

MODULI OF CURVES OF GENUS ONE WITH TWISTED FIELDS*

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Abstract. We construct a smooth Artin stack parameterizing the stable weighted curves of genus one with twisted fields and prove that it is isomorphic to the blowup stack of the moduli of genus one weighted curves studied by Hu and Li. This leads to a blowup-free construction of Vakil-Zinger’s desingularization of the moduli of genus one stable maps to projective spaces. This construction provides the cornerstone of the theory of stacks with twisted fields, which is thoroughly studied in [8] and leads to a blowup-free resolution of the stable map moduli of genus two.

Key words. moduli of weighted curves, twisted fields, blowup-free desingularization.

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1. Introduction. Moduli problems are of central importance in algebraic geometry. Many moduli spaces possess arbitrary singularities [12]. Among them, the moduli $\overline{M}_g(\mathbb{P}^n, d)$ of degree d stable maps from genus g nodal curves into projective spaces \mathbb{P}^n are particularly important. We aim to resolve singularities of $\overline{M}_g(\mathbb{P}^n, d)$, that is, to construct a new Deligne-Mumford stack $\widetilde{M}_g(\mathbb{P}^n, d)$ admitting a proper morphism onto $\overline{M}_g(\mathbb{P}^n, d)$ and having smooth irreducible components with at worst normal crossing singularities such that the primary component of $\widetilde{M}_g(\mathbb{P}^n, d)$ dominates the primary component of $\overline{M}_g(\mathbb{P}^n, d)$ birationally, at least, when d is sufficiently large. The problem of resolution of singularities is arguably among the hardest ones in algebraic geometry [3, 4, 9, 10].

The stable map moduli are smooth if $g = 0$ and singular if $g \geq 1$ and $n \geq 2$. For $g = 1$, a resolution was constructed by Vakil and Zinger [13], followed by an algebraic approach of Hu and Li [5]. The latter is achieved by constructing a canonical smooth blowup $\widetilde{\mathfrak{M}}_1^{\text{wt}}$ of the Artin stack $\mathfrak{M}_1^{\text{wt}}$ of stable weighted nodal curves of genus one. The method of [5] was further developed in [7] to finally establish a resolution in the case of $g = 2$. The resolution of [7] is achieved by constructing a canonical smooth blowup $\widetilde{\mathfrak{P}}_2$ of the relative Picard stack \mathfrak{P}_2 of nodal curves of genus two.

In higher genus cases, the construction of a possible resolution of the stable map moduli may seem formidable. The constructions of the explicit resolutions in [13, 5, 7] rely on certain precise knowledge on the singularities of the moduli. For arbitrary genus, it calls for a more abstract and geometric approach. As advocated by the first author, every singular moduli space should admit a resolution which itself is also a moduli. Following this principle, we interpret the blowup stack $\widetilde{\mathfrak{M}}_1^{\text{wt}}$ of [5] as a smooth algebraic stack of stable weighted nodal curves of genus one with twisted fields, and consequently, the resolution $\widetilde{M}_1(\mathbb{P}^n, d)$ of $\overline{M}_1(\mathbb{P}^n, d)$ as a Deligne-Mumford stack of genus one stable maps with twisted fields. The results in this paper are the first step to tackle the arbitrary genus case.

The main theorem of this paper is the following:

THEOREM 1.1. *There exists a smooth Artin stack $\mathfrak{M}_1^{\text{tf}}$ parameterizing the weighted nodal curves of genus one with twisted fields, along with a universal family $\mathcal{C}^{\text{tf}} \rightarrow$*

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$\mathfrak{M}_1^{\text{tf}}$ and a proper and birational forgetful morphism $\varpi : \mathfrak{M}_1^{\text{tf}} \rightarrow \mathfrak{M}_1^{\text{wt}}$. Moreover, $\mathfrak{M}_1^{\text{tf}}/\mathfrak{M}_1^{\text{wt}}$ is isomorphic to the blowup stack $\widetilde{\mathfrak{M}}_1^{\text{wt}}/\mathfrak{M}_1^{\text{wt}}$.

We remark that both $\mathfrak{M}_1^{\text{tf}}$ and $\mathfrak{M}_1^{\text{wt}}$ are the disjoint unions of infinitely many smooth components, each of which is of finite type and indexed by the total weight of the weighted curves. By saying $\mathfrak{M}_1^{\text{tf}} \rightarrow \mathfrak{M}_1^{\text{wt}}$ is birational, we mean it is birational on each connected component.

The outline of the proof of Theorem 1.1 is as follows. We first construct the strata of $\mathfrak{M}_1^{\text{tf}}$ and the forgetful map ϖ in §2; see (2.14). We then glue the strata of $\mathfrak{M}_1^{\text{tf}}$ together using smooth charts in §3 and conclude that $\mathfrak{M}_1^{\text{tf}}$ is a smooth Artin stack and is birational to $\mathfrak{M}_1^{\text{wt}}$ in Corollary 3.8. The universal family $\mathcal{C}^{\text{tf}} \rightarrow \mathfrak{M}_1^{\text{tf}}$ is described in Proposition 3.12. We finally show that $\mathfrak{M}_1^{\text{tf}}/\mathfrak{M}_1^{\text{wt}}$ is isomorphic to $\widetilde{\mathfrak{M}}_1^{\text{wt}}/\mathfrak{M}_1^{\text{wt}}$ in Proposition 4.4, which implies the properness of ϖ . These results together establish Theorem 1.1.

We remark that a direct approach to the properness of ϖ (i.e. without the comparison with the blowup $\widetilde{\mathfrak{M}}_1^{\text{wt}}/\mathfrak{M}_1^{\text{wt}}$) is provided in the proof of [8, Theorem 2.19(p₁)], in a more general setting.

We also point out that there should exist a groupoid, represented by $\mathfrak{M}_1^{\text{tf}}$, that sends any scheme S to the set of the flat families of stable weighted nodal curves of genus 1 with twisted fields over S as in (3.34); see Remark 3.13 for some details.

According to [5], the resolution $\widetilde{M}_1(\mathbb{P}^n, d)$ of $\overline{M}_1(\mathbb{P}^n, d)$ is given by

$$\widetilde{M}_1(\mathbb{P}^n, d) = \overline{M}_1(\mathbb{P}^n, d) \times_{\mathfrak{M}_1^{\text{wt}}} \widetilde{\mathfrak{M}}_1^{\text{wt}},$$

where

$$\overline{M}_1(\mathbb{P}^n, d) \rightarrow \mathfrak{M}_1^{\text{wt}}, \quad [C, \mathbf{u}] \mapsto [C, c_1(\mathbf{u}^* \mathcal{O}_{\mathbb{P}^n}(1))] \tag{1.1}$$

and $\widetilde{\mathfrak{M}}_1^{\text{wt}} \rightarrow \mathfrak{M}_1^{\text{wt}}$ is the canonical blowup. Analogously, we take

$$\widetilde{M}_1^{\text{tf}}(\mathbb{P}^n, d) := \overline{M}_1(\mathbb{P}^n, d) \times_{\mathfrak{M}_1^{\text{wt}}} \mathfrak{M}_1^{\text{tf}},$$

where $\mathfrak{M}_1^{\text{tf}} \rightarrow \mathfrak{M}_1^{\text{wt}}$ is the forgetful morphism aforementioned. Theorem 1.1 then leads to the following conclusion immediately.

COROLLARY 1.2. *$\widetilde{M}_1^{\text{tf}}(\mathbb{P}^n, d)$ is a proper Deligne-Mumford stack and is isomorphic to $\widetilde{M}_1(\mathbb{P}^n, d)$.*

Via the above isomorphism and applying [5], one sees that the stack $\widetilde{M}_1^{\text{tf}}(\mathbb{P}^n, d)$ provides a resolution of $\overline{M}_1(\mathbb{P}^n, d)$. Nonetheless, without relating to $\widetilde{M}_1(\mathbb{P}^n, d)$, we can directly prove the resolution property of $\widetilde{M}_1^{\text{tf}}(\mathbb{P}^n, d)$ by investigating the local equations of $\overline{M}_1(\mathbb{P}^n, d)$ in [5] and their pullbacks to $\widetilde{M}_1^{\text{tf}}(\mathbb{P}^n, d)$; see Remark 3.9.

The methods and ideas of this paper are essential to the development in [8] and forthcoming works. Based on the construction of $\mathfrak{M}_1^{\text{tf}}$, we introduce the theory of stacks with twisted fields (STF) in [8, Theorem 2.19]. To be somewhat more informative, we work on a smooth stack \mathfrak{M} that has a stratification indexed by a set Γ of graphs similar to (2.10); see [8, Definition 2.15]. The graphs in Γ need not to come from the dual graphs as in (2.10), but the stratification of \mathfrak{M} should resemble (3.2) locally. Moreover, Γ need not to consist of trees, but it should contain necessary information on the notion of the (weighted) level trees in Definition 2.1 so that we can add the twisted fields to the strata of \mathfrak{M} parallel to (2.13) and obtain a new stack

\mathfrak{M}^{tf} ; see [8, Definition 2.17]. Such \mathfrak{M}^{tf} enjoys desirable properties as in Corollary 3.8 and Remark 3.9. As an application of the STF theory, in [8], we construct a smooth Artin stack $\mathfrak{P}_2^{\text{tf}}$ of genus 2 nodal curves with line bundles and twisted fields, along with a proper and birational forgetful morphism $\mathfrak{P}_2^{\text{tf}} \rightarrow \mathfrak{P}_2$, such that

$$\widetilde{M}_2^{\text{tf}}(\mathbb{P}^n, d) = \overline{M}_2(\mathbb{P}^n, d) \times_{\mathfrak{P}_2} \mathfrak{P}_2^{\text{tf}} \rightarrow \overline{M}_2(\mathbb{P}^n, d)$$

provides a resolution. Further, we expect that they can be extended to arbitrary genus, as far as the existence of moduli of nodal curves with twisted fields is concerned. This is the main motivation of the current article.

In a related work [11], D. Ranganathan, K. Santos-Parker, and J. Wise provide a different modular perspective of $\widetilde{M}_1(\mathbb{P}^n, d)$ using logarithmic geometry.

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Convention. The subscript “1” of the relevant stacks indicating the genus appears only in §1 and will be omitted starting §2, as we only deal with the genus 1 case in this paper. In particular, we will denote by

$$\mathfrak{M}^{\text{wt}} \quad \text{and} \quad \mathfrak{M}^{\text{tf}}$$

the aforementioned stacks $\mathfrak{M}_1^{\text{wt}}$ and $\mathfrak{M}_1^{\text{tf}}$, respectively.

2. Set-theoretic descriptions. In §2.1, we discuss the combinatorics of the dual graphs of nodal curves and introduce the notion of the weighted level trees. They will be used to define \mathfrak{M}^{tf} set-theoretically in §2.2.

2.1. Weighted level trees. Let γ be a **rooted tree**, i.e. a connected finite graph that contains no cycles, along with a special vertex o , called the **root**. The sets of the vertices and the edges of γ are denoted by

$$\text{Ver}(\gamma) \quad \text{and} \quad \text{Edg}(\gamma),$$

respectively. The set $\text{Ver}(\gamma)$ is endowed with a partial order $>_\gamma$, called the **tree order**, so that $v >_\gamma v'$ if and only if $v \neq v'$ and v belongs to a path between o and v' . When the context is clear, we drop the subscript γ and simply denote by $>$ the tree order. We write $v \geq v'$ if either $v > v'$ or $v = v'$. The root o is thus the unique maximal element of $\text{Ver}(\gamma)$ with respect to the tree order.

For each $e \in \text{Edg}(\gamma)$, we denote by $v_e^\pm \in \text{Ver}(\gamma)$ the endpoints of e such that $v_e^+ > v_e^-$. Then, every vertex $v \in \text{Ver}(\gamma) \setminus \{o\}$ corresponds to a unique

$$e_v \in \text{Edg}(\gamma) \quad \text{satisfying} \quad v_{e_v}^- = v.$$

The tree order on $\text{Ver}(\gamma)$ induces a partial order on $\text{Edg}(\gamma)$, still called the **tree order**, so that

$$e > e' \iff v_e^- \geq v_{e'}^+.$$

We write $e \geq e'$ if either $e > e'$ or $e = e'$.

We call a pair $\tau = (\gamma, \mathbf{w})$ consisting of a rooted tree γ and a function

$$\mathbf{w} : \text{Ver}(\gamma) \rightarrow \mathbb{Z}_{\geq 0}$$

a weighted tree. For such τ , we write $\text{Ver}(\tau) = \text{Ver}(\gamma)$ and $\text{Edg}(\tau) = \text{Edg}(\gamma)$. The set of all the weighted trees is denoted by $\mathcal{T}_{\mathbb{R}}^{\text{wt}}$, which will be used to index the stratification 2.10.

Next, we introduce the extra structures needed for \mathfrak{M}^{tf} . We call a map $\ell: \text{Ver}(\gamma) \rightarrow \mathbb{R}_{\leq 0}$ satisfying

$$\ell^{-1}(0) = \{o\} \quad \text{and} \quad \ell(v) > \ell(v') \quad \text{whenever} \quad v > v'$$

a level map. For each $i \in \ell(\text{Ver}(\gamma)) \setminus \{0\}$, let

$$i^\# = \min \{ k \in \ell(\text{Ver}(\gamma)) : k > i \}, \tag{2.1}$$

i.e. the level $i^\#$ is right “above” the level i ; see Figure 1. We remark that a rooted tree along with a level map is called a *level graph* with the root as the unique *top level* vertex in [1, §1.5].

DEFINITION 2.1. *We call the tuple*

$$\mathfrak{t} = (\gamma, \mathbf{w}: \text{Ver}(\gamma) \rightarrow \mathbb{Z}_{\geq 0}, \ell: \text{Ver}(\gamma) \rightarrow \mathbb{R}_{\leq 0})$$

a weighted level tree if $(\gamma, \mathbf{w}) \in \mathcal{T}_{\mathbb{R}}^{\text{wt}}$ and ℓ is a level map.

For every weighted level tree \mathfrak{t} as above, we write $\text{Ver}(\mathfrak{t}) = \text{Ver}(\gamma)$ and $\text{Edg}(\mathfrak{t}) = \text{Edg}(\gamma)$. Set

$$\begin{aligned} \mathbf{m} &= \mathbf{m}(\mathfrak{t}) = \max \{ \ell(v) : v \in \text{Ver}(\mathfrak{t}), \mathbf{w}(v) > 0 \} && (\leq 0), \\ \widehat{\text{Edg}}(\mathfrak{t}) &= \{ e \in \text{Edg}(\mathfrak{t}) : \ell(v_e^+) > \mathbf{m} \} && (\subset \text{Edg}(\mathfrak{t})). \end{aligned}$$

For any two levels $i, j \in \mathbb{R}_{\leq 0}$, we write

$$([i, j])_{\mathfrak{t}} = \ell(\text{Ver}(\mathfrak{t})) \cap (i, j), \quad \llbracket i, j \rrbracket_{\mathfrak{t}} = \ell(\text{Ver}(\mathfrak{t})) \cap [i, j]. \tag{2.2}$$

For every $e \in \widehat{\text{Edg}}(\mathfrak{t})$, let

$$\ell(e) = \max \{ \ell(v_e^-), \mathbf{m} \} \quad (\in \llbracket \mathbf{m}, 0 \rrbracket_{\mathfrak{t}}).$$

For each level $i \in \llbracket \mathbf{m}, 0 \rrbracket_{\mathfrak{t}}$, we set

$$\mathfrak{E}_i = \mathfrak{E}_i(\mathfrak{t}) = \{ e \in \widehat{\text{Edg}}(\mathfrak{t}) : \ell(e) \leq i < \ell(v_e^+) \}. \tag{2.3}$$

Intuitively, \mathfrak{E}_i consists of all the edges crossing the gap between the levels i and $i^\#$.

We point out that all the notions in the preceding paragraph depend on the weighted level tree \mathfrak{t} , and we may hereafter omit \mathfrak{t} in any of such notions when the context is clear.

REMARK 2.2. The weighted level trees will be used to label the strata $\mathfrak{M}_{[\mathfrak{t}]}^{\text{tf}}$ of \mathfrak{M}^{tf} . Locally speaking, the edges in $\widehat{\text{Edg}}(\mathfrak{t})$ correspond to the normal directions of the blowup centers of $\widetilde{\mathfrak{M}}^{\text{wt}} \rightarrow \mathfrak{M}^{\text{wt}}$ in [5]; each level $i (\geq \mathbf{m})$ corresponds to an exceptional divisor \widetilde{E}'_i of the sequential blowup; \mathfrak{E}_i correspond to the components of \widetilde{E}'_i ; the edges $e \in \mathfrak{E}_i$ with $\ell(v_e^-) = i$ correspond to the non-zero components of \widetilde{E}'_i ; the level \mathbf{m} indicates when the sequential blowup terminates.

Every weighted level tree \mathfrak{t} determines a unique index set

$$\begin{aligned} \mathbb{I}(\mathfrak{t}) &= \mathbb{I}_+(\mathfrak{t}) \sqcup \mathbb{I}_{\mathbf{m}}(\mathfrak{t}) \sqcup \mathbb{I}_-(\mathfrak{t}), & \text{where } \mathbb{I}_+(\mathfrak{t}) &= \llbracket \mathbf{m}, 0 \rrbracket_{\mathfrak{t}}, \\ \mathbb{I}_{\mathbf{m}}(\mathfrak{t}) &= \{e \in \widehat{\text{Edg}}(\mathfrak{t}) : \ell(v_e^-) < \mathbf{m}\}, & \mathbb{I}_-(\mathfrak{t}) &= (\text{Edg}(\mathfrak{t}) \setminus \widehat{\text{Edg}}(\mathfrak{t})). \end{aligned} \tag{2.4}$$

The set $\mathbb{I}_+(\mathfrak{t})$ becomes empty if $\mathbf{m} = 0$, i.e. the root o is positively weighted. As mentioned before, we may simply write

$$\mathbb{I} = \mathbb{I}(\mathfrak{t}), \quad \mathbb{I}_{\pm} = \mathbb{I}_{\pm}(\mathfrak{t}), \quad \mathbb{I}_{\mathbf{m}} = \mathbb{I}_{\mathbf{m}}(\mathfrak{t})$$

when the context is clear. We point out that \mathbb{I}_+ consists of levels, whereas $\mathbb{I}_{\mathbf{m}}$ and \mathbb{I}_- consist of edges.

For each $I \subset \mathbb{I}$ (possibly empty), let

$$I_{\mathbf{m}} = I \cap \mathbb{I}_{\mathbf{m}}, \quad I_{\pm} = I \cap \mathbb{I}_{\pm}.$$

The subset I determines a weighted level tree

$$\mathfrak{t}_{(I)} = (\tau_{(I)}, \ell_{(I)}) = (\gamma_{(I)}, \mathbf{w}_{(I)}, \ell_{(I)}) \tag{2.5}$$

intuitively as follows: the weighted level tree $\mathfrak{t}_{(I)}$ is obtained from \mathfrak{t} by contracting all the edges labeled by I_- , then lifting all the vertices v with $e_v \in I_{\mathbf{m}}$ to the level \mathbf{m} , and finally contracting all the levels in I_+ ; see Figures 2 and 3 for illustration. Such $\mathfrak{t}_{(I)}$ will be used to describe the local structure of \mathfrak{M}^{tf} in §3.

Below we present the formal definition of $\mathfrak{t}_{(I)}$:

- the rooted tree $\gamma_{(I)}$ is obtained via the edge contraction

$$\pi_{(I)} : \text{Ver}(\gamma) \rightarrow \text{Ver}(\gamma_{(I)})$$

such that the set of the contracted edges is

$$\text{Edg}(\mathfrak{t}) \setminus \text{Edg}(\mathfrak{t}_{(I)}) = \{e \in (\widehat{\text{Edg}}(\mathfrak{t}) \setminus \mathbb{I}_{\mathbf{m}}) \sqcup I_{\mathbf{m}} : \llbracket \ell(e), \ell(v_e^+) \rrbracket_{\mathfrak{t}} \subset I_+\} \sqcup I_-; \tag{2.6}$$

- the weight function $\mathbf{w}_{(I)}$ is given by

$$\mathbf{w}_{(I)} : \text{Ver}(\gamma_{(I)}) \rightarrow \mathbb{Z}_{\geq 0}, \quad \mathbf{w}_{(I)}(v) = \sum_{v' \in \pi_{(I)}^{-1}(v)} \mathbf{w}(v');$$

- the level map $\ell_{(I)}$ is such that for any $e \in \text{Edg}(\gamma_{(I)})$ ($\subset \text{Edg}(\mathfrak{t})$),

$$\ell_{(I)}(v_e^-) = \begin{cases} \min\{i \in \mathbb{I}_+ \setminus I_+ : i \geq \ell(v) \ \forall v \in \pi_{(I)}^{-1}(v_e^-)\} & \text{if } e \in \widehat{\text{Edg}}(\mathfrak{t}) \setminus \mathbb{I}_{\mathbf{m}}, \\ \min(\mathbb{I}_+ \setminus I_+) & \text{if } e \in I_{\mathbf{m}}, \\ \max\{\ell(v) : v \in \pi_{(I)}^{-1}(v_e^-)\} & \text{if } e \in (\mathbb{I}_{\mathbf{m}} \setminus I_{\mathbf{m}}) \sqcup \mathbb{I}_-. \end{cases}$$

It is a direct check that $\tau_{(I)}$ is a weighted tree and $\ell_{(I)}$ satisfies the criteria of a level map, hence (2.5) gives a well defined weighted level tree.

The construction of $\mathfrak{t}_{(I)}$ implies

$$\begin{aligned} \mathbb{I}_+(\mathfrak{t}_{(I)}) &= \mathbb{I}_+ \setminus I_+, & \mathbf{m}(\mathfrak{t}_{(I)}) &= \min((\mathbb{I}_+ \setminus I_+) \sqcup \{0\}), \\ \mathbb{I}_{\mathbf{m}}(\mathfrak{t}_{(I)}) &= \{e \in \mathbb{I}_{\mathbf{m}} \setminus I_{\mathbf{m}} : \ell(v_e^+) > \mathbf{m}(\mathfrak{t}_{(I)})\}, \\ \mathbb{I}_-(\mathfrak{t}_{(I)}) &= \mathbb{I}_- \setminus I_- \sqcup \{e \in \mathbb{I}_{\mathbf{m}} \setminus I_{\mathbf{m}} : \ell(v_e^+) \leq \mathbf{m}(\mathfrak{t}_{(I)})\}. \end{aligned} \tag{2.7}$$

DEFINITION 2.3. *Two weighted level trees $\mathfrak{t} = (\gamma, \mathbf{w}, \ell)$ and $\mathfrak{t}' = (\gamma', \mathbf{w}', \ell')$ are said to be equivalent, written as $\mathfrak{t} \sim \mathfrak{t}'$, if*

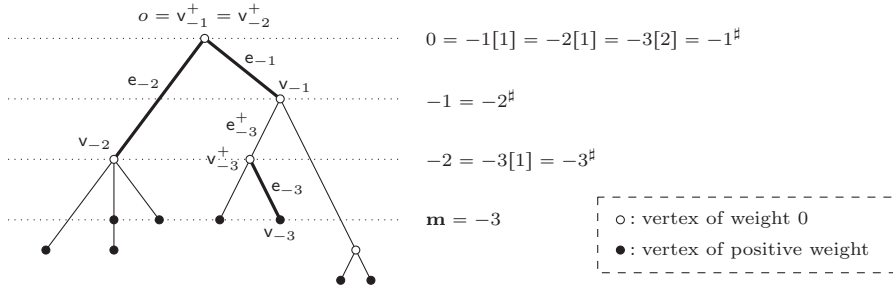


FIG. 1. A weighted level tree with chosen v_{-1}, v_{-2} , and v_{-3}

- (E1) $(\gamma, \mathbf{w}) = (\gamma', \mathbf{w}')$ as weighted trees;
- (E2) $\ell^{-1}[\mathbf{m}(t), 0]_t = (\ell')^{-1}[\mathbf{m}(t'), 0]_{t'}$;
- (E3) there exists a (unique) order preserving bijection $\alpha: [\mathbf{m}(t), 0]_t \rightarrow [\mathbf{m}(t'), 0]_{t'}$ so that $\alpha \circ \ell = \ell'$ on $\ell^{-1}[\mathbf{m}(t), 0]_t$.

It is a direct check that \sim is an equivalence relation on the set of weighted level trees. Intuitively, this equivalence relation records the relative positions of the vertices *above* or *in* the level $\mathbf{m}(t)$; see Figure 1 for illustration. We do not need to know the relative positions of the vertices below $\mathbf{m}(t)$, because from the blowup point of view, the level $\mathbf{m}(t)$ corresponds to when the blowup $\widetilde{\mathfrak{M}}^{\text{wt}} \rightarrow \mathfrak{M}^{\text{wt}}$ stops locally; c.f. Remark 2.2.

We denote by $\mathcal{T}_L^{\text{wt}}$ the set of the *equivalence classes* of the weighted level trees. There is a natural forgetful map

$$f: \mathcal{T}_L^{\text{wt}} \rightarrow \mathcal{T}_R^{\text{wt}}, \quad [\gamma, \mathbf{w}, \ell] \mapsto (\gamma, \mathbf{w}), \tag{2.8}$$

which is well defined by Condition (E1) of Definition 2.3.

2.2. Twisted fields. For every genus 1 nodal curve C , its dual graph γ_C^* has either a unique vertex o corresponding to the genus 1 irreducible component of C or a unique loop. In the former case, γ_C^* can be considered as a rooted tree with the root o ; in the latter case, we contract the loop to a single vertex o and obtain a rooted tree with the root o . Such defined rooted tree is denoted by γ_C and called the **reduced dual tree** of C (c.f. [5, §3.4]). We call the minimal connected genus 1 subcurve of C the **core** and denote it by C_o . Other irreducible components of C are smooth rational curves and denoted by $C_v, v \in \text{Ver}(\gamma_C) \setminus \{o\}$. For every incident pair (v, e) , let

$$q_{v,e} \in C_v \tag{2.9}$$

be the nodal point corresponding to the edge e .

Let \mathfrak{M}^{wt} be the Artin stack of genus 1 stable weighted curves introduced in [5, §2.1]. Here the subscript “1” indicating the genus is omitted as per our convention. The stack \mathfrak{M}^{wt} consists of the pairs (C, \mathbf{w}) of genus 1 nodal curves C with non-negative weights $\mathbf{w} \in H^2(C, \mathbb{Z})$, meaning that $\mathbf{w}(\Sigma) \geq 0$ for all irreducible $\Sigma \subset C$. Here (C, \mathbf{w}) is said to be **stable** if every rational irreducible component of weight 0 contains at least three nodal points. The weight of the core $\mathbf{w}(C_o)$ is defined as the sum of the weights of all irreducible components of the core.

Every $(C, \mathbf{w}) \in \mathfrak{M}^{\text{wt}}$ uniquely determines a function

$$\mathbf{w}: \text{Ver}(\gamma_C) \rightarrow \mathbb{Z}_{\geq 0}, \quad v \mapsto \mathbf{w}(C_v),$$

which makes the pair (γ_C, \mathbf{w}) a weighted tree, called the weighted dual tree. Thus, the stack \mathfrak{M}^{wt} can be stratified as

$$\mathfrak{M}^{\text{wt}} = \bigsqcup_{\tau \in \mathcal{T}_R^{\text{wt}}} \mathfrak{M}_\tau^{\text{wt}} = \bigsqcup_{\tau \in \mathcal{T}_R^{\text{wt}}} \{ (C, \mathbf{w}) \in \mathfrak{M}^{\text{wt}} : (\gamma_C, \mathbf{w}) = \tau \}. \tag{2.10}$$

If the sum of the weights of all vertices is fixed, the stability condition of \mathfrak{M}^{wt} then guarantees there are only finitely many $\tau \in \mathcal{T}_R^{\text{wt}}$ so that $\mathfrak{M}_\tau^{\text{wt}}$ is non-empty.

Given $\tau = (\gamma, \mathbf{w}) \in \mathcal{T}_R^{\text{wt}}$ and $e \in \text{Edg}(\tau)$, let

$$L_e^\pm \longrightarrow \mathfrak{M}_\tau^{\text{wt}}$$

be the line bundles whose fibers over a weighted curve (C, \mathbf{w}) are the tangent vectors of the irreducible components $C_{v_e^\pm}$ at the nodal points $q_{v_e^\pm; e}$, respectively. We take

$$L_e = L_e^+ \otimes L_e^- \longrightarrow \mathfrak{M}_\tau^{\text{wt}}, \quad L_e^\geq = \bigotimes_{e' \in \text{Edg}(\tau), e' \geq e} L_{e'} \longrightarrow \mathfrak{M}_\tau^{\text{wt}}. \tag{2.11}$$

The line bundles L_e describe normal directions of $\mathfrak{M}_\tau^{\text{wt}}$; c.f. Lemma 3.1 and its proof.

REMARK 2.4. The motivation of L_e^\geq is twofold: one comes from the comparison with the blowup construction $\widetilde{\mathfrak{M}}^{\text{wt}} \longrightarrow \mathfrak{M}^{\text{wt}}$, while the other from close examination of the local equations of $\overline{M}_1(\mathbb{P}^n, d)$. The former will be elaborated in (4.5) and the paragraph after it; we now explain the latter. Given $(C, u) \in \overline{M}_1(\mathbb{P}^n, d)$, we denote by (C, \mathbf{w}) its image in \mathfrak{M}^{wt} under (1.1). By (2.10), there exists a unique $\tau \in \mathcal{T}_R^{\text{wt}}$ such that $(C, \mathbf{w}) \in \mathfrak{M}_\tau^{\text{wt}}$. W.l.o.g. we assume that $u|_{C_o}$ is constant (i.e. $\mathbf{w}(o) = 0$), for otherwise $\overline{M}_1(\mathbb{P}^n, d)$ is already smooth near (C, u) . As shown in [5, §5.2], over an affine smooth chart $\mathcal{V} \longrightarrow \mathfrak{M}^{\text{wt}}$ containing (C, \mathbf{w}) , one can identify an open neighborhood of $(C, u) \in \overline{M}_1(\mathbb{P}^n, d)$ with the kernel of a homomorphism of the trivial vector bundles: $(\mathcal{O}_{\mathcal{V}}^{\oplus d})^{\oplus n} \longrightarrow \mathcal{O}_{\mathcal{V}}^{\oplus n}$. Under suitable trivialization, the vector bundle homomorphism can be written as a matrix. Each entry of the matrix is (up to a unit on \mathcal{V}) in the form $\prod_{e' \geq e} \zeta_{e'}$, where $e \in \text{Edg}(\tau)$ satisfies $\mathbf{w}(v_e^-) > 0$, and the local parameters $\zeta_{e'}$ correspond to the smoothing of the nodes (see (3.1) for exact definition).

To desingularize $\overline{M}_1(\mathbb{P}^n, d)$, one needs to compare the products $\prod_{e' \geq e} \zeta_{e'}$ for different edges e near $\mathfrak{M}_\tau^{\text{wt}}$ (where all $\zeta_{e'} = 0$). Given two edges e_1 and e_2 , the limit of the ratio $[(\prod_{e' \geq e_1} \zeta_{e'}) : (\prod_{e'' \geq e_2} \zeta_{e''})]$ at $\mathfrak{M}_\tau^{\text{wt}}$ depends on the choice of the local parameters ζ_e in (3.1) unless e_1 and e_2 are comparable. Nonetheless, with ∂_{ζ_e} as before Lemma 3.1, the limit of the ratio

$$\left[\bigotimes_{e' \geq e_1} \zeta_{e'} \partial_{\zeta_{e'}} : \bigotimes_{e'' \geq e_2} \zeta_{e''} \partial_{\zeta_{e''}} \right]$$

at $\mathfrak{M}_\tau^{\text{wt}}$ does not depend on the choice of ζ_e ; c.f. (3.31). By Lemma 3.1, the restriction of each ∂_{ζ_e} to $\mathfrak{M}_\tau^{\text{wt}}$ is a nowhere vanishing section of $L_e/(\mathcal{V} \cap \mathfrak{M}_\tau^{\text{wt}})$, which suggests that L_e^\geq should be a key part of (2.13).

For any direct sum of line bundles $V = \bigoplus_m L'_m$ (over any base), we write

$$\mathring{\mathbb{P}}(V) := \{ (x, [v_m]) \in \mathbb{P}(V) : v_m \neq 0 \ \forall m \}. \tag{2.12}$$

For any vector bundle V' over the same base, by abuse of notation, we identify V with $V \oplus 0 \subset V \oplus V'$ and identify $\mathring{\mathbb{P}}(V)$ with its image in $\mathbb{P}(V \oplus V')$. For any morphisms $M_1, \dots, M_k \longrightarrow S$, we write

$$\prod_{1 \leq i \leq k} (M_i/S) := M_1 \times_S M_2 \times_S \dots \times_S M_k.$$

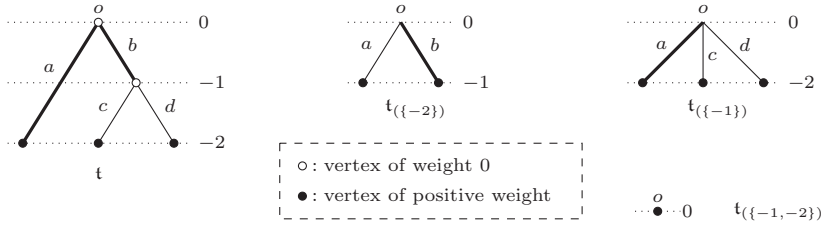


FIG. 2. *Weighted level trees in Example 2.6*

With notation as above, given $\tau \in \mathcal{S}_R^{\text{wt}}$ and $[\mathfrak{t}] = [\tau, \ell] \in \mathcal{S}_L^{\text{wt}}$, let

$$\begin{aligned} \varpi : \mathfrak{M}_{[\mathfrak{t}]}^{\text{tf}} &= \left(\prod_{i \in \mathbb{I}_+(\mathfrak{t})} \left(\left(\mathbb{P} \left(\bigoplus_{e \in \text{Edg}(\mathfrak{t}), \ell(v_e^-) = i} L_e^{\geq} \right) \right) / \mathfrak{M}_\tau^{\text{wt}} \right) \right) \longrightarrow \mathfrak{M}_\tau^{\text{wt}}, \\ \mathcal{E}_{[\mathfrak{t}]} &= \left(\prod_{i \in \mathbb{I}_+(\mathfrak{t})} \left(\left(\mathbb{P} \left(\bigoplus_{e \in \mathfrak{E}_i} L_e^{\geq} \right) \right) / \mathfrak{M}_\tau^{\text{wt}} \right) \right) \longrightarrow \mathfrak{M}_\tau^{\text{wt}}, \end{aligned} \tag{2.13}$$

where $\mathbb{I}_+(\mathfrak{t})$, L_e^{\geq} , and \mathfrak{E}_i are as in (2.4), (2.11), and (2.3), respectively. It is straightforward that both bundles in (2.13) are independent of the choice of the weighted level tree \mathfrak{t} representing $[\mathfrak{t}]$. Since

$$\{e \in \text{Edg}(\mathfrak{t}) : \ell(v_e^-) = i\} \subset \mathfrak{E}_i \quad \forall i \in \mathbb{I}_+(\mathfrak{t}),$$

we see that $\mathfrak{M}_{[\mathfrak{t}]}^{\text{tf}}$ is a subset of $\mathcal{E}_{[\mathfrak{t}]}$. In addition, since each stratum $\mathfrak{M}_\tau^{\text{wt}}$ is an algebraic stack, so are $\mathfrak{M}_{[\mathfrak{t}]}^{\text{tf}}$ and $\mathcal{E}_{[\mathfrak{t}]}$.

Using (2.13) and (2.10), we define

$$\mathfrak{M}^{\text{tf}} := \bigsqcup_{[\mathfrak{t}] \in \mathcal{S}_L^{\text{wt}}} \mathfrak{M}_{[\mathfrak{t}]}^{\text{tf}} \xrightarrow{\varpi} \mathfrak{M}^{\text{wt}}. \tag{2.14}$$

This is the set-theoretic definition of the proposed stack \mathfrak{M}^{tf} as well as the forgetful map in Theorem 1.1. For any $x \in \mathfrak{M}_\tau^{\text{wt}}$, the points of the fiber $\mathfrak{M}_{[\mathfrak{t}]}^{\text{tf}}|_x$ are called the twisted fields over x .

REMARK 2.5. By (2.14), $\widetilde{M}_1^{\text{tf}}(\mathbb{P}^n, d)$ in Corollary 1.2 consists of the tuples

$$(C, \mathbf{u}, [\mathfrak{t}], \underline{\eta}),$$

where (C, \mathbf{u}) are stable maps in $\overline{M}_1(\mathbb{P}^n, d)$, $[\mathfrak{t}]$ are the equivalence classes of weighted level trees satisfying $f[\mathfrak{t}] = (\gamma_C, c_1(\mathbf{u}^* \mathcal{O}_{\mathbb{P}^n}(1)))$, and $\underline{\eta}$ are twisted fields over $(C, c_1(\mathbf{u}^* \mathcal{O}_{\mathbb{P}^n}(1)))$.

2.3. Examples on the strata of \mathfrak{M}^{tf} . In this subsection, we use two examples to illustrate the strata of \mathfrak{M}^{tf} . These two examples will be revisited in §3.3 to illustrate how the strata are glued together by the twisted charts (3.16).

EXAMPLE 2.6. Let $[\mathfrak{t}] = [\tau, \ell] \in \mathcal{S}_L^{\text{wt}}$ be given by the leftmost diagram in Figure 2. Then,

$$\text{Edg}(\mathfrak{t}) = \{a, b, c, d\}, \quad \mathbf{m} = -2, \quad \mathbb{I} = \mathbb{I}_+ = \{-1, -2\}, \quad \mathbb{I}_{\mathbf{m}} = \mathbb{I}_- = \emptyset.$$

As illustrated in Figure 2, each $I \subset \mathbb{I}$ determines a weighted level tree $\mathfrak{t}_{(I)}$, whose underlying weighted tree is denoted by $\tau_{(I)}$ as in (2.5). Each $\mathfrak{t}_{(I)}$ in turn gives rise to a stratum of \mathfrak{M}^{tf} as follows:

$$\begin{aligned} \mathfrak{M}_{[\mathfrak{t}]}^{\text{tf}} &= \mathbb{P}(L_b) \times_{\mathfrak{M}_{\tau}^{\text{wt}}} \mathring{\mathbb{P}}(L_a \oplus L_b \otimes L_c \oplus L_b \otimes L_d), & \mathfrak{M}_{[\mathfrak{t}_{\{-2\}}]}^{\text{tf}} &= \mathring{\mathbb{P}}(L_a \oplus L_b) / \mathfrak{M}_{\tau_{\{-2\}}}^{\text{wt}}, \\ \mathfrak{M}_{[\mathfrak{t}_{\{-1\}}]}^{\text{tf}} &= \mathring{\mathbb{P}}(L_a \oplus L_c \oplus L_d) / \mathfrak{M}_{\tau_{\{-1\}}}^{\text{wt}}, & \mathfrak{M}_{[\mathfrak{t}_{\{-1,-2\}}]}^{\text{tf}} &= \mathfrak{M}_{\tau_{\{-1,-2\}}}^{\text{wt}}. \end{aligned}$$

In Example 3.10, we will demonstrate how to glue these strata by charts.

EXAMPLE 2.7. Let $[\mathfrak{t}] = [\tau, \ell] \in \mathcal{T}_{\mathbb{L}}^{\text{wt}}$ be given by the leftmost diagram in the first row of Figure 3. Then,

$$\text{Edg}(\mathfrak{t}) = \{a, b, c, d\}, \quad \mathbf{m} = -2, \quad \mathbb{I} = \{-1, -2, a\}, \quad \mathbb{I}_+ = \{-1, -2\}, \quad \mathbb{I}_{\mathbf{m}} = \{a\}, \quad \mathbb{I}_- = \emptyset.$$

As illustrated in Figure 3, each $I \subset \mathbb{I}$ determines a weighted level tree $\mathfrak{t}_{(I)}$, whose underlying weighted tree is denoted by $\tau_{(I)}$ as in (2.5). Each $\mathfrak{t}_{(I)}$ in turn gives rise to a stratum of \mathfrak{M}^{tf} as follows:

$$\begin{aligned} \mathfrak{M}_{[\mathfrak{t}]}^{\text{tf}} &= \mathbb{P}(L_b) \times_{\mathfrak{M}_{\tau}^{\text{wt}}} \mathring{\mathbb{P}}(L_b \otimes L_c \oplus L_b \otimes L_d), \\ \mathfrak{M}_{[\mathfrak{t}_{\{-2\}}]}^{\text{tf}} &= \mathbb{P}(L_b) / \mathfrak{M}_{\tau_{\{-2\}}}^{\text{wt}}, \\ \mathfrak{M}_{[\mathfrak{t}_{\{-1\}}]}^{\text{tf}} &= \mathring{\mathbb{P}}(L_c \oplus L_d) / \mathfrak{M}_{\tau_{\{-1\}}}^{\text{wt}}, \\ \mathfrak{M}_{[\mathfrak{t}_{\{a\}}]}^{\text{tf}} &= \mathbb{P}(L_b) \times_{\mathfrak{M}_{\tau_{\{a\}}}^{\text{wt}}} \mathring{\mathbb{P}}(L_a \oplus L_b \otimes L_c \oplus L_b \otimes L_d), \\ \mathfrak{M}_{[\mathfrak{t}_{\{-2,a\}}]}^{\text{tf}} &= \mathring{\mathbb{P}}(L_a \oplus L_b) / \mathfrak{M}_{\tau_{\{-2,a\}}}^{\text{wt}}, \\ \mathfrak{M}_{[\mathfrak{t}_{\{-1,a\}}]}^{\text{tf}} &= \mathring{\mathbb{P}}(L_a \oplus L_c \oplus L_d) / \mathfrak{M}_{\tau_{\{-1,a\}}}^{\text{wt}}, \\ \mathfrak{M}_{[\mathfrak{t}_{\{-1,-2\}}]}^{\text{tf}} &= \mathfrak{M}_{\tau_{\{-1,-2\}}}^{\text{wt}}, \\ \mathfrak{M}_{[\mathfrak{t}_{\{-1,-2,a\}}]}^{\text{tf}} &= \mathfrak{M}_{\tau_{\{-1,-2,a\}}}^{\text{wt}}. \end{aligned}$$

In Example 3.11, we will demonstrate how to glue these strata by charts.

3. The stack structure of \mathfrak{M}^{tf} . In §3, we show \mathfrak{M}^{tf} is naturally a smooth Artin stack and describe its universal family. To help understand the combinatorial notation in §3.1 and §3.2, the readers may consult the examples in §3.3.

3.1. Twisted charts. We first fix $[\mathfrak{t}] = [\gamma, \mathbf{w}, \ell] \in \mathcal{T}_{\mathbb{L}}^{\text{wt}}$ and $x \in \mathfrak{M}_{[\mathfrak{t}]}^{\text{tf}}$, and write

$$\tau = \mathfrak{f}[\mathfrak{t}] = (\gamma, \mathbf{w}) \in \mathcal{T}_{\mathbb{R}}^{\text{wt}}, \quad (C, \mathbf{w}) = \varpi(x) \in \mathfrak{M}_{\tau}^{\text{wt}}.$$

Since \mathfrak{M}^{wt} is smooth, we take an affine smooth chart

$$\mathcal{V} = \mathcal{V}_{\varpi(x)} \longrightarrow \mathfrak{M}^{\text{wt}}$$

containing (C, \mathbf{w}) .

As in [5, §4.3] and [7, §2.5], there exists a set of regular functions

$$\zeta_e \in \Gamma(\mathcal{O}_{\mathcal{V}}) \quad \text{with} \quad e \in \text{Edg}(\gamma), \tag{3.1}$$

called the modular parameters, so that for each $e \in \text{Edg}(\gamma)$, the locus

$$\mathcal{Z}_e = \{\zeta_e = 0\} \subset \mathcal{V}$$

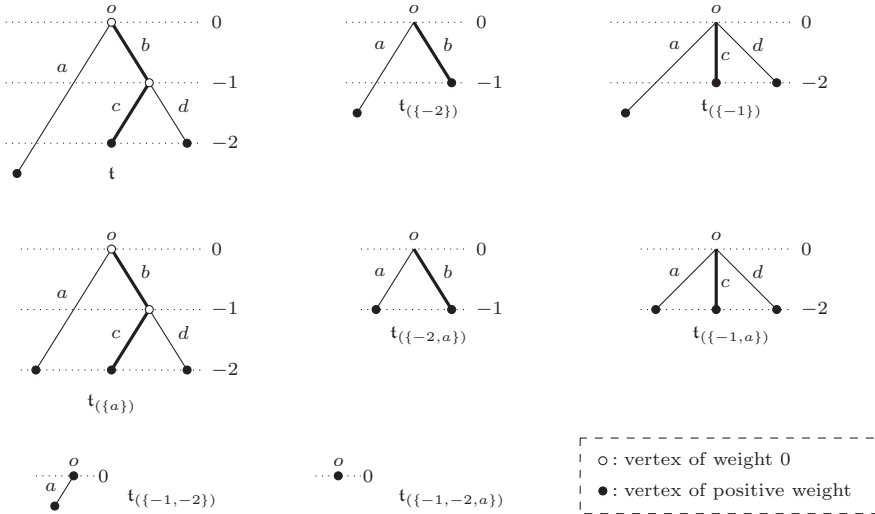


FIG. 3. *Weighted level trees in Example 2.7*

is the irreducible smooth Cartier divisor on \mathcal{V} where the node labeled by e is not smoothed. For any $I \subset \mathbb{I} = \mathbb{I}(\mathfrak{t})$, let

$$\begin{aligned} \mathcal{V}_{(I)}^\circ &:= \{ \zeta_{e'} \neq 0 : e' \in (\text{Edg}(\gamma) \setminus \text{Edg}(\gamma_{(I)})) \} && \subset \mathcal{V}, \\ \mathcal{V}_{\tau(I)} &:= \mathcal{V}_{(I)}^\circ \cap \{ \zeta_e = 0 : e \in \text{Edg}(\gamma_{(I)}) \} && \subset \mathcal{V}_{(I)}^\circ \subset \mathcal{V}. \end{aligned}$$

Then, $\mathcal{V}_{(I)}^\circ$ is an open subset of \mathcal{V} . Shrinking \mathcal{V} if necessary, we see that

$$\mathcal{V}_{\tau(I)} \in \pi_0(\mathfrak{M}_{\tau(I)}^{\text{wt}} \cap \mathcal{V}), \tag{3.2}$$

where π_0 denotes the set of the connected components. In particular, $\mathcal{V}_{\tau(I)}$ can be considered as a smooth chart of the stratum $\mathfrak{M}_{\tau(I)}^{\text{wt}}$. Rigorously, the sets \mathbb{I} and I depend on the choice of the weighted level tree \mathfrak{t} representing $[t]$, however, $\mathcal{V}_{(I)}^\circ$ and $\mathcal{V}_{\tau(I)}$ are independent of such choice.

Given a set of the modular parameters as in (3.1), we may extend it to a set of local parameters on \mathcal{V} centered at (C, \mathbf{w}) :

$$\{ \zeta_e \}_{e \in \text{Edg}(\gamma)} \sqcup \{ \varsigma_j \}_{j \in J} \quad \text{with} \quad (C, \mathbf{w}) = \underline{0} := (0, \dots, 0), \tag{3.3}$$

where J is a finite set. We do not impose other conditions on ς_j .

For each $e \in \text{Edg}(\gamma)$, let $\partial_{\zeta_e} \in \Gamma(\mathcal{V}; T\mathfrak{M}^{\text{wt}})$ be such that

$$\langle \partial_{\zeta_e}, d\zeta_e \rangle = 1, \quad \langle \partial_{\zeta_e}, d\zeta_{e'} \rangle = 0 \quad \forall e' \in \text{Edg}(\gamma) \setminus \{e\}, \quad \langle \partial_{\zeta_e}, d\varsigma_j \rangle = 0 \quad \forall j \in J.$$

For $\mathcal{Z}_e = \{ \zeta_e = 0 \}$, we denote by $\bar{\partial}_{\zeta_e}$ the image of $\partial_{\zeta_e}|_{\mathcal{Z}_e}$ in the normal bundle of \mathcal{Z}_e , i.e.

$$\bar{\partial}_{\zeta_e}|_{\mathfrak{z}} := \partial_{\zeta_e}|_{\mathfrak{z}} + T_{\mathfrak{z}}\mathcal{Z}_e \quad \forall \mathfrak{z} \in \mathcal{Z}_e.$$

LEMMA 3.1. *For every $I \subset \mathbb{I}$ and every $e \in \text{Edg}(\gamma_{(I)})$, the restriction of $\bar{\partial}_{\zeta_e}$ to $\mathcal{V}_{\tau(I)}$ is a nowhere vanishing section of the restriction of the line bundle L_e in (2.11) to $\mathcal{V}_{\tau(I)}$.*

Proof. Since the restriction of $d\zeta_e$ to $\mathcal{Z}_e (\supset \mathcal{V}_{\tau(I)})$ is identically zero, we observe that $\tilde{\partial}_{\zeta_e}$ is a nowhere vanishing section of the normal bundle of \mathcal{Z}_e . It is a well-known fact of the moduli of curves that the normal bundle of \mathcal{Z}_e is L_e ; see [2, Proposition 3.31]. \square

For each level $i \in \mathbb{I}_+ = \mathbb{I}_+(\mathfrak{t})$, we choose a special vertex

$$v_i \in \text{Ver}(\gamma) \quad \text{s.t.} \quad \ell(v_i) = i. \tag{3.4}$$

Recall that for every $v \in \text{Ver}(\mathfrak{t}) \setminus \{o\}$, the edge e_v “above” v is given by $v_{e_v}^- = v$. We then denote by $e_i, e_i^+,$ and v_i^+ respectively the edges and the vertex satisfying

$$e_i = e_{v_i}, \quad v_i^+ = v_{e_i}^+, \quad e_i^+ = e_{v_i^+}; \tag{3.5}$$

see Figure 1. We remark that e_i^+ is not defined if $v_i^+ = o$. Each $i \in \mathbb{I}_+$ determines a strictly increasing sequence

$$i[0] := i < i[1] := \ell(v_i^+) < i[2] := \ell(v_{i[1]}^+) < \dots \tag{3.6}$$

We would like to remark that $i[1]$ and i^\sharp in (2.1) need not to be the same; see Figure 1 for illustration. This sequence is finite, as there is a unique step h satisfying $i[h] = 0$.

REMARK 3.2. As mentioned in Remark 2.2, for each level $i \in \mathbb{I}_+$, the vertices v with $\ell(v) = i$ (or equivalently the edges e with $\ell(v_e^-) = i$) locally correspond to the non-zero components of an exceptional divisor in the blowup $\widetilde{\mathfrak{M}}^{\text{wt}} \rightarrow \mathfrak{M}^{\text{wt}}$ of [5]. Thus, choosing v_i (or equivalently e_i) means selecting a standard chart for the exceptional divisor. Such choices affect the expression (3.14), hence affect the twisted chart (3.16). Nonetheless, Lemma 3.5 guarantees that they do not affect the stack structure of \mathfrak{M}^{tf} .

By Lemma 3.1, there exist $\lambda_e \in \mathbb{A}$ with $e \in \widehat{\text{Edg}}(\mathfrak{t})$ so that the fixed $x \in \mathfrak{M}_{[\mathfrak{t}]}^{\text{tf}}$ over $(C, \mathbf{w}) = \underline{0} \in \mathcal{V}_\tau$ can uniquely be written as

$$x = \left(\underline{0}; \prod_{i \in \mathbb{I}_+} \left[\lambda_e \cdot \left(\bigotimes_{e' \geq e} \tilde{\partial}_{\zeta_{e'}}|_{\underline{0}} \right) : \ell(e) = i \right] \right) \\ \in \mathcal{V}_\tau \times_{\mathfrak{M}_\tau^{\text{wt}}} \mathfrak{M}_{[\mathfrak{t}]}^{\text{tf}} = \prod_{i \in \mathbb{I}_+(\mathfrak{t})} \left(\left(\mathbb{P}^1 \left(\bigoplus_{e \in \text{Edg}(\mathfrak{t}), \ell(e)=i} L_e^\geq \right) \right) / \mathcal{V}_\tau \right), \tag{3.7}$$

where $\lambda_e \neq 0 \quad \forall e \in \widehat{\text{Edg}}(\mathfrak{t}) \setminus \mathbb{I}_\mathbf{m}, \quad \lambda_{e_i} = 1 \quad \forall i \in \mathbb{I}_+, \quad \lambda_e = 0 \quad \forall e \in \mathbb{I}_\mathbf{m}.$

With the index set J as in (3.3) (which indexes the local parameters ζ_j on \mathcal{V} other than the modular parameters ζ_e), let

$$\mathfrak{U}_x \subset \mathbb{A}^{\mathbb{I}_+} \times \mathbb{A}^{\widehat{\text{Edg}}(\mathfrak{t}) \setminus \{e_i : i \in \mathbb{I}_+\}} \times \mathbb{A}^{\mathbb{I}_-} \times \mathbb{A}^J$$

be an open subset containing the point

$$y_x := (\underline{0}, (\lambda_e)_{e \in \widehat{\text{Edg}}(\mathfrak{t}) \setminus \{e_i : i \in \mathbb{I}_+\}}, \underline{0}, \underline{0}). \tag{3.8}$$

The coordinates on \mathfrak{U}_x are denoted by

$$((\varepsilon_i)_{i \in \mathbb{I}_+}, (u_e)_{e \in \widehat{\text{Edg}}(\mathfrak{t}) \setminus \{e_i : i \in \mathbb{I}_+\}}, (z_e)_{e \in \mathbb{I}_-}, (w_j)_{j \in J}). \tag{3.9}$$

For any $I \subset \mathbb{I}$, we set

$$\begin{aligned} \mathfrak{U}_{x;(I)}^\circ &= \{ \varepsilon_i \neq 0 \ \forall i \in I_+ ; u_e \neq 0 \ \forall e \in I_{\mathbf{m}} ; z_e \neq 0 \ \forall e \in I_- \} \subset \mathfrak{U}_x, \\ \mathfrak{U}_{x;[t_{(I)}]} &= \mathfrak{U}_{x;(I)}^\circ \cap \{ \varepsilon_i = 0 \ \forall i \in \mathbb{I}_+ \setminus I_+ ; u_e = 0 \ \forall e \in \mathbb{I}_{\mathbf{m}} \setminus I_{\mathbf{m}} ; z_e = 0 \ \forall e \in \mathbb{I}_- \setminus I_- \}. \end{aligned}$$

This gives rise to a stratification

$$\mathfrak{U}_x = \bigsqcup_{I \subset \mathbb{I}} \mathfrak{U}_{x;[t_{(I)}]}. \tag{3.10}$$

We remark that neither \mathfrak{U}_x nor its stratification (3.10) depends on the choice of the weighted level tree \mathfrak{t} representing $[\mathfrak{t}]$, even though the sets \mathbb{I} and I depend on such choice. We also notice that $\mathfrak{U}_{x;(I)}^\circ$ is an open subset of \mathfrak{U}_x , but the strata $\mathfrak{U}_{x;[t_{(I)}]}$ are not open unless $I = \mathbb{I}$.

For each $i \in \mathbb{I}_+$, we take

$$u_{e_i} := 1.$$

By (3.7), shrinking \mathfrak{U}_x if necessary, we have

$$u_e \in \Gamma(\mathcal{O}_{\mathfrak{U}_x}^*) \quad \forall e \in \widehat{\text{Edg}}(\mathfrak{t}) \setminus \mathbb{I}_{\mathbf{m}}. \tag{3.11}$$

With the local parameters ζ_e and ς_j as in (3.3), we construct a morphism

$$\theta_x : \mathfrak{U}_x \longrightarrow \mathcal{V} \quad (\longrightarrow \mathfrak{M}^{\text{wt}})$$

given by

$$\begin{aligned} \theta_x^* \zeta_e &= \frac{u_e \cdot u_{e_{\ell(e)}}^+ \cdot u_{e_{\ell(e)[1]}}^+ \cdots}{u_{e_{v_e^+}} \cdot u_{e_{\ell(v_e^+)}}^+ \cdot u_{e_{\ell(v_e^+)[1]}}^+ \cdots} \cdot \prod_{i \in [\ell(e), \ell(v_e^+)]} \varepsilon_i \quad \forall e \in \widehat{\text{Edg}}(\mathfrak{t}); \\ \theta_x^* \zeta_e &= z_e \quad \forall e \in \mathbb{I}_- = \text{Edg}(\mathfrak{t}) \setminus \widehat{\text{Edg}}(\mathfrak{t}); \quad \theta_x^* \varsigma_j = w_j \quad \forall j \in J. \end{aligned} \tag{3.12}$$

The numerator and the denominator in the first line of (3.12) are both finite products, because (3.6) is always a finite sequence.

For any $I \subset \mathbb{I}$, it follows from (3.11) and (3.12) that

$$\theta_x(\mathfrak{U}_{x;(I)}^\circ) \subset \mathcal{V}_{(I)}^\circ, \quad \theta_x(\mathfrak{U}_{x;[t_{(I)}]}) \subset \mathcal{V}_{\tau_{(I)}}, \tag{3.13}$$

where $\mathcal{V}_{(I)}^\circ$ and $\mathcal{V}_{\tau_{(I)}}$ are described before (3.2).

Fix $I \subset \mathbb{I}$ (I may be empty). With

$$[t_{(I)}] = [\tau_{(I)}, \ell_{(I)}] \in \mathcal{T}_{\mathfrak{L}}^{\text{wt}}$$

as in (2.5), $\mathfrak{M}_{[t_{(I)}]}^{\text{tf}}$ as in (2.14), and the chart $\mathcal{V}_{\tau_{(I)}} \longrightarrow \mathfrak{M}_{\tau_{(I)}}^{\text{wt}}$ as in (3.2), let

$$\begin{aligned} \Phi_{x;(I)} &: \mathfrak{U}_{x;[t_{(I)}]} \longrightarrow \mathcal{V}_{\tau_{(I)}} \times_{\mathfrak{M}_{\tau_{(I)}}^{\text{wt}}} \mathfrak{M}_{[t_{(I)}]}^{\text{tf}} \\ &= \prod_{i \in \mathbb{I}_+ \setminus (t_{(I)})} \left(\left(\mathbb{P} \left(\bigoplus_{e \in \text{Edg}(t_{(I)}), \ell_{(I)}(e)=i} L_e^{\geq \gamma(I)} \right) \right) / \mathcal{V}_{\tau_{(I)}} \right) \quad (\longrightarrow \mathfrak{M}_{[t_{(I)}]}^{\text{tf}}) \end{aligned}$$

be the morphism so that for any $y \in \mathfrak{U}_{x;t_{(I)}}$,

$$\Phi_{x;(I)}(y) = \left(\theta_x(y); \prod_{i \in \mathbb{I}_+ \setminus I_+} \left[\mu_{e;i;I}(y) \cdot \left(\bigotimes_{\substack{e \geq e, \\ [\ell(e), \ell(v_e^+)] \not\subset I}} \mathfrak{D}_{\zeta_e} \Big|_{\theta_x(y)} \right) : e \in \mathfrak{E}_i \right] \right), \tag{3.14}$$

where

$$\mu_{e;i;I} = \left(u_e \cdot \frac{u_{e_{\ell(e)}}^+ \cdot u_{e_{\ell(e)[1]}}^+ \cdots}{u_{e_i^+} \cdot u_{e_{i[1]}^+} \cdots} \right) \left(\frac{\prod_{\mathbf{e} > \mathbf{e}_i; \llbracket \ell(\mathbf{e}), \ell(v_e^+) \rrbracket \subset I} \theta_x^* \zeta_{\mathbf{e}}}{\prod_{\mathbf{e}' > \mathbf{e}; \llbracket \ell(\mathbf{e}'), \ell(v_{e'}^+) \rrbracket \subset I} \theta_x^* \zeta_{\mathbf{e}'}} \right) \left(\prod_{h \in \llbracket \ell(e), i \rrbracket} \varepsilon_h \right)$$

for all $i \in \mathbb{I}_+ \setminus I_+$ and $e \in \mathfrak{E}_i$. Similar to (3.12), the products in the first pair of parentheses above are both finite products.

By (3.13), the description of $\mathcal{V}_{(I)}^\circ$ above (3.2), and (2.6), we see that

$$\mu_{e;i;I} \in \Gamma(\mathcal{O}_{\mathfrak{U}_{x;(I)}^\circ}) \quad \forall I \subset \mathbb{I}, i \in \mathbb{I}_+ \setminus I_+, e \in \mathfrak{E}_i.$$

Moreover, by (3.11),

$$\mu_{e;i;I} |_{\mathfrak{U}_{x;[\mathfrak{t}_{(I)}]}} \begin{cases} = 0 & \text{if } \ell_{(I)}(v_e^-) < i, \\ \in \Gamma(\mathcal{O}_{\mathfrak{U}_{x;[\mathfrak{t}_{(I)}]}^*}) & \text{if } \ell_{(I)}(v_e^-) = i. \end{cases} \tag{3.15}$$

This, along with (3.13), Lemma 3.1, and (2.13), implies $\Phi_{x;(I)}$ is well-defined.

The morphisms $\Phi_{x;(I)}$, $I \subset \mathbb{I}$, together determine

$$\Phi_x : \mathfrak{U}_x \longrightarrow \mathfrak{M}^{\text{tf}}, \quad \Phi_x(y) = \Phi_{x;(I)}(y) \quad \text{if } y \in \mathfrak{U}_{x;[\mathfrak{t}_{(I)}]}. \tag{3.16}$$

We remark that $\Phi_{x;(I)}$ and Φ_x are also independent of the choice of the weighted level tree \mathfrak{t} representing $[\mathfrak{t}]$. Moreover, we observe that

$$\Phi_x(y_x) = \Phi_{x;(\emptyset)}(y_x) = x \quad \forall x \in \mathfrak{M}^{\text{tf}}, \tag{3.17}$$

where $y_x \in \mathfrak{U}_x$ is given in (3.8).

A priori Φ_x is just a *map*, for the set-theoretic definition (2.14) of \mathfrak{M}^{tf} does not describe its stack structure, although each $\mathfrak{M}_{[\mathfrak{t}]}^{\text{tf}}$ is a stack. In §3.2, we will show such Φ_x patch together to endow \mathfrak{M}^{tf} with a smooth stack structure. Each Φ_x will hereafter be called a *twisted chart centered at x* (lying over $\mathcal{V} \longrightarrow \mathfrak{M}^{\text{wt}}$), although rigorously it becomes a *chart* of \mathfrak{M}^{tf} only after Corollary 3.8 is established.

LEMMA 3.3. *For every $I \subset \mathbb{I}$, $\Phi_{x;(I)} : \mathfrak{U}_{x;[\mathfrak{t}_{(I)}]} \longrightarrow \mathfrak{M}_{[\mathfrak{t}_{(I)}]}^{\text{tf}}$ of (3.14) is an isomorphism to an open subset of $\mathfrak{M}_{[\mathfrak{t}_{(I)}]}^{\text{tf}}$.*

Proof. For any $i \in \mathbb{I}_+(\mathfrak{t}_{(I)}) = \mathbb{I}_+ \setminus I_+$, notice that every edge in \mathfrak{E}_i of the weighted level tree \mathfrak{t} is not contracted in the construction of $\mathfrak{t}_{(I)}$ (c.f. (2.6)). Thus,

$$\mathfrak{E}_i \subset \text{Edg}(\mathfrak{t}_{(I)}) \quad \forall i \in \mathbb{I}_+(\mathfrak{t}_{(I)}).$$

In particular, the edges e_i , $i \in \mathbb{I}_+(\mathfrak{t}_{(I)})$, can be used as the special edges of $\mathfrak{t}_{(I)}$. For conciseness, let

$$\begin{aligned} \mathbf{E}[\mathfrak{t}_{(I)}] &:= \widehat{\text{Edg}}(\mathfrak{t}_{(I)}) \setminus (\{e_i : i \in \mathbb{I}_+(\mathfrak{t}_{(I)})\} \sqcup \mathbb{I}_{\mathbf{m}}(\mathfrak{t}_{(I)})) \\ &= \bigsqcup_{i \in \mathbb{I}_+(\mathfrak{t}_{(I)})} \{e \in \text{Edg}(\mathfrak{t}_{(I)}) : \ell_{(I)}(v_e^-) = i, e \neq e_i\} \quad (\subset \widehat{\text{Edg}}(\mathfrak{t}_{(I)}) \subset \widehat{\text{Edg}}(\mathfrak{t})); \end{aligned}$$

see (2.7) for notation.

Let $\{\zeta_e\}_{e \in \text{Edg}(\gamma)} \cup \{\zeta_j\}_{j \in J}$ be a set of the local parameters on \mathcal{V} centered at $\varpi(x)$ as in (3.3). By the definition of $\mathcal{V}_{\tau_{(I)}}$ above (3.2),

$$\{\zeta_e\}_{e \in \text{Edg}(\mathfrak{t}) \setminus \text{Edg}(\mathfrak{t}_{(I)})} \sqcup \{\zeta_j\}_{j \in J} \tag{3.18}$$

is a set of local parameters of $\mathcal{V}_{\tau(I)}$.

Recall that there exist $\lambda_e \in \mathbb{A}^*$, $e \in \widehat{\text{Edg}}(\mathfrak{t})$, such that

$$x = \left(\underline{0} ; \prod_{i \in \mathbb{I}_+} \left[\lambda_e \cdot \left(\bigotimes_{\mathfrak{e} \geq e} \partial_{\zeta_{\mathfrak{e}}} \mid \underline{0} \right) : \ell(v_e^-) = i \right] \right)$$

as in (3.7). Let $U_{x; \mathbf{E}[t(I)]}$ be an open subset of $(\mathbb{A}^*)^{\mathbf{E}[t(I)]}$ such that

$$(\lambda_e)_{e \in \mathbf{E}[t(I)]} \in U_{x; \mathbf{E}[t(I)]} \subset (\mathbb{A}^*)^{\mathbf{E}[t(I)]}.$$

The coordinates of $U_{x; \mathbf{E}[t(I)]}$ are denoted by

$$(\mu_e)_{e \in \mathbf{E}[t(I)]}.$$

In addition, we set

$$\mu_{e_i} = 1 \quad \forall i \in \mathbb{I}_+(t(I)), \quad \mu_e = 0 \quad \forall e \in \mathbb{I}_{\mathbf{m}}(t(I)).$$

Thus, the function μ_e is defined for all $e \in \widehat{\text{Edg}}(t(I))$, and is nowhere vanishing on $U_{x; \mathbf{E}(t(I))}$ for all $e \in \widehat{\text{Edg}}(t(I)) \setminus \mathbb{I}_{\mathbf{m}}(t(I))$.

The smooth chart $\mathcal{V}_{\tau(I)} \rightarrow \mathfrak{M}_{\tau(I)}^{\text{wt}}$ in (3.2) induces a smooth chart

$$\mathcal{U}'_{x; [t(I)]} := \mathcal{V}_{\tau(I)} \times U_{x; \mathbf{E}[t(I)]} \rightarrow \mathfrak{M}_{[t(I)]}^{\text{tf}}$$

given by

$$\left(\mathfrak{z}, (\mu_e)_{e \in \mathbf{E}(t(I))} \right) \mapsto \left(\mathfrak{z}, \prod_{i \in \mathbb{I}_+(t(I))} \left[\mu_e \left(\bigotimes_{\mathfrak{e} \in \text{Edg}(t(I)), \mathfrak{e} \geq e} \partial_{\zeta_{\mathfrak{e}}} \mid \mathfrak{z} \right) : \ell_{(I)}(v_e^-) = i \right] \right).$$

We will construct a morphism

$$\Psi_{x; (I)} : \mathcal{U}'_{x; [t(I)]} \rightarrow \mathfrak{U}_{x; [t(I)]} \tag{3.19}$$

such that $\Phi_{x; (I)} \circ \Psi_{x; (I)}$ and $\Psi_{x; (I)} \circ \Phi_{x; (I)}$ are both the identity morphisms, which will then establish Lemma 3.3.

Given $(\mathfrak{z}, (\mu_e)_{e \in \mathbf{E}(t(I))}) \in \mathcal{U}'_{x; t(I)}$, we denote its image by

$$y := \Psi_{x; (I)}(\mathfrak{z}, (\mu_e)_{e \in \mathbf{E}(t(I))}) \in \mathfrak{U}_{x; [t(I)]},$$

which is to be constructed. With the coordinates on \mathfrak{U}_x as in (3.9), we set

$$z_e(y) = \zeta_e(\mathfrak{z}) \quad \forall e \in \mathbb{I}_-, \quad w_j(y) = \varsigma_j(\mathfrak{z}) \quad \forall j \in J. \tag{3.20}$$

By (3.2), we see that

$$z_e(y) = \zeta_e(\mathfrak{z}) = 0 \iff e \in \mathbb{I}_- \setminus I_- \quad (\subset \mathbb{I}_-(t(I))). \tag{3.21}$$

The rest of the coordinates of y are much more complicated; we describe them by induction over the levels in $\mathbb{I}_+ = \mathbb{I}_+(\mathfrak{t})$. More precisely, we will show that $\varepsilon_i(y)$ with $i \in \mathbb{I}_+$ and $u_e(y)$ with $e \in \widehat{\text{Edg}}(\mathfrak{t}) \setminus \{\mathfrak{e}_i : i \in \mathbb{I}_+\}$ are all rational functions in $\zeta_{e'}(\mathfrak{z})$ and $\mu_{e''}$, satisfying

$$\left(\varepsilon_i(y) = 0 \iff i \in \mathbb{I}_+ \setminus I_+ \right) \quad \text{and} \quad \left(u_e(y) = 0 \iff e \in \mathbb{I}_{\mathbf{m}} \setminus I_{\mathbf{m}} \right). \tag{3.22}$$

In particular, (3.22) and (3.21) imply $y \in \mathfrak{U}_{x;[\mathfrak{t}_{(I)}]}$, i.e. $\Psi_{x;(I)}$ is well-defined.

The base case of the induction is for the level

$$i_0 := \max \mathbb{I}_+(\mathfrak{t}).$$

We take

$$\varepsilon_{i_0}(y) = \zeta_{\mathbf{e}_{i_0}}(\mathfrak{z}). \tag{3.23}$$

By (3.2), we see that $\varepsilon_{i_0}(y)$ satisfies (3.22). We take

$$u_{\mathbf{e}_{i_0}}(y) = 1.$$

For any $e \neq \mathbf{e}_{i_0}$ with $\ell(e) = i_0$, we set

$$u_e(y) = \begin{cases} \mu_e & \text{if } i_0 \notin I_+ \text{ (i.e. } i_0 \in \mathbb{I}_+(\mathfrak{t}_{(I)}), e \in \mathbf{E}(\mathfrak{t}_{(I)}) \text{)} \\ \frac{\zeta_e(\mathfrak{z})}{\zeta_{\mathbf{e}_{i_0}}(\mathfrak{z})} & \text{if } i_0 \in I_+ \text{ (i.e. } \mathbf{e}_{i_0} \in \text{Edg}(\mathfrak{t}) \setminus \text{Edg}(\mathfrak{t}_{(I)}) \text{)} \end{cases}. \tag{3.24}$$

If $i_0 \in I_+$, then by (3.18) and (3.2), we have $\zeta_e(\mathfrak{z}) = 0$ if and only if $(\mathbf{m} = i_0 \text{ and } e \in \mathbb{I}_{\mathbf{m}} \setminus I_{\mathbf{m}})$. If $i_0 \notin I_+$, then $\mu_e = 0$ if and only if $(\mathbf{m} = i_0 \text{ and } e \in \widehat{\text{Edg}}(\mathfrak{t}) \setminus I_{\mathbf{m}})$. We thus conclude that $u_e(y)$ satisfies (3.22) for all $e \in \widehat{\text{Edg}}(\mathfrak{t})$ with $\ell(e) = i_0$. Moreover, such $\varepsilon_{i_0}(y)$ and $u_e(y)$ are obviously rational functions in $\zeta_{e'}(\mathfrak{z})$ and $\mu_{e''}$. Hence, the base case is complete.

Next, for any $i \in \mathbb{I}_+$, assume that all $\varepsilon_k(y)$ with $k > i$ and all $u_e(y)$ with $e \in \widehat{\text{Edg}}(\mathfrak{t})$ and $\ell(e) > i$ have been expressed as rational functions in $\zeta_{e'}(\mathfrak{z})$ and $\mu_{e''}$, satisfying (3.22).

For the level i , we first construct $\varepsilon_i(y)$. The construction is subdivided into three cases.

Case 1. If $i \notin I_+$, then set

$$\varepsilon_i(y) = 0.$$

Obviously this satisfies (3.22).

Case 2. If $i \in I_+$ and $[[i, i[1]]] \not\subset I_+$, then $\mathbf{e}_i \in \mathbf{E}(\mathfrak{t}_{(I)})$, hence $\mu_{\mathbf{e}_i} \neq 0$. Let

$$\widehat{i} := \min ([[i, i[1]]] \setminus I_+) \in \mathbb{I}_+(\mathfrak{t}_{(I)}).$$

Intuitively, \widehat{i} is the level containing the image of \mathbf{v}_i in $\mathfrak{t}_{(I)}$. Thus,

$$\ell_{(I)}(\mathbf{e}_i) = \widehat{i}.$$

Let $\varepsilon_i(y)$ be given by

$$\mu_{\mathbf{e}_i} = \varepsilon_i(y) \cdot \frac{u_{\mathbf{e}_i^+}(y) \cdot u_{\mathbf{e}_{i[1]}^+}(y) \cdots \prod_{\mathbf{e} > \mathbf{e}_{\widehat{i}}, \ell(\mathbf{e}), \ell(\mathbf{v}_{\mathbf{e}}^+)} \mathfrak{t} \subset I \zeta_{\mathbf{e}}(\mathfrak{z})}{u_{\mathbf{e}_{\widehat{i}}^+}(y) \cdot u_{\mathbf{e}_{\widehat{i}[1]}^+}(y) \cdots \prod_{\mathbf{e}' > \mathbf{e}_i, \ell(\mathbf{e}'), \ell(\mathbf{v}_{\mathbf{e}'}^+)} \mathfrak{t} \subset I \zeta_{\mathbf{e}'}(\mathfrak{z})} \cdot \prod_{h \in ([i, \widehat{i}]_{\mathfrak{t}})} \varepsilon_h(y). \tag{3.25}$$

The inductive assumption implies that all $\varepsilon_h(y)$ with $h \in ([i, \widehat{i}]_{\mathfrak{t}})$, as well as all $u_{\mathbf{e}_{i[h]}^+}(y)$ and $u_{\mathbf{e}_{i[h]}^+}(y)$ with $h \geq 0$, are non-zero and are rational functions in $\zeta_{e'}(\mathfrak{z})$ and $\mu_{e''}$. By (3.18) and (2.6), we also see that all $\zeta_{\mathbf{e}}(\mathfrak{z})$ and $\zeta_{\mathbf{e}'}(\mathfrak{z})$ in (3.25) are non-zero. Therefore, such defined $\varepsilon_i(y)$ is a rational function in $\zeta_{e'}(\mathfrak{z})$ and $\mu_{e''}$, satisfying (3.22).

Case 3. If $\llbracket i, i[1] \rrbracket \subset I_+$ (hence $i \in I_+$), then we see $e_i \in \text{Edg}(\mathfrak{t}) \setminus \text{Edg}(\mathfrak{t}_{(I)})$; c.f. (2.6). Intuitively, this means e_i is contracted in the construction of $\mathfrak{t}_{(I)}$. By the description of $\mathcal{V}_{\tau_{(I)}}$ above (3.2), we see that $\zeta_{e_i}(\mathfrak{z}) \neq 0$. Let $\varepsilon_i(y)$ be given by

$$\zeta_{e_i}(\mathfrak{z}) = \varepsilon_i(y) \cdot \prod_{h \in \langle i, \hat{i} \rangle_{\mathfrak{t}}} \varepsilon_h(y). \tag{3.26}$$

Mimicking the argument in Case 2, we conclude that $\varepsilon_i(y)$ is a rational function in $\zeta_{e'}(\mathfrak{z})$ and $\mu_{e''}$, satisfying (3.22).

Next, we construct $u_e(y)$ for $e \in \widehat{\text{Edg}}(\mathfrak{t})$ with $\ell(e) = i$. Set

$$u_{e_i}(y) = 1.$$

For $e \neq e_i$, the construction is subdivided into two cases.

Case A. If $\llbracket i, \ell(v_e^+) \rrbracket \not\subset I_+$, then

$$e \in \mathbf{E}(\mathfrak{t}_{(I)}) \sqcup \mathbb{I}_{\mathbf{m}}(\mathfrak{t}_{(I)}) \quad (= \widehat{\text{Edg}}(\mathfrak{t}_{(I)}) \setminus \{e_i : i \in \mathbb{I}_+(\mathfrak{t}_{(I)})\}),$$

hence μ_e exists, and $\mu_e = 0$ if and only if $e \in \mathbb{I}_{\mathbf{m}}(\mathfrak{t}_{(I)})$. In Case A, since $\llbracket i, \ell(v_e^+) \rrbracket \not\subset I_+$, (2.7) further implies

$$\mu_e = 0 \iff e \in \mathbb{I}_{\mathbf{m}} \setminus I_{\mathbf{m}}. \tag{3.27}$$

Let

$$\kappa = \kappa_e := \min(\llbracket i, \ell(v_e^+) \rrbracket \setminus I_+) \quad (\in \mathbb{I}_+(\mathfrak{t}_{(I)})).$$

Since $\llbracket i, \kappa \rrbracket \subset I_+$, we have

$$\ell_{(I)}(e) = \kappa.$$

Let $u_e(y)$ be given by

$$\mu_e = u_e(y) \cdot \frac{u_{e_i^+}(y) \cdot u_{e_{i[1]}^+}(y) \cdots \prod_{\mathfrak{e} > e_{\kappa}, \llbracket \ell(\mathfrak{e}), \ell(v_{\mathfrak{e}}^+) \rrbracket_{\mathfrak{t}} \subset I} \zeta_{\mathfrak{e}}(\mathfrak{z})}{u_{e_{\kappa^+}^+}(y) \cdot u_{e_{\kappa^+[1]}^+}(y) \cdots \prod_{e' > e, \llbracket \ell(e'), \ell(v_{e'}^+) \rrbracket_{\mathfrak{t}} \subset I} \zeta_{e'}(\mathfrak{z})} \cdot \prod_{h \in \llbracket i, \kappa \rrbracket_{\mathfrak{t}}} \varepsilon_h(y). \tag{3.28}$$

We observe that if $i \notin I_+$, then $\kappa = i$ and hence $\prod_{h \in \llbracket i, \kappa \rrbracket_{\mathfrak{t}}} \varepsilon_h(y) = 1$; if $i \in I_+$, then the previous construction of $\varepsilon_i(y)$, along with the inductive assumption, guarantees $\prod_{h \in \llbracket i, \kappa \rrbracket_{\mathfrak{t}}} \varepsilon_h(y) \neq 0$. Mimicking the argument of Case 2 of the construction of $\varepsilon_i(y)$ and taking (3.27) into account, we see that $u_e(y)$ determined by (3.28) is a rational function in $\zeta_{e'}(\mathfrak{z})$ and $\mu_{e''}$, and it satisfies (3.22).

Case B. If $\llbracket i, \ell(v_e^+) \rrbracket \subset I_+$, then (2.6) gives

$$e \in \text{Edg}(\mathfrak{t}_{(I)}) \quad (\text{i.e. } \zeta_e(\mathfrak{z}) = 0) \iff e \in \mathbb{I}_{\mathbf{m}} \setminus I_{\mathbf{m}}. \tag{3.29}$$

Let $u_e(y)$ be given by

$$\zeta_e(\mathfrak{z}) = u_e(y) \cdot \frac{u_{e_i^+}(y) \cdot u_{e_{i[1]}^+}(y) \cdots}{u_{e_{v_e^+}^+}(y) \cdot u_{e_{\ell(v_e^+)}^+}(y) \cdot u_{e_{\ell(v_e^+)[1]}^+}(y) \cdots} \cdot \prod_{h \in \llbracket i, \ell(v_e^+) \rrbracket_{\mathfrak{t}}} \varepsilon_h(y). \tag{3.30}$$

Once again, mimicking the argument of Case A of the construction of $u_e(y)$, and taking (3.29) as well as the description of $\mathcal{V}_{\tau_{(I)}}$ right before (3.2) into account, we see that $u_e(y)$ determined by (3.30) is a rational function in $\zeta_{e'}(\mathfrak{z})$ and $\mu_{e''}$, and it satisfies (3.22).

The cases 1-3, A, and B together complete the inductive construction of $\Psi_{x; (I)}$. Moreover, comparing

- (3.20) with the second line of (3.12),
- (3.23), the second case of (3.24), (3.26), and (3.30) with the first line (3.12),
- the first case of (3.24), (3.25), and (3.28) with the expressions of $\mu_{e;i;I}$ right after (3.14),

we observe that $\Psi_{x;(I)}$ is the inverse of $\Phi_{x;(I)}$. \square

COROLLARY 3.4. $\Phi_x : \mathcal{U}_x \longrightarrow \mathfrak{M}^{\text{tf}}$ is injective.

Proof. This follows from Lemma 3.3 and the stratification (3.10) and (2.14) directly. \square

3.2. Stack structure. In this subsection, we will show the twisted charts Φ_x patch together to endow \mathfrak{M}^{tf} with a smooth stack structure; c.f. Proposition 3.7 and Corollary 3.8. Note that a priori, Φ_x depends on the choices of the special vertices $\{v_i\}_{i \in \mathbb{I}_+}$ (3.4) and of the local parameters (3.3). Nonetheless, Lemmas 3.5 and 3.6 below will guarantee that such choices do not affect the proposed stack structure of \mathfrak{M}^{tf} , hence will make the proof of Proposition 3.7 more concise.

Let $\{w_i : i \in \mathbb{I}_+\}$ be another set of the special vertices satisfying (3.4), and w_i^+ , d_i , and d_i^+ be the analogues of v_i^+ , e_i , and e_i^+ in (3.5), respectively. As in (3.6), each level $i \in \mathbb{I}_+$ similarly determines a finite sequence

$$i\langle 0 \rangle = i < i\langle 1 \rangle = \ell(w_i^+) < i\langle 2 \rangle = \ell(w_{i\langle 1 \rangle}^+) < \dots$$

We take an open subset

$$\mathcal{U}_x^{\mathbf{a}} \subset \mathbb{A}^{\mathbb{I}_+} \times \widehat{\mathbb{A}^{\text{Edg}(t)} \setminus \{d_i : i \in \mathbb{I}_+\}} \times \mathbb{A}^{\mathbb{I}_-} \times \mathbb{A}^J$$

with the coordinates

$$((\delta_i)_{i \in \mathbb{I}_+}, (u_e^{\mathbf{a}})_{e \in \widehat{\text{Edg}(t)} \setminus \{d_i : i \in \mathbb{I}_+\}}, (z_e^{\mathbf{a}})_{e \in \mathbb{I}_-}, (w_j^{\mathbf{a}})_{j \in J}).$$

as in (3.9), and then construct

$$\theta_x^{\mathbf{a}} : \mathcal{U}_x^{\mathbf{a}} \longrightarrow \mathcal{V}, \quad \mu_{e;i;I}^{\mathbf{a}} \in \Gamma(\mathcal{O}_{\mathcal{U}_x^{\mathbf{a}, \circ}}), \quad \Phi_x^{\mathbf{a}} : \mathcal{U}_x^{\mathbf{a}} \longrightarrow \mathfrak{M}^{\text{tf}}$$

parallel to (3.12) and (3.14). Let $\mathcal{U} = \Phi_x^{\mathbf{a}}(\mathcal{U}_x^{\mathbf{a}}) \cap \Phi_x(\mathcal{U}_x)$.

LEMMA 3.5. *The transition map*

$$(\Phi_x^{\mathbf{a}})^{-1} \circ \Phi_x : \Phi_x^{-1}(\mathcal{U}) \longrightarrow (\Phi_x^{\mathbf{a}})^{-1}(\mathcal{U})$$

is an isomorphism.

Proof. Let $g : \Phi_x^{-1}(\mathcal{U}) \longrightarrow (\Phi_x^{\mathbf{a}})^{-1}(\mathcal{U})$ be the isomorphism given by

$$g^* \delta_i = \varepsilon_i \cdot \frac{u_{e_i^+} \cdot u_{e_{i[1]}^+} \cdots u_{d_i} \cdot u_{d_{i\langle 1 \rangle}} \cdots u_{d_{i\#}^+} \cdot u_{d_{i\# \langle 1 \rangle}^+} \cdots}{u_{e_{i\#}^+} \cdot u_{e_{i\#[1]}^+} \cdots u_{d_{i\#}} \cdot u_{d_{i\#\langle 1 \rangle}} \cdots u_{d_i^+} \cdot u_{d_{i\langle 1 \rangle}^+} \cdots} \quad \forall i \in \mathbb{I}_+,$$

$$g^* u_e^{\mathbf{a}} = \frac{u_e}{u_{d_{\ell(e)}}} \quad \forall e \in \widehat{\text{Edg}(t)} \setminus \{d_i\}_{i \in \mathbb{I}_+}, \quad g^* z_e^{\mathbf{a}} = z_e \quad \forall e \in \mathbb{I}_-, \quad g^* w_j^{\mathbf{a}} = w_j \quad \forall j \in J.$$

The fact that g is an isomorphism can be shown by constructing its inverse explicitly, which is similar to the proof of Lemma 3.3, but is simpler. The key point of the construction is that

$$u_{d_i} = \frac{1}{g^* u_{e_i}^{\mathbf{a}}} \quad \forall i \in \mathbb{I}_+, \quad u_e = \frac{g^* u_e^{\mathbf{a}}}{g^* u_{e_i}^{\mathbf{a}}} \quad \forall e \in \widehat{\text{Edg}(t)} \text{ with } \ell(e) = i,$$

and each ε_i is a product of $g^*\delta_i$ and a rational function of u_e with $e \in \widehat{\text{Edg}}(\mathfrak{t})$.

It is a direct check that the isomorphism g satisfies

$$\theta_x^a \circ g = \theta_x \quad \text{and} \quad g^* \mu_{e;i;I}^a = \frac{\mu_{e;i;I}}{\mu_{d_i;i;I}} \quad \forall I \subset \mathbb{I}, \quad i \in \mathbb{I}_+ \setminus I_+, \quad e \in \mathfrak{E}_i.$$

Thus, $(\Phi_x^a)^{-1} \circ \Phi_x = g$ and hence is an isomorphism. \square

Let

$$\{\widehat{\zeta}_e : e \in \text{Edg}(\gamma)\} \sqcup \{\widehat{\zeta}_j : j \in J\}$$

be another set of extended modular parameters centered at x on the same chart $\mathcal{V} \rightarrow \mathfrak{M}^{\text{wt}}$; see (3.3). We use this set of local parameters to construct another twisted chart $\widehat{\Phi}_x : \widehat{\mathfrak{U}}_x \rightarrow \mathfrak{M}^{\text{tf}}$; in particular, we have $\widehat{\theta}_x : \widehat{\mathfrak{U}}_x \rightarrow \mathcal{V}$ and $\widehat{\mu}_{e;i;I} \in \Gamma(\mathcal{O}_{\widehat{\mathfrak{U}}_x(I)})$ as in (3.12) and (3.14), respectively. Parallel to (3.9), the coordinates on $\widehat{\mathfrak{U}}_x$ are denoted by

$$((\widehat{\varepsilon}_i)_{i \in \mathbb{I}_+}, (\widehat{u}_e)_{e \in \widehat{\text{Edg}}(\mathfrak{t}) \setminus \{e_i : i \in \mathbb{I}_+\}}, (\widehat{z}_e)_{e \in \mathbb{I}_-}, (\widehat{w}_j)_{j \in J}).$$

Let $\mathcal{U} = \Phi_x(\mathfrak{U}_x) \cap \widehat{\Phi}_x(\widehat{\mathfrak{U}}_x)$.

LEMMA 3.6. *The transition map*

$$(\widehat{\Phi}_x)^{-1} \circ \Phi_x : \Phi_x^{-1}(\mathcal{U}) \rightarrow \widehat{\Phi}_x^{-1}(\mathcal{U})$$

is an isomorphism.

Proof. By Lemma 3.5, it suffices to use the same set of the special vertices $\{v_i\}_{i \in \mathbb{I}_+}$ for both Φ_x and $\widehat{\Phi}_x$. For any $e \in \text{Edg}(\gamma)$, the local parameters $\widehat{\zeta}_e$ and ζ_e defines the same locus $\mathcal{Z}_e = \{\zeta_e = 0\} = \{\widehat{\zeta}_e = 0\}$, hence there exists $f_e \in \Gamma(\mathcal{O}_{\mathcal{V}}^*)$ such that

$$\widehat{\zeta}_e = f_e \cdot \zeta_e.$$

Since

$$\partial_{\zeta_{e'}}|_{\mathcal{Z}_e} \in \Gamma(T\mathcal{Z}_e) \quad \forall e' \in \text{Edg}(\gamma) \setminus \{e\}, \quad \partial_{\zeta_j}|_{\mathcal{Z}_e} \in \Gamma(T\mathcal{Z}_e) \quad \forall j \in J$$

(where ∂_{ζ_j} are defined in the same way as $\partial_{\zeta_{e'}}$), we have

$$\bar{\partial}_{\widehat{\zeta}_e} = \frac{1}{f_e} \bar{\partial}_{\zeta_e} \tag{3.31}$$

as nowhere vanishing sections of the normal bundle $L_e \rightarrow \mathcal{Z}_e$.

Let $g : \Phi_x^{-1}(\mathcal{U}) \rightarrow \widehat{\Phi}_x^{-1}(\mathcal{U})$ be the isomorphism given by

$$g^* \widehat{\varepsilon}_i = \varepsilon_i \cdot \theta_x^* \left(\frac{f_{e_i} \cdot f_{e_i[1]} \cdots}{f_{e_i\#} \cdot f_{e_i\#[1]} \cdots} \right) \quad \forall i \in \mathbb{I}_+, \quad g^* \widehat{z}_e = z_e \cdot (\theta_x^* f_e) \quad \forall e \in \mathbb{I}_-, \quad g^* \widehat{w}_j = \theta_x^* \widehat{\zeta}_j \quad \forall j \in J,$$

$$g^* \widehat{u}_e = u_e \cdot \theta_x^* \left(\frac{\prod_{e' \geq e} f_{e'}}{\prod_{e' \geq e_\ell(e)} f_{e'}} \right) \quad \forall e \in \widehat{\text{Edg}}(\mathfrak{t}) \setminus \{e_i : i \in \mathbb{I}_+\}.$$

Shrinking \mathcal{U} if necessary, we see that such defined g is invertible, because

$$((\varepsilon_i)_{i \in \mathbb{I}_+}, (u_e)_{e \in \widehat{\text{Edg}}(\mathfrak{t}) \setminus \{e_i : i \in \mathbb{I}_+\}}, (z_e)_{e \in \mathbb{I}_-}, (\theta_x^* \widehat{\zeta}_j)_{j \in J})$$

can be considered as coordinates on $\Phi_x^{-1}(\mathcal{U})$.

It is a direct check that $\widehat{\theta}_x \circ g = \theta_x$ and

$$\frac{g^* \widehat{\mu}_{e;i;I}}{\theta_x^* \left(\prod_{e \geq e} \llbracket \ell(e), \ell(v_e^+) \rrbracket_{\Phi I} f_e \right)} = \frac{\mu_{e;i;I}}{\theta_x^* \left(\prod_{e \geq e_i} \llbracket \ell(e), \ell(v_e^+) \rrbracket_{\Phi I} f_e \right)} \quad \forall I \subset \mathbb{I}, \quad i \in \mathbb{I}_+ \setminus I_+, \quad e \in \mathfrak{E}_i.$$

Taking (3.31) into account, we conclude that $(\widehat{\Phi}_x)^{-1} \circ \Phi_x = g$ and hence is an isomorphism. \square

Given $I \subset \mathbb{I}$ and $x' \in \Phi_x(\mathfrak{U}_{x;[\mathfrak{t}_{(I)}]})$, let $\mathcal{V}_{\varpi(x')} \rightarrow \mathfrak{M}^{\text{wt}}$ be a chart containing $\varpi(x')$ and $\Phi_{x'} : \mathfrak{U}_{x'} \rightarrow \mathfrak{M}^{\text{tf}}$ be a twisted chart centered at x' over $\mathcal{V}_{\varpi(x')} \rightarrow \mathfrak{M}^{\text{wt}}$. Let $\mathcal{U} = \Phi_x(\mathfrak{U}_x) \cap \Phi_{x'}(\mathfrak{U}_{x'})$.

PROPOSITION 3.7. *The transition map*

$$\Phi_{x'}^{-1} \circ \Phi_x : \Phi_x^{-1}(\mathcal{U}) \rightarrow \Phi_{x'}^{-1}(\mathcal{U})$$

is an isomorphism.

Proof. Since $x' \in \Phi_x(\mathfrak{U}_{x;[\mathfrak{t}_{(I)}]}) \subset \Phi_x(\mathfrak{U}_x)$, its underlying weighted curve satisfies

$$\varpi(x') \in \mathcal{V}_{(I)}^\circ \quad \left(= \{ \zeta_e \neq 0 : e \in \text{Edg}(\mathfrak{t}) \setminus \text{Edg}(\mathfrak{t}_{(I)}) \} \subset \mathcal{V} \right).$$

Thus, replacing $\mathcal{V}_{\varpi(x')}$ by $\mathcal{V}_{\varpi(x')} \cap \mathcal{V}_{(I)}^\circ$ if necessary, we may assume

$$\varpi(x') \in \mathcal{V}_{\varpi(x')} \subset \mathcal{V}_{(I)}^\circ.$$

Moreover, the following modular parameters on \mathcal{V} :

$$\zeta_e, \quad e \in \text{Edg}(\mathfrak{t}_{(I)}) \subset \text{Edg}(\mathfrak{t})$$

also serve as modular parameters on $\mathcal{V}_{\varpi(x')}$. Thus by Lemma 3.6, we may assume $\Phi_{x'}$ is constructed using the local parameters on $\mathcal{V}_{\varpi(x')}$:

$$\{ \zeta_e \}_{e \in \text{Edg}(\mathfrak{t}_{(I)})} \sqcup \left(\{ \zeta_j \}_{j \in J} \sqcup \{ \zeta_e \}_{e \in \text{Edg}(\mathfrak{t}) \setminus \text{Edg}(\mathfrak{t}_{(I)})} \right)$$

as the analogue of (3.3).

Let the special vertices v_i and edges e_i of \mathfrak{t} be respectively as in (3.4) and (3.5). By Lemma 3.5, we may further assume that the special vertices and edges of $\mathfrak{t}_{(I)}$ are respectively

$$v_i \quad \text{and} \quad e_i, \quad i \in \mathbb{I}_+(\mathfrak{t}_{(I)}) = \mathbb{I}_+ \setminus I_+.$$

For any $i \in \mathbb{I}_+ \setminus I_+$ and $h \in \mathbb{Z}_{\geq 0}$, let i^\uparrow , e_i^\dagger , and $i(h)$ be the analogues of i^\sharp , e_i^+ , and $i[h]$, respectively, for the weighted level tree $\mathfrak{t}_{(I)}$ instead of \mathfrak{t} ; see (2.1), (3.5), and (3.6) for notation.

Recall that $\mathbb{I}_-(\mathfrak{t}_{(I)}) = \mathbb{I}_- \setminus I_- \sqcup \{ e \in \mathbb{I}_m \setminus I_m : \ell(v_e^+) \leq \mathfrak{m}(\mathfrak{t}_{(I)}) \}$. We denote by

$$\left((\varepsilon'_i)_{i \in \mathbb{I}_+ \setminus I_+}, (u'_e)_{e \in \widehat{\text{Edg}}(\mathfrak{t}_{(I)}) \setminus \{ e_i : i \in \mathbb{I}_+ \setminus I_+ \}}, (z'_e)_{e \in \mathbb{I}_-(\mathfrak{t}_{(I)})}, (w'_j)_{j \in J \sqcup (\text{Edg}(\mathfrak{t}) \setminus \text{Edg}(\mathfrak{t}_{(I)}))} \right)$$

the coordinates on $\mathfrak{U}_{x'}$ parallel to (3.9), and construct

$$\theta_{x'} : \mathfrak{U}_{x'} \rightarrow \mathcal{V}_{\varpi(x')} \quad \text{and} \quad \mu'_{e;i;I'} \in \Gamma(\mathcal{O}_{\mathfrak{U}_{x';(I')}}^\circ), \quad I' \subset \mathbb{I} \setminus I, \quad i \in \mathbb{I}_+ \setminus (I_+ \sqcup I'), \quad e \in \mathfrak{E}_i$$

parallel to

$$\theta_x : \mathfrak{U}_x \longrightarrow \mathcal{V} \quad \text{and} \quad \mu_{e; i; I} \in \Gamma(\mathcal{O}_{\mathfrak{U}_{x; (I)}}^\circ), \quad i \in \mathbb{I}_+ \setminus I_+, \quad e \in \mathfrak{E}_i,$$

of (3.12) and (3.14), respectively. In this way, $\Phi_{x'} : \mathfrak{U}_{x'} \longrightarrow \mathfrak{M}^{\text{tf}}$ is constructed analogously to Φ_x .

Let $g : \Phi_x^{-1}(\mathcal{U}) \longrightarrow \Phi_{x'}^{-1}(\mathcal{U})$ be the isomorphism given by

$$g^* \varepsilon'_i = \varepsilon_i \cdot \left(\prod_{h \in \langle i, i^\dagger \rangle_t} \varepsilon_h \right) \cdot \frac{\prod_{\mathfrak{e} >_t \mathfrak{e}_{i^\dagger}, \llbracket \ell(\mathfrak{e}), \ell(v_\mathfrak{e}^+) \rrbracket_t \subset I} \theta_x^* \zeta_\mathfrak{e}}{\prod_{\mathfrak{e} >_t \mathfrak{e}_i, \llbracket \ell(\mathfrak{e}), \ell(v_\mathfrak{e}^+) \rrbracket_t \subset I} \theta_x^* \zeta_\mathfrak{e}} \cdot \frac{u_{\mathfrak{e}_i^+} \cdot u_{\mathfrak{e}_{i^\dagger[1]}^+} \cdots}{u_{\mathfrak{e}_{i^\dagger}^+} \cdot u_{\mathfrak{e}_{i^\dagger[1]}^+} \cdots} \\ \cdot \frac{\mu_{\mathfrak{e}_{i^\dagger}^+; i^\dagger(1); I} \cdot \mu_{\mathfrak{e}_{i^\dagger(1)}^+; i^\dagger(2); I} \cdots}{\mu_{\mathfrak{e}_i^+; i(1); I} \cdot \mu_{\mathfrak{e}_{i(1)}^+; i(2); I} \cdots} \quad \forall i \in \mathbb{I}_+ \setminus I_+$$

and

$$g^* u'_e = \mu_{e; \ell_{(I)}(e); I} \quad \forall e \in \widehat{\text{Edg}}(\mathfrak{t}_{(I)}) \setminus \{\mathfrak{e}_i : i \in \mathbb{I}_+ \setminus I_+\}, \\ g^* z'_e = u_e \quad \forall e \in \{e \in \mathbb{I}_\mathfrak{m} \setminus I_\mathfrak{m} : \ell(v_e^+) \leq \mathfrak{m}(\mathfrak{t}_{(I)})\}, \quad g^* z'_e = z_e \quad \forall e \in \mathbb{I}_- \setminus I_-, \\ g^* w'_j = w_j \quad \forall j \in J, \quad g^* w'_e = \theta_x^* \zeta_e \quad \forall e \in \text{Edg}(\mathfrak{t}) \setminus \text{Edg}(\mathfrak{t}_{(I)}).$$

To see g is well defined, notice that $\mathcal{V}_{\varpi(x')} \subset \mathcal{V}_{(I)}^\circ$ implies that

$$\Phi_x^{-1}(\mathcal{U}) \subset \mathfrak{U}_{x; (I)}^\circ. \quad (3.32)$$

Thus, every $\mu_{e; i; I}$ above can be considered as a function on $\Phi_x^{-1}(\mathcal{U})$. By (3.32) and (3.13), the function $\theta_x^* \zeta_\mathfrak{e}$ with $\mathfrak{e} \in \widehat{\text{Edg}}(\mathfrak{t}) \setminus \mathbb{I}_\mathfrak{m}$ is nowhere vanishing on $\Phi_x^{-1}(\mathcal{U})$ whenever $\llbracket \ell(\mathfrak{e}), \ell(v_\mathfrak{e}^+) \rrbracket_t \subset I$. Taking (3.11) and (3.15) into account, we conclude that g is well defined.

Once again, the explicit expression of g implies it is invertible; see the parallel argument in the proof of Lemma 3.5. Moreover, it is a direct check that $\theta_{x'} \circ g = \theta_x$ and

$$g^* \mu'_{e; i; I'} = \mu_{e; i; I \sqcup I'} \in \Gamma(\mathcal{O}_{\mathfrak{U}_{x; (I \sqcup I')}}^\circ \cap \Phi_x^{-1}(\mathcal{U})) \quad \forall I' \subset \mathbb{I} \setminus I, \quad i \in \mathbb{I}_+ \setminus (I_+ \sqcup I'), \quad e \in \mathfrak{E}_i.$$

Thus, $\Phi_{x'}^{-1} \circ \Phi_x = g$ and hence is an isomorphism. \square

COROLLARY 3.8. \mathfrak{M}^{tf} is a smooth Artin stack that is birational to \mathfrak{M}^{wt} , with $\{\Phi_x : \mathfrak{U}_x \longrightarrow \mathfrak{M}^{\text{tf}}\}_{x \in \mathfrak{M}^{\text{tf}}}$ as smooth charts. Moreover, the structure of the stratification (2.14) is locally identical to the one induced by (3.10), and each stratum $\mathfrak{M}_{[t]}^{\text{tf}}$ is of codimension $|\mathbb{I}(t)|$. Furthermore, for any $x \in \mathfrak{M}^{\text{tf}}$, any chart $\mathcal{V} \longrightarrow \mathfrak{M}^{\text{wt}}$ containing $\varpi(x) \in \mathfrak{M}^{\text{wt}}$, and any twisted chart $\Phi_x : \mathfrak{U}_x \longrightarrow \mathfrak{M}^{\text{tf}}$ centered at x lying over $\mathcal{V} \longrightarrow \mathfrak{M}^{\text{wt}}$, we have the Cartesian diagram

$$\begin{array}{ccc} \mathfrak{U}_x & \longrightarrow & \mathfrak{M}^{\text{tf}} \\ \theta_x \downarrow & \searrow \Phi_x & \downarrow \varpi \\ \mathcal{V} & \longrightarrow & \mathfrak{M}^{\text{wt}} \end{array}$$

where ϖ is the forgetful morphism as in (2.14) and θ_x is as in (3.12).

Proof. The first statement follows from Proposition 3.7, (3.17), and the fact that ϖ restricts to the identity map on the preimage of the open subset

$$\{(C, \mathbf{w}) \in \mathfrak{M}^{\text{wt}} : \mathbf{w}(C_o) > 0\} \subset \mathfrak{M}^{\text{wt}}.$$

Lemma 3.3 then implies for every $[t] \in \mathcal{T}_1^{\text{wt}}$, the stack structure of $\mathfrak{M}_{[t]}^{\text{tf}}$ is the same as that induced from the inclusion $\mathfrak{M}_{[t]}^{\text{tf}} \hookrightarrow \mathfrak{M}^{\text{tf}}$. Besides, the $I = \emptyset$ case of Lemma 3.3, along with the description of $\mathfrak{U}_{x;[t(I)]}$ before (3.10) and the definition of the index set $\mathbb{I}(t)$ in (2.4), gives rise to the codimension of $\mathfrak{M}_{[t]}^{\text{tf}}$. The last statement follows from (3.14). \square

REMARK 3.9. By (3.12) and Corollary 3.8, one sees that on an arbitrary twisted chart \mathfrak{U}_x of \mathfrak{M}^{tf} ,

$$\begin{aligned} \varpi^* \left(\prod_{e' \geq e_{\mathbf{m}}} \zeta_{e'} \right) &= (u_{e_{\mathbf{m}}^+} u_{e_{\mathbf{m}[1]}^+} \cdots) \prod_{i \in \mathbb{I}_{\mathbf{m}}} \varepsilon_i, \\ \varpi^* \left(\prod_{e' \geq e} \zeta_{e'} \right) &= (u_e u_{e_{\mathbf{m}}^+} u_{e_{\mathbf{m}[1]}^+} \cdots) \prod_{i \in \mathbb{I}_{\mathbf{m}}} \varepsilon_i = u_e \cdot \varpi^* \left(\prod_{e' \geq e_{\mathbf{m}}} \zeta_{e'} \right) \quad \forall e \in \mathfrak{E}_{\mathbf{m}}. \end{aligned}$$

This, along with the local equations of $\overline{M}_1(\mathbb{P}^n, d)$ in [5, §5.2], implies that $\widetilde{M}_1^{\text{tf}}(\mathbb{P}^n, d)$ has smooth irreducible components and contains at worst normal crossing singularities. This observation should be useful for the cases of higher genera.

3.3. Examples on the twisted charts. In this subsection, we revisit the two examples of §2.3 to illustrate how the strata of \mathfrak{M}^{tf} are glued together by the twisted charts.

EXAMPLE 3.10. We continue with the setup of Example 2.6. Let $x \in \mathfrak{M}_{[t]}^{\text{tf}}$ be a weighted curve of genus 1 with twisted fields over $(C, \mathbf{w}) \in \mathfrak{M}^{\text{wt}}$. The core and the nodes of C are labeled by o and by a, b, c, d , respectively.

Let $\mathcal{V} \rightarrow \mathfrak{M}^{\text{wt}}$ be an affine smooth chart containing (C, \mathbf{w}) , with a set of local parameters

$$\{\zeta_a, \zeta_b, \zeta_c, \zeta_d\} \sqcup \{\zeta_j\}_{j \in J}$$

centered at (C, \mathbf{w}) as in (3.3), where ζ_a, \dots, ζ_d correspond to the smoothing of the nodes. There then exist *non-zero* λ_c and λ_d such that

$$x = \left(\underline{0}; [0, \bar{\partial}_{\zeta_b}|_{\underline{0}}], [\bar{\partial}_{\zeta_a}|_{\underline{0}}, \lambda_c \cdot (\bar{\partial}_{\zeta_b} \otimes \bar{\partial}_{\zeta_c})|_{\underline{0}}, \lambda_d \cdot (\bar{\partial}_{\zeta_b} \otimes \bar{\partial}_{\zeta_d})|_{\underline{0}}] \right) \in \mathcal{V}_\tau \times_{\mathfrak{M}^{\text{wt}}} \mathfrak{M}_{[t]}^{\text{tf}}.$$

Here we identify $[0, \bar{\partial}_{\zeta_b}|_{\underline{0}}]$ with $[\bar{\partial}_{\zeta_b}|_{\underline{0}}]$, as mentioned below (2.12).

We choose the special edges (3.5) of \mathfrak{t} to be $e_1 = b$ and $e_2 = a$. Let

$$\mathfrak{U}_x \subset \mathbb{A}^{\{-1, -2\}} \times \mathbb{A}^{\{c, d\}} \times \mathbb{A}^J$$

be an open subset containing the point

$$y_x = (0, 0, \lambda_c, \lambda_d, 0, \dots, 0).$$

The coordinates of \mathfrak{U}_x are denoted by

$$\varepsilon_{-1}, \varepsilon_{-2}, u_c, u_d, \quad \text{and} \quad w_j \quad \text{with} \quad j \in J,$$

where J is the same as the index set for the local parameters ζ_j on \mathcal{V} . Since $\lambda_c, \lambda_d \neq 0$, by shrinking \mathfrak{U}_x if necessary we assume that $\mathfrak{U}_x \subset \{u_c \neq 0, u_d \neq 0\}$.

By Corollary 3.8 and (3.12), the forgetful morphism $\varpi: \mathfrak{M}^{\text{tf}} \rightarrow \mathfrak{M}^{\text{wt}}$ can locally be written as $\theta_x: \mathfrak{U}_x \rightarrow \mathcal{V}$ such that

$$\theta_x(\varepsilon_{-1}, \varepsilon_{-2}, u_c, u_d, (w_j)) = \left(\underbrace{\varepsilon_{-1} \cdot \varepsilon_{-2}}_{\zeta_a}, \underbrace{\varepsilon_{-1}}_{\zeta_b}, \underbrace{\varepsilon_{-2} \cdot u_c}_{\zeta_c}, \underbrace{\varepsilon_{-2} \cdot u_d}_{\zeta_d}, (w_j) \right).$$

Considering all possible subsets I of \mathbb{I} in (3.14), we obtain a twisted chart $\Phi_x: \mathfrak{U}_x \rightarrow \mathfrak{M}^{\text{tf}}$ centered at x over $\mathcal{V} \rightarrow \mathfrak{M}^{\text{wt}}$ so that for any

$$y = (\varepsilon_{-1}, \varepsilon_{-2}, u_c, u_d, (w_j)) \in \mathfrak{U}_x,$$

- if $\varepsilon_{-1} = \varepsilon_{-2} = 0$ (i.e. $y \in \mathfrak{U}_{x;[\{j\}]}$), then

$$\begin{aligned} \Phi_x(y) &= \left((0, 0, 0, 0, (w_j)); \right. \\ &\quad \left. [0, \bar{\partial}_{\zeta_b}|_{\theta_x(y)}], [\bar{\partial}_{\zeta_a}|_{\theta_x(y)}, u_c(\bar{\partial}_{\zeta_b} \otimes \bar{\partial}_{\zeta_c})|_{\theta_x(y)}, u_d(\bar{\partial}_{\zeta_b} \otimes \bar{\partial}_{\zeta_d})|_{\theta_x(y)}] \right) \\ &\in \mathcal{V}_\tau \times \mathfrak{M}_{\tau}^{\text{wt}} \mathfrak{M}_{[\{j\}]}^{\text{tf}}; \end{aligned}$$

- if $\varepsilon_{-1} \neq 0$ and $\varepsilon_{-2} = 0$ (i.e. $y \in \mathfrak{U}_{x;[\{(-1)\}]}$), then

$$\begin{aligned} \Phi_x(y) &= \left((0, \varepsilon_{-1}, 0, 0, (w_j)); [\bar{\partial}_{\zeta_a}|_{\theta_x(y)}, \frac{u_c}{\varepsilon_{-1}} \cdot \bar{\partial}_{\zeta_c}|_{\theta_x(y)}, \frac{u_d}{\varepsilon_{-1}} \cdot \bar{\partial}_{\zeta_d}|_{\theta_x(y)}] \right) \\ &\in \mathcal{V}_{\tau(\{-1\})} \times \mathfrak{M}_{\tau(\{-1\})}^{\text{wt}} \mathfrak{M}_{[\{(-1)\}]}^{\text{tf}}; \end{aligned}$$

- if $\varepsilon_{-1} = 0$ and $\varepsilon_{-2} \neq 0$ (i.e. $y \in \mathfrak{U}_{x;[\{(-2)\}]}$), then

$$\begin{aligned} \Phi_x(y) &= \left((0, 0, \varepsilon_{-2} \cdot u_c, \varepsilon_{-2} \cdot u_d, (w_j)); [\varepsilon_{-2} \cdot \bar{\partial}_{\zeta_a}|_{\theta_x(y)}, \bar{\partial}_{\zeta_b}|_{\theta_x(y)}] \right) \\ &\in \mathcal{V}_{\tau(\{-2\})} \times \mathfrak{M}_{\tau(\{-2\})}^{\text{wt}} \mathfrak{M}_{[\{(-2)\}]}^{\text{tf}}; \end{aligned}$$

- if $\varepsilon_{-1} \neq 0$ and $\varepsilon_{-2} \neq 0$ (i.e. $y \in \mathfrak{U}_{x;[\{(-1, -2)\}]}$), then

$$\Phi_x(y) = \theta_x(y) = (\varepsilon_{-1} \cdot \varepsilon_{-2}, \varepsilon_{-1}, \varepsilon_{-2} \cdot u_c, \varepsilon_{-2} \cdot u_d, (w_j)) \in \mathcal{V}_{\tau(\{-1, -2\})}.$$

When a point $y \in \mathfrak{U}_x$ satisfying $\varepsilon_{-1}(y) \neq 0$ and $\varepsilon_{-2}(y) = 0$ approaches a point $y' \in \mathfrak{U}_x$ satisfying $\varepsilon_{-1}(y') = \varepsilon_{-2}(y') = 0$, it may seem that the limit of the expression of $\Phi_x(y)$ does not match $\Phi_x(y')$, however, as suggested in Remark 2.4, $[\bar{\partial}_{\zeta_a}|_{\theta_x(y')}, u_c(\bar{\partial}_{\zeta_b} \otimes \bar{\partial}_{\zeta_c})|_{\theta_x(y')}, u_d(\bar{\partial}_{\zeta_b} \otimes \bar{\partial}_{\zeta_d})|_{\theta_x(y')}]$ of $\Phi_x(y')$ captures the limit of the ratio $[\zeta_a : \zeta_b \zeta_c : \zeta_b \zeta_d]$, whereas $[\bar{\partial}_{\zeta_a}|_{\theta_x(y)}, \frac{u_c}{\varepsilon_{-1}} \cdot \bar{\partial}_{\zeta_c}|_{\theta_x(y)}, \frac{u_d}{\varepsilon_{-1}} \cdot \bar{\partial}_{\zeta_d}|_{\theta_x(y)}]$ of $\Phi_x(y)$ captures the limit of the ratio $[\zeta_a : \zeta_c : \zeta_d]$. If we multiply the last two components of the twisted field of $\Phi_x(y)$ by ε_{-1} (i.e. ζ_b) and then take $\varepsilon_{-1} \rightarrow 0$, we can recover $\Phi_x(y')$.

With the expressions of Φ_x as above, it is straightforward to check

$$\Phi_x(0, 0, \lambda_c, \lambda_d, 0, \dots, 0) = x,$$

as well as to verify the statements of Lemmas 3.3, 3.5, 3.6, and Proposition 3.7 in this situation.

EXAMPLE 3.11. We continue with the setup of Example 2.7. Let $x \in \mathfrak{M}_{[\{j\}]}^{\text{tf}}$ be a weighted curve of genus 1 with twisted fields over $(C, \mathbf{w}) \in \mathfrak{M}^{\text{wt}}$. The core and the nodes of C are still labeled by o and by a, b, c, d , respectively.

Let $\mathcal{V} \longrightarrow \mathfrak{M}^{\text{wt}}$ be an affine smooth chart containing (C, \mathbf{w}) , with a set of local parameters

$$\{\zeta_a, \zeta_b, \zeta_c, \zeta_d\} \sqcup \{\zeta_j\}_{j \in J}$$

centered at (C, \mathbf{w}) as in Example 3.10. There then exists a *non-zero* λ_d such that

$$x = \left(\underline{0}; [0, \bar{\partial}_{\zeta_b}|_{\underline{0}}], [0, (\bar{\partial}_{\zeta_b} \otimes \bar{\partial}_{\zeta_c})|_{\underline{0}}], \lambda_d \cdot (\bar{\partial}_{\zeta_b} \otimes \bar{\partial}_{\zeta_d})|_{\underline{0}} \right) \in \mathcal{V}_\tau \times_{\mathfrak{M}^{\text{wt}}} \mathfrak{M}_{[\mathfrak{t}]}^{\text{tf}}.$$

We choose the special edges (3.5) of \mathfrak{t} to be $e_1 = b$ and $e_2 = c$. Let

$$\mathfrak{U}_x \subset \mathbb{A}^{\{-1, -2\}} \times \mathbb{A}^{\{a, d\}} \times \mathbb{A}^J$$

be an open subset containing the point

$$y_x = (0, 0, 0, \lambda_d, 0, \dots, 0).$$

The coordinates of \mathfrak{U}_x are denoted by

$$\varepsilon_{-1}, \varepsilon_{-2}, u_a, u_d, \quad \text{and} \quad w_j \text{ with } j \in J,$$

where J is the same as the index set for the local parameters ζ_j on \mathcal{V} . Since $\lambda_d \neq 0$, by shrinking \mathfrak{U}_x if necessary we assume that $\mathfrak{U}_x \subset \{u_d \neq 0\}$.

By Corollary 3.8 and (3.12), the forgetful morphism $\varpi: \mathfrak{M}^{\text{tf}} \longrightarrow \mathfrak{M}^{\text{wt}}$ can locally be written as $\theta_x: \mathfrak{U}_x \longrightarrow \mathcal{V}$ such that

$$\theta_x(\varepsilon_{-1}, \varepsilon_{-2}, u_a, u_d, (w_j)) = \left(\underbrace{\varepsilon_{-1} \cdot \varepsilon_{-2} \cdot u_a}_{\zeta_a}, \underbrace{\varepsilon_{-1}}_{\zeta_b}, \underbrace{\varepsilon_{-2}}_{\zeta_c}, \underbrace{\varepsilon_{-2} \cdot u_d}_{\zeta_d}, (w_j) \right).$$

Considering all possible subsets I of \mathbb{I} in (3.14), we obtain a twisted chart $\Phi_x: \mathfrak{U}_x \longrightarrow \mathfrak{M}^{\text{tf}}$ centered at x over $\mathcal{V} \longrightarrow \mathfrak{M}^{\text{wt}}$ so that for any

$$y = (\varepsilon_{-1}, \varepsilon_{-2}, u_a, u_d, (w_j)) \in \mathfrak{U}_x,$$

- if $\varepsilon_{-1} = \varepsilon_{-2} = u_a = 0$ (i.e. $y \in \mathfrak{U}_{x; [\mathfrak{t}]}$), then

$$\begin{aligned} \Phi_x(y) &= \left((0, 0, 0, 0, (w_j)); [0, \bar{\partial}_{\zeta_b}|_{\theta_x(y)}], [0, (\bar{\partial}_{\zeta_b} \otimes \bar{\partial}_{\zeta_c})|_{\theta_x(y)}], u_d(\bar{\partial}_{\zeta_b} \otimes \bar{\partial}_{\zeta_d})|_{\theta_x(y)} \right) \\ &\in \mathcal{V}_\tau \times_{\mathfrak{M}^{\text{wt}}} \mathfrak{M}_{[\mathfrak{t}]}^{\text{tf}}; \end{aligned}$$

- if $\varepsilon_{-1} = u_a = 0$ and $\varepsilon_{-2} \neq 0$ (i.e. $y \in \mathfrak{U}_{x; [\mathfrak{t}_{\{-2\}}]}$), then

$$\Phi_x(y) = \left((0, 0, \varepsilon_{-2}, \varepsilon_{-2} \cdot u_d, (w_j)); [0, \bar{\partial}_{\zeta_b}|_{\theta_x(y)}] \right) \in \mathcal{V}_{\tau_{\{-2\}}} \times_{\mathfrak{M}^{\text{wt}}_{\tau_{\{-2\}}}} \mathfrak{M}_{[\mathfrak{t}_{\{-2\}}]}^{\text{tf}};$$

- if $\varepsilon_{-1} \neq 0$ and $\varepsilon_{-2} = u_a = 0$ (i.e. $y \in \mathfrak{U}_{x; [\mathfrak{t}_{\{-1\}}]}$), then

$$\Phi_x(y) = \left((0, \varepsilon_{-1}, 0, 0, (w_j)); [0, \bar{\partial}_{\zeta_c}|_{\theta_x(y)}], u_d \bar{\partial}_{\zeta_d}|_{\theta_x(y)} \right) \in \mathcal{V}_{\tau_{\{-1\}}} \times_{\mathfrak{M}^{\text{wt}}_{\tau_{\{-1\}}}} \mathfrak{M}_{[\mathfrak{t}_{\{-1\}}]}^{\text{tf}};$$

- if $\varepsilon_{-1} = \varepsilon_{-2} = 0$ and $u_a \neq 0$ (i.e. $y \in \mathfrak{U}_{x; [\mathfrak{t}_{\{a\}}]}$), then

$$\begin{aligned} \Phi_x(y) &= \left((0, 0, 0, 0, (w_j)); \right. \\ &\quad \left. [0, \bar{\partial}_{\zeta_b}|_{\theta_x(y)}], [u_a \cdot \bar{\partial}_{\zeta_a}|_{\theta_x(y)}], (\bar{\partial}_{\zeta_b} \otimes \bar{\partial}_{\zeta_c})|_{\theta_x(y)}, u_d(\bar{\partial}_{\zeta_b} \otimes \bar{\partial}_{\zeta_d})|_{\theta_x(y)} \right) \\ &\in \mathcal{V}_{\tau_{\{a\}}} \times_{\mathfrak{M}^{\text{wt}}_{\tau_{\{a\}}}} \mathfrak{M}_{[\mathfrak{t}_{\{a\}}]}^{\text{tf}}; \end{aligned}$$

- if $\varepsilon_{-1} = 0$, $\varepsilon_{-2} \neq 0$, and $u_a \neq 0$, (i.e. $y \in \mathfrak{U}_{x;[\mathfrak{t}_{\{-2,a\}}]}$), then

$$\begin{aligned} \Phi_x(y) &= \left((0, 0, \varepsilon_{-2}, \varepsilon_{-2} \cdot u_d, (w_j)); [\varepsilon_{-2} \cdot u_a \cdot \check{\partial}_{\zeta_a}|_{\theta_x(y)}, \check{\partial}_{\zeta_b}|_{\theta_x(y)}] \right) \\ &\in \mathcal{V}_{\tau_{\{-2,a\}}} \times \mathfrak{M}_{\tau_{\{-2,a\}}}^{\text{wt}} \mathfrak{M}_{[\mathfrak{t}_{\{-2,a\}}]}^{\text{tf}}; \end{aligned}$$

- if $\varepsilon_{-1} \neq 0$, $u_a \neq 0$, and $\varepsilon_{-2} = 0$ (i.e. $y \in \mathfrak{U}_{x;[\mathfrak{t}_{\{-1,a\}}]}$), then

$$\begin{aligned} \Phi_x(y) &= \left((0, \varepsilon_{-1}, 0, 0, (w_j)); \right. \\ &\quad \left. [u_a \cdot \check{\partial}_{\zeta_a}|_{\theta_x(y)}, \frac{1}{\varepsilon_{-1}} \cdot \check{\partial}_{\zeta_c}|_{\theta_x(y)}, \frac{u_d}{\varepsilon_{-1}} \cdot \check{\partial}_{\zeta_d}|_{\theta_x(y)}] \right) \\ &\in \mathcal{V}_{\tau_{\{-1,a\}}} \times \mathfrak{M}_{\tau_{\{-1,a\}}}^{\text{wt}} \mathfrak{M}_{[\mathfrak{t}_{\{-1,a\}}]}^{\text{tf}}; \end{aligned}$$

- if $\varepsilon_{-1} \neq 0$, $\varepsilon_{-2} \neq 0$, and $u_a = 0$, (i.e. $y \in \mathfrak{U}_{x;[\mathfrak{t}_{\{-1,-2\}}]}$), then

$$\Phi_x(y) = \theta_x(y) = (0, \varepsilon_{-1}, \varepsilon_{-2}, \varepsilon_{-2} \cdot u_d, (w_j)) \in \mathcal{V}_{\tau_{\{-1,-2\}}};$$

- if $\varepsilon_{-1} \neq 0$, $\varepsilon_{-2} \neq 0$ and $u_a \neq 0$, (i.e. $y \in \mathfrak{U}_{x;[\mathfrak{t}_{\{-1,-2,a\}}]}$), then

$$\Phi_x(y) = \theta_x(y) = (\varepsilon_{-1} \cdot \varepsilon_{-2} \cdot u_a, \varepsilon_{-1}, \varepsilon_{-2}, \varepsilon_{-2} \cdot u_d, (w_j)) \in \mathcal{V}_{\tau_{\{-1,-2,a\}}}.$$

With the expressions of Φ_x as above, it is straightforward to check

$$\Phi_x(0, 0, 0, \lambda_d, 0, \dots, 0) = x,$$

as well as to verify the statements of Lemmas 3.3, 3.5, 3.6, and Proposition 3.7 in this situation.

3.4. Universal family. Let $\pi^{\text{wt}} : \mathcal{C}^{\text{wt}} \rightarrow \mathfrak{M}^{\text{wt}}$ be the universal weighted nodal curves of genus 1. The stratification (2.10) gives rise to a stratification

$$\mathcal{C}^{\text{wt}} = \bigsqcup_{\tau \in \mathcal{T}_R^{\text{wt}}} \mathcal{C}_\tau^{\text{wt}} \quad \text{satisfying} \quad \pi^{\text{wt}}(\mathcal{C}_\tau^{\text{wt}}) = \mathfrak{M}_\tau^{\text{wt}} \quad \forall \tau \in \mathcal{T}_R^{\text{wt}}.$$

Parallel to (2.13) and (2.14), we set

$$\begin{aligned} \mathcal{C}_{[\mathfrak{t}]}^{\text{tf}} &= \left(\prod_{i \in \mathbb{I}_+(\mathfrak{t})} \left(\mathbb{P} \left(\bigoplus_{e \in \text{Edg}(\mathfrak{t}), \ell(v_e^-) = i} (\pi^{\text{wt}})^* L_e^{\geq} \right) / \mathcal{C}_{[\mathfrak{t}]}^{\text{wt}} \right) \right) \rightarrow \mathcal{C}_{[\mathfrak{t}]}^{\text{wt}} \quad \forall [\mathfrak{t}] \in \mathcal{T}_L^{\text{wt}}, \\ \mathcal{C}^{\text{tf}} &= \bigsqcup_{[\mathfrak{t}] \in \mathcal{T}_L^{\text{wt}}} \mathcal{C}_{[\mathfrak{t}]}^{\text{tf}} \rightarrow \mathcal{C}^{\text{wt}}. \end{aligned}$$

Mimicking the construction of the stack structure of \mathfrak{M}^{tf} in §3.1 and §3.2, we can endow \mathcal{C}^{tf} with a stack structure analogously. Furthermore, the projection $\pi^{\text{wt}} : \mathcal{C}^{\text{wt}} \rightarrow \mathfrak{M}^{\text{wt}}$ induces a unique projection

$$\pi^{\text{tf}} : \mathcal{C}^{\text{tf}} \rightarrow \mathfrak{M}^{\text{tf}}. \tag{3.33}$$

It is straightforward that

$$\mathcal{C}^{\text{tf}} \cong \mathcal{C}^{\text{wt}} \times_{\mathfrak{M}^{\text{wt}}} \mathfrak{M}^{\text{tf}} \rightarrow \mathfrak{M}^{\text{tf}}.$$

For any scheme S , a flat family \mathcal{Z}/S of stable weighted nodal curves of genus 1 with twisted fields corresponds to a morphism $f: S \rightarrow \mathfrak{M}^{\text{tf}}$ such that \mathcal{Z}/S is the pullback of (3.33):

$$\mathcal{Z}/S \cong (S \times_{\mathfrak{M}^{\text{tf}}} \mathcal{C}^{\text{tf}})/S. \tag{3.34}$$

This leads to the following statement.

PROPOSITION 3.12. $\mathcal{C}^{\text{tf}} \rightarrow \mathfrak{M}^{\text{tf}}$ in (3.33) gives the universal family of \mathfrak{M}^{tf} .

REMARK 3.13. One may establish a moduli interpretation of \mathfrak{M}^{tf} as follows: for any scheme S , every flat family \mathcal{Z}/S of stable weighted nodal curves of genus 1 with twisted fields can be constructed directly as follows. A priori, \mathcal{Z}/S should be over a flat family \mathcal{C}_S/S of stable weighted curves, thus by the universality of the moduli \mathfrak{M}^{wt} , there exists a morphism

$$\alpha: S \rightarrow \mathfrak{M}^{\text{wt}} = \bigsqcup_{\tau \in \mathcal{S}_R^{\text{wt}}} \mathfrak{M}_\tau^{\text{wt}}$$

such that \mathcal{C}_S/S is the pullback of $\mathcal{C}^{\text{wt}}/\mathfrak{M}^{\text{wt}}$ via α . This induces a stratification of the scheme S :

$$S = \bigsqcup_{\tau \in \mathcal{S}_R^{\text{wt}}} S_\tau \quad \text{satisfying} \quad \alpha(S_\tau) \subset \mathfrak{M}_\tau^{\text{wt}} \quad \forall \tau \in \mathcal{S}_R^{\text{wt}}. \tag{3.35}$$

We take

$$S_{[\mathfrak{t}]}^{\text{tf}} = \left(\prod_{i \in \mathbb{I}_+(\mathfrak{t})} \left(\mathbb{P}^1 \left(\bigoplus_{e \in \text{Edg}(\mathfrak{t}), \ell(v_e^-) = i} \alpha^* L_e^{\geq} \right) \right) / S_{[\mathfrak{t}]} \right) \xrightarrow{\pi_S} S_{[\mathfrak{t}]} \quad \forall [\mathfrak{t}] \in \mathcal{S}_L^{\text{wt}},$$

$$S^{\text{tf}} = \bigsqcup_{[\mathfrak{t}] \in \mathcal{S}_L^{\text{wt}}} S_{[\mathfrak{t}]}^{\text{tf}} \xrightarrow{\pi_S} S.$$

For any chart $\mathcal{V}_S \rightarrow S$, shrinking it if necessary, we see there exists a (smooth) chart $\mathcal{V} \rightarrow \mathfrak{M}^{\text{wt}}$ such that

$$\begin{array}{ccc} \mathcal{V}_S & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathfrak{M}^{\text{wt}} \end{array}$$

commutes. The modular parameters ζ_e on \mathcal{V} pull back to regular functions on \mathcal{V}_S , which are denoted by ζ_e^S . By (3.35), we have

$$S_\tau \cap \mathcal{V}_S = \{\zeta_e^S = 0 : e \in \text{Edg}(\tau)\} \cap \{\zeta_{e'}^S \neq 0 : e' \notin \text{Edg}(\tau)\}.$$

Mimicking the construction in §3.1 and §3.2, we can thus endow S^{tf} with a scheme structure.

We say \mathcal{Z}/S is a flat family of stable weighted nodal curves of genus 1 with twisted fields if and only if there exists a section σ^{tf} of $\pi_S: S^{\text{tf}} \rightarrow S$ such that

$$\mathcal{Z} = \mathcal{C}_S \times_S (\sigma^{\text{tf}}(S)).$$

This construction is consistent with (3.34). One can check that the groupoid sending any scheme S to the set of all such defined flat families \mathcal{Z}/S is represented by \mathfrak{M}^{tf} .

We would like to remark that a more succinct definition of a flat family of stable weighted nodal curves of genus 1 with twisted fields should be desirable.

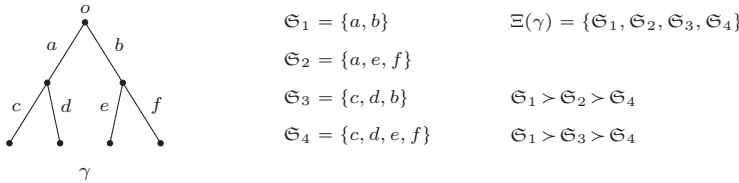


FIG. 4. An example of traverse sections

4. Comparison with Hu-Li’s blowup stack $\widetilde{\mathfrak{M}}^{\text{wt}}$. Let $\pi : \widetilde{\mathfrak{M}}^{\text{wt}} \rightarrow \mathfrak{M}^{\text{wt}}$ be the sequential blowup constructed in [5, §2.2]. More precisely, let $Z_k \subset \mathfrak{M}^{\text{wt}}$, $k \in \mathbb{Z}_{>0}$, be the closed locus whose *general* point is obtained by attaching k smooth positively-weighted rational curves to the smooth 0-weighted elliptic core at pairwise distinct points. Then $\widetilde{\mathfrak{M}}^{\text{wt}}$ is obtained by blowing up \mathfrak{M}^{wt} along the proper transforms of Z_1, Z_2, \dots sequentially. Since \mathfrak{M}^{wt} is a smooth Artin stack and the blowup centers are all smooth, so is $\widetilde{\mathfrak{M}}^{\text{wt}}$. As per the convention of this paper, we omit the subscript indicating the genus. In Proposition 4.4, we show that \mathfrak{M}^{tf} is isomorphic to $\widetilde{\mathfrak{M}}^{\text{wt}}$. Lemma 4.3 is rather technical; it is only used in the proof of Proposition 4.4.

We briefly recall the notion of the locally tree-compatible blowups described in [7, §3]. Let \mathfrak{M} be a smooth stack, γ be a rooted tree, and \mathcal{V} be an affine smooth chart of \mathfrak{M} . If there exists a set of local parameters on \mathcal{V} so that a subset of which can be written as

$$\{z_e \in \Gamma(\mathcal{O}_{\mathcal{V}}) : e \in \text{Edg}(\gamma)\},$$

then the set is called a γ -labeled subset of local parameters on \mathcal{V} . For example, if $\mathfrak{M} = \mathfrak{M}^{\text{wt}}$ and \mathcal{V} is a chart centered at a weighted curve whose reduced dual tree is γ , then the set of the modular parameters $\{\zeta_e\}_{e \in \text{Edg}(\gamma)}$ is a γ -labeled subset of local parameters.

Let $\text{Ver}(\gamma)_{\min}$ be the set of the minimal vertices of γ with respect to the tree order. We call a subset \mathfrak{S} of $\text{Edg}(\gamma)$ a **traverse section** if for any $v \in \text{Ver}(\gamma)_{\min}$, the path between o and v contains exactly one element of \mathfrak{S} . For example, the subsets \mathfrak{E}_i of $\text{Edg}(\gamma)$ as in (2.3) are traverse sections. Let $\Xi(\gamma)$ be the set of the traverse sections. The tree order on $\text{Edg}(\gamma)$ induces a partial order on $\Xi(\gamma)$ such that

$$\mathfrak{S} > \mathfrak{S}' \iff (\mathfrak{S} \neq \mathfrak{S}') \text{ and } (\forall e \in \mathfrak{S}, \exists e' \in \mathfrak{S}' \text{ s.t. } e \geq e').$$

We remark that the tree order on $\text{Edg}(\gamma)$ and the induced order on $\Xi(\gamma)$ in this paper are both *opposite* to those in [7], in order to be consistent with the order of the levels of the weighted level trees.

EXAMPLE 4.1. Let γ be the rooted tree in Figure 4. The set $\Xi(\gamma)$ consists of four traverse sections, as listed in Figure 4. The partial order on $\Xi(\gamma)$ is given by $\mathfrak{S}_1 > \mathfrak{S}_2 > \mathfrak{S}_4$ and $\mathfrak{S}_1 > \mathfrak{S}_3 > \mathfrak{S}_4$; the traverse sections \mathfrak{S}_2 and \mathfrak{S}_3 are not comparable.

Let $\widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ be the sequential blowup of \mathfrak{M} successively along the proper transforms of the closed substacks Z_1, Z_2, \dots of \mathfrak{M} .

DEFINITION 4.2 ([7, Definitions 3.2.4 & 3.2.1]). *The blowup $\widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ above is said to be locally tree-compatible if there exists an étale cover $\{\mathcal{V}\}$ of \mathfrak{M} such that for*

each $\mathcal{V} \in \{\mathcal{V}\}$, there exist a rooted tree γ , a partition of $\Xi(\gamma)$:

$$\Xi(\gamma) = \bigsqcup_{k \geq 1} \Xi_k(\gamma)$$

and a γ -labeled subset of local parameters on \mathcal{V} such that

- for every $k \geq 1$,

$$Z_k \cap \mathcal{V} = \bigcup_{\mathfrak{S} \in \Xi_k(\gamma)} \{z_e = 0 : e \in \mathfrak{S}\};$$

- if $\mathfrak{S}' \in \Xi_{k'}(\gamma)$, $\mathfrak{S}'' \in \Xi_{k''}(\gamma)$, and $\mathfrak{S}' > \mathfrak{S}''$, then $k' < k''$.

If a sequential blowup $\widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ is locally tree-compatible, then the blowup procedure is finite on each $\mathcal{V} \in \{\mathcal{V}\}$, because the set $\Xi(\gamma)$ is finite.

LEMMA 4.3. *If the blowup $\widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ successively along the proper transforms of the closed substacks Z_1, Z_2, \dots of \mathfrak{M} is locally tree-compatible, then the blowup $\widetilde{\mathfrak{M}}' \rightarrow \mathfrak{M}$ successively along the total transforms of*

$$Y_1 = Z_1, \quad Y_2 = Z_1 \cup Z_2, \quad Y_3 = Z_1 \cup Z_2 \cup Z_3, \quad \dots$$

yields the same space, i.e. $\widetilde{\mathfrak{M}}' = \widetilde{\mathfrak{M}}$.

Proof. We prove the statement by induction. For each $h \geq 1$, we will show that after the h -th step, the blowup stacks $\widetilde{\mathfrak{M}}'_{(h)}$ of \mathfrak{M} along the total transforms of Y_1, \dots, Y_h is the same as the blowup $\widetilde{\mathfrak{M}}_{(h)}$ of \mathfrak{M} along the proper transforms of Z_1, \dots, Z_h .

The base case of the induction is trivial. Suppose the blowup $\widetilde{\mathfrak{M}}'_{(k)} = \widetilde{\mathfrak{M}}_{(k)}$. We will show that for any $x \in \mathfrak{M}$ and any lift \tilde{x} of x after the k -th step, the blowup along the total transform \check{Y}_{k+1} of Y_{k+1} has the same effect as that along the proper transform \check{Z}_{k+1} of Z_{k+1} near \tilde{x} . Since x and \tilde{x} are arbitrary, this will establish the $(k+1)$ -th step of the induction.

W.l.o.g. we may assume $x \in \bigcap_{i=1}^{k+1} Z_i$ (otherwise we simply omit the loci Z_i not containing x and change the indices of Z_i and Y_i accordingly). The blowup $\widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ is locally tree-compatible, hence there exist a rooted tree γ , an affine smooth chart \mathcal{V} containing x , and a γ -labeled subset of local parameters $z_e, e \in \text{Edg}(\gamma)$ on \mathcal{V} such that

$$x \in \{z_e = 0 : e \in \text{Edg}(\gamma)\}.$$

As shown in [7, Lemma 3.3.2], there exist traverse sections $\mathfrak{S}_{(k)} \in \Xi_k(\gamma)$ and $\mathfrak{S}_{(k+1)} \in \Xi_{k+1}(\gamma)$ (c.f. Definition 4.2), an affine smooth chart $\check{\mathcal{V}}_{\tilde{x}}$, and a subset of local parameters

$$\check{\varepsilon}_1, \dots, \check{\varepsilon}_k; \quad \check{z}_e \text{ with } e \in \mathfrak{S}_{(k+1)} \setminus \mathfrak{S}_{(k)}; \quad \check{z}_e \text{ with } e \in \mathfrak{S}_{(k+1)} \cap \mathfrak{S}_{(k)}$$

on $\check{\mathcal{V}}_{\tilde{x}}$ so that \check{Z}_{k+1} is locally given by

$$\check{Z}_{k+1} \cap \check{\mathcal{V}}_{\tilde{x}} = \{\check{z}_e = 0 : e \in \mathfrak{S}_{(k+1)} \setminus \mathfrak{S}_{(k)}, \check{z}_e = 0 : e \in \mathfrak{S}_{(k+1)} \cap \mathfrak{S}_{(k)}\}.$$

Moreover, by [7, (3.13)], the total transform of each Z_i with $1 \leq i \leq k$ is locally given by $\{\check{\varepsilon}_i = 0\}$. Thus, \check{Y}_{k+1} is locally given by

$$\check{Y}_{k+1} \cap \check{\mathcal{V}}_{\tilde{x}} = (\check{Z}_{k+1} \cap \check{\mathcal{V}}_{\tilde{x}}) \cup \left\{ \prod_{1 \leq i \leq k} \check{\varepsilon}_i = 0 \right\}.$$

That is, on the chart $\tilde{Y}_{\tilde{x}}$, \tilde{Z}_{k+1} and \tilde{Y}_{k+1} are defined by the ideals

$$\mathcal{I}_{\tilde{Z}_{k+1}} = \langle \tilde{z}_e : e \in \mathfrak{S}_{(k+1)} \setminus \mathfrak{S}_{(k)}, \tilde{z}_e : e \in \mathfrak{S}_{(k+1)} \cap \mathfrak{S}_{(k)} \rangle \quad \text{and} \quad \mathcal{I}_{\tilde{Y}_{k+1}} \left(\prod_{1 \leq i \leq k} \tilde{\varepsilon}_i \right),$$

respectively. Therefore, blowing up along \tilde{Z}_{k+1} has the same effect on $\tilde{Y}_{\tilde{x}}$ as that along \tilde{Y}_{k+1} . \square

PROPOSITION 4.4. $\mathfrak{M}^{\text{tf}}/\mathfrak{M}^{\text{wt}}$ is isomorphic to $\tilde{\mathfrak{M}}^{\text{wt}}/\mathfrak{M}^{\text{wt}}$. In particular, $\varpi : \mathfrak{M}^{\text{tf}} \rightarrow \mathfrak{M}^{\text{wt}}$ is proper.

Proof. Our goal is to construct two morphisms ψ_1 and ψ_2 between $\tilde{\mathfrak{M}}^{\text{wt}}$ and \mathfrak{M}^{tf} so that the following diagram

$$\begin{array}{ccc} \tilde{\mathfrak{M}}^{\text{wt}} & \begin{array}{c} \xrightarrow{\psi_2} \\ \xleftarrow{\psi_1} \end{array} & \mathfrak{M}^{\text{tf}} \\ & \begin{array}{c} \searrow \pi \\ \swarrow \varpi \end{array} & \\ & \mathfrak{M}^{\text{wt}} & \end{array}$$

commutes. Since π and ϖ restrict to the identity map on the preimages of the open subset

$$\{(C, \mathbf{w}) \in \mathfrak{M}^{\text{wt}} : \mathbf{w}(C_o) > 0\} \subset \mathfrak{M}^{\text{wt}},$$

respectively, we see that ψ_1 and ψ_2 are the identity maps over the open subsets, respectively, hence $\psi_2 \circ \psi_1$ and $\psi_1 \circ \psi_2$ are the identity maps. This then implies the former statement of Proposition 4.4. The latter statement follows from the former as well as the properness of the blowup $\tilde{\mathfrak{M}}^{\text{wt}} \rightarrow \mathfrak{M}^{\text{wt}}$.

We first construct ψ_1 . For each $k \in \mathbb{Z}_{>0}$, let $Z_k \subset \mathfrak{M}^{\text{wt}}$ be the closed locus whose general point is obtained by attaching k smooth positively-weighted rational curves to the smooth 0-weighted elliptic core at pairwise distinct points. By Lemma 4.3, the blowup $\pi : \tilde{\mathfrak{M}}^{\text{wt}} \rightarrow \mathfrak{M}^{\text{wt}}$ successively along the proper transforms of Z_1, Z_2, \dots can be identified with the blowup of \mathfrak{M}^{wt} successively along the total transforms of

$$Y_1 = Z_1, \quad Y_2 = Z_1 \cup Z_2, \quad Y_3 = Z_1 \cup Z_2 \cup Z_3, \quad \dots$$

We observe that for each $k \in \mathbb{Z}_{>0}$, the pullback $\varpi^{-1}(Y_k)$ to \mathfrak{M}^{tf} is a Cartier divisor. In fact, for any $[t] = [\gamma, \mathbf{w}, \ell] \in \mathcal{T}_{\mathbb{L}}^{\text{wt}}$ and $x \in \mathfrak{M}_{[t]}^{\text{tf}}$, let $\mathfrak{U} \rightarrow \mathfrak{M}^{\text{tf}}$ be a twisted chart centered at x , lying over a chart $\mathcal{V} \rightarrow \mathfrak{M}^{\text{wt}}$. In [5], the blowup π locally on \mathcal{V} is proved to be compatible with the weighted tree $(\tilde{\gamma}, \tilde{\mathbf{w}})$ obtained by contracting all the edges e of γ as long as there exists $v \geq v_e^+$ satisfying $\mathbf{w}(v) > 0$. Let $\{\zeta_e : e \in \text{Edg}(\gamma)\}$ be a set of modular parameters on \mathcal{V} as in (3.1) and

$$\{\varepsilon_i\}_{i \in \mathbb{I}_+} \cup \{u_e\}_{e \in \widehat{\text{Edg}}(t) \setminus \{e_i : i \in \mathbb{I}_+\}} \cup \{z_e\}_{e \in \mathbb{I}_-}$$

be the subset of the parameters (3.9) on \mathfrak{U} . We claim that

$$\varpi^{-1}(Y_k) \cap \mathfrak{U} = \left\{ \prod_{i \in \mathbb{I}_+, |\mathfrak{E}_i| \leq k} \varepsilon_i = 0 \right\}. \tag{4.1}$$

To show (4.1), we first notice that $\varpi^{-1}(Y_k) \cap \mathfrak{U} = \varpi^{-1}(Y_k \cap \mathcal{V})$ by Corollary 3.8. Every irreducible component of $Y_k \cap \mathcal{V}$ can be written in the form

$$Y_{k, \mathfrak{S}} := \{\zeta_e = 0 : e \in \mathfrak{S}\} \quad \text{with} \quad \mathfrak{S} \in \Xi(\tilde{\gamma}), \quad |\mathfrak{S}| \leq k, \quad \mathfrak{S} \cap (\widehat{\text{Edg}}(t) \setminus \mathbb{I}_{\mathbf{m}}) \neq \emptyset.$$

For each irreducible component $Y_{k,\mathfrak{S}}$ of $Y_k \cap \mathcal{V}$, the local expression θ_x of ϖ as in (3.12) implies the pullback $\varpi^{-1}(Y_{k,\mathfrak{S}})$ can be written as

$$\left\{ \begin{aligned} \prod_{h \in \llbracket \ell(e), \ell(v_e^+) \rrbracket} \varepsilon_h = 0 : e \in \mathfrak{S} \cap (\widehat{\text{Edg}}(\mathfrak{t}) \setminus \mathbb{I}_{\mathbf{m}}), \\ (u_e \cdot \prod_{h \in \llbracket \ell(e), \ell(v_e^+) \rrbracket} \varepsilon_h) = 0 : e \in \mathfrak{S} \cap \mathbb{I}_{\mathbf{m}}, \quad z_e = 0 : e \in \mathfrak{S} \cap \mathbb{I}_{-} \end{aligned} \right\}. \tag{4.2}$$

Since $\mathfrak{S} \cap (\widehat{\text{Edg}}(\mathfrak{t}) \setminus \mathbb{I}_{\mathbf{m}}) \neq \emptyset$ and $|\mathfrak{S}| \leq k$, we can always find $e \in \mathfrak{S} \cap (\widehat{\text{Edg}}(\mathfrak{t}) \setminus \mathbb{I}_{\mathbf{m}})$ such that $|\mathfrak{E}_h| \leq |\mathfrak{S}| \leq k$ for all $h \in \llbracket \ell(e), \ell(v_e^+) \rrbracket$. By (4.1) and (4.2), the pullback $\varpi^{-1}(Y_{k,\mathfrak{S}})$ is thus a sub-locus of the right-hand side of (4.1). Moreover, it is a direct check that

$$\varpi^{-1}(Y_{k,\mathfrak{E}_i}) \cap \mathfrak{U} = \{\varepsilon_i = 0\} \quad \forall i \in \mathbb{I}_+ \text{ with } |\mathfrak{E}_i| \leq k.$$

Therefore, (4.1) holds.

Since every $\varpi^{-1}(Y_k)$ is a Cartier divisor of \mathfrak{M}^{tf} , by the universality of the blowup $\pi: \widetilde{\mathfrak{M}}^{\text{wt}} \rightarrow \mathfrak{M}^{\text{wt}}$, we obtain a unique morphism

$$\psi_1 : \mathfrak{M}^{\text{tf}} \rightarrow \widetilde{\mathfrak{M}}^{\text{wt}}$$

that $\varpi: \mathfrak{M}^{\text{tf}} \rightarrow \mathfrak{M}^{\text{wt}}$ factors through.

We next construct ψ_2 explicitly. For any $\tilde{x} \in \widetilde{\mathfrak{M}}^{\text{wt}}$, let (C, \mathbf{w}) be its image in \mathfrak{M}^{wt} . As shown in [7, §3.3], there exists a unique maximal sequence of exceptional divisors

$$\tilde{E}_{i_1}, \dots, \tilde{E}_{i_k} \subset \widetilde{\mathfrak{M}}^{\text{wt}}, \quad 1 \leq i_1 < \dots < i_k$$

containing \tilde{x} . Each \tilde{E}_{i_j} is obtained from blowing up along the proper transform of Z_{i_j} . Note that k is possibly 0, which means (C, \mathbf{w}) is not in the blowup loci. The weighted dual tree $\tau = (\gamma_C, \mathbf{w})$, along with the exceptional divisors $\tilde{E}_{i_1}, \dots, \tilde{E}_{i_k}$, uniquely determines a weighted level tree $\mathfrak{t}_{\tilde{x}}$ such that

$$\mathbb{I}_+ = \mathbb{I}_+(\mathfrak{t}_{\tilde{x}}) = \{-i_k, \dots, -i_1\}.$$

In particular, $\mathbf{m} = \mathbf{m}(\mathfrak{t}_{\tilde{x}}) = -i_k$.

With the line bundles $L_e, e \in \text{Edg}(\mathfrak{t}_{\tilde{x}})$, as in (2.11), the notation $(\cdot, \cdot)_{\mathfrak{t}_{\tilde{x}}}$ and $\llbracket \cdot, \cdot \rrbracket_{\mathfrak{t}_{\tilde{x}}}$ as in (2.2), and the notation $i[h]$ as in (3.6), the line bundles

$$\mathfrak{L}_i = L_{e_i} \otimes \bigotimes_{j \in \langle i, i[1] \rangle_{\mathfrak{t}_{\tilde{x}}}} \mathfrak{L}_j^\vee \rightarrow \mathfrak{M}_\tau^{\text{wt}}, \quad i \in \mathbb{I}_+, \tag{4.3}$$

can be constructed inductively over \mathbb{I}_+ . Then, we take

$$\mathfrak{L}_e = L_e \otimes \bigotimes_{j \in \langle \ell(e), \ell(v_e^+) \rangle_{\mathfrak{t}_{\tilde{x}}}} \mathfrak{L}_j^\vee \rightarrow \mathfrak{M}_\tau^{\text{wt}}, \quad e \in \widehat{\text{Edg}}(\mathfrak{t}_{\tilde{x}}). \tag{4.4}$$

In particular, $\mathfrak{L}_{e_i} = \mathfrak{L}_i$. For each $e \in \text{Edg}(\mathfrak{t}_{\tilde{x}})$, (4.3) and (4.4) imply

$$\mathfrak{L}_e \otimes \bigotimes_{e' > e} (\mathfrak{L}_{e'} \otimes \mathfrak{L}_{\ell(e')}^\vee) = L_e^{\geq} \otimes \left(\bigotimes_{j \in \langle \ell(e), 0 \rangle_{\mathfrak{t}_{\tilde{x}}}} \mathfrak{L}_j^\vee \right).$$

Hence for each $i \in \mathbb{I}_+$,

$$\mathring{\mathbb{P}} \left(\bigoplus_{\ell(v_e^-) = i} (\mathfrak{L}_e \otimes \bigotimes_{e' > e} (\mathfrak{L}_{e'} \otimes \mathfrak{L}_{\ell(e')}^\vee)) \right) = \mathring{\mathbb{P}} \left(\bigoplus_{\ell(v_e^-) = i} L_e^{\geq} \right). \tag{4.5}$$

For $h \geq 1$, let $\tilde{x}_{(h)}$ be the image of \tilde{x} in the exceptional divisor of the h -th step. Given $i \in \mathbb{I}_+$, The proper transform of Z_{-i} after the first $-i-1$ steps of the blowup may have several connected components; see [7, Lemma 3.3.2]. The normal bundle of the component containing $\tilde{x}_{(-i-1)}$ is the pullback $\pi_{(-i-1)}^* \bigoplus_{e \in \mathfrak{E}_i} \mathfrak{L}_e$, where $\pi_{(h)} : \widetilde{\mathfrak{M}}_{(h)}^{\text{wt}} \rightarrow \mathfrak{M}^{\text{wt}}$ is the blowup after the h -th step.

Notice that the non-zero entries of $\tilde{x}_{(-i)}$ exactly correspond to the edges $e \in \mathfrak{E}_i$ satisfying $\ell(v_e^-) = i$. Therefore,

$$\tilde{x}_{(-i)} \in \pi_{(-i-1)}^* \mathring{\mathbb{P}} \left(\bigoplus_{\ell(v_e^-) = i} \mathfrak{L}_e \right).$$

Then, $\tilde{x}_{(-j)}$ with $j \in \llbracket i, 0 \rrbracket_{\mathfrak{t}_{\tilde{x}}}$ together determine a unique

$$\eta_i(\tilde{x}) \in \mathring{\mathbb{P}} \left(\bigoplus_{\ell(v_e^-) = i} (\mathfrak{L}_e \otimes \bigotimes_{e' > e} (\mathfrak{L}_{e'} \otimes \mathfrak{L}_{\ell(e')}^\vee)) \right) = \mathring{\mathbb{P}} \left(\bigoplus_{\ell(v_e^-) = i} L_e^{\geq} \right).$$

The last equality above follows from (4.5). We then set

$$\psi_2(\tilde{x}) = \left((C, \mathbf{w}), [\mathfrak{t}_{\tilde{x}}], (\eta_i(\tilde{x}) : i \in \mathbb{I}_+(\mathfrak{t}_{\tilde{x}})) \right) \in \mathfrak{M}_{[\mathfrak{t}_{\tilde{x}}]}^{\text{tf}}.$$

Obviously, this implies $\varpi \circ \psi_2 = \pi$.

It remains to verify such defined ψ_2 is a morphism. Let $\mathcal{V} \rightarrow \mathfrak{M}^{\text{wt}}$ be a smooth chart containing (C, \mathbf{w}) , and $\{\zeta_e : e \in \text{Edg}(\mathfrak{t}_{\tilde{x}})\} \sqcup \{\zeta_j : j \in J\}$ be a set of local parameters centered at (C, \mathbf{w}) as in (3.3). As shown in [7, §3.1&§3.3], there exists a chart $\tilde{\mathcal{V}}_{\tilde{x}} \rightarrow \widetilde{\mathfrak{M}}^{\text{wt}}$ containing \tilde{x} with local parameters

$$\begin{aligned} \tilde{\varepsilon}_i, i \in \mathbb{I}_+; \quad \rho_e, e \in \widehat{\text{Edg}}(\mathfrak{t}_{\tilde{x}}) \setminus (\mathbb{I}_{\mathbf{m}} \sqcup \{\mathbf{e}_i : i \in \mathbb{I}_+\}); \\ \tilde{z}_e, e \in \mathbb{I}_{\mathbf{m}}; \quad \tilde{z}_e, e \in \mathbb{I}_-; \quad s_j, j \in J. \end{aligned}$$

All ρ_e are nowhere vanishing on $\tilde{\mathcal{V}}_{\tilde{x}}$. Moreover, with $\pi : \tilde{\mathcal{V}}_{\tilde{x}} \rightarrow \mathcal{V}$ denoting the blowup, we have

$$\begin{aligned} \pi^* \zeta_{\mathbf{e}_i} &= \prod_{h \in \llbracket i, 1 \rrbracket_{\mathfrak{t}_{\tilde{x}}}} \tilde{\varepsilon}_h \quad \forall i \in \mathbb{I}_+; \quad \pi^* \zeta_e = \rho_e \prod_{i \in \llbracket \ell(e), \ell(v_e^+) \rrbracket_{\mathfrak{t}_{\tilde{x}}}} \tilde{\varepsilon}_i \quad \forall e \in \widehat{\text{Edg}}(\mathfrak{t}_{\tilde{x}}) \setminus (\mathbb{I}_{\mathbf{m}} \sqcup \{\mathbf{e}_i : i \in \mathbb{I}_+\}); \\ \pi^* \zeta_e &= \tilde{z}_e \prod_{i \in \llbracket \ell(e), \ell(v_e^+) \rrbracket_{\mathfrak{t}_{\tilde{x}}}} \tilde{\varepsilon}_i \quad \forall e \in \mathbb{I}_{\mathbf{m}}; \quad \pi^* \zeta_e = \tilde{z}_e \quad \forall e \in \mathbb{I}_-; \quad \pi^* \zeta_j = s_j \quad \forall j \in J. \end{aligned}$$

For $e \in \{\mathbf{e}_i : i \in \mathbb{I}_+\}$, we set $\rho_e = 1$. Then,

$$\rho_e \in \Gamma(\mathcal{O}_{\tilde{\mathcal{V}}_{\tilde{x}}}^*) \quad \forall e \in \widehat{\text{Edg}}(\mathfrak{t}_{\tilde{x}}) \setminus \mathbb{I}_{\mathbf{m}}.$$

Let $\mathfrak{U}_{\psi_2(\tilde{x})} \rightarrow \mathfrak{M}^{\text{tf}}$ be a twisted chart centered at $\psi_2(\tilde{x})$, lying over $\mathcal{V} \rightarrow \mathfrak{M}^{\text{wt}}$. The parameters on $\mathfrak{U}_{\psi_2(\tilde{x})}$ are as in (3.3). It is a direct check that the point-wise defined ψ_2 can locally be written as

$$\psi_2 : \tilde{\mathcal{V}}_{\tilde{x}} \rightarrow \mathfrak{U}_{\psi_2(\tilde{x})}$$

such that

$$\begin{aligned} \psi_2^* \varepsilon_i &= \tilde{\varepsilon}_i \quad \forall i \in \mathbb{I}_+; \quad \psi_2^* u_e = \frac{\prod_{\mathbf{e} \geq e} \rho_{\mathbf{e}}}{\prod_{\mathbf{e} \geq \mathbf{e}_{\ell(e)}} \rho_{\mathbf{e}}} \quad \forall e \in \widehat{\text{Edg}}(\mathfrak{t}) \setminus (\mathbb{I}_{\mathbf{m}} \sqcup \{\mathbf{e}_i : i \in \mathbb{I}_+\}); \\ \psi_2^* u_e &= \tilde{z}_e \cdot \frac{\prod_{\mathbf{e} > e} \rho_{\mathbf{e}}}{\prod_{\mathbf{e} \geq \mathbf{e}_{\ell(e)}} \rho_{\mathbf{e}}} \quad \forall e \in \mathbb{I}_{\mathbf{m}}; \quad \psi_2^* z_e = \tilde{z}_e \quad \forall e \in \mathbb{I}_-; \quad \psi_2^* w_j = s_j \quad \forall j \in J. \end{aligned}$$

This shows $\psi_2: \widetilde{\mathfrak{M}}^{\text{wt}} \rightarrow \mathfrak{M}^{\text{tf}}$ is a morphism. \square

REMARK 4.5. In [6], another resolution $\widetilde{\mathfrak{M}}^{\text{dr}} \rightarrow \mathfrak{M}^{\text{wt}}$, called the derived resolution of \mathfrak{M}^{wt} , is constructed for the purpose of diagonalizing certain direct image sheaves. That resolution is “smaller” in the sense that the resolution $\widetilde{\mathfrak{M}}^{\text{wt}} \rightarrow \mathfrak{M}^{\text{wt}}$ of [5] factors through $\widetilde{\mathfrak{M}}^{\text{dr}} \rightarrow \mathfrak{M}^{\text{wt}}$. Mimicking the approach of §3, we may construct a moduli stack

$$\mathfrak{N} = \bigsqcup_{[t] \in \mathcal{T}_L^{\text{wt}}} \mathfrak{N}_{[t]}, \quad \mathfrak{N}_{[t]} = \mathbb{P} \left(\bigoplus_{e \in \text{Edg}(t), \ell(v_e^-) = \mathbf{m}(t)} L_e^{\geq} \right) \rightarrow \mathfrak{M}_{j[t]}^{\text{wt}}.$$

This moduli should be isomorphic to $\widetilde{\mathfrak{M}}^{\text{dr}}$.

REFERENCES

- [1] M. BAINBRIDGE, D. CHEN, Q. GENDRON, S. GRUSHEVSKY, AND M. MÖLLER, *Compactification of strata of Abelian differentials*, *Duke Math. J.*, 167:12 (2018), pp. 2347–2416.
- [2] J. HARRIS AND I. MORRISON, *Moduli of curves*, *Graduate Texts in Math.* vol. 187, Springer-Verlag, New York, 1998.
- [3] H. HIRONAKA, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I*, *Ann. of Math.* 2, 79:1 (1964), pp. 109–203.
- [4] H. HIRONAKA, *Resolution of singularities of an algebraic variety over a field of characteristic zero. II*, *Ann. of Math.* 2, 79:1 (1964), pp. 205–326.
- [5] Y. HU AND J. LI, *Genus-one stable maps, local equations and Vakil-Zinger’s desingularization*, *Math. Ann.*, 348:4 (2010), pp. 929–963.
- [6] Y. HU AND J. LI, *Derived resolution property for stacks, Euler classes and applications*, *Math. Res. Lett.*, 18:4 (2011), pp. 677–690.
- [7] Y. HU, J. LI, AND J. NIU, *Genus two stable maps, local equations and modular resolutions*, *math/1201.2427v3*.
- [8] Y. HU AND J. NIU, *A theory of stacks with twisted fields and resolution of moduli of genus two stable maps*, *math/2005.03384*.
- [9] A. DE JONG, *Smoothness, semi-stability and alterations*, *Publ. Math. IHES*, 83 (1996), pp. 51–93.
- [10] J. KOLLÁR, *Lectures on resolution of singularities*, *Annals of Mathematics Studies* 166, (2007).
- [11] D. RANGANATHAN, K. SANTOS-PARKER, AND J. WISE, *Moduli of stable maps in genus one \mathcal{E} logarithmic geometry I*, *Geometry & Topology*, 23 (2019), pp. 3315–3366.
- [12] R. VAKIL, *Murphy’s law in algebraic geometry: Badly-behaved deformation spaces*, *Invent. Math.*, 164:3 (2006), pp. 569–590.
- [13] R. VAKIL AND A. ZINGER, *A desingularization of the main component of the moduli space of genus-one stable maps into \mathbb{P}^n* , *Geom. Topol.*, 12:1 (2008), pp. 1–95.

