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ON THE ANALYTIC CLASSIFICATION OF IRREDUCIBLE PLANE CURVE SINGULARITIES*

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Abstract. We present new results regarding which Puiseux coefficients the analytic type of a complex irreducible plane curve singularity depends on.

Key words. Analytic classification, irreducible plane curve singularity, analytic invariant.

Mathematics Subject Classification. 14H20, 32S10.

1. Introduction. There are two different classifications of complex analytic plane curve singularities. The first one, named *equisingularity* or *topological equivalence*, results from either of the following equivalent criteria:

(a) same behaviour when intersecting with other curves (Enriques-Chisini,[8], Libro IV, Chap. I),
(b) same structure of the resolution by blowing up (Enriques-Chisini, [8], Libro IV, Chap. II, Zariski [20]), and
(c) same topology as germs of subsets of $\mathbb{C}^2$ (Brauner [2], Zariski [19]).

Equisingularity has been considered since at least the early years of the past century, although the first precise definition of the equivalence, as well as the name *equisingularity*, were given by Zariski in 1965 ([20]). The second classification –*analytic classification*– takes two plane curve singularities as equivalent if and only if there is an analytic isomorphism between open sets of their planes mapping one singularity to the other. It is finer than equisingularity.

Focusing on irreducible singularities, any irreducible germ of plane curve γ is given, using suitable local coordinates x, y, by an equality

$$y = \sum_{i \geq n} c_i x^{i/n}, \quad c_i \in \mathbb{C}, \tag{1}$$

whose second member is a convergent fractionary power series called a Puiseux series of γ. The equisingularity class of γ determines and is determined by the characteristic exponents of the Puiseux series: among the i/n with $c_i \neq 0$, these are the first fractionary exponent and, inductively, each exponent that cannot be reduced to the minimal common denominator of the former ones. By contrast, the analytic class –also called *analytic type*– of γ depends also on the values of some of the Puiseux coefficients c_i.

Besides some interesting comments in Semple and Kneebone's [17] and the pioneering work by Kasner and DeCicco [11], Ebey [7] and Wolffhardt [18], analytic equivalence of curve singularities does not seem to have attracted much attention until the fundamental Zariski memoir [22] (in the sequel we will refer to its more recent English translation [23]). In it, Zariski introduces moduli spaces parameterizing the analytic classes of irreducible singularities with fixed equisingularity class and works out very interesting examples. After the publication of Zariski's memoir, most of the

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attention was placed on the study of these moduli spaces, which showed to be quite hard to deal with. To this respect, the reader may see [6], [3], [12], [9] and [14].

In this paper we address the problem of determining which Puiseux coefficients do affect the analytic type of an irreducible germ of analytic plane curve with a single characteristic exponent, and the way in which they do so. These coefficients are finitely many due to a result of Samuel [16], but the problem has not a simple solution because the variation of a coefficient may or may not modify the analytic class depending on the values of former coefficients, as already shown by Zariski in [23].

The approach used here is new, based on considering the coefficients of an equation of the germ rather than those of a Puiseux series. The fact is that each coefficient of the equation has associated a coefficient of the Puiseux series in such a way that both may be taken as coordinates on a certain (infinitesimal) first neighbourhood: as a consequence, they are linearly related and the variation of one of them causes a variation of the analytic type if and only if so does the variation of the other (5.8). When this is the case we say that either of the coefficients is *relevant*. Using the coefficients of the equation leads to simpler –rational– computations and allows an easy presentation of our results about relevance, which are summarized next.

Assume that γ is an irreducible germ of plane curve with single characteristic exponent m/n. Assume also –which is not restrictive– that γ is given by an equation of the form

$$y^n - x^m + \sum_{ni+mj>nm} A_{i,j}x^iy^j = 0$$

and take each coefficient $A_{i,j}$ as corresponding to the point (i,j) of a real plane (*Newton plane*). There is a well known numerical discrete invariant of irreducible germs, called the *Zariski invariant*; we assume the Zariski invariant σ of γ to be fixed and finite, the latter condition excluding only the germs analytically equivalent to $y^n - x^m = 0$. Figure 1 is a representation of the Newton plane. In it $\omega = (\omega_1, \omega_2)$ –the *Zariski point*– is the only point in the first quadrant with integral coordinates and satisfying of $n\omega_1 + m\omega_2 = nm + \sigma$; ℓ is the line through ω with slope $-n/m$; from the two lines parallel to ℓ through the other two vertices of the rectangle with opposite vertices ω and $(m-1, n-1)$, ℓ' is the one closest to the origin. Note thus that, depending on ω, the black triangle may as well lie inside the other grey triangle.

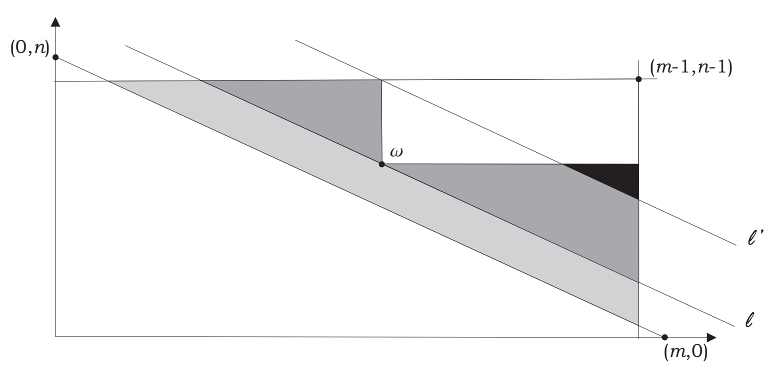

FIG. 1. *The zones of the Newton plane whose points correspond to the different types of possibly relevant coefficients, as explained in the text.*

Then it holds:

(1) The coefficients corresponding to ω or to a point in the interior of the light grey zone (*critical coefficients*) are all relevant. The variation of any of these coefficients gives rise to a different analytic type only when the coefficient takes a certain value (*critical value*). See Section 7.
(2) The coefficients corresponding to a point in the interior of the dark grey zone (*continuous invariants*) are all relevant. Different values of any of these coefficients give rise to different analytic types, but for values congruent by the action of a certain finite group of homotheties of $\mathbb{C}$. See Section 11.
(3) The coefficients corresponding to a point in the interior of the black triangle (*conditional invariants*) may or may not be relevant, depending on the values of former (invariant) coefficients of the germ. When relevant, they behave as the continuous invariants of item (2) above and will be also called continuous invariants. See Section 12, which contains also some constraints on the number of those coefficients that may be continuous invariants (12.17).
(4) All other coefficients are irrelevant. See Sections 6 and 8.

The reader may also see Examples 12.14 and 12.18.

A Puiseux series of γ having the form $x^{m/n} + \sum_{r>0} c_r x^{(m+r)/n}$, the Puiseux coefficient corresponding to $A_{i,j}$ is $c_{ni+mj-nm}$, after which the above may be read in terms of Puiseux coefficients. Our approach here being new, we will proceed from scratch, recovering on the way many already known results such as the Zariski elimination criteria (6.1 and 8.1); this makes the presentation reasonably self-contained regarding the analytic aspects.

We will not address here the relationship between the analytic type of a germ of plane curve $f = 0$ and the differentials of its local ring or, equivalently, the curves defined by the elements of its Jacobian ideal $\mathbf{J} = (f, \partial_x f, \partial_y f)$, in particular its polar curves. This is a rich subject that has received many contributions; among them and besides those in [23], there are [21], [15], [13], in which the analytical type is proved to be determined by the quotient algebra $\mathbb{C}\{x, y\}/\mathbf{J}$, [4] and the more recent [10], in which the degrees of the Puiseux coefficients which are continuous invariants are characterized as being the gaps of the orders of the differentials (up to a suitable shifting).

I would like to express here my high appreciation of Prof. O. Zariski's work on the subject: understanding the situations unveiled by his hardly worked examples in [23] has been a strong motivation.

2. Preliminary definitions and results. The reader is referred to [5] for known facts regarding plane curve singularities and, in particular, infinitely near points. Surface will mean smooth complex analytic surface. All our considerations being local, we will consider analytic curves on a surface $\mathbf{S}$, defined in open neighbourhoods of a point $O \in \mathbf{S}$, and their germs at –or with origin– O. In the sequel, these germs of curve will be often called just *germs*. After fixing local coordinates x, y on $\mathbf{S}$ with origin at O, any germ ξ at O is given by a non-zero power series $f = f(x, y) \in \mathbb{C}\{x, y\}$, which is in turn determined by the germ up to multiplication by an invertible series. The series f will be referred to as an *equation* of ξ, its zeros in a suitable neighbourhood of O are the points of a representative of ξ. We will write $\xi : f = 0$ to indicate that the germ ξ has equation f, ξ being then said to be the germ $f = 0$. The multiplicity of the germ ξ, also called the multiplicity of O on ξ, will be noted $e_O(\xi)$

We will study here irreducible germs γ with a single characteristic exponent m/n (m, n coprime, $m > n > 1$). For each such germ, there is a convergent fractionary power series S such that mapping $x \mapsto (x, S(x))$, for x in an open neighbourhood of O in $\mathbb{C}$, parameterizes a representative of γ; using suitable coordinates, such a series has the form

$$S = x^{m/n} + \sum_{r>0} c_r x^{(m+r)/n}. \tag{2}$$

The polynomial in y

$$\prod_{\varepsilon^n=1} \left(y - \varepsilon^m x^{m/n} - \sum_{r>0} c_r \varepsilon^{m+r} x^{(m+r)/n} \right)$$

is an equation of γ which –the coordinates being fixed– is uniquely determined by γ and called the *Weierstrass equation* of γ. Note that $n = e_O(\gamma)$. Usually, any of the roots of the Weierstrass equation is called a Puiseux series of γ. Nevertheless, here we will fix our attention on S –the only root with initial coefficient 1– and refer to it as *the* Puiseux series of γ. Both the above parameterization $x \mapsto (x, S(x))$ and the equivalent integral one $t \mapsto (t^n, S(t^n))$, will be called Puiseux parameterizations of γ.

Any equation of γ being the product of its Weierstrass equation by an invertible series, an easy computation shows that, up to a constant factor, any equation of γ has the form

$$f = y^n - x^m + \sum_{ni+mj>nm} A_{i,j} x^i y^j, \tag{3}$$

with $A_{i,j} \in \mathbb{C}$.

Conversely, the Newton-Puiseux algorithm ([5], 1.4, for instance) shows that any germ of curve γ defined by an equation (3), with $m > n > 1$ and $\gcd(n, m) = 1$, is irreducible, has single characteristic exponent m/n and a Puiseux series as (2) above.

Remark 2.1. Assuming to have fixed a surface $\mathbf{S}$, a point $O \in \mathbf{S}$ and local coordinates x, y on $\mathbf{S}$ with origin at O, in the sequel we will consider the irreducible germs of curve γ on $\mathbf{S}$ at O which are defined by an equation of the form (3) with $m > n > 1$ and $\gcd(n, m) = 1$, or, equivalently, have a Puiseux series of the form (2), still with $m > n > 1$ and $\gcd(n, m) = 1$. Since germs of curve having the same equation are analytically equivalent, these germs provide representatives of all analytic equivalence classes of irreducible germs of plane curve with a single characteristic exponent.

The germ γ being as in 2.1 above, we will refer to the coefficients c_r, of its Puiseux series (2), as *Puiseux coefficients*, while the coefficients $A_{i,j}$ of its equation (3) will be called just *coefficients*.

The pair of coprime integers n, m being fixed, we define $ni + mj$ as being the *twisted degree* of any monomial ax^iy^j, $a \in \mathbb{C} - \{0\}$, sometimes also referred to as the *twisted degree* of the coefficient a if this causes no confusion. Then, as usual, the *twisted degree* $\mathrm{td}(g)$ of a non-zero polynomial $g \in \mathbb{C}[x, y]$ will be the maximal twisted degree of its non zero monomials, while the *twisted order* $\mathrm{to}(f)$ of a non-zero series $f \in \mathbb{C}\{x, y\}$ will be the minimal twisted degree of its non-zero monomials. The twisted order of the series 0 is taken to be ∞. Needless to say,

$$\mathrm{td}(gg') = \mathrm{td}(g) + \mathrm{td}(g') \quad , \quad \mathrm{td}(g + g') \leq \max\{\mathrm{td}(g), \mathrm{td}(g')\}$$

and

$$\mathrm{to}(ff') = \mathrm{to}(f) + \mathrm{to}(f')) \quad , \quad \mathrm{to}(f + f') \geq \min\{\mathrm{to}(f), \mathrm{to}(f')\}$$

for any non-zero polynomials g, g' and series f, f'.

Among the coefficients of a power series $\sum_{i,j\geq 0} A_{i,j}x^iy^j$, those *preceding* a given $A_{k,h}$ will be taken to be the $A_{i,j}$ with $ni + mj \leq nk + mh$ and $(i,j) \neq (k,h)$. Also dots $\ldots$ placed at the end of a sum of monomials in a single variable will represent, as usual, terms of higher degree, while when placed after a polynomial expression in two variables, usually x, y, the dots will represent terms of higher twisted degree.

Still assume that the irreducible germ γ is given by an equation such as (3) above. As customary, in a real plane with fixed coordinates $\mathcal{N} = \mathbb{R}^2$, named the *Newton plane* in the sequel, we consider the subset $\mathbb{N}^2$ of the points with non-negative integral coordinates and take the point $(i,j) \in \mathbb{N}^2$ as *corresponding* to the monomial $A_{i,j}x^iy^j$ of (3) –or just to its coefficient $A_{i,j}$. If $p = (i,j)$, we will sometimes write $A_p = A_{i,j}$. We will say that $\mathrm{td}(p) = ni + mj$ is the twisted degree of the point $p = (i,j)$: a monomial and the point representing it have thus the same twisted degree. The points with twisted degree d are those on the line $ni + mj = d$ of the Newton plane. In particular the monomials of minimal twisted degree, y^n and $-x^m$, are represented by the points $(0,n)$ and $(m,0)$, which span the line $ni+mj = mn$ and are the vertices of the only side of the Newton polygon of γ. The next lemma sets a few elementary facts that will be important in the sequel.

Lemma 2.2. *Consider the following sets of points of the Newton plane:*

$$\begin{aligned}
W &= \{(i,j) \in \mathbb{N}^2 \mid ni + mj > nm,\ j < n\}\\
T_0 &= \{(i,j) \in \mathbb{N}^2 \mid ni + mj > nm,\ i < m \text{ and } j < n\}\\
T_1 &= \{(i,j) \in \mathbb{N}^2 \mid ni + mj < 2nm \text{ and } j \geq n\}\\
T_2 &= \{(i,j) \in \mathbb{N}^2 \mid ni + mj < 2nm \text{ and } i \geq m\}.
\end{aligned}$$

Then:

(a) *For any integer $d > nm$, there is one and only one point in W with twisted degree d.*

(b) *For any $(i,j) \in T_0$, no point with non-negative integral coordinates other than (i,j) itself has twisted degree $ni + mj$.*

(c) *Mapping $(i,j) \mapsto (i + m, j - n)$ is a bijection between T_1 and T_2. Each $(i,j) \in T_1$ and its image $(i + m, j - n)$ are the only points with integral non-negative coordinates which have twisted degree $ni + mj$.*

Proof. Just note that all solutions of a Diophantine equation $nx + my = d$ result from adding to a given one the integral multiples of $(-m.n)$, and take $d = ni + mj$. □

We will consider a second copy of $\mathbb{R}^2$ as a vector space whose elements –called vectors– act on the points of the Newton plane, as usual, by translation: $p + v = (i+a, j+b)$ for any point $p = (i,j) \in \mathcal{N}$ and any vector $v = (a,b) \in \mathbb{R}^2$. If the twisted degree is defined for vectors by the same rule as for points, namely $\mathrm{td}(a,b) = na+mb$, then, clearly, $\mathrm{td}(p+v) = \mathrm{td}(p)+\mathrm{td}(v)$ for any $p \in \mathcal{N}$ and any $v \in \mathbb{R}^2$. Non-zero vectors $v = (a,b)$ with $a \geq 0$ and $b \geq 0$ will be called *positive*, written $v > 0$. Obviously, positive vectors have positive twisted degree.

We recall next a few basic facts and set some notations regarding blowing-ups and infinitely near points; the reader is referred to [5], 3.1, 3.2 for further information.

The blowing-up of a point O on a surface $\mathbf{S}$ is an analytic morphism of surfaces $\pi : \mathbf{S}' \to \mathbf{S}$ which restricts to an isomorphism $\mathbf{S}' - \pi^{-1}(O) \simeq \mathbf{S} - \{O\}$. If U is an open neighbourhood of O in S where local coordinates x, y are defined, then $\pi^{-1}(U)$ is covered by two open sets V, V', with coordinates $\bar{x}, z$ and $\bar{y}, z'$ respectively, in such a way that coordinates of the same point are related by the equalities $\bar{y} = \bar{x}z$ and $z' = z^{-1}$, and π is given by the rules $(\bar{x}, z) \mapsto (\bar{x}, \bar{x}z)$ and $(\bar{y}, z') \mapsto (\bar{y}z', \bar{y})$.

The *exceptional divisor* of π, $E = \pi^{-1}(O)$ is a projective line ([5], 3.1,3) which has equation $\bar{x} = 0$ in V and $\bar{y} = 0$ in V'. On E, z (and hence also $z' = z^{-1}$) may be taken as an absolute coordinate. The following fact will be important in the sequel, see [5], 3.2.2:

LEMMA 2.3. *Mapping the direction of the vector (α, β) to the point with coordinates $\bar{x} = 0$, $z = \beta/\alpha$ if $\alpha \neq 0$, and to the point with $\bar{y} = 0$, $z' = \alpha/\beta = 0$ if $\beta \neq 0$, is a projectivity ϕ between the projectivized of the tangent space to $\mathbf{S}$ at O and E.*

In the sequel, tangent directions will be identified to their images by ϕ. Also, we will write x and y for $\bar{x} = x \circ \pi$ and $\bar{y} = y \circ \pi$, respectively, as no confusion will result.

As usual, O is taken as the only point in its 0-*th neighbourhood* and, inductively, the points in the r-*th neighbourhood* of O, $r > 0$, are those in the exceptional divisor of blowing-up some point in the $(r-1)$-th neighbourhood of O. The points in any r-th neighbourhood of O, $r > 0$, are called *points infinitely near to O*. By its definition, each point p infinitely near to O lies on a well defined smooth surface $\mathbf{S}_p$ related to $\mathbf{S}$ by the composition $\pi_p : \mathbf{S}_p \to \mathbf{S}$ of a well determined sequence of blowing-ups whose centers are said to *precede* p: if p is a double point of the exceptional divisor $\pi_p^{-1}(O)$ of π_p, then p is said to be a *satellite point*; otherwise it is said to be a *free point*. A satellite point p is determined by the two irreducible components of the exceptional divisor of π_p it belongs to, or, equivalently, by the two previously blown up points these components correspond to (the two points p is *proximate* to, see [5] 3.3 and 3.5 for the details). If p is a point equal or infinitely near to O, then the first neighbourhood E_p of p is a projective line whose points correspond by a projectivity to the tangent directions to $\mathbf{S}_p$ at p. The first neighbourhood of O contains no satellite points. If p is a satellite point, then E_p contains exactly two satellite points, which correspond to the tangent directions to the two components of the exceptional divisor that meet at p. If p is a free point, then E_p contains a single satellite point, corresponding to the direction of the only component of the exceptional divisor through p. From now on, the first neighbourhood of any free point p will be seen as an affine line with improper point the satellite point; its affine group will be denoted $\mathrm{Aff}(E_p)$.

If ξ is a germ at O, then the origins of the iterated strict transforms of γ by the successive blowing-ups are called the *infinitely near points belonging to*, or *lying on*, ξ; the germ ξ is then said *to contain* or *to go through* all these points. The multiplicity on ξ of such a point p, noted $e_p(\xi)$, is its multiplicity on the corresponding strict transform. The intersection multiplicity of two germs of curve ξ, ξ' with origin at O is then

$$[\xi \cdot \xi'] = \sum e_p(\xi) e_p(\xi'), \tag{4}$$

the summation running on the points p equal or infinitely near to O and belonging to both germs (Noether's formula, see [5], 3.3.1).

Assume that γ is a germ of curve as in 2.1. The germ γ being irreducible, it contains a single point in each neighbourhood of O. Take $h = [m/n]$ as before. The points on γ in the first, second, $\ldots$, h-th neighbourhood of O are those belonging to the first axis $x = 0$; all of them are free and are followed on γ by a group of finitely many consecutive satellite points, referred to in the sequel as *the satellite points on* γ because there is no other. The multiplicities and positions of the satellite points on γ depend only on m/n and are fully described in [5], 5.2.2. Anyway, here we do not need the full details, but just the following facts:

LEMMA 2.4. *The satellite points on* γ *and the free points preceding them, as well as their multiplicities, are the same for all germs* γ *as in 2.1. The last satellite point has multiplicity one. Furthermore, the sum of the squares of the multiplicities of* O *and all points infinitely near to* O *on* γ *until that last satellite, equals* nm.

Proof. Follows from [5], 5.2.2 and a direct computation for the last claim, see [5], Exercise 5.6. □

The last satellite point on γ having multiplicity one, all points following it on γ are free and have also multiplicity one. They make a totally ordered sequence of consecutive infinitely near points and will be referred to as the *points on* γ *after the last satellite.* These points vary with γ.

REMARK 2.5. Due to Noether's formula (4) and 2.4, the number of points after the last satellite shared by two germs γ and γ', both as in 2.1, is $[\gamma \cdot \gamma'] - nm$.

REMARK 2.6. The germs γ and γ' being as in 2.1, using for one of them the Puiseux parameterization $x = t^n$, $y = t^m + \dots$ shows that $[\gamma \cdot \gamma'] > nm$. In particular, by 2.5, the first point after the last satellite is shared by all germs γ in 2.1.

3. Local analytic isomorphisms. Given points O, on a surface $\mathbf{S}$, and O', on a surface $\mathbf{S}'$, we will consider the germs $\varphi : (\mathbf{S}, O) \to (\mathbf{S}', O')$ of analytic maps $\bar{\varphi} : U \to \mathbf{S}'$, for U an open neighbourhood of O in $\mathbf{S}$, that map O to O' and are isomorphisms locally at O, that is, that have non-zero Jacobian determinant at O: they will be called *local (analytic) isomorphisms* at O. We will often write $O' = \varphi(O)$.

Each local isomorphism $\varphi : (\mathbf{S}, O) \to (\mathbf{S}', O')$ is determined by the equations giving the coordinates x^*, y^* of $\bar{\varphi}(p)$ as convergent series in the coordinates x, y of an arbitrary point p in a certain neighbourhood of O, namely

$$x^* = \sum_{(i,j)\in\mathbb{N}^2-\{(0,0)\}} a_{i,j}x^iy^j, \quad y^* = \sum_{(i,j)\in\mathbb{N}^2-\{(0,0)\}} b_{i,j}x^iy^j, \tag{5}$$

with $a_{1,0}b_{0,1} - a_{0,1}b_{1,0} \neq 0$. Any two such series define a local isomorphism. The equations (5) and their coefficients will be referred to as the *equations* and *coefficients* of φ.

If $\xi : g = 0$ is a germ of curve on $\mathbf{S}'$ at O', its *inverse image* by φ is $\varphi^*(\xi) : g(x^*(x,y), y^*(x,y)) = 0$. The (direct) *image* of a germ η at O is then $\varphi(\eta) = (\varphi^{-1})^*(\eta)$.

In case of being $\mathbf{S} = \mathbf{S}'$ and $O = O'$, the local isomorphisms above will be called *local automorphisms* of $\mathbf{S}$ at O. For fixed $\mathbf{S}$ and O, the local automorphisms of $\mathbf{S}$ at O with the composition make a group, called the *analytic group of* $\mathbf{S}$ *at* O and denoted $\mathcal{A}_O(\mathbf{S})$ in the sequel.

After identifying tangent directions at O and O' to their corresponding points in their first neighbourhoods, the tangent map to φ at O induces a projectivity $\hat{\varphi}$, between the first neighbourhoods of O and O', which will be called the *action of* φ *on* *the first neighbourhood.* Using the absolute coordinate z introduced above, $\hat{\varphi}$ is given by the rule

$$z \longmapsto \frac{b_{1,0} + b_{0,1}z}{a_{1,0} + a_{0,1}z}.$$

REMARK 3.1. If γ is any irreducible germ at O, since the tangent map to φ maps any vector tangent to γ to a vector tangent to $\varphi(\gamma)$, $\hat{\varphi}$ maps the point on γ in the first neighbourhood of O to the point on $\varphi(\gamma)$ in the first neighbourhood of O'.

The local isomorphisms lift to local isomorphisms between the blown up surfaces and the lifted local isomorphisms map the germ of the exceptional divisor to the germ of the exceptional divisor and strict transforms to strict transforms. More precisely:

LEMMA 3.2. *Let p be a point in the first neighbourhood of O and $\varphi : (\mathbf{S}, O) \to (\mathbf{S}', O')$ a local isomorphism at O. Write $\bar{\pi} : \mathbf{S}_1 \to \mathbf{S}$ and $\bar{\pi}' : \mathbf{S}'_1 \to \mathbf{S}'$ the blowing-ups of O and O'. If $\hat{\varphi}(p) = p'$, then there is a uniquely determined local isomorphism $\varphi_1 : (\mathbf{S}_1, p) \to (\mathbf{S}'_1, p')$ that satisfies $\pi' \circ \varphi_1 = \varphi \circ \pi$, where π and π' are the germs of $\bar{\pi}$ and $\bar{\pi}'$ at p and p'. The germ at p of the exceptional divisor E of π is mapped by φ_1 to the germ at p' of the exceptional divisor of π', and the restriction of φ_1 to E is the germ of $\hat{\varphi}$. For any irreducible germ of curve γ at O going through p it holds*

$$\varphi_1(\tilde{\gamma}) = \widetilde{\varphi(\gamma)},$$

where $\tilde{}$ denotes strict transform.

Proof. After a suitable choice of the local coordinates at O and O' we may assume p and p' to be the points on the first axes: then they both have absolute coordinate $z = 0$. If φ is given by the equations (5), being $\hat{\varphi}(p) = p'$ is equivalent to $b_{1,0} = 0$, after which $a_{1,0} \neq 0$ and $b_{0,1} \neq 0$. These conditions being satisfied, the reader may easily check that, using x, z as local coordinates at p,

$$x^* = \sum a_{i,j} x^{i+j} z^j, \quad z^* = \frac{\sum b_{i,j} x^{i+j-1} z^j}{\sum a_{i,j} x^{i+j-1} z^j},$$

all summations for $(i,j) \in \mathbb{N}^2 - \{(0,0)\}$, define a local isomorphism as wanted. The uniqueness follows from being $\pi_{|\mathbf{S}_1 - \pi^{-1}(O)}$ and $\pi'_{|\mathbf{S}'_1 - (\pi')^{-1}(O')}$ isomorphisms, while the other claims are direct. □

Assume given a sequence $O = p_0, p_1, \dots p_{i-1}, p_i$, $i > 0$ of points, each p_j in the first neighbourhood of p_{j-1} for $0 < j \leq i$. Take $\mathbf{S}_0 = \mathbf{S}$, $\mathbf{S}'_0 = \mathbf{S}'$, $\varphi_0 = \varphi$ and $\widehat{\varphi_{-1}}(O) = O'$. Denote by π_j the germ at p_j of the composition $\bar{\pi}_j : \mathbf{S}_j \to \mathbf{S}$ of the blowing-ups of $p_{j-1}, \dots, p_0$. Then, directly from 3.2, we have:

PROPOSITION 3.3. *Inductively on j, for $0 < j \leq i$, there is a uniquely determined local isomorphism*

$$\varphi_j : (\mathbf{S}_j, p_j) \longrightarrow (\mathbf{S}'_j, \widehat{\varphi_{j-1}}(p_j))$$

satisfying $\varphi \circ \pi_j = \pi'_j \circ \varphi_j$, where π'_j is the germ at $\widehat{\varphi_{j-1}}(p_j)$ of the composition $\bar{\pi}'_j : \mathbf{S}'_j \to \mathbf{S}'$ of the blowing-ups of $\widehat{\varphi_{j-2}}(p_{j-1}), \dots, \widehat{\varphi_0}(p_1), O'$. Such a φ_j maps:

(1) *the germ of the exceptional divisor of blowing up p_{j-1} to the germ of the exceptional divisor of blowing up $\widehat{\varphi_{j-2}}(p_{j-1})$,*

(2) *for $1 \leq \ell < j$, the strict transform of the germ of the exceptional divisor of blowing up $p_{\ell-1}$ to the strict transform of the germ of the exceptional divisor of blowing up $\widehat{\varphi_{\ell-2}}(p_{\ell-1})$, and*

(3) *the strict transform by $\bar{\pi}_j$ of any irreducible germ γ at O to the the strict transform by $\bar{\pi}'_j$ of its image $\varphi(\gamma)$.*

In the remarks that follow we keep the hypothesis and notations as in 3.3.

REMARK 3.4. Since the restrictions of $\bar{\pi}_i$ and $\bar{\pi}'_i$ to $\mathbf{S}_i - \varphi_i^{-1}(O)$ and $\mathbf{S}'_i - (\varphi')_i^{-1}(O)$, respectively, are isomorphisms, φ_i determines φ .

REMARK 3.5. Due to the uniqueness claimed in 3.3, if $\psi : (\mathbf{S}', O') \to (\mathbf{S}'', O'')$ is a local isomorphism at O', then $\varphi_i \circ \psi_i = (\varphi \circ \psi)_i$, and therefore also $\widehat{\varphi}_i \circ \widehat{\psi}_i = \widehat{(\varphi \circ \psi)}_i$.

For $j = 0, \ldots, i$, the action $\widehat{\varphi_j}$ of φ_j on the first neighbourhood of p_j will be called the *action of φ on the first neighbourhood of p_j*, we will write $\varphi(p) = \widehat{\varphi_j}(p)$ for any p in the first neighbourhood of p_j. If φ is a local automorphism at O and $p_j = \varphi(p_j)$ for $j = 1, \ldots, i$, then the points invariant by $\widehat{\varphi}_i$ will be said to be *invariant by φ*. In the sequel, when saying that a local automorphism φ at O leaves invariant a point p in the first neighbourhood of p_i, it will be implicitly assumed that all the p_j, $1 \leq j \leq i$, are also invariant by φ.

REMARK 3.6. As a result of claims (1),(2) of 3.3, each $\widehat{\varphi_j}$ maps satellite points to satellite points and free points to free points. In particular, if p_i is a free point, so is $\varphi(p_i)$. Then, if the first neighbourhoods of p_i and $\varphi(p_i)$ are seen as affine lines with improper points the satellite points, $\widehat{\varphi}_i$ is an affinity.

REMARK 3.7. Due to 3.1, if p is the point on γ in the first neighbourhood of p_i, then $\varphi(p)$ is the point on $\varphi(\gamma)$ in the first neighbourhood of $\varphi(p_i)$ and $e_p(\gamma) = e_{\varphi(p)}(\varphi(\gamma))$. In particular p is fixed by φ if and only if p belongs to $\varphi(\gamma)$.

REMARK 3.8. By 3.5, the local automorphisms leaving p_i invariant compose a subgrup $\mathcal{A}_O(\mathbf{S}, p_i)$ of $\mathcal{A}_O(\mathbf{S})$, mapping $\varphi \mapsto \varphi_i$ is a group monomorphism $\mathcal{A}_O(\mathbf{S}, p_i) \to \mathcal{A}_{p_i}(\mathbf{S_i})$, while mapping $\varphi \mapsto \widehat{\varphi}_i$ is a group homomorphism $\mathcal{A}_O(\mathbf{S}, p_i) \to \mathrm{Aff}(E_{p_i})$.

4. Coefficients and coordinates. Let γ be a germ as in 2.1. An old result (see for instance [8], Book IV) assures that the Puiseux coefficients may be taken as coordinates of the infinitely near points in the first neighbourhoods they belong to. Here is a precise claim adapted to our case; a more general version is proved in [5], 5.7.1.

PROPOSITION 4.1. *Fix complex numbers $c_1, \ldots, c_{s-1}$, $s \geq 1$. All germs whose Puiseux series has partial sum*

$$x^{m/n} + \sum_{r=1}^{s-1} c_r x^{(m+r)/n}$$

share all their infinitely near points up to the point q_s in the s-th neighbourhood of the last satellite. Furthermore there is an absolute coordinate ϑ in the first neighbourhood of q_s for which:

(a) *The satellite point has coordinate $\vartheta = \infty$.*

(b) *For any $c_s \in \mathbb{C}$, germ with Puiseux series*

$$S = x^{m/n} + \sum_{r=1}^{s-1} c_r x^{(m+r)/n} + c_s x^{(m+s)/n} + \dots$$

goes through the point with coordinate ϑ if and only if $c_s = \vartheta$.

When using the coordinate ϑ of 4.1, we will say that the Puisieux coefficient c_s is being taken as coordinate.

Remark 4.2. Inductive use of the second claim of 4.1 shows that the converse of the first claim is also true, that is, if irreducible germs of curve with Puiseux series $x^{m/n} + \sum_{r\geq 1} c_r x^{(m+r)/n}$ and $x^{m/n} + \sum_{r\geq 1} c'_r x^{(m+r)/n}$ share their points up to the one in the s-th neighbourhood of the last satellite, then $c_r = c'_r$ for $r = 1, \dots, s-1$.

We will see next that also the coefficients of the equations may be used as coordinates:

Proposition 4.3. *Fix a pair of non-negative integers k, h with $d = nk + mh > nm$ and complex numbers $A_{i,j}$ for $nm < ni + mj \leq d$ and $(i,j) \neq (k,h)$. Write $s = d - nm$. It holds:*

(1) *All germs which have an equation with partial sum*

$$y^n - x^m + \sum_{nm<ni+mj<d} A_{i,j} x^i y^j$$

share all their infinitely near points up to the point q_s in the s-th neighbourhood of the last satellite.

(2) *There is an absolute coordinate θ in the first neighbourhood of q_s for which:*
   (a) *The satellite point has coordinate $\theta = \infty$, and*
   (b) *a germ with equation*

$$y^n - x^m + \sum_{nm<ni+mj\leq d} A_{i,j} x^i y^j + \dots,$$

where $A_{k,h}$ is an arbitrary complex number, goes through the point with coordinate θ if and only if $A_{k,h} = \theta$.

Proof. For the first claim, assume that the germs γ and γ' have equations f and f', both of the form (3) and with equal partial sums of twisted degree $d-1$. Then, either $f = f'$, in which case the claim is obvious, or $f - f'$ has $\mathrm{to}(f - f') \geq d$ and defines a germ of curve ξ at O. Then on one hand $[\gamma \cdot \gamma'] = [\gamma \cdot \xi]$, while on the other, using the Puiseux parameterization $x = t^n$, $y = t^m + \dots$ of γ, $[\gamma \cdot \xi] \geq d$. All together, $[\gamma \cdot \gamma'] \geq d$, after which the first claim follows from 2.5.

Regarding the second claim, take

$$f_0 = y^n - x^m + \sum_{\substack{nm<ni+mj\leq d\\(i,j)\neq(k,h)}} A_{i,j} x^i y^j$$

and consider the pencil of germs at O composed of the germs

$$\xi_\theta : \lambda f_0 + \mu x^k y^h = 0, \quad \theta = \mu/\lambda \in \mathbb{C} \cup \{\infty\}.$$

Computing as above, $[\xi_0 \cdot \xi_\infty] = d$, from which any two different germs in the pencil have intersection multiplicity d. Using 2.5 shows that any two different germs ξ_θ, $\xi_{\theta'}$, $\theta, \theta' \neq \infty$, contain the point q_s and share no point infinitely near to it. Since each point preceding q_s has the same multiplicity on all germs ξ_θ, $\theta \neq \infty$ (by 2.4), the strict transform with origin at q_s of each ξ_θ, $\theta \neq \infty$, is defined by an equation of the form

$$\bar{f}_0 + \theta g = 0, \quad \theta \in \mathbb{C}$$

where $\bar{f}_0$ is an equation of the strict transform of ξ_0 with origin at q_s and g is independent of θ. The above strict transforms of the ξ_θ, $\theta \neq \infty$, are thus all but one of the germs

$$\bar{\xi}_\theta : \lambda \bar{f}_0 + \mu g = 0, \quad \theta = \mu/\lambda \in \mathbb{C} \cup \{\infty\},$$

which in turn compose a pencil Λ. We know that the point q_s belongs with multiplicity one to both ξ_0 and ξ_1, which in turn share no point infinitely near to q_s. The strict transforms $\bar{\xi}_0, \bar{\xi}_1$ are thus smooth germs with different tangents, after which Λ is a pencil composed of smooth germs with variable tangent. It follows that λ, μ may be taken as homogeneous coordinates of the tangent line to $\bar{\xi}_\theta$, $\theta = \mu/\lambda$, in the pencil of tangent lines at q_s, and therefore, by 2.3, also as homogeneous coordinates of the point of $\bar{\xi}_\theta$ in the first neighbourhood of q_s. If $\theta \neq \infty$, this point is the point on ξ_θ in the first neighbourhood of q_s while, all these points being free, the point with coordinates $(0, 1)$ cannot be other than the satellite point in the first neighbourhood of q_s.

Up to now, we have proved that there is an absolute coordinate θ in the first neighbourhood of q_s such that the satellite point has coordinate $\theta = \infty$ and, for any $\theta \neq \infty$, the germ ξ_θ goes through the point with coordinate θ. Now, still for any $\theta \neq \infty$, using the first claim with $d + 1$ in the place of d, any germ with equation

$$y^n - x^m + \sum_{nm < ni + mj \leq d} A_{i,j} x^i y^j + \dots$$

and $A_{k,h} = \theta$ shares with ξ_θ all points up to the one in the first neighborhood of q_s; this completes the proof. □

When using the coordinate θ of 4.3, we will say that the coefficient $A_{k,h}$ is being taken as coordinate.

Remark 4.4. A Weierstrass equation

$$y^n - x^m + \sum_{\substack{ni + mj > nm \\ j < n}} A_{i,j} x^i y^j \tag{6}$$

has a single coefficient for each twisted degree higher than nm, by 2.2(a). Then, inductive use of the second claim of 4.3 shows that the converse of the first claim holds true for Weierstrass equations, namely, if two germs of curve are defined by Weierstrass equations, such as (6) above, and share all points up to the one in the s-th neighbourhood of the last satellite, then the Weierstrass equations have equal partial sums of degree $d - 1 = nm + s - 1$. The same claim does not hold true for arbitrary equations because once there are two or more coefficients of the same twisted degree $nm+s$, then they may be simultaneously modified without varying the point in the first neighbourhood of q_s, see 4.5 and 4.7 below.

Anyway, regarding the equations of the irreducible germs going through a prescribed infinitely near point we have:

PROPOSITION 4.5. *Let q_s be the point in the s-th neighbourhood, $s \geq 1$, of the last satellite on the germ γ, defined by the equation*

$$f = y^n - x^m + \sum_{ni+mj>nm} A_{i,j} x^i y^j,$$

and take $d = nm+s$. Any irreducible germ γ' going through q_s has an equation which:

(a) *has the same partial sum of twisted degree $d-1$ as f, and*

(b) *has all its coefficients of twisted degree d but a single one taking arbitrary already prescribed values.*

Proof. Let g be the Weierstrass equation of γ: it is $f = ug$ where u is an invertible convergent series. If g' is the Weierstrass equation of γ', then by 4.4, $\mathrm{to}(g-g') \geq d$. It follows $\mathrm{to}(ug-ug') \geq d$ and therefore $f' = ug'$ is an equation of γ' and has the same partial sum of twisted degree $d-1$ as f. Write

$$f' = y^n - x^m + \sum_{nm<ni+mj<d} A_{i,j} x^i y^j + \sum_{d \leq ni+mj} A'_{i,j} x^i y^j.$$

Being $d > nm$, we may take (k,h) to be the pair of non-negative integers with minimal h satisfying $nk+mh = d$, and then $\ell = [k/m]$. After this, the coefficients of twisted degree d of f' are $A'_{k-im,h+in}$, $0 \leq i \leq \ell$. Take

$$u = \sum_{1 \leq i \leq \ell} \alpha_i x^{k-im} y^{h+(i-1)n},$$

where the α_i are complex numbers to be determined. Since u is homogeneous of twisted degree $d-mn$, it is clear that the partial sums of twisted degree $d-1$ of $f'' = (1+u)f'$ and f' are equal. The homogeneous part of twisted degree d of f'' is

$$\begin{aligned} &\sum_{i=0}^{\ell} A'_{k-im,h+in} x^{k-im} y^{h+in} + (y^n - x^m)\left(\sum_{i=1}^{\ell} \alpha_i x^{k-im} y^{h+(i-1)n}\right) \\ &= \sum_{i=0}^{\ell} (A'_{k-im,h+in} + \alpha_i - \alpha_{i+1}) x^{k-im} y^{h+in} \qquad (7) \end{aligned}$$

where α_0 and $\alpha_{\ell+1}$ are taken to be zero. It is clear that all but one of the coefficients of the right hand side of (7) take arbitrarily prescribed values for a suitable choice of the α_i, $i = 1, \dots, \ell$. Thus, for such a choice of the α_i, f'' is an equation of γ' that fulfils the conditions of the claim. ☐

REMARK 4.6. The coordinates of 4.1 and 4.3 are of course linearly related. In order to make their relationship more explicit, fix as above $d > nm$, take $s = d - nm$ and assume that γ_0 is a germ of curve whose Puiseux series has partial sum

$$S_0 = x^{m/n} + \sum_{r=1}^{s-1} c_r x^{(m+r)/n}$$

and which has an equation with partial sum

$$f_0 = y^n - x^m + \sum_{nm<ni+mj<d} A_{i,j} x^i y^j.$$

Let q_s be the point on γ_0 in the s-th neighbourhood of the last satellite point. then, on one hand, according to 4.1, for an arbitrary $c_s \in \mathbb{C}$, the germ γ, with Puiseux series

$$S = x^{m/n} + \sum_{r=1}^{s-1} c_r x^{(m+r)/n} + c_s x^{(m+s)/n}$$

goes through the point of coordinate $\vartheta = c_s$ in the first neighbourhood of q_s.

On the other hand, fix k, h as in 4.3, with $nk + mh = d$, and fix also complex numbers $A_{i,j}$ for any pair of non-negative integers i, j such that $ni + mj = d$ and $(i, j) \neq (k, h)$. According to 4.3, for an arbitrary $A_{k,h} \in \mathbb{C}$, the germ γ', with equation

$$f = y^n - x^m + \sum_{nm<ni+mj\leq d} A_{i,j} x^i y^j$$

goes through the point of coordinate $\theta = A_{k,h}$ in the first neighbourhood of q_s.

Since the germs γ and γ' are both going through the point q_s, by 2.5 it holds $[\gamma \cdot \gamma'] \geq d$, and the inequality is strict if and only if the point of coordinate $\vartheta = c_s$ and the point of coordinate $\theta = A_{k,h}$, both in the first neighbourhood of q_s, do agree. Therefore, the result of replacing the Puiseux parameterization of γ in the equation of γ', ordered by increasing powers of t, needs to be

$$f(t^n, S(t^n)) = Bt^d + \dots$$

and $B = 0$ if and only if $\vartheta = c_s$ and $\theta = A_{k,h}$ are coordinates of the same point. It is straightforward to check that B has the form

$$B = \sum_{ni+mj=d} A_{i,j} + nc_s + B' \tag{8}$$

where B' is a polynomial expression in the c_r, $r < s$ and the $A_{i,j}$, $ni + mj < d$. This in particular reproves that the condition for $\vartheta = c_s$ and $\theta = A_{k,h}$ to be coordinates of the same point, namely

$$B = \left[\frac{f(t^n, S(t^n))}{t^d}\right]_{t=0} = 0,$$

is linear in both coordinates.

REMARK 4.7. The equality (8) above shows that if the coefficients $A_{i,j}$, $ni+mj = d$, of the equation of γ', are modified keeping their sum constant, then the point on γ' in the first neighbourhood of q_s remains the same.

EXAMPLE 4.8. In case $n = 6$, $m = 7$, coefficients $A_{i,j}$ of the Weierstrass equation are related to their corresponding Puiseux coefficients $c_{6i+7j-42}$ by (8) as follows:

$$A_{5,2} = -6c_2, \quad A_{4,3} = -6c_3, \quad A_{3,4} = -6c_4 - 3c_2^2,$$
$$A_{5,3} = -6c_9 - 2c_3^3 - 12c_2c_3c_4, \quad A_{4,4} = -6c_{10}, \quad A_{5,4} = -6c_{16}.$$

Note that there are cases in which $B' \neq 0$.

REMARK 4.9. The computation of 4.6 may be used to prove either of the second claims of 4.1 and 4.3 from the other.

5. Relevant coefficients. Still considering irreducible germs of curve at a point O of a surface $\mathbf{S}$, assume to have fixed complex numbers $c_1, \ldots, c_{s-1}$, $s \geq 1$. By 4.2, all irreducible germs whose Puiseux series has partial sum

$$S_0 = x^{m/n} + \sum_{r=1}^{s-1} c_r x^{(m+r)/n}$$

share their infinitely near points up to the point q_s in the s-th neighbourhood of the last satellite. According to 4.1, take the s-th Puiseux coefficient as coordinate in the first neighbourhood of q_s. Then it holds:

LEMMA 5.1. *If ϑ and ϑ' are complex numbers, then the three claims below are equivalent:*

(i) *For any germ γ, with Puiseux series $S_0 + \vartheta x^{(m+s)/n} + \ldots$, there is a germ γ' which has Puiseux series $S_0 + \vartheta' x^{(m+s)/n} + \ldots$ and is analytically equivalent to γ*

(ii) *There are analytically equivalent germs γ, with Puiseux series $S_0 + \vartheta x^{(m+s)/n} + \ldots$ and γ', with Puiseux series $S_0 + \vartheta' x^{(m+s)/n} + \ldots$.*

(iii) *There is an analytic automorphism of $\mathbf{S}$ at O that leaves q_s invariant and whose action in the first neighbourhood of q_s maps the point of coordinate ϑ to the point of coordinate ϑ'.*

Proof. (i) $\Rightarrow$ (ii) is clear. For (ii) $\Rightarrow$ (iii), the local automorphism φ at O that maps γ to γ' satisfies the condition of (iii) by 3.7 and 4.1. To close, for (iii) $\Rightarrow$ (i), the local automorphism φ at O that satisfies the condition of (iii) maps γ to an analytically equivalent γ' with Puiseux series as claimed, also by 3.7 and 4.1. □

Assume as above to have fixed complex numbers $c_1, \ldots, c_{s-1}$, $s \geq 1$. We will say that the s-th Puiseux coefficient c_s of a Puiseux series

$$S = x^{m/n} + \sum_{r \geq 1} c_r x^{(m+r)/n}$$

is *relevant* –or *analytically relevant*– if and only if there are complex numbers ϑ and ϑ' such that no germs γ, given by a Puiseux series

$$x^{m/n} + \sum_{r=1}^{s-1} c_r x^{(m+r)/n} + \vartheta x^{(m+s)/n} + \ldots,$$

and γ', given by

$$x^{m/n} + \sum_{r=1}^{s-1} c_r x^{(m+r)/n} + \vartheta' x^{(m+s)/n} + \ldots,$$

are analytically equivalent. Otherwise c_s is said to be *irrelevant*. Obviously, the condition depends only on the partial sum of degree $(m+s-1)/n$ of S.

The relevance of a Puiseux coefficient is equivalent to an intrinsic geometric condition:

PROPOSITION 5.2. *Assume to have fixed complex numbers $c_1, \ldots, c_{s-1}$, $s \geq 1$. By 4.2, all irreducible germs with a Puiseux series*

$$S = x^{m/n} + \sum_{r=1}^{s-1} c_r x^{(m+r)/n} + \ldots.$$

share the point q_s in the s-th neighbourhood of the last satellite. Then the s-th Puiseux coefficient c_s of S is irrelevant if and only if the local automorphisms leaving invariant q_s act transitively on the set of free points in the first neighbourhood of q_s.

Proof. Direct from 4.1 and 5.1. □

REMARK 5.3. Hypothesis and notations being as in 5.2, Lemma 5.1 assures that if the Puiseux coefficient c_s is irrelevant, then, for any $\vartheta, \vartheta' \in \mathbb{C}$, any germ γ with Puiseux series

$$x^{m/n} + \sum_{r=1}^{s-1} c_r x^{(m+r)/n} + \vartheta x^{(m+s)/n} + \dots$$

is analytically equivalent to a germ γ' which has Puiseux series

$$x^{m/n} + \sum_{r=1}^{s-1} c_r x^{(m+r)/n} + \vartheta' x^{(m+s)/n} + \dots.$$

Let γ be an irreducible germ of plane curve with single characteristic exponent m/n. We will say that the s-th Puiseux coefficient is *relevant* –or *analytically relevant*– for γ if and only if, after fixing local coordinates relative to which γ has Puiseux series

$$S = x^{m/n} + \sum_{r\geq 1} c_r x^{(m+r)/n},$$

the s-th Puiseux coefficient c_s of S is relevant. Otherwise we will say that the s-th Puiseux coefficient is *irrelevant* for γ. By 5.2, the condition depends only on the point in the s-th neighbourhood of the last satellite point on γ, and so it is in particular independent of the choice of the local coordinates. Using the results of Section 3, it is also clear that if γ' is any irreducible germ of plane curve analytically equivalent to γ, then the s-th Puiseux coefficient is relevant for γ if and only if it is relevant for γ'.

Now we will proceed similarly with the coefficients of the equations. Fix non-negative integers k, h with $d = nk + mh > nm$, assume given complex numbers $A_{i,j}$, for $nm < ni + mj \leq d$, $(i,j) \neq (k,h)$ and write

$$f_0 = y^n - x^m + \sum_{\substack{nm<ni+mj\leq d\\ (i,j)\neq(k,h)}} A_{i,j} x^i y^j.$$

If $s = d - nm$, let q_s be the point in the s-the neighbourhood of the last satellite on any germ defined by an equation

$$y^n - x^m + \sum_{nm<ni+mj<d} A_{i,j} x^i y^j + \dots$$

(4.3) and take in the first neighbourhood of q_s the coefficient $A_{k,h}$ as coordinate. Then we have:

LEMMA 5.4. *If θ and θ' are complex numbers, then the three claims below are equivalent:*

(i) *For any germ γ, with equation $f = f_0 + \theta x^k y^h + \dots$, there is a germ γ' which has equation $f' = f_0 + \theta' x^k y^h + \dots$ and is analytically equivalent to γ.*

(ii) *There are analytically equivalent germs* γ*, with equation* $f = f_0 + \theta x^k y^h + \dots$*, and* γ'*, with equation* $f' = f_0 + \theta' x^k y^h + \dots$*.*

(iii) *There is an analytic automorphism of* $\mathbf{S}$ *at* O *that leaves* q_s *invariant and whose action in the first neighbourhood of* q_s *maps the point of coordinate* θ *to the point of coordinate* θ'*.*

Proof. Again (i) $\Rightarrow$ (ii) is clear. For (ii) $\Rightarrow$ (iii), the local automorphism φ at O that maps γ to γ' satisfies the condition of (iii) by 3.7 and 4.3. For (iii) $\Rightarrow$ (i), the local automorphism φ at O that satisfies the condition of (iii) maps γ to an analytically equivalent γ' which goes through the point of coordinate θ' by 3.7. Proposition 4.5 assures that γ' has an equation of the form

$$y^n - x^m + \sum_{\substack{nm < ni+mj \leq d \\ (i,j) \neq (k,h)}} A_{i,j} x^i y^j + \beta x^k y^h + \dots.$$

and then $\beta = \theta'$ by 4.3. □

Still let k, h be non-negative integers, with $d = nk + mh > nm$ and assume to have fixed the coefficients $A_{i,j}$, $ni + mj \leq d$, $(i,j) \neq (k,h)$ of an equation

$$f = y^n - x^m + \sum_{ni+mj > nm} A_{i,j} x^i y^j,$$

the other $A_{i,j}$ being left undetermined. We will say that the coefficient $A_{k,h}$, of f, is *relevant* –or *analytically relevant*– if and only if there are complex numbers θ and θ' such that no germs γ, defined by an equation

$$y^n - x^m + \sum_{\substack{nm < ni+mj \leq d \\ (i,j) \neq (k,h)}} A_{i,j} x^i y^j + \theta x^k y^h + \dots$$

and γ', defined by

$$y^n - x^m + \sum_{\substack{nm < ni+mj \leq d \\ (i,j) \neq (k,h)}} A_{i,j} x^i y^j + \theta' x^k y^h + \dots$$

are analytically equivalent. Otherwise $A_{k,h}$ is said to be *irrelevant.* The condition depends only on the fixed coefficients $A_{i,j}$, $ni + mj \leq d$, $(i,j) \neq (k,h)$.

PROPOSITION 5.5. *Non-negative integers* k, h *with* $d = nk + mh > nm$ *and complex numbers* $A_{i,j}$*, for* $nm < ni + mj \leq d$*,* $(i,j) \neq (k,h)$ *being fixed, still take* $s = d - mn$*, so that, by 4.3, the irreducible germs defined by an equation*

$$f = y^n - x^m + \sum_{nm < ni+mj \leq d-1} A_{i,j} x^i y^j + \dots.$$

share their infinitely near points up to the point q_s *in the* s*-th neighbourhood of the last satellite. Then the coefficient* $A_{k,h}$ *of* f *is irrelevant if and only if the local automorphisms leaving invariant* q_s *act transitively on the set of free points in the first neighbourhood of* q_s*.*

Proof. Direct from 5.4. □

REMARK 5.6. By 5.4, if the coefficient $A_{k,h}$ is irrelevant for the equation

$$y^n - x^m + \sum_{ni+mj>nm} A_{i,j}x^iy^j.$$

of a germ γ, then for any $B \in \mathbb{C}$ there is a germ analytically equivalent to γ which has an equation of the form

$$y^n - x^m + \sum_{\substack{nm<ni+mj\leq d\\(i,j)\neq(k,h)}} A_{i,j}x^iy^j + Bx^ky^h + \dots.$$

Still let γ be an irreducible germ of plane curve with single characteristic exponent m/n. Fix an integer d, $d > nm$. We will say that the coefficients of twisted degree d are *relevant* –or *analytically relevant*– for γ if and only if, after fixing local coordinates relative to which γ has an equation

$$f = y^n - x^m + \sum_{ni+mj>nm} A_{i,j}x^iy^j,$$

a coefficient of twisted degree d of f is relevant. Otherwise we will say that the coefficients of twisted degree d are *irrelevant* for γ. By 5.5, the condition depends only on the point in the $(d-nm)$-th neighbourhood of the last satellite point on γ, and so it is in particular independent of the choice of the local coordinates, of the choice of the equation of γ, and also of the choice of the particular coefficient of twisted degree d.

REMARK 5.7. It also follows from 5.5 that if γ' is any irreducible germ of plane curve analytically equivalent to γ, then the coefficients of twisted degree d are relevant for γ if and only if they are relevant for γ'.

All together, the transitivity conditions in 5.2 and 5.5 being the same, the theorem below needs no proof.

THEOREM 5.8. *If γ is an irreducible germ of plane curve on a surface $\mathbf{S}$ with origin O and single characteristic exponent m/n, then for any positive integer s the following conditions are equivalent:*

(i) *The s-th Puiseux coefficient is relevant for γ.*
(ii) *The coefficients of twisted degree $d = mn + s$ are relevant for γ.*
(iii) *The local automorphisms of $\mathbf{S}$ at O that leave invariant the point q_s of γ in the s-th neighbourhood of the last satellite, do not act transitively on the set of the free points in the first neighbourhood of q_s.*

In the process of obtaining distinguished representatives of the analytic classes of irreducible germs, often a local automorphism is used to turn a Puiseux coefficient into zero leaving the preceding coefficients unmodified. When this has been done, it is said that the Puiseux coefficient has been *eliminated*. It is clear from 5.4 that any irrelevant Puiseux coefficient may be eliminated, but Puiseux coefficients that already are zero may be relevant. An irrelevant coefficient of an equation may be similarly eliminated, this time by 5.6, but still relevant coefficients may already be zero. Further, since a coefficient and its corresponding Puiseux coefficient are not, in general, proportional (see 4.8), eliminating a coefficient does not necessarily eliminate the corresponding Puiseux coefficient, and conversely.

6. Standing coefficients. From now on we will mainly concentrate on the relevance of coefficients. The results may be reformulated in terms of relevance of Puiseux coefficients, or, more intrinsecaly, in terms of actions on first neighbourhoods, using 5.8. As before, we asssume the germ of curve γ to be as in 2.1, and in particular with equation (3). The proposition below shows in particular that all but finitely many coefficients $A_{i,j}$ are irrelevant for γ.

PROPOSITION 6.1. *If the coefficient $A_{k,h}$ has $k \geq m-1$ or $h \geq n-1$, then it is irrelevant for γ.*

Proof. Assume $k \geq m-1$. For an arbitrary $a \in \mathbb{C}$, consider the local automorphism φ_a that has equations

$$x^* = x + ax^{k-m+1}y^h, \quad y^* = y.$$

Substitution in the equation of γ gives $(A_{k,h} + ma)x^k y^h$ as the new (k,h)-monomial, while all other monomials of twisted degree less than or equal to $nk + mh$ remain unchanged; this proves the irrelevance of $A_{k,h}$. The same argument applies to the case $h \geq n-1$, this time using the analytic automorphism defined by

$$x^* = x, \quad y^* = y + bx^k y^{h-n+1}.$$

□

If Γ is the additive semigroup generated by m, n, the reader may easily check that the Puiseux exponents $m+s = nk+mj-nm+m$ corresponding to the pairs of indices k, h with $k \geq m-1$ or $h \geq n-1$ are exactly the elements of $\Gamma - n$. After this, for irreducible germs with a single characteristic exponent, Proposition 6.1 is equivalent to Zariski's first (III.1.2) and second (IV.2.6) elimination criteria in [23].

After 6.1, the coefficients corresponding to points that do not belong to the finite subset of the Newton plane

$$\mathbf{T} : \{(i,j) \in \mathbb{N}^2 \mid ni + mj > nm,\ i < m-1 \text{ and } j < n-1\}$$

are irrelevant for γ, and in particular so are all coefficients corresponding to points with twisted degree $n(m-1) + m(n-1) = 2nm - n - m + 1$ or higher. The points in $\mathbf{T}$ will be called *standing points*, and their corresponding coefficients and monomials, *standing coefficients* and *standing monomials*. The points $(i,j) \in \mathbb{N}^2 - \mathbf{T}$ and with $ni + mj > nm$, as well as their corresponding coefficients and monomials, will be referred to as *non-standing*, see Figure 2 in Section 8.

REMARK 6.2. If γ is any irreducible germ with single characteristic exponent m/n, iterated use of 6.1 and 5.6 proves the existence of an analytically equivalent germ given by an equation

$$y^n - x^m + \sum_{ni+mj>nm} A_{i,j}x^i y^j, \tag{9}$$

all whose non-standing coefficients $A_{i,j}$ up to an arbitrary already fixed twisted degree (or degree) are zero.

REMARK 6.3. Actually, as a consequence of a result of Samuel ([16], see also [5],7.7.2), for any irreducible germ of curve $\xi : f = 0$, there is a positive integer ρ such that the partial sum of degree ρ of f defines a germ of curve analytically equivalent to ξ. This, together with 6.2 proves that any irreducible germ with single characteristic exponent m/n has an analytically equivalent germ given by an equation such as (9) above, all whose non-standing coefficients $A_{i,j}$ are zero.

By 2.2, different standing points have different twisted degrees: from now on, the standing points, as well as their corresponding coefficients and monomials, will be taken with the ordering induced by the twisted degree, they composing thus totally ordered finite sequences.

7. Critical coefficients. Notations and hypothesis being as in the preceding section, we have seen there that all non-standing coefficients $A_{i,j}$ are irrelevant for $\gamma : f = 0$. In this section and the forthcoming ones, our goal is to determine which standing coefficients are relevant. The first relevant coefficients we will find have a particular property, namely that their variation avoiding a certain value does not affect the analytic type of the germ. Being more precise, assume fixed a standing point (k, h) and write $d = nk + mh$ its twisted degree; as noted before, no other $(i, j) \in \mathbb{N}^2$ has twisted degree d. Consider the partial sum of f

$$f_d = y^n - x^m + \sum_{nm<ni+mj<d} A_{i,j} x^i y^j .$$

We will say that $A_{k,h}$ is a critical coefficient of the equation f if and only if $A_{k,h}$ is relevant for f and there is $\theta_0 \in \mathbb{C}$ such that for any $\theta, \theta' \in \mathbb{C}$, $\theta, \theta' \neq \theta_0$, there are analytically equivalent germs γ, defined by an equation $f_d + \theta x^k y^h + \dots$, and γ', defined by an equation $f_d + \theta' x^k y^h + \dots$.

Remark 7.1. If $A_{k,h}$, as above, is critical, then no germ $\gamma_0 : f_d + \theta_0 x^k y^h + \cdots = 0$ is equivalent to a germ $\gamma : f_d + \theta x^k y^h + \cdots = 0$ if $\theta \neq \theta_0$, as otherwise, by the transitivity of the analytic equivalence, $A_{k,h}$ would be irrelevant.

It follows in particular that the complex number θ_0 in the above definition of critical coefficient is uniquely determined: θ_0 will be called the critical value of $A_{k,h}$. In the sequel, when saying that a coefficient $A_{k,h}$ has a certain critical value, we will implicitly assume that $A_{k,h}$ is critical.

Notations and hypothesis being as above, take $s = d - nm$, let q_s be the point on $\gamma : f = 0$ in the s-th neighbourhood of the last satellite and take the coefficient $A_{k,h}$ as coordinate on the first neighbourhood of q_s (4.3). Then 5.4 directly gives:

Proposition 7.2. *The following conditions are equivalent.*

(i) $A_{k,h}$ *is a critical coefficient of* f *and has critical value* θ_0.

(ii) *The action of* $\mathcal{A}_O(\mathbf{S}, q_s)$ *on the set of free points in the first neighbourhood of* q_s *has exactly two orbits, one of which has the point of coordinate* θ_0 *as its only point.*

Remark 7.3. It follows also from 5.4 that if $A_{k,h}$ is critical and has critical value θ_0, then for any complex numbers $\theta, \theta' \neq \theta_0$ and any germ $\gamma : f_d + \theta x^k y^h + \cdots = 0$, there is a germ $\gamma' : f_d + \theta' x^k y^h + \cdots = 0$ analytically equivalent to γ.

Remark 7.4. The germ γ being fixed, the condition of being $A_{k,h}$ critical depends only on the point q_s on the s-th neighbourhood of the last satellite on γ. It is in particular independent of the choices of the equation f of γ and the coordinates, and therefore it makes sense to say that $A_{k,h}$ is a critical coefficient for γ.

REMARK 7.5. Also from 7.2, if $\gamma' = \varphi(\gamma)$ is a germ analytically equivalent to γ and is given by an equation

$$f' = y^n - x^m + \sum_{ni+mj>nm} A'_{i,j} x^i y^j,$$

then the coefficient $A_{k,h}$ is critical for γ if and only if $A'_{k,h}$ is critical for γ'. Furthermore, the coefficients $A_{h,k}$ and $A'_{k,h}$ being taken as coordinates in the first neighbourhoods of q_s and $\varphi(q_s)$, if q is the point in the first neighbourhood of q_s whose coordinate is the critical value of $A_{k,h}$ for γ, then the critical value of $A'_{k,h}$ for γ' is the coordinate of $\varphi(q)$.

Before continuing, we need to set a couple of facts about the local automorphisms that leave certain points invariant. Assume that φ is a local automorphism at O given by equations

$$x^* = \sum_{(i,j)\in\mathbb{N}^2-\{(0,0)\}} a_{i,j} x^i y^j, \quad y^* = \sum_{(i,j)\in\mathbb{N}^2-\{(0,0)\}} b_{i,j} x^i y^j, \tag{10}$$

where, of course, $a_{1,0}b_{0,1} - a_{0,1}b_{1,0} \neq 0$.

LEMMA 7.6. *Let γ be as in 2.1. Then the automorphism φ keeps invariant the point q_1 on γ in the first neighbourhood of the last satellite if and only if $b_{i,0} = 0$ for $i < m/n$ and $a_{1,0}^m = b_{0,1}^n$.*

Proof. By Noether's formula (4) and 3.7,

$$[\gamma \cdot \varphi^*(\gamma)] = \sum_p e_p(\gamma) e_p(\varphi^*(\gamma)) = \sum_p e_p(\gamma)^2,$$

the summations running on O and the points infinitely near to O shared by γ and $\varphi^*(\gamma)$, or, which is the same by 3.7, the points infinitely near to O on γ that are invariant by φ. As a consequence, by 2.4, q_1 is among these invariant points if and only if $[\gamma \cdot \varphi^*(\gamma)] > nm$. We will use the latter condition.

If $i < m/n$ is minimal among the indices for which $b_{i,0} \neq 0$, then the equation of $\varphi^*(\gamma)$, $f(x^*, y^*)$ has twisted initial form $b_{i,0}^n x^{ni}$ which, after replacing with the Puiseux parameterization $x = t^n$, $y = t^m + \dots$, of γ, gives $[\gamma \cdot \varphi^*(\gamma)] = n^2 i < nm$ and q_1 is not invariant.

If the index i above does not exist, in particular it is $b_{1,0} = 0$, which assures $a_{1,0} \neq 0$ and $b_{0,1} \neq 0$; then the twisted initial form of $f(x^*, y^*)$ is $b_{0,1}^n y^n - a_{1,0}^m x^m$. If $a_{1,0}^m \neq b_{0,1}^n$, it results $[\gamma \cdot \varphi^*(\gamma)] = nm$ and, again q_1 is not invariant.

If otherwise, $b_{i,0} = 0$ for $i < m/n$ and $a_{1,0}^m = b_{0,1}^n$, then the computation above gives $[\gamma \cdot \varphi^*(\gamma)] > nm$ and q_1 is invariant, as claimed. □

REMARK 7.7. After 7.6, if φ leaves q_1 invariant, then $\mathrm{to}(x^*) \geq n$, $\mathrm{to}(y^*) \geq m$ and there is $\lambda \in \mathbb{C} - \{0\}$ such that $a_{1,0} = \lambda^n$ and $b_{0,1} = \lambda^m$.

LEMMA 7.8. *Assume given an integer d, $nm < d < 2nm - n - m + 1$, and still let γ be as in 2.1. Take $s = d - nm$. If the automorphism φ keeps invariant the point q_s on γ in the s-th neighbourhood of the last satellite, then $a_{0,j} = 0$ for $j < (n+s)/m$ and $b_{i,0} = 0$ for $i < (m+s)/n$.*

Proof. Take $h = \min\{j \mid a_{0,j} \neq 0\}$, $h = \infty$ if $a_{0,j} = 0$ for all j, and, similarly, $k = \min\{i \mid b_{i,0} \neq 0\}$, $k = \infty$ if $b_{i,0} = 0$ for all i. Assume that the claim fails, and therefore either $h < (n+s)/m$, which is equivalent to $n(m-1)+mh < d$, or $k < (m+s)/n$, which is equivalent $nk + m(n-1) < d$. In the first case $h < n-1$, as having $h \geq n-1$ would contradict the hypothesis $d < 2nm - n - m + 1$. Similarly, in the second case $k < m-1$. Therefore, in no case $n(m-1)+mh = nk+m(n-1)$, because the points $(m-1,h)$ and $(k,n-1)$ are different and have different twisted degrees by 2.2(b). In the sequel we will argue assuming $n(m-1)+mh < nk+m(n-1)$, which in particular assures that $n(m-1)+mh < d$; similar arguments apply to the case $n(m-1)+mh > nk+m(n-1)$.

Take $f^* = f(x^*, y^*)$ as an equation of $\varphi^*(\gamma)$. In it, there is the monomial $M = -ma_{1,0}^{m-1}a_{0,h}x^{m-1}y^h$ coming from $-(x^*)^m$; its corresponding point on the Newton plane is $(m-1,h)$. We will check next that the following claim holds true:

Claim. any non-zero monomial Bx^iy^j other than M, produced by replacing x^* for x and y^* for y in a monomial of f, has either $i > m-1$, or $j > h$, or $ni + mj > nm - n + mh = \mathrm{td}(M)$.

Proof of the claim. If Bx^iy^j comes from $(x^*)^m$ and is not M, then it has either $i > m-1$, if it involves no $a_{0,r}$, or, by the definition of h, $j > h$ in case it involves some $a_{0,r}$. Due to 7.7, all monomials coming from $A_{s,t}(x^*)^s(y^*)^t$ have twisted degree non less than $ns+mt$, and, by the hypothesis, $ns+mt \geq d > n(m-1)+mh = \mathrm{td}(M)$. Regarding the monomials coming from $(y^*)^n$, those involving no $b_{s,0}$ have degree in y at least n, and in turn $n(m-1)+mh < d < 2nm - n - m + 1$ forces $n > h$. Assume that a monomial coming from $(y^*)^n$ has $r > 0$ non-zero factors $b_{s,0}$. By the definition of k, it is $s \geq k$, after which such a monomial has degree in x at least rk and degree in y at least $n-r$, hence twisted degree $\rho \geq nrk + m(n-r)$. Using 7.6, $nk > m$ and therefore

$$\rho \geq nrk + m(n-r) = nm + r(nk-m) \geq nk + m(n-1) > n(m-1) + mh.$$

□

It follows in particular from the claim that the monomial M, which corresponds to the point $(m-1,h)$, is not cancelled in f^*. Consider now an arbitrary equation of $\varphi^*(\gamma)$, namely uf^* with u invertible. Let N be a non-zero monomial of f^* corresponding to the point $(i,j) \in \mathbb{N}^2$. As it is clear, uN still has a non-zero monomial corresponding to (i,j) and no further monomial other than those corresponding to points $(i,j)+v$, v a positive vector. According to the already proved claim, for no N as above the vector $(m-1,h)-(i,j)$ is positive, after which still uf^* has a non-zero monomial corresponding to the point $(m-1,h)$. Since this contradicts 4.5 (a), the proof is complete. □

A first set of relevant coefficients, all critical, arises as follows:

Theorem 7.9. *If γ is a germ of curve given by an equation*

$$f = y^n - x^m + \sum_{ni+mj>nm} A_{i,j}x^iy^j,$$

then the first standing coefficient is critical for γ. Proceeding inductively, as far as all standing coefficients preceding a standing coefficient $A_{k,h}$ are critical and take their corresponding critical values, $A_{k,h}$ is critical.

The proof requires two lemmas; in both of them γ is as in 7.9:

Lemma 7.10. *Assume that (k,h) is a standing point, $A_{h,k} \neq 0$ and $A_{i,j} = 0$ for $ni + mj < d = nk + mh$. Then:*

(a) *for any* $B \neq 0$ *there is a germ*

$$\gamma' : y^n - x^m + Bx^k y^h + \cdots = 0$$

analytically equivalent to γ.

(b) *No germ*

$$\xi : g = y^n - x^m + \sum_{ni+mj>d} A_{i,j} x^i y^j = 0, \quad A_{i,j} \in \mathbb{C}$$

is analytically equivalent to γ.

Proof. Claim (a) is clear: just take $s = d - nm$, λ any s-th root of $B/A_{k,h}$, φ the local automorphism defined by $x^* = \lambda^n x$, $y^* = \lambda^m y$ and $\gamma' = \varphi^*(\gamma)$. Regarding claim (b), take any ξ as in the claim and note first that γ and ξ share all points up to the point q_s in the s-th neighbourhood of the last satellite, by 4.3. Therefore, any analytic automorphism φ satisfying $\varphi(\gamma) = \xi$ leaves invariant q_s. By 7.8, such a φ is given by equations

$$x^* = \sum_{(i,j)\in\mathbb{N}^2-\{(0,0)\}} a_{i,j} x^i y^j, \quad y^* = \sum_{(i,j)\in\mathbb{N}^2-\{(0,0)\}} b_{i,j} x^i y^j$$

where $a_{0,j} = 0$ for $j < (n+s)/m$ and $b_{i,0} = 0$ for $i < (m+s)/n$. Take $g^* = g(x^*, y^*)$ as an equation of $\varphi^*(\xi)$. We will see next that g^* has no non-zero standing monomial of twisted degree d or less. Clearly, by 7.7, no such a monomial comes from $\sum_{ni+mj>d} A_{i,j}(x^*)^i(y^*)^j$. The non-zero monomials coming from $(x^*)^m$ and involving none or just one of the coefficients $a_{0,j}$ have degree in x at least $m-1$ and therefore are not standing monomials; those involving $r \geq 2$ of these coefficients have degree in x at least $m-r$ and degree in y at least $r(n+s)/m$: this gives twisted degree at least $n(m-r) + r(n+s) = nm + rs > d$. A similar argument applies to the monomials coming from $(y^*)^n$. Now, an arbitrary equation of $\varphi^*(\xi)$ is ug^* with u an invertible series. Since all non-zero monomials of ug^* correspond to points $p+v$, where p corresponds to a non-zero monomial of g^* and v is zero or a positive vector, neither ug^* has a non-zero standing monomial of twisted degree d or less. In particular the monomial of ug^* corresponding to (k,h) is zero, hence $ug^* \neq f$ for any invertible u, against the assumption $\varphi(\gamma) = \xi$. ☐

LEMMA 7.11. *Still assume that* (k,h) *is a standing point and* $A_{i,j} = 0$ *for* $ni + mj < d = nk + mh$. *Then* $A_{k,h}$ *and all standing coefficients preceding it are critical coefficients and have critical value zero.*

Proof. Just use 7.10 and induction on d. ☐

Proof of 7.9. By 6.2 and 7.5 we may replace γ with an analytically equivalent germ satisfying the hypothesis of 7.11, and such a germ satisfies the claim just due to 7.11. ☐

When the hypothesis of 7.11 fails, the critical values need not be zero:

EXAMPLE 7.12. The coefficient corresponding to the point $(5,2)$ is critical for the germ $\gamma : y^5 - x^7 - 7x^6y - 21x^5y^2 = 0$ and takes its critical value, which is therefore -21. Indeed, taking φ given by $x^* = x - y$ and $y^* = y$ eliminates the irrelevant coefficient corresponding to $(6,1)$: $\varphi^*(\gamma)$ has equation $y^5 - x^7 + \dots$, the dots representing terms of twisted degree 41 or higher. By 7.11, the coefficient –zero–

corresponding to $(5,2)$ is critical for $\varphi^*(\gamma)$ and takes its critical value; the same holds thus for γ, by 7.5.

PROPOSITION 7.13. *A germ $\gamma : y^n - x^m + \sum_{ni+mj\geq nm} A_{i,j}x^iy^j = 0$ has all its standing coefficients critical and taking their critical values if and only if γ is analytically equivalent to the germ $\delta : y^n - x^m = 0$.*

Proof. By 7.11, all standing coefficients are critical for the germ δ and take their critical values, and therefore the same occurs for any germ analytically equivalent to δ. Conversely, assume that all standing coefficients are critical for γ and take their critical values. By 6.3, there is a germ γ', analytically equivalent to γ, whose equation has all non-standing coefficients equal to zero. By 7.11, all standing coefficients of the equation of γ' are zero too, hence $\gamma' = \delta$ and γ is analytically equivalent to δ. □

The germs of curve analytically equivalent to the germ δ of 7.13 are often called *quasihomogeneous germs*. By 7.13, all standing coefficients of a quasihomogeneous germ are critical, and hence relevant.

If γ is not quasihomogeneous, by 7.9 and 7.13, there is a well determined minimal critical coefficient A_ω which does not take its critical value: we will name its corresponding point $\omega \in \mathbb{N}^2$ the *Zariski point* of γ. The difference $\sigma(\gamma) = \mathrm{td}(\omega) - nm$ is an analytic invariant that determines ω (by 2.2); it is called the *Zariski invariant* of γ. The Zariski invariant actually describes the whole of the critical coefficients of γ: on one hand, all standing coefficients with twisted degree less than or equal to $\sigma(\gamma) + nm$ are critical for γ, and all of them take their critial values but the last one, which does not; on the other, it will turn out in the sequel that γ has no other critical coefficient, see 12.11. For quasihomogeneous germs, the Zariski point is left undefined and the Zariski invariant is taken to be ∞. In what follows, when the Zariski point of a germ γ is mentioned, we will implicitly assume that γ is not quasihomogeneous.

REMARK 7.14. Still by 7.11, if ω is the Zariski point, then A_ω is the first standing coefficient that remains non-zero after eliminating all the non-standing monomials of lesser twisted degree.

PROPOSITION 7.15. *Assume that γ is a non-quasihomogeneous irreducible germ, with Zariski point $\omega = (\omega_1, \omega_2)$ and Zariski invariant $\sigma = n\omega_1 + m\omega_2 - nm$. Then there is a germ γ', analytically equivalent to γ, which is given by an equation of the form*

$$y^n - x^m + x^{\omega_1}y^{\omega_2} + \sum_{ni+mj>nm+\sigma} A_{i,j}x^iy^j,$$

Proof. By 6.2, there is a germ γ'', analytically equivalent to γ, given by an equation

$$y^n - x^m + \sum_{ni+mj>nm} B_{i,j}x^iy^j$$

with all non-standing $B_{i,j}$, $ni + mj < nm + \sigma$, equal to zero. Then, by 7.14 and the definition of σ, also all standing $B_{i,j}$ with $ni + mj < nm + \sigma$ are zero. The above equation is thus

$$y^n - x^m + B_\omega x^{\omega_1}y^{\omega_2} + \sum_{ni+mj>nm+\sigma} B_{i,j}x^iy^j$$

with $B_\omega \neq 0$, after which it suffices to apply 7.10(a). ☐

The contents of this section may of course be reformulated in terms of Puiseux coefficients. In that form, the essential results already appeared in [23] and [3]. Leaving the details to the reader, the Puiseux coefficients c_s, $s = ni + mj$, that correspond to a (necessarily unique, by 2.2) standing coefficient $A_{i,j}$, $(i,j) \in \mathbf{T}$, may be called *standing Puiseux coefficients*, and the corresponding monomials in the Puiseux series, *standing monomials*; the non-standing Puiseux coefficients are irrelevant by 6.1 and 5.8. The definition of critical coefficients applies without changes to the Puiseux coefficients, a claim similar to 7.2 does hold and in particular proves that a coefficient is critical if and only if the corresponding Puiseux coefficient is. From 7.14, it is direct to check that $m + \sigma(\gamma)$ is the degree of the first non-zero standing monomial that remains in the Puiseux series after eliminating all non-standing monomials preceding it; this is the way the Zariski invariant was introduced in [23].

8. Further irrelevant coefficients. The relevance of the coefficients for a quasihomogeneous germ being clear, from now on we will fix our attention on non-quasihomogeneous germs. The Zariski point of such a germ allows to determine a new set of irrelevant coefficients, namely:

PROPOSITION 8.1. *If γ is as in 2.1, non-quasihomogeneous and with Zariski point ω, then all the standing coefficients of the form $A_{\omega+v}$, $v > 0$, are irrelevant for γ.*

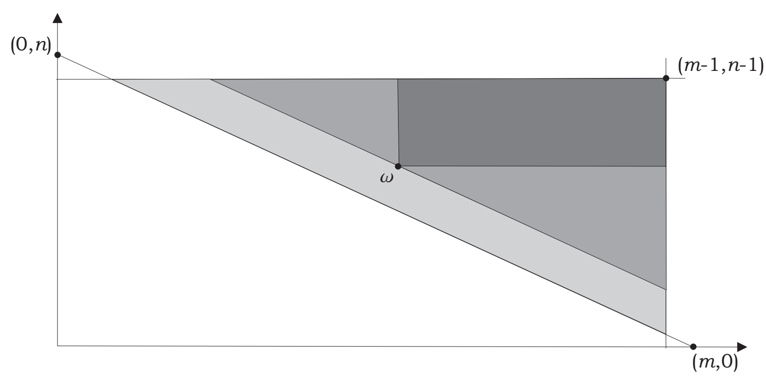

FIG. 2. *The standing points are the integral points in the interior of the shaded area; coefficients corresponding to other points are all irrelevant by 6.1. If ω is the Zariski point of γ, then ω and the points in the interior of the light grey area correspond to critical coefficients of γ. By 8.1, the coefficients corresponding to the standing points other than ω in the dark grey area are irrelevant.*

Proof. Take $d = \mathrm{td}(\omega + v)$. Up to replacing γ with an analytically equivalent germ, it is enough to prove the claim assuming that the non-standing coefficients of f of twisted degree less than d are zero and $A_\omega = 1$, by 7.15 and 5.7.

Let θ, θ' be arbitrary complex numbers and assume $A_{\omega+v} = \theta$. We will prove the existence of a germ γ', analytically equivalent to γ, and an equation f' of γ' which has the same partial sum of twisted degree $d-1$ as f and the coefficient corresponding to $\omega + v$ equal to θ'. Since $\omega + v$ is a standing point, $A_{\omega+v}$ is the only coefficient of twisted degree d (by 2.2) and therefore this will prove it to be irrelevant.

Write $\omega = (k,h)$ and $v = (r,s)$, after which $d = n(k+r) + m(h+s)$. First, we use the analytic automorphism φ, defined by

$$x^* = x + ax^{r+1}y^s, \quad y^* = y + bx^r y^{s+1},$$

where a, b are complex numbers that will be determined in a while. The germ $\varphi^*(\gamma)$ has equation $f^* = f(x^*, y^*)$. The replacement of x, y with x^*, y^* in f produces monomials identical to those already in f, and other which are as follows:

(a) Monomials coming from an $A_{i,j}(x^*)^i(y^*)^j$ with $ni + mj > nm + \sigma$: they involve at least one factor $ax^{r+1}y^s$ or bx^ry^{s+1} and therefore have twisted degree at least $ni + mj + nr + ms > d$.

(b) Monomials coming from $(x^*)^k(y^*)^h$: we get $kax^{k+r}y^{h+s}$, $hbx^{k+r}y^{h+s}$, both of twisted degree d, and other having higher twisted degree.

(c) Monomials coming from $-(x^*)^m$: there is $-max^{m+r}y^s$ and then further monomials, all corresponding to points of the form $(m, 0) + w$, where w is a positive vector of twisted degree strictly higher than $nr + ms$.

(d) Monomials coming from $(y^*)^n$: there is nbx^ry^{n+s} and then further monomials, all corresponding to points of the form $(0, n)+w$, where w is a positive vector of twisted degree strictly higher than $nr + ms$.

We now multiply f^* by the invertible factor $1-nbx^sy^r$; by doing so, each non-zero monomial Bx^iy^j of f^* gives rise to a new monomial $-nbBx^{i+r}y^{j+s}$ which is added to f^*. The new monomial given rise to by y^n just cancels nbx^ry^{n+s}, while the one given rise to by $-x^m$ turns $-max^{m+r}y^s$ into $(-ma+nb)x^{m+r}y^s$. The new monomial given rise to by x^ky^h turns $(\theta + ka + hb)x^{k+r}y^{h+s}$ into $(\theta + ka + (h - n)bx^{k+r}y^{h+s}$. The equations in a, b

$$ka + (h - n)b = \theta' - \theta \quad , \quad -ma + nb = 0,$$

have determinant $-nk-mh+nm < 0$, and therefore a unique solution. We take a, b to be such a solution. This assures that the new equation $g = (1-nbx^sy^r)f^*$ of $\varphi^*(\gamma)$ has the monomials corresponding to the points $(0, n) + (r, s)$ and $(m, 0) + (r, s)$ cancelled, and θ' as coefficient of the monomial corresponding to $(k + r, h + s)$. Regarding the other new monomials of g, those given rise to by monomials of f^* of twisted degree higher than $nm + \sigma$ have twisted degree higher than d. To close, the new monomials given rise to by monomials of f^* corresponding to points of the form $(m, 0) + w$ or $(0, n)+w$, where w is a positive vector, also correspond to points of the form $(m, 0)+w$ or $(0, n) + w$ with w positive, and for all of them it is $\mathrm{td}(w) > nr + ms$.

Summarizing, up to now we have obtained an equation g that defines a germ γ' analytically equivalent to γ and has $\theta' x^{k+r}y^{h+s}$ as its only monomial of twisted degree d. Furthermore, the partial sums of twisted degree $d - 1$ of f and g, let us call them f_{d-1} and g_{d-1}, differ in a sum of non-standing monomials that do not appear in f, all corresponding to points of the form $(m, 0) + w$ or $(0, n) + w$, w a positive vector and $\mathrm{td}(w) > nr + ms$. Since f_{d-1} has no non-standing monomials, these are the only non-standing monomials of g_{d-1}; of course, if all of them are zero, $f_{d-1} = g_{d-1}$ and the proof is complete. Otherwise, we will eliminate them by using finitely many local automorphisms that preserve the other monomials.

Let d' be the minimal twisted degree for which g_{d-1} has a non-zero momomial corresponding to either a point $(m, 0)+w$ or to a point $(0, n)+w$, w a positive vector and $\mathrm{td}(w) > nr + ms$. Then $\mathrm{td}(w) < d' - nm < d - nm < nm$, after which w is uniquely determined by d'. Write $w = (i, j)$ and assume the monomials of g_{d-1} corresponding to $(m, 0)+w$ and $(0, n)+w$ to be $mCx^{m+i}y^j$ and $nC'x^iy^{n+j}$. Consider the local automorphism ψ given by

$$x^* = x + Cx^{i+1}y^j \quad , \quad y^* = y.$$

and the equation $g^* = g(x^*, y^*)$ of $\psi^*(\gamma')$. Taking in account that the standing monomials of g_{d-1} have twisted degree non-less than $nk+mh$ and $ni+mj = \mathrm{td}(w) >$

$nr+ms$, after the replacement no standing monomial of g_{d-1} gives rise to a new term of twisted degree d or less. After this, it is direct to check that, comparing to g, g^* has:

- all standing monomials of twisted degree d or less unmodified,
- the monomial corresponding to $(m+i, j)$ cancelled,
- the monomials corresponding to $(m, 0)$, $(0, n)$ and $(i, n+j)$ unmodified and
- all other non-zero and non-standing monomials of twisted degree less than d corresponding to points $(m, 0) + w'$ or $(0, n) + w'$, for w' a positive vector with $\mathrm{td}(w') > ni + mj$.

The monomial $nC'x^iy^{n+j}$ is then cancelled similarly using the local automorphism

$$x^* = x \quad , \quad y^* = y + C'x^iy^{j+1}.$$

Repeating the procedure above cancels all non-standing monomials of g which have twisted degree less than d, thus giving rise to an equation of a germ analytically equivalent to γ which has f_{d-1} as partial sum of twisted degree $d-1$ and $\theta' x^{k+r}y^{h+s}$ as only monomial of twisted degree d, as wanted. □

As it is easy to check, 8.1 is equivalent to Zariski's third elimination criterium ([23], IV.3.1). If still $\mathbf{T}$ denotes the set of the standing points, take

$$\mathbf{Z} = \{p \in \mathbf{T} \mid p = \omega + v \text{ for } v \text{ positive or zero}\}.$$

From 8.1, arguing as in 6.3, it directly follows:

COROLLARY 8.2. *If γ is a non-quasihomogeneous irreducible germ with single characteristic exponent m/n and Zariski point $\omega = (k, h)$, then there is a germ analytically equivalent to γ which has equation*

$$y^n - x^m + x^ky^h + \sum_{\substack{ni+mj>nk+mh \\ (i,j)\in\mathbf{T}-\mathbf{Z}}} A_{i,j}x^iy^j,$$

where the $A_{i,j}$ are complex numbers.

9. Principal automorphisms and inverse images. In this section we will compute equations of inverse images of germs by a restricted class of local automorphisms. First of all we have:

LEMMA 9.1. *Let γ be a germ as in 2.1 and with Zariski invariant $\sigma \neq \infty$. If a local automorphism φ, given by equations*

$$x^* = \sum_{(i,j)\in\mathbb{N}^2-\{(0,0)\}} a_{i,j}x^iy^j, \quad y^* = \sum_{(i,j)\in\mathbb{N}^2-\{(0,0)\}} b_{i,j}x^iy^j,$$

leaves invariant the point $q_{\sigma+1}$ on γ in the $(\sigma+1)$-th neighbourhood of the last satellite, then there is a σ-th root of unity ε such that $a_{1,0} = \varepsilon^m$ and $b_{0,1} = \varepsilon^n$.

Proof. Assume $\omega = (k, h)$ and therefore $\sigma = nk + hm - nm$. First of all, note that $b_{1,0} = 0$ by 7.6, and therefore $a_{1,0}$ and $b_{0,1}$ are the eigenvalues of the tangent map to φ; in case of being different, $a_{1,0}$ is the one corresponding to the direction tangent to γ. If $\psi(\gamma)$ is any germ analytically equivalent to γ, ψ an analytic isomorphism, then $\psi \circ \varphi \circ \psi^{-1}$ leaves invariant the point in the $(\sigma+1)$-th neighbourhood of the last satellite on $\psi(\gamma)$ and still its tangent map has eigenvalues $a_{1,0}$ and $b_{0,1}$, $a_{1,0}$ being the one corresponding to the direction tangent to $\psi(\gamma)$ if they are different. By 7.15, this shows that it suffices to prove the claim for a germ γ with equation

$$f = y^n - x^m + x^k y^h + \sum_{ni+mj>nm+\sigma} A_{i,j} x^i y^j.$$

According to 7.8 and 7.7, the equations of φ have $a_{0,j} = 0$ for $j < (n+\sigma)/m$, $b_{i,0} = 0$ for $i < (m+\sigma)/n$ and $a_{1,0} = \lambda^n$ and $b_{0,1} = \lambda^m$ for some $\lambda \in \mathbb{C} - \{0\}$.

We will examine the non-zero monomials of $f^* = f(x^*, y^*)$ other than the evident $\lambda^{nm} y^n$ and $-\lambda^{nm} x^m$. From them, all those coming from a term $A_{i,j}(x^*)^i (y^*)^j$, $ni + mj > nm + \sigma$ have twisted degree higher than $nm + \sigma$ due to 7.7. The term $(x^*)^k (y^*)^h$ gives rise to the monomial $\lambda^{nm+\sigma} x^k y^h$ and other monomials with twisted degree strictly higher than $nm + \sigma$. The non-zero monomials coming from $-(x^*)^m$ and involving at most one $a_{0,j}$ have all degree in x at least $m - 1$ and are therefore non-standing; the non-zero monomials involving $r \geq 2$ non-zero $a_{0,j}$, due to being $j \geq (n+\sigma)/m$, have twisted degree non-less than $(m-r)n + r(n+\sigma)$ which in turn is strictly higher than $mn + \sigma$. Similarly, all non-zero standing monomials given rise to by $(y^*)^n$ have twisted degree strictly higher than $mn + \sigma$. All together $g = \lambda^{-nm} f^*$ has the form $g = y^n - x^m + \ldots$ and has $\lambda^\sigma x^k y^h$ as as the only non-zero standing monomial of twisted degree $nm + \sigma = nk + mj$ or less.

Since φ is assumed to leave invariant the point q_σ in the σ-th neighbourhood of the last satellite on γ, q_σ belongs also to $\varphi^*(\gamma)$ and therefore, by 4.5, $\varphi^*(\gamma)$ has an equation g' with the same partial sum of twisted degree $nm+\sigma-1$ as f. In view of the terms of minimal twisted degree, such an equation needs to be $g' = (1+u)g$, where u is a series with $u(0,0) = 0$. If $B_{i,j} x^i y^j$ is a non-standing monomial of g, all monomials of $u B_{i,j} x^i y^j$ correspond to points $(i,j) + v$, v a positive vector, and therefore no one is a standing monomial. As a consequence $\lambda^\sigma x^k y^h$, the non-zero standing monomial of minimal twisted degree of g, remains unchanged after multiplication by $1 + u$ and so

$$g' = y^n - x^m + \lambda^\sigma x^k y^h + \ldots$$

Using the equations f and g', and the coefficient corresponding to (k,h) as coordinate in the first neighbourhood of q_σ, the point $q_{\sigma+1}$ has coordinate 1, while $\varphi^{-1}(q_{\sigma+1})$ –the point on $\varphi^*(\gamma)$– has coordinate λ^σ. The invariance of $q_{\sigma+1}$ forces thus $\lambda^\sigma = 1$ and the proof is complete. □

From now on we will mainly consider local automorphisms at O which leave invariant the point q_1, in the first neighbourhood of the last satellite on all germs γ of 2.1, and whose tangent map has both eigenvalues equal to one: they will be called *principal automorphisms at O*. Directly from 7.6 we have

LEMMA 9.2. *A local automorphism φ, given by equations*

$$x^* = \sum_{(i,j)\in\mathbb{N}^2-\{(0,0)\}} a_{i,j} x^i y^j, \quad y^* = \sum_{(i,j)\in\mathbb{N}^2-\{(0,0)\}} b_{i,j} x^i y^j,$$

is principal if and only if $b_{i,0} = 0$ for $i < m/n$ and $a_{1,0} = b_{0,1} = 1$.

In particular the tangent map to a principal automorphism has matrix

$$\begin{pmatrix} 1 & a_{0,1} \\ 0 & 1 \end{pmatrix}.$$

It is direct to check that the principal automorphisms at O compose a subgroup of $\mathcal{A}_O(S)$ which in the sequel will be denoted $\mathcal{PA}_O(S)$, or just $\mathcal{PA}$ if S and O are clear.

REMARK 9.3. Still let q_s be the point on γ in the s-th neighbourhood of the last satellite; denote by $\mathbb{A}_1$ the affine line whose points are the free points in the first neighbourhood of q_s. The reason for considering only principal automorphisms is that it will turn out in the sequel (see 12.11) that, if $s > \sigma$, then the principal automorphisms leaving q_s invariant either act transitively on $\mathbb{A}_1$, or, otherwise, they all leave all points of $\mathbb{A}_1$ invariant. In the first case the coefficient of twisted degree $nm+s$ is irrelevant for γ due to 5.8. In the second case, if φ is any local automorphism at O leaving q_s invariant, its tangent map has a triangular matrix, by 7.6, and eigenvalues $\varepsilon^n, \varepsilon^m$, $\varepsilon^\sigma = 1$, by 9.1. After this, φ^σ is principal and therefore the action $\widehat{\varphi}$ of φ on $\mathbb{A}_1$ has order a divisor of σ; in particular $\widehat{\varphi}$ cannot be a non-trivial translation. After an elementary argument, the $\widehat{\varphi}$, $\varphi \in \mathcal{A}_O(\mathbf{S}, q_s)$, describe a finite group H, composed of ν homotheties, all with the same center, and the action of $\mathcal{A}_O(\mathbf{S}, q_s)$ on $\mathbb{A}_1$ has orbits of at most ν elements. Then any coefficient of twisted degree $d = nm + s$ is relevant for γ, by 5.8; this in particular assures that there is a single coefficient of twisted degree d, by 6.1 and 2.2. Furthermore, by 5.4, the partial sums of twisted degree $d-1$ being kept constant, the coefficient of twisted degree d takes finitely many –at most ν– values on each analytic class of germs. In such a situation, namely when all principal automorphisms leaving invariant q_s leave also invariant all free points in the first neighbourhood of q_s, we will say that the coefficient of twisted degree d, $A_{\alpha,\beta}$, is a *continuous invariant*; actually, when $\nu > 1$, the true invariant is $(A_{\alpha,\beta} - a)^\nu$, a the coordinate of the common centre of the elements of H.

REMARK 9.4. It is direct to check from the above definition that if germs γ and γ', both as in 2.1, are analytically equivalent, then the coefficient of the equation of γ corresponding to a certain $p \in \mathcal{N}$ is a continuous invariant if and only if so is the coefficient of the equation of γ' corresponding to p.

We have already dealt with the monomials of equations $\varphi^*(f) = f^* = f(x^*, y^*)$ for certain local automorphisms φ. Now we will consider the more general case in which still γ is as in 2.1 and φ is an arbitrary principal automorphism. By 9.2, φ is given by equations

$$x^* = \sum_{(i,j)\in\mathbb{N}^2-\{(0,0)\}} a_{i,j}x^iy^j, \quad y^* = \sum_{(i,j)\in\mathbb{N}^2-\{(0,0)\}} b_{i,j}x^iy^j, \tag{11}$$

with $a_{1,0} = b_{0,1} = 1$ and $b_{i,0} = 0$ for $i < m/n$. The other coefficients $a_{i,j}, b_{i,j}$ will be initially taken as free variables and we will impose to them the algebraic relations resulting from the invariance by φ of successive points on γ.

In the sequel we will call *x-admissible* the vectors in the set

$$\mathbf{A} = \{v \in \mathbb{N}^2 \mid \text{ either } v > 0 \text{ or } v = (-1, j) \text{ and } j > 0\}.$$

From them, those of the form $(-1, j)$ will be called *negative*. Similarly, we will call *y-admissible* the vectors in the set

$$\mathbf{B} = \{v \in \mathbb{N}^2 \mid \text{ either } v > 0 \text{ or } v = (i, -1) \text{ and } i > m/n\},$$

those of the form $(i, -1)$ being also called *negative*. We will refer to the elements of $\mathbf{A} \cup \mathbf{B}$ as *admissible vectors* and write $\mathbf{A}_s$ and $\mathbf{B}_s$ for the sets of, respectively, the x-admissible and the y-admissible vectors of twisted degree s or less.

We associate to each $v = (i, j) \in \mathbf{A}$ the coefficient $a_v = a_{i+1,j}$ of the first of the equations (11), and to each $v = (i, j) \in \mathbf{B}$ the coefficient $b_v = b_{i,j+1}$ of the second of the equations (11). Note that these are different conventions, one for admissible x-vectors and the other for admissible y-vectors, and both are different from the convention we are using for the coefficients of the equations of germs, namely $A_p = A_{r,s}$ if $p = (r, s) \in \mathbb{N}^2$. Taking the associated coefficients induces bijections between $\mathbf{A}$ and the set of the coefficients of the first of the equations (11) other than $a_{1,0}$, and between $\mathbf{B}$ and the set of the coefficients of the second of the equations (11) other than $b_{0,1}$ and the $b_{i,0}$ with $i < m/n$. We will say that an admissible vector is *cancelled* when its associated coefficient has been fixed to be zero after imposing the invariance of some point on γ.

Direct computation yields:

$$(x^*)^\alpha (y^*)^\beta =$$

$$x^\alpha y^\beta \sum_{\substack{r=0,\dots,\alpha\\ s=0,\dots,\beta}} \binom{\alpha}{r}\binom{\beta}{s} \left(\sum_{\substack{v_1,\dots,v_r \in \mathbf{A}\\ w_1,\dots,w_s \in \mathbf{B}}} a_{v_1} \dots a_{v_r} b_{w_1} \dots b_{w_s} (xy)^{v_1+\dots+v_r+w_1+\dots+w_s} \right),$$

where for $v = (i, j) \in \mathbb{N}^2$ we have written $(xy)^v = x^i y^j$.

The monomial of bidegree (α', β') in $A_{\alpha,\beta}(x^*)^\alpha (y^*)^\beta$ (the *contribution* of $A_{\alpha,\beta} x^\alpha y^\beta$ to the monomial of bidegree (α', β') of f^*) is then

$$A_{\alpha,\beta} \left(\sum \binom{\alpha}{r}\binom{\beta}{s} a_{v_1} \dots a_{v_r} b_{w_1} \dots b_{w_s} \right) x^{\alpha'} y^{\beta'}. \tag{12}$$

where the summation runs on all ordered sets $v_1, \dots, v_r$ of x-admissible vectors and all ordered sets $w_1, \dots, w_s$ of y-admissible vectors, all non-cancelled, with $0 \le r \le \alpha$, $0 \le s \le \beta$ and $v_1+\dots+v_r+w_1+\dots+w_s = (\alpha', \beta')-(\alpha, \beta)$. When these conditions are satisfied we will say that the ordered set $v_1, \dots, v_r, w_1, \dots, w_s$ is a *decomposition* of the vector $(\alpha', \beta')-(\alpha, \beta)$ in admissible vectors. All admissible vectors having positive twisted degree, the value of (12) is zero unless $\mathrm{td}(\alpha, \beta) \le \mathrm{td}(\alpha', \beta')$. It follows:

Lemma 9.5. *The coefficient of the monomial of bidegree (α', β') of $f^* = f(x^*, y^*)$ is*

$$\sum_{nm \le \mathrm{td}(\alpha,\beta) \le \mathrm{td}(\alpha',\beta')} A_{\alpha,\beta} \left(\sum \binom{\alpha}{r}\binom{\beta}{s} a_{v_1} \dots a_{v_r} b_{w_1} \dots b_{w_s} \right), \tag{13}$$

where the innermost summation runs as the summation in (12) and we take $A_{m,0} = -1$ and $A_{0,n} = 1$.

Remark 9.6. In (13), there is the summand $A_{\alpha',\beta'}$ corresponding to the empty decomposition of $0 = (\alpha', \beta') - (\alpha', \beta')$ in admissible vectors.

Remark 9.7. It is clear form 9.5 that the coefficient of bidegree (α', β') of f^* is a polynomial function of the $A_{\alpha,\beta}$, $\mathrm{td}(\alpha, \beta) \le \mathrm{td}(\alpha', \beta')$, and the a_v, b_v, $\mathrm{td}(v) \le \mathrm{td}(\alpha', \beta') - nm$, linear in the $A_{\alpha,\beta}$.

Remark 9.8. For any decomposition $v_1, \dots, v_r, w_1, \dots, w_s$ appearing in (12),

$$\sum_{h=1}^{r} \mathrm{td}(v_h) + \sum_{h=1}^{s} \mathrm{td}(w_h) = \mathrm{td}(\alpha', \beta') - \mathrm{td}(\alpha, \beta)$$

In particular, if all admissible vectors v with $\mathrm{td}(v) \leq \mathrm{td}(\alpha', \beta') - \mathrm{td}(\alpha, \beta)$ are cancelled, and $(\alpha', \beta') \neq (\alpha, \beta)$ so that the empty decomposition does not occur, then the monomial (12) is zero.

REMARK 9.9. If a coefficient a_v (resp. b_w) effectively appears in 13, then there is $A_{\alpha,\beta} \neq 0$ and a decomposition of $(\alpha', \beta') - (\alpha, \beta)$ in non-cancelled admissible vectors, one of which is v (resp. w).

REMARK 9.10. If the vector $v = (\alpha', \beta') - (m, 0)$ is x-admissible (resp. $w = (\alpha', \beta') - (0, n)$ is y-admissible), then in (13) there is the summand $-ma_v$ (resp. nb_w) and no other summand involves a_v (resp. b_w), because $\mathrm{td}(v)$ (resp. $\mathrm{td}(w)$) equals the maximal value $\mathrm{td}(\alpha', \beta') - nm$ of all differences $\mathrm{td}(\alpha', \beta') - \mathrm{td}(\alpha, \beta)$, $A_{\alpha,\beta} \neq 0$.

10. Leaving further points invariant. This section is devoted to proving the next result:

PROPOSITION 10.1. *Let γ be an irreducible germ with single characteristic exponent m/n. Assume it is not quasihomogeneous, has Zariski point $\omega = (\omega_1, \omega_2)$ (and hence Zariski invariant $\sigma = n\omega_1 + m\omega_2 - nm$) and equation*

$$f = y^n - x^m + x^{\omega_1} y^{\omega_2} + \sum_{n\alpha + m\beta > nm + \sigma} A_{\alpha,\beta} x^\alpha y^\beta.$$

Take $\kappa = \min\{\mathrm{td}(m-1, \omega_2), \mathrm{td}(\omega_1, n-1)\} - nm$, fix an integer ρ, $\sigma < \rho < \kappa$, and let q_ρ be the point on γ in the ρ-th neighbourhood of the last satellite.

If φ, given by equations

$$x^* = \sum_{(i,j) \in \mathbb{N}^2 - \{(0,0)\}} a_{i,j} x^i y^j, \quad y^* = \sum_{(i,j) \in \mathbb{N}^2 - \{(0,0)\}} b_{i,j} x^i y^j,$$

is a principal automorphism leaving invariant the point on γ in the first neighbourhood of q_ρ, then, as far as they are defined, $a_v = b_v = 0$ for

– *all positive admissible vectors v with* $\mathrm{td}(v) \leq \rho - \sigma$, *and*
– *all negative admissible vectors v with* $\mathrm{td}(v) \leq \rho$.

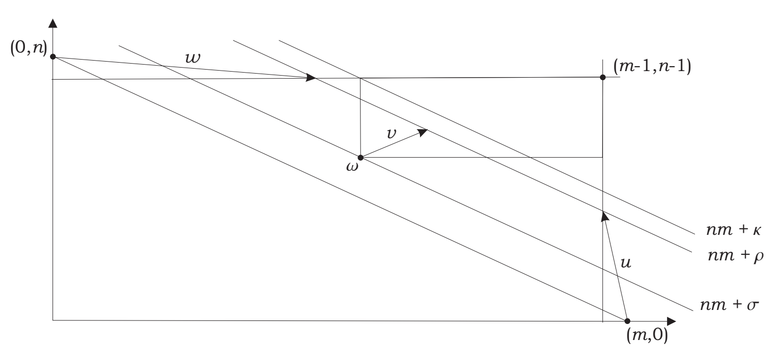

FIG. 3. *There are shown the Zariski point ω and the level lines of twisted degrees $nm + \sigma$, $nm + \rho$ and $nm + \kappa$. The vector v is admissible and positive, and has twisted degree $\rho - \sigma$. The vectors u and w are x-admissible and y-admissible, respectively; both are negative and have twisted degree ρ.*

Proof. The claim for negative vectors v with $\mathrm{td}(v) < \sigma$ has been already proved in 7.8, with $d = \sigma + mn$ there. Furthermore, there are no negative admissible vectors v of twisted degree σ: indeed, if $v = (-1, j)$ and $\mathrm{td}(v) = \sigma$, then the point $(m, 0) + v = (m - 1, j)$ has twisted degree $nm + \sigma$, after which, using that $nm + \sigma < n(m-1) + m(n-1)$, it has $j < n - 1$; since $(m-1, j) \neq \omega$ and $\mathrm{td}(\omega) = nm + \sigma$, this contradicts 2.2. A similar argument applies to the case $v = (i, -1)$.

Since there are no positive admissible vectors with twisted degree negative or zero, the claim regarding them is true in all cases for $\rho \leq \sigma$ and so we will proceed by induction on ρ, starting at $\rho = \sigma$. The case $\rho = \sigma$ being proved, we assume $\rho > \sigma$ and, by the induction hypothesis, $a_v = b_v = 0$ for

– all positive vectors v with $\mathrm{td}(v) < \rho - \sigma$, and also for
– all negative vectors v with $\mathrm{td}(v) < \rho$.

If there is not a positive admissible vector v with $\mathrm{td}(v) = \rho - \sigma$, and neither is a negative admissible vector v with $\mathrm{td}(v) = \rho$, then there is nothing to prove.

An admissible positive vector v, of twisted degree $\rho - \sigma$ gives the point $\omega + v$, which has twisted degree $\rho + mn$. If u is a x-admissible negative vector of twisted degree ρ, the point $(m, 0) + u$ has twisted degree $\rho + mn$, and the same occurs with the point $(0, n) + w$ if w is a negative y-admissible vector of twisted degree ρ. These points are pairwise different and all belong to the set T_0 of 2.2 (see Figure 3). Hence, by 2.2, at most one of them may exist and therefore we will end the proof by considering the following three cases, from which each excludes the other two.

Case 1: There is an admissible positive vector v with $\mathrm{td}(v) = \rho - \sigma$. Assume $v = (i, j)$ and therefore $\rho - \sigma = ni + mj$. Since all admissible vectors with twisted degree strictly less than $\rho - \sigma$ are cancelled by the induction hypothesis, the difference $f^* - f$ has no terms of twisted degree strictly less than $nm + \rho - \sigma$ due to 9.8. By 2.2, there are two possible monomials of twisted degree $nm + \rho - \sigma$, which obviously are those corresponding to the points $p = (m + i, j)$ and $p' = (i, n + j)$. Again by 9.8, no monomial of f with twisted degree strictly higher than nm, other than the one corresponding to p, has a non-zero contribution to the monomial of f^* corresponding to p. Neither has y^n, because a decomposition in admissible vectors of $p - (0, n)$ needs to involve at least one negative y-admissible vector e (due to being $j < n$), and e is cancelled because $\mathrm{td}(e) \leq \mathrm{td}(p - (0, n)) = \rho - \sigma < \rho$. The contribution of $-x^m$ being $-ma_v x^{m+i} y^j$, the monomial of f^* corresponding to p is then $(A_p - ma_v) x^{m+i} y^j$. Similarly, the monomial of f^* corresponding to p' is $(A_{p'} + nb_v) x^i y^{n+j}$. Now it is direct to check that

$$(1 - nb_v x^i y^j) f^* - f = (-ma_v - nb_v) x^{m+i} y^j + \dots.$$

By taking $\bar{f} = (1 - nb_v x^i y^j) f^*$ as equation of $\varphi^*(\gamma)$ and using the coefficient corresponding to p as coordinate, according to 4.3, the invariance by φ of the point on γ in the $(\rho - \sigma + 1)$-th neighbourhood of the last satellite forces $ma_v - nb_v = 0$, after which $\mathrm{to}(\bar{f} - f) > nm + \rho - \sigma$.

We fix now our attention on the monomial M of f^* corresponding to the point $p'' = \omega + v = (\omega_1 + i, \omega_2 + j)$; it has twisted degree $nm + \rho$. We will examine the contributions to M of the different monomials of f using the equality (12) of section 9. By the definition of κ, the abscissa of p'' is at most $m - 2$. Therefore, a decomposition of $p'' - (m, 0)$ in admissible vectors needs to involve at least two negative x-admissible vectors: each of them has thus twisted degree strictly less than $\mathrm{td}(p'' - (m, 0)) = \rho$ and therefore is cancelled. As a consequence the contribution of $-x^m$ to M is zero. The same occurs with the contribution of y^n, by a similar argument. Next non-trivial monomial of f is the one corresponding to the Zariski point ω: we have two decompositions $p'' - \omega = v + 0 = 0 + v$, where v is taken as x-admissible and y-admissible, respectively. Any other decomposition involves at least two non-zero vectors which, as in former cases, are cancelled because its twisted degree is strictly less than $\mathrm{td}(p'' - \omega) = \rho - \sigma$. The contribution of $x^{\omega_1} y^{\omega_2}$ is thus

$$(\omega_1 a_v + \omega_2 b_v) x^{\omega_1 + i} y^{\omega_2 + j}.$$

Any other non-trivial monomial $A_{\alpha,\beta} x^\alpha y^\beta$ of f, $(\alpha, \beta) \neq p''$, has twisted degree strictly higher than $nm + \sigma$, after which $\mathrm{td}(p'' - (\alpha, \beta)) < \rho - \sigma$ and any (necessarily non-empty) decomposition of $p'' - (\alpha, \beta)$ in admissible vectors is composed of cancelled vectors. There is thus no other non-trivial contribution but the one of $A_{p''} x^{\omega_1 + i} y^{\omega_2 + j}$, which equals the monomial itself. All together, the monomial of f^* corresponding to p'' is

$$(A_{p''} + \omega_1 a_v + \omega_2 b_v) x^{\omega_1 + i} y^{\omega_2 + j}.$$

It is now direct to check that the monomial of $\bar{f} = (1 - n b_v x^i y^j) f^*$ corresponding to p'' is

$$(A_{p''} + \omega_1 a_v + \omega_2 b_v - n b_v) x^{\omega_1 + i} y^{\omega_2 + j}.$$

Furthermore, according to 4.5, the invariance of q_ρ assures that there is an equation $\hat{f}$ of $\varphi^*(\gamma)$ that satisfies $\mathrm{to}(\hat{f} - f) \geq nm + \rho$. Obviously such a $\hat{f}$ needs to have the form $\hat{f} = (1 + g)\bar{f}$, where $g \in \mathbb{C}\{x, y\}$, $g(0,0) = 0$. The above inequality and the already obtained $\mathrm{to}(\bar{f} - f) > nm + \rho - \sigma$ give

$$\mathrm{to}(g\bar{f}) = \mathrm{to}(\hat{f} - \bar{f}) > nm + \rho - \sigma$$

and therefore

$$\mathrm{to}(g) > \rho - \sigma.$$

After this it is direct to check that $g\bar{f}$ has the monomial corresponding to p'' equal to zero and hence $\hat{f}$ has the same monomial corresponding to p'' as $\bar{f}$, namely

$$(A_{p''} + \omega_1 a_v + \omega_2 b_v - n b_v) x^{\omega_1 + i} y^{\omega_2 + j},$$

Now we take the equations f of γ and $\hat{f}$ of $\varphi^*(\gamma)$: they have the same partial sum of twisted degree $nm + \rho - 1$ and each has the monomial corresponding to p'' as its only monomial with twisted degree $nm + \rho$, by 2.2. The coefficients of these monomials being taken as coordinates (4.3), the invariance of the point of γ in the first neighbourhood of q_ρ forces

$$\omega_1 a_v + (\omega_2 - n) b_v = 0.$$

Since we already know that

$$m a_v - n b_v = 0$$

and

$$\begin{vmatrix} m & -n \\ \omega_1 & \omega_2 - n \end{vmatrix} = \mathrm{td}(\omega) - nm > 0,$$

it is $a_v = b_v = 0$, as wanted.

Case 2: There is a x-admissible negative vector u with $\mathrm{td}(u) = \rho$. In this case b_u is not defined and we need to prove just $a_u = 0$. Assume $u = (-1, j)$. Then $\mathrm{td}(u) = -n + mj = \rho < \kappa$ gives $j < \omega_2$. Take $p = (m, 0) + u = (m-1, j)$. We will examine first the monomial of f^* corresponding to p. Any decomposition of $p-(m, 0)$ in two or more admissible vectors involves a x-admissible negative vector which, being of twisted degree strictly less than $\mathrm{td}(p-(m, 0)) = \rho$, is cancelled. It remains only the decomposition $p-(m, 0) = u$ and so the contribution of $-x^m$ is $-ma_u x^{m-1}y^j$. Being $j < \omega_2$ assures that $j - n \leq -2$, after which the contribution of y^n is zero: indeed, any decomposition of $p-(0, n) = (m-1, j-n)$ involves at least two negative y-admissible vectors which, as in former cases, need to be cancelled because each has twisted degree strictly less than ρ. Also the contribution of $x^{\omega_1}y^{\omega_2}$ is zero, because, being $j < \omega_2$, any decomposition of $p - \omega$ involves a negative y-admissible vector e and

$$\mathrm{td}(e) \leq \mathrm{td}(p - \omega) = \rho - \sigma < \rho,$$

which forces e to be cancelled. Any other monomial $A_{\alpha,\beta}x^\alpha y^\beta$ of f, other than the one corresponding to p, has $\mathrm{td}(p - (\alpha, \beta)) < \rho - \sigma$ which forces any vector in a decomposition of $p - (\alpha, \beta)$ to be cancelled, and hence the contribution of $A_{\alpha,\beta}x^\alpha y^\beta$ to be zero. Taking in account the contribution of the monomial of f corresponding to p itself, the coefficient of f^* corresponding to p results to be $A_p - ma_u$.

By 9.8 and the induction hypothesis, f^* and f have the same partial sum of twisted degree $nm+\rho-\sigma-1$, that is, $\mathrm{to}(f^* - f) \geq nm+\rho-\sigma$. Furthermore, for any point $p' = (\alpha', \beta')$ with $\alpha' \leq m-1$, $\beta' \leq n-1$ and $\mathrm{td}(p') < nm + \rho$, the monomials of f and f^* corresponding to p' are equal. Indeed, this follows from arguments similar to the ones used above for the point p, using that this time $\mathrm{td}(p' - (m, 0)) = \mathrm{td}(p' - (0, n)) < \rho$; the details are left to the reader.

Again, by 4.6 and the invariance of q_ρ, there is an equation $\hat{f} = (1+g)f^*$ of $\varphi^*(\gamma)$ for which $\mathrm{to}(\hat{f} - f) \geq nm + \rho$, $g \in \mathbb{C}\{x, y\}$, $g(0, 0) = 0$. This inequality together with the already obtained $\mathrm{to}(f^* - f) \geq nm + \rho - \sigma$, easily give $\mathrm{to}(g) \geq \rho - \sigma$.

We will check now that the monomial of gf^* corresponding to p is zero. Indeed, otherwise there is a non-zero monomial of f^*, say corresponding to a point $p' = (\alpha', \beta')$, for which the vector $p - p'$ is positive and has twisted degree

$$\mathrm{td}(p - p') \geq \mathrm{to}(g) \geq \rho - \sigma.$$

Such a p' needs thus to have $\mathrm{td}(p') \leq nm+\sigma$, and, furthermore, $\alpha' \leq m-1$, $\beta' \leq n-1$, as otherwise the vector $p - p'$ is negative. Since $nm + \sigma < nm + \rho$, the monomials of f and f^* corresponding to p' do agree, and so the former is not zero: this forces $p' = \omega$, but the vector $p - \omega$ being also negative, p' does not exist.

Now, the monomial corresponding to p of gf^* being zero, $\hat{f}$ has the same monomial corresponding to p as f^*, namely $A_p - ma_u$. Again, by 2.2, the monomial corresponding to p is the only one with twisted degree $nm + \rho$: taking it as a coordinate, the points on γ and $\varphi^*(\gamma)$ in the first neighbourhood of q_ρ have coordinates A_p and $A_p - ma_u$, respectively. The hypothesis of invariance of the point on γ in the first neighbourhood of q_ρ forces thus $a_u = 0$, as wanted.

Case 3: There is an y-admissible negative vector w with $\mathrm{td}(w) = \rho$. Follows from the same arguments used in Case 2 by switching over the roles of $(m, 0)$ and $(0, n)$. The details are left to the reader. ☐

11. Continuous invariants. We are now able to show the existence of a number of continuous invariants:

THEOREM 11.1. *Assume that γ is a non-quasihomogeneous irreducible germ with single characteristic exponent m/n, given by an equation*

$$f = y^n - x^m + \sum_{n\alpha+m\beta>nm} A_{\alpha,\beta}x^\alpha y^\beta,$$

and has Zariski point $\omega = (\omega_1, \omega_2)$. Let $\sigma = \mathrm{td}(\omega_1, \omega_2) - nm$ be the Zariski invariant of γ and take $\kappa = \min\{\mathrm{td}(m-1, \omega_2), \mathrm{td}(\omega_1, n-1)\} - nm$. Then all coefficients $A_{k,h}$ with $\sigma < \mathrm{td}(k,h) - nm < \kappa$ and either $k < \omega_1$ or $h < \omega_2$, are continuous invariants of γ.

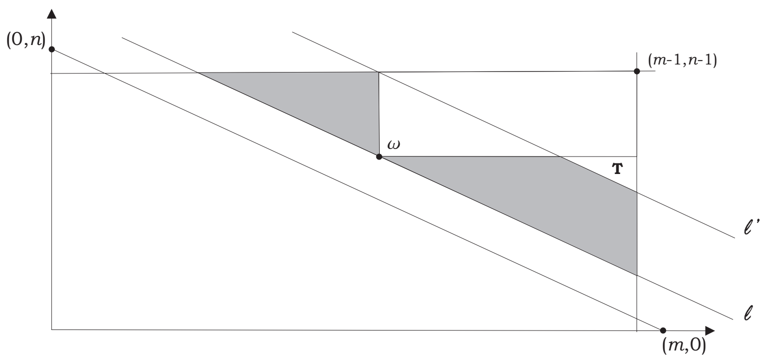

FIG. 4. *Theorem 11.1: the coefficients corresponding to points in the interior of the shaded area are continuous invariants. Using coordinates α, β, the lines ℓ and ℓ' are $n\alpha + m\beta = \sigma + nm$ and $n\alpha + m\beta = \kappa + nm$, respectively. The triangle* **T** *will appear in section 12.*

Proof. By 9.4 and 7.15, we may assume that

$$f = y^n - x^m + x^{\omega_1}y^{\omega_2} + \sum_{n\alpha+m\beta>nm+\sigma} A_{\alpha,\beta}x^\alpha y^\beta.$$

Fix $(k,h) = p$ satisfying the conditions of the claim and write $\rho = nk + mh - nm$: it is thus $\sigma < \rho < \kappa$. Assume that φ, given by equations

$$x^* = x + \sum_{(i,j)\in\mathbb{N}^2-\{(0,0),(1,0)\}} a_{i,j}x^i y^j,$$
$$y^* = y + \sum_{(i,j)\in\mathbb{N}^2-\{(0,0),(0,1)\}} b_{i,j}x^i y^j,$$

is a principal automorphism leaving invariant the point q_ρ in the ρ-th neighbourhood of the last satellite point on γ and take $f^* = f(x^*, y^*)$.

First we will show that no monomial of f other than $A_{k,h}x^k y^h$ gives a non-zero contribution to the monomial of f^* corresponding to p and therefore the later is just $A_{k,h}x^k y^h$. Indeed, there is no non-zero contribution of the monomials $-x^m$ or y^n because a decomposition in admissible vectors of either $p-(m,0)$ or $p-(0,n)$ involves at least two negative vectors, each of which needs, by 9.8, to have twisted degree strictly less than $\rho = \mathrm{td}(p-(m,0)) = \mathrm{td}(p-(0,n))$, and such a vector is cancelled by 10.1. Neither the monomial corresponding to ω has a non-zero contribution by a similar reason: $p-\omega$ having a negative component, any decomposition of it involves a negative vector e, as before $\mathrm{td}(e) \leq \mathrm{td}(p-\omega) = \rho - \sigma < \rho$ and again by 10.1, e is cancelled. To close, any other non-zero monomial of f corresponding to a point $\bar{p}$ preceding p has $\sigma + mn < \mathrm{td}(\bar{p}) < \rho + nm$, hence $\mathrm{td}(p-\bar{p}) < \rho - \sigma$, and any vector in a decomposition is cancelled, once again by 9.8 and 10.1.

Similarly, there is no non-zero contribution to the monomial of f^* corresponding to a standing point p', $\mathrm{td}(p') < nm + \rho$, other than the contribution of the monomial of f corresponding to p'. For, the point p being a standing point, a decomposition of $p' - (m, 0)$ involves a negative vector which is cancelled because its twisted degree is at most $\mathrm{td}(p' - (m, 0)) < \rho$, so there is no contribution of $-x^m$. The same argument works for y^n. Any other non-zero monomial of f corresponds to a point p'' with $\mathrm{td}(p' - p'') < \rho - \sigma$ which, as above, forces all vectors in a decomposition of $p' - p''$ to be cancelled.

As a consequence, f and f^* share the monomial corresponding to p, as well as all those corresponding to the standing points preceding p, and of course also $-x^m$ and y^n. By 4.5, there is an equation $\hat{f}$ of $\varphi^*(\gamma)$ such that the difference $\hat{f} - f$ has twisted order $nm+\rho$ or higher. The series $\hat{f}$ and f^* have thus the same standing monomials of twisted degree $nm+\rho-1$ or less. If, as in in former occasions, we write $\hat{f} = (1+g)f^*$, with $g(0,0) = 0$, gf^* has equal to zero all its standing monomials of twisted degree less than or equal to $nm+\rho-1$. Take $\delta = \mathrm{to}(g)$ and call g_δ the twisted initial form of g: its monomials have all twisted degree δ. Assume $\delta < \rho - \sigma$; then it is easy to check (using that $\rho < \kappa$) that all non-zero monomials of $x^{\omega_1} y^{\omega_2} g_\delta$ are standing monomials. Since there are no standing monomials in $-x^m g$ or $y^n g$, the non-zero monomials of $x^{\omega_1} y^{\omega_2} g_\delta$ does not cancel in gf^*, against the fact that they are standing monomials of twisted degree $\delta + \sigma + nm \leq nm + \rho - 1$. It is thus $\delta \geq \rho - \sigma$. This set, neither of the monomials $-x^m$, y^n or $x^{\omega_1} y^{\omega_2}$ gives rise to a non-zero monomial of gf^* corresponding to p, because neither of the vectors $p - (m, 0)$, $p - (0, n)$ and $p - \omega$ is positive, and neither does any other non-zero monomial M of f^*, because such a monomial has twisted degree strictly higher than $nm + \sigma$ and hence $\mathrm{to}(gM) > nm + \rho = \mathrm{td}(p)$. All together, the monomial of gf^* corresponding to p is zero and therefore those of f and $\hat{f}$ do agree.

We have seen above that the equations f, of γ, and $\hat{f}$, of $\varphi^*(\gamma)$, have coincident partial sums of degree $\rho + mn$. By 4.3, γ and $\varphi^*(\gamma)$ have the same point q in the first neighbourhood of q_ρ and therefore q is invariant by φ. Since the arguments used so far hold for arbitrary values of the coefficient $A_{k,h}$, again by 4.3 the invariance of q holds for any free point q in the first neighbourhood of q_ρ and the proof is complete. □

12. Conditional invariants. If still γ is a non-quasihomogeneous irreducible germ with single characteristic exponent m/n and Zariski point $\omega = (\omega_1, \omega_2)$, take, as in 11.1, $\kappa = \min\{\mathrm{td}(m-1, \omega_2), \mathrm{td}(\omega_1, n-1)\} - nm$. We have so far examined the relevance of all coefficients A_p of an equation of γ with $\mathrm{td}(p) < \kappa + mn$. Also the coefficients $A_{\alpha,\beta}$ with either $\alpha \geq m-1$, or $\beta \geq n-1$, or $\alpha \geq \omega_1$ and $\beta \geq \omega_2$, have been seen to be irrelevant in 6.1 and 8.1. Yet, it remains undecided the relevance of the coefficients corresponding to the points in the interior of a triangle, which is either

$$\mathbf{T} = \{(\alpha, \beta) \mid n\alpha + m\beta \geq \kappa + mn, \alpha \leq m-1, \beta \leq \omega_2\},$$

if $n\omega_1 + m(n-1) < n(m-1) + \omega_2$, or

$$\mathbf{T} = \{(\alpha, \beta) \mid n\alpha + m\beta \geq \kappa + mn, \alpha \leq \omega_1, \beta \leq n-1\}$$

otherwise. Figure 4 shows the first case. Note that in no case there are integral points on the hypothenuse of $\mathbf{T}$ by 2.2. The relevance of the coefficients A_p with p in the interior of $\mathbf{T}$ is not so easy to describe, because, as already mentioned in the introduction, each of them may or may not be a continuous invariant, depending on the former continuous invariants of the germ. We call them *conditional invariants.*

All coefficients A_p with $\mathrm{td}(p) \geq 2nm - n - m + 1 = \mathrm{td}((m-1, n-1))$ being irrelevant, in this section we will give a description which applies to all coefficients A_p with $\mathrm{td}(p) < 2nm - n - m + 1$. It applies thus to the conditional invariants, namely the A_p with p in the interior of $\mathbf{T}$, and also to other coefficients, but for the latter more precise information has been obtained before.

First of all, we need to set the following:

LEMMA 12.1. *Fix an integer s, $0 < s < nm - n - m + 1$. Then there are polynomials $U_{i,j}(Z_{\alpha,\beta}, T_{\alpha,\beta})$, $(i,j) \in \mathbb{N}^2$, $0 \leq ni + mj < s$, with coefficients in $\mathbb{Z}$ and variables $Z_{\alpha,\beta}, T_{\alpha,\beta}$, $(\alpha, \beta) \in \mathbb{N}^2$, $n\alpha + m\beta \leq nm + s - 1$, such that for any two series $f, f' \in \mathbb{C}\{x, y\}$, of the form*

$$f = y^n - x^m + \sum_{n\alpha+m\beta>nm} A_{\alpha,\beta} x^\alpha y^\beta, \quad f' = y^n - x^m + \sum_{n\alpha+m\beta>nm} B_{\alpha,\beta} x^\alpha y^\beta,$$

if there is a series $u = \sum_{i,j} u_{i,j} x^i y^j \in \mathbb{C}\{x, y\}$ for which uf' and f have equal partial sums of twisted degree $nm + s - 1$, then it holds

$$u_{i,j} = U_{i,j}(A_{\alpha,\beta}, B_{\alpha,\beta})$$

for all i, j, $(i, j) \in \mathbb{N}^2$, $0 \leq ni + mj < s$.

Proof. The case $s = 1$ is obviously satisfied by taking $U_{0,0} = 1$. So, by induction, we will assume $s > 1$ and the polynomials $U_{i,j}$ determined for $ni + mj < s - 1$. Being $s < nm - n - m + 1$, by 2.2 there is at most a single point with integral non-negative coordinates and twisted degree $s - 1$. If such a point does not exist, then no polynomial $U_{i,j}$ of twisted degree $s - 1$ is required. Otherwise, assume that point to be (k, h). By equating the monomials corresponding to $(m + k, h)$ in f and uf', it results

$$A_{m+k,h} = -u_{k,h} + \sum_{\substack{0\leq i\leq k\\ 0\leq j\leq h\\ (i,j)\neq(k,h)}} u_{i,j} B_{m+k-i,h-j},$$

After which it suffices to take

$$U_{k,h} = -Z_{m+k,h} + \sum_{\substack{0\leq i\leq k\\ 0\leq j\leq h\\ (i,j)\neq(k,h)}} U_{i,j} T_{m+k-i,h-j},$$

the $U_{i,j}$ in the summation being determined by the induction hypothesis. ☐

PROPOSITION 12.2. *Assume fixed an integer s, $0 < s < nm - n - m$. There exists a polynomial $\Psi_s \in \mathbb{Z}[X_v, Y_w, Z_{\alpha,\beta}]$, in the variables X_v, $v \in \mathbf{A}_s$, Y_w, $w \in \mathbf{B}_s$ and $Z_{\alpha,\beta}$, $(\alpha, \beta) \in \mathbb{N}^2$, $nm < \mathrm{td}(\alpha, \beta) < nm + s$, such that:*

(a) *For any irreducible germ of curve γ with equation*

$$f = y^n - x^m + \sum_{n\alpha+m\beta>nm} A_{\alpha,\beta} x^\alpha y^\beta,$$

and any principal automorphism φ at the origin of γ, with equations

$$x^* = x\left(1 + \sum_{v\in\mathbf{A}} a_v (xy)^v\right), \quad y^* = y\left(1 + \sum_{w\in\mathbf{B}} b_w (xy)^w\right),$$

leaving invariant the point q_s on γ in the s-th neighbourhood of the last satellite, the action of φ in the first neighbourhood of q_s is the translation

$$\theta \longmapsto \theta + \Psi_s(a_v, b_w, A_{\alpha,\beta}),$$

where θ is a certain affine coordinate in the first neighbourhood of q_s.

(b) *If there is $v \in \mathbf{A}$ (resp. $v \in \mathbf{B}$) with $\mathrm{td}(v) = s$, then Ψ_s has a monomial cX_v (resp. cY_v), $c \in \mathbb{Z} - \{0\}$, and no other monomial of Ψ_s involves X_v (resp. Y_v).*

Proof. Take any $(\alpha', \beta') \in \mathbb{N}^2$ with $nm < \mathrm{td}(\alpha', \beta') \le nm + s$ and

$$P_{\alpha',\beta'} = \sum_{nm\le \mathrm{td}(\alpha,\beta)<\mathrm{td}(\alpha',\beta')} Z_{\alpha,\beta}\left(\sum \binom{\alpha}{r}\binom{\beta}{h} X_{v_1}\dots X_{v_r} Y_{w_1}\dots Y_{w_h}\right),$$

where the innermost summation runs on all ordered sets $v_1, \dots, v_r$ of x-admissible vectors and all ordered sets $w_1, \dots, w_h$ of y-admissible vectors with $0 \le r \le \alpha$, $0 \le h \le \beta$ and $v_1 + \cdots + v_r + w_1 + \cdots + w_h = (\alpha', \beta') - (\alpha, \beta)$. Take any germ γ and any local automorphism φ as described in the claim. According to 9.5, the (α', β')-coefficient of the equation $f^* = f(x^*, y^*)$ of $\varphi^*(\gamma)$ is

$$A^*_{\alpha',\beta'} = A_{\alpha',\beta'} + P_{\alpha',\beta'}(a_v, b_w, A_{\alpha,\beta}).$$

We know from 4.5 that there is an invertible series $u = \sum_{i,j\ge 0} u_{i,j}x^iy^j$ such that f and uf^* have he same partial sum of degree $nm + s - 1$, which in particular forces $u_{0,0} = 1$. Then the (α', β')-coefficient of uf^* is

$$\begin{aligned}\bar{A}_{\alpha',\beta'} &= \sum_{\substack{0\le i\le \alpha'\\ 0\le j\le\beta'\\ (i,j)\ne(0,0)}} u_{i,j}A^*_{\alpha'-i,\beta'-j} + A^*_{\alpha',\beta'}\\ &= \sum_{\substack{0\le i\le \alpha'\\ 0\le j\le\beta'\\ (i,j)\ne(0,0)}} u_{i,j}\left(P_{\alpha'-i,\beta'-j}(a_u, b_v, A_{\alpha,\beta}) + A_{\alpha'-i,\beta'-j}\right)\\ &\qquad + P_{\alpha',\beta'}(a_v, b_w, A_{\alpha,\beta}) + A_{\alpha',\beta'}.\end{aligned}$$

Therefore, using 12.1 and the polynomials $U_{i,j}$ therein, it suffices to take

$$Q_{\alpha',\beta'} = \sum_{\substack{0\le i\le \alpha'\\ 0\le j\le\beta'\\ (i,j)\ne(0,0)}} \bar{U}_{i,j}\cdot(P_{\alpha'-i,\beta'-j} + Z_{\alpha'-i,\beta'-j}) + P_{\alpha',\beta'},$$

where

$$\bar{U}_{i,j} = U_{i,j}(Z_{\alpha,\beta}, Z_{\alpha,\beta} + P_{\alpha,\beta}),$$

in order to have

$$\bar{A}_{\alpha',\beta'} = Q_{\alpha',\beta'}(a_u, b_v, A_{\alpha,\beta}) + A_{\alpha',\beta'}. \tag{14}$$

Now, by 2.2, there are either one or two points in $\mathbb{N}^2$ with twisted degree $d = nm + s$. Assume to be in the first case and let (α', β') to be the only point with twisted degree d. The coefficients $A_{\alpha,\beta}$, $nm < n\alpha + m\beta < d$, being fixed, we take the coefficient corresponding to (α', β') as coordinate in the first neighbourhood of q_s. Then, since f and uf^* have the same partial sum of twisted degree $d-1$ and a single coefficient of twisted degree d, for any value of $A_{\alpha',\beta'}$, the points on γ and $\varphi^*(\gamma)$ in the first neighbourhood of q_s have coordinates, respectively, $A_{\alpha',\beta'}$ and $\bar{A}_{\alpha',\beta'}$, the latter given by (14) above. Condition (a) is thus satisfied by taking $\Psi_s = Q_{\alpha',\beta'}$.

Assume now to have two points with twisted degree d. Let (α', β') be the one with minimal second coordinate, after which the other is $(\alpha'', \beta'') = (\alpha' - m, \beta' + n)$. It is direct to check that

$$\left(1 - (\bar{A}_{\alpha'',\beta''} - A_{\alpha'',\beta''})x^{\alpha''}y^{\beta''-n}\right) uf^*$$

has the same partial sum of twisted degree $d-1$ as f and uf^*, the coefficient corresponding to (α'', β'') equal to $A_{\alpha'',\beta''}$ and the coefficient corresponding to (α', β') equal to $\bar{A}_{\alpha'',\beta''} - A_{\alpha'',\beta''} + \bar{A}_{\alpha',\beta'}$. Using the same arguments as above and the equality (14) for both $\bar{A}_{\alpha'',\beta''}$ and $\bar{A}_{\alpha',\beta'}$, shows that $\Psi_s = Q_{\alpha'',\beta''} + Q_{\alpha',\beta'}$ fulfills condition (a).

Regarding condition (b), it is clear from its definition that the variables X_v Y_v, $\mathrm{td}(v) = s$ do not appear in $P_{\alpha',\beta'}$ if $n\alpha' + m\beta' - nm < s$, while $P_{\alpha',\beta'}$ satisfies the condition claimed for Ψ_s in (b) in case of being $n\alpha' + m\alpha' - nm = s$. After this, a direct checking shows that Ψ_s itself satisfies condition (b). ☐

REMARK 12.3. It follows from 12.2 that the action of φ in the first neighbourhood of q_s is independent of the coefficients a_v, b_v, $\mathrm{td}(v) > s$, of the equations of φ.

In order to introduce a twisted version of the jets of principal automorphisms, for any non-negative integer s we will consider the set $\mathcal{H}_s$ of all principal automorphisms ψ given by equations

$$x^* = x\left(1 + \sum_{v\in\mathbf{A}} a_v(xy)^v\right), \quad y^* = y\left(1 + \sum_{w\in\mathbf{B}} b_w(xy)^w\right),$$

in which $a_v = b_w = 0$ for all v, w with $\mathrm{td}(v) \leq s$, $\mathrm{td}(w) \leq s$. All admissible vectors having positive twisted degree, $\mathcal{H}_0 = \mathcal{PA}$

REMARK 12.4. As it follows from the definition above, $\psi \in \mathcal{H}_s$ if and only if

$$\mathrm{to}(x \circ \psi - x) > n + s \quad \text{and} \quad \mathrm{to}(y \circ \psi - y) > m + s.$$

If this is the case, a direct computation shows that

$$\mathrm{to}(g \circ \psi - g) > \mathrm{to}(g) + s$$

for any $g \in \mathbb{C}\{x, y\}$.

REMARK 12.5. Still let γ be as in 12.2 and $s > 0$. Since the identical automorphism leaves invariant the point on γ in the s-th neighbourhood of the last satellite and leaves also invariant all points in its first neighbourhood, so does any principal automorphism $\psi \in \mathcal{H}_s$, due to 12.3 used inductively on s.

As before, denote by $\mathcal{PA}$ the group of all principal automorphisms at O. We have:

LEMMA 12.6. *$\mathcal{H}_s$ is a normal subgroup of $\mathcal{PA}$.*

Proof. Note first that for any principal automorphism φ and any $g \in \mathbb{C}\{x, y\}$, $\mathrm{to}(g \circ \varphi) = \mathrm{to}(g)$. Then, for any $\psi, \psi_1, \psi_2 \in H_s$, any $\varphi \in \mathcal{PA}$ and any $g \in \mathbb{C}\{x, y\}$,

$$\begin{aligned}\mathrm{to}(g \circ \psi_1 \circ \psi_2^{-1} - g)) &= \mathrm{to}((g \circ \psi_1 - g \circ \psi_2) \circ \psi_2^{-1}) \\ &= \mathrm{to}(g \circ \psi_1 - g + g - g \circ \psi_2) > \mathrm{to}(g) + s\end{aligned}$$

and

$$\begin{aligned}\mathrm{to}(g \circ \varphi^{-1} \circ \psi \circ \varphi - g) &= \mathrm{to}((g \circ \varphi^{-1} \circ \psi - g \circ \varphi^{-1}) \circ \varphi) \\ &= \mathrm{to}(g \circ \varphi^{-1} \circ \psi - g \circ \varphi^{-1}) > \mathrm{to}(g) + s\end{aligned}$$

after which the claim follows from 12.4 after taking $g = x, y$. □

We will write $\mathcal{B}_s$ the quotient group $\mathcal{B}_s = \mathcal{PA}/\mathcal{H}_s$. The class in $\mathcal{B}_s$ of any $\varphi \in \mathcal{PA}$ will be denoted $[\varphi]_s$ and called the *twisted s-jet* of φ. We have:

LEMMA 12.7. *If principal automorphisms φ and $\bar{\varphi}$ are given, respectively, by equations*

$$x^* = x\left(1 + \sum_{v \in \mathbf{A}} a_v (xy)^v\right), \quad y^* = y\left(1 + \sum_{w \in \mathbf{B}} b_w (xy)^w\right) \tag{15}$$

and

$$x^* = x\left(1 + \sum_{v \in \mathbf{A}} \bar{a}_v (xy)^v\right), \quad y^* = y\left(1 + \sum_{w \in \mathbf{B}} \bar{b}_w (xy)^w\right),$$

then $[\varphi]_s = [\bar{\varphi}]_s$ if and only if $a_v = \bar{a}_v$ and $b_v = \bar{b}_v$ for $\mathrm{td}(v) \leq s$.

Proof. For any $g \in \mathbb{C}\{x, y\}$, it holds

$$\mathrm{to}(g \circ \varphi \circ \bar{\varphi}^{-1} - g) = \mathrm{to}(g \circ \varphi - g \circ \bar{\varphi})$$

and so one of the sides is strictly higher than $\mathrm{to}(g) + s$ if and only if so is the other. By taking $g = x, y$ the claim follows using 12.4. □

After 12.7, $\mathcal{B}_s$ may be identified to an affine space with coordinates X_v, $v \in \mathbf{A}_s$ and Y_w, $w \in \mathbf{B}_s$, the class of a principal automorphism φ being identified to the point whose coordinates are the corresponding coefficients of the equations of φ written as in (15) above. In this way $\mathcal{B}_s$ becomes an algebraic group and the natural morphism between quotient groups $\varpi_s : \mathcal{B}_s \to \mathcal{B}_{s-1}$, $s > 1$, appears as the projection forgetting the coordinates X_v, Y_v with $\mathrm{td}(v) = s$. Note that ϖ_s is the equality when there are no admissible vectors of twisted degree s. The dimension of $\mathcal{B}_s$ is not easy to compute, see for instance [1], 2.7.

Still assume γ to be the germ of curve at O defined by

$$f = y^n - x^m + \sum_{n\alpha + m\beta > nm} A_{\alpha,\beta} x^\alpha y^\beta, \quad (n, m) = (1), \tag{16}$$

and, as before, for $s > 0$, call q_s the point on γ in the s-th neighbourhood of the last satellite. We will denote by $E_s = E_{q_s}$ the first neighbourhood of q_s. By 3.8, the principal automorphisms leaving q_s invariant describe a subgroup $\mathcal{PA}_s$ of $\mathcal{PA}$ and $\mathcal{PA}_s$ contains $\mathcal{H}_{s-1}$ due to 12.5. The quotient group $\mathcal{W}_s = \mathcal{PA}_s/\mathcal{H}_{s-1}$ is of course a subgroup of $\mathcal{B}_{s-1}$ and $\varpi_{s-1}(\mathcal{W}_s) \subset \mathcal{W}_{s-1}$ if $s > 1$. We will take

$$\overline{\mathcal{W}}_s = \mathcal{PA}_s/\mathcal{H}_s = \varpi_s^{-1}(\mathcal{W}_s),$$

which is a subgroup of $\mathcal{B}_s$.

By 12.5, the action on E_s of any $\varphi \in \mathcal{PA}_s$ depends only on its class $[\varphi]_s \in \overline{\mathcal{W}}_s$, it will be also referred to as the action of $[\varphi]_s$. By 3.8, mapping $[\varphi]_s$ to its action on E_s is a group homomorphism $\delta_s : \overline{\mathcal{W}}_s \to \mathrm{Aff}(E_s)$.

REMARK 12.8. Assume that $\varphi \in \mathcal{PA}_s$ is defined by the equations (15) above; then, by 12.2, $\delta_s([\varphi]_s)$ is the translation

$$\theta \longmapsto \theta + \Psi_s(a_v, b_w, A_{\alpha,\beta}).$$

In particular δ_s is a morphism of algebraic groups and the following three conditions are equivalent:

(i) a free point $q \in E_s$ is invariant by φ,
(ii) all points in E_s are invariant by φ,
(iii) $\Psi_s(a_v, b_w, A_{\alpha,\beta}) = 0$.

PROPOSITION 12.9. *For $s \geq 1$, $\mathcal{W}_s$ (resp. $\overline{\mathcal{W}}_s$) is the locus of zeros in $\mathcal{B}_{s-1}$ (resp. $\mathcal{B}_s$) of the polynomials*

$$\overline{\Psi}_j(X_v, Y_w) = \Psi_j(X_v, Y_w, A_{\alpha,\beta}), \quad j = 1, \ldots, s-1,$$

where the Ψ_j are the polynomials of 12.2.

Proof. Clearly, it suffices to prove the claim relative to $\mathcal{W}_s$. For $s = 2$ the proof is direct from 12.8; after this, inductive use of the same argument gives the general case. □

REMARK 12.10. By 12.9, $\mathcal{W}_s$ and $\overline{\mathcal{W}}_s$ are (Zariski) closed subgroups of $\mathcal{B}_{s-1}$ and $\mathcal{B}_s$, respectively. In particular they are pure-dimensional affine varieties and, as a such, $\overline{\mathcal{W}}_s \simeq \mathcal{W}_s \times \mathbb{C}^{\rho_s}$, $\rho_s = \dim \mathcal{B}_s - \dim \mathcal{B}_{s-1}$, because they are defined by the same equations.

REMARK 12.11. The image $\delta_s(\overline{\mathcal{W}}_s)$ being a closed subgroup of the group of translations of E_s, it is either $\{Id_{E_s}\}$ or the entire group of translations. In particular, as advanced in 9.3, the action of the elements of $\mathcal{PA}_s$ on the set of free points of E_s either is transitive, or leaves all points invariant.

Write m/n as an odd-order continued fraction $m/n = [h, h_1, \ldots, h_{2r+1}]$ and take m'/n' to be its last reduced fraction, namely $m'/n' = [h, h_1, \ldots, h_{2r}]$, with m', n' positive and coprime. Then, by the properties of the continued fractions, the vector $e = (-m', n')$ has twisted degree one. We will use e to define a finite sequence $\Sigma = \{p_s\}$, $s \in \mathbb{Z}$, $0 \leq s \leq nm-n-m+1$ of points of the Newton plane with $\mathrm{td}(p_s) = nm+s$: they are thus one for each twisted degree in the interval $[nm, 2nm - n - m]$. First we take $p_0 = (m, 0)$ and $p_s = p_{s-1} + e$ as far as the ordinate of p_s remains strictly less than n (which in this case occurs for $s < n/n'$). If p_{s_1} is the last of these points, then we take $p_{s_1+1} = p_{s_1} + e + (m, -n)$ and repeat the procedure using p_{s_1+1} in the place of p_0, so again taking $p_s = p_{s-1} + e$ for $s > s_1$ as far as the ordinate remains strictly less than n; then, if the last of these points is p_{s_2}, we take $p_{s_2+1} = p_{s_2} + e + (m, -n)$, and so on. Part of the sequence Σ is represented in Figure 6 below for $m/n = 13/6$. It is direct to check that all points in Σ have non-negative integral coordinates and $\mathrm{td}(p_s) = nm + s$ for all s. We stop the sequence at the point $p_{nm-m-n+1}$: since $(n-1, m-1)$ is the only point in $\mathbb{N}^2$ with twisted degree $2nm - n - m + 1$ (by 2.2), $p_{nm-m-n+1} = (m-1, n-1)$.

Note that all coefficients A_p whose relevance has not yet been decided, namely those with p in the interior of $\mathbf{T}$, have $p \in \Sigma$. The next theorem partially describes the groups $\mathcal{B}_s$ and $\mathcal{W}_s$ for the points q_s on γ, $0 < s \leq nm - n - m$, as well as the action of the elements of $\mathcal{PA}_s$ on the first neighbourhood of q_s.

The non-standing points (α, β) with either $\alpha = m-1$ and $\beta < n-1$ or $\alpha < m-1$ and $\beta = n-1$ will be called in the sequel *border points*, while the other non-standing points will be called *external points*.

THEOREM 12.12. *For any integer s, $0 < s < nm - n - m$:*

(a) *If $p_s \in \Sigma$ is an external point, then $\dim \mathcal{B}_s = \dim \mathcal{B}_{s-1} + 2$, $\dim \mathcal{W}_{s+1} = \dim \mathcal{W}_s + 1$ and $\mathcal{PA}_s$ acts transitively on the set of free points of E_s.*

(b) *If $p_s \in \Sigma$ is a border point, then $\dim \mathcal{B}_s = \dim \mathcal{B}_{s-1} + 1$, $\dim \mathcal{W}_{s+1} = \dim \mathcal{W}_s$ and $\mathcal{PA}_s$ acts transitively on the set of free points of E_s.*

(c) *If $p_s \in \Sigma$ is a standing point, then $\mathcal{B}_s = \mathcal{B}_{s-1}$ (and so $\overline{\mathcal{W}}_s = \mathcal{W}_s$) and either*

(c.1) *the restriction of $\overline{\Psi}_s$ to $\mathcal{W}_s$ is not constant, $\dim \mathcal{W}_{s+1} = \dim \mathcal{W}_s - 1$ and $\mathcal{PA}_s$ acts transitively on the set of free points of E_s, or else*

(c.2) *$\overline{\Psi}_s$ is zero on $\mathcal{W}_s$, $\mathcal{W}_{s+1} = \mathcal{W}_s$ and the elements of $\mathcal{PA}_s$ leave fixed all points of E_s.*

Proof. In case (a), the vector $v = p_s - (m, 0)$ is both x- and y-admissible. By 2.2, no admissible vector other than v has twisted degree s, after which $\mathcal{B}_s = \mathcal{B}_{s-1} \times \mathbb{C}^2$, where ϖ_s is the first projection and X_v, Y_v are coordinates on the second factor. In particular, $\dim \mathcal{B}_s = \dim \mathcal{B}_{s-1} + 2$. Then $\overline{\mathcal{W}}_s = \mathcal{W}_s \times \mathbb{C}^2$, and by 12.9, $\mathcal{W}_{s+1}$ is the locus of zeros of $\overline{\Psi}_s$ on $\overline{\mathcal{W}}_s$. Since both X_v and Y_v effectively appear in $\overline{\Psi}_s$, each in a single monomial which has degree one (12.2,(b)), on one hand the equality $\dim \mathcal{W}_{s+1} = \dim \mathcal{W}_s + 1$ is clear, while on the other $\overline{\Psi}_s$ is not constant on $\overline{\mathcal{W}}_s$, after which the transitivity follows from 12.8 and 12.11.

In case (b), either $p_s = (m-1, r)$, $0 < r < n-1$, or $p_s = (\ell, n-1)$, $0 < \ell < m-1$. In the first case $v = p - (m, 0)$ is x-admissible, but not y-admissible, while in the other $w = p - (0, n)$ is y-admissible, but not x-admissible. Again by 2.2, in no case there is another admissible vector of twisted degree s, so just one new variable, either X_v or Y_w, appears; after this, arguments similar to the ones used in case (a) give the proof.

In case (c), once again by 2.2, there is no admissible vector of twisted degree s. As a consequence, $\mathcal{B}_s = \mathcal{B}_{s-1}$ and so $\overline{\mathcal{W}}_s = \mathcal{W}_s$. Then $\mathcal{W}_{s+1}$ is the locus of zeros of $\overline{\Psi}_s$ in $\mathcal{W}_s$ and the claim follows arguing as in case (a). □

REMARK 12.13. By 5.8, in cases (a), (b) and (c.1) of 12.12 the coefficients of twisted degree $nm + s$ of γ are irrelevant (which, for cases (a) and (b), was already seen in 6.1). By 9.3, in case (c.2) the coefficient of twisted degree $nm + s$ of γ is a continuous invariant provided $s > \sigma$.

EXAMPLE 12.14. (Zariski, [23], V.5) Assume the germ γ above to have $m/n = 7/6$ and minimal Zariski invariant $\sigma = 2$. Then (see Figure 5) the Zariski point is

$\omega = (5, 2)$, the coefficients corresponding to the points $(4, 3)$ and $(3, 4)$ are continuous invariants by 11.1, those corresponding to (5.4) and (5.5) are irrelevant by 8.1 and the only point in the interior of the triangle $\mathbf{T}$ is $(4, 4)$.

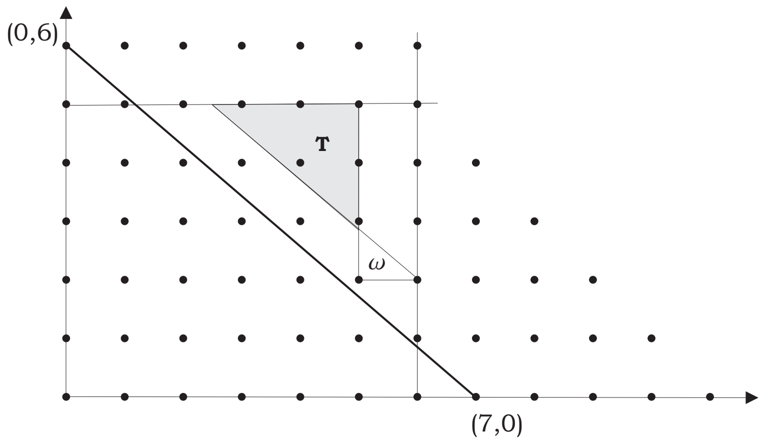

FIG. 5. *The case of Example 12.14: $m/n = 7/6$ and minimal Zariski invariant.*

We will see that the coefficient $A_{4,4}$ may or may not be a continous invariant depending on the values of $A_{4,3}$ and $A_{3,4}$. By 6.1 and 8.1, up to replacing γ with an analytically equivalent germ there is no restriction in assuming that the equation of γ is

$$y^6 - x^7 + x^5y^2 + A_{4,3}x^4y^3 + A_{3,4}x^3y^4 + A_{4,4}x^4y^4 + \dots,$$

where, if wanted, even the dots may be assumed to be zero (6.3). We will use the ordinary numerical indexing –not the vectorial one as before– for the coefficients of a principal automorphism φ, namely we will write $a_{i,j} = a_v$ and $b_{i,j} = b_w$ for $v = (i-1, j)$ and $w = (i, j-1)$. Note that $b_{1,0}$ is not allowed as a coefficient, by 7.6. By a direct computation, the conditions $\overline{\Psi}_s(a_{i,j}, b_{h,\ell}) = 0$, to be successively imposed for the invariance of the points on γ, are:

- q_1: none.
- q_2: $a_{0,1} = 0$.
- q_3, q_4, q_5: none, according to the relevance of $A_{5,2}, A_{4,3}, A_{3,4}$.
- q_6: $b_{2,0} = 0$.
- q_7: $7a_{2,0} - 6b_{1,1} = 0$, used in the sequel to eliminate $b_{1,1}$.
- q_8: $7a_{1,1} - 6b_{0,2} = 0$, used in the sequel to eliminate $b_{0,2}$.
- q_9:$-a_{2,0} + 21a_{0,2} = 0$.
- q_{10}: $3A_{4,3}a_{2,0} + 2a_{1,1} = 0$.

The condition for the invariance of all points in the first neighbourhood of q_{10} is then $4A_{3,4}a_{2,0} + 3A_{4,3}a_{1,1} + 30a_{0,2} = 0$, which is a consequence of the preceding ones if and only if

$$\begin{vmatrix} -1 & 0 & 21 \\ 3A_{4,3} & 2 & 0 \\ 4A_{3,4} & 3A_{4,3} & 30 \end{vmatrix} = 3(63A_{4,3}^2 - 56A_{3,4} - 20) = 0.$$

The above is thus the necessary and sufficient condition for the coefficient $A_{4,4}$ of γ to be relevant, and hence a continuous invariant (12.12, (c.2)). Using 4.8, the condition may be written in terms of Puiseux coefficients, and in this form it is Zariski's condition $\Delta = 0$ in [23], 5.3.

REMARK 12.15. The proof of 12.12 gives some further precisions: in case (a) $\mathcal{W}_{s+1}$ is isomorphic to $\mathcal{W}_s \times \mathbb{C}$ as varieties over $\mathcal{W}_s$, while in case (b), ϖ_{s+1} induces an isomorphism between $\mathcal{W}_{s+1}$ and $\mathcal{W}_s$; in all cases but (c.2), $\delta_s(\overline{\mathcal{W}}_s)$ is the group of translations of E_s.

Let us write ε_s for the number of external points $p_t \in \Sigma$, $0 < t < s$, and τ_s for the number of standing points $p_t \in \Sigma$, $0 < t < s$, for which A_{p_t} is irrelevant. Since obviously $\mathcal{B}_0 = \mathcal{W}_1$ is a single point, by just adding-up from 12.12 it results:

COROLLARY 12.16. *For any integer s, $0 < s \leq nm - n - m$,*

$$\dim \mathcal{W}_s = \varepsilon_s - \tau_s.$$

A constraint on the number of points corresponding to irrelevant coefficients in a sequence of consecutive standing points $p_s \in \Sigma$ follows from 12.16:

COROLLARY 12.17. *Assume that $p_s, \dots, p_{s'} \in \Sigma$ are consecutive standing points. Then the number of them that correspond to irrelevant coefficients is less than or equal to* $\dim \mathcal{W}_s = \varepsilon_s - \tau_s$.

Proof. Just write, using 12.16,

$$0 \leq \dim \mathcal{W}_{s'+1} = \dim \mathcal{W}_s + \varepsilon_{s'+1} - \varepsilon_s - \tau_{s'+1} + \tau_s$$

and note that $\varepsilon_{s'+1} - \varepsilon_s = 0$ because $p_s, \dots, p_{s'}$ are standing points, while $\tau_{s'+1} - \tau_s$ is the number that appears in the claim. □

Here is a case in which the constraint above is not obvious:

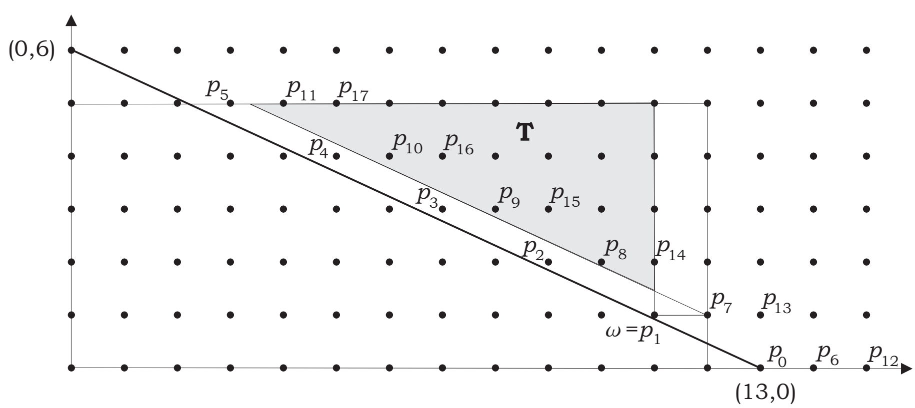

FIG. 6. *The case of Example 12.18: $m/n = 13/6$ and minimal Zariski invariant. Part of the sequence Σ is shown.*

EXAMPLE 12.18. Let γ be any irreducible germ with single characteristic exponent $13/6$ and minimal Zariski invariant $\sigma = 1$, see Figure 6. The points p_1 to p_4 correspond to relevant coefficients: the first one is the Zariski point, while the other correspond to continuous invariants due to 11.1. The points p_5, p_6 and p_7

being, respectively, a border point, an external point and a border point, by 12.16 it is $\dim \mathcal{W}_8 = 1$ and then, by 12.17, at least two of the standing points p_8, p_9, p_{10} correspond to continuous invariants. In case of these being exactly two, by a similar argument, at least one of the points p_{15}, p_{16} corresponds to a continuous invariant.

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