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# DEFORMATIONS OF CR MAPS AND APPLICATIONS*

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**Abstract.** We study the deformation theory of CR maps in the positive codimensional case. In particular, we study structural properties of the *mapping locus* $E$ of (germs of nondegenerate) holomorphic maps $H\colon (M,p) \to M'$ between generic real submanifolds $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$, defined to be the set of points $p' \in M'$ which admit such a map with $H(p) = p'$. We show that this set $E$ is semi-analytic and provide examples for which $E$ possesses (prescribed) singularities.

**Key words.** CR maps, deformations of CR manifolds, mapping locus, jet parametrization property, semi-analytic sets.

**Mathematics Subject Classification.** 32H02, 32V40.

**1. Introduction.** A classical counting argument due to Poincaré [22] shows that for "most" germs of real-analytic submanifolds $(M,p) \subset (\mathbb{C}^N,p)$ and $(M',p') \subset (\mathbb{C}^N,p)$, it is impossible to move one into the other by means of a germ of a holomorphic map $H\colon (\mathbb{C}^N,p) \to (\mathbb{C}^N,p)$. This rigidity phenomenon is by now well understood in the context of real submanifolds of the same dimension by means of *jet parametrization results*; we mention here e.g. the paper [15] for a thorough discussion of those.

An extension of the techniques developed in that paper allowed us to study in [8] the *equivalence locus* $E(p)$ of $p \in M$, which is defined as the set of points $q \in M$ for which there exists a biholomorphism taking $(M,p)$ into $(M,q)$ and show that this set is locally a real-analytic submanifold of $M$. In the present paper, we are interested in studying the positive codimensional case, and extending results obtained in [9, 10]. Observe the following general rigidity phenomena for mappings $H\colon (\mathbb{C}^N,p) \to (\mathbb{C}^{N'},p')$ where $N' > N$: Given a germ of a real-analytic submanifold $(M,p) \subset (\mathbb{C}^N,p)$ and a subvariety $M' \subset \mathbb{C}^{N'}$, only few points $p' \in M'$ allow for a (nondegenerate) holomorphic map $H\colon (\mathbb{C}^N,p) \to (\mathbb{C}^{N'},p')$ satisfying $H(M) \subset M'$. The general *mapping locus* is defined as

$$E = \left\{ p' \in M' \colon \exists H\colon (\mathbb{C}^N,p) \to (\mathbb{C}^{N'},p') \text{ holomorphic}, H(M) \subset M', H \not\equiv p' \right\};$$

for our results, we shall have to restrict ourselves to the mapping locus defined for classes of maps for which we have so-called jet parametrization results. We will see that in the positive codimensional setting, these mapping loci can only be shown to be *semi-analytic* and that there exist examples for which we actually obtain a mapping locus with singularities.

Before we proceed, we need to introduce some definitions. We denote the ideal associated to $M'$ at the point $p'$, consisting of all germs of real analytic functions vanishing on $M'$, by $\mathcal{I}_{p'}(M') \subset \mathbb{C}\{w - p', \overline{w - p'}\}$, and the set of all germs of real-analytic CR vector fields tangent to $M$ near $p$ by $\Gamma_p(M)$. We say that such a map $H$

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is $\ell$-*finitely nondegenerate* at $p$ if

$$\dim_{\mathbb{C}}\left\{\bar{L}_1\cdots\bar{L}_k\varrho'_w(H(z),\overline{H(z)})|_{z=p}\colon \bar{L}_j\in\Gamma_p(M), k\leq\ell,\ \varrho'\in\mathcal{I}_{H(p)}(M')\right\}=N'.$$

Given $M$ and $M'$ as above, and $\ell\in\mathbb{N}$, we define the ($\ell$-finitely nondegenerate) *mapping locus* $E_\ell\subset E\subset M'$ consisting of all points $p'\in M'$ with the property that there exists an $\ell$-finitely nondegenerate map $H\colon(M,p)\to M'$ with $H(p)=p'$.

The rigidity properties of real objects with respect to holomorphic maps already alluded to above lead to interesting structural properties of this set. Apart from this, there are several reasons motivating the study of $E$. One of the main possible applications is in the study of the moduli space of CR maps with respect to the actions of the automorphism groups of $M$ and $M'$ from the right or the left, respectively. This is particularly interesting (and has been studied a lot) in the case where $M$ and $M'$ are spheres. We refer the interested reader to the survey of Huang and Ji [14] on this matter, and also note that there is a notion of homotopy equivalence in this setting introduced by D'Angelo and Lebl [7].

Another motivation is that when one studies deformations of *proper maps* between domains with real-analytic boundaries, then the existence of such maps finds obstructions in the existence of maps between the boundaries very naturally, as all such maps extend holomorphically across the boundary in many settings (see e.g. the paper by Mir [21]). Let us now state our first theorem:

Theorem 1. *If $M$ is real-analytic CR manifold and $M'\subset\mathbb{C}^{N'}$ is a real-analytic subvariety, then for every $\ell\in\mathbb{N}$ the mapping locus $E_\ell\subset M'$ is locally a semi-analytic set.*

In particular, we recover one of the main results of [8] for the equivalence locus $E(p)$: since it is also homogeneous (by definition), $E(p)\subset M$ is necessarily a locally closed submanifold. We remark that the level of generality obtained in the equidimensional case remains currently out of reach in the positive codimensional case; i.e. our paper makes no claim (and offers no conjectures) regarding the full mapping locus $E$.

One might wonder whether the mapping locus $E_\ell$, in contrast to the equidimensional case, can have singularities if the codimension $N'-N$ is positive. We construct an example showing that this is actually the case (even if the source manifold is assumed to be very nice).

Theorem 2. *Let $M$ be the unit sphere in $\mathbb{C}^N$. Then there is a real hypersurface $M'\subset\mathbb{C}^{N+1}$ such that the mapping locus $E_\ell$, for any $\ell\geq 2$, is a singular real-analytic subset of $M'$.*

Our approach to studying the mapping locus is to consider the variation of the image point $p'\in M'$ as a deformation of $M'$ and to deduce the semi-analyticity result from a more general semi-analyticity result valid for general deformations of $M'$.

This approach allows us to shed additional light on the mapping locus in some interesting cases. We point out one instance of this here: the degeneration of the mapping locus to a point can be checked by a sufficient linear criterion, which allows us to recover a statement already implicit in results of [11] from our considerations of deformations.

We recall the necessary definitions: Assume that $H\colon(M,p)\to(M',p')$ is an $\ell$-finitely nondegenerate map. We say that $Y(z)\in\mathbb{C}\{z-p\}^{N'}$ is an infinitesimal deformation of $H$ if $Y(p) = 0$ and if

$$\operatorname{Re}\left(\varrho_w(H(z), \overline{H(z)}) \cdot Y(z)\right)\Big|_{z\in M} = 0.$$

We note that we shall see later that the set of infinitesimal deformations of $H$ is a (finite-dimensional) real subspace $\mathfrak{hol}(H)$ of $\mathbb{C}\{z-p\}^{N'}$, which is tightly related to the tangent space of the set of possible maps of $(M, p)$ into $M'$.

THEOREM 3. *Let $M$ be a real-analytic CR manifold and $M' \subset \mathbb{C}^{N'}$ a real-analytic subvariety. Assume $H$ is $\ell$-finitely nondegenerate and that there exist no nontrivial infinitesimal deformations of $H$, i.e.* $\dim_{\mathbb{R}} \mathfrak{hol}(H) = 0$. *Then there exists a neighborhood $U$ of $p'$ such that $E_\ell \cap U = \{p'\}$.*

Theorem 1 and Theorem 3 are obtained by a more general study of analytic deformations of the target manifold $M'$ together with the jet parametrization technique for CR maps, see Section 2 for all relevant definitions. Their proofs are given in Section 3 and Section 4. The proof of Theorem 2 is given in Section 5. Finally, Section 6 contains a list of examples illustrating various properties of the mapping locus.

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**2. Preliminaries and further results.** This section introduces all the relevant notions and notations used throughout the paper; we also state the more general theorems from which we will deduce the theorems stated in the introduction.

**2.1. Manifolds, maps and deformations.** Let $\mathcal{H}((\mathbb{C}^N, p), \mathbb{C}^{N'})$ be the set of germs at $p$ of maps from $\mathbb{C}^N$ to $\mathbb{C}^{N'}$. This space is endowed with the inductive limit topology with respect to the Banach spaces $\mathcal{H}((\overline{B_R(p)}, p), \mathbb{C}^{N'})$, where $B_R(p)$ denotes the ball of radius $R > 0$ in $\mathbb{C}^N$ centered at $p$. In the following every subspace of $\mathcal{H}((\mathbb{C}^N, p), \mathbb{C}^{N'})$ will be equipped with the induced topology.

We define $\mathcal{H}((\mathbb{C}^N, p), (\mathbb{C}^{N'}, p')) \subset \mathcal{H}((\mathbb{C}^N, p), \mathbb{C}^{N'})$ as the subset of $\mathcal{H}((\mathbb{C}^N, p), \mathbb{C}^{N'})$ of maps $H$ satisfying $H(p) = p'$.

Let $M \subset \mathbb{C}^N$ be a generic real-analytic submanifold and $M' \subset \mathbb{C}^{N'}$ a real-analytic subvariety. A holomorphic map $H : M \to M'$ can be considered as the restriction of a holomorphic map $H$ defined on a neighborhood of $M$. We denote by $\mathcal{H}(M, M')$ the collection of all holomorphic maps sending $M$ into $M'$.

If $(M, p) \subset \mathbb{C}^N$ is a germ of a real-analytic submanifold and $(M', p') \subset \mathbb{C}^{N'}$ is a germ of a real-analytic subvariety we denote by $\mathcal{H}((M, p), (M', p'))$ the collection of holomorphic maps $H$ sending $(M, p)$ into $(M', p')$ (in particular $H(p) = p'$).

Often we need to consider subsets of $\mathcal{H}(M, M')$ or $\mathcal{H}((M, p), (M', p'))$ satisfying certain in some sense good geometric and analytic properties, especially those admitting a jet parametrization. An example is given by the class of *finitely nondegenerate maps* (see Section 4). We will generically denote such a subset of maps by $\mathcal{F} \subset \mathcal{H}(M, M')$ or $\mathcal{F} \subset \mathcal{H}((M, p), (M', p'))$.

In the paper we will need to treat not only the case of a fixed target set $M'$, but also the case of a *deformation* of $M'$; that is a family of subvarieties $M'_\epsilon$ including $M'$. More precisely, we extend a definition taken from [8]; we assume that the reader is acquainted with the basics of semi-analytic geometry, but refer to Section 2.6 for a discussion of the notions we use here.

Definition 4. Let $X$ be a semi-analytic compact set in some $\mathbb{R}^m$. Let $(M',p')$ be a germ of a real-analytic subvariety of $\mathbb{C}^{N'}$. A *deformation* $\mathfrak{D} = (M'_\epsilon, p')_{\epsilon\in X}$ of $(M',p')$ is given by a family of subvarieties $M'_\epsilon \subset \mathbb{C}^{N'}$, depending analytically on $\epsilon \in X$ in the sense that there exist $\varrho_1(w,\bar w,\epsilon),\dots,\varrho_d(w,\bar w,\epsilon) \in \mathcal{C}^\omega(X)\{w-p', \overline{w-p'}\}$ such that

$$\mathcal{I}_{p'}(M'_\varepsilon) = (\varrho_1(w,\bar w,\epsilon),\dots,\varrho_d(w,\bar w,\epsilon)),$$

and $M'_{\epsilon_0} = M'$ for some $\epsilon_0 \in X$. We write $\mathcal{I}(\mathfrak{D})$ for the set of all $\varrho(w,\bar w,\epsilon) \in \mathcal{C}^\omega(X)\{w-p',\overline{w-p'}\}$ satisfying $\varrho(\cdot,\cdot,\epsilon) \in \mathcal{I}_{p'}(M'_\epsilon)$ for every $\epsilon \in X$.

A *base-point-type deformation* of $M'$ is a deformation obtained in the following way: We choose $r>0$ small enough, take $X = M' \cap \overline{B_r(p')}$, and for all $q' \in X$ we define the germ $(M'_{q'}, p')$ as $(M' + p' - q', p')$, where $M' + w' = \{v' + w' : v' \in M'\}$, which is a deformation in our sense, as we can use $\varrho_j(w,\bar w,q') = \varrho_j(w-(p'-q'), \overline{w-(p'-q')})$ for any generating set $\varrho_1,\dots,\varrho_d \in \mathcal{I}_{p'}(M')$.

It is useful to study the infinitesimal notion corresponding to deformations of maps.

Let $H : M \to \mathbb{C}^{N'}$ be a real-analytic CR map satisfying $H(M) \subset M'$. We denote by $\Gamma_H = \Gamma_{CR}(H^*(\mathbb{C}T(\mathbb{C}^{N'})))$ the space of real-analytic CR sections of the pull back bundle of $\mathbb{C}T(\mathbb{C}^{N'})$ with respect to $H$, cf. [11].

Definition 5. Let $(M,p)$ and $\mathfrak{D} = ((M'_\epsilon)_{\epsilon\in X}, p')$ be as above with $M'_\epsilon = \{\varrho(\cdot,\cdot,\epsilon) = 0\}$. Let $\epsilon_0 \in X^{\mathrm{reg}}$ and $H_{\epsilon_0} : (M,p) \to (M'_{\epsilon_0}, p')$ be in $\mathcal{F}$. We say that an element $(v,Y) \in T_{\epsilon_0}X \times \Gamma_{H_{\epsilon_0}}$ is an *infinitesimal deformation* of $H_{\epsilon_0}$ into $\mathfrak{D}$ if $Y(p) = 0$ and the following equations are satisfied:

$$2\,\mathrm{Re}\left(\varrho_w\left(H_{\epsilon_0}(Z), \bar H_{\epsilon_0}(\bar Z), \epsilon_0\right)\cdot Y(Z)\right) + \varrho_\epsilon\left(H_{\epsilon_0}(Z), \bar H_{\epsilon_0}(\bar Z), \epsilon_0\right)\cdot v = 0,$$

for all $\varrho \in \mathcal{I}(\mathfrak{D})$ and $Z \in M$. We denote the space of all infinitesimal deformations of $H_{\epsilon_0}$ into $\mathfrak{D}$ by $\mathfrak{hol}(H_{\epsilon_0}, \mathfrak{D})$.

Remark 6. If we consider a curve $(\epsilon(t), H(t))$ with $H(t) : (M,p) \to (M'_{\epsilon(t)}, p')$ for $\epsilon(t) \in X$ then $(v,Y) = (\epsilon'(0), \frac{d}{dt}|_{t=0} H(t))$ (note that $(\frac{d}{dt}|_{t=0}H(t))(p) = 0$) belongs to $\mathfrak{hol}(H(0),\mathfrak{D})$. The proof is the same as [10, Lemma 21].

Definition 7. Let $H(t) \subset \mathcal{H}(M,\mathbb{C}^{N'})$ be a smooth curve such that $H(0) \in \mathcal{H}(M, M'_{\epsilon_0})$ and $\epsilon(t)$ a smooth curve in $X$ with $\epsilon(0) = \epsilon_0$. We say that $(\epsilon(t), H(t))$ is *tangent to* $\mathcal{H}(M, M'_\epsilon)$ *to order* $r$ at $(\epsilon_0, H(0))$ if for any local parametrization $Z(s)$ of $M$ we have that $\varrho(H(Z(s),t), \overline{H(Z(s),t)}, \epsilon(t)) = O(t^{r+1})$ for any $\varrho \in \mathcal{I}_{H(0)}(M'_\epsilon)$. We denote the set of such parametrized curves by $\mathfrak{P}^r$ (or $\mathfrak{P}^r_{(\epsilon_0,H)}$ if we need to emphasize that $\epsilon(0) = \epsilon_0$ and $H = H(0)$).

Definition 8. Let $(M,p)$ and $\mathfrak{D} = ((M'_\epsilon)_{\epsilon\in X}, p')$ be as above. Let $(\epsilon, H) \in X^{\mathrm{reg}} \times \mathcal{F}$. We say that $(w,Y) \in (T_{\epsilon_0}X)^k \times \Gamma_H^k$, where $\Gamma_H^k = \Gamma_H \times \dots \times \Gamma_H$, is an *infinitesimal deformation of* $H$ *of order* $k$, and write $(w,Y) \in \mathfrak{hol}^k(H,\mathfrak{D})$ if $\tau_k(w,Y) := (\epsilon + tw_1 + \cdots t^k w_k, H + tY_1 + \cdots + t^k Y_k) \in \mathcal{H}_{(\epsilon,H)}[t] \cap \mathfrak{P}^k_{(\epsilon,H)}$.

Note that for $k=1$ we recover $\mathfrak{hol}(H,\mathfrak{D})$ given in Definition 5.

Next, we want to make the connection of infinitesimal deformations of a map $H : M \to M'$, introduced and studied in [6, 9, 10, 11], to infinitesimal deformations into base-point-type deformations.

Definition 9. Let $H \in \mathcal{H}((M,p),(M',p'))$. A CR section $V \in \Gamma_H$ is called an *infinitesimal deformation of* $H$ if the real part of $V$ is tangent to $M'$ along $H(M)$. More precisely, this means that for every real-analytic function $\varrho = \varrho(w,\bar{w})$ defined in a neighborhood of $H(p)$ and vanishing on $M'$ we have

$$\mathrm{Re}\sum_{j=1}^{N'} V_j(Z)\varrho_{w_j}(H(Z),\overline{H(Z)}) = 0, \qquad \forall\, Z \in M \cap U,$$

for some open neighborhood $U$ of $p$. Here, $\varrho_w = (\varrho_{w_1},\dots,\varrho_{w_{N'}})$ denotes the complex gradient of $\varrho$. The space of infinitesimal deformations of $H$ is denoted by $\mathfrak{hol}(H)$.

We will use the following identity in the next lemma: If $\sigma$ is a real-valued function on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ we can write $\sigma(p) = \tilde{\sigma}(p,\bar{p})$ for $p \in \mathbb{C}^n$. Then for any $v \in \mathbb{C}^n$ we have $\sigma_p \cdot v = \tilde{\sigma}_p \cdot v + \tilde{\sigma}_{\bar{p}} \cdot \bar{v}$, where the first $\cdot$ denotes the inner product in $\mathbb{R}^{2n}$ (hence $v$ has been written as a vector in $\mathbb{R}^{2n}$) and the second and third $\cdot$ are given by $a \cdot b := a_1 b_1 + \cdots + a_n b_n$ for $a = (a_1,\dots,a_n) \in \mathbb{C}^n$ and $b = (b_1,\dots,b_n) \in \mathbb{C}^n$.

Lemma 10. *Let $(M,p)$ be as above, and consider the base-point-type deformation of $M'$ and $H : M \to M'$ a holomorphic map. Then $(v,Y) \in \mathfrak{hol}(H,\mathfrak{D})$ if and only if $Y + v \in \mathfrak{hol}(H)$. Moreover, the map $\mathfrak{hol}(H,\mathfrak{D}) \ni (v,Y) \mapsto Y + v \in \mathfrak{hol}(H)$ is an isomorphism.*

*Proof.* Assume $p = 0$ and $p' = 0$ and denote by $\varrho_1,\dots,\varrho_d$ a set of generators of $\mathcal{I}_0(M')$. By definition $\rho_j(Z',\bar{Z}',\epsilon) = \varrho_j(Z'+\epsilon,\bar{Z}'+\bar{\epsilon})$ for $\epsilon \in M'$ and $1 \le j \le d$ is a set of generators for $\mathcal{I}_0(M'_\epsilon)$. Since $\rho_{j\,Z'} = \varrho_{j\,Z'}$ and $\rho_{j\,\epsilon} = 2\,\mathrm{Re}(\varrho_{j\,Z'})$ it holds that $(v,Y) \in \mathfrak{hol}(H,\mathfrak{D})$ if and only if for $1 \le j \le d$

$$\begin{aligned} &2\,\mathrm{Re}\left(\rho_{j\,Z'}\left(H(Z),\bar{H}(\bar{Z}),\epsilon\right)\cdot Y(Z)\right) + \rho_{j\,\epsilon}\left(H(Z),\bar{H}(\bar{Z}),\epsilon\right)\cdot v = 0, \quad Z \in M,\\ \iff\ &2\,\mathrm{Re}\left(\varrho_{j\,Z'}\left(H(Z),\bar{H}(\bar{Z})\right)\cdot(Y(Z)+v)\right) = 0, \quad Z \in M, 1 \le j \le d. \end{aligned}$$

The last equation is satisfied if and only if $Y + v \in \mathfrak{hol}(H)$. The last statement follows from the fact that $\mathfrak{hol}(H) \ni W \mapsto (W(0), W + W(0)) \in \mathfrak{hol}(H,\mathfrak{D})$ is an inverse to the map given in the hypothesis. $\square$

Corollary 11. *If $\pi_1$ is the projection on the first factor, then $\pi_1(\mathfrak{hol}(H,\mathfrak{D})) = \mathfrak{hol}(H)(p) = \{X(p) : X \in \mathfrak{hol}(H)\}$.*

We will be particularly interested in studying the set of deformation parameters $\epsilon$ for which a map between $M$ and $M'_\epsilon$ exists. We will call this set the *mapping locus*, more precisely we have the following definition, cf. the definition of equivalence locus from [8] for the equidimensional case.

Definition 12. Let $(M,p)$ be a germ of submanifold of $\mathbb{C}^N$, and let $(M'_\epsilon,p')_{\epsilon\in X}$ be a deformation of $(M',p') = (M'_{\epsilon_0},p') \in \mathbb{C}^{N'}$. Let $\mathcal{F}_\epsilon \subset \mathcal{H}((M,p),(M'_\epsilon,p'))$ and $\mathcal{F} = (\mathcal{F}_\epsilon)_{\epsilon\in X}$. We define the $\mathcal{F}$*-mapping locus* as the set $E_\mathcal{F} \subset X$ given by

$$E_\mathcal{F} = \{\epsilon \in X | \mathcal{F}_\epsilon \neq \emptyset\}.$$

In other words, $\epsilon \in E_\mathcal{F}$ if and only if there exists a holomorphic map $H : M \to M'_\epsilon$ with $H(p) = p'$ and $H \in \mathcal{F}_\epsilon$. In particular if we consider base-point-type deformations, then $E_\mathcal{F} \subset M'$.

**2.2. Jet spaces.** We will work with maps by means of their jets through suitable parametrization results. The following definitions are very standard and here we mainly aim at establishing the notation used later in the paper. For all $p = (p_1, \ldots, p_N) \in \mathbb{C}^N$ we define the space of $k$-jets at $p$ of holomorphic maps $\mathbb{C}^N \to \mathbb{C}^{N'}$ as follows:

$$J_p^k = \mathbb{C}\{Z-p\}^{N'} \big/ \mathfrak{m}_p^{k+1},$$

where $\mathfrak{m}_p = (Z_1 - p_1, \ldots, Z_N - p_N)$ is the maximal ideal of the ring of power series centered at $p$, and $j_p^k$ denotes the natural projection. For a given $k$, we will denote by $\Lambda$ the coordinates in $J_p^k$.

**2.3. Jet parametrization.** It turns out that our structural results hold in higher generality than the setting discussed in the introduction. The methods apply equally well to understanding the structure of $\mathcal{F}$ if we assume that $\mathcal{F}$ satisfies the following *jet parametrization property* JPP (see Definition 13).

Jet parametrization results can be proved in a variety of different contexts and have been used widely in the study of the structure of CR mappings, see e.g. [2, 3, 15, 18, 19]. In the following definition we abstractly define what we need from such a parametrization to obtain the desired structural results. In Section 4 we are going to consider classes of maps which satisfy the jet parametrization property.

DEFINITION 13. Let $(M, p)$ be a germ of submanifold of $\mathbb{C}^N$, and let $\mathfrak{D} = (M'_\epsilon, p')_{\epsilon \in X}$ be a germ of real-analytic deformation of $(M', p') = (M'_{\epsilon_0}, p') \in \mathbb{C}^{N'}$, where $\epsilon_0$ is a distinguished parameter in $X$. For all $\epsilon \in X$ let $\mathcal{F}_\epsilon \subset \mathcal{H}(M, M'_\epsilon)$ be an open subset of maps. We say that $\mathcal{F} = (\mathcal{F}_\epsilon)_{\epsilon \in X}$ satisfies the *jet parametrization property of order* $\mathbf{t}_0 \in \mathbb{N}$ if the following holds.

**JPP:** *There exists an open neighborhood $V$ of $p$ in $\mathbb{C}^N$, an open neighborhood $W$ of $\epsilon_0$ in $X$, a finite index set $J$, real-analytic functions $q_j : W \times J_p^{\mathbf{t}_0} \to \mathbb{R}, j \in J$ such that $q_j(\epsilon, \Lambda)$ is polynomial in $\Lambda$, and a holomorphic map $\Phi_j : \mathcal{U}_j \to \mathbb{C}^{N'}$ (where $\mathcal{U}_j = V \times U_j$ and $U_j = \{q_j(\epsilon, \Lambda) \neq 0\} \subset W \times J_p^{\mathbf{t}_0}$) of the form*

$$\Phi_j(Z, \epsilon, \Lambda) = \sum_{\alpha \in \mathbb{N}_0^N} \frac{p_j^\alpha(\epsilon, \Lambda)}{q_j(\epsilon, \Lambda)^{d_\alpha^j}} Z^\alpha, \quad p_j^\alpha \in \mathbb{C}\{\epsilon\}[\Lambda], \quad d_\alpha^j \in \mathbb{N}_0, \quad j \in J, \qquad (1)$$

*such that for every map $t \mapsto (\epsilon(t), H(t))$ belonging to $\mathfrak{P}^r_{(\epsilon(0), H(0))}$ there exists $j \in J$ such that the following holds for all $t$ close enough to $0$:*

(a) $(\epsilon(t), j_p^{\mathbf{t}_0} H(t)) \in U_j$,

(b) $H(Z, t)|_V = \Phi_j(Z, \epsilon(t), j_p^{\mathbf{t}_0} H(t)) + O(t^{r+1})$.

*In particular, there exist real-analytic functions $c_i^j : W \times J_p^{\mathbf{t}_0} \to \mathbb{R}$, $i \in \mathbb{N}$, polynomial in $\Lambda$ such that*

$$A_\epsilon := j_p^{\mathbf{t}_0}(\mathcal{F}_\epsilon) = \bigcup_{j \in J} \{\Lambda \in J_p^{\mathbf{t}_0} : q_j(\epsilon, \Lambda) \neq 0, \, c_i^j(\epsilon, \Lambda, \bar{\Lambda}) = 0\}. \qquad (2)$$

*Define $A \subset X \times J_p^{\mathbf{t}_0}$ as $A := \bigcup_\epsilon (\{\epsilon\} \times A_\epsilon)$ and set $A_j = A \cap \mathcal{U}_j$. Then $\Lambda$ is the $k_0$-jet of a map $\mathcal{F}_\epsilon \ni H : (M, p) \to (M'_\epsilon, p')$ if and only if $(\epsilon, \Lambda) \in A$.*
*Furthermore for any $(\epsilon(t), H(t)) \in \mathfrak{P}^r_{(\epsilon, H)}$ with $\Lambda(t) = j_{p_k}^{\mathbf{t}_0} H(t)$ we have for small enough $t$:*

$$c_i^j(\epsilon(t), \Lambda(t), \bar{\Lambda}(t)) = O(t^{r+1}), \qquad i, j \in \mathbb{N}. \qquad (3)$$

Remark 14. Since $\mathcal{F} \subset \bigcup_{\epsilon \in X} \mathcal{H}(M, M'_\epsilon) \subset X \times \mathcal{H}(M, \mathbb{C}^{N'})$ we can equip $\mathcal{F}$ with the induced topology. Similarly as in [10] one can show that $\Phi_j : A_j \to \mathcal{F}$ is locally a homeomorphism. For more details we refer to [10, Lemma 19].

Remark 15. Let $(\epsilon, \Lambda) \in A_j^{\mathrm{reg}}$ and $w \in T_{(\epsilon,\Lambda)} A_j^{\mathrm{reg}}$ and consider a curve $c(t) = (\epsilon(t), \Lambda(t))$ in $A_j^{\mathrm{reg}}$ with $c(0) = (\epsilon, \Lambda)$ and $c'(0) = w$. Then a similar computation as in Remark 6 applied to $(\epsilon(t), H(t))$ with $H(t) = \Phi_j(., \epsilon(t), \Lambda(t))$ shows that $D\Phi_j(T_{(\epsilon,\Lambda)} A_j^{\mathrm{reg}}) \subseteq \mathfrak{hol}(H, \mathfrak{D})$ for any $(\epsilon, \Lambda) \in A_j^{\mathrm{reg}}$.

Remark 16. In a similar way as in [11, Remark 18] one can deduce a jet parametrization for $\mathfrak{hol}^k(H, \mathfrak{D})$, which for $k = 1$ implies that $\mathfrak{hol}(H, \mathfrak{D})$ is finite dimensional. As in [10, Cor. 32] one can deduce that $X \times \mathcal{F}_\epsilon \ni (\epsilon, H) \mapsto \dim(\mathfrak{hol}(H, \mathfrak{D}))$ is upper semicontinuous. Applying this fact to a base-point-type deformation and using the last statement of Lemma 10 shows that $p \mapsto \dim \mathfrak{hol}(H)(p)$ is upper semicontinuous.

**2.4. Further results.** The results in the introduction actually hold in a more general setting. In particular Theorem 1 can be formulated for mappings which satisfy JPP.

Theorem 17. *Let $(M, p) \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be generic real-analytic submanifolds and assume that $\mathcal{F}$ satisfies JPP. Then $E_{\mathcal{F}}$ is locally a semi-analytic subset of $M'$.*

Theorem 1 now follows from Theorem 17 and Theorem 28, which shows that the class of $\ell$-finitely nondegenerate maps satisfies JPP. Note that Theorem 17 can be regarded as a result for base-point type deformations and hence can be considered as a special case of the following more general result:

Theorem 18. *Let $(M, p)$, $(M'_\epsilon, p')$ and $\mathcal{F}$ be as in Definition 13. Then $E_{\mathcal{F}}$ is locally a semi-analytic subset of $X$.*

Theorem 18 is a partial generalization of [8, Theorem 2], its proof is given in Section 3 below.

The parametrization method can also be used to provide a sufficient linear criterion to show that a given map is isolated in $\mathcal{F}$. The following theorem is a generalization of Theorem 3 from the introduction:

Theorem 19. *Let $M, \mathfrak{D}$ and $\mathcal{F}$ be as in JPP. Fix $H : M \to M'_{\epsilon_0}$ with $H \in \mathcal{F}_{\epsilon_0}$. Suppose that* $\dim \mathfrak{hol}(H, \mathfrak{D}) = 0$*, then $H$ is isolated in $\mathcal{F}$.*

The following corollary is an immediate consequence of Theorem 19 and Corollary 11.

Corollary 20. *If* $\dim \mathfrak{hol}(H) = 0$*, then $H$ is isolated in $\mathcal{H}(M, M')$.*

The result can be proved as a consequence of Remarks 15 and 16 in an analogous way as in [10]. In the following we are only outlining the main steps and refer to [10, Lemma 23]. We need the following Lemma:

Lemma 21. *Let $(\epsilon_0, \Lambda_0) \in A_j$, and suppose that* $\dim \mathfrak{hol}(\Phi_j(\epsilon_0, \Lambda_0), \mathfrak{D}) = \ell$*. Then there exists a neighborhood $U$ of $(\epsilon_0, \Lambda_0) \in X \times J_p^{\mathbf{t}_0}$ such that, if $N \subset A_j$ is a submanifold with $N \cap U \neq \emptyset$, then* $\dim N \leq \ell$*.*

*Proof.* Let $U$ be a neighborhood of $(\epsilon_0, \Lambda_0)$ such that $\dim \mathfrak{hol}(\Phi_j(\epsilon, \Lambda)) \leq \ell$ for all $(\epsilon, \Lambda) \in U$. The existence of $U$ is guaranteed by the upper semicontinuity property given in Remark 16 and the continuity of $\Phi_j$. Let $N$ be a submanifold of $A_j$ intersecting $U$. Using the rank theorem for Banach spaces as in [10, Lemma 23] we conclude that $D\Phi_j(\epsilon, \Lambda)$ is injective on $T_{(\epsilon,\Lambda)}N$ for $(\epsilon, \Lambda)$ belonging to a dense open set in $N$. By Remark 15 we obtain the following inequalities:

$$\dim N = \dim D\Phi_j(\epsilon, \Lambda)(T_{(\epsilon,\Lambda)}N) \leq \dim \mathfrak{hol}(\Phi_j(\epsilon, \Lambda)) \leq \ell,$$

which concludes the proof. □

*Proof of Theorem 19.* Define $\Lambda_0 = j_p^{\mathbf{t}_0} H$. There exists $j \in J$ such that $(\epsilon_0, \Lambda_0) \in A_j$. By Lemma 21 there exists a neighborhood $U$ of $(\epsilon_0, \Lambda_0)$ such that for any submanifold $N$ in $U \cap A_j$ it holds that $\dim N = 0$. Then the dimension of $U \cap A_j$ is zero, and thus $U \cap A_j$ consists of isolated points. By Remark 14 the proof is concluded. □

**2.5. CR geometry.** In this subsection we briefly introduce some standard notation from CR geometry; more details can be found e.g. in [4]. Let $M$ be a generic real-analytic CR submanifold of $\mathbb{C}^N$. It is well known (see [4]) that one can choose *normal coordinates* $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d = \mathbb{C}^N$ in such a way that $M$ is written as

$$w = Q(z, \bar{z}, \bar{w}), \quad (\text{or equivalently} : \bar{w} = \overline{Q}(\bar{z}, z, w)),$$

where $Q$ is a germ of a holomorphic map $Q : \mathbb{C}^{2n+d} \to \mathbb{C}^d$ satisfying $Q(z, 0, \bar{w}) \equiv Q(0, \bar{z}, \bar{w}) \equiv \bar{w}$ and $Q(z, \bar{z}, \overline{Q}(\bar{z}, z, w)) \equiv w$.

In the proof of the parametrization results, the notion of *Segre maps* is also needed. For $j \in \mathbb{N}$ let $(x_1, \ldots, x_j)$ be coordinates of $\mathbb{C}^{nj}$ ($x_\ell \in \mathbb{C}^n$ for $\ell = 1, \ldots, j$); we also write $x^{[j;k]} := (x_j, \ldots, x_k)$. The *Segre map* of order $q \in \mathbb{N}$ is the map $S_0^q : \mathbb{C}^{nq} \to \mathbb{C}^N$ defined as follows:

$$S_0^1(x_1) := (x_1, 0), \quad S_0^q(x^{[1;q]}) := \left(x_1, Q\left(x_1, \overline{S}_0^{q-1}(x^{[2;q]})\right)\right).$$

We say that $M$ is *minimal at* $p \in M$ if it does not contain any germ of a CR submanifold $\widetilde{M} \subsetneq M$ of $\mathbb{C}^N$ through $p$ having the same CR dimension as $M$ at $p$. The minimality criterion obtained in [1] states that if $M$ is minimal at 0, then $S_0^q$ is generically of full rank for sufficiently large $q$, and moreover, in this case, for every neighborhood $U \subset \mathbb{C}^{2qn}$ there exists $x^0 \in U$ sucht that $S_0^{2q}(0) = 0$ and such that $S_0^{2q}$ is of full rank at $x^0$ (see e.g. [3]).

**2.6. Real-analytic geometry.** In order to prove our theorems we work within the framework of subanalytic and semi-analytic sets; to this end, we recall some basic notions and results.

A set $A \subset \mathbb{R}^n$ is called *semi-analytic* if it is a finite union of intersections of sets defined by real-analytic equations and inequalities:

$$A = \bigcup_{i=1}^{k} \bigcap_{j=1}^{N(i)} A_{ij}, \tag{4}$$

where $A_{ij}$ is either of the form $\{h_{ij} = 0\}$ or $\{h_{ij} > 0\}$ for some real-analytic $h_{ij} \in \mathbb{R}\{x_1, \ldots, x_n\}$. The notion of semi-analytic set is modeled on the notion of semi-algebraic sets, which are defined in a similar way (with polynomial functions instead of analytic ones) and are closed under projections (Tarski-Seidenberg theorem). On the other hand, semi-analytic sets do not enjoy the same property, and therefore we will need some more subtle results.

If $\mathcal{R}$ is any ring of real functions over a set $E$, a subset $A \subset E$ is called *definable over* $\mathcal{R}$ if it can be expressed as in (4), with $A_{ij}$ being either of the form $\{f_{ij} = 0\}$ or $\{f_{ij} > 0\}$ for some $f_{ij} \in \mathcal{R}$. We have the following (see e.g. [5]):

THEOREM 22 (Łojasiewicz). *Let $X$ be an analytic manifold, let $A \subset X \times \mathbb{R}^k$ be definable over the ring $C^\omega(X)[x_1, \ldots, x_k]$ and let $\pi : X \times \mathbb{R}^k \to X$ be the projection on the first factor. Then $\pi(A)$ is semi-analytic, i.e., definable over $C^\omega(X)$.*

The family of semi-analytic sets and maps (which are defined as maps whose graphs are semi-analytic sets) is not closed under projections. If one wants to work with images of semi-analytic sets, one encounters the subanalytic sets.

A set $A \subset \mathbb{R}^n$ is called *subanalytic* if locally $A$ is a projection of a (relatively compact) semi-analytic set $B \subset \mathbb{R}^n$. Similar as for semi-analytic sets, subanalytic sets enjoy the following properties: Finite unions and intersections of subanalytic sets as well as the complement of a subanalytic set are subanalytic. In fact, the class of subanalytic sets agrees with the class of sets obtained by taking finite unions, intersections and complements of closed images of proper, real-analytic maps of closed real-analytic sets. A map $f : A \to \mathbb{R}^m$, where $A \subset \mathbb{R}^n$ is a subanalytic set, is called *subanalytic* if its graph is a subanalytic subset of $\mathbb{R}^n \times \mathbb{R}^m$. For more information on subanalytic sets we refer the reader to [5].

A (subanalytic) *cell decomposition* of a subanalytic set $A$ is a finite collection of subsets $\{C_j^q\}$ such that each $C_j^q$ is subanalytically homeomorphic to the ball $B^q = \{x \in \mathbb{R}^q : |x| < 1\}$ ($C_j^q$ is then called a *cell of dimension* $q$) and satisfies the following properties:

(1) $A = \bigcup_{j,q} C_j^q$

(2) The closure $\overline{C}_j^q$ is the union of $C_j^q$ and cells of strictly smaller dimension.

In this case we say that the sets $C_j^q$ form a *stratification*.

We will need the following local triviality result of Hardt [13] for bounded subanalytic maps.

THEOREM 23 (Hardt). *Let $X$ and $Y$ be bounded subanalytic sets and $f : X \to Y$ be a continuous subanalytic map. Then there exist a finite subanalytic stratification $\{Y_1, \ldots, Y_k\}$ of $Y$, a collection $\{F_1, \ldots, F_k\}$ of bounded subanalytic sets, and subanalytic homeomorphisms $g_j \colon f^{-1}(Y_j) \to Y_j \times F_j$, such that $f|_{f^{-1}(Y_j)} = \pi \circ g_j$, where $\pi$ denotes the projection $\pi : Y_j \times F_j \to Y_j$.*

**3. Basic properties of the mapping locus.** In this section we describe in more details some main properties of the mapping locus. In particular we are going to provide the proof of Theorem 18.

*Proof of Theorem 18.* We follow the same steps as in [8]. By [12], for any bounded semi-analytic set $B \subset X$ the ring $C^\omega(\overline{B})$ is Noetherian. It follows that the ring $C^\omega(\overline{B})[\Lambda]$ is also Noetherian. To prove that $E_{\mathcal{F}}$ is locally semi-analytic for any $p \in E_{\mathcal{F}}$ we consider an open, bounded semi-analytic neighborhood $B$ of $p$. We will show that $E_{\mathcal{F}} \cap B$ is semi-analytic.

Let $A$ be as in (2). Notice that by construction, it holds that $E_{\mathcal{F}} \cap B$ is precisely equal to the projection of $A \cap (B \times J_p^{\mathbf{t}_0})$. Write now $A = \bigcup_{j \in J} A_j$, where $A_j = A \cap U_j$. Then since $E_{\mathcal{F}} \cap B$ is the union of the projections of $A_j \cap (B \times J_p^{\mathbf{t}_0})$ and the finite union of semi-analytic sets is semi-analytic, it is enough to prove that each projection of $A_j \cap (B \times J_p^{\mathbf{t}_0})$ is semi-analytic. In the following we fix $j \in J$ and omit the dependence on $j$ notationally. Furthermore $c_i \in C^\omega(\overline{B})$ for all $i \in \mathbb{N}$. Hence there exist finitely many indices $i_1, \dots, i_\ell$ such that

$$A \cap (\overline{B} \times J_p^{\mathbf{t}_0}) = \{c_{i_1} = c_{i_2} = \ldots = c_{i_\ell} = 0\}.$$

It follows that $A \cap (\overline{B} \times J_p^{\mathbf{t}_0})$ is definable in the sense of Lojasiewicz. By Lojasiewicz's Theorem (cf. [8]), the projection of $A \cap (\overline{B} \times J_p^{\mathbf{t}_0})$ onto $X$ is a semi-analytic subset of $X$. □

The next result gives more information about maps $H_\epsilon : (M, p) \to (M'_\epsilon, p')$ with $\epsilon \in E_{\mathcal{F}}$. More specifically, such maps can be (at least generically) selected to depend on $\epsilon$ in an analytic way. The following result holds, cf. [8, Lemma 8] for the equidimensional case:

Lemma 24. *Suppose $E_{\mathcal{F}}$ contains a real-analytic submanifold $R \subset X$. Then, on an open dense set of $R$, the jet of the maps $H_\delta$ can be taken to depend analytically on $\delta$.*

*More precisely, the following holds: let $\epsilon \in R$ and fix a neighborhood $U \subset R$ of $\epsilon$. Then there exist $\delta_1 \in U$, a neighborhood $V$ of $\delta_1$, a real-analytic map $L : V \cap R \to J_0^{\mathbf{t}_0}$ and maps $\mathcal{F} \ni H^\delta \colon (M, 0) \to (M'_\delta, 0)$ such that $j_0^{\mathbf{t}_0} H^\delta = L(\delta)$ for all $\delta \in V$.*

*Proof.* Denote by $\pi \colon X \times J_0^{\mathbf{t}_0} \to X$ the projection onto the first factor, and define $\mathcal{R} := \pi^{-1}(R) \cap A$. Then $\mathcal{R}$ is a semi-analytic subset of $X \times J_0^{\mathbf{t}_0}$. Let $\mathcal{R}^{\mathrm{reg}}$ be the regular part of $\mathcal{R}$, which is an open dense smooth semi-analytic subset of $\mathcal{R}$. For any point $a \in \mathcal{R}^{\mathrm{reg}}$ define $r(a) := \operatorname{rank}(\pi|_{\mathcal{R}^{\mathrm{reg}}}(a))$ and let $r_0 := \max_{a \in \mathcal{R}^{\mathrm{reg}}} r(a)$. Define $\tilde{\mathcal{R}} := \{a \in \mathcal{R}^{\mathrm{reg}} : r(a) = r_0\}$, then $\tilde{\mathcal{R}}$ is an open dense subset of $\mathcal{R}^{\mathrm{reg}}$, and thus of $\mathcal{R}$. Since $\pi(\mathcal{R}) = R$ by assumption, we note that it holds that $\tilde{R} := \pi(\tilde{\mathcal{R}})$ is open and dense in $R$. Let $\delta_1 \in U \cap \tilde{R}$ and let $a_1 \in \tilde{\mathcal{R}}$ such that $\pi(a_1) = \delta_1$, i.e. $a_1 = (\delta_1, \Lambda_1)$ for a certain $\Lambda_1 \in J_0^{\mathbf{t}_0}$. By the constant rank theorem there is a neighborhood $\mathcal{V}$ of $a_1 \in \tilde{\mathcal{R}}$, a neighborhood $V$ of $\delta_1$ in $\tilde{R}$, a ball $B$ in some $\mathbb{R}^m$ and an analytic diffeomorphism $\psi : \mathcal{V} \to V \times B$, such that $\psi(a_1) = (\delta_1, 0)$, and such that $\pi \circ \psi = \pi$. Define $N := V \times \{0\} \subset V \times B$ and consider $L := \pi_2 \circ \psi^{-1}|_N$, where $\pi_2$ is the projection on the second factor, the jet space. Then $L$ is a real-analytic map and the proof is concluded by setting $H^\delta := \Phi(\delta, L(\delta))$ for all $\delta \in V$ where $\Phi$ is given by (1). □

The previous selection lemma can be used to generically estimate the dimension of the mapping locus by means of the space of infinitesimal deformations, cf. [8, Theorem 6] for the equidimensional case.

Theorem 25. *Let $(M'_\epsilon, 0)$, $\epsilon \in X$ be a deformation of $(M', 0)$ as before. Let $S$ be a semi-analytic subset of $E_{\mathcal{F}}$ and $S^{\mathrm{reg}}$ be the set of regular points in $S$. Then the tangent space of $S^{\mathrm{reg}}$ at $\delta$ is contained in the space of infinitesimal deformations of a map realizing $\delta$, for almost all $\delta \in S^{\mathrm{reg}}$.*

*More precisely, there exists an open dense subset $D$ of $S^{\mathrm{reg}}$ with the following properties:*

(i) *For every $\epsilon \in D$ there exists a neighborhood $U$ of $\epsilon$ in $S^{\mathrm{reg}}$ and an analytic map $\phi : U \to \mathcal{F}$ such that for every $\delta \in U$ we have $\phi(\delta) : (M, 0) \to (M'_\delta, 0)$ (note that $U$ as an open subset of $S^{\mathrm{reg}}$ is an analytic submanifold).*

(ii) *For all $\delta \in D$ we have:*

$$T_\delta S^{\mathrm{reg}} \subset \pi_1(\mathfrak{hol}(\phi(\delta), \mathfrak{D})).$$

(iii) *If $M'_\epsilon$ is a base-point-type deformation (in particular $X = M'$), then for all $\delta \in D$ we have*

$$T_\delta S^{\mathrm{reg}} \subset \mathfrak{hol}(\psi(\delta))(0)$$

*where we define $\psi(\delta) = \phi(\delta) + \delta$.*

*Proof.* Note that since $S$ is semi-analytic, by [5, 20] $S^{\mathrm{reg}}$ is a dense semi-analytic subset of $S$. Let $D$ be the set of the points satisfying (i) and (ii): We will show that $D$ is an open, dense subset of $S^{\mathrm{reg}}$. Let $\epsilon_0 \in S^{\mathrm{reg}}$ and let $O$ be a neighborhood of $\epsilon_0$, and let $\delta_1 \in X$ be the point given by Lemma 24 (with $R = O \cap S^{\mathrm{reg}}$): we will show that $\delta_1 \in D$. Indeed, take $U = V$, where $V$ is given by Lemma 24. To show that property (i) is satisfied, define $\phi$ as $\phi(\delta) = H^\delta$ for all $\delta \in U$, where $H^\delta$ is the one from Lemma 24. To establish (ii) let $v \in T_{\delta_1} S^{\mathrm{reg}}$. Let $\delta(t)$ be a smooth curve in (a neighborhood of $\delta_1$ in) $D$ such that $\delta(0) = \delta_1$ and $\delta'(0) = v$. We define a smooth curve of maps $c_t : (M, 0) \to (M'_\delta(t), 0)$ as $c_t = \phi(\delta(t))$. Let $Y$ be the derivative $\frac{dc_t}{dt}|_{t=0}$ of $c_t$ at $t = 0$, then $(v, Y) \in \mathfrak{hol}(c_0, \mathfrak{D}) = \mathfrak{hol}(\phi(\delta(0)), \mathfrak{D}) = \mathfrak{hol}(\phi(\delta_1), \mathfrak{D})$ (cf. Remark 6). Thus $v = \pi_1(v, Y) \in \pi(\mathfrak{hol}(\phi(\delta_1), \mathfrak{D}))$. Since $v$ is an arbitrary vector of $T_{\delta_1} S^{\mathrm{reg}}$, we conclude that $T_{\delta_1} S^{\mathrm{reg}} \subset \pi_1(\mathfrak{hol}(\psi(\delta_1), \mathfrak{D}))$. In the case of the base-point-type deformation, (iii) is a direct consequence of Corollary 11. This shows that $\delta_1 \in D$. Repeating the same arguments for any $\delta \in U$ shows that $\delta \in D$, hence $D$ is an open subset of $S^{\mathrm{reg}}$. □

It is worth noting that a stronger selection lemma than in Lemma 24 can be proved using more advanced results of real-analytic geometry. The following lemma is the analogous of [8, Lemma 6].

LEMMA 26. *A generic map $H^{\delta_0} : (M, 0) \to (M'_{\delta_0}, 0)$ with $\delta_0 \in E_{\mathcal{F}}$ can be deformed continuously to a family of maps $H^\delta : (M, 0) \to (M'_\delta, 0)$, for $\delta$ close to $\delta_0$.*

*A more precise formulation can be given as follows: Let $\epsilon \in E_{\mathcal{F}}$, and fix a neighborhood $U$ of $\epsilon$. Then there exists $\delta_0 \in E_{\mathcal{F}} \cap U$ such that for every map $\mathcal{F} \ni H^{\delta_0} : (M, 0) \to (M'_{\delta_0}, 0)$ and every neighborhood $W$ of $j_0^{\mathbf{t}_0} H^{\delta_0}$ there exists a neighborhood $V$ of $\delta_0$ such that the following holds: For all $\delta \in V \cap E_{\mathcal{F}}$ there exists a map $\mathcal{F} \ni H^\delta : (M, 0) \to (M'_\delta, 0)$ such that $j_0^{\mathbf{t}_0} H^\delta \in W$.*

*Proof.* Let $A$ be as in (2). It is enough to show the conclusion for each $A_j$, which we denote by $A$ by an abuse of notation. Just as in the proof of Theorem 18, for a small ball $B \subset U$ we can write $A \cap (\overline{B} \times J_p^{\mathbf{t}_0})$ as the intersection of the vanishing set of finitely many $c_i$. Using the same argument as Hardt (see [13], Step I of the proof of the main theorem) we can reduce to the case of bounded subanalytic sets, by a fibrewise projectivization of $A$; this is possible since $A$ is defined by functions which are polynomial in the jet variable. By the version of Hardt's theorem for subanalytic sets (see [13], in particular the remarks starting at the end of page 291), there exists a partition of $\pi(A \cap B)$ into subanalytic sets $C_1, \dots, C_m$ in such a way that $\pi|_{\pi^{-1}(C_j)}$ is trivial for $1 \le j \le m$. Furthermore we can find a stratification $C_j^i$ of $\pi(A \cap B)$ by smooth subanalytic sets respecting $\{C_1, \dots, C_m\}$ (i.e. $\overline{C}_j^i$ is the union of $C_j^i$ and strata of strictly smaller dimension): see [20]. Let $d = \max\{d' : \exists \delta \in E_{\mathcal{F}} \cap B \text{ s.t. } \delta \in C_j^i \text{ and } \dim C_j^i = d'\}$. Let $\delta_0 \in E_{\mathcal{F}} \cap B$ such that $\delta_0 \in C_{j_0}^{i_0}$ with $\dim C_{j_0}^{i_0} = d$. Then, by the stratification property, there exists a neighborhood $V'$ of $\delta_0$ such that $E_{\mathcal{F}} \cap V' \subset C_{j_0}^{i_0}$ (see for instance [8, proof of Lemma 6]). By Hardt's theorem there exist a subanalytic set $Y \subset J_0^{\mathbf{t}_0}$ and a subanalytic homeomorphism $\psi : \pi^{-1}(C_{j_0}^{i_0}) \to C_{j_0}^{i_0} \times Y$

such that $\pi \circ \psi = \pi$. Let $H^{\delta_0}$ be a map $(M, 0) \to (M'_{\delta_0}, 0)$ and $W$ a neighborhood of $j_0^{\mathbf{t}_0} H^{\delta_0}$. Define $O := \psi((V' \times W) \cap \pi^{-1}(C_{j_0}^{i_0}))$, which is an open neighborhood of $(\delta_0, j_0^{\mathbf{t}_0} H^{\delta_0})$ in $C_{j_0}^{i_0} \times Y$. Then there is an open set $V \subset V'$ such that for every $\delta \in V$ it holds that $(\delta, j_0^{\mathbf{t}_0} H^{\delta_0}) \in O$. Then $(\delta, j_\delta) := \psi^{-1}(\delta, j_0^{\mathbf{t}_0} H^{\delta_0})$ has the property that $j_\delta \in W$ and setting $H^\delta := \Phi(\delta, j_\delta)$, where $\Phi$ is from (1), shows the claim. □

**4. A class of maps satisfying JPP.** We are now going to define a class of maps satisfying JPP from Definition 13, namely the *finitely nondegenerate maps* ([17]). The following definition is taken almost verbatim from [11]; since some adaptations to the setting of this paper are needed we state the definition again for the convenience of the reader.

Definition 27. Let $M'$ be a real-analytic subvariety of $\mathbb{C}^{N'}$, $p' \in M'$ and $\varrho_1, \dots, \varrho_d$ generators of $\mathcal{I}_{p'}(M')$. Let $(L_1, \dots, L_n)$ be a basis of CR vector fields of $(M, p)$. For a multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we write $L^\alpha = L_1^{\alpha_1} \cdots L_n^{\alpha_n}$. Given a holomorphic map $H = (H_1, \dots, H_{N'}) \in \mathcal{H}((M, p), (M', p'))$ with $H(p) = p'$, and a fixed sequence $\iota = (\iota_1, \dots, \iota_{N'})$ of multiindices $\iota_m \in \mathbb{N}_0^n$ and $N'$-tuple of integers $\ell = (\ell^1, \dots, \ell^{N'})$ with $1 \le \ell^k \le d'$, we consider the determinant

$$s_H^{\iota,\ell}(Z) = \det\left(\left(L^{\iota_j} \varrho_{\ell^j, Z'_k}(H(Z), \overline{H}(\bar{Z}))\right)_{1 \le j,k \le N'}\right). \tag{5}$$

We define the open set $\mathcal{F}_k((M, p), (M', p')) \subset \mathcal{H}((M, p), (M', p'))$ as the set of maps $H$ for which there exists such a sequence of multiindices $\iota = (\iota_1, \dots, \iota_{N'})$ satisfying $k = \max_{1 \le m \le N'} |\iota_m|$ and $N'$-tuple of integers $\ell = (\ell^1, \dots, \ell^{N'})$ as above such that $s_H^{\iota,\ell}(p) \neq 0$. We define $J_{k_0}$ as the set of all pairs $j = (\iota, \ell)$, where $\iota = (\iota_1, \dots, \iota_{N'})$ is a sequence of multiindices with $k_0 = \max_{1 \le m \le N'} |\iota_m|$ and $\ell = (\ell^1, \dots, \ell^{N'})$ is as above. We will say that $H$ with $H(M) \subset M'$ is $k_0$-*nondegenerate* at $p$ if $k_0 = \min\{k \colon H \in \mathcal{F}_k((M, p), (M', p'))\}$ is a finite number. We write $\mathcal{F}_{k_0}((M, p), (M', p'))$ for the (open) subset of $\mathcal{H}((M, p), (M', p'))$ containing all $k_0$-nondegenerate maps. A holomorphic map $H$ with $H(M) \subset M'$ is called $\ell$-*finitely nondegenerate* if for each $p \in M$ the map $H$ is $k(p)$-nondegenerate at $p$ and $\ell = \max\{k(p) : p \in M\}$.

Let us stress that the distinction between $k$-nondegeneracy and $\ell$-finite nondegeneracy is important: A map is $\ell$-finitely nondegenerate if and only if it is $k$-nondegenerate for some $k \le \ell$. Our next two results show that both $k$-nondegenerate and $\ell$-finitely nondegenerate maps satisfy the JPP.

Theorem 28. *Let $(M, p) \subset \mathbb{C}^N$ be a germ of a generic minimal real-analytic submanifold of $(\mathbb{C}^N, p)$ and $(M'_\epsilon, p')_{\epsilon \in X}$ be a deformation. Fix $k_0 \in \mathbb{N}$ and let $\mathbf{t}$ be the minimum integer, such that the Segre map $S_p^{\mathbf{t}}$ of order $\mathbf{t}$ associated to $M$ is generically of full rank. Then $\mathcal{F}_{k_0,\epsilon} := \mathcal{F}_{k_0}((M, p), (M'_\epsilon, p'))$ satisfies Definition 13 with $\mathbf{t}_0 = 2\mathbf{t}k_0$.*

The following corollary is immediate from the definition of the JPP and the preceding theorem.

Corollary 29. *Let $(M, p) \subset \mathbb{C}^N$ be a germ of a generic minimal real-analytic submanifold of $(\mathbb{C}^N, p)$ and $(M'_\epsilon, p')_{\epsilon \in X}$ be a deformation. Fix $\ell \in \mathbb{N}$ and let $\mathbf{t}$ be the minimum integer, such that the Segre map $S_p^{\mathbf{t}}$ of order $\mathbf{t}$ associated to $M$ is generically of full rank. Then the family $(\mathcal{F}_\epsilon^\ell)_{\epsilon \in X}$, where $\mathcal{F}_\epsilon^\ell = \bigcup_{k=0}^\ell \mathcal{F}_{k,\epsilon}$, satisfies Definition 13 with $\mathbf{t}_0 = 2\mathbf{t}\ell$.*

In order to show Theorem 28 we follow the same line of thought as in [11]. The next Lemma is an immediate consequence of [17, Prop. 25, Cor. 26] and of standard parametrization techniques. The key fact we need to use is that in the basic identity of [17, Prop. 25] the map $\Psi$ will depend analytically on $\epsilon$, since the $\Phi_j$ appearing in the proof depend polynomially on finitely many derivatives of the defining function of $M'$. Furthermore the implicit function theorem used to obtain $\Psi$ from the $\Phi_j$ preserves analyticity in $\epsilon$; using [11, Prop. 37] allows to prove the following.

Lemma 30. *Under the assumptions of Theorem 28 the following holds: For all $\ell \in \mathbb{N}$ and $j \in J_{k_0}$ there exists a holomorphic mapping $\Psi^j_\ell : X \times \mathbb{C}^N \times \mathbb{C}^N \times \mathbb{C}^{K(k_0+\ell)N'} \to \mathbb{C}^{N'}$ such that for every curve $(\epsilon(t), H(t)) \in \mathfrak{P}^r_{(\epsilon,H)}$ with $H \in \mathcal{F}_{k_0,\epsilon}$ such that $s^j_H(p) \neq 0$, where $s^j_H$ is given as in (5), we have for sufficiently small $t$*

$$\partial^\ell H(Z,t) = \Psi^j_\ell(\epsilon(t), Z, \zeta, \partial^{k_0+\ell}\bar{H}(\zeta,t)) + O(t^{r+1}), \tag{6}$$

*for $(Z,\zeta)$ in a neighborhood of $(p,\bar{p})$ in $\mathcal{M}$, where $\partial^\ell$ denotes the collection of all derivatives up to order $\ell$. Furthermore there exist polynomials $P^{\ell,j}_{\alpha,\beta}, Q_{\ell,j}$ and integers $e^{\ell,j}_{\alpha,\beta}$ such that*

$$\Psi_{\ell,j}(\epsilon, Z, \zeta, W) = \sum_{\alpha,\beta\in\mathbb{N}_0^N} \frac{P^{\ell,j}_{\alpha,\beta}(\epsilon, W)}{Q_{\ell,j}^{e^{\ell,j}_{\alpha,\beta}}(\epsilon, W)} Z^\alpha \zeta^\beta. \tag{7}$$

The next step is a well-used technique in the study of CR maps, see [11, section 5]. In order to use the minimality criterion from [1], we need to evaluate (6) along the image of the Segre map of order $q$ and perform an iteration. We proceed as in the proof of [11, Corollary 41] to obtain the following result. Notice that the Segre maps involved do not depend on $\epsilon$, so the analytic dependence on $\epsilon$ is preserved, when equations of the form (6) are evaluated along the image of the Segre map of order $q$.

Corollary 31. *For fixed $j \in J, q \in \mathbb{N}$ with $q$ even there exists a holomorphic mapping $\varphi^j_q : X \times \mathbb{C}^{qn} \times \mathbb{C}^{K(qk_0)N'} \to \mathbb{C}^{N'}$ such that for every curve $(\epsilon(t), H(t)) \in \mathfrak{P}^r_{(\epsilon,H)}$ with $H \in \mathcal{F}_{k_0,\epsilon}$ such that $s^j_H(p) \neq 0$, where $s^j_H$ is given as in (5), we have for sufficiently small $t$*

$$H(S^q_p(x^{[1;q]}), t) = \varphi^j_q(\epsilon(t), x^{[1;q]}, j^{qk_0}_p H(t)) + O(t^{r+1}). \tag{8}$$

*Furthermore there exist (holomorphic) polynomials $R^{q,j}_\gamma, S_{q,j}$ and integers $m^{q,j}_\gamma$ such that*

$$\varphi^j_q(\epsilon, x^{[1;q]}, \Lambda) = \sum_{\gamma\in\mathbb{N}_0^{qn}} \frac{R^{q,j}_\gamma(\epsilon,\Lambda)}{S_{q,j}^{m^{q,j}_\gamma}(\epsilon,\Lambda)} (x^{[1;q]})^\gamma. \tag{9}$$

*Proof of Theorem 28.* The following is an adaptation of the proof as given in [11, proof of Thm. 36]. By the choice of $\mathbf{t} \leq d+1$, the Segre map $S^{\mathbf{t}}_p$ is generically of maximal rank. By Lemma 4.1.3 in [3], the Segre map $S^{2\mathbf{t}}_p$ is of maximal rank at $p$. Using the constant rank theorem, there exists a neighborhood $\mathcal{V}$ of $S^{2\mathbf{t}}_p$ in $(\mathbb{C}\{x^{[1;2\mathbf{t}]}\})^N$ and a map $T : \mathcal{V} \to (\mathbb{C}\{Z\})^{2\mathbf{t}n}$ such that $A \circ T(A) = Id$ for all $A \in \mathcal{V}$. We now define the holomorphic map

$$\phi : \mathcal{V} \times (\mathbb{C}\{x^{[1;2\mathbf{t}]}\})^{N'} \to (\mathbb{C}\{Z\})^{N'}, \quad \phi(A,\psi) = \psi(T(A)). \tag{10}$$

Thus we have that $\phi(A, h \circ A) = h(A(T(A))) = h$ for all $A \in \mathcal{V}$ for all $h \in (\mathbb{C}\{Z\})^{N'}$. We define $\Phi_j(\epsilon, \cdot, \Lambda) = \phi(S_p^{2\mathbf{t}}, \varphi_{2\mathbf{t}}^j(\epsilon, \cdot, \Lambda))$. Note that $\Phi_j$ depends analytically on $\epsilon$, applying $\phi(S_p^{2\mathbf{t}}, \cdot)$ to both sides of equation (8) with $q = 2\mathbf{t}$ we get

$$\begin{aligned} H(t) &= \phi(S_0^{2\mathbf{t}}, H(t) \circ S_0^{2\mathbf{t}}) = \phi(S_0^{2\mathbf{t}}, \varphi_{2\mathbf{t}}^j(\epsilon(t), \cdot, j_0^{2\mathbf{t}k_0} H(t)) + O(t^{r+1})) \\ &= \Phi_j(\epsilon(t), \cdot, j_0^{2\mathbf{t}k_0} H(t))) + O(t^{r+1}), \end{aligned}$$

which gives (b) in JPP. By setting $q_j(\epsilon, \Lambda) = S_{2\mathbf{t},j}(\epsilon, \Lambda)$, where $S_{2\mathbf{t},j}$ is given in (9), a direct computation using (9) and (10) allows to derive the expansion in (1). Let $\mathcal{U}_j$ be a neighborhood of $\{p\} \times U_j$ in $\mathbb{C}^N$ such that $\Phi_j$ is convergent on $\mathcal{U}_j$. By applying the usual procedure of plugging the form (1) into the mapping equation (after choosing a parametrization $t \mapsto \Sigma(t)$ of $M$) and developing in powers of $t$ we obtain (2). The $c_i^j$ appear as coefficients of the powers of $t$ in the expansion and depend analytically on $\epsilon$, because $\Phi_j$ and every generator of $\mathcal{I}_{p'}(M'_\epsilon)$ do. The remaining fact about $c_i^j$ follows in the same way as in [11, proof of Thm. 36]. □

**5. Construction of a singular mapping locus.** In this section we are going to provide an example of a mapping locus whose irreducible components are singular. This phenomenon cannot happen in the equidimensional case and highlights one difference with the positive codimensional case.

Let $N, N' \in \mathbb{N}$ with $2N > N'$. We write coordinates on $\mathbb{C}^{N+1}$ as $(Z, w)$ with $Z = (z_1, z_2, \ldots, z_N)$, and on $\mathbb{C}^{N'+1}$ as $(Z', w')$ with $Z' = (z'_1, z'_2, \ldots, z'_{N'})$. For any $j, k \in \mathbb{N}$, we denote by $\mathfrak{N}(j,k)$ the number of monomials in $j$ variables of order less than or equal to $k$.

Let $M = \{\operatorname{Im} w = |z_1|^2 + |z_2|^2 + \ldots + |z_N|^2\} \subset \mathbb{C}^{N+1}$. Let $\mathfrak{J}_0^{k,d}$ be the space of $k$-jets at 0 of real maps $\rho' : \mathbb{C}^{N'+1} \to \mathbb{R}^d$ such that $\rho'(0) = 0$, and let $\mathcal{J}_0^k$ be the space of $k$-jets of holomorphic maps $\psi : \mathbb{C}^{N'+1} \to \mathbb{C}^{N'+1}$ such that $\psi(0) = 0$. We can define a real-algebraic action of $\mathcal{J}_0^k$ on $\mathfrak{J}_0^{k,d}$ in the natural way: if $\psi \in \mathcal{J}_0^k$ and $\rho' \in \mathfrak{J}_0^{k,d}$, then $\psi \cdot \rho' = j_0^k(\rho' \circ \psi)$.

Our construction is based on the following crucial observation, which gives stringent conditions on possible submanifolds containing the holomorphic image of a unit sphere. We mention here the related paper of Kossovskiy and Xiao [16] which shows a similar restriction holds "the other way round": only few submanifolds of $\mathbb{C}^N$ can be embedded into a hyperquadric in $\mathbb{C}^{N'}$.

Lemma 32. *There exists $k_0 \in \mathbb{N}$ and a semi-algebraic subset $B \subset \mathfrak{J}_0^{k_0,d}$ of positive codimension such that if $\rho' : \mathbb{C}^{N'+1} \to \mathbb{R}^d$ is a local defining function, having the property that $M$ admits a holomorphic embedding in $\{\rho' = 0\}$ passing through 0, then $j_0^{k_0}\rho' \in B$.*

(With an abuse of notation, we will identify with $B$ the analogous set $B_p$ contained in the space $\mathfrak{J}_p^{k_0,d}$ of jets about $p \neq 0$.)

*Proof.* For any $k \in \mathbb{N}$, consider the subset $B'_k \subset \mathfrak{J}_0^{k,d}$ consisting of the $k$-jets of functions $\rho' : \mathbb{C}^{N'+1} \to \mathbb{R}^d$ satisfying the relation

$$\rho'(Z, 0, u + i\|Z\|^2) = O(k+1). \tag{11}$$

For any multi-indices $J \in \mathbb{N}^{2N+1}$ and $K \in \mathbb{N}^{2N'+1}$, where $J = (j, \bar{j}, \ell)$ with $j = (j_1, \ldots, j_N)$, $\bar{j} = (\bar{j}_1, \ldots, \bar{j}_N)$ and $K = (k, \bar{k}, m)$ with $k = (k_1, \ldots, k_{N'})$,

$\overline{k} = (\overline{k}_1, \ldots, \overline{k}_{N'})$ let

$$D_J = \frac{\partial^{|J|}}{\partial Z^j \partial \overline{Z}^{\overline{j}} \partial u^\ell}, \quad D'_K = \frac{\partial^{|K|}}{\partial Z'^k \partial \overline{Z'}^{\overline{k}} \partial u'^m}.$$

Applying the $D_J$ derivative (with $|J| \leq k$) to the left hand side of (11), using the chain rule and evaluating at 0 we obtain

$$D'_J \rho'(0) + (\text{algebraic expression in lower order derivatives}) = 0 \tag{12}$$

The system of equations in $\mathfrak{J}_0^{k,d}$ given by (12) (with $|J| \leq k$) has a Jacobian of full rank at 0, as can be easily verified by taking in account the triangular structure of the system. Hence $B'_k$ is a smooth algebraic submanifold of $\mathfrak{J}_0^{k,d}$, of codimension $d \cdot \mathfrak{N}(2N+1, k)$.

Note that the image $H(M)$ of any holomorphic embedding of $M$ into $\mathbb{C}^{N'+1}$ can be turned into the manifold

$$\{z'_{N+1} = \ldots = z'_{N'} = 0, \ \operatorname{Im} w' = |z'_1|^2 + \ldots + |z'_N|^2\} \tag{13}$$

by a local change of coordinates. Indeed, if $H : M \to H(M)$ is such an embedding, we can first apply a change of coordinates in $\mathbb{C}^{N'+1}$ which sends $H(\mathbb{C}^{N+1})$ to the complex $(N+1)$-plane $\{z'_{N+1} = \ldots = z'_{N'} = 0\}$. After this, we can apply a new coordinate change involving only $z'_1, \ldots, z'_N$ which sends $H(M)$ to the manifold described by (13).

Thus, for any $k \in \mathbb{N}$ the set $B_k := \mathcal{J}_0^k \cdot B'_k$ (i.e. the orbit of $B'_k$ by the action of $\mathcal{J}_0^k$) contains all the $k$-jets of local defining functions $\rho'$ such that $\{\rho' = 0\}$ admits a local embedding of $M$. Moreover, $B_k$ is a semi-algebraic subset of $\mathfrak{J}_0^{k,d}$ because it is the image of a (real)-algebraic map defined on (real)-algebraic manifolds. Since the dimension of $\mathcal{J}_0^k$ is $2\mathfrak{N}(N'+1, k)$, the dimension of $B_k$ is at most $\dim(B'_k) + 2\mathfrak{N}(N'+1, k)$, and its codimension is at least $d\mathfrak{N}(2N+1, k) - 2\mathfrak{N}(N'+1, k)$. Since $2N > N'$, there exists $k_0$ such that $d\mathfrak{N}(2N+1, k_0) - 2\mathfrak{N}(N'+1, k_0) > 0$. The proof of the lemma is concluded by putting $B = B_{k_0}$. ☐

*Proof of Theorem 2.* Write $n = N - 1$. Let $u, Z = (z_1, \ldots, z_n), \tau$ be coordinates in $\mathbb{R} \times \mathbb{C}^n \times \mathbb{C}$ with $Z = (z_1, \ldots, z_n)$, $z_j = x_j + iy_j$, $\tau = s + it$. Let $Y \subset \mathbb{C}$ be a singular real-analytic subset defined as $Y = \{r(s,t) = 0\}$ with $r$ vanishing at 0. For certain (real) polynomial functions $\alpha_1, \ldots, \alpha_n, \beta, \gamma : \mathbb{R} \times \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$, to be determined later, we define the following map $\phi : \mathbb{R} \times \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^{N+1}$:

$$\phi(u, Z, \tau) = (z_1, \ldots, z_n, z_1^2 + \ldots + z_n^2 + \tau, u + i|Z|^2) + r(\tau, \bar{\tau})\,(\alpha_1, \ldots, \alpha_n, \beta, \gamma)\,.$$

Then $d\phi$ has (real) rank $2n+3$ (at least around the origin) and $M' = \phi(\mathbb{R} \times \mathbb{C}^n \times \mathbb{C}) \subset \mathbb{C}^{N+1}$ is a real-analytic hypersurface (at least around the origin). We denote by $\rho'$ the (uniquely determined) defining function of $M'$ of the form $\rho' = \operatorname{Im} w' - f(Z', \operatorname{Re} w')$.

For any $\tau_0 \in Y$, the map $\psi_{\tau_0} : \mathbb{C}^N \to \mathbb{C}^{N+1}$ defined as

$$\psi_{\tau_0}(Z, w) = (z_1, \ldots, z_n, z_1^2 + \ldots + z_n^2 + \tau_0, w)$$

is an embedding of $M$ into $M'$. Define

$$C := \bigcup_{\tau_0 \in Y} \psi_{\tau_0}(M).$$

Note that $C$ is a real-analytic subset of $M'$ of dimension $2N$ and $C \cong Y \times M$. Let $E \subset M'$ be the mapping locus, i.e. the set of the points of $M'$ admitting a local holomorphic mapping of $M$. By construction and the homogeneity of $M$ it holds that $C \subset E$. We want to show that for a suitable choice of $\alpha_1, \dots, \alpha_n, \beta, \gamma$ the set $E$ is contained in a proper semi-analytic subset $\widetilde{E}$ of $M'$ of dimension $2N$. This proves the theorem, since if $C \subset E \subset \widetilde{E}$, where the dimension of $\widetilde{E}$ is equal to the dimension of $C$, then $E$ is a singular semi-analytic subset having $C$ as an irreducible component.

Let $\widetilde{E}$ be the set of the $p \in M'$ such that $j_p^{k_0}\rho' \in B$, where $B$ is given in Lemma 32 with $d = 1$ and $N' = N$; then $\widetilde{E}$ is a semi-analytic subset of $M'$, as a preimage of the semi-algebraic set $B$ under the analytic map $p \mapsto j_p^{k_0}\rho'$. Moreover, by definition $E \subset \widetilde{E}$. Note that, since $B$ is semi-algebraic and of positive codimension, it is contained in the union of finitely many proper algebraic subvarieties $B_1, \dots, B_\ell$ (this can be seen by disregarding the inequalities in the definition of $B$). To show that $\widetilde{E}$ is a semi-analytic subset of positive codimension, then it will be enough to check that for a suitable choice of $\alpha_1, \dots, \alpha_n, \beta, \gamma$ and of $\tau_0$ close enough to 0, putting $\phi(0, \tau_0) = p_0$ we have $j_{p_0}^{k_0}\rho' \notin B_j$ for all $j = 1, \dots, \ell$. In this case $\widetilde{E}$ is contained in the union of the real-analytic sets $X_j = \{p \in M' : j_p^{k_0}\rho' \in B_j\}$, each of them proper since $p_0 \notin X_j$, and hence it has positive codimension.

To show the claim, choose $\tau_0 = s_0 + it_0$ close enough to 0 such that $\tau_0 \notin Y$, consider the change of coordinates $s' = s - s_0, t' = t - t_0$, and let $q$ be such that $q(s', t') = r(s, t)$. We have $q(0, 0) \neq 0$, so that we can consider the real power series centered at 0 defining $1/q$. Let $\delta$ be any complex-valued power series in the variables $u, Z, \tau' = s' + it'$, and let $\alpha = j^{k_0}(\delta\frac{1}{q})$: then $\alpha$ is a polynomial in $u, Z, \tau'$ and

$$j^{k_0}(q\alpha) = j^{k_0}\left(j^{k_0}q j^{k_0}\left(\delta\frac{1}{q}\right)\right) = j^{k_0}\left(j^{k_0}(\delta) j^{k_0}(q) j^{k_0}\left(\frac{1}{q}\right)\right) = j^{k_0}\delta.$$

In other words we can obtain any prescribed jet of $\phi$ at $(0, \tau_0)$, and hence any prescribed jet of $\rho'$ at $\phi(0, \tau_0) = p_0$, by the appropriate choice of $\alpha_1, \dots, \alpha_n, \beta, \gamma$: in particular we can choose them in such a way that $j_{p_0}^{k_0}\rho' \notin \bigcup_{j=1}^{\ell} B_j$. $\square$

Remark 33. More specifically, the proof of Theorem 2 shows that any given singular set $Y$ of $\mathbb{C}$ can be realized in the following sense: We can construct a hypersurface $M' \subset \mathbb{C}^{N+1}$ such that there exists an irreducible component $C$ of $E_{\mathcal{F}}$ with the property that $C \cong Y \times M$.

**6. Examples.** We wish to present a list of examples showing the various phenomena that can appear in the geometric structure of the mapping locus. Similar behaviors can be seen in the (equidimensional) case of the equivalence locus, cf. [8, §7]. Indeed that one is a special case of our setting. However we would like to construct examples in the positive codimensional case.

Example 1. Let $(M, 0)$ be the germ at 0 of $M = \{\operatorname{Im} w = |z|^2\} \subset \mathbb{C}^2$, and $M' = \{\operatorname{Im} w' = |z_1'|^4 + |z_2'|^4\}$. In this case we take $\mathcal{F} = \mathcal{F}_k$ as the set of $k$-nondegenerate maps from $M$ into $M'$. Then the mapping locus $E_{\mathcal{F}}$ is equal to $M' \setminus Y$, where $Y = (\{z_1 = 0\} \cup \{z_2 = 0\}) \cap M'$. Indeed, there are no $k$-nondegenerate maps $H$ from $M$ into $M'$, such that $H(0) \in Y$. This follows from the fact that $M'$ is not finitely nondegenerate at the points of $Y$ and that the image of $k$-nondegenerate maps is contained in the set of points of $M'$ at which $M'$ is at most $k$-nondegenerate, cf. [17, §3]. Moreover $M' \setminus Y \subset E_{\mathcal{F}}$: Let $p' = (\hat{z}_1', \hat{z}_2', \hat{w}') \in M' \setminus Y$ and $\psi(z_1', z_2', w') = ({z_1'}^2, {z_2'}^2, w')$. Denote

$H^5 = \{\operatorname{Im} w' = |z_1'|^2 + |z_2'|^2\}$. Since $p' \notin Y$ there is a neighborhood $U'$ of $p'$ such that $\psi : U' \to V' = \psi(U')$ is a biholomorphism and $\psi(U' \cap M') = H^5 \cap V'$. A map $H : M \to M'$ with $H(0) = p'$ can then be constructed by choosing a 2-nondegenerate map $F$ from $M$ into $H^5$ such that $F(0) = \psi(p')$ and taking $H = \psi^{-1} \circ F$.

This example shows that the mapping locus need not be an analytic variety, but in general it is just semi-analytic, that is, inequalities can indeed occur. The following example illustrates the situation of Theorem 25.

Example 2. Let $(M, 0)$ be the germ at 0 of $M = \{\operatorname{Im} w = |z|^2 + |z|^4\} \subset \mathbb{C}^2$ and $M' = \{\operatorname{Im} w' = |z_1'|^2 + |z_1'|^4 + (\operatorname{Re}(z_1'))^2 \operatorname{Im}(z_2')\} \subset \mathbb{C}^3$. Let $\mathcal{F} = \mathcal{F}_2$. The map $H_{s,t} : (z, w) \mapsto (z, t, w + s)$ sends $(M, 0)$ into $M'$ for all $(s, t) \in \mathbb{R}^2$ and belongs to $\mathcal{F}$, hence $S = \{(0, t, s) : (s, t) \in \mathbb{R}^2\} \subset E_{\mathcal{F}}$ and $T_pS \subset \mathfrak{hol}(H_{s,t})(0)$, where $p = (0, t, s)$, cf. Theorem 25. Indeed, a computation shows that $T_pS = \mathfrak{hol}(H_{s,t})(0)$.

Example 3. Let $M, M', H_{0,t}$ and $\mathcal{F}$ be as in Example 2. For a closed subset $Y \subset \mathbb{R}$ choose a $C^\infty$ function $\Phi : \mathbb{R} \to \mathbb{R}$ vanishing exactly on $Y$. Define the $C^\infty$ submanifold $\widetilde{M}'$ by

$$\operatorname{Im} w' = |z_1'|^2 + |z_1'|^4 + (\operatorname{Re}(z_1'))^2 \operatorname{Im}(z_2') + \Phi(\operatorname{Re}(z_2'))K(z_1', z_2', \operatorname{Re} w'),$$

where we set $U := \widetilde{M}' \cap \{\operatorname{Re}(z_2') \notin Y\}$ and choose $K$ a $C^\infty$-function in such a way that $U$ does not contain any analytic subvariety. Then for $t \in Y$ it holds that $H_{0,t}(M) \subset \widetilde{M}'$, which implies that $Y' := \bigcup_{t \in Y}\{H_{0,t}(0)\} \subset E_{\mathcal{F}}$. We would like to argue that $E_{\mathcal{F}} = Y'$: Indeed any holomorphic embedding of $M$ into $\widetilde{M}'$ has the property that its image has empty intersection with $U$, because $U$ does not contain any analytic submanifold.

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