{D_i \in \mathcal{T}' | f(D_i) = \min(f(\mathcal{T}))\}) < \sharp(\mathcal{T}_{\min(f)})$ *if* $f(D_{i_0}) = \min(f(\mathcal{T}))$.

Proof. If

$$\min(f(\mathcal{T})) < f(D_{i_0} - E_+) < f(D_{i_0} + E_-) \leq \max(f(\mathcal{T})), \tag{4.3}$$

by Lemma 4.6, Lemma 4.9 and Lemma 4.10, we can replace $\mathcal{O}(D_{i_0})$ with $\mathcal{O}(D_{i_0} - E_+)$ or $\mathcal{O}(D_{i_0} + E_-)$ to reach the result.

If Equation 4.3 fails, there are two cases to consider.

Case $f(D_{i_0} - E_+) \leq \min(f(\mathcal{T}))$. We have $f(D_{i_0} + E_-) \leq \max(f(\mathcal{T}))$ by the assumption that $\max(f(\mathcal{T})) - \min(f(\mathcal{T})) \geq f(E_+ + E_-) = 1$. We show that replacing $\mathcal{O}(D_{i_0})$ with $\mathcal{O}(D_{i_0} + E_-)$ is possible and will achieve our goal, see (2) of Figure 3. We have $\mathrm{Ext}^{\mathrm{rk}(N)}(D_{i_0} + E_L, D_j) = 0$ for all $j \in \{1, \ldots, m\}$ and $L \subsetneqq I_-$. Otherwise, we get

$$D_j - D_{i_0} - E_L \in \sum_{i \in \{1,\ldots,m\}} \mathbb{Z}_{<0} E_i = -E_+ - E_- + \sum_{i \in \{1,\ldots,m\}} \mathbb{Z}_{\leq 0} E_i$$

for some j and some $L \subsetneqq I_-$. Thus,

$$D_j - D_{i_0} + E_+ \in (E_L - E_-) + \sum_{i \in \{1,\ldots,m\}} \mathbb{Z}_{\leq 0} E_i.$$

Then $f(D_j - D_{i_0} + E_+) = f((E_L - E_-) + \sum_{i \in \{1,\ldots,m\}} \mathbb{Z}_{\leq 0} E_i) < 0$ which contradicts $f(D_{i_0} - E_+) \leq \min(f(\mathcal{T}))$. Then by Lemma 4.8, the line bundle $\mathcal{O}(D_{i_0} + E_-) \in \mathcal{D}\langle\mathcal{T}\rangle$.

Also, we have $\mathrm{Ext}^{\mathrm{rk}(N)}(D_{i_0} + E_-, D_j) = 0$ for all $j \in \{1, \ldots, m\}$. Otherwise, we get

$$D_j - D_{i_0} - E_- \in \sum_{i \in \{1,\ldots,m\}} \mathbb{Z}_{<0} E_i = -E_+ - E_- + \sum_{i \in \{1,\ldots,m\}} \mathbb{Z}_{\leq 0} E_i$$

for some j. Thus

$$D_j - D_{i_0} \in -E_+ + \sum_{i \in \{1,\ldots,m\}} \mathbb{Z}_{\leq 0} E_i.$$

If the coefficients in $\sum_{i \in \{1,\ldots,m\}} \mathbb{Z}_{\leq 0} E_i$ are not all zero, then $f(D_j - D_{i_0} + E_+) = f(\sum_{i \in \{1,\ldots,m\}} \mathbb{Z}_{\leq 0} E_i) < 0$, which contradicts that $f(D_{i_0} - E_+) \leq \min(f(\mathcal{T}))$. If the coefficients in $\sum_{i\in\{1,\dots,m\}}\mathbb{Z}_{\leq 0}E_i$ are all zero, then $D_j - D_{i_0} = -E_+$. This implies $\operatorname{Ext}^{a_-}(\mathcal{O}(D_{i_0}),\mathcal{O}(D_j)) \neq 0$ which contradicts the assumption that $\mathcal{T}$ is a strong exceptional collection.

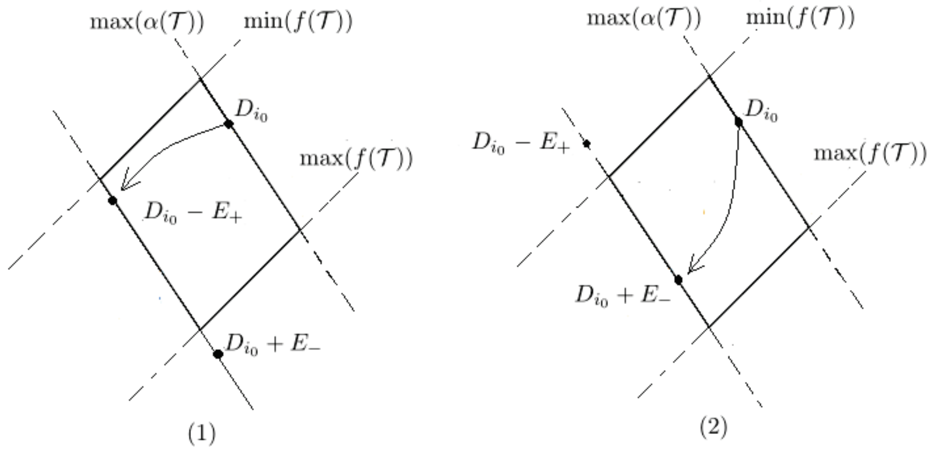

FIG. 3.

Then by Lemma 4.9, we get a strong exceptional collection $\mathcal{T}'$ in $\mathcal{D}\langle\mathcal{T}\rangle$ by replacing D_{i_0} with $\mathcal{O}(D_{i_0}+E_-)$ which satisfies (2), (3) and (4) of this Proposition. Since $f(D_{i_0}+E_-) \leq \max(f(\mathcal{T}))$, then $\max(f(\mathcal{T}')) \leq \max(f(\mathcal{T}))$.

Case $f(D_{i_0}+E_-) > \max(f(\mathcal{T}))$. We have $f(D_{i_0}-E_+) > \min(f(\mathcal{T}))$. By the same arguments, we can get a strong exceptional collection $\mathcal{T}'$ in $\mathcal{D}\langle\mathcal{T}\rangle$ by replacing D_{i_0} with $\mathcal{O}(D_{i_0}-E_+)$ which satisfies (1), (3) and (4) of this Proposition, see (1) of Figure 3. Since $f(D_{i_0}-E_+) > \min(f(\mathcal{T}))$, then $\min(f(\mathcal{T}')) \geq \min(f(\mathcal{T}))$. □

REMARK 4.13. *Let* $\mathcal{T} = \{\mathcal{O}(D_1),\dots,\mathcal{O}(D_n)\}$ *be a strong exceptional collection of line bundles with length* $n = \operatorname{rk}(K_0(\mathbb{P}_{\mathbf{\Sigma}}))$. *Assume all line bundles in* $\mathcal{T}$ *are within a strip of* α *with width less than* $\alpha(E_+)$ *and* $\max(f(\mathcal{T}))-\min(f(\mathcal{T})) \geq f(E_++E_-) = 1$. *After doing the move in Proposition 4.12, we can guarantee that all line bundles in the new strong exceptional collection is within a strip of* α *with width less or equal to* $\alpha(E_+)$. *After replacing all* D_j *in* $\mathcal{T}$ *such that* $\alpha(D_j) = \max(\alpha(\mathcal{T}))$, *we get the width of the strip of* α *to be less than* $\alpha(E_+)$.

THEOREM 4.14. *Let* $\mathbb{P}_{\mathbf{\Sigma}}$ *be a Fano toric DM stack with* $\operatorname{rank}(\operatorname{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 2$. *Assume* $\mathcal{T} = \{\mathcal{O}(D_1),\dots,\mathcal{O}(D_n)\}$ *be a strong exceptional collection of line bundles with length* $n = \operatorname{rk}(K_0(\mathbb{P}_{\mathbf{\Sigma}}))$. *Then* $\mathcal{T}$ *is a full strong exceptional collection.*

Proof. Without of loss of generality, we can assume that $\max(\alpha(\mathcal{T})) - \min(\alpha(\mathcal{T})) < \alpha(E_+) = \alpha(-E_-)$ by Proposition 4.11.

Let D_j be an element in $\mathcal{T}$ such that $f(D_j) = \min(f(\mathcal{T}))$. If $\alpha(D_j) = \max(\alpha(\mathcal{T}))$, then by Proposition 4.12, after replacing $\mathcal{O}(D_j)$ with $\mathcal{O}(D_j - E_+)$ or $\mathcal{O}(D_j + E_-)$, we get another strong exceptional collection $\mathcal{T}'$ such that $\sharp(\{D_i \in \mathcal{T}' | f(D_i) = \min(f(\mathcal{T}))\}) < \sharp(\mathcal{T}_{\min(f)})$. If $\alpha(D_j) < \max(\alpha(\mathcal{T}))$, then by repeating the process in Proposition 4.12 several times, we will get to the situation that α takes maximal value at D_j, see Figure 4.

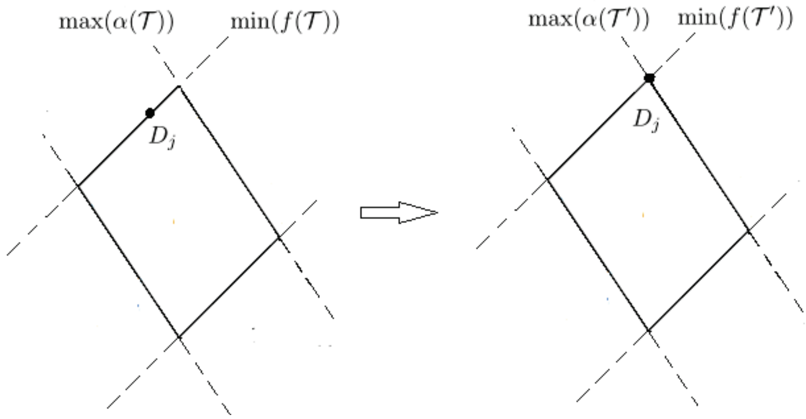

FIG. 4.

After replacing all elements in $\mathcal{T}_{\min(f)}$, we get $\min(f(\mathcal{T}))$ increase. Then we continue to apply Proposition 4.12. During the process, we assure that $\max(f(\mathcal{T}))$ does not increase and $\min(f(\mathcal{T}))$ increases. Thus $\max(f(\mathcal{T})) - \min(f(\mathcal{T}))$ decreases. Since f takes rational values, these numbers have a fixed denominator, so the process terminates. Eventually we will be in the situation $\max(f(\mathcal{T})) - \min(f(\mathcal{T})) < 1$.

Also, by Remark 4.13, we get a new strong exceptional collection $\mathcal{S}$ of line bundles in $\mathcal{D}\langle\mathcal{T}\rangle$ such that $\max(\alpha(\mathcal{S})) - \min(\alpha(\mathcal{S})) < \alpha(E_+)$ and $\max(f(\mathcal{S})) - \min(f(\mathcal{S})) < 1$. So $\mathcal{S}$ is a full strong exceptional collection by Proposition 4.2. Thus $\mathcal{D}\langle\mathcal{T}\rangle \supseteq \mathcal{D}\langle\mathcal{S}\rangle = \mathbf{D}^b(coh(\mathbb{P}_{\mathbf{\Sigma}}))$. □

REMARK 4.15. *When* $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ *has torsion, the arguments of this section go through without significant change. The details are left to the reader.*

REMARK 4.16. *In our proofs when we replace* $j_0 \in \{1, \dots, n\}$ *such that* $\alpha(D_{j_0}) = \max(\alpha(\mathcal{T}))$ *with* $\mathcal{O}(D_{i_0} - E_+)$ *or* $\mathcal{O}(D_{i_0} + E_-)$*, the strong exceptional collection "shrinks" in* $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$*. We would like to find a more geometric meaning of this phenomenon. It was pointed out to us by the referee that this operation might be related to flop autoequivalences of the total space of the canonical bundle of* $\mathbb{P}_{\mathbf{\Sigma}}$*.*

5. Comments. We expect our main result to be valid without the assumption on the rank of the Picard group, as stated in Conjecture 1.1. Also, in the case of $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 1$, we know that any exceptional collection of line bundles is a strong exceptional collection by Remark 3.2. However, in the case of $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 2$, Theorem 4.14 does not tell us that every exceptional collection of maximal length is a full exceptional collection. Thus we hope we can drop strong assumption to ask whether every exceptional collection of maximal length is a full exceptional collection. The possible future directions include dimension two $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 3$ Fano case, and dimension two non-Fano case. We hope that techniques of this paper can be modified to settle them.

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