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SPECTRA RELATED TO THE LENGTH SPECTRUM*

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Abstract. We extend the Covering Spectrum (CS) of Sormani-Wei to two spectra, called the Extended Covering Spectrum (ECS) and Entourage Spectrum (ES) that are new metric invariants related to the Length Spectrum for Riemannian manifolds. These spectra measure the "size" of certain covering maps called *entourage covers* that generalize the δ-covers of Sormani-Wei. For Riemannian manifolds M of dimension at least 3, we topologically characterize entourage covers as those covers corresponding to subgroups of $\pi_1(M)$ that are the normal closures of finite subsets. We show that CS⊂ES⊂MLS, where MLS is the set of lengths of closed curves that are shortest in their free homotopy classes. For Riemannian manifolds these inclusions can be strict. Finally, we give equivalent definitions for any metric on a Peano continuum, including resistance metrics on fractals, which have a Laplace Spectrum, opening new fronts in the old problem of the relationship between the Laplace Spectrum and the Length Spectrum.

Key words. length spectrum, covering spectrum, Laplace spectrum, resistance metrics on fractals.

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1. Introduction. The Length Spectrum (LS) of a Riemannian manifold is the set of lengths of closed geodesics, with various notions of multiplicity. The notion goes back at least to Huber's papers in the late 1950's ([16], [17]) in which the notion of LS is defined for what seems to be the first time. He showed that for compact Riemann surfaces, LS and the Laplace Spectrum (LaS) determine one another. Put another way, two Riemann surfaces have the same LS if and only if they are isospectral. These results have been followed by a decades-long investigation into the relationship between LS and LaS. We will not give a detailed history here, but will mention the fundamental open question of whether isospectral compact Riemannian manifolds have the same Weak Length Spectrum (defined as LS ignoring multiplicities, also called the Absolute Length Spectrum). For Riemann surfaces it is also known that LS and LaS are each completely determined by a finite subset, the size of which is bounded in the first case by the injectivity radius (Theorem 10.1.4, [7]), and in the second case by the injectivity radius and the genus (Theorem 14.10.1, [7]). This result suggests that there may be interesting relationships among geometrically and topologically significant subsets of LS, LaS, and other spectra from geometric analysis.

One of the most important subsets of LS is what Carolyn Gordon called the [L]-spectrum ([12]) and Christina Sormani and Guofang Wei called the Minimum Length Spectrum (MLS) in [28]. MLS is the set of lengths of curves that are shortest in their free homotopy classes. As is well-known (more on this later), in a compact Riemannian manifold there is always a shortest curve in every free homotopy class and that curve must be a closed geodesic, hence MLS $\subset$ LS. In [12], Gordon showed that there are isospectral manifolds with distinct MLS, considering multiplicity as the number of distinct free homotopy classes.

In [28], Sormani-Wei introduced a subset of $\frac{1}{2}$MLS called the Covering Spectrum (CS). The existence of isospectral compact Riemannian manifolds with different CS was established in 2010 by Bart de Smit, Ruth Gornet and Craig Sutton ([10], dimen-

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sions ≥ 3 and [11], surfaces). Athough CS and MLS are not “spectral invariants”, these spectra have mathematical applications, some of which we will mention below, and they may have nicer properties than LS. For example MLS is discrete and CS is finite for any compact Riemannian manifold, but LS may not be discrete ([25]).

In this paper we show how to extend CS to two spectra, called the Extended Covering Spectrum (ECS) and the Entourage Spectrum (ES), that are new even for compact Riemannian manifolds. ECS is discrete for arbitrary metrics on Peano Continua (compact, connected, locally path connected spaces), but may contain arbitrarily small values and so generally is not contained in $\frac{1}{2}$LS. ES contains 2CS and is contained in MLS, and therefore is discrete for Riemannian manifolds, but we do not know whether it is discrete for arbitrary compact geodesic spaces (metric spaces in which every x, y are joined by a curve, called a geodesic, of length equal to $d(x, y)$).

We show by example that for compact Riemannian manifolds, CS may be properly contained in ECS and ES, and ES may be properly contained in MLS; that is, in general all of these spectra are distinct. Counting multiplicity, the cardinality of ES, like MLS but unlike CS, is a topological invariant, independent of any metric.

We also show how to extend the notion of MLS to compact geodesic spaces in general, where, contrary to statements in [13] and [14], there may be no shortest curve in a free homotopy class, and when there is one it may not be a closed geodesic ([3]). For all of these length-type spectra except LS itself, we give equivalent alternative definitions for geodesic spaces that don't actually involve lengths of curves. That is, we define “length spectra” when there may be no length. Of particular interest are resistance metrics on self-similar fractals such as the Sierpinski Gasket and Carpet (which are Peano continua). These metric spaces have a meaningful notion of Laplacian (see [18], [35] for general references), but they are generally far from being geodesic spaces. In fact there may be no non-constant rectifiable curves at all, and hence empty LS. On the other hand, we show in [22] that all of the generalized spectra discussed in the present paper have infinitely many values for these resistance metric spaces, hence they provide good proxies for questions about the relationship between subsets of LS and LaS.

There are at least five characterizations of CS for compact geodesic spaces, most of which we will use at some point in this paper, and we will add two more. The original definition of CS due to Sormani-Wei ([28]) begins with a general construction of Spanier ([34]) that produces a regular covering map determined by an open cover $\mathcal{U}$ of a connected, locally path connected space X. In $\pi_1(X)$, let $\Gamma(\mathcal{U})$ be the (normal) subgroup generated by the set of homotopy classes of loops of the form $\overline{\alpha} * \lambda * \alpha$ where λ lies entirely in one of the open sets in $\mathcal{U}$ and α starts at the basepoint. Here $*$ denotes concatenation and $\overline{\alpha}$ is the reverse parameterization of α. According to Spanier there is a regular covering map of X such that $\Gamma(\mathcal{U})$ is the image of $\pi_1(X)$ via the homomorphism induced by the covering map. The deck group $\pi_1(X)/\Gamma(\mathcal{U})$ of the covering map is in a sense a “fundamental group at the scale of $\mathcal{U}$” because it “ignores small holes” contained in elements of $\mathcal{U}$ when modding out by $\Gamma(\mathcal{U})$. Sormani-Wei took for $\mathcal{U}$ the open cover of X by (open) δ-balls $B(x, \delta)$, and called the resulting covering map the δ-cover of X. In the case when X is a compact geodesic space, Sormani-Wei showed that the equivalence type of δ-covers changes at certain values, which they called the Covering Spectrum, that are discrete in $(0, \infty)$. Viewing δ as a parameter, as δ shrinks from the diameter of X to 0, X “unrolls” more and more at the discrete values in CS. If X has a universal cover $\widetilde{X}$, then $\widetilde{X}$ is the δ-cover for all sufficiently small $\delta > 0$ and CS is finite. By “universal cover” we mean in a categorical sense, which for compact geodesic spaces is equivalent to finiteness of CS (see Theorem 3.4 in [28] and [36] for related equivalent conditions).

Another way to characterize CS uses the discrete homotopy methods of Berestovskii-Plaut, developed in 2001 for topological groups ([4]) then uniform spaces in 2007 ([5]). In 2010, Plaut-Wilkins ([20]) focused on the special case of metric spaces, where discrete homotopy theory means replacing continuous curves and homotopies by discrete sequences and homotopies called ε-chains and ε-homotopies, respectively. An ε-chain is a finite sequence of points $\{x_0, ..., x_n\}$ in a metric space such that for all i, $d(x_i, x_{i+1}) < \varepsilon$. Discrete homotopies consist of finitely many steps adding or removing a single point in an ε-chain (fixing the endpoints) so that the sequence remains an ε-chain at each step. Discrete homotopies "ignore small holes" simply by skipping over them. As in [4] and [5], one can imitate the classical construction of the universal covering space, substituting ε-chains for curves and ε-homotopies for homotopies. This produces what Plaut-Wilkins called ε-covers $\phi_\varepsilon : X_\varepsilon \to X$. The fact that both ways of "ignoring small holes" are essentially the same (despite the very different constructions) was shown by Plaut-Wilkins in [21]: for compact geodesic spaces, the Sormani-Wei δ-covers are equivalent to the Plaut-Wilkins ε-covers when $\varepsilon = \frac{2}{3}\delta$. Plaut-Wilkins also defined the Homotopy Critical Spectrum (HCS) in [20] to be the set of all ε such that there is an ε-loop that is not ε-null (homotopic) but is δ-null when considered as a δ-chain for any $\delta > \varepsilon$. They also showed in [21] that $\mathrm{CS} = \frac{3}{2}\mathrm{HCS}$ for compact geodesic spaces. In this paper, since we will refer frequently to [20] and [21], we will generally use the notation of ε-covers. Plaut-Wilkins also defined special closed geodesics called "essential circles" whose lengths are precisely three times the values of HCS, discussed in more detail later in this paper.

ECS, ES, and our generalized definition of MLS involve expanding the class of ε-covers to a larger class of covering spaces called *entourage covers.* Entourage covers are defined using the original construction of Berestovskii-Plaut for uniform spaces ([5]). The present paper is written so that no special knowledge of uniform spaces is required, and but the language and framework of uniform spaces are useful. If E is an "entourage" in a uniform space X, which is a special symmetric set containing an open subset of the diagonal in $X \times X$, one may define "E-chains" to be sequences $\{x_0, ..., x_n\}$ such that for all i, $(x_i, x_{i+1}) \in E$. Then E-homotopies and the corresponding covering map $\phi_E : X_E \to X$ may be defined analogous to ε-homotopies and ε-covers. We will provide more details in the next section. Compact topological spaces have a unique uniform structure in which entourages are just any symmetric subsets of $X \times X$ containing an open set containing the diagonal, and therefore the maps ϕ_E are determined only by the topology (whereas ε-covers are determined by the metric).

In metric spaces there are *metric entourages* $E_\varepsilon := \{(x, y) : d(x, y) < \varepsilon\}$ for $\varepsilon > 0$. E_ε-chains and E_ε-homotopies are precisely the ε-chains and ε-homotopies previously described. Metric entourages form a *basis* for a uniform structure uniquely determined by the metric (although there may be other uniform structures compatible with the topology). That is, entourages in a metric space are simply symmetric subsets of $X \times X$ that contain some E_ε. We also use an analogous notation for E-balls: $B(x, E) := \{y \in X : (x, y) \in E\}$. If an E-loop is E-homotopic to the trivial chain, we say it is *E-null.*

In general the covering maps ϕ_E can be problematic, especially if the balls $B(x, E)$ are not connected. For example, if X is a compact metric space that is not geodesic, the ε-covers many have infinitely many components and HCS may not only not be discrete, it may even be dense in $[0, 1]$ ([8], [37]). Disconnected metric balls also may occur in resistance metrics on finite graphs ([2], Remark 7.19) and have been numerically verified by Cucuringu-Strichartz for certain resistance metrics on the Sierpinski Gasket ([9], Section 4). For non-geodesic spaces, it is not clear whether focusing on metric entourages is likely to be a successful strategy. In some sense, these problems occur because there is a disconnect between the metric and the underlying topology, which is improperly "viewed" by metric balls.

We address this problem by restricting attention to what we call "chained entourages": an entourage E is *chained* if it is contained in the closure of its interior, which is assumed compact if the space is not, and whenever $(x, y) \in E$, x, y may be joined by an F-chain that lies entirely in $B(x, E) \cap B(y, E)$, for any entourage F. That is, x and y may be joined in $B(x, E) \cap B(y, E)$ by "arbitrarily fine" chains. *Entourage covers* are by definition those covers ϕ_E such that E is a chained entourage; we will sometimes call it the E-cover of X. In a geodesic space, metric entourages are always chained. This is true because by the triangle inequality, any geodesic joining x, y must stay inside $B(x, \varepsilon) \cap B(y, \varepsilon)$. Then one may simply take arbitrarily fine chains along the geodesic.

One advantage of discrete methods is that they are amenable to counting arguments. For example, while Sormani-Wei showed using convergence methods that the size of CS is bounded in any Gromov-Hausdorff precompact class, in [20] we actually give an explicit bound. Moreover, since for compact Riemannian manifolds $\pi_1(M) = \pi_\varepsilon(M)$ for all sufficiently small $\varepsilon > 0$, by building a simplicial model of the space, Plaut-Wilkins were able to give an explicit fundamental group finiteness theorem generalizing those of Anderson ([1]) and Shen-Wei ([26]). Similarly, we are able to prove the following explicit finiteness theorem, where $C(X, \varepsilon)$ denotes the number of ε-balls needed to cover X and $\sigma(E) := \sup\{\varepsilon : E_\varepsilon \subset E\}$ (which is a measure of "size" of E).

Theorem 1. *Let X be a compact geodesic space and $\varepsilon > 0$. Then the number $NC(\varepsilon)$ of equivalence classes of E-covers $\phi_E : X_E \to X$ such that $\sigma(E) \geq \varepsilon$ is at most*

$$2^{C(X,\frac{\varepsilon}{4})^{40C(X,\frac{\varepsilon}{2})}}.$$

In order to apply the above theorem in more generality, recall that the Bing-Moise Theorem ([6], [19]) says, in modern terminology, that every Peano continuum has a compatible geodesic metric. "Compatible" means precisely that every metric entourage in the original metric contains a metric entourage in the geodesic metric, and vice versa. We immediately obtain:

Corollary 2. *If X is a Peano Continuum with a given (possibly non-geodesic) metric and $\varepsilon > 0$, then $NC(\varepsilon) < \infty$.*

We are now in a position to modify the original Sormani-Wei definition of CS to apply to entourage covers. The only complication is that E-covers, unlike ε-covers, are not totally ordered by the relation of one space covering another. But by Corollary 2 there are certain discrete values of ε such that $NC(\varepsilon)$ *strictly increases*, and we define ECS to be those values. We may also define the *multiplicity* of a value ε in ECS to be $NC(\varepsilon) - NC(\delta)$ for $\delta < \varepsilon$ sufficiently close to ε. With a little more effort we show:

Theorem 3. *If X is a compact geodesic space then CS $\subset$ ECS. For some compact Riemannian manifolds, ECS may be infinite (while CS is always finite) and hence this inclusion may be proper.*

An immediate consequence of Corollary 2 is that ECS is discrete in $(0, \infty)$ for any metric on a Peano continuum.

REMARK 4. *By Gromov's Precompactness Criterion ([13], [14]), a corollary of Theorem 1 is that $NC(\varepsilon)$, and hence the number of elements of ECS greater than ε, is uniformly bounded below for any fixed ε in any Gromov-Hausdorff precompact class of compact geodesic spaces. This extends the corresponding statement about CS.*

We aleady know that any ε-cover for any geodesic metric is an entourage cover, and a natural question is: in general, which regular covers are entourage covers? We have the following necessary algebraic condition. The meaning of "covering map corresponding to N" is standard from algebraic topology and will be reviewed as part of the proof.

THEOREM 5. *Let X be a semi-locally simply connected Peano continuum and $N \subset \pi_1(X)$ be a normal subgroup. If the covering map corresponding to N is an entourage cover, then N is the normal closure of a finite set.*

REMARK 6. *The normal closure of a subset of a group is by definition the smallest normal subgroup containing it. Being the normal closure of a finite set is intimately connected with the study of finitely presented groups, since a quotient of a finitely presented group is finitely presented if and only if the kernel is the normal closure of a finite set. We are unable to find a reference for this equivalence, although the proof of one implication may be found in [15], Lemma 3, which cites Siebenmann's dissertation for the statement. The other implication is an exercise in basic algebra using the formal definition of "finitely presented".*

REMARK 7. *Suppose that G is a finitely presented group with a quotient that is not finitely presented (such G are well-known to exist). Then as is also classically known, one may construct a compact 4-manifold M with G as its fundamental group. Therefore M must have a regular cover that is not an entourage cover.*

For manifolds of dimension at least 3, the condition in Theorem 5 is also sufficient:

THEOREM 8. *Let M be any compact smooth manifold of dimension at least 3. If $G \subset \pi_1(M)$ is the normal closure of a finite set then M has a Riemannian metric for which the cover corresponding to G is an ε-cover for the metric.*

COROLLARY 9. *If M is a smooth manifold of dimension at least 3, a normal subgroup G of $\pi_1(X)$ corresponds to an E-cover if and only if G is the normal closure of a finite set. Moreover, we may always take E to be an open entourage.*

For closed manifolds of dimension 1 there is a simple statement: the only entourage covers of the circle are the trivial and universal covers (which are always entourage covers for manifolds), see Example 70. We do not know much about the situation for closed surfaces, including, for example, exactly which covers of the 2-torus are entourage covers. See also Example 71 concerning the Möbius Band.

"Circle covers" were introduced in [21] as a way to precisely describe the limits of ε-covers, following a line of inquiry of Sormani-Wei ([28]). Since circle covers are defined as quotients of ε-covers via the normal closures of finite sets (Definition 22 in [21]), the following corollary is immediate.

COROLLARY 10. *If M is a compact Riemannian manifold of dimension at least 3 then any circle cover of M is an entourage cover.*

Remark 11. *There are various interesting questions at this point. Is Corollary 10 more generally true for compact geodesic spaces, even without the dimension restriction? Suppose that $X_i \to X$ are compact geodesic spaces converging in the Gromov-Hausdorff metric, E_i is a chained entourage in X_i, and for some $\varepsilon > 0$, $E_\varepsilon \subset E_i$ for all i. Is it true that, taking a subsequence if necessary, X_{E_i} converges in the pointed Gromov-Hausdorff metric to X_E for some chained entourage E? Due to issues of dimension, Corollary 9 is not true for compact geodesic spaces in general (attach circle to a 3-sphere at one point, for example). Nonetheless, one may ask whether the theorem is be true for some special classes of Gromov-Hausdorff limits of compact Riemannian manifolds of dimension at least 3, for example with Ricci curvature and volume uniformly bounded below and diameter uniformly bounded above.*

We will define later in this paper (Definition 47) a notion of E-homotopy (or free E-homotopy) of curves, which essentially means the two curves may be "E-subdivided" into E-chains that are E-homotopic (or freely E-homotopic). Equivalently, the lifts of the curve to a given point in X_E have the same endpoints (Lemma 48). We say a chained entourage is "essential" if there is an E-loop that is not E-null. The next proposition (correctly) generalizes the classical statement mentioned above about shortest curves in free homotopy classes in Riemannian manifolds. We will use the term "ε-geodesic" to describe a curve that is arclength parameterized and minimizing on all intervals of length ε, i.e. the distance between the endpoints of such segments is ε. This is similar to the concept of $\frac{1}{k}$-geodesic of Sormani ([27]) but we obtain more precise results by allowing arbitrary values of ε. A closed curve c that is an ε-geodesic for some $\varepsilon > 0$ for any reparameterization involving a parameter shift is called a "closed ε-geodesic" or simply a "closed geodesic" if a particular ε isn't specified. Briefly, we express this by saying c is minimal on all segments of length ε, understanding that when c is closed this includes segments that have the common start/end point in the interior.

Proposition 12. *Let X be a compact geodesic space, E be a chained entourage in X, and c be a closed curve in X. Then c has a shortest curve $\overline{c}$ in its free E-homotopy class, and for any such $\overline{c}$,*

1. *$\overline{c}$ is non-constant if and only if E is essential, and*
2. *if $\overline{c}$ is non-constant then $\overline{c}$ is a closed $\frac{3\varepsilon}{2}$ geodesic whenever $E_\varepsilon \subset E$.*

We now have the following characterizations of MLS and CS for Riemannian manifolds. Each pair consists of a statement involving lengths of curves (which in a general metric space might be infinite), and another that makes sense and is always finite in any metric space. By the length $L(\alpha)$ of a finite chain $\alpha = \{x_0, ...x_n\}$ in a metric space we mean $\sum_{i=1}^{n} d(x_{i-1}, x_i)$.

Theorem 13. *Suppose that M is a compact Riemannian manifold. Then over the set of all essential entourages E,*

1. *MLS is the set of*
   (a) *lengths of non-constant closed curves that are shortest in their free E-homotopy class.*
   (b) *lengths of non-trivial E-loops that are shortest in their free E-homotopy class.*
2. *CS is the set of*
   (a) *half the shortest lengths of closed curves that are not freely E-null.*
   (b) *half the shortest lengths of E-loops that are not freely E-null.*

We may now simply use Theorem 13.1.a as the *definition* of MLS for compact geodesic spaces, and with this definition Theorem 13 is true for any compact geodesic space (Theorem 69). We may use Theorem 13.1b and Theorem 13.2b as the definitions for arbitrary metric spaces, although if the underlying space is not a Peano continuum then there may not be many (or any!) essential entourages.

REMARK 14. *We do not know whether, for arbitrary metric spaces, the definition using Theorem 13.1b is equivalent to the definition of CS for metric spaces given in [10], namely that CS is the collection of all $\varepsilon > 0$ such that some covering map is maximally evenly covered on all ε-balls.*

While CS $= \frac{3}{2}$HCS for compact geodesic spaces, the approaches of Sormani-Wei and Plaut-Wilkins diverge when more general E-covers are added to the mix. The Sormani-Wei playbook leads to ECS as we have already described. To apply the Plaut-Wilkins approach, for an open, chained entourage E in a metric space X we define an E-loop λ (resp. curve loop c) to be *E-critical* if λ (resp. c) is not E-null but is $\overline{E}$-null, where $\overline{E}$ is the closure of E. If E has a critical E-loop then we will say that E is *critical*, and we let $\psi(E) := \inf\{L(\lambda) : \lambda \text{ is an } E\text{-critical } E\text{-loop}\}$. We define the Entourage Spectrum ES to be the set of $\psi(E)$ for all critical entourages E. Note that this definition does not involve lengths of curves. We show:

THEOREM 15. *Let X be a compact geodesic space. Then*

1. *For every open, chained entourage E there is a critical E-loop λ if and only if there is a critical loop c.*
2. *If E is a critical entourage then there is a critical E-loop (resp. critical loop c) such that $L(c) = L(\lambda) = \psi(E)$.*
3. *$2CS = 3HCS \subset ES \subset MLS$.*

Moreover, there are compact Riemannian manifolds for which both of the above inclusions are proper. There also are pairs of diffeomorphic Riemannian manifolds that have the same CS but different ES, and pairs that have the same ES but different MLS.

REMARK 16. *The proof of Theorem 15 uses in essential ways the fact that the metric is geodesic–in particular by lifting the metric to a geodesic metric on X_E. However, the "lifted metric" defined later in this paper is defined for any metric, and in particular some of these methods may be modified for arbitrary metrics on Peano continua ([22]).*

REMARK 17. *It seems there are interesting questions involving these metric invariants akin to classical results in the metric geometry of Riemannian manifolds. For example, suppose that M is a compact Riemannian manifold. Are there metrics having a particular fixed spectrum (pick one of CS, ECS, ES, MLS) that are optimal with respect to some other geometric parameters, for example having minimal volume with fixed bounds on sectional curvature? And if so what is the regularity of those optimal metrics? In effect this question fixes the size of certain significant "holes" in the space and asks how the space can minimally be stretched to maintain those sizes while constraining curvature.*

REMARK 18. *Sormani-Wei have explored ways in which to extend ideas related to CS to non-compact spaces ([30], [31], [32]). Certainly CS has meaning for non-compact geodesic spaces, for example, but it will "miss" any loops that are homotopic* *to arbitrarily small loops, for example in the surface obtained by revolving the graph of $y = e^x$ around the x-axis. Along these lines, we note that some of the basic and technical results in this paper only require a kind of uniform local compactness (Remark 39)–which is why we state in the definition of "chained" that the closure of the entourage is compact when the space is not. An alternative approach to understand non-compact spaces might be to consider all possible uniform structures that are compatible with the underlying topology of a given metric on the space, rather than just the uniform structure induced by the particular metric. For example, for the surface mentioned above, the uniform structure compatible with the metric does not "see" the fundamental group but the uniform structure of the same space metrized as a flat cylinder has the universal cover as an ε-cover.*

2. Basic Constructions. This section has a mixture of background from [5], extensions of some results in [20], and a completely new basic result called the Ball Continuity Lemma. The length of a curve is defined in the standard way for metric spaces and it is a classical result that curves having finite length (i.e. rectifiable curves) in metric spaces always have monotone reparameterizations proportional to arclength. We will always assume rectifiable curves are parameterized this way. See [23] for a review, with references, of many basic concepts from metric geometry.

We now recall a bit of basic terminology for uniform spaces. One should keep in mind the two fundamental examples mentioned in the Introduction: metric spaces and compact topological spaces. We have already defined the metric entourage E_ε in a metric space X. In general, a *uniform structure* on an (always Hausdorff) topological space X is a collection of symmetric subsets of $X \times X$ that contain an open set containing the diagonal, which are called *entourages.* Moreover, entourages have the following properties: (UA) Their intersection is the diagonal (equivalent to Hausdorff), and (UB) for every entourage E there exists an entourage F such that

$$F^2 := \{(x, z) : \text{for some } y, (x, y), (y, z) \in F\}$$

is contained in E. For metric entourages we note that it follows from the triangle inequality that $\left(E_{\frac{\varepsilon}{2}}\right)^2 \subset E_\varepsilon$. We may also iteratively define, for any entourage F, F^n. Equivalently, in the terminology from the Introduction, F^n consists of all $(x, y) \in X \times X$ such that there is an F-chain $\{x = x_0, ..., x_n = y\}$.

As mentioned in the Introduction, for an entourage E in a uniform space X, an E-chain consists of a finite sequence $\alpha = \{x_0, ..., x_n\}$ in X such that for all i, $(x_i, x_{i+1}) \in E$. We define $\nu(\alpha) = n$. The concatenation of two chains $\alpha = \{x_0, ..., x_n\}$ and $\beta = \{y_0 = x_n, y_1, ..., y_m\}$ is the chain $\alpha * \beta := \{x_0, ..., x_n = y_0, y_1, ..., y_m\}$ and the reversal of α is the chain $\overline{\alpha} = \{x_n, ..., x_0\}$. An E-homotopy between E-chains α and β consists of a finite sequence $\{\alpha = \eta_0, ..., \eta_n = \beta\}$ of E-chains η_i all having the same endpoints such that for all i, η_i differs from η_{i+1} by one of the following two *basic moves*:

(1) *Insert* a point x between x_i and x_{i+1}, which we will denote by

$$\{x_0, ..., x_i, \overbrace{x}, x_{i+1}, ..., x_n\}$$

and which is "legal" provided $(x_i, x) \in E$ and $(x, x_{i+1}) \in E$.

(2) *Remove* a point x_i (but never an endpoint!), which we will denote by

$$\{x_0, ..., x_{i-1}, \underbrace{x_i}, x_{i+1}, ..., x_n\}$$

and which is legal provided $(x_{i-1}, x_{i+1}) \in E$.

The E-homotopy equivalence class of an E-chain α is denoted by $[\alpha]_E$. We will sometimes abuse notation by dropping brackets, for example writing $[x_0, ..., x_n]_E$ rather than $[\{x_0, ..., x_n\}]_E$. We note that if $E \subset F$ then α may also be considered as an F-chain, and $[\alpha]_F$ also makes sense. Fixing a basepoint $*$, the collection of all $[\alpha]_E$ such that the first point of α is $*$ is called X_E. For any entourage $F \subset E$ in X, we define $F^* \subset X_E \times X_E$ to be the set of all ordered pairs $([\alpha]_E, [\beta]_E)$ such that $[\overline{\alpha} * \beta]_E = [x_n, y_m]_E$, where x_n, y_m are the endpoints of α, β, respectively, and $(x_n, y_m) \in F$. The sets F^* form (a basis of) a uniform structure on X_E, which we will call the "lifted uniform structure".

Remark 19. *It is an easy exercise, worthwhile for the unfamiliar reader, to check that this definition of F^* is equivalent to the more cumbersome but sometimes useful original from [5], namely that up to E-homotopy we may write $\alpha = \{* = x_0, ..., x_{n-1}, x_n\}$ and $\beta = \{* = x_0, ..., x_{n-1}, y_n\}$ with $(x_n, y_n) \in F$.*

Since E-homotopies don't change endpoints, the endpoint map $\phi_E : X_E \to X$, $\phi_E([x_0, ..., x_n]_E) := x_n$ is well-defined, and its restriction to any E^*-ball $B([\alpha]_E, E^*)$ is a bijection onto its image $B(x, E)$ under ϕ_E, where x is the endpoint of α. Since ϕ_E is a local bijection, X_E has a unique topology such that ϕ_E is a local homeomorphism, with a basis given by all F^*-balls with $F \subset E$. This topology is compatible with the lifted uniform structure.

Concatenation is compatible with E-homotopies–that is, if α_1, β_1 are E-homotopic to α_2, β_2, respectively, then $[\alpha_2 * \beta_2]_E = [\alpha_1 * \beta_1]_E$. Concatenation induces a group structure on the set of all E-homotopy classes $\pi_E(X)$ of E-loops starting and ending at the basepoint $*$. That is, $[\lambda_1]_E[\lambda_2]_E = [\lambda_1 * \lambda_2]_E$ and $[\lambda]_E^{-1} = [\overline{\lambda}]_E$, with $[*]_E$ as identity. The group $\pi_E(X)$ acts on X_E induced by pre-concatenation of any loop to an E-chain starting at $*$, and the resulting maps are uniform homeomorphisms (i.e. preserve the uniform structure). With this topology, ϕ_E is a regular covering map with deck group $\pi_E(X)$ such that the E-balls are evenly covered by disjoint unions of E^*-balls. Moreover, X is identified with the quotient space $X_E/\pi_E(X)$. Two E-loops λ_1 and λ_2 are said to be *freely* E-homotopic if there exist E-chains α and β starting at a common point x_0, to the initial points of λ_1 and λ_2, respectively, such that

$$\overline{\alpha} * \lambda_1 * \alpha \text{ is } E\text{-homotopic to } \overline{\beta} * \lambda_2 * \beta. \tag{1}$$

It is easy to check, and we will use without reference, the following facts: If we can satisfy Formula (1) for some x_0 then we can do it for any other point, including the basepoint $*$. Likewise λ_1 and λ_2 are freely E-homotopic if and only if given an E-chain α from x_0 to the initial point of λ_1 then we can always find a β so that Formula 1 is satisfied.

Remark 20. *One can equivalently define free E-homotopies of E-loops by adding a special "homotopy step" to enable one to move the common start/end point. That is, "double it" by adding a repeat of the start/end point, then add a new point (which becomes the new start/end point) between the doubled points. However, this does not seem simpler in all of its details, or more useful than the definition we have given.*

Whenever $E \subset F$ there is a natural covering map $\phi_{FE} : X_E \to X_F$ that simply treats an E-chain as an F-chain. That is, $\phi_{FE}([\alpha]_E) = [\alpha]_F$. Given $D \subset E \subset F$, by definition $\phi_{FD} = \phi_{FE} \circ \phi_{ED}$. The restriction of ϕ_{EF} to $\pi_F(X)$ is a homomorphism denoted by $\theta_{EF} : \pi_F(X) \to \pi_E(X)$. This homomorphism is injective (resp. surjective) if and only if ϕ_{EF} is injective (resp. surjective), and plays a critical role in this paper. Note that the mapping ϕ_{EF} may be identified with the quotient mapping from X_F to $X_F/\ker\theta_{EF} = X_E$. In the special case of metric entourages, we denote $\phi_{E_\varepsilon E_\delta}$ by $\phi_{\varepsilon\delta}$ as in [20].

We will need the following general lemma, which partly justifies why we require that the balls in a chained entourage be contained in the closure of their interior.

Lemma 21 (Ball Continuity). *Suppose E is an entourage in a metric space X and $x_i \to x$ in X. Then*

1. *If $y_i \to y$ in X and $(x_i, y_i) \in E$ for all i, then $(x, y) \in \overline{E}$.*
2. *If E has compact closure then $B(x_i, E)$ is Hausdorff convergent to a subset of $B(x, \overline{E})$.*
3. *If E is open then $B(x, E)$ is the union of the sets $B(x, E) \cap B(x_i, E)$.*
4. *If E is contained in the compact closure of its interior then $B(x_i, E) \underset{H}{\to} B(x, E)$ (meaning convergence in the Hausdorff metric).*

Proof. The first part is true in any uniform space: just note that a if U and V are open sets containing x and y, respectively, then $U \times V$ eventually contains some $(x_i, y_i) \in E$.

For the second part, let A be the (compact) set of all limits of convergent sequences (z_i) such that $z_i \in B(x_i, E)$. By the first part, $A \subset B(x, \overline{E})$. For any $\varepsilon > 0$, we may cover A by finitely many balls $B(p_1, \frac{\varepsilon}{2}), ..., B(p_m, \frac{\varepsilon}{2})$. Then for sufficiently large k there are points $p_j' \in B(x_k, E)$ such that $d(p_j, p_j') < \frac{\varepsilon}{2}$, and from the triangle inequality it follows that the ε-neighborhood of $B(x_k, E)$ contains A. Now suppose that for all k, the ε-neighborhood of A does not contain $B(x_k, E)$. That is, for all k there exist $w_k \in B(x_k, E)$ such that $d(w_k, z) \geq \varepsilon$ for all $z \in A$. Since $(w_k, x_k) \in E$ and E has compact closure, by taking a subsequence if necessary we may assume that (w_k, x_k) is convergent to $(w, x) \in \overline{E}$. By definition, $w \in A$, so $d(w_k, w) \geq \varepsilon$ for all k, a contradiction to $w_k \to w$.

For the third part, let $y \in B(x, E)$. Since E is open there exist open U, V in X such that $(x, y) \in U \times V \subset E$. Then for large enough i, $x_i \in U$ and therefore $(x_i, y) \in U \times V \subset E$, i.e. $y \in B(x_i, E)$.

Finally, in the special case when E is open with compact closure, the fourth part follows from the second and third parts. The proof in general is now finished by observing that in general if $A_i \underset{H}{\to} A$ then any sequence of dense subsets of A_i is Hausdorff convergent to any dense subset of A. □

We will now extend the idea of the "lifted metric" from [20] to this more general situation. The following definition is the same as the corresponding part of Definition 12, [20], with "E" replacing "ε":

Definition 22. *Let X be a metric space and $[\alpha]_E, [\beta]_E \in X_E$. We define*

$$|[\alpha]_E| := \inf\{L(\kappa) : [\alpha]_E = [\kappa]_E\}$$

and

$$d([\alpha]_E, [\beta]_E) = \left|[\overline{\alpha} * \beta]_E\right|.$$

The proof that d is a metric on X_E is essentially identical to the proof of this statement (Proposition 13) in [20]. Likewise, one may prove that the deck group $\pi_E(X)$ acts as isometries. However in [20], Proposition 14, we also proved that ϕ_ε

preserves all distances less than ε. We will verify here an important analog of this statement, namely:

LEMMA 23. *If X is a metric space, E is an entourage, and $([\alpha]_E, [\beta]_E) \in E^*$ then $d([\alpha]_E, [\beta]_E) = d(\phi_E([\alpha]_E), \phi_E([\beta]_E))$. In particular, if $F^2 \subset E$ then the restriction of ϕ_E to any F^*-ball is an isometry onto an F-ball.*

Proof. If $([\alpha]_E, [\beta]_E) \in E^*$ then by definition of E^*, $[\overline{\alpha} * \beta]_E = [x_n, y_m]_E$, where x_n, y_m are the endpoints of α, β, respectively. Since $[x_n, y_m]_E$ is the shortest possible E-chain joining x_n, y_m, $d([\alpha]_E, [\beta]_E) = d(x_n, y_m) = d(\phi_E([\alpha]_E), \phi_E([\beta]_E))$.

To prove the second part, suppose that $[\alpha]_E, [\beta]_E \in B([\gamma]_E, F^*)$. By definition, $([\alpha]_E, [\gamma]_E) \in F^*$ and $([\beta]_E, [\gamma]_E) \in F^*$. This in turn means that if the endpoints of α, β, γ are x_n, y_m, z_k, respectively, then $[\overline{\alpha} * \gamma]_E = [x_n, z_k]_E$ and $[\overline{\gamma} * \beta]_E = [z_k, y_m]_E$ with $(x_n, z_k), (z_k, y_m) \in F$. This implies that $(x_n, y_m) \in F^2 \subset E$. Next note that

$$[\overline{\alpha} * \beta]_E = [\overline{\alpha} * \gamma * \overline{\gamma} * \beta]_E = [x_n, z_k, z_k, y_m]_E = [x_n, z_k, y_m]_E.$$

Since $(x_n, y_m) \in E$, removing z_k is a legal move for an E-homotopy, proving that $([\alpha]_E, [\beta]_E) \in E^*$. By the first part of this lemma, $d([\alpha]_E, [\beta]_E)$ is preserved by ϕ_E. We know from [20] (and it is easy to verify) that the restriction of ϕ_E to any F^*-ball is a bijection onto an F-ball, finishing the proof. □

Since ϕ_E is a local isometry, it preserves the lengths of curves. If X is a geodesic space then the lifted metric is a geodesic metric and ϕ_E is distance non-increasing. In fact, one can check that in this case the metric we have defined is the unique metric with these properties (cf. [20], Proposition 23).

3. Refinement and Approximation. The underlying assumption for the main results in [20] and [21] is that the space in question is a geodesic space. Two fundamental issues appear when attempting to extend results from metric entourages in geodesic spaces to entourages in general. First, there is the issue of refinement. In a geodesic space, when $0 < \delta < \varepsilon$, any ε-chain $\alpha = \{x_0, ..., x_n\}$ can always be "refined" into a δ-chain that is in the same ε-homotopy class as α. This is accomplished simply by subdividing a geodesic joining x_i and x_{i+1}. Lack of some method of refinement in some sense "causes" the problems observed in [8] and [37] concerning the HCS in non-geodesic metric spaces. As we will see, restricting to chained entourages solves this problem.

The second issue in this generality is that metric entourages are totally ordered by inclusion, but entourages in general are not. This issue is unavoidable and has many implications. For example, as soon as there are two entourage covers of a space, neither of which covers the other, then it is impossible that those two entourage covers may be simultaneously an ε_1-cover and an ε_2-cover for a single geodesic metric. Hence it is generally impossible to realize all entourage covers as ε-covers of a single geodesic metric. On the other hand, as we will see, Theorem 8 shows that for smooth manifolds of dimension ≥ 3, every entourage cover is realized as an ε-cover for *some* Riemannian metric.

DEFINITION 24. *Let X be a uniform space, $E \subset F$ be entourages, and $\alpha = \{x_0, ..., x_n\}$ be an F-chain. An E-refinement of α is an E-chain of the form*

$$\{x_0 = m_{00}, ..., m_{0k_0} = x_1, ..., x_r = m_{r0}, ..., m_{rk_r} = x_{r+1}, ..., x_n\}$$

such that each E-chain $\{x_j = m_{j0}, ..., m_{jk_j} = x_{j+1}\}$ lies entirely in $B(x_j, F) \cap B(x_{j+1}, F)$.

Remark 25. *As mentioned above, in [20] we defined "ε-refinement" of a δ-chain α in a geodesic space using subdivisions of geodesics joining sucessive pairs of points in α. An ε-refinement as defined in that paper is an E_ε-refinement in the sense of the present paper, but due to the special method of construction, not every E_ε-refinement in the present sense (even in a geodesic space) is an ε-refinement in the sense of [20]. For this reason, we will maintain a distinction between ε-refinements and E_ε-refinements. Note also that we cannot expect existence of ε-refinements even in a geodesic space if the refinement involves a non-metric entourage E. This is because it is possible that no geodesic joining a pair $(x, y) \in E$ stays inside $B(x, E) \cap B(y, E)$.*

Remark 26. *There is some potential for confusion because any F-chain α may be considered as a D-chain when $F \subset D$, and the notion of E-refinement depends on whether we consider α as an F-chain or a D-chain. That is, an E-refinement when α is considered as an F-chain is always an E-refinement when α is considered as a D-chain, but not always conversely.*

Remark 27. *Note that it is immediate from the defintion in the Introduction that if E is a chained entourage and $F \subset E$ then every E-chain has an F-refinement. We will see later (Lemmas 33 and 34) that F-refinements do not change the E-homotopy class and have a size that can be uniformly controlled in any Gromov-Hausdorff precompact class.*

Recall that a subset A of a uniform space X is called *chain connected* if for every pair of points $x, y \in A$ and entourage E there is an E-chain in A joining x and y (see [5] and note that this definition is equivalent to what is sometimes known as *uniformly connected* in the literature). It is easy to check that connected implies chain connected but the converse is not true (e.g. the rational numbers). For compact subsets of uniform spaces, connected and chain connected are equivalent. It is also easy to check that if X has a basis consisting of entourages with open, connected balls then X is connected if and only if X is chain connected. In particular, any chain connected geodesic space is connected.

Remark 28. *We are using chain connectedness rather than connectedness not simply to gain a little extra generality. Chain connectedness is a far more natural condition in the context of these discrete methods; many arguments are simplified using it even when the sets in question are ultimately known to be connected; see for example Lemmas 32 and 33. Two easy-to-check and useful properties of chain connected sets that are not true for connected sets are: the closure $\overline{A}$ of a set A is chain connected if and only if A is chain connected; and the Hausdorff limit of a sequence of chain connected subsets of a metric space is chain connected.*

The next lemma complements the definition in the Introduction, and we will use it without reference.

Lemma 29. *An entourage E in a uniform space is chained if and only if E is contained in the closure of its interior and for every $(x, y) \in E$ there is a chain connected set $C \subset B(x, E) \cap B(y, E)$ that contains x and y.*

Proof. Sufficiency is obvious. Suppose that for every $(x, y) \in E$ and entourage F there is an F-chain joining x and y in $B(x, E) \cap B(y, E)$. Let C denote the set of all points z in $B(x, E) \cap B(y, E)$ such that for every entourage F, z is joined to x and to y by an F-chain in $B(x, E) \cap B(y, E)$. If $v, w \in C$ then for any F, we may join v

to x with an F-chain, then x to y with an F-chain, then y to w with an F-chain, all of them lying in $B(x,E) \cap B(y,E)$. In other words, C is a chain connected subset of $B(x,E) \cap B(y,E)$, and since it also contains x, y, the proof is complete. □

REMARK 30. *Note that the above lemma implies that every $B(x,E)$ is chain connected. This in turn has another important consequence: since E-balls are chain connected, X_E is chain connected and for any $F \subset E$, $\phi_{EF} : X_F \to X_E$ is surjective (Proposition 71 in [5]).*

We will use the following lemma only in the case of a union of two entourages but this statement isn't much harder to prove:

LEMMA 31. *If $\{E_\alpha\}$ is a collection of entourages in a uniform space X then $E := \bigcup_\alpha E_\alpha$ is an entourage and for any $x \in X$, $B(x,E) = \bigcup_\alpha B(x,E_\alpha)$. In addition, if each E_α is chained then E is chained.*

Proof. Clearly E is an entourage. Now $y \in B(x,E)$ if and only if $(x,y) \in E_\alpha$ for some α, which is equivalent to $y \in B(x,E_\alpha)$ for some α. This proves $B(x,E) = \bigcup_\alpha B(x,E_\alpha)$. Next observe that $B(x,E) \cap B(y,E)$ contains $B(x,E_\beta) \cap B(y,E_\beta)$ for any β, which itself contains a chain connected set containing x and y because E_β is chain connected. It is an exercise in elementary point set topology to show that E is contained in the closure of its interior. □

In a geodesic space, not only does every δ-chain α have an ε-refinement when $\delta < \varepsilon$, we can control its size in terms of $\nu(\alpha)$. This is because we may choose points of distance arbitrarily close to δ along a geodesic joining x_i, x_{i+1}, which has length less than ε. So we may always refine by adding at most $\frac{\delta}{\varepsilon}$ points between any point and its successor. For more general chained entourages, we are only able to control this number in compact spaces (but uniformly in any Gromov-Hausdorff precompact class).

LEMMA 32. *Let X be a compact metric space and $\varepsilon > 0$ and A be a subset of X. If two points in A may be joined by an ε-chain in A then they may be joined by an ε-chain in A having at most $2C(X, \frac{\varepsilon}{2})$ points.*

Proof. If $x, y \in A$ may be joined by an ε-chain in A then there is an ε-chain $\{x = x_0, ..., x_n = y\}$ in A with a minimal number of points. Let Z be an $\frac{\varepsilon}{2}$-dense in X having $C(X, \frac{\varepsilon}{2})$ points. For each x_i, choose some z_i such that $d(x_i, z_i) < \frac{\varepsilon}{2}$. We claim that each element of Z can be paired in this way at most twice, which completes the proof. To prove the claim, suppose that $z_i = z_j$ for some $i < j$. Then by the triangle inequality $d(x_i, x_j) < \varepsilon$. If $j > i + 1$ then we could reduce the size of the chain by eliminating the points $x_{i+1}, ..., x_{j-1}$, contradicting minimality. In other words, if a point of Z is used more than once, it must be used for precisely two adjacent points in the chain. □

LEMMA 33. *Suppose that E is a chained entourage in a compact metric space X, and $E_\varepsilon \subset F \subset E$ for some entourage F and $\varepsilon > 0$. Then every E-chain α has an F-refinement α' such that $\nu(\alpha') \leq 2\nu(\alpha) \cdot C(X, \frac{\varepsilon}{2})$.*

Proof. It suffices to show that an E-chain $\{x_0, x_1\}$ has an E_ε-refinement of at most $2C(X, \frac{\varepsilon}{2})$ points; such refinements are also F-refinements and may then be concatenated for a longer chain. By definition of chained entourage, x_0 and x_1 lie in a chain connected set A contained in $B(x_0, E) \cap B(x_1, E)$, and we may apply Lemma 32 to find an ε-chain $\{x_0 = z_0, ..., z_n = x_1\}$ with $n \leq 2C(X, \frac{\varepsilon}{2})$. ◻

The next lemma, while simple, is important because it says that all refinements stay in the same E-homotopy class.

Lemma 34. *Let X be a uniform space and $F \subset E$ be entourages. If $\alpha = \{x_0, ..., x_n\}$ is an F-chain in $B(x_0, E)$ then α is E-homotopic to $\{x_0, x_n\}$. In particular, every F-refinement of an E-chain β is E-homotopic to β and any two F-refinements of β are E-homotopic.*

Proof. Since $x_2 \in B(x_0, E)$, $\{x_0, \widehat{x_1}, x_2 ... x_n\}$ is a legal move. Likewise $\{x_0, \widehat{x_2}, x_3, ... x_n\}$ is a legal move and the proof of the first statement is finished after finitely many steps. The last statements are immediate consequences of the first and the definitions. ◻

The next lemma formalizes a process that is a discrete version of the Arzela-Ascoli Theorem, used several times in this paper and originating in [20].

Lemma 35 (Chain Normalizing). *Let X be a compact metric space, E be an entourage in X with $E_\varepsilon \subset E$, and $\alpha_i = \{x_{i0}, x_{i1}, ..., x_{in_i}\}$ be a sequence of E-chains of length $\leq L$. Then*

1. *Up to E-homotopy we may assume that $n_i \leq \frac{4L}{\varepsilon}$ or $n_i = n := \lceil \frac{4L}{\varepsilon} \rceil$ for all i.*
2. *By choosing a subsequence if necessary we may assume that for all $0 \leq j \leq n$, $x_{ij} \to x_j$ for some $x_j \in X$.*

Proof. For any i, choose some representative of $[\alpha_i]_E$ of length $L_i \leq L$. We may assume that the maximum number of values $d(x_{ij}, x_{i(j+1)})$ that are smaller than $\frac{\varepsilon}{2}$ is at most $\frac{n_i}{2}$. Otherwise there would have to be two consecutive distances smaller than $\frac{\varepsilon}{2}$ and we could remove one point while staying in the same E-homotopy class and not increasing length. In other words, $L \geq L_i \geq \frac{n_i}{2} \cdot \frac{\varepsilon}{2}$ and we conclude that $n_i \leq \frac{4L}{\varepsilon}$ for all i. Now by adding repeated points, if needed, we may ensure that $\nu(\alpha_i) = \lceil \frac{4L}{\varepsilon} \rceil$. The second part is an immediate consequence of compactness. ◻

For many problems it is important to have some version of "close E-chains are E-homotopic". Proposition 36 is analogous to Proposition 15 in [20], using $E = E_\varepsilon$ and $F \subset E_{\frac{\varepsilon}{2}}$. The proof here uses the same homotopy steps as the one in [20], but due to the differences in assumptions we write out a full proof here.

Proposition 36. *Let X be a uniform space, E be an entourage, and F be an entourage such that $F^2 \subset E$. If $\alpha = \{x_0, ..., x_n\}$ is an F-chain and $\beta = \{x_0 = y_0, ..., y_n = x_n\}$ is a chain such $(x_i, y_i) \in F$ for all i, then β is an E-chain that is E-homotopic to α.*

Proof. We will construct an E-homotopy η from α to β, using the fact that $F^2 \subset E$ to see that in each step that the resulting chain is an E-chain, i.e. the step is legal. For example, the first step in the second line below is justified by the fact that $(x_0, x_1) \in F$ and $(x_1, y_1) \in F$, so $(x_0, y_1) \in F^2 \subset E$. The remaining justifications are similar.

$$\begin{aligned}\alpha = \{x_0, x_1, ..., x_n\} &\to \{x_0, \overbrace{x_1}, x_1, ..., x_n\} \to \{x_0, x_1, \overbrace{y_1}, x_1, ..., x_n\}\\ &\to \{x_0, \underbrace{x_1}, y_1, x_1, ..., x_n\} \to \{x_0, y_1, \underbrace{x_1}, x_2, ..., x_n\}\\ &\to \{x_0, y_1, \overbrace{x_2}, x_2, ..., x_n\} \to \{x_0, y_1, x_2, \overbrace{y_2}, x_2, ..., x_n\}\\ &\to \{x_0, y_1, \underbrace{x_2}, y_2, x_2, ..., x_n\} \to \{x_0, y_1, y_2, \underbrace{x_2}, x_3, ..., x_n\} \to \cdots \to \beta\end{aligned}$$

□

PROPOSITION 37. *Let X be a metric space and E be a chained entourage. Suppose that $\alpha_i := \{x_0 = x_{i0}, ..., x_{in} = x_n\}$ is a sequence of E-chains converging to $\alpha = \{x_0, ..., x_n\}$. Then α is an $\overline{E}$-chain such that for all sufficiently large i, α_i is E-homotopic to some (hence any) E-refinement of α.*

Proof. The Ball Continuity Lemma (Lemma 21) implies that α is an $\overline{E}$-chain. Note that in any metric space and for any $0 < \varepsilon < \delta$, by continuity of the distance function,

$$\overline{E_\varepsilon} \subset \{(x, y) : d(x, y) \le \varepsilon\} \subset E_\delta.$$

In particular we may find some *closed* entourage F such that $F^2 \subset E$. Let α_i' be an F-refinement of α for every i. Since $\nu(\alpha_i) = n$ for all i, $L(\alpha_i) \to L(\alpha)$ and hence $\{L(\alpha_i)\}$ is bounded. By the Chain Normalizing Lemma (Lemma 35) we may suppose that for some fixed k, and for all i,

$$\alpha_i' = \{x_{i0} = m^i_{00}, ..., m^i_{0k} = x_{i1}, ..., x_{ir} = m^i_{r0}, ..., m^i_{rk} = x_{r+1}, ..., x_n\}$$

and that for all j, m, $m^i_{mj} \to m_{mj}$, for some $m_j \in X$. In addition,

$$\alpha' := \{x_0 = m_{00}, ..., m_{0k} = x_1, ..., x_r = m_{r0}, ..., m_{rk} = x_{r+1}, ..., x_n\}$$

is an F-chain since F is closed. By the Ball Continuity Lemma (Lemma 21), each m_{jk} lies in $B(x_j, \overline{E}) \cap B(x_{j+1}, \overline{E})$ and therefore α' is an F-refinement, hence an E-refinement, of the $\overline{E}$-chain α. For all large enough i and all j, m, $(m^i_{mj}, m_{mj}) \in F$, and Proposition 36 tells us that α_i' is E-homotopic to α'. Since each α_i is E-homotopic to α_i' by Lemma 34, the proof is complete. □

COROLLARY 38. *Let X be a metric space and E be a chained entourage. Suppose that $\alpha_i := \{x_{i0}, ..., x_{in} = x_{i0}\}$ is a sequence of E-loops converging to an $\overline{E}$-loop $\alpha = \{x_0, ..., x_n = x_0\}$. Then for all sufficiently large i, α_i is freely E-homotopic to some (hence any) E-refinement of α.*

Proof. Let $\beta = \{* = y_0, ..., y_m = x_0\}$ be any E-chain. Then for large enough i, $(x_0, x_{i0}) \in E$ and therefore

$$\alpha_i' := \{* = y_0, ..., x_0, x_{i0}\} * \alpha_i * \{x_{i0}, x_0, ..., y_0 = *\}$$

is an E-loop at $*$. Then $\alpha_i' \to \alpha' := \overline{\beta} * \alpha * \beta$ and by Proposition 37, for all large i, α_i' is E-homotopic to any E-refinement of the $\overline{E}$-loop α'. By definition of freely E-homotopic, the proof is finished. □

REMARK 39. *While we will not need it here, in analogy with the notion of "proper metric space" (meaning all closed metric balls are compact), one could define a "proper* *uniform space" to be a uniform space Y such that $Y \times Y$ has a basis of, and is the countable union of compact entourages. Of course covering spaces of a compact space X may not be compact, but one can show that X_E is still a proper uniform space for any chained entourage E. Proposition 37 and various other results in this paper may be applied to proper uniform spaces.*

4. Curves and the Fundamental Group. Many of the next few results extend basic theorems concerning covering space theory both discrete and classical, with ϕ_E playing the role of the universal covering map. The standard proofs of some of these results in the classical theory use the homotopy lifting property, which one could try to emulate here: i.e. lifting F-homotopies to F^*-homotopies in X_E. However, it is not clear whether this approach is worth the effort. For the results we need, we are able to get by only with chain lifting. See also [36] for some lifting results concerning ε-chains in geodesic spaces.

LEMMA 40 (Chain Lifting). *Let X be a uniform space and E be an entourage. Suppose that $\beta := \{x_0, ..., x_n\}$ is an E-chain and $[\alpha]_E$ is such that $\phi_E([\alpha]_E) = x_0$. Let $y_i := [\alpha * \{x_0, ..., x_i\}]_E$. Then $\widetilde{\beta} := \{y_0 = [\alpha]_E, y_1, ..., y_n = [\alpha * \beta]_E\}$ is the unique "lift" of β to $[\alpha]_E$. That is, $\widetilde{\beta}$ is the unique E^*-chain in X_E starting at $[\alpha]_E$ such that $\phi_E(\widetilde{\beta}) = \beta$.*

Proof. Since the endpoint of $\alpha * \{x_0, ..., x_i\}$ is x_i, $\phi_E(y_i) = x_i$; i.e., $\phi_E(\widetilde{\beta}) = \beta$. Also, by definition of E^*, $\widetilde{\beta}$ is an E^*-chain (in fact this is easiest to see using the original definition from [5], see Remark 19). So we need only show uniqueness, which we will prove by induction on n. For $n = 0$, the starting point $[\alpha]_E$ is determined by assumption, so $\{[\alpha]_E\}$ is the unique lift. Suppose we have proved the statement for $n - 1$. Then given $\beta = \{x_0, ..., x_n\}$ and using the above notation we know that $\{y_0, ..., y_{n-1}\}$ is the unique lift of $\{x_0, ..., x_{n-1}\}$. But we already know that ϕ_E is injective from $B(y_{n-1}, E^*)$ onto $B(x_{n-1}, E)$ and therefore x_n has a unique preimage in $B(y_{n-1}, E^*)$. Therefore y_n, which lies in $B(y_{n-1}, E^*)$ and satisfies $\phi_E(y_n) = x_n$, is the only possibility. □

LEMMA 41. *Let X be a uniform space and E be an entourage. Let $[\lambda]_E \in \pi_E(X)$ and $[\alpha]_E \in X_E$. Then considering $[\lambda]_E$ as a deck transformation of X_E, $[\lambda]_E([\alpha]_E)$ is the endpoint of the lift of α starting at $[\lambda]_E$ in X_E. In particular, the unique lift of α starting at the basepoint ends at $[\alpha]_E$.*

Proof. By definition, $[\lambda]_E([\alpha]_E) = [\lambda * \alpha]_E$, which by the Chain Lifting Lemma (Lemma 40) is the endpoint of the lift of $\lambda * \alpha$ to $[*]_E$. But by uniqueness this is the endpoint of the lift of α starting at the endpoint z of the lift $\widetilde{\lambda}$ of λ starting at $[*]_E$. By the Chain Lifting Lemma, $z = [\lambda]_E$, completing the proof. □

We also need to extend the definition of "stringing" from [20], which among other things allows us to understand the relationship between the fundamental group and the groups $\pi_E(X)$. Note that the definition below is slightly stronger than the one in [20], but the new one is more natural and necessary for the current paper; replacing the definition in [20] by the new one would have essentially no impact on [20].

DEFINITION 42. *Let $\alpha := \{x_0, ..., x_n\}$ be an E-chain in a metric space X, where E is an entourage in a uniform space. A stringing of α consists of a path $\widehat{\alpha}$ formed by concatenating paths γ_i from x_i to x_{i+1} where each path γ_i lies entirely in $B(x_i, E) \cap B(x_{i+1}, E)$. Conversely, let $c : [0, 1] \to X$ be a curve. An E-subdivision of c is an*

E-chain $\beta = \{c(t_0), ..., c(t_n)\}$, where $0 = t_0 \leq \cdots \leq t_n = 1$ are such that c is a stringing of β.

REMARK 43. *Note that, unlike the typical definition of partitions, our definition of E-subdivision allows "repeated points" (as do chains in general) due to the inequality $t_i \leq t_{i+1}$. This simplifies matters, for example when considering limits of chains and curves.*

LEMMA 44. *If X is a metric space, $c : [0, 1] \to X$ is a curve and E is an entourage in X, then c has an E-subdivision. Moreover, any two E-subdivisions of c are E-homotopic and curves c_1 and c_2 with the same endpoints are E-homotopic if and only if any E-subdivision of c_1 is E-homotopic to any E-subdivision of c_2.*

Proof. For existence, it suffices to find an E_ε-subdivision for some $E_\varepsilon \subset E$. Since c is uniformly continuous there exists some $\delta > 0$ such that if $|s - t| < \delta$ then $d(c(s), c(t)) < \varepsilon$. Now subdivide $[0, 1]$ into intervals of length less than δ with endpoints $t_0 = 0 < t_1 < \cdots < t_n = 1$. Then the image of the restriction of c to any interval $[t_i, t_{i+1}]$ lies entirely in $B(c(t_i), \varepsilon) \cap B(c(t_{i+1}), \varepsilon)$ and hence the ε-chain $\{c(t_0), ..., c(t_n)\}$ is an E_ε-subdivision.

For the second statement, suppose that α_1, α_2 are E-subdivisions of c, corresponding to partitions τ_1, τ_2 of $[0, 1]$. We claim that $s = \tau_1 \cup \tau_2$ has the property that if $s = \{s_0 \leq \cdots \leq s_m\}$ then $\alpha := \{c(s_0), ..., c(s_m)\}$ is an E-refinement of both α_1 and α_2. This will complete the proof by Lemma 34. Moreover, by symmetry it suffices to show that α is an E-refinement of α_1. Consider some $t_i \leq t_{i+1} \in \tau_1$, and suppose that for some $j \leq k$, $t_i \leq s_j \leq \cdots \leq s_k < t_{i+1}$ with $s_m \in \tau_2$. Since α_1 is an E-subdivision of c, the restriction of c to $[t_i, t_{i+1}]$ lies entirely in $B(c(t_i), E) \cap B(c(t_{i+1}), E)$ and hence each of the points $c(s_m)$, which are the added points of α_2, lie in $B(c(t_i), E) \cap B(c(t_{i+1}), E)$. This shows that α is an E-refinement of α_1, as required. □

LEMMA 45. *Suppose that X is a Peano continuum, E is a chained entourage and α is an E-chain. Then for some chained entourage $F \subset E$, α has an F-refinement with a stringing. In particular, $[\alpha]_E$ contains a representative with a stringing.*

Proof. Let X have a geodesic metric (by the Bing-Moise Theorem). Since E is a chained entourage, there is some E_ε-refinement $\beta = \{x_0, ..., x_n\}$ of α when $E_\varepsilon \subset E$. Now x_i and x_{i+1} may be joined by a geodesic, which remains inside $B(x_i, E_\varepsilon) \cap B(x_{i+1}, E_\varepsilon) \subset B(x_i, E) \cap B(x_{i+1}, E)$. The concatenation of these geodesics is an F-stringing, hence an E-stringing of β. □

REMARK 46. *Note that the stringing of β in the above lemma may not be a stringing of α since there is no reason why the concatenated curve must lie inside $B(y_i, E) \cap B(y_{i+1}, E)$, where y_i and y_{i+1} are consecutive points in α. Note also that the proof of the lemma only requires that X be a uniform space with a uniformly equivalent geodesic metric–for example any smooth, possibly non-compact, manifold with the uniform structure given by a particular Riemannian metric.*

DEFINITION 47. *If c_1, c_2 are curves in a uniform space starting and ending at the same point,and E is an entourage, we say that c_1 and c_2 are E-homotopic (resp. freely E-homotopic) if there exist E-subdivisions κ_i of c_i, that are E-homotopic (resp. freely E-homotopic).*

LEMMA 48. *Let E be an entourage in a uniform space X and c_1, c_2 be curves starting and ending at the same point. Then c_1 and c_2 are E-homotopic if and only* *if one lift (equivalently any lifts) of c_1 and c_2 to X_E starting at the same point end at the same point. In particular, a curve c lifts as a loop to X_E if and only if c is freely E-null.*

Proof. By arguments analogous to standard ones for covering space theory involving concatenations and uniqueness of lifts, the entire statement reduces to showing the following: If c is a loop at the basepoint then c lifts as a loop at the basepoint if and only if it is E-null. Let $\lambda := \{* = c(t_0), ..., c(t_n) = *\}$ be a E-subdivision of c. By the Chain Lifting Lemma its unique lift $\widetilde{\lambda}$ starting at the basepoint ends at $[\lambda]_E$, i.e. $\widetilde{\lambda}$ is a loop if and only if λ, hence c, is E-null. □

The next two statements are extensions of [20], Proposition 20 and Corollary 21, replacing ε by E or E^* as appropriate. Note only that E is not assumed to be chained, so that X_E may not be connected (and we are not guaranteed that stringings exist). But this does not affect the statement or arguments.

Proposition 49. *Let E be an entourage in a uniform space X and α be an E-chain starting at the basepoint. Then the unique lift of any stringing $\widehat{\alpha}$ starting at the basepoint $[*]_E$ in X_E has $[\alpha]_E$ as its endpoint.*

Proof. By uniqueness in the Chain Lifting Lemma and the second statement in Lemma 41, it suffices to show that if $c : [0, 1] \to X$ is a stringing of $\alpha := \{x_0, ..., x_n\}$ with $x_i = c(t_i)$, and $\widetilde{c}$ is the unique lift of c to the basepoint, then $\{\widetilde{c}(t_0), ..., \widetilde{c}(t_n)\}$ is an E^*-chain. But by definition, for any i, the segment of c from x_i to x_{i+1} lies entirely in $B(x_i, E)$, so lifts into $B(\widetilde{c}(t_i), E^*)$, proving that $(\widetilde{c}(t_i), \widetilde{c}(t_{i+1})) \in E^*$. □

The next corollary now follows from the homotopy lifting property for covering spaces, verifying that homotopies of curves are "stronger than" E-homotopies. Recall that even with a chained entourage in a geodesic space we are not guaranteed that stringings exist, but Lemma 45 tells us we can always find a stringing of an E-homotopic E-chain.

Corollary 50. *Let E be an entourage in a metric space X and α, β be E-chains. If there exist stringings $\widehat{\alpha}$ and $\widehat{\beta}$ that are path homotopic then α and β are E-homotopic.*

Corollary 51. *Let X be a Peano continuum, $\lambda = \{x_0, ..., x_n\}$ be an E-loop, and c_i be a path from x_i to x_{i+1}, denoting by c the concatenation of those paths. If the lift of c to any point $[\alpha]_E$ in X_E ends at $[\alpha * \lambda]_E$, then c is E-homotopic to some, hence any stringing of λ.*

Proof. The statement and the proof use the Chain Lifting Lemma (Lemma 40), and we will use it without further explicit reference. First, in order for the hypothesis to make sense, α must be an E-chain from the basepoint to x_0. By Lemma 45, up to E-homotopy we may let $\widehat{\alpha}$ be a stringing of α. Then $\widehat{\lambda} * \widehat{\alpha}$ is a stringing of $\lambda * \alpha$. According to Proposition 49, the endpoint of the unique lift $\widetilde{\widehat{\lambda} * \widehat{\alpha}}$ of $\widehat{\lambda} * \widehat{\alpha}$ is $[\lambda * \alpha]_E$. By assumption, the lift $\widetilde{c}$ of c to $[\alpha]_E$ also ends in $[\lambda * \alpha]_E$. By Lemma 48, $\widehat{\lambda} * \widehat{\alpha}$ is E-homotopic to $c * \widehat{\alpha}$ and therefore c is E-homotopic to $\widehat{\lambda}$ as required. □

The above Corollary is useful because it "frees us" from having to rely on stringings in some situations; for example the next proposition is useful in the proof of Theorem 69.

Proposition 52. *Let X be a geodesic space and E be a chained entourage. Then*

1. *If λ is an E-loop then there is a curve loop c such that some (hence any) E-subdivision of c is E-homotopic to λ and $L(c) = L(\lambda)$.*
2. *Conversely, if c is a curve loop then there is an E-loop λ such that $L(c) = L(\lambda)$ and λ is E-homotopic to some (hence any) E-subdivision of c.*

Proof. For the first part, let X_E have the lifted geodesic metric (see the end of the second section). Let $\widetilde{\lambda}$ be the unique lift of λ at a point $[\alpha]_E \in X_E$, which is an E^*-chain. By Lemma 23, $L(\widetilde{\lambda}) = L(\lambda)$. Since X_E is a geodesic space we may join each point in $\widetilde{\lambda}$ to its successor by a geodesic, concatenating to produce a curve $\widetilde{c}$ such that $L(\widetilde{c}) = L(\widetilde{\lambda}) = L(\lambda)$. Since ϕ_E preserves the lengths of curves, $c := \phi_E(\widetilde{c})$ has the same length as λ. Moreover, the endpoint of $\widetilde{c}$, which by uniqueness is the lift of c at $[\alpha]_E$, is by construction the same as the endpoint of $\widetilde{\lambda}$. That endpoint is equal to $[\alpha * \lambda]_E$ by Proposition 49.

Now let λ' be any E-subdivision of c; that is, c is a stringing of λ'. Again by uniqueness of lifts, the lift $\widetilde{\lambda'}$ of λ' at $[\alpha]_E$ lies on c and also ends at $[\alpha * \lambda]_E$. By Proposition 49, $[\alpha * \lambda']_E = [\alpha * \lambda]_E$, and hence $[\lambda']_E = [\lambda]_E$, completing the proof of the first part.

For the second part, let F be an open entourage contained in E begin with any F-subdivision ω of c. By definition of the length of a curve $L(\omega) \leq L(c)$. If the lengths are equal, we are finished. Otherwise, suppose that $\delta := L(c) - L(\omega) > 0$ and choose any point x_i and its successor x_{i+1} on ω. Since F is open we may choose n large enough that, setting $\varepsilon := \frac{\delta}{2n}$, $B(x_i, F) \cap B(x_{i+1}, F)$ contains $B(x_i, 2\varepsilon)$. Because the space is geodesic, we may pick a point $z \in B(x_i, 2\varepsilon)$ so that $d(x_i, z) = \frac{\delta}{2n}$ (for example just apply the Intermediate Value Theorem to any geodesic from x_i to x_{i+1}). Now we make the following legal moves for an E-homotopy:

$$\{x_i, \overbrace{z}, x_{i+1} \to \{x_i, z, \overbrace{x_i}, x_{i+1} \to \{x_i, z, x_i, \overbrace{z}, x_{i+1}$$
$$\to \{x_i, z, x_i, z, \overbrace{x_i}, x_{i+1} \to \cdots$$

Each pair of moves, adding z then x_i, increases the length of λ by precisely 2δ. Therefore after n such moves we have attained precisely $L(c)$. □

Note that the next statement is not true when replacing "E-homotopic" with "homotopic", for example if the space is not semi-locally simply connected.

Proposition 53. *Let $c_i : [0, 1] \to X$ be a sequence of curves in a compact metric space X of uniformly bounded length, uniformly convergent to $c : [0, 1] \to X$, and let E be any chained entourage.*

1. *If the c_i have the same start and endpoints then c_i is E-homotopic to c for all large i.*
2. *If the c_i are closed then c_i is freely E-homotopic to c for all large i.*

Proof. If the statement were not true then we could, by taking a subsequence, assume that for all i, c_i is never E-homomotopic to c. Let $\varepsilon > 0$ be such that $\overline{E_\varepsilon} \subset E$. For every i, let $\lambda_i = \{x_{0i} = c_i(0), x_{1i} = c_i(t_{1i}), ..., x_{n_i i} = c_i(1)\}$ be an E_ε-subdivision of c_i, where $0 \leq t_{1i} \leq \cdots \leq t_{(n_i-1)i} \leq 1$. By Lemma 33, we may assume that these chains have bounded length, and hence we may apply the Chain Normalizing Lemma (Lemma 35). That is, we may assume that $ni = n$ for some fixed n and all i, and $x_{ji} \to x_j$ for some x_j, for all j. Since c_i converges uniformly to c, $x_j = c(t_j)$ for some $t_j \in [0, 1]$ with $0 \leq t_1 \leq \cdots \leq t_n = 1$. We will next show that $\lambda = \{x_0, ..., x_n\}$ is an $\overline{E_\varepsilon}$-subdivision, hence an E-subdivision, of c. Let c_{ji} denote the restriction of c_i to $[t_{ji}, t_{(j+1)i}]$; by definition of subdivision, the image of c_{ji} lies in $B(x_{ji}, E) \cap B(x_{(j+1)i}, E)$. Since $\{c_i\}$ converges uniformly, the Ball Continuity Lemma (Lemma 21) implies that if c_j denotes the restriction of c to $[t_j, t_{j+1}]$ then c_j lies in $B(x_j, \overline{E_\varepsilon}) \cap B(x_{j+1}, \overline{E_\varepsilon})$. This shows that λ is an $\overline{E_\varepsilon}$-subdivision of c. It now follows from Proposition 37 that λ is an $\overline{E_\varepsilon}$-loop, hence an E-loop that is ε-homotopic (hence E-homotopic) to λ_i for all large i. That is, c_i is E-homotopic to c, completing the proof of the first part.

The proof of the second part is the same as the proof of the first, using Corollary 38 rather than Proposition 37. □

Remark 54. *We do not need it for this paper, but the above statement may be proved with a little more work without the condition that the curves have uniformly bounded length. The idea is that even if the curves do not have uniformly bounded length, the fact that the convergence is uniform allows one to divide any c_i into a concatenation of segments, the number of which is uniformly bounded and each of which lies in an $\frac{\varepsilon}{2}$-ball. Each of those segments then has an E_ε-subdivision consisting only of two points, and those together provide an E_ε-subdivision of c_i with length bounded independent of i.*

The next definition extends a notion from [20]:

Definition 55. *If X is a uniform space and E is an entourage, an E-loop of the form $\lambda = \alpha * \tau * \overline{\alpha}$, where $\nu(\tau) = 3$, will be called E-small. Note that this notation includes the case when α consists of a single point–i.e. $\lambda = \tau$. In this case, we will call λ an E-triad.*

Note that any E-small loop is E-null since two of the points in τ may be removed one by one, followed by the points in α and $\overline{\alpha}$.

The next essential proposition is an extension of Proposition 29 in [20], with a similar proof (essencially replacing ε by E and δ by D). However, we write out the complete proof here so it is clear how our new definition of refinement is used (including the obvious fact that the reversal of a refinement is a refinement).

Proposition 56. *Let X be a uniform space, D be a chained entourage in X and $E \subset D$ be an entourage. Suppose α, β are E-chains and $\langle \gamma_0, ..., \gamma_n \rangle$ is a D-homotopy such that $\gamma_0 = \alpha$ and $\gamma_n = \beta$. Then $[\beta]_E = [\lambda_1 * \cdots * \lambda_r * \alpha * \lambda_{r+1} * \cdots * \lambda_n]_E$, where each λ_i is an E-refinement of a D-small loop.*

Proof. We will prove by induction that for every $k \le n$, an E-refinement γ_k' of γ_k is E-homotopic to $\lambda_1 * \cdots * \alpha * \cdots * \lambda_k$, where each λ_i is an E-refinement of a D-small loop. Since the E-refinement $\gamma_n'$of $\gamma_n = \beta$ is E-homotopic to β (Lemma 34), this will complete the proof.

The case $k = 0$ is trivial. Suppose the statement is true for some $0 \le k < n$. The points required to E-refine γ_k to γ_k' will be denoted by m_i. Suppose that γ_{k+1} is obtained from γ_k by adding a point x between x_i and x_{i+1}. Let $\{x_i, a_1, ..., a_k, x\}$ be an E-refinement of $\{x_i, x\}$ and $\{x, b_1, ..., b_m, x_{i+1}\}$ be an E-refinement of $\{x, x_{i+1}\}$, so

$$\gamma_{k+1}' := \{x_0, m_0, ..., x_i, a_1, ..., a_k, x, b_1, ..., b_m, x_{i+1}, m_r, ..., x_j\}$$

is an E-refinement of γ_{k+1}. Defining $\mu_{k+1} := \{x_0, m_0, ..., x_i\}$ and

$$\kappa_{k+1} = \{x_i, a_1, ..., a_k, x, b_1, ..., b_m, x_{i+1}, m_r, ..., x_i\}$$

we have

$$\left[\gamma'_{k+1}\right]_E = [\mu_{k+1} * \kappa_{k+1} * \overline{\mu_{k+1}} * \gamma'_k]_E$$

and since the homotopy is a D-homotopy, $\lambda_{k+1} := \mu_{k+1}*\kappa_{k+1}*\overline{\mu_{k+1}}$ is an E-refinement of a D-small loop. The case when a point is removed from γ_k is similar, except that the refined D-small loop is multiplied on the right. □

Corollary 57. *Let X be a uniform space, D be a chained entourage in X and $E \subset D$ be an entourage. Then $\ker \theta_{DE}$ is equal to the subgroup of $\pi_E(X)$ generated by all E-homotopy classes of E-refinements of D-small loops.*

Proof. It follows from Proposition 56 that every element of $\ker \theta_{ED}$ (i.e. an E-homotopy class of an E-loop that is D-null) is a product of E-homotopy classes of E-refinements of D-small loops. On the other hand, any concatenation of E-refinements of D-small loops is D-null and hence its E-homotopy class is in $\ker \theta_{DE}$. □

Further extending the results of [20] we define for any entourage E in a metric space X, a map from fixed-endpoint homotopy classes of continous paths to E-homotopy classes of E-chains as follows. Suppose $c : [0, 1] \to X$ is continuous. We set $\Lambda_E([c]) := [\alpha]_E$, where α is any E-subdivision of c. By Lemma 44, Λ_E is well-defined. Note that if E is a chained entourage then by Lemma 45 every E-chain α may be assumed, up to E-homotopy, to have a stringing $\overline{\alpha}$. By definition $\Lambda_E(\overline{\alpha}) = [\alpha]_E$; that is, Λ_E is surjective. Restricting Λ_E to the fundamental group at any base point yields a homomorphism $\pi_1(X) \to \pi_E(X)$ that we will also refer to as Λ_E.

Continuing to assume that E is a chained entourage, fix a basepoint and suppose that $[c] \in \ker \Lambda_E$. In other words, any E-subdivision $\{c(t_0), ..., c(t_n) = c(t_0)\} := \lambda$ is E-null. Taking $D = E$ in Proposition 56, we see that λ is E-homotopic to a product of E-small E-loops. Since Λ_E is a homomorphism, c is homotopic to the concatenation of stringings of E-small loops. We have shown:

Theorem 58. *Let X be a Peano continuum that has a (compatible) geodesic metric and E be a chained entourage. Then for any basepoint, $\Lambda_E : \pi_1(X) \to \pi_E(X)$ is a surjective homomorphism and $\ker \Lambda_E$ is the subgroup of $\pi_1(X)$ generated by homotopy classes of stringings of E-small loops.*

Remark 59. *The above theorem actually only requires that X have a geodesic metric compatible with the uniform structure, see Remark 46.*

Remark 60. *Note that one may also restrict Λ_E to the set of equivalence classes of curves starting a fixed basepoint, which is by definition the universal covering space of X when X is semi-locally simply connected. According to Theorem 26.2 in [20], if X is a compact, semi-locally simply connected geodesic space, then $\Lambda_{E_\varepsilon} : \pi_1(X) \to \pi_\varepsilon(X)$ is length preserving when ε is a lower bound for HCS. That is, Λ_{E_ε} restricts to a bijection from the universal cover $\widetilde{X}$ of X to X_ε. That is, we may identify the universal cover $\widetilde{X}$ of X with X_ε and $\pi_1(X)$ with $\pi_\varepsilon(X)$.*

Remark 61. *In the above theorem we see a hint of the relationship between our construction and the construction of Spanier used by Sormani-Wei, referred to in the Introduction. For a metric entourage E_ε in a geodesic space, we may take stringings using geodesics, and such geodesics always remain in $B(x_i, \frac{3\varepsilon}{2})$. That is, $\ker \Lambda_{E_\varepsilon}$ is precisely the Spanier subgroup used by Sormani-Wei, and this is how the equivalence* *of ε-covers and δ-covers was proved in [21]. One could take the Sormani-Wei use of Spanier a bit farther by applying it to the covering of a space by entourage balls for a fixed entourage, but it seems unlikely that the resulting covering maps would not be equivalent to entourage covers as we have defined them.*

Remark 62. *As for basepoints, as was observed in [5], as long as X is chain connected, the various groups and homomorphisms defined above are independent of the basepoint, up to natural isomorphisms induced by basepoint change. This is why we have not included basepoints in our notation. When necessary we can always assume that all mappings are basepoint-preserving (take basepoints to basepoints).*

5. Properties of Entourage Covers. Two chained entourages E_1, E_2 in a uniform space will be called *equivalent if* ϕ_{E_1} and ϕ_{E_2} are equivalent as covering maps.

Proposition 63. *Let X be a uniform space, $D \subset E$ be chained entourages, G be a normal subgroup of $\pi_D(X)$, and $\pi : X_D \to X_D/G = Y$ be the quotient covering map. Then there is a covering map $h : X_E \to Y$ such that $h \circ \phi_{ED} = \pi$ if and only if* $\ker\theta_{ED} \subset G$. *In particular, the cover $\phi_E : X_E \to X$ and the induced cover $\phi : Y \to X$ are equivalent if and only if $G = \ker\theta_{ED}$.*

Proof. If $\ker\theta_{ED} \subset G$ then as mentioned in the background section just prior to Lemma 21, we have $X_E = X_D/\ker\theta_{ED}$ and so $G/\ker\theta_{ED}$ acts properly discontinuously on X_E with quotient space naturally identified with X_D/G (cf. Theorem 1.6.11 in [34]). That is, the quotient map

$$h : X_E = X_D/\ker\theta_{ED} \to X_E/(G/\ker\theta_{ED}) = X_D/G = Y$$

is a (regular) covering map that by definition satisfies $h \circ \phi_{ED} = \pi$.

Conversely, suppose that there is a covering map $h : X_E \to Y$ such that $h \circ \phi_{ED} = \pi$. By composing with a covering equivalence we may suppose that h is basepoint preserving. Now suppose that $\theta_{ED}([\lambda]_D) = [*]_E$. Then since h is basepoint preserving, $h \circ \theta_{ED}([\lambda]_D) = \pi([*]_D)$. That is, $[\lambda]_D \in \pi^{-1}(\pi([*]_E) = G$. □

Corollary 64. *Let D, E, F be chained entourages in a uniform space X with $D \subset E \cap F$. Then*

1. *E and F are equivalent if and only if* $\ker\theta_{ED} = \ker\theta_{FD}$.
2. *There is a non-trivial covering map $h : X_E \to X_F$ if and only if there is some F-triad with a D-refinement that is not E-null.*

Proof. The first statement is an obvious consequence of Proposition 63, which also says that there is a non-trivial covering map $h : X_E \to X_F$ if and only if $\ker\theta_{ED}$ is a proper subset of $\ker\theta_{FD}$. Equivalently, there is a D-loop λ that is F-null but not E-null. Equivalently, by Proposition 56,

$$[\lambda]_D = [\lambda_1 * \cdots * \lambda_n]_D$$

where each λ_i is a D-refinement of an F-small loop, at least one of which is not E-null. The proof is now finished by the definition of F-small. □

Remark 65. *Corollary 64 can be considered as an extension of Corollary 31 in [5] to include the situation when neither E nor F may be contained in the other, but there is a covering equivalence between ϕ_E and ϕ_F.*

THEOREM 66. *If X is a geodesic space and $\varepsilon > 0$ then there is a set $S \subset \pi_\varepsilon(X)$ with $|S| \leq C(X, \frac{\varepsilon}{4})^{40C(X,\frac{\varepsilon}{2})}$ such that if E is a chained entourage with $E_\varepsilon \subset E$ then $\ker \theta_{EE_\varepsilon}$ is the normal closure of some subset of S.*

Proof. Let S be the set of all $[\beta]_\varepsilon \subset \pi_\varepsilon(X)$ such that $L(\beta) \leq 10\varepsilon C(X, \frac{\varepsilon}{2})$. By Theorem 3.2 of [20] $|S| \leq C(X, \frac{\varepsilon}{4})^{40C(X,\frac{\varepsilon}{2})}$. Therefore we need only prove that for any such E, $\ker \theta_{EE_\varepsilon}$ is the normal closure of a set of elements of length at most $10\varepsilon C(X, \frac{\varepsilon}{2})$. By Corollary 57, $\ker \theta_{EE_\varepsilon}$ is equal to the subgroup of $\pi_\varepsilon(X)$ generated by the collection of all $[\alpha * \tau * \overline{\alpha}]_\varepsilon$ where τ is an E_ε-refinement of an E-triad. But it is an easy algebraic argument that this means that $\ker \theta_{EE_\varepsilon}$ is the normal closure of elements of the form $[\overline{\alpha_\tau} * \tau * \alpha_\tau]_\varepsilon$, where (1) τ is an element of a set Γ of E_ε-refinements of E-triads that contains exactly one representitive of each free E_ε-homotopy class of E_ε-refinements of E-triads, and (2) α_τ is any ε-chain from the basepoint to the start/end point of τ. In fact, any generator of $\ker \theta_{EE_\varepsilon}$ is conjugate to some such $[\overline{\alpha_\tau} * \tau * \alpha_\tau]_\varepsilon$.

According to Lemma 32 the chains α_τ from the above paragraph may be chosen to have at most $2C(X, \frac{\varepsilon}{2})$ points. Likewise, we may produce an E_ε-subdivision of τ having at most $6C(X, \frac{\varepsilon}{2})$, and therefore there are E_ε-refinements of the E-small loops $\overline{\alpha_\tau} * \tau * \alpha_\tau$ having at most $10C(X, \frac{\varepsilon}{2})$ points and hence length at most $10\varepsilon C(X, \frac{\varepsilon}{2})$. □

Theorem 1 now follows from Theorem 66 and Proposition 64.

Proof of Theorem 3. Suppose that σ is any value of CS. By definition the corresponding σ-cover is not equivalent to any δ-cover for $\delta > \sigma$. Now suppose that E is an entourage that contains E_δ for some $\delta > \sigma$. Then $\phi_{EE_\sigma} = \phi_{E_\delta E} \circ \phi_{E_\delta E_\sigma}$ and since $\phi_{E_\delta E_\sigma}$ is not injective (and all maps are surjective), ϕ_{EE_σ} is not either. That is, there is at least one chained entourage (namely E_σ) that contains E_σ and is not equivalent to any E with $\sigma(E) > \sigma$, hence $\sigma \in$ ECS.

Now let X be a flat torus of dimension ≥ 3. Then $\pi_1(X)$ contains infinitely distinct finitely generated subgroups, each of which is is the normal closure of a finite set (since $\pi_1(X)$ is abelian). According to Corollary 9, X has infinitely many entourage covers. Theorem 1 now implies that ECS is infinite. But since X is a compact Riemannian manifold, CS is finite. □

Proof of Theorem 5. We may assume that X is a geodesic space by the Bing-Moise Theorem. Since X is semilocally simply connected, Corollary 43 of [20] implies that for all sufficiently small ε, $\phi_\varepsilon : X_\varepsilon \to X$ is the universal covering map of X and $\pi_\varepsilon(X) = \pi_1(X)$. By definition, ϕ_E is the covering map corresponding to $\ker_{EE_\varepsilon}$ and the proof of Theorem 5 is finished by Theorem 66. □

REMARK 67. *In [5], Theorem 37, Berestovskii-Plaut showed that for any entourage E in a compact uniform space, if ϕ_E is chain connected (which is true when E is chained according to our new definition) then $\pi_E(X)$ is finitely generated. Essentially the same argument as the proof of Theorem 5, together with Remark 6, shows that if X is a Peano continuum and E is a chained entourage then $\pi_E(X)$ is in fact finitely presented.*

We need some facts from [20] and [21] concerning essential circles, along with a new result (Proposition 68). Essential circles are defined (Definition 5, [20]) to be continuous paths of length L that are not ε-null for $\varepsilon = \frac{L}{3}$ (we may also refer to it as an essential ε-circle). According to Lemma 33, [20] this means that essential circles are characterized as curves of positive length that are shortest in their ε-homotopy class for some ε. Essential circles are special closed geodesics that are miminal on half their length ("2-geodesics" in the parlance of Sormani-Wei, $\frac{L}{2}$-geodesics in our terminology) whose lengths are three times the lengths of the values in HCS, or equivalently $\frac{2}{3}$ the values of CS ([20], Theorem 6). If C is an essential ε-circle then any triple of points $T = \{x_0, x_1, x_2\}$ on C such that $d(x_i, x_j) = \varepsilon$ when $i \neq j$ is an *essential ε-triad* (or just *essential triad* when ε is unspecified), meaning that no ε-subdivision of T is ε-null. Essential triads are characterized by the fact that if the points on them are joined by geodesics the resulting curve is an essential circle. To summarize, essential triads are precisely the discrete analogs of essential circles: adding "edges" to an essential triad creates an essential circle, and any triad of equally spaced points on an essential circle is an essential triad.

Two essential triads τ_1, τ_2 are said to be *equivalent* if they are both essential ε-triads for some $\varepsilon > 0$, and some, hence any ε-subdivision of τ_1 is ε-homotopic to an ε-subdivision of τ_2 or $\overline{\tau_2}$. Two essential circles are said to be *equivalent* if the corresponding essential triads are equivalent (see [20] for more details). As we will now show, the smallest essential circles are generally easiest to find.

PROPOSITION 68. *Let X be a compact geodesic space that is semilocally simply connected. If c is a path loop that is not null-homotopic and whose length is equal to the* 1*-systole σ_1 of X (i.e. the length of the shortest curve that is not null-homotopic) then c a shortest essential circle. Moreover, any two shortest essential circles are equivalent if and only if one is freely homotopic to the other or its reversal.*

Proof. The fact σ_1 is positive and is the shortest possible length of any essential circle is part of Corollary 43 in [20]. It follows from Theorem 6 in [20] that there is at least one essential circle of length σ_1. Now suppose that c is not null-homotopic of length σ_1. Taking $\varepsilon := \frac{\sigma_1}{3}$, to show c is an essential circle we need only show that it is not ε-null. Suppose it were; that is, some ε-subdivision λ of c must be ε-null. Taking $\delta = \varepsilon$ in Proposition 30 of [20] (which is analgous to Proposition 56 in the present paper), for some choice of α, $\overline{\alpha} * \lambda * \alpha$ is ε-homotopic to a product of ε-small ε-loops, one of which must not be ε-null. But recall that an ε-small loop is of the form $\overline{\beta} * \tau * \beta$, where $\tau = \{x_0, x_1, x_2, x_0\}$ satisfies $d(x_i, x_j) < \varepsilon$ for all i, j. But such a τ is necessarily ε-null, a contradiction.

Next, suppose that c_1, c_2 are equivalent shortest essential circles. Reversing c_2, if necessary, this means that for $\varepsilon = \frac{\sigma_1}{3}$, some essential triad τ_i on c_i has an ε-subdivision τ_i' that is ε-homotopic to τ_j' for $j \neq i$. That is, c_1 is ε-homotopic to c_2. By Theorem 26 of [20], X_ε is the universal covering space of X, meaning that two loops lift to a loop in X_ε if and only if they are homotopic. But the same statement is true for ε-homotopic curves by Proposition 49, completing the proof. □

In [21], Theorem 27, when $0 < \delta < \varepsilon$, we showed that $\phi_{\varepsilon\delta} : X_\delta \to X_\varepsilon$ is characterized as the quotient map of X_δ via the (normal) subgroup $K_\varepsilon(\mathcal{T})$ of $\pi_\varepsilon(X)$ generated by all ε-loops of the form $\overline{\alpha} * \tau_i * \alpha$, where is in a set $\mathcal{T} = \{\tau_1, ..., \tau_k\}$ containing exactly one essential triad τ_i representing each equivalence class of essential μ-triads with $\varepsilon \leq \mu < \delta$. Via the argument in the proof of Theorem 66, $K_\varepsilon(\mathcal{T})$ is in fact the normal closure of the finite set of all $\alpha_i * \tau_i * \overline{\alpha_i}$ for any specific choice of ε-chain α_i.

Proof of Theorem 8. In [10], DeSmit, Gornet, and Sutton introduced (Definition 2.3) the notion of a *length map* on a group H with identity 1, namely a function $m : H \to R^+$ such that (a) m is positive except $m(1) = 0$, (b) $m(hgh^{-1}) = m(g)$ for all $g, h \in H$, and (c) $m(g^k) \leq |k|\, m(g)$ for all $g \in H$ and $k \in \mathbb{Z}$. A particular example of a length map is what Sormani-Wei ([28]) called the Minimum Marked Length Map m_g on a manifold with Riemannian metric g: m_g assigns to each element of the fundamental group the length of the shortest curve its free homotopy class. So the values of m_g are precisely MLS. We will need Theorem 2.9 of [10], which we will describe in a weaker simplified form using Example 5.5 [10], and we will shift the subscripts to improve the exposition for our purposes. Let M be a Riemannian manifold of dimension at least 3. Let $\mathcal{F}(M)$ denote the collection of un-oriented free homotopy classes of loops in M. Suppose that $\{c_0, ..., c_k\}$ is a set of distinct elements of $\mathcal{F}(M)$ with c_0 trivial, and $l_0 = 0 < l_1 \leq \cdots \leq l_k$ is a sequence of real numbers such that $2l_1 \geq l_k$. Then by Theorem 2.9 and Example 5.5 of [10] there is a Riemannian metric g on M such that $m_g(c_i) = l_i$ for all i and $m_g(c) \geq l_k$ otherwise. In other words, one may prescribe the length of the shortest geodesic in the free homotopy class of all c_i, and force all other values of MLS to be at least l_k.

Suppose that G is the normal closure of a finite set $\{g_1, ..., g_{k-1}\}$, all distinct and none of which is trivial. If G is equal to $\pi_1(M)$ then the corresponding covering map is trivial, hence an ε-cover for any Riemannian metric. Assume there is some nontrivial $g_k \notin G$. Now let c_0 be the trivial free homotopy class and for each $1 \leq i \leq k$ let c_i be the free homotopy class of some, hence any loop in g_i. Define $l_0 := 0$, $l_k = 1.5$ and $l_i = 1$ for $i = 1, ..., k-1$. By Example 5.5 in [10] (noting that our indexing begins with 0 rather than 1) and Theorem 2.9 of [10] there is Riemannian metric g on M such that $m_g(c_i) = 1$ for all $i \neq 0, k$ and $m_g(c) \geq 1.5$ for every free homotopy class $c \neq c_i$ for any $i < k$. Let κ_i be shortest curves representing each c_i with $1 \leq i \leq k-1$. By Proposition 68, each κ_i is a shortest essential circle, and two κ_i are non-equivalent as essential circles if and only if they lie in different c_i. Moreover, any other essential circle, being shortest in its homotopy class, must have length at least 1.5.

As discussed in Remark 60, we may suppose that $\varepsilon > 0$ is small enough that we may identify, via the function Λ_{E_ε}, M_ε with the universal cover $\widetilde{M}$ of M and identify G with a normal subgroup of $\pi_\varepsilon(M) = \pi_1(M)$. Under this correspondence, G is the normal closure of $\{[\lambda_i]_\varepsilon\}$, where λ_i is any E_ε-subdivision of a loop of the form $\overline{f_i} * c_i * f_i$, where f_i is any curve from the basepoint to the start point of the essential circle κ_i. That is, G is precisely equal to $K_\sigma(\mathcal{T})$ (see discussion prior to this proof), where $\mathcal{T} := \{\tau_i\}_{i=1}^{k-1}$ is a collection of essential triads representing all equivalence classes of smallest essential circles, with exactly one representative τ_i for each free homotopy class. On the other hand, letting $\delta := \frac{1.5}{3} = \frac{1}{2}$, Theorem 27 of [21] states that since $\mathcal{T}$ contains a representative for each essential σ-triad with $\frac{1}{2} < \sigma < \varepsilon$, $K_\varepsilon(\mathcal{T}) = \ker \theta_{\delta\varepsilon}$. That the covering space $\widetilde{M}/G$ is equivalent to M_δ now follows from Proposition 63. □

Proof of Proposition 12. The statements about inessential E are obvious. The proof of the rest is very similar to the proof of the statement for Riemannian manifolds involving free homotopy classes, replacing the fact that small loops are null-homotopic by the fact that small loops are E-null. Suppose that E is essential, c is not freely E-null and let $c_i : [0, 1] \to X$ be E-loops that are freely E-homotopic to c, parameterized proportional to arclength, with lengths converging to

$$L := \inf\{L(f) : f \text{ is a curve loop that is } E\text{-homotopic to } c\}.$$

Since the lengths of the c_i are bounded, a standard application of the Ascoli-Arzela Theorem shows we may assume, taking a subsequence if needed, that c_i converges uniformly to some $\overline{c} : [0, 1] \to X$ with $L(c) \leq L$. By Proposition 53, for all large i, $\overline{c}$ is freely E-homotopic to c_i hence to c. So $\overline{c}$ is the desired shortest curve. To see that any such $\overline{c}$ is a closed $\frac{3\varepsilon}{2}$-geodesic, suppose that a segment S of length $\leq \frac{3\varepsilon}{2}$ is not minimal. By definition we may join its endpoints by a new curve S' of length less than $\frac{3\varepsilon}{2}$. Then S and S' together form a loop of length less than 3ε, which is ε-null

(Lemma 33, [20]), hence E-null. Therefore the curve obtained by replacing S by S' has length shorter than L and is E-homotopic to c, a contradiction. □

Proof of Theorem 13. In any geodesic space, if c is shortest in its E-homotopy class, it is shortest in its homotopy class by Corollary 50, showing that the set described in Part 1a is contained in MLS. On the other hand, suppose that c is shortest in its homotopy class. As in the proof of Proposition 68, let $\varepsilon > 0$ be small enough that X_ε is the universal covering space of X, so that curves are freely homotopic if and only if they are freely ε-homotopic. That is, c is shortest in its ε-homotopy class, i.e. its E-homotopy class for $E = E_\varepsilon$. This proves Part 1a. The remaining parts are true for any compact geodesic space, as will be shown next. □

Theorem 69. *Theorem 13 holds when M is simply assumed to be a compact geodesic space, using Theorem 13.1a as the definition of MLS.*

Proof. If c is non-constant and shortest in its free E-homotopy class then according to Proposition 52.1 there is some E-loop λ having the same length as c that is E-homotopic to any E-subdivision of c. But Proposition 52.2 now shows that λ must also be shortest in its E-homotopy class. An analogous argument shows that if λ' is shortest in its E-homotopy class then there must be a curve of the same length as λ' that is shortest in its E-homotopy class. That is, the quantities described by Parts 1a and 1b in Theorem 69 are the same.

For the second part, note that in [10], Section 3, de Smit, Gornet, and Sutton gave the following equivalent definition of CS in any compact geodesic space X: CS consists of half the lengths of loops that lift as a non-loop to *any* covering space of X. For the covering spaces X_E, Lemma 48 implies that this is precisely the length of a shortest loop that is not E-null, so any value in Part 2a is contained in CS. On the other hand, CS $= \frac{3}{2}$ HCS consists of $\frac{3}{2}$ the lengths of all essential circles. Since every essential ε-circle has length 3ε and curves of length less than 3ε are ε-null (Lemma 33, [20]), then essential circles are the shortest possible loops that are not E-null for $E = E_\varepsilon$. This completes the proof that CS is consists of the values in Part 2a.

Note that if c is a shortest loop that is not E-null then c must be shortest in its E-homotopy class. Proposition 52.1 tells us that there is an E-loop of the same length as c that is E-homotopic to any E-subdivision of c. By Proposition 52.2, there cannot be a shorter such loop. This justifies the term "shortest" in 2b. Moreover, as in the proof of Part 1, Proposition 52 tells us that the values in 2a and 2b are the same. □

Proof of Theorem 15. For the first part, note that if c is E-critical then any E-subdivision λ of c, being also an $\overline{E}$-subdivision of c, is also E-critical. Likewise, any stringing $\widehat{\lambda}$ of an E-critical E-loop λ is also a stringing of λ when considered as an $\overline{E}$-chain. It is immediate that $\widehat{\lambda}$ is an E-critical loop.

The proof of the second part is similar to the proof of Proposition 12 and we will only state the essential steps for the argument involving c. Let $c_i : [0,1] \to X$ be E-loops that are E-critical, parameterized proportional to arclength, with lengths converging to $\psi(E)$. We may assume that c_i converges uniformly to some $c : [0,1] \to X$ with $L(c) \leq \psi(E)$. For all large i, c is freely E-homotopic and $\overline{E}$-homotopic to c_i. That is, c is also E-critical and has shortest length. Proposition 15 now finishes the proof.

For the third part, recall that HCS consists of $\frac{1}{3}$ the lengths of essential triads. But any essential ε-triad T, having length 3ε, is $\overline{E_\varepsilon}$-null, but by defininition has no ε-null ε-subdivision. That is, any ε-subdivision of T (which also has length 3ε) is

E_ε-critical. By definition, CS $\subset$ ES. On the other hand, if c is a shortest E-critical curve then c must be shortest in its free E-homotopy class. If not, it would be E-homotopic, hence $\overline{E}$-homotopic to a shorter curve–but that curve would still be E-critical, a contradiction. The final two statements are shown in Example 73. □

6. Examples.

EXAMPLE 70. *Let X be a circle with the unique geodesic metric of circumference 1 and let E be a chained entourage. According to Theorem 58,* $\ker \Lambda_E$ *is generated by homotopy classes of stringings of E-small loops. We will show that every E-triad has a stringing that represents either the trivial homotopy class or the class of a generator of $\pi_1(X) = \mathbb{Z}$. From this it follows that the only two possible E-covers of the circle are the trivial cover and the universal cover. This shows that Corollary 9 fails in dimension* 1.

If an E-triad $\tau = \{x_0, x_1, x_2\}$ has no E-null stringing then in particular all three points must be distinct. We assume that the points are ordered in the clockwise direction and let A_i denote the arc in the clockwise direction from x_i to x_{i+1}. Since E is chained, each $B(x_i, E) \cap B(x_{i+1})$ must contain A_i or $A_j \cup A_k$ with $j, k \neq i$. If $B(x_i, E) \cap B(x_{i+1})$ only contains A_i for all i, then the only possible stringings of τ are E-homotopic to the circle itself, meaning that the ϕ_E is trivial.

Otherwise, without loss of generality we may suppose that $B(x_0, E) \cap B(x_1, E)$ contains $A_1 \cup A_2$. Case 1: Suppose that $A_2 \subset B(x_2, E) \cap B(x_0, E)$. There are two subcases. 1a: $A_1 \subset B(x_1, E) \cap B(x_2, E)$. In this case (slightly abusing notation by considering each A_i as a path), $\overline{A_2} * \overline{A_1} * A_1 * A_2$ is a stringing of τ that is clearly E-null. 1b: $A_0 \cup A_2 \subset B(x_1, E) \cap B(x_2, E)$. In this case, $\overline{A_2} * \overline{A_1} * \overline{A_0} * \overline{A_2} * A_2$ is a stringing of τ that represents a generator of $\pi_1(X)$. Now observe that we have considered the three essential cases (up to re-ordering): Each $B(x_i, E) \cap B(x_{i+1})$ contains only A_i, exactly two of the $B(x_i, E) \cap B(x_{i+1})$ contain only A_i, or exactly one of the $B(x_i, E) \cap B(x_{i+1})$ contains only A_i, so the proof is complete.

The next examples are related to the question of identifying entourage covers for 2-dimensional manifolds (with or without boundary).

EXAMPLE 71. *Let M be the Moebius Band. First note that M is homeomorphic to $\mathbb{RP}^2$ with a small disk removed. Therefore we may Gromov-Hausdorff approximate $\mathbb{RP}^2$ by Mobius bands, taking the standard metric on $\mathbb{RP}^2$ and induced geodesic metric on M with smaller and smaller disks removed. Since the double cover of $\mathbb{RP}^2$ is its universal cover, hence an ε-cover for all sufficiently small ε, the double cover of M is also an ε-cover for small enough ε with respect to the induced metrics with small enough disks removed. This follows from the convergence result Proposition 37, [21]– since small enough $\varepsilon > 0$ is not a homotopy critical value of $\mathbb{RP}^2$. Note that M deformation retracts onto the circle and yet the double cover of the circle is not an entourage cover by Example 70. This example shows one way that convergence can be used to identify entourage covers but also shows the limitations of "enlarging" spaces to try to find entourage covers.*

REMARK 72. *Let X be a compact geodesic space such that HCS has n elements, counting multiplicity as defined in [20]. In other words, there are a total of n equivalence classes of essential circles. As pointed out in the proof of Theorem 15, every essential ε-circle is E_ε-critical, and as defined in [20], the multiplicity is the number of distinct E_ε-homotopy classes of ε-circles. Since being E-critical is a topological property these essential circles will still be critical loops in any other metric on X,* *and by definition their lengths will be elements of ES. That is, ES in any metric will always have at least n elements, but the lengths of curves, and hence the values and multiplicities of ES will be different. In particular, as soon as there are two geodesic metrics with CS having different sizes, then the space with the smaller CS must have CS strictly contained in ES, counting multiplicity. In the next example we essentially carry this out in a carefully controlled setting, allowing us to insure that the multiplicities are 1 and hence control the absolute size of the spectra.*

EXAMPLE 73. *Let M be a compact smooth manifold of dimension 3 or higher with fundamental group $\mathbb{Z}_4$, which we will denote by $\{[c_0],[c_1],[c_2],[c_3]\}$. (There are many other possibilities for $\pi_1(M)$, but we are using $\mathbb{Z}_4$ for simplicity.) Denote the double cover of M (corresponding to the subgroup generated by $[c_2]$) by M' and the universal cover of M by M''. By standard covering space theory, c_2 is the only loop, up to free homotopy, that lifts as a loop to M'. We now make the following assignments: $[c_1] \to 1.2$, $[c_2] \to 1.1$, $[c_3] \to 1.2$ and apply Theorem 2.9 of [10]. In the resulting metric, the shortest loop lifting to a non-loop in M' must be freely homotopic to either c_1 or c_3 and hence has length 1.2. According to Proposition 68, M'' is equivalent to $M_{\frac{1.1}{3}}$. In particular, $HCS = \{\frac{1.1}{3}, \frac{1.2}{3}\}$ and $CS = \{\frac{1.1}{2}, \frac{1.2}{2}\}$. Moreover, as pointed out in the proof of Theorem 15, if c is an essential $\frac{1.2}{3}$-circle then c is E-critical, where $E := E_{\frac{1.2}{3}}$. Note that since c lifts as a non-loop to M_E, c is freely homotopic to c_2.*

Now use the following assignments: $[c_1] \to 1.1$, $[c_2] \to 1.2$, $[c_3] \to 1.3$ and apply Theorem 2.9 of [10]. With this metric, any shortest loop lifting as a non-loop to either M' and M'' has length 1.1.That is, $HCS = \{\frac{1.1}{3}\}$ and $CS = \{\frac{1.1}{2}\}$. Since being E-critical is a topological property, the loop c is still E-critical and by definition the length of the shortest curve c' in its free E-homotopy class is an element of ES. But any curve in the free E-homotopy class of c must lift to a loop in $M' = M_E$. Since c_2 is the only such curve, up to free homotopy, c' must be freely homotopic to c_2 and therefore has length 1.2. That is, ES contains $\{1.1, 1.2\}$ and in particular ES strictly contains $3HCS = 2CS$. Moreover, since we may change the value assigned to $[c_2]$ by any small amount, it is possible for different Riemannian metrics on M to have the same CS but different ES. Finally, since there are only two non-trivial entourage covers of M, hence at most two elements in ES, $ES = \{1.1, 1.2\}$. But 1.3 is an element of MLS by definition, so ES is strictly contained in MLS. Again, the value assigned to $[c_3]$ may be changed by a small amount without impacting ES, and therefore one obtains Riemannian metrics on M that have the same ES but different MLS. This example verifies Theorem 15.2.d-e.

EXAMPLE 74. *In [3], Berestovskii-Plaut-Stallman gave two examples of compact geodesic spaces with free homotopy classes having no closed geodesics in them. The first is 1-dimensional, consisting of a circle in the plane with smaller and smaller straight segments added to join points around the perimeter circle, with the induced geodesic metric. The circle itself is not a closed geodesic because any segment may be "bypassed" by a shorter straight segment, but it is, up to monotone reparametization, the only curve, hence the shortest curve, in its homotopy class. Now for any entourage E, the circle is E-homotopic to a shorter curve that is a piecewise segment and is shortest in its E-homotopy class. Note that although this segment visually has "corners" in the construction, it is still a closed geodesic with the induced geodesic metric.*

The other example is the infinite torus, the countable product T^∞ of circles with the Tychonoff topology. This space can be metrized by making the sizes of the circles *square summable and taking the geometric product metric (see [23]). This space is of course a compact topological group and this metric is bi-invariant. For this example, Stallman proved in his dissertation that (1) there is a unique 1-parameter subgroup (i.e. a homomorphism $\theta : \mathbb{R} \to T^\infty$) in each free homotopy class of a curve, (2) there are free homotopy classes containing no rectifiable curves (3) if there are rectifiable curves then the shortest one is the unique 1-parameter subgroup in it, but (4) the 1-parameter subgroup may not be a closed geodesic. It would be interesting to see whether the shortest curves in free E-homotopy classes given by Proposition 12 are also 1-parameter subgroups.*

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