TWISTING LEMMA FOR Λ-ADIC MODULES[*]

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Abstract. A classical twisting lemma says that given a finitely generated torsion module M over the Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$ with $\Gamma \cong \mathbb{Z}_p$, there exists a continuous character $\theta : \Gamma \to \mathbb{Z}_p^\times$ such that, the Γ^{p^n}-Euler characteristic of the twist $M(\theta)$ is finite for every n. This twisting lemma has been generalized for the Iwasawa algebra of a general compact p-adic Lie group G. In this article, we consider a further generalization of the twisting lemma to $\mathcal{T}[[G]]$ modules, where G is a compact p-adic Lie group and $\mathcal{T}$ is a finite extension of $\mathbb{Z}_p[[X]]$. Such modules naturally occur in Hida theory. We also indicate arithmetic applications by considering the 'big' Selmer (respectively fine Selmer) group of a Λ-adic form over a p-adic Lie extension.

Key words. Iwasawa theory, Selmer groups, Λ-adic form, G-Euler characteristic.

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Introduction. In this article, we discuss some topics in non-commutative Iwasawa theory for modules over $\mathcal{T}[[G]]$; the completed group ring of a compact p-adic Lie group G with coefficients in the ring $\mathcal{T}$, where $\mathcal{T}$ is a finite extension of $\mathbb{Z}_p[[X]]$.

We fix an odd prime p throughout. Let B be any commutative, complete, noetherian local domain of characteristic 0 with finite residue field of characteristic p. For a profinite group $\mathcal{G}$, recall the Iwasawa algebra of $\mathcal{G}$ over B is defined as $B[[\mathcal{G}]] := \varprojlim_U B[\mathcal{G}/U]$, where U varies over open normal subgroups of $\mathcal{G}$ and the inverse limit is taken with respect to the canonical projection maps.

Let G be a compact p-adic Lie group with a closed normal subgroup H such that $\Gamma := G/H \cong \mathbb{Z}_p$. We will denote by O, the ring of integers of a finite extension of $\mathbb{Q}_p$. For a left $O[[G]]$ module M and a continuous character $\theta : \Gamma \to \mathbb{Z}_p^\times$, denote by $M(\theta)$ the $O[[G]]$-module $M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\theta)$ with diagonal G-action. We will assume throughout that G has no p-torsion element. Recall that for a compact p-adic Lie group G with no element of order p, the Iwasawa algebra $O[[G]]$ has finite global dimension [Br, La].

DEFINITION. Let G be a compact p-adic Lie group without any element of order p. For a finitely generated $O[[G]]$-module M, we say that the G-Euler characteristic of M exists if the homology groups $H_i(G, M)$ are all finite and we define it as

$$\chi(G, M) := \prod_{i \geq 0} (\#H_i(G, M))^{(-1)^i} .$$

Given an $O[[G]]$-module M, $\chi(G, M)$ is an invariant attached to M. A natural example of $O[[G]]$ modules in arithmetic comes from Selmer group attached to a motive over p-adic Lie extension of a number field, with G being the corresponding Galois group. The Euler characteristic of a Selmer group naturally carries arithmetical information. For any number field K, let K_{cyc} denote the cyclotomic $\mathbb{Z}_p$ extension and set $\Gamma = \text{Gal}(K_{\text{cyc}}/K)$. Let $E/\mathbb{Q}$ be an elliptic curve with good, ordinary reduction at p and let $X(E/\mathbb{Q}_{\text{cyc}})$ denote the dual p^∞-Selmer group of E over $\mathbb{Q}_{\text{cyc}}$. Then it

is known that, under suitable condition, the p-adic valuation of $\chi(\Gamma, X(E/\mathbb{Q}_{\text{cyc}}))$ is related to the p-adic valuation of the special value $\frac{L_E(1)}{\Omega_E}(\#\tilde{E}(\mathbb{F}_p)(p))^2$ of the complex L-function of E over $\mathbb{Q}$. (see [JS, Introduction])

On the other hand, consider a p-adic Lie extension $K \subset K_{\text{cyc}} \subset K_\infty$ of a number field K which is unramified outside a finite set of places of K and set $G = \text{Gal}(K_\infty/K)$ and $H = \text{Gal}(K_\infty/K_{\text{cyc}})$. Then $\Gamma := G/H \cong \mathbb{Z}_p$. For such a general p-adic Lie extension K_∞/K, the twisted Euler characteristic of $\mathbb{Z}_p[[G]]$ modules M, such that $M/M(p)$ is finitely generated over $\mathbb{Z}_p[[H]]$, has been discussed in [CFKSV]. Also for an elliptic curve $E/\mathbb{Q}$, the conjectural relation between $\chi(G, X(E/K_\infty)(\theta))$ and twisted L-values are studied (cf. [CFKSV, Theorem 3.6], [JS]).

For $G = \Gamma$, following classical twisting lemma is well known in Iwasawa theory and can be found in the works of Greenberg [Gr] and Perrin-Riou [Pr]: For any finitely generated torsion $O[[\Gamma]]$-module M, there exists a continuous character $\theta : \Gamma \to \mathbb{Z}_p^\times$ such that the largest Γ^{p^n}-coinvariant quotient $H_0(\Gamma^{p^n}, M(\theta)) = (M(\theta))_{\Gamma^{p^n}}$ is finite for every $n \in \mathbb{N}$. Note that $H_0(\Gamma^{p^n}, M(\theta))$ is finite if and only if $\chi(\Gamma^{p^n}, M(\theta))$ is finite.

Let R be a ring and M be a left R module. Define, $M(p) := \bigcup_{r \geq 1} M[p^r]$, where $M[p^r]$ is the set of p^r torsion points of M. Unless stated otherwise, when we consider a module M over a ring R, we mean M is a left module over R.

The twisting Lemma in the non-commutative setting was established in [JOZ]:

THEOREM(JOZ). *Let G be a compact p-adic Lie group and H be a closed normal subgroup of G such that $\Gamma := G/H \cong \mathbb{Z}_p$. Let M be an $O[[G]]$ module which is finitely generated over $O[[H]]$. Then, there exists a continuous character $\theta : \Gamma \to \mathbb{Z}_p^\times$ such that $M(\theta)_U = H_0(U, M(\theta))$ is finite for every open normal subgroup U of G.*

Note that for a general compact p-adic Lie group G and an $O[[G]]$ module M, $H_0(G, M)$ is finite does not necessarily imply $\chi(G, M)$ exists (i.e. finite) [JS, see Remark 1.5]. In [JS], Theorem(JOZ) was extended to the following result on the twisted Euler characteristic.

THEOREM(JS). *Let G be a compact p-adic Lie group without any element of order p and H be a closed normal subgroup of G such that $\Gamma := G/H \cong \mathbb{Z}_p$. Let M be a finitely generated $O[[G]]$ module such that $M/M(p)$ is a finitely generated $O[[H]]$ module. Then there exists a continuous character $\theta : \Gamma \to \mathbb{Z}_p^\times$ such that $\chi(U, M(\theta))$ exists for every open normal subgroup U of G.*

Moreover, it is shown that for a given M, there is a countable subset S_M of all continuous characters from Γ to $\mathbb{Z}_p^\times$, such that for any choice of a continuous $\theta : \Gamma$ to $\mathbb{Z}_p^\times$ outside S_M, Theorem(JOZ) and Theorem(JS) hold.

Congruence of modular forms is an important topic in number theory and it naturally leads to the study of modules over 'two variable' Iwasawa algebra where the coefficient ring of the Iwasawa algebra is a certain universal ordinary deformation ring (cf. [Hi], [Wi]). These universal ordinary deformation rings are typically finite extensions of $\mathbb{Z}_p[[X]]$. Thus it is natural to ask for a generalization of twisting Lemma for modules over $\mathcal{T}[[G]]$, where G is a compact p-adic Lie group and $\mathcal{T}$ is finite over $\mathbb{Z}_p[[X]]$. The main result of the article is the following:

THEOREM 0.1. *Let G be a compact p-adic Lie group without any element of order p and H be a closed normal subgroup of G with $\Gamma := G/H \cong \mathbb{Z}_p$. Let $\mathcal{T}$ be a commutative, complete local domain which is finite over $\mathbb{Z}_p[[X]]$. Let M be a finitely generated $\mathcal{T}[[G]]$ module such that $M/M(p)$ is finitely generated over $\mathcal{T}[[H]]$.*

Then there exists a continuous character $\theta : \Gamma \to \mathbb{Z}_p^\times$ and a countable set $C_{M,\theta}$ of height 1 prime ideals of $\mathcal{T}$ such that if we choose any height 1 prime ideal $Q \notin C_{M,\theta}$, then $\chi\left(U, \frac{M}{QM}(\theta)\right)$ is finite for every open normal subgroup U of G.

DEFINITION 0.2. *Define* $S := \{\theta | \theta$ *is a continuous character from* Γ *to* $\mathbb{Z}_p^\times\}$ *and set* $C = C^{\mathcal{T}} := \{Q : Q$ *is a height 1 prime ideal of* $\mathcal{T}, Q \nmid (p)\}$.

REMARK 0.3. In fact, in the proof of Theorem 0.1, we will establish a slightly stronger result by showing there exists a countable subset S_M of S such that for any choice of $\theta \in S \setminus S_M$, Theorem 0.1 holds.

REMARK 0.4. In Theorem 0.1, we consider finitely generated $\mathcal{T}[[G]]$ modules M such that $M/M(p)$ is finitely generated over $\mathcal{T}[[H]]$. This comes from the formulation of non-commutative GL_2 Iwasawa main conjecture in [CFKSV], where it is suggested, many arithmetic modules, including the dual Selmer group of an elliptic curve over a p-adic Lie extension would satisfy this condition.

REMARK 0.5. Let us keep the setting and hypotheses of Theorem 0.1. Also for simplicity, take $\mathcal{T} = O[[X]]$. Then, from Theorem 0.1, one may naturally ask the following questions:

Question 1: Is it possible that for all but finitely many height 1 prime ideals in $O[[X]]$, $\left(\frac{M}{QM}\right)_U = H_0\left(U, \frac{M}{QM}\right)$ is finite, for every open normal subgroup U of G ?

Question 2: Does there exist a continuous character $\theta : \Gamma \to \mathbb{Z}_p^\times$ and a finite set C' of C such that if we choose and fix any $Q \in C \setminus C'$ then $\left(\frac{M}{QM}(\theta)\right)_U$ is finite for every open normal subgroup U of G ?

Question 3: Does there exist a countable subset S' of S and a countable subset C' of C such that for any choice of $\theta \in S \setminus S'$ and for any choice of a height 1 prime $Q \in C \setminus C'$, $\left(\frac{M}{QM}(\theta)\right)_U$ is finite for every open normal subgroup U of G ?

Question 4: Does there exist a countable set $S' \times C' \subset S \times C$, such that for any choice of $(\theta, Q) \notin S' \times C'$, $\left(\frac{M}{QM}(\theta)\right)_U$ is finite, for every open normal subgroup U of G ?

Let S' and C' be countable subsets of S and C respectively. We observe that $(S \setminus S') \times (C \setminus C') \subset (S \times C) \setminus (S' \times C')$. From this it follows that a negative answer to question 3 gives a negative answer to question 4.

For questions 1 and 2, we will find O, G, H, an $O[[X]][[G]]$ module M and for every $\theta \in S$, a countably infinite subset C' of C such that for every $Q \in C'$, $\left(\frac{M}{QM}(\theta)\right)_{U_0}$ is infinite for some open normal subgroup U_0 of G. This will provide a negative answer to both question 1 and question 2. Further, we will show the same module M will give a negative answer to question 3 (and hence question 4).

EXAMPLE. Let $G = \Gamma = <\gamma>$, $H = \{1\}$ and $O = \mathbb{Z}_p$. Let $M = \dfrac{\mathbb{Z}_p[[X]][[T]]}{(X - T)}$, where $1 + T$ corresponds to γ. Let $Q \neq (p)$ be a height 1 prime ideal of $\mathbb{Z}_p[[X]]$. Write $Q = (g(X))$, where $(g(X))$ is an irreducible Weierstrass polynomial.

For a continuous character $\theta : \Gamma \longrightarrow \mathbb{Z}_p^\times$, we can write $\theta(\gamma^{-1}) = 1 + p\lambda$ for some (fixed) $\lambda \in \mathbb{Z}_p$. Then $M(\theta) \cong \dfrac{\mathbb{Z}_p[[X]][[T]]}{((1 + p\lambda)(T + 1) - (X + 1))}$. Now, if $Q \neq (p)$ is a prime

ideal of height 1 in $\mathbb{Z}_p[[X]]$, then

$$\big(\frac{M}{QM}(\theta)\big)_{\Gamma^{p^n}} \cong \frac{\mathbb{Z}_p[[X]][[T]]}{((1+p\lambda)(T+1)-(X+1),(T+1)^{p^n}-1,Q)}.$$

is finite if and only if $(Q,(X+1)^{p^n}-(1+p\lambda)^{p^n})$ has height 2. Let

$$C^\theta_{p^n} := \{Q \text{ is a height 1 prime in } \mathbb{Z}_p[[X]] \ : \ Q \neq (p) \text{ and } \big(\frac{M}{QM}(\theta)\big)_{\Gamma^{p^n}} \text{ is infinite}\}.$$

Then $C^\theta_{p^n} = \{(g(X)) \mid g(X) \text{ is an irreducible divisor of } (X+1)^{p^n}-(1+p\lambda)^{p^n}\}$. For $m > n$, by Weierstrass preparation theorem, there exists an irreducible factor $\ell^\theta(X)$ of $\frac{(X+1)^{p^m}-(1+p\lambda)^{p^m}}{(X+1)^{p^n}-(1+p\lambda)^{p^n}}$. Then, $(\ell^\theta(X)) \in C_{p^m} \setminus C_{p^n}$ and hence for $m > n$, $C^\theta_{p^n} \subsetneq C^\theta_{p^m}$ and $C'_\theta := \bigcup_{n \geq 1} C^\theta_{p^n}$ is an infinite set. Now, for each $Q \in C'_\theta$, there exists an integer n_0 such that $\big(\frac{M}{QM}(\theta)\big)_{\Gamma^{p^{n_0}}}$ is infinite. This shows that the answer to question 2 in Remark 0.5 is negative.

Now recall the sets S and C defined in Definition 0.2. If possible, assume that the answer to question 3 in Remark 0.5 is true. Then, there exists a countable subset S' of S and C' of C such that for $\theta \in S \setminus S'$ and $Q \in C \setminus C'$, $H_0(\Gamma^{p^n}, \frac{M}{QM}(\theta))$ is finite for every $n \geq 0$.

For every $\theta \in S$, we can write $\theta(\gamma^{-1}) = 1+p\lambda_\theta$ for some (fixed) $\lambda = \lambda_\theta \in \mathbb{Z}_p$. Then $\big(\frac{M}{QM}(\theta)\big)_{\Gamma^{p^n}} \cong \frac{\mathbb{Z}_p[[X]]}{((X+1)^{p^n}-(1+p\lambda)^{p^n},Q)}$. Observe that if $Q = (X-p\lambda)$ then $\big(\frac{M}{QM}(\theta)\big)_{\Gamma^{p^n}}$ is infinite for every $n \in \mathbb{N}$.

Now let us choose a $\theta \in S \setminus S'$. Then by our assumption in Statement 3, $\big(\frac{M}{QM}(\theta)\big)_{\Gamma^{p^n}}$ is finite for every $Q \in C \setminus C'$. Thus the height 1 prime $(X-p\lambda)$ must be in C'. Hence for every $\theta \in S \setminus S'$, we get an element $Q_\theta = (X+1-\theta(\gamma)) \in C'$. Note for $\theta_1, \theta_2 \in S \setminus S'$ with $\theta_1 \neq \theta_2$, $Q_{\theta_1} \neq Q_{\theta_2}$. Since $S \setminus S'$ is uncountable, the set C' is also uncountable. This is a contradiction and hence question 3 has a negative answer as well.

However, the following variant of the Theorem 0.1 holds true.

PROPOSITION 0.6. *Let $G, H, \mathcal{T}$ and M be as in Theorem 0.1. Let C' be a countable subset of C. Then there exists a countable set S_M of S, such that for any choice of $\theta \in S \setminus S_M$ and for any choice of $Q \in C'$, $\chi(U, \frac{M}{QM}(\theta))$ exists, for every open normal subgroup U of G.*

Proof. Let us enumerate $C' = \{Q_1, Q_2, \cdots\}$. For each Q_i, note that $\frac{\mathcal{T}}{Q_i\mathcal{T}}[[G]] \cong O_i[[G]]$ and $\frac{\mathcal{T}}{Q_i\mathcal{T}}[[H]] \cong O_i[[H]]$, where O_i is the ring of integers of a finite extension L^i of $\mathbb{Q}_p$. Then from Theorem(JS), there exists a countable subset $S_{Q_i,M}$ of all the continuous character from Γ to $\mathbb{Z}_p^\times$, such that for any choice of $\theta \notin S_{Q_i,M}$, $\chi(U, \frac{M}{Q_iM}(\theta))$ exists for every open normal subgroup U of G. Now choose and fix any $\theta : \Gamma \to \mathbb{Z}_p^\times$ outside the countable set $S_M := \bigcup_{i=1}^\infty S_{Q_i,M}$. Then it follows that for any choice of $Q_i \in C'$, $\chi(U, \frac{M}{Q_iM}(\theta))$ exists for every open normal subgroup U of G. $\square$

Let μ_{p^∞} denote the group of p power roots of unity. Define a homomorphism $\nu_{k,\zeta} : \mathbb{Z}_p[[X]] \to \bar{\mathbb{Q}}_p^\times$ by $\nu_{k,\zeta}(1+X) = \zeta(1+p)^k$, where $\zeta \in \mu_{p^\infty}$ and $k \in \mathbb{N}$. Define $A_{\text{arith}}(\mathbb{Z}_p[[X]]) := \{Q|Q = \ker(\nu_{k,\zeta}) \text{ for some } k \geq 1 \text{ and some } \zeta \in \mu_{p^\infty}\}$. Clearly, elements of $A_{\text{arith}}(\mathbb{Z}_p[[X]])$ are height 1 prime ideals of $\mathbb{Z}_p[[X]]$. Similarly, we define $A_{\text{arith}}(\mathcal{T})$ as the set of those height 1 prime ideals of $\mathcal{T}$ which divide some

height 1 prime ideal in $A_{\mathrm{arith}}(\mathbb{Z}_p[[X]])$. The elements of $A_{\mathrm{arith}}(\mathcal{T})$ are called the arithmetic primes or classical primes of $\mathcal{T}$ (see [Wi]). As an immediate corollary of Proposition 0.6, we deduce the following:

COROLLARY 0.7. *Let us keep the hypotheses and setting of Proposition 0.6. Then, there exists a countable subset S_M of S, such that for any choice of $\theta \in S \setminus S_M$ and for any $Q \in A_{arith}(\mathcal{T})$, $\chi(U, \frac{M}{QM}(\theta))$ exists for every open normal subgroup U of G.*

Significance in Arithmetic. We set up a necessary framework that is needed to describe applications of our results in arithmetic. Consider a p-adic Lie extension $K \subset K_{\mathrm{cyc}} \subset K_{\infty}$ of a number field K, such that K_{∞} is unramified outside finitely many places of K and $G = \mathrm{Gal}(K_{\infty}/K)$ has no element of order p. Put $H = \mathrm{Gal}(K_{\infty}/K_{\mathrm{cyc}})$ and note $\Gamma := G/H = \mathrm{Gal}(K_{\infty}/K) \cong \mathbb{Z}_p$. For any open normal subgroup U of G, the fixed field $K_U := K_{\infty}^U$ defines a finite extension of K inside K_{∞}.

Let f be a p-ordinary newform of weight ≥ 2. We denote by L_f, a lattice of V_f, the p-adic Galois representation associated to f and O_f will denote the ring of integers of the p-adic field defined by the Fourier coefficients of f.

Let $\mathcal{T} = \mathcal{T}_{\mathcal{F}}$ be the quotient of the universal ordinary Hecke algebra that corresponds to an ordinary Λ-adic newform $\mathcal{F}$ (see [Hi]). The algebra $\mathcal{T}_{\mathcal{F}}$ is a local domain and is finite flat over $\mathbb{Z}_p[[X]]$. By a celebrated result of Hida ([Hi], [Wi]), there exists a 'large' continuous irreducible representation $\rho_{\mathcal{F}} : G_{\mathbb{Q}} \to \mathrm{Aut}_{\mathcal{T}_{\mathcal{F}}}(\mathcal{L}_{\mathcal{F}})$, where $\mathcal{L}_{\mathcal{F}}$ is a finitely generated, torsion-free module of generic rank 2 over $\mathcal{T}_{\mathcal{F}}$. We assume that the residual representation $\bar{\rho}_{\mathcal{F}}$ associated to $\rho_{\mathcal{F}}$ (see [Hi]) is absolutely irreducible. For each $Q \in A_{\mathrm{arith}}(\mathcal{T}_{\mathcal{F}})$, there exists a p-ordinary, p-stabilized newform f_Q, such that the quotient $\mathcal{L}_{\mathcal{F}}/Q\mathcal{L}_F$ is isomorphic to the (unique) lattice L_{f_Q}. To simplify the notation, we will write $L_Q = L_{f_Q}$ and $O_Q = O_{f_Q}$.

In this setting of p-ordinarity, the dual Selmer group $X(L_f/K_{\infty})$ [JO, Definition 1.11] (respectively $\mathcal{X}(\mathcal{L}_{\mathcal{F}}/K_{\infty})$ [Jh, §3]) of f (respectively $\mathcal{F}$) over K_{∞} is defined and is a finitely generated $O_f[[G]]$ module (respectively $\mathcal{T}[[G]]$ module). Similarly, with f as above, for the twisted Galois representation $V_f(\theta)$, the dual Selmer group $X(L_f(\theta)/K_{\infty})$ is also defined and it is a finitely generated $O_f[[G]]$ module. A control theorem, originally proved by Mazur, is a widely used tool in Iwasawa theory. In this setting, under suitable condition, we deduce by a control theorem, that the kernel and cokernel of $X(L_f(\theta)/K_{\infty})_U \longrightarrow X(L_f(\theta)/K_U)$ are finite for every U (see [JO, Theorem 0.1]).

(I) Finiteness of the twisted Selmer groups over K_U. Assume that $\frac{X(L_f/K_{\infty})}{X(L_f/K_{\infty})(p)}$ is finitely generated over $O_f[[H]]$. Then Theorem(JOZ) along with a control theorem shows there exists, $\theta : \Gamma \to \mathbb{Z}_p^{\times}$ such that the twisted Selmer group $X(L_f(\theta)/K_U)$ is finite for every finite extension K_U inside K_{∞}.

Now, our Corollary 0.7 in particular shows that there exists, $\theta \in S$ such that for each $Q \in A_{\mathrm{arith}}(\mathcal{T})$, $\left(\frac{M}{QM}(\theta)\right)_U$ is finite for every U. Take $M = \mathcal{X}(\mathcal{L}_{\mathcal{F}}/K_{\infty})$ and assume $\frac{\mathcal{X}(\mathcal{L}_{\mathcal{F}}/K_{\infty})}{\mathcal{X}(\mathcal{L}_{\mathcal{F}}/K_{\infty})(p)}$ is a finitely generated $\mathcal{T}_{\mathcal{F}}[[H]]$ module. Then by estimating the kernel and cokernel of $\frac{\mathcal{X}(\mathcal{L}_{\mathcal{F}}/K_{\infty})}{Q\mathcal{X}(\mathcal{L}_{\mathcal{F}}/K_{\infty})} \to X(L_Q/K_{\infty})$ [Jh, §3], we can deduce the following: For any chosen $\theta \in S \setminus S_{\mathcal{X}(\mathcal{L}_{\mathcal{F}}/K_{\infty})}$, the twisted Selmer groups $X(L_Q(\theta)/K_U)$ of the congruent family of cuspforms f_Q, $Q \in A_{\mathrm{arith}}(\mathcal{T})$ are all simultaneously finite when K_U varies over every finite extension of K inside K_{∞}.

Algebraic functional equation. For a given f, the finiteness of $X(L_f(\theta)/K_U)$ for a fixed θ, where K_U varies over every intermediate finite extension of K in-

side K_∞, can be useful in various arithmetical situations; for example, finiteness of $X(L_f(\theta)/K_U)$ is crucial in the proof of algebraic functional equation for $X(L_f/K_\infty)$ (see [JO, §5, Theorem 0.3]). Theorem(JS) was also used in the proof in [JO]. Indeed, this is a generalization of the fact that the classical twisting lemma is crucial in Greenberg's [Gr, Theorem 2] and Perrin-Riou's [Pr, Theorem 4.2.1] proof of algebraic functional equation.

Similarly, the finiteness of $X(L_Q(\theta)/K_U)$ where Q and U both vary, can be used, for example, to establish an algebraic functional equation for the 'big' Selmer group $\mathcal{X}(\mathcal{L}_\mathcal{F}/K_\infty)$ of $\mathcal{F}$ over K_∞.

(II) Fine Selmer group. The dual fine Selmer group $Y(L_f/K_\infty)$ (resp. $\mathcal{Y}(\mathcal{L}_\mathcal{F}/K_\infty)$) of f (resp. $\mathcal{F}$) over K_∞, is a quotient of $X(L_f/K_\infty)$ (resp. $\mathcal{X}(\mathcal{L}_\mathcal{F}/K_\infty)$) [Jh, §2]) and is a finitely generated $O_f[[G]]$ (resp. $\mathcal{T}[[G]]$) module. It has interesting arithmetic properties. Following conjectures of Coates-Sujatha [CS, Conjectures A & B], it is believed that $Y(L_f/K_{\text{cyc}})$ is a finitely generated $\mathbb{Z}_p$ module and if $\dim G > 1$, then $Y(L_f/K_\infty)$ is a pseudonull $O_f[[G]]$ module, for any f. For $\dim G > 1$, it is not easy to determine if $\chi(G, Y(L_f/K_\infty))$ exists, even if we assume $Y(L_f/K)$ is finite and not much is known in the literature. Thus it is reasonable to discuss, if at least after a twist, the Euler characteristic of the fine Selmer group exists and also makes sense to consider the Euler characteristic of a twist of the fine Selmer groups in a congruent family.

COROLLARY 0.8. *Assume* $\frac{\mathcal{Y}(\mathcal{L}_\mathcal{F}/K_\infty)}{\mathcal{Y}(\mathcal{L}_\mathcal{F}/K_\infty)(p)}$ *is a finitely generated* $\mathcal{T}_\mathcal{F}[[H]]$ *module. Then applying Corollary 0.7, we deduce that there exists a countable subset* $S_{\mathcal{Y}(\mathcal{L}_\mathcal{F}/K_\infty)}$ *of* S *such that for any chosen* $\theta \in S \setminus S_{\mathcal{Y}(\mathcal{L}_\mathcal{F}/K_\infty)}$ *and for every* $Q \in A_{arith}(\mathcal{T}_\mathcal{F})$, $\chi(U, \frac{\mathcal{Y}(\mathcal{L}_\mathcal{F}/K_\infty)}{Q\mathcal{Y}(\mathcal{L}_\mathcal{F}/K_\infty)}(\theta))$ *is finite for every open normal subgroup* U *of* G.

Let m be any p-power free integer. Now consider a particular example of a p-adic Lie extension ('false Tate curve' extension) given by $K = \mathbb{Q}(\mu_p)$ and $K_\infty = \cup_n \mathbb{Q}(\mu_{p^\infty})(m^{1/p^n})$. Assume that $\mathcal{T}_\mathcal{F} = O[[X]]$. Then the kernel and the cokernel of the natural map $\frac{\mathcal{Y}(\mathcal{L}_\mathcal{F}/K_\infty)}{Q\mathcal{Y}(\mathcal{L}_\mathcal{F}/K_\infty)} \longrightarrow Y(L_Q/K_\infty)$ are finitely generated O_Q modules [Jh, Remark 11]. Also assume for some $Q_0 \in A_{\text{arith}}(\mathcal{T}_\mathcal{F})$, $Y(L_{Q_0}/\mathbb{Q}(\mu_{p^\infty}))$ is a finitely generated O_{Q_0} module. Then $\mathcal{Y}(\mathcal{L}_\mathcal{F}/K_\infty)$ and $Y(L_Q/K_\infty)$ are finitely generated modules respectively over $\mathcal{T}_\mathcal{F}[[H]]$ and $O_Q[[H]]$, for every $Q \in A_{\text{arith}}(\mathcal{T}_\mathcal{F})$ [Jh, §2]. From the proof of Proposition 0.6 and Corollary 0.8, we obtain the following result on the family of fine Selmer groups $\{Y(L_Q/K_\infty) : Q \in A_{\text{arith}}(\mathcal{T}_\mathcal{F})\}$.

THEOREM 0.9. *Let* $K_\infty/\mathbb{Q}(\mu_p)$ *be the false Tate curve extension. Let* $\mathcal{F}$ *be a* Λ-*adic newform. Assume that* $\mathcal{T}_\mathcal{F} \cong O[[X]]$ *is a power series ring and there exists* $Q_0 \in A_{arith}(\mathcal{T}_\mathcal{F})$ *such that* $Y(L_{Q_0}/\mathbb{Q}(\mu_{p^\infty}))$ *is a finitely generated* O_{Q_0} *module.*
Then there exists a countable subset $S_{\mathcal{Y}(\mathcal{L}_\mathcal{F}/K_\infty)}$ *of* S, *such that for any chosen* $\theta \in S \setminus S_{\mathcal{Y}(\mathcal{L}_\mathcal{F}/K_\infty)}$ *and for every* f_Q, $Q \in A_{arith}(\mathcal{T}_\mathcal{F})$, *the* U-*Euler characteristic* $\chi(U, Y(L_Q/K_\infty)(\theta))$ *exists, for every open normal subgroup* U *of* G. $\square$

REMARK 0.10. A result similar to Corollary 0.8 and Theorem 0.9 respectively for the usual 'big' dual Selmer group $\mathcal{X}(\mathcal{L}_\mathcal{F}/K_\infty)$ (see [Jh, §3]) and for the Selmer group $X(L_Q/K_\infty)$ of f_Q for $Q \in A_{\text{arith}}(\mathcal{T}_\mathcal{F})$ can also be obtained.

(III) $\mathbb{Z}_p[[H]]$ free Selmer group. Let E be the elliptic curve of conductor 121 defined by the equation $y^2 + y = x^3 - x^2 - 887x - 10143$. Let us consider a

particular false Tate curve extension by taking $p = 5$ and $m = 11$. Then $G \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_5$. Let $X(E/K_\infty)$ (respectively $X(E/K_{\mathrm{cyc}})$) denote the dual of the 5^∞-Selmer group of E over K_∞ (respectively K_{cyc}). Then using the fact that $X(E/K_{cyc})$ has μ-invariant zero together with [HV, Theorem 3.1], there exists an injective $\mathbb{Z}_5[[H]]$ module homomorphism $f : X(E/K_\infty) \longrightarrow (\mathbb{Z}_5[[H]])^t$, for some integer t. In fact, using $\tilde{E}(\mathbb{F}_5)[5] = 0$, from the proof of [HV, Theorem 3.1] we can show that f is an isomorphism. We explain it here briefly. From [HV, Theorem 3.1(ii)], $A :=$ cokernel(f) is a $\mathbb{Z}_5[[H]]$ module of finite cardinality. Taking coinvariance by H, we get that $H_1(H, A)$ is a subgroup of $X(E/K_\infty)_H$. Now it follows from [HV, equation 3.6] and the assumption $\tilde{E}(\mathbb{F}_5)[5] = 0$ that $X(E/K_\infty)_H$ is a free $\mathbb{Z}_p$-module. This implies that $H_1(H, A) = 0$. Since, A is a finite and H is pro-cyclic, we deduce $H_0(H, A) = 0$. Hence, by Nakayama's Lemma, $A = 0$. Therefore $X(E/K_\infty)$ is a free $\mathbb{Z}_p[[H]]$-module.

Let $Q_0 \in A_{\mathrm{arith}}(\mathcal{T}_\mathcal{F})$ be such that the weight 2 specialization f_{Q_0} corresponds to E via modularity. Since E has CM, in this case $\mathcal{T} \cong \mathbb{Z}_5[[X]]$. Then following an argument similar to [Sh, page 412], we can deduce that the natural map $\mathcal{X}(\mathcal{L}_\mathcal{F}/K_\infty)/Q_0 \longrightarrow X(E/K_\infty)$ is an isomorphism. Now it is well known that $\mathcal{X}(\mathcal{L}_\mathcal{F}/K_\infty)$ has no non-trivial pseudonull $\mathcal{T}[[G]]$ submodule. In particular, $\mathcal{X}(\mathcal{L}_\mathcal{F}/K_\infty)[Q_0] = 0$. Since $X(E/K_\infty)$ is a free $\mathbb{Z}_5[[H]]$-module of finite rank, by Nakayama's lemma $\mathcal{X}(\mathcal{L}_\mathcal{F}/K_\infty)$ is also a free $\mathcal{T}[[H]]$-module of finite rank. This also implies that $\mathcal{X}(\mathcal{L}_\mathcal{F}/K_\infty)/Q \cong X(L_Q/K_\infty)$ is free $O_Q[[H]]$-module of finite rank for every $Q \in A_{\mathrm{arith}}(\mathcal{T}_\mathcal{F})$. We notice that as the rank of $E(\mathbb{Q}) = 1$, the U-Euler characteristics of $X(E/K_\infty)$ does not exist for any open normal subgroup U of G.

Write $M = \mathcal{X}(\mathcal{L}_\mathcal{F}/K_\infty)$. Then applying Theorem 0.1, we get a countable subset S_M of S and for each $\theta \in S \setminus S_M$, there exists a countable subset $C_{M,\theta}$ of C such that the following holds: For every $\theta \in S \setminus S_M$ and for any $Q \in C \setminus C_{M,\theta}$, the U-Euler characteristic $\chi(U, \frac{M}{QM}(\theta))$ exists for every U. Further, by Lemma 1.10, it is given by $\chi(U, \frac{M}{QM}(\theta)) = \#(\frac{M}{QM}(\theta))_U$.

Now consider $Q \in A_{\mathrm{arith}}(\mathcal{T}_\mathcal{F})$. Then by Corollary 0.7, there exists a countable subset S_M of S such that for any $\theta \in S \setminus S_M$ and for every $Q \in A_{\mathrm{arith}}(\mathcal{T}_\mathcal{F})$, $\chi(U, \frac{M}{QM}(\theta))$ exists for every U. Moreover, by Lemma 1.10, it is given by $\chi(U, \frac{M}{QM}(\theta)) = \#(\frac{M}{QM}(\theta))_U = \#(X(L_Q/K_\infty)(\theta))_U = \#(X(E/K_\infty)(\theta))_U$. In particular, for every $Q \in A_{\mathrm{arith}}(\mathcal{T}_\mathcal{F})$, $\chi(U, X(L_Q/K_\infty)(\theta))$ is finite and $= \#(X(E/K_\infty)(\theta))_U$, for every open normal subgroup U of G.

Now, we give a proof of Theorem 0.1. Also see Remarks 1.13 and 1.14.

1. Proof of Theorem 0.1. This proof of Theorem 0.1 is divided into several remarks, lemmas and propositions. We begin with various reduction steps.

REMARK 1.1. Let $G, H, \mathcal{T}$ be as in Theorem 0.1. Let N be a finitely generated $\mathcal{T}[[G]]$ module. Then consider the module $N(p)$. Note that $\mathcal{T}[[G]]$ is noetherian and hence $N(p) = N[p^r]$ for some $r \in \mathbb{N}$. Thus from the short exact sequence $0 \longrightarrow N[p^r] \longrightarrow N \longrightarrow \frac{N}{N[p^r]} \longrightarrow 0$, we get another right exact sequence

$$\frac{N[p^r]}{QN[p^r]} \longrightarrow \frac{N}{QN} \longrightarrow \frac{\frac{N}{N[p^r]}}{Q\frac{N}{N[p^r]}} \longrightarrow 0,$$

for any $Q \in C$. Now let $I_{N,Q}$ be the image of $\frac{N[p^r]}{QN[p^r]}$ in $\frac{N}{QN}$. Then $I_{N,Q}$ is a finitely generated p^r-torsion $\frac{\mathcal{T}}{Q\mathcal{T}}[[G]]$ module and hence $\chi(U, I_{N,Q})$ exists for every U [Ho,

Proposition 1.6]. Thus, $\chi(U, \frac{N}{QN})$ is finite if and only if $\chi\left(U, \frac{\frac{N}{N[p^r]}}{Q\frac{N}{N[p^r]}}\right)$ is finite.

We can apply this observation for $N = M(\theta)$, with M, θ as in Theorem 0.1 and without any loss of generality, we may assume that M in Theorem 0.1 is a finitely generated $\mathcal{T}[[H]]$ module.

Next we consider finitely generated $\mathcal{T}[[G]]$ modules which are $\mathcal{T}$-torsion.

PROPOSITION 1.2. *Let G, H be as in Theorem 0.1. Let N be a finitely generated $\mathcal{T}[[G]]$ module such that N is also $\mathcal{T}$ torsion. Also assume $\mathcal{T}$ is a regular (local) ring. Then there exists a finite subset C_N of C such that for any $Q \in C \setminus C_N$, $\chi(U, \frac{N}{QN})$ exists for every open normal subgroup U of G.*

Proof. By Remark 1.1, we may assume N is p-torsion free. Let N be generated by the elements $x_1, \cdots, x_n$ as a $T[[G]]$-module. As N is $\mathcal{T}$ torsion, there exists $r_i \in \mathcal{T}$ such that $r_i x_i = 0$. Set $r := r_1 r_2 .... r_n$ and notice that r being an element of $\mathcal{T}$ commutes with the elements of $\mathcal{T}[[G]]$. Thus we get that $rN = 0$. As $\mathcal{T}$ is a UFD, let $r = p_1^{n_1} p_2^{n_2} ... p_t^{n_t}$ be the unique factorization of r where p_i, $1 \le i \le t$ are height 1 primes in $\mathcal{T}$. Set $C_N := \{p_1, p_2, ... p_t\}$. Then for any $Q \in C \setminus C_N$, $\frac{N}{QN}$ is annihilated by the height 2 ideal (r, Q). Now $\mathcal{T}$ is a regular local ring of dimension 2. Thus for some $n_0 \in \mathbb{N}$, $p^{n_0} \in (r, Q)$. This shows $p^{n_0} \frac{N}{QN} = 0$. Hence for any choice of $Q \in C \setminus C_N$, $\chi(U, \frac{N}{QN})$ exists for every U [Ho, Prop. 1.6]. $\square$

COROLLARY 1.3. *Let us keep the setting of Proposition 1.2. Then in the proof of Theorem 0.1, we may assume that M is $\mathcal{T}$-torsion free.*

Proof. Take any $\theta \in S$ and write $N = M(\theta)$. Let $N_\mathcal{T}$ denote the $\mathcal{T}$-torsion submodule of N. Consider the exact sequence $0 \to N_\mathcal{T} \longrightarrow N \longrightarrow N/N_\mathcal{T} \to 0$. Then for any $Q \in C$, we get the induced exact sequence

$$0 \to \frac{N_\mathcal{T}}{QN_\mathcal{T}} \longrightarrow \frac{N}{QN} \longrightarrow \frac{\frac{N}{N_\mathcal{T}}}{Q\frac{N}{N_\mathcal{T}}} \to 0.$$

By Proposition 1.2, there exists a finite subset $C_N = C_{M,\theta}$ of C such that for any $Q \in C \setminus C_N$, $\chi(U, \frac{N_\mathcal{T}}{QN_\mathcal{T}})$ exists for every U. Thus the statement of Theorem 0.1 holds for M if and only if it holds for $M/M_\mathcal{T}$. $\square$

Recall the following well known result, which can be conveniently found in [SS, Lemma 6.15].

LEMMA 1.4. *Let $\mathcal{T}$ be a commutative, complete local domain and let $\mathcal{T}$ be finite over $\mathbb{Z}_p[[X]]$. Let Q be a height 1 prime ideal in $\mathcal{T}$. Put $\mathfrak{q} = Q \cap \mathbb{Z}_p[[X]]$. Let $Q = Q_1, Q_2, \cdots Q_d$ be the height 1 prime ideals in $\mathcal{T}$ lying above $\mathfrak{q}$. If q is unramified in $\mathcal{T}$, then the kernel and cokernel of the natural map $\mathcal{T}/\mathfrak{q}\mathcal{T} \longrightarrow \bigoplus_{1 \le i \le d} \mathcal{T}/Q_i\mathcal{T}$ are finite of p-power order.* $\square$

Using Lemma 1.4, we further reduce Theorem 0.1 to the case $\mathcal{T} = Z_p[[X]]$.

COROLLARY 1.5. *Let us keep the hypotheses and setting of Theorem 0.1. Then in Theorem 0.1, without any loss of generality, we may assume $\mathcal{T} = \mathbb{Z}_p[[X]]$.*

Proof. Since the Kälher differential $\Omega_{\mathcal{T}/\mathbb{Z}_p[[X]]}$ is supported outside finitely many height 1 prime ideals in $\mathbb{Z}_p[[X]]$, it follows that only finitely many height 1 primes of $\mathbb{Z}_p[[X]]$ can ramify in $\mathcal{T}$. Let $C_1^{\mathbb{Z}_p[[X]],\mathcal{T}}$ be the finite set of height 1 prime ideals in $\mathbb{Z}_p[[X]]$ that ramify in $\mathcal{T}$. Let $\mathfrak{q} \in C^{\mathbb{Z}_p[[X]]} \setminus C_1^{\mathbb{Z}_p[[X]],\mathcal{T}}$ and let $Q = Q_1, Q_2, \cdots Q_d$ be

the height 1 primes in $\mathcal{T}$ dividing $\mathfrak{q}$. Recall M in Theorem 0.1 is a finitely generated $\mathcal{T}[[G]]$ module. By Lemma 1.4, for any $\theta \in S$, there exists an exact sequence

$$0 \to K_{\mathfrak{q}}^{\theta} \longrightarrow \frac{M(\theta)}{\mathfrak{q}M(\theta)} \longrightarrow \bigoplus_{1 \leq i \leq d} \frac{M(\theta)}{Q_i M(\theta)} \longrightarrow CK_{\mathfrak{q}}^{\theta} \to 0, \tag{1}$$

where $K_{\mathfrak{q}}^{\theta}$ and $CK_{\mathfrak{q}}^{\theta}$ are finitely generated p-power torsion $\frac{\mathcal{T}}{\mathfrak{q}\mathcal{T}}[[G]]$ modules. In particular, $K_{\mathfrak{q}}^{\theta}$ and $CK_{\mathfrak{q}}^{\theta}$ are finitely generated p-power torsion $\mathbb{Z}_p[[G]]$ modules. Now, as explained in Remark 1.1, $\chi(U, K_{\mathfrak{q}}^{\theta})$ and $\chi(U, CK_{\mathfrak{q}}^{\theta})$ always exist for any open normal subgroup U of G.

Now assume Theorem 0.1 holds for $\mathcal{T} = \mathbb{Z}_p[[X]]$. Then there exists a countable subset S_M of S and for any $\theta \in S \setminus S_M$ there exists a countable subset $C_{M,\theta}^{\mathbb{Z}_p[[X]]}$ of $C^{\mathbb{Z}_p[[X]]}$ such that the following holds: For any $\theta \in S \setminus S_M$ and for any $Q \in C^{\mathbb{Z}_p[[X]]} \setminus C_{M,\theta}^{\mathbb{Z}_p[[X]]}$, the U-Euler characteristic $\chi\left(U, \frac{M}{\mathfrak{q}M}(\theta)\right)$ exists for every open normal subgroup U of G. Note $C_{M,\theta}^{\mathbb{Z}_p[[X]]} \cup C_1^{\mathbb{Z}_p[[X]],\mathcal{T}}$ is countable and hence $C_{M(\theta)}^0 := \{Q \in C^{\mathcal{T}} : Q \cap \mathbb{Z}_p[[X]] \in C_{M,\theta}^{\mathbb{Z}_p[[X]]} \cup C_1^{\mathbb{Z}_p[[X]],\mathcal{T}}\}$ is a countable subset of $C^{\mathcal{T}}$. Now from (1), for any $\theta \in S \setminus S_M$ and for any $Q \in C \setminus C_{M,\theta}^0$, $\chi(U, \frac{M}{QM}(\theta))$ is finite for every U. Thus Theorem 0.1 holds for a general $\mathcal{T}$. $\square$

REMARK 1.6. As explained in [JS, Remark 2.3], we may also assume without any loss of generality, that G is a compact, pro-p, p-adic Lie group without any element of order p. It should be noted [JS, Remark 2.3] uses fundamental work of Lazard [La].

LEMMA 1.7. *Let G, H be in the setting of Theorem 0.1 and assume $\mathcal{T} = \mathbb{Z}_p[[X]]$. Let M be a finitely generated $\mathbb{Z}_p[[X]][[G]]$ module which is finitely generated over $\mathbb{Z}_p[[X]][[H]]$. Also assume there exists a height 1 prime ideal Q_0 in $\mathbb{Z}_p[[X]]$ with $\mathbb{Q}_0 \neq (p)$, such that $\#(\frac{M}{Q_0 M})_U$ is finite for every open normal subgroup U of G. Then for all but countably many height 1 prime ideals Q in $\mathbb{Z}_p[[X]]$, $\#(\frac{M}{QM})_U$ is finite for every open normal subgroup U of G.*

Proof. Let U be an open normal subgroup of G. As $[G : U] < \infty$, M_U is a finitely generated $\mathbb{Z}_p[[X]]$ module. Now, from the structure theorem for finitely generated $\mathbb{Z}_p[[X]]$ modules, we know that there exists a $\mathbb{Z}_p[[X]]$ module homomorphism:

$$M_U \to \mathbb{Z}_p[[X]]^{r_U} \oplus \bigoplus_{i=1}^{s_U} \frac{\mathbb{Z}_p[[X]]}{(Q_{i,U}^{n_i})} \tag{2}$$

with finite kernel and cokernel, where $r_U, n_i, s_U \in \mathbb{N} \cup \{0\}$ and $Q_{i,U}$'s are height 1 prime ideals in $\mathbb{Z}_p[[X]]$. Note that $(\frac{M}{Q_0 M})_U \cong \frac{M_U}{Q_0 M_U}$. Hence using the fact that the cardinality of $(\frac{M}{Q_0 M})_U$ is finite, we deduce from (2) that $r_U = 0$. So, if we choose $Q \in C$ such that $Q \notin C_{M,U} := \bigcup_{i=1}^{s_U} Q_{i,U}$, then $\#\left(\frac{M}{QM}\right)_U$ is finite. Since G is a profinite group, it has countable base at identity. Thus we can take the set $\mathcal{U}$ of open normal subgroups U of G to be countable. Then for any $Q \in C$ which does not belong to the countable set $C_M := \bigcup_{U \in \mathcal{U}} C_{M,U}$, we get $\#\left(\frac{M}{QM}\right)_U$ is finite for every U. This completes the proof of the lemma. $\square$

We next prove the following proposition:

PROPOSITION 1.8. *Let G, H be as in the setting of Theorem 0.1. Let M be a finitely generated $\mathbb{Z}_p[[X]][[G]]$ module which is finitely generated over $\mathbb{Z}_p[[X]][[H]]$.*

Then there exists a countable subset S_M of S and further, for each $\theta \in S \setminus S_M$, there exists a countable subset $C_{M,\theta}$ of C such the the following holds: If we choose any $\theta \in S \setminus S_M$ and any $Q \in C \setminus C_{M,\theta}$, then $\#\big(\frac{M}{QM}(\theta)\big)_U$ is finite for every open normal subgroup U of G.

Proof. Note that $\mathbb{Z}_p[[X]][[G]] \cong \mathbb{Z}_p[[G_1 \times G]]$, where $G_1 \cong \mathbb{Z}_p$. Let U be any open normal subgroup of G. Then $U' := G_1 \times U$ is an open normal subgroup of $G_1 \times G$ that maps onto U under the projection map $\pi : G_1 \times G \longrightarrow G$. Now by Theorem(JOZ), there is a countable subset S_M of S, such that for any $\theta \in S \setminus S_M$, we have $(M(\theta))_{U'} = (M(\theta))_{G_1 \times U}$ is finite for every U. Therefore, $(M(\theta))_{U'} = (M(\theta)_U)_{U'/U} \cong \frac{M(\theta)_U}{(X)}$ is finite for every U.

So writing $N = M(\theta)$ and $Q_0 = (X)$, we have $\big(\frac{N}{Q_0 N}\big)_U$ is finite for every U open normal in G. Now applying Lemma 1.7, there is a countable subset $C_{M,\theta} \subset C$ such that for any $Q \in C \setminus C_{M,\theta}$, we have $(\frac{N}{QN})_U = \big(\frac{M}{QM}(\theta)\big)_U$ is finite for every open normal subgroup U. $\square$

The next lemma is easy to prove and used later.

LEMMA 1.9. *Let G, H be as in Theorem 0.1. Let G_1 be a p-adic Lie group isomorphic to $\mathbb{Z}_p$. Let K be an open subgroup of $G_1 \times G$ such that $G_1 \times H \subset K$. Then $K = G_1 \times G^0$, where G^0 is an open subgroup of G.* $\square$

Using Lemma 1.9, we deduce the following result.

LEMMA 1.10. *Let G be a compact, pro-p, p-adic Lie group without any p-torsion element. Let H be a closed normal subgroup of G with $\Gamma := G/H \cong \mathbb{Z}_p$. Let M be a finitely generated $\mathbb{Z}_p[[X]][[G]]$ module which is also finitely generated over $\mathbb{Z}_p[[X]][[H]]$. Then, there exists an open subgroup G^0 of G with $H \subset G^0$ and a resolution*

$$0 \to N_k \to N_{k-1} \to \cdots \to N_1 \to M \to 0$$

of M by finitely generated $\mathbb{Z}_p[[X]][[G^0]]$ modules N_i, $1 \leq i \leq k$ such that each N_i is a free $\mathbb{Z}_p[[X]][[H]]$ module of finite rank.

Proof. Note that $\mathbb{Z}_p[[X]][[G]] \cong \mathbb{Z}_p[[G_1 \times G]]$, where $G_1 \cong \mathbb{Z}_p$. Also $\frac{G_1 \times G}{G_1 \times H} \cong \Gamma$. Using [JS, Lemma 2.4], there exists an open subgroup G^{00} of $G_1 \times G$ such that $G_1 \times H \subset G^{00}$ and a resolution

$$0 \to N_k \to N_{k-1} \to \cdots \to N_1 \to M \to 0$$

of M by finitely generated $\mathbb{Z}_p[[G^{00}]]$ modules N_i, $1 \leq i \leq k$ such that each N_i is a free $\mathbb{Z}_p[[G_1 \times H]]$ module of finite rank. By Lemma 1.9, $G^{00} = G_1 \times G^0$, where G^0 is an open subgroup of G containing H. $\square$

REMARK 1.11. Let us again consider the false Tate curve extension given by $K = \mathbb{Q}(\mu_p)$, $K_\infty = \bigcup_n \mathbb{Q}(\mu_{p^\infty})(m^{1/p^n})$ and $G = \mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p \rtimes \mathbb{Z}_p$. Then it follows from a result of Zábrádi [Za, Lemma 4.3], that for this particular G, one can, in fact take $G^0 = G$ in [JS, Lemma 2.4] and hence we can also assume G^0 to be equal to G in Lemma 1.10.

Next we calculate the Euler characteristic of a free $\mathbb{Z}_p[[X]][[H]]$ module. Recall, for any $Q \in C^{\mathbb{Z}_p[[X]]}$, $\frac{\mathbb{Z}_p[[X]]}{Q} \cong O_Q$, the ring of integers of certain finite extension of $\mathbb{Q}_p$.

PROPOSITION 1.12. *Let G be a compact, pro-p, p-adic Lie group without any p-torsion element. Let H be a closed normal subgroup of G with $\Gamma := G/H \cong \mathbb{Z}_p$. Let N be a finitely generated $\mathbb{Z}_p[[X]][[G]]$ module which is also a finitely generated free $\mathbb{Z}_p[[X]][[H]]$ module of rank d. Then there exists a countable subset S_N of S and for any $\theta \in S \setminus S_N$ there exists a countable subset $C_{N,\theta}$ of C such that the following holds: For any $\theta \in S \setminus S_N$ and for any $Q \in C \setminus C_{N,\theta}$, the U-Euler characteristic $\chi\left(U, \frac{N}{QN}(\theta)\right)$ exists for every open normal subgroup U of G. Moreover, $\chi\left(U, \frac{N}{QN}(\theta)\right) = \#\left(\frac{N}{QN}(\theta)\right)_U$.*

Proof. By Proposition 1.8, there exists a countable subset S_N of S and for any $\theta \in S \setminus S_N$ there exists a countable subset $C_{N,\theta}$ of C such that the following holds: For any $\theta \in S \setminus S_N$ and for any $Q \in C \setminus C_{N,\theta}$, $H_0\left(U, \frac{N}{QN}(\theta)\right)$ is finite for every open normal subgroup U of G. Now as $\frac{N}{QN}$ is a free $O_Q[[H]]$ module of finite rank, the result follows directly from [JS, Proposition 2.7]. □

REMARK 1.13. Let us keep the setting and hypotheses of Lemma 1.10. Then by Lemma 1.10, there exists an open normal subgroup $H \subset G^0 \subset G$ a resolution of M by $\Lambda(G^0)$-module,

$$0 \longrightarrow N_k \xrightarrow{f_k} N_{k-1} \xrightarrow{f_{k-1}} \cdots \xrightarrow{f_2} N_1 \xrightarrow{f_1} M \to 0 \tag{3}$$

such that each N_i is a finitely generated, free $\mathbb{Z}_p[[X]][[H]]$-module. Let U be an open normal subgroup of G^0. Take $Q = (X+1) - \zeta(1+p)^k \in A_{\mathrm{arith}}(\mathbb{Z}_p[[X]])$. By Corollary 0.7, we can choose a $\theta \in S$ such that for every $Q \in A_{\mathrm{arith}}(\mathbb{Z}_p[[X]])$, $\chi(U, \frac{N_i}{QN_i}(\theta))$, $1 \leq i \leq k$ and $\chi(U, \frac{M}{QM}(\theta))$ are all finite. An easy computation shows that for every $Q \in A_{\mathrm{arith}}(\mathbb{Z}_p[[X]])$, the U-Euler characteristics $M(\theta)[Q]$ also exists and $\chi(U, \frac{M}{QM}(\theta))/\chi(U, M(\theta)[Q]) = \prod_i \left(\chi(U, \frac{N_i}{QN_i}(\theta))\right)^{(-1)^{i+1}}$.

We also have $\chi(U, \frac{N_i}{QN_i}(\theta)) = \#\left(\frac{N_i}{QN_i}(\theta)\right)_U = \#\frac{\left(N_i(\theta)\right)_U}{Q\left(N_i(\theta)\right)_U}$, for every $Q \in A_{\mathrm{arith}}(\mathbb{Z}_p[[X]])$. Let $f_i^\theta(X) \in \mathbb{Z}_p[[X]]$ denote the $\mathbb{Z}_p[[X]]$ characteristic element of $\left(N_i(\theta)\right)_U$. Since $\frac{\left(N_i(\theta)\right)_U}{Q\left(N_i(\theta)\right)_U}$ is finite for every i, Q does not divide $f_i^\theta(X)$ and we have $\chi(U, \frac{N_i}{QN_i}(\theta)) = f_i^\theta(Q) := f_i^\theta(\zeta(1+p)^k - 1) = \#\frac{\mathbb{Z}_p[[X]]}{(Q, f_i^\theta)}$, for every $Q \in A_{\mathrm{arith}}(\mathbb{Z}_p[[X]])$. Thus

$$\frac{\chi(U, \frac{M}{QM}(\theta))}{\chi(U, M(\theta)[Q])} = \prod_i \left(\#\frac{\mathbb{Z}_p[[X]]}{(Q, f_i^\theta(X))}\right)^{(-1)^{i+1}}. \tag{4}$$

Now in this setting, if for some Q, $M[Q] = 0$, then using (4) we can compute $\chi(U, \frac{M}{QM}(\theta))$. On the other hand, it is also natural to ask if $\chi(U, \frac{M}{Q_0 M}(\theta)) = \chi(U, M(\theta)[Q_0])$ for some $Q_0 \in A_{\mathrm{arith}}(\mathbb{Z}_p[[X]])$, then can we say $\chi(U, \frac{M}{QM}(\theta)) = \chi(U, M(\theta)[Q])$ for every $Q \in A_{\mathrm{arith}}(\mathbb{Z}_p[[X]])$? Unfortunately, we do not know the answer of this. The reason being, we do not know if $\prod_i \left(f_i^\theta(X)\right)^{(-1)^{i+1}} \in \mathbb{Z}_p[[X]]$ or not.

REMARK 1.14. Let E/K be an elliptic curve with good, ordinary reduction at primes of K dividing p. Let p-cohomological dimension $cd_p(G)$ of $G = \mathrm{Gal}(K_\infty/K)$ be ≥ 3. Also assume (i) for any prime v of K dividing p, $cd_p(G_v) < cd_p(G)$ and (ii) for any prime u of K not dividing p, such that either u ramifies in K_∞ or u is a bad prime

of E, $cd_p(G_u) \geq 2$. Further assume, $X(E/K_{\mathrm{cyc}})$ is a finitely generated $\mathbb{Z}_p$ module. Then by [OV, Proposition 5.2], the $\mathbb{Z}_p[[G]]$ projective dimension of $X(E/K_\infty)$ is $= cd_p(G) - 1$ if $E(K)[p] \neq 0$ and $= cd_p(G) - 2$ if $E(K)[p] = 0$. Now, as $X(E/K_{\mathrm{cyc}})$ is a finitely generated $\mathbb{Z}_p$ module, applying a control theorem, we can deduce $X(E/K_\infty)$ is a finitely generated $\mathbb{Z}_p[[H]]$ module. Since $cd_p(H) = cd_p(G) - 1$, $\mathbb{Z}_p[[H]]$ projective dimension of $X(E/K_\infty)$ is $cd_p(H) - 1$ if $E(K)[p] \neq 0$ and $= cd_p(H) - 2$ if $E(K)[p] = 0$. In particular, if $cd_p(H) \geq 3$, then $X(E/K_\infty)$ cannot be a free $\mathbb{Z}_p[[H]]$ module. Nevertheless, there exists an open subgroup G^0 of G containing H and a $\mathbb{Z}_p[[G^0]]$ resolution of $X(E/K_\infty)$ of length k by finitely generated $\mathbb{Z}_p[[H]]$ modules [JS, Lemma 2.4]. Moreover, the length of the resolution is given by $k = cd_p(H) - 1$ if $E(K)[p] \neq 0$ and $k = cd_p(H) - 2$ if $E(K)[p] = 0$. A similar assertion holds for the 'big' Selmer group $\mathcal{X}(\mathcal{T}_\mathcal{F}/K_\infty)$ by using Lemma 1.10.

Now we are ready to prove Theorem 0.1.

Proof of Theorem 0.1. First of all, using Corollary 1.5, we can assume without any loss of generality, that $\mathcal{T} = \mathbb{Z}_p[[X]]$. Next, by Remark 1.1, we can assume M is a finitely generated $\mathbb{Z}_p[[X]][[H]]$ module. Further, as $\mathbb{Z}_p[[X]]$ is regular, we can assume using Corollary 1.3 that M is $\mathcal{T}$-torsion free. Moreover, following Remark 1.6, we will assume that G is a compact, pro-p, p-adic Lie group without any element of order p.

By Lemma 1.10, there exists an open normal subgroup G^0 of G with $H \subset G^0$ and a resolution

$$0 \longrightarrow N_k \xrightarrow{f_k} N_{k-1} \xrightarrow{f_{k-1}} \cdots \xrightarrow{f_2} N_1 \xrightarrow{f_1} M \to 0 \tag{5}$$

of M by finitely generated $\mathbb{Z}_p[[X]][[G^0]]$ modules N_i, $1 \leq i \leq k$ such that each N_i is a free $\mathbb{Z}_p[[H]]$ module of finite rank.

Set $\Gamma^0 := G^0/H$. Then as explained in the proof of [JS, Theorem 1.2], for any given $\theta \in S$, $Q \in C$ and any open normal subgroup U of G, if $\chi(U \cap G^0, \frac{M}{QM}(\theta|_{\Gamma^0}))$ is finite, then we can deduce $\chi(U, \frac{M}{QM}(\theta))$ is also finite. Thus, for the rest of the proof, we will only discuss the finiteness of $\chi(U \cap G^0, \frac{M}{QM}(\theta|_{\Gamma^0}))$; and further, to ease the burden of notation, will write $G^0 = G$ and $\Gamma^0 = \Gamma$ for the rest of the proof.

We will proceed by induction on k in (5). For $k = 1$, M/QM is a free $\frac{\mathbb{Z}_p[[X]]}{Q}[[H]] \cong O_Q[[H]]$ module of finite rank, for any $Q \in C$. Hence by Proposition 1.12, there exists a countable subset S_M of S and a countable subset $C_{M,\theta}$ of C, such that the following holds: For any $\theta \in S \setminus S_M$ and for any $Q \in C \setminus C_{M,\theta}$, $\chi\left(U, \frac{N}{QN}(\theta)\right)$ exists for every open normal subgroup U of G.

Next, pick any $Q \in C$. Then $\mathbb{Z}_p[[X]]$ being a regular local ring, we get $Q = (q)$ and hence $M[q] = 0$ as M is $\mathcal{T}$-torsion free. Thus (5) gives rise to another exact sequence of $O_Q[[G]]$ modules

$$0 \longrightarrow \mathrm{Img}(f_2)/Q\mathrm{Img}(f_2) \longrightarrow N_1/QN_1 \xrightarrow{f_1} M/QM \to 0. \tag{6}$$

By induction, there exists a countable subset S_2 of S and a countable subset $C_{2,\theta}$ of C, such that the following holds: For any $\theta \in S \setminus S_2$ and for any $Q \in C \setminus C_{2,\theta}$, $\chi\left(U, \frac{Img(f_2)}{QImg(f_2)}(\theta)\right)$ exists for every open normal subgroup U of G. Similarly, N_1/QN_1 is a free $O_Q[[H]]$ module of finite rank and hence there exists a countable subset S_1 of S and a countable subset $C_{1,\theta}$ of C, such that the following holds: For any $\theta \in S \setminus S_1$ and for any $Q \in C \setminus C_{1,\theta}$, $\chi\left(U, \frac{N_1}{QN_1}(\theta)\right)$ exists for every open normal subgroup U

of G. Define $S_M := S_1 \cup S_2$, and for any $\theta \in S \setminus S_M$, set $C_{M,\theta} = C_{1,\theta} \cup C_{2,\theta}$. Then from (6), for any $\theta \in S \setminus S_M$ and any $Q \in C \setminus C_{M,\theta}$, $\chi\left(U, \frac{M}{QM}(\theta)\right)$ is finite for every U. This completes the proof of Theorem 0.1. $\square$

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