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# EXPLICIT DESCRIPTION OF GENERALIZED WEIGHT MODULES OF THE ALGEBRA OF POLYNOMIAL INTEGRO-DIFFERENTIAL OPERATORS $\mathbb{I}_n$*

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**Abstract.** For the algebra $\mathbb{I}_n = K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, \int_1, \ldots, \int_n\rangle$ of polynomial integro-differential operators over a field $K$ of characteristic zero, a classification of simple weight and generalized weight (left and right) $\mathbb{I}_n$-modules is given. It is proven that the category of weight $\mathbb{I}_n$-modules is semisimple. An explicit description of generalized weight $\mathbb{I}_n$-modules is given and using it a criterion is obtained for the problem of classification of indecomposable generalized weight $\mathbb{I}_n$-modules to be of finite representation type, tame or wild. In the tame case, a classification of indecomposable generalized weight $\mathbb{I}_n$-modules is given. In the wild case 'natural' tame subcategories are considered with explicit description of indecomposable modules. For an arbitrary ring $R$, we introduce the concept of *absolutely prime* $R$-module (a nonzero $R$-module $M$ is absolutely prime if all nonzero subfactors of $M$ have the same annihilator). It is proven that every generalized weight $\mathbb{I}_n$-module is a unique sum of absolutely prime modules. It is also shown that every indecomposable generalized weight $\mathbb{I}_n$-module is equidimensional. A criterion is given for a generalized weight $\mathbb{I}_n$-module to be finitely generated.

**Key words.** the algebra of polynomial integro-differential operators, weight and generalized weight modules, indecomposable module, simple module, finite representation type, tame and wild.

**Mathematics Subject Classification.** 16D60, 16D70, 16P50, 16U20.

**1. Introduction.** Throughout, ring means an associative ring with 1; module means a left module; $\mathbb{N} := \{0, 1, \ldots\}$ is the set of natural numbers; $\mathbb{N}_+ := \{1, 2, \ldots\}$ and $\mathbb{Z}_{\leq 0} := -\mathbb{N}$; $K$ is a field of characteristic zero and $K^*$ is its group of units; $\otimes = \otimes_K$; $P_n := K[x_1, \ldots, x_n]$ is a polynomial algebra over $K$; $\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n}$ are the partial derivatives ($K$-linear derivations) of $P_n$; $\mathrm{End}_K(P_n)$ is the algebra of all $K$-linear maps from $P_n$ to $P_n$; the subalgebras

$$A_n := K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle \text{ and } \mathbb{I}_n := K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, \int_1, \ldots, \int_n\rangle$$

of the algebra $\mathrm{End}_K(P_n)$ are called the $n$'th *Weyl algebra* and the *algebra of polynomial integro-differential operators*, respectively.

The Weyl algebras $A_n$ are Noetherian algebras and domains. The algebras $\mathbb{I}_n$ were introduced in [7, 8] and they are neither left nor right Noetherian and not domains. Moreover, they contain infinite direct sums of nonzero left and right ideals [7]. The algebra $\mathbb{I}_n$ contains a polynomial algebra

$$D_n = K[H_1, \ldots, H_n], \text{ where } H_1 := \partial_1 x_1, \ldots, H_n := \partial_n x_n,$$

which is a maximal commutative subalgebra of $\mathbb{I}_n$, [7]. An $\mathbb{I}_n$-module $M$ is called a *weight module* if it is a *semisimple* $D_n$-module provided the field $K$ is an algebraically

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closed field. For an arbitrary field, a weight $\mathbb{I}_n$-module is a direct sum of common eigenspaces for the commuting elements $H_1, \dots, H_n$, by definition. An $\mathbb{I}_n$-module $M$ is called a *generalized weight module* if for all elements $m \in M$, $\dim_K(D_n m) < \infty$, i.e., $M$ is a *locally finite dimensional* $D_n$-module provided the field $K$ is an algebraically closed field. Every weight module is a generalized weight module but not vice versa. The $\mathbb{I}_n$-module $P_n$ is a simple weight module. Introduction of (generalized) weight modules for the algebras $\mathbb{I}_n$ was inspired by a similar concept for semisimple finite dimensional Lie algebras. For a semisimple finite dimensional Lie algebra $\mathcal{G}$, a classification of *simple* generalized weight modules is known only when $\mathcal{G} = \mathrm{sl}_2$. Furthermore, a classification of *indecomposable* generalized weight $\mathrm{sl}_2$-modules was done in [5] and the problem of classification turned out to be tame provided the Casimir element acts as a scalar. In fact, in [5] in the tame case a classification of indecomposable generalized weight modules was obtained for a large class of algebras, the, so-called, *generalized Weyl algebras* (the universal enveloping algebra $U(\mathrm{sl}_2)$ and the Weyl algebras are examples of generalized Weyl algebras as well as many other quantum groups are). Recently, for some non-semisimple Lie algebras $\mathcal{G}$ and their quantum analogues, classifications of simple weight modules are given: the Schrödinger algebra, [15]; $\mathrm{sl}_2 \ltimes V_2$, [16], where $V_2$ is the simple 2-dimensional $\mathrm{sl}_2$-module; the enveloping algebra of the Euclidean algebra, [14]; $\mathbb{K}_q[X,Y] \rtimes U_q(\mathrm{sl}_2)$, [17]; the quantum spatial ageing algebra, [13].

In the paper, explicit descriptions of weight and generalized weight $\mathbb{I}_n$-modules are given (Theorem 3.2, Theorem 3.6 and (9), (19)). They are too technical to explain them in the Introduction. Classifications of simple weight and simple generalized weight $\mathbb{I}_n$-modules are obtained (Theorem 2.3). It is proven that the category of weight $\mathbb{I}_n$-modules is a semisimple category (Theorem 2.5). In [6], a classification of indecomposable generalized weight $\mathbb{I}_1$-modules of finite length is given. This classification is used in the present paper in order to obtain the general case. Usually, the case $n = 1$ serves as the base of induction for the case $n > 1$. Using Theorem 2.3, a criterion is given to decide whether the problem of classification of indecomposable generalized weight $\mathbb{I}_n$-modules is of finite representation type, tame or wild (Theorem 3.9). In the case $n = 1$, the problem is tame, [6]. In the tame case, a classification of indecomposable generalized weight $\mathbb{I}_n$-modules is given. It is shown that every indecomposable generalized weight $\mathbb{I}_n$-module is equidimensional (Corollary 3.7). A criterion is given for a generalized weight $\mathbb{I}_n$-module to be finitely generated (Corollary 3.8). In the wild case, 'natural' tame subcategories are considered with explicit description of indecomposable modules, see the end of Section 2. In particular, descriptions of categories $\mathrm{ind}(D_2, \mathfrak{m}^2)$, $\mathrm{ind}_f(\Gamma)$ and $\mathrm{ind}_f(A)$ are obtained. In Section 4, similar results are proven for generalized weight *right* $\mathbb{I}_n$-modules.

Properties of the algebras $\mathbb{I}_n$ are studied in [7, 9, 10]. In the case $n = 1$, for a more general setting see also [25]. The simple $\mathbb{I}_1$-modules are classified in [9]. Simple $A_1$-modules were classified in [19] (see also [2, 3] for some generalized Weyl algebras including $A_1$). The automorphism groups $\mathrm{Aut}_{K-\mathrm{alg}}(\mathbb{I}_n)$ are found in [8]. The weak homological dimension of the algebra $\mathbb{I}_n$ is $n$, [7]. Futhermore, the weak homological dimension is $n$ for all the factor algebras of $\mathbb{I}_n$, [12].

Finite dimensionality of Ext-groups of simple modules over the (first) Weyl algebra $A_1$ was proven in [30]. Finite dimensionality of Ext-groups of simple modules over the generalized Weyl algebras was proven in [1]. Simple modules over certain generalized Weyl algebras were classified in [2]. In [6], Ext-groups are described between indecomposable generalized weight $\mathbb{I}_1$-modules, it is shown that they are finite dimensional vector spaces. In [10], it is proven that the algebra $\mathbb{I}_n$ is a left coherent algebra iff the algebra $\mathbb{I}_n$ is a right coherent iff $n = 1$; the algebra $\mathbb{I}_n$ is a maximal left (resp., right) order in the largest left (resp., right) quotient ring $Q_l(\mathbb{I}_n)$ (resp., $Q_r(\mathbb{I}_n)$) of $\mathbb{I}_n$. The (left and right) global dimension of the algebra $\mathbb{I}_n$ and all prime factor algebras of $\mathbb{I}_n$ is equal to $n$, [12].

Classifications of (various classes of) simple weight modules over algebras that are close to the (generalized) Weyl algebras are given in [28, 18, 26, 32, 27, 22, 29, 23, 11].

**2. Classification of simple (generalized) weight $\mathbb{I}_n$-modules.** In this section, a classification of simple generalized weight and simple weight $\mathbb{I}_n$-modules is given (Theorem 2.3). It is proven that the category of weight $\mathbb{I}_n$-modules is a semisimple category (Theorem 2.5). At the beginning of the section, we collect some results about the algebras $\mathbb{I}_n$ that are used in the paper. In the case when $n = 1$, we drop the subscript '1' in order to simplify the notation.

As an abstract algebra, the algebra $\mathbb{I}_1$ is generated by the elements $\partial$, $H := \partial x$ and $\int$ (since $x = \int H$) that satisfy the defining relations, [7, Proposition 2.2] (where $[a, b] := ab - ba$):

$$\partial \int = 1, \;\; [H, \int] = \int, \;\; [H, \partial] = -\partial, \;\; H(1 - \int \partial) = (1 - \int \partial)H = 1 - \int \partial. \quad (1)$$

Since $\mathbb{I}_n = \mathbb{I}_1 \otimes \cdots \otimes \mathbb{I}_1$ ($n$ times), the defining relations of the algebra $\mathbb{I}_n$ is the union of the defining relations (1) for each index $i = 1, \ldots, n$ and the relations $a_i a_j = a_j a_i$ for all $i \neq j$ where $a_i \in \{\partial_i, H_i, \int_i\}$. The elements of the algebra $\mathbb{I}_1$,

$$e_{ij} := \int^i \partial^j - \int^{i+1} \partial^{j+1}, \;\; i, j \in \mathbb{N}, \quad (2)$$

satisfy the relations $e_{ij} e_{kl} = \delta_{jk} e_{il}$ where $\delta_{jk}$ is the Kronecker delta function. Notice that $e_{ij} = \int^i e_{00} \partial^j$. The matrices of the linear maps $e_{ij} \in \mathrm{End}_K(K[x])$ with respect to the basis $\{x^{[s]} := \frac{x^s}{s!}\}_{s \in \mathbb{N}}$ of the polynomial algebra $K[x]$ are the elementary matrices, i.e. $e_{ij}$ acts on polynomials as follows:

$$e_{ij} * x^{[s]} = \begin{cases} x^{[i]} & \text{if } j = s, \\ 0 & \text{if } j \neq s. \end{cases}$$

Let $E_{ij} \in \mathrm{End}_K(K[x])$ be the usual matrix units, i.e. $E_{ij} * x^s = \delta_{js} x^i$ for all $i, j, s \in \mathbb{N}$. Then

$$e_{ij} = \frac{j!}{i!} E_{ij}, \quad (3)$$

$K e_{ij} = K E_{ij}$, and $F := \bigoplus_{i,j \geq 0} K e_{ij} = \bigoplus_{i,j \geq 0} K E_{ij} \simeq M_\infty(K)$, the algebra (without 1) of infinite dimensional matrices. The algebra $\mathbb{I}_n = \mathbb{I}_1(1) \otimes \cdots \otimes \mathbb{I}_1(n) \simeq \mathbb{I}_1^{\otimes n}$ where $\mathbb{I}_1(i) = K\langle \partial_i, H_i, \int_i \rangle$ for $i = 1, \ldots, n$.

**$\mathbb{Z}^n$-grading on the algebra $\mathbb{I}_n$ and the canonical form of an integro-differential operator, [7].** The algebra $\mathbb{I}_1 = \bigoplus_{i \in \mathbb{Z}} \mathbb{I}_{1,i}$ is a $\mathbb{Z}$-graded algebra ($\mathbb{I}_{1,i} \mathbb{I}_{1,j} \subseteq \mathbb{I}_{1,i+j}$ for all $i, j \in \mathbb{Z}$) where

$$\mathbb{I}_{1,i} = \begin{cases} D_1' \int^i = \int^i D_1' & \text{if } i > 0, \\ D_1' & \text{if } i = 0, \\ \partial^{|i|} D_1' = D_1' \partial^{|i|} & \text{if } i < 0, \end{cases} \quad (4)$$

the algebra $D_1' := K[H] \bigoplus \bigoplus_{i\in\mathbb{N}} Ke_{ii}$ is a commutative non-Noetherian subalgebra of $\mathbb{I}_1$, $He_{ii} = e_{ii}H = (i+1)e_{ii}$ for $i \in \mathbb{N}$ (notice that $\bigoplus_{i\in\mathbb{N}} Ke_{ii}$ is the direct sum of non-zero ideals of $D_1'$);

$$\left(\int^i D_1'\right)_{D_1'} \simeq D_1', \ \int^i d \mapsto d; \ {}_{D_1'}\left(D_1'\partial^i\right) \simeq D_1',$$
$$d\partial^i \mapsto d \ \text{ for all } \ i \geq 0 \ \text{ and all } \ d \in D_1'$$

since $\partial^i \int^i = 1$ (where the notation ${}_RM$ (resp., $M_R$) means that $M$ is a left (resp., right) module over a ring $R$). Notice that the maps

$$\cdot \int^i : D_1' \to D_1' \int^i, \ \ d \mapsto d\int^i \ \ \text{and} \ \ \partial^i \cdot : D_1' \to \partial^i D_1', \ \ d \mapsto \partial^i d$$

have the same kernel $\bigoplus_{j=0}^{i-1} Ke_{jj}$. The algebra

$$\mathbb{I}_n = \bigoplus_{\alpha\in\mathbb{Z}^n} \mathbb{I}_{n,\alpha}$$

is a $\mathbb{Z}^n$-graded algebra ($\mathbb{I}_{n,\alpha}\mathbb{I}_{n,\beta} \subseteq \mathbb{I}_{n,\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}^n$) where $\mathbb{I}_{n,\alpha} = \otimes_{i=1}^n \mathbb{I}_1(i)_{\alpha_i}$ and $\alpha = (\alpha_1, \dots, \alpha_n)$.

Each element $a$ of the algebra $\mathbb{I}_1$ is a unique finite sum

$$a = \sum_{i>0} a_{-i}\partial^i + a_0 + \sum_{i>0} \int^i a_i + \sum_{i,j\in\mathbb{N}} \lambda_{ij} e_{ij} \tag{5}$$

where $a_k \in K[H]$ and $\lambda_{ij} \in K$. This is the *canonical form* of the polynomial integro-differential operator, [7]. Let

$$v_i := \begin{cases} \int^i & \text{if } i > 0, \\ 1 & \text{if } i = 0, \\ \partial^{|i|} & \text{if } i < 0. \end{cases}$$

Then $\mathbb{I}_{1,i} = D_1' v_i = v_i D_1'$ and an element $a \in \mathbb{I}_1$ is the unique finite sum

$$a = \sum_{i\in\mathbb{Z}} b_i v_i + \sum_{i,j\in\mathbb{N}} \lambda_{ij} e_{ij} \tag{6}$$

where $b_i \in K[H]$ and $\lambda_{ij} \in K$. So, the set

$$\{H^j\partial^i, H^j, \int^i H^j, e_{st} \,|\, i \geq 1; j, s, t \geq 0\}$$

is a $K$-basis for the algebra $\mathbb{I}_1$. The tensor product of these bases is a basis for the algebra $\mathbb{I}_n$. The multiplication in the algebra $\mathbb{I}_1$ is given by the rule:

$$\int H = (H-1)\int, \ \ H\partial = \partial(H-1), \ \ \int e_{ij} = e_{i+1,j},$$

$$e_{ij}\int = e_{i,j-1}, \ \ \partial e_{ij} = e_{i-1,j}, \ \ e_{ij}\partial = \partial e_{i,j+1},$$

$$He_{ii} = e_{ii}H = (i+1)e_{ii}, \ \ i \in \mathbb{N},$$

where $e_{-1,j} := 0$ and $e_{i,-1} := 0$. The algebra $\mathbb{I}_1$ has the only proper ideal

$$F = \bigoplus_{i,j \in \mathbb{N}} Ke_{ij} \simeq M_\infty(K), \ \text{ and } \ F^2 = F.$$

The factor algebra $\mathbb{I}_1/F$ is canonically isomorphic to the *skew Laurent polynomial algebra*

$$B_1 := K[H][\partial, \partial^{-1}; \sigma^{-1}], \ \ \sigma(H) = H - 1, \ \text{ via } \ \partial \mapsto \partial, \ \ \int \mapsto \partial^{-1}, \ \ H \mapsto H$$

(where $\partial^{\pm 1}\alpha = \sigma^{\mp 1}(\alpha)\partial^{\pm 1}$ for all elements $\alpha \in K[H]$). The algebra $B_1$ is canonically isomorphic to the (left and right) localization $A_{1,\partial}$ of the Weyl algebra $A_1$ at the powers of the element $\partial$ (notice that $x = \partial^{-1}H$).

Recall that the algebra of polynomial integro-differential operators over a field $K$ of characteristic zero,

$$\mathbb{I}_n = K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, \int_1, \ldots, \int_n \rangle = \mathbb{I}_1(1) \otimes \cdots \otimes \mathbb{I}_1(n),$$

is the tensor product of algebras $\mathbb{I}_1(i) = K\langle x_i, \partial_i, \int_i \rangle \simeq \mathbb{I}_1$. Each algebra $\mathbb{I}_1(i)$ contains a unique proper ideal

$$F(i) = \bigoplus_{s,t \in \mathbb{N}} Ke_{st}(i) \ \text{ where } \ e_{st}(i) = \int_i^s \partial_i^t - \int_i^{s+1} \partial_i^{t+1},$$

$F(i)^2 = F(i)$ and

$$B_1(i) := \mathbb{I}(i)/F(i) \simeq K[H_i][\partial_i, \partial_i^{-1}; \sigma_i^{-1}] \ \text{ where } \ \sigma_i(H_i) = H_i - 1.$$

The algebra $\mathbb{I}_n$ is a local algebra where the unique maximal ideal $\mathfrak{a}_n$ is generated by $F(1), \ldots, F(n)$ and the factor algebra $\mathbb{I}_n/\mathfrak{a}_n$ is isomorphic to the skew Laurent polynomial algebra

$$B_n = D_n[\partial_1^{\pm 1}, \ldots, \partial_n^{\pm 1}; \sigma_1^{-1}, \ldots, \sigma_n^{-1}] \ \text{ where } \ \sigma_i(H_j) = H_j - \delta_{ij},$$

see [7]. Furthermore, the algebra $B_n$ is the only left/right Noetherian factor algebra of $\mathbb{I}_n$, [7].

A classification of all ideals (including prime ideals) of the algebra $\mathbb{I}_n$ is obtained in [7]. There are precisely $n$ height 1 prime ideals:

$$\mathfrak{p}_1 = F \otimes \mathbb{I}_{n-1}, \ \ \mathfrak{p}_2 = \mathbb{I}_1 \otimes F \otimes \mathbb{I}_{n-2}, \ldots, \mathfrak{p}_n = \mathbb{I}_{n-1} \otimes F,$$

see [7]. The algebra $\mathbb{I}_n$ is a prime algebra (0 is a prime ideal of $\mathbb{I}_n$). Every nonzero prime ideal $\mathfrak{p}$ is a unique sum

$$\mathfrak{p}_I = \sum_{i \in I} \mathfrak{p}_i$$

of height 1 prime ideals where $I \subseteq \{1, \ldots, n\}$ is a unique set for the ideal $\mathfrak{p} = \mathfrak{p}_I$, and $\mathrm{ht}(\mathfrak{p}) = |I|$ where $\mathrm{ht}(\mathfrak{p})$ if the height of the ideal $\mathfrak{p}$. Every ideal of $\mathbb{I}_n$ is an idempotent ideal ($\mathfrak{a}^2 = \mathfrak{a}$), ideals of $\mathbb{I}_n$ commute ($\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}$) and the ideal $\mathfrak{a}_n = \mathfrak{p}_1 + \cdots + \mathfrak{p}_n$ is the only maximal ideal of the algebra $\mathbb{I}_n$.

**Generalized weight $\mathbb{I}_n$-modules.** The group $\mathbb{Z}^n$ acts on the vector space $K^n$ by addition. For an element $\lambda = (\lambda_1, \dots, \lambda_n) \in K^n$,

$$\mathcal{O}(\lambda) = \lambda + \mathbb{Z}^n$$

is its orbit. The set of all $\mathbb{Z}^n$-orbits is isomorphic to the factor group $K^n/\mathbb{Z}^n$, $\mathcal{O}(\lambda) \leftrightarrow \lambda + \mathbb{Z}^n$. In particular, two orbits are equal, $\mathcal{O}(\lambda) = \mathcal{O}(\lambda')$ iff $\lambda - \lambda' \in \mathbb{Z}^n$. Let $e_1, \dots, e_n$ be the standard basis for the vector space $K^n$. Then $\mathbb{Z}^n = \oplus_{i=1}^n \mathbb{Z}e_i \subseteq K^n = \oplus_{i=1}^n Ke_i$.

The $K$-automorphisms $\sigma_1, \dots, \sigma_n$ of the polynomial algebra $D_n$ commute, $\sigma_i\sigma_j = \sigma_j\sigma_i$, since $\sigma_i(H_j) = H_j - \delta_{ij}$ where $\delta_{ij}$ is the Kronecker delta. The subgroup

$$G = \langle \sigma_1, \dots, \sigma_n \rangle$$

of the group of $K$-algebra automorphisms $\mathrm{Aut}_K(D_n)$ generated by the automorphisms $\sigma_1, \dots, \sigma_n$ is an abelian group isomorphic to $\mathbb{Z}^n$ via the isomorphism $G \to \mathbb{Z}^n$, $\sigma_1 \mapsto e_1, \dots, \sigma_n \mapsto e_n$.

Let $\mathcal{M}_n$ be the set of all maximal ideals of the algebra $D_n$ of the type

$$\mathfrak{m} = \mathfrak{m}_\lambda = (H_1 - \lambda_1, \dots, H_n - \lambda_n) \text{ where } \lambda = (\lambda_1, \dots, \lambda_n) \in K^n.$$

The group $\mathrm{Aut}_K(D_n)$ acts on the set $\mathcal{M}_n$, $(\sigma, \mathfrak{m}) \mapsto \sigma(\mathfrak{m})$. Recall that the group $\mathbb{Z}^n$ acts on $K^n$ in the obvious way:

$$\mathbb{Z}^n \times K^n \to K^n, \ \ (i, \lambda) \mapsto i + \lambda.$$

In a similar way, the group $G$ acts on $\mathcal{M}_n$, $(\sigma_1^{i_1} \cdots \sigma_n^{i_n}, \mathfrak{m}) \mapsto \sigma_1^{i_1} \cdots \sigma_n^{i_n}(\mathfrak{m})$ where $i = (i_1, \cdots, i_n) \in \mathbb{Z}^n$. When we identify the set $\mathcal{M}_n$ with $K^n$ via the bijection

$$\mathcal{M}_n \to K^n, \ \ \mathfrak{m}_\lambda \mapsto \lambda$$

and the group $G$ with $\mathbb{Z}^n$ via the group isomorphism $G \to \mathbb{Z}^n$, $\sigma_i \mapsto e_i$ $(i = 1, \dots, n)$, the set of $G$-orbits $\mathcal{M}_n/G$ is identified with the factor group $K^n/\mathbb{Z}^n$ via the bijection

$$\mathcal{M}_n/G \to K^n/\mathbb{Z}^n, \ \ G\mathfrak{m}_\lambda \mapsto \lambda + \mathbb{Z}^n,$$

and the action $\sigma_1^{i_1} \cdots \sigma_n^{i_n}(\mathfrak{m}_\lambda)$ is identified with the action $\lambda + i$ where $i = (i_1, \dots, i_n) \in \mathbb{Z}^n$.

The polynomial algebra $D_n = K[H_1, \dots, H_n]$ is a maximal commutative subalgebra of $\mathbb{I}_n$. For each $\lambda = (\lambda_1, \dots, \lambda_n) \in K^n$, the field

$$K_\mathfrak{m} := D_n/\mathfrak{m}_\lambda = D_n/(H_1 - \lambda_1, \dots, H_n - \lambda_n) = K_\lambda \simeq K$$

is a unique simple $D_n$-module that is annihilated by the maximal ideal $\mathfrak{m}_\lambda = (H_1 - \lambda_1, \dots, H_n - \lambda_n)$. Let $\mathrm{Max}(D_n)$ be the maximal spectrum of the algebra $D_n$ (the set of maximal ideals of $D_n$). Notice that if $K$ is an algebraically closed field then $\mathcal{M}_n = \mathrm{Max}(D_n)$.

An $\mathbb{I}_n$-module $M$ is called a *weight* module if

$$M = \bigoplus_{\mathfrak{m} \in \mathcal{M}_n} M_\mathfrak{m} \text{ where } M_\mathfrak{m} = \mathrm{ann}_M(\mathfrak{m}) := \{m \in M \mid \mathfrak{m}m = 0\}.$$

If $\mathfrak{m} = \mathfrak{m}_\lambda = (H_1 - \lambda_1, \dots, H_n - \lambda_n)$ for some $\lambda \in K^n$ then

$$M_\lambda := M_{\mathfrak{m}_\lambda} = \{m \in M \mid H_1 m = \lambda_1 m, \dots, H_n m = \lambda_n m\}.$$

The set $\mathrm{Supp}(M) = \{\mathfrak{m} \in \mathcal{M}_n \mid M_\mathfrak{m} \neq 0\}$ is called the *support* of the weight $\mathbb{I}_n$-module $M$. So, an $\mathbb{I}_n$-module $M$ is weight iff it is a (direct) sum of common eigen-spaces for the commuting elements $H_1, \dots, H_n$ of the algebra $\mathbb{I}_n$.

An $\mathbb{I}_n$-module $M$ is called a *generalized weight* module if

$$M = \bigoplus_{\mathfrak{m} \in \mathcal{M}_n} M^\mathfrak{m} \text{ where } M^\mathfrak{m} = \{m \in M \mid \mathfrak{m}^i m = 0 \text{ for some } i \geq 0\} = \bigcup_{i \geq 0} \mathrm{ann}_M(\mathfrak{m}^i).$$

If $\mathfrak{m} = \mathfrak{m}_\lambda$ for some $\lambda \in K^n$ then

$$M^\mathfrak{m} = \{m \in M \mid (H_1 - \lambda_1)^i m = 0, \dots, (H_n - \lambda_n)^i m = 0 \text{ for some } i = i(m) \geq 1\}.$$

The set $\mathrm{Supp}(M) = \{\mathfrak{m} \in \mathcal{M}_n \mid M^\mathfrak{m} \neq 0\}$ is called the *support* of the generalized weight $\mathbb{I}_n$-module $M$. Recall that we identified $(G, \mathcal{M}_n, \mathcal{M}_n/G)$ with $(\mathbb{Z}^n, K^n, K^n/\mathbb{Z}^n)$. Therefore, the maximal ideal $\mathfrak{m}_\lambda$ is identified with vector $\lambda \in K^n$, $M_\lambda := M_{\mathfrak{m}_\lambda}$ and $M^\lambda := M^{\mathfrak{m}_\lambda}$. So,

$$\mathrm{Supp}(M) = \{\lambda \in K^n \mid M^\lambda \neq 0\}.$$

If, in addition, the field $K$ is algebraically closed then the set $\mathrm{Max}(D_n)$ of maximal ideals of the ring $D_n$ is equal to $\mathcal{M}_n$, every weight $\mathbb{I}_n$-module is an $\mathbb{I}_n$-module which is a semisimple $D_n$-module (and vice versa), and every generalized weight $\mathbb{I}_n$-module is an $\mathbb{I}_n$-module which is a *locally finite dimensional* $D_n$-module (and vice versa).

We denote by $\mathrm{W}(\mathbb{I}_\mathrm{n})$ (resp., $\mathrm{GW}(\mathbb{I}_\mathrm{n})$) the category of weight (resp., generalized weight) $\mathbb{I}_n$-modules. Clearly,

$$\mathrm{W}(\mathbb{I}_\mathrm{n}) \subseteq \mathrm{GW}(\mathbb{I}_\mathrm{n}) \subseteq \mathbb{I}_n\text{-Mod}$$

are inclusions of categories where $\mathbb{I}_n$-Mod is the category of all left $\mathbb{I}_n$-modules. The category $\mathrm{GW}(\mathbb{I}_\mathrm{n})$ is a full subcategory of $\mathbb{I}_n$-Mod, it is closed under arbitrary direct sums, extensions, submodules and factor modules. The category $\mathrm{W}(\mathbb{I}_\mathrm{n})$ is closed under direct sums, submodules and factor modules but not under extensions, see [6, Theorem 2.5].

Let $M$ be a generalized weight $\mathbb{I}_n$-module. It follows from the defining relations of the algebra $\mathbb{I}_n$, [7, Proposition 2.2] or (1), that for all $\mathfrak{m} \in \mathcal{M}_n$ and $i = 1, \dots, n$,

$$x_i M^\mathfrak{m} \subseteq M^{\sigma_i(\mathfrak{m})}, \quad \int_i M^\mathfrak{m} \subseteq M^{\sigma_i(\mathfrak{m})}, \quad \partial_i M^\mathfrak{m} \subseteq M^{\sigma_i^{-1}(\mathfrak{m})},$$

$$x_i M_\mathfrak{m} \subseteq M_{\sigma_i(\mathfrak{m})}, \quad \int_i M_\mathfrak{m} \subseteq M_{\sigma_i(\mathfrak{m})}, \quad \partial_i M_\mathfrak{m} \subseteq M_{\sigma_i^{-1}(\mathfrak{m})}.$$

So, the generalized weight $\mathbb{I}_n$-module $M$ is a direct sum of its generalized weight submodules $M^\mathcal{O}$,

$$M = \bigoplus_{\mathcal{O} \in \mathcal{M}_n/G} M^\mathcal{O} \text{ where } M^\mathcal{O} := \bigoplus_{\mathfrak{m} \in \mathcal{O}} M^\mathfrak{m}. \tag{7}$$

Similarly, a weight $\mathbb{I}_n$-module $M$ is a direct sum of its weight submodules $M_\mathcal{O}$,

$$M = \bigoplus_{\mathcal{O} \in \mathcal{M}_n/G} M_\mathcal{O} \text{ where } M_\mathcal{O} := \bigoplus_{\mathfrak{m} \in \mathcal{O}} M_\mathfrak{m}. \tag{8}$$

For each orbit $\mathcal{O} \in \mathcal{M}_n/G$, let $\mathrm{W}(\mathbb{I}_n, \mathcal{O})$ (resp., $\mathrm{GW}(\mathbb{I}_n, \mathcal{O})$) be the subcategory of weight (resp., generalized weight) $\mathbb{I}_n$-modules $M$ with $\mathrm{Supp}(M) \subseteq \mathcal{O}$. By (7) and (8),

$$\mathrm{W}(\mathbb{I}_n) = \bigoplus_{\mathcal{O}\in\mathcal{M}_n/G} \mathrm{W}(\mathbb{I}_n, \mathcal{O}) \text{ and } \mathrm{GW}(\mathbb{I}_n) = \bigoplus_{\mathcal{O}\in\mathcal{M}_n/G} \mathrm{GW}(\mathbb{I}_n, \mathcal{O}), \tag{9}$$

direct sum of full subcategories of $\mathrm{W}(\mathbb{I}_n)$ and $\mathrm{GW}(\mathbb{I}_n)$, respectively.

So, the problem of classification of indecomposable weight or generalized weight $\mathbb{I}_n$-modules is reduced to the case when the support of a module belongs to a single orbit.

Let $0 \to N \to M \to L \to 0$ be a short exact sequence of $\mathbb{I}_n$-modules. Then $M$ is a generalized weight module iff so are the modules $N$ and $L$, and in this case,

$$\mathrm{Supp}(M) = \mathrm{Supp}(N) \cup \mathrm{Supp}(L). \tag{10}$$

**The simple weight $\mathbb{I}_n$-module $P_n$.** By the definition, the algebra $\mathbb{I}_n$ is a subalgebra of the algebra $\mathrm{End}_K(P_n)$ of all $K$-endomorphisms of the vector space $P_n$. So, the polynomial algebra $P_n$ is a (left) $\mathbb{I}_n$-module. Since $A_n \subset \mathbb{I}_n$, the $A_n$-module $P_n$ is a simple faithful $A_n$-module. Hence, the $\mathbb{I}_n$-module $P_n$ is also simple and faithful (by the very definition, the algebras $A_n$ and $\mathbb{I}_n$ are subalgebras of $\mathrm{End}_K(P_n)$). The action of the elements $x_i, \partial_i, H_i$ and $\int_i$ on $P_n$ is given by the rule: For all elements $p \in P_n$,

$$\partial_i p = \frac{\partial p}{\partial x_i}, \quad H_i p = \frac{\partial}{\partial x_i}(x_i p), \quad \int_i p = \int_0^{x_i} p dx_i \text{ and } x_i p = x_i \cdot p \text{ (multiplication by } x_i).$$

For all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $i = 1, \ldots, n$,

$$H_i x^\alpha = (\alpha_i + 1)x^\alpha \text{ where } x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Therefore, the $\mathbb{I}_n$-module $P_n$ is a weight module with $\mathrm{Supp}(P_n) = \mathbb{N}_+^n$. The polynomial algebra $P_n = K[x_1] \otimes \cdots \otimes K[x_n]$ is a tensor product of polynomial algebras $K[x_i]$. Furthermore, it is a tensor product of simple weight $\mathbb{I}_1(i)$-modules $K[x_i]$ where $\mathbb{I}_1(i) = K\langle \partial_i, H_i, \int_i \rangle$.

**The indecomposable $\mathbb{I}_1$-modules $M(s, \lambda)$, [6].** For $\lambda \in K$ and a natural number $s \geq 1$, let us consider the induced $B_1$-module

$$M(s, \lambda) := B_1 \otimes_{K[H]} K[H]/(H - \lambda)^s. \tag{11}$$

Clearly,

$$M(s, \lambda) \simeq B_1/B_1(H - \lambda)^s \simeq \mathbb{I}_1/(F + \mathbb{I}_1(H - \lambda)^s). \tag{12}$$

The $\mathbb{I}_1$-module/$B_1$-module $M(s, \lambda)$ is a generalized weight module with $\mathrm{Supp} M(s, \lambda) = \lambda + \mathbb{Z}$,

$$M(s, \lambda) = \bigoplus_{i\in\mathbb{Z}} M(s, \lambda)^{\lambda+i} \text{ and } \dim M(s, \lambda)^{\lambda+i} = s \text{ for all } i \in \mathbb{Z}. \tag{13}$$

LEMMA 2.1.

(1) *Each simple generalized weight $\mathbb{I}_1$-module is a simple weight $\mathbb{I}_1$-module, and vice versa.*

(2) *Each simple generalized weight* $\mathbb{I}_1$*-module is isomorphic to one of the modules:* $K[x]$ *or* $M(1,\lambda)$, $\lambda \in \Lambda$ *(where* $\Lambda$ *is any fixed subset of* $K$ *such that the map* $\Lambda \to K/\mathbb{Z}$, $\lambda \mapsto \lambda + \mathbb{Z}$ *is a bijection), they are pairwise non-isomorphic.*
(3) $\mathrm{End}_{\mathbb{I}_1}(M) \simeq K$ *for all simple weight* $\mathbb{I}_1$*-modules.*

*Proof.* 1 and 2. Statements 1 and 2 follow from [6, Theorem 2.5].
3. Let $\mathrm{id}_{K[x]}$ be the identity map on $K[x]$. Then

$$K\mathrm{id}_{K[x]} \subseteq \mathrm{End}_{\mathbb{I}_1}(K[x]) \subseteq \mathrm{End}_{A_1}(K[x]) \simeq \mathrm{Ker}_{K[x]}(\partial\cdot) \simeq K$$

where

$$\partial\cdot : K[x] \to K[x], \quad p \mapsto \frac{dp}{dx},$$

and $A_1 = K\langle x, \partial \,|\, \partial x - x\partial = 1\rangle$ is the first Weyl algebra. Therefore, $\mathrm{End}_{\mathbb{I}_1}(K[x]) \simeq K$. Similarly,

$$\mathrm{End}_{\mathbb{I}_1}(M(1,\lambda)) \simeq \mathrm{End}_{B_1}(B_1/(B_1(H-\lambda))) \simeq \mathrm{Ker}_{M(1,\lambda)}((H-\lambda)\cdot) \simeq K. \quad \square$$

Lemma 2.2.
(1) *[7, Proposition 6.1.(1)] The* $\mathbb{I}_1$*-module* $K[x]$ *is isomorphic to* $\mathbb{I}_1/\mathbb{I}_1\partial$.
(2) *[6, Eq. (12)] For all elements* $\lambda \in K\backslash\mathbb{N}_+$*, the* $\mathbb{I}_1$*-module* $M(1,\lambda)$ *is isomorphic to the* $\mathbb{I}_1$*-module* $\mathbb{I}_1/\mathbb{I}_1(H-\lambda)$.

For all $n \in \mathbb{N}_+$,

$$F + \mathbb{I}_1(H_1 - n) = E_{*,n-1} \oplus \mathbb{I}_1(H_1 - n) \text{ where } E_{*,n-1} := \bigoplus_{i \geq 0} Ke_{i,n-1} \simeq_{\mathbb{I}_1} K[x].$$

Hence, $M(1,n) = \mathbb{I}_1/\big(E_{*,n-1} \oplus \mathbb{I}_1(H_1-n)\big)$ and there is a short exact sequence of $\mathbb{I}_1$-modules

$$0 \to K[x_1] \simeq E_{*,n-1} \to \mathbb{I}_1/\mathbb{I}_1(H_1-n) \to M(1,n) \to 0. \tag{14}$$

In fact, it splits (Theorem 2.5).

Let $A$ be an algebra and $A$-Mod be the category of left $A$-modules. A subcategory $\mathcal{C}$ of $A$-Mod is called a *semisimple* category if every module of $\mathcal{C}$ is a direct sum of its simple submodules in $\mathcal{C}$. The category $\mathrm{GW}(\mathbb{I}_\mathrm{n})$ of generalized weight $\mathbb{I}_n$-modules is a subcategory of the category $\mathbb{I}_n$-Mod of all left $\mathbb{I}_n$-modules. The category $\mathrm{W}(\mathbb{I}_\mathrm{n})$ of weight $\mathbb{I}_n$-modules is a subcategory of $\mathrm{GW}(\mathbb{I}_\mathrm{n})$.

**Classification of simple weight $\mathbb{I}_n$-modules.** We denote by $\widehat{\mathbb{I}}_n$(weight) (resp., $\widehat{\mathbb{I}}_n$(gen. weight)) the set of isomorphism classes of simple weight (resp., generalized weight) $\mathbb{I}_n$-modules. The next theorem classifies (up to isomorphism) all the simple weight $\mathbb{I}_n$-modules.

Theorem 2.3.
(1)

$$\widehat{\mathbb{I}}_n(\textit{gen. weight}) = \widehat{\mathbb{I}}_n(\textit{weight}) = \widehat{\mathbb{I}}_1(\textit{weight})^{\otimes n},$$

*i.e., every simple generalized weight* $\mathbb{I}_n$*-module is a simple weight* $\mathbb{I}_n$*-module, and vice versa; every simple weight* $\mathbb{I}_n$*-module* $M$ *is isomorphic to the tensor* *product* $M_1 \otimes \cdots \otimes M_n$ *of simple weight* $\mathbb{I}_1$*-modules and two such modules are isomorphic over* $\mathbb{I}_n$,

$$M_1 \otimes \cdots \otimes M_n \simeq M_1' \otimes \cdots \otimes M_n',$$

*iff for each* $i = 1, \dots, n$, *the* $\mathbb{I}_1$*-modules* $M_i$ *and* $M_i'$ *are isomorphic.*

(2) *For each simple weight* $\mathbb{I}_n$*-module* $M = \bigotimes_{i=1}^n M_i$, $\mathrm{Supp}(M) = \prod_{i=1}^n \mathrm{Supp}(M_i)$.

*Proof.* To prove the theorem we use induction on $n$. The case $n = 1$ is true, see [6, Theorem 2.5].

Suppose that $n > 1$ and that the theorem is true for all $n' < n$. Let $M$ be a simple generalized weight $\mathbb{I}_n$-module. By (7), $\mathrm{Supp}(M) \subseteq \mathcal{O}$ for some orbit $\mathcal{O} \in K^n/\mathbb{Z}^n$. Since

$$\mathbb{I}_n = \mathbb{I}_1 \otimes \mathbb{I}_{n-1},$$

the $\mathbb{I}_1$-module $M$ is a generalized weight $\mathbb{I}_1$-module with $\mathrm{Supp}_{\mathbb{I}_1}(M) \subseteq \lambda_1 + \mathbb{Z}$ where $\lambda_1 \in K$ such that $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathrm{Supp}_{\mathbb{I}_n}(M)$ for some $\lambda_2, \dots, \lambda_n \in K$. Fix a nonzero vector $v \in M^\lambda$ such that $(H_1 - \lambda_1)v = 0$. The $\mathbb{I}_1$-submodule $\mathbb{I}_1 v$ of $M$ is an epimorphic image of the $\mathbb{I}_1$-module

$$N = \mathbb{I}_1/\mathbb{I}_1(H_1 - \lambda_1).$$

By [9, Theorem 3.6.(2)] (or by Lemma 2.2.(2) and Eq. (14)), the $\mathbb{I}_1$-module $N$ has finite length. Hence, so does the $\mathbb{I}_1$-module $\mathbb{I}_1 v$. Changing the element $v$, if necessary, we can assume that the $\mathbb{I}_1$-module $M_1 = \mathbb{I}_1 v$ is a simple weight $\mathbb{I}_1$-module. The $\mathbb{I}_n$-module $M$ is a simple module. The $\mathbb{I}_n$-module homomorphism

$$N \otimes \mathbb{I}_{n-1} = \mathbb{I}_1/\mathbb{I}_1(H_1 - \lambda_1) \otimes \mathbb{I}_{n-1} \longrightarrow M, \ \ \overline{1} \otimes 1 \mapsto v \ \ (\text{where } \ \overline{1} = 1 + \mathbb{I}_1(H_1 - \lambda_1))$$

is an epimorphism. By Lemma 2.1.(3), $\mathrm{End}_{\mathbb{I}_1}(N) \simeq K$. By [4],

$$M \simeq N \otimes M'$$

for some simple $\mathbb{I}_{n-1}$-module $M'$. The $\mathbb{I}_n$-module $M$ is a generalized weight $\mathbb{I}_n$-module. Hence, the $\mathbb{I}_{n-1}$-module $M'$ is a generalized weight $\mathbb{I}_{n-1}$-module. Now, the result follows by induction on $n$. □

**The category** $\mathrm{W}(\mathbb{I}_n)$ **of weight** $\mathbb{I}_n$**-modules is a semisimple category.** Let $n = 1$. For each orbit $\mathcal{O} \in K/\mathbb{Z}$, we fix an element $\lambda_\mathcal{O} \in K$ such that $\lambda_\mathcal{O} \in \mathcal{O}$. In particular, $\lambda_\mathcal{O} + \mathbb{Z} = \mathcal{O}$. For $\mathcal{O} = \mathbb{Z}$ let $\lambda_\mathbb{Z} := 0$. Let $n > 1$. For each orbit $\mathcal{O} = \mathcal{O}_1 \times \cdots \times \mathcal{O}_n \in K^n/\mathbb{Z}^n$, let

$$\lambda_\mathcal{O} := (\lambda_{\mathcal{O}_1}, \dots, \lambda_{\mathcal{O}_n}) \in K^n.$$

In particular $\lambda_\mathcal{O} + \mathbb{Z}^n = \mathcal{O}$. The map $\mathcal{O} \mapsto \lambda_\mathcal{O}$ is a bijection, by definition. For the orbit $\mathcal{O} = \mathcal{O}_1 \times \cdots \times \mathcal{O}_n$, let

$$\mathbb{D}_\mathcal{O} := \{i \in \{1, \dots, n\} \,|\, \mathcal{O}_i = \mathbb{Z}\}.$$

Then $\{1, \dots, n\} \setminus \mathbb{D}_\mathcal{O} = \{j \in \{1, \dots, n\} \,|\, \mathcal{O}_j \neq \mathbb{Z}\}$. Let $\mathcal{D}_\mathcal{O}$ be any subset of $\mathbb{D}_\mathcal{O}$ (eg., $\mathcal{D}_\mathcal{O} = \emptyset$). Then

$$\{1, \dots, n\} = \mathcal{D}_\mathcal{O} \sqcup \mathcal{N}_\mathcal{O} \tag{15}$$

is a disjoint union where $\mathcal{N}_{\mathcal{O}} := \{1, \dots, n\} \setminus \mathcal{D}_{\mathcal{O}}$. For each pair $(\mathcal{O}, \mathcal{D}_{\mathcal{O}})$, let us define the $\mathbb{I}_n$-module

$$M(\mathcal{D}_{\mathcal{O}}) := \bigotimes_{i=1}^{n} M(\mathcal{D}_{\mathcal{O}_i}) \text{ where } M(\mathcal{D}_{\mathcal{O}_i}) = \begin{cases} K[x_i] & \text{if } i \in \mathcal{D}_{\mathcal{O}}, \\ M(1, \lambda_{\mathcal{O}_i}) & \text{if } i \notin \mathcal{D}_{\mathcal{O}}. \end{cases} \tag{16}$$

For each choice of the set $\mathcal{D}_{\mathcal{O}}$,

$$\mathcal{O} = \mathcal{O}_{\text{deg}} \times \mathcal{O}_{\text{ndeg}} \text{ where } \mathcal{O}_{\text{deg}} := \prod_{i \in \mathcal{D}_{\mathcal{O}}} \mathbb{N}_+ = \mathbb{N}_+^{|\mathcal{D}_{\mathcal{O}}|}$$

is called the *degenerate part* of the pair $(\mathcal{O}, \mathcal{D}_{\mathcal{O}})$ and

$$\mathcal{O}_{\text{ndeg}} := \prod_{j \in \mathcal{N}_{\mathcal{O}}} \mathcal{O}_j$$

is called the *non-degenerate part* of the pair $(\mathcal{O}, \mathcal{D}_{\mathcal{O}})$. The elements of the set $\mathcal{D}_{\mathcal{O}}$ (resp., $\mathcal{N}_{\mathcal{O}}$) are called the *degenerate* (resp., *non-degenerate*) *indices* with respect to the pair $(\mathcal{O}, \mathcal{D}_{\mathcal{O}})$, or, simply, the $\mathcal{D}_{\mathcal{O}}$*-degenerate* (resp., $\mathcal{D}_{\mathcal{O}}$*-non-degenerate*) *indices*.

For each subset $I$ of $\{1, \dots, n\}$, let $\mathfrak{a}_n(I)$ be the ideal of $\mathbb{I}_n$ generated by the ideals $F(i)$ of $\mathbb{I}_1(i)$ where $i \in I$. If $I = \emptyset$ we set $\mathfrak{a}_n(\emptyset) := 0$. Clearly,

$$\mathfrak{a}_n(I) = \sum_{i \in I} \mathfrak{p}_i.$$

If $I = \{1, \dots, n\}$ then $\mathfrak{a}_n(I) = \mathfrak{a}_n$ is the maximal ideal of $\mathbb{I}_n$. The factor algebra

$$\mathbb{I}_n(\mathcal{D}_{\mathcal{O}}) := \mathbb{I}_n / \mathfrak{a}_n(\mathcal{D}_{\mathcal{O}}) \simeq B(\mathcal{O}_{\text{deg}}) \otimes \mathbb{I}_l(\mathcal{O}_{\text{ndeg}})$$

is a tensor product of algebras where

$$B(\mathcal{O}_{\text{deg}}) := \bigotimes_{i \in \mathcal{D}_{\mathcal{O}}} B_1(i), \quad \mathbb{I}_l(\mathcal{O}_{\text{ndeg}}) := \bigotimes_{i \in \mathcal{N}_{\mathcal{O}}} \mathbb{I}_1(i) \text{ and } l = |\mathcal{N}_{\mathcal{O}}|.$$

For an algebra $A$ and an $A$-module $M$, $\text{ann}_A(M) := \{a \in A \,|\, aM = 0\}$ is the *annihilator* of the $A$-module $M$.

LEMMA 2.4. *Let $\mathcal{O} \in K^n/\mathbb{Z}^n$. Then the set* $\widehat{\text{GW}(\mathbb{I}_{\text{n}}, \mathcal{O})}$ *of isomorphisms classes of simple (generalized) weight $\mathbb{I}_n$-modules in the category* $\text{GW}(\mathbb{I}_{\text{n}}, \mathcal{O})$ *is equal to the set*

$$\{M(\mathcal{D}_{\mathcal{O}}) \,|\, \mathcal{D}_{\mathcal{O}} \subseteq \mathbb{D}_{\mathcal{O}}\},$$

*and the number of elements in this set is $2^{|\mathbb{D}_{\mathcal{O}}|}$. Furthermore,*

(1) *The $\mathbb{I}_n$-module $M(\mathcal{D}_{\mathcal{O}})$ is faithful iff $\mathcal{D}_{\mathcal{O}} = \{1, \dots, n\}$.*
(2) *If $\mathcal{D}_{\mathcal{O}} \neq \{1, \dots, n\}$ then* $\text{ann}_{\mathbb{I}_n}(M(\mathcal{D}_{\mathcal{O}})) = \mathfrak{a}_n(\mathcal{N}_{\mathcal{O}})$.
(3) *The modules in the set* $\widehat{\text{GW}(\mathbb{I}_{\text{n}}, \mathcal{O})}$ *are uniquely determined by their annihilators, i.e., the map $M(\mathcal{D}_{\mathcal{O}}) \mapsto \text{ann}_{\mathbb{I}_n}(M(\mathcal{D}_{\mathcal{O}}))$ is a bijection.*

*Proof.* Since $\mathbb{I}_n = \otimes_{i=1}^n \mathbb{I}_1(i)$ and $M(\mathcal{D}_{\mathcal{O}}) = \otimes_{i=1}^n M(\mathcal{D}_{\mathcal{O}_i})$, we have

$$\text{ann}_{\mathbb{I}_n}(M(\mathcal{D}_{\mathcal{O}})) = \mathfrak{a}_n(\mathcal{N}_{\mathcal{O}}),$$

and statements 1 and 2 follow. Then statement 2 implies statement 3, and the rest follows. □

Clearly, for $\mathcal{O}=\prod_{i=1}^n \mathcal{O}_i$,

$$S(\mathcal{D}_\mathcal{O}) := \mathrm{Supp}(M(\mathcal{D}_\mathcal{O})) = \mathbb{N}_+^{|\mathcal{D}_\mathcal{O}|} \times \prod_{i\in\mathcal{N}_\mathcal{O}} \mathcal{O}_i. \qquad (17)$$

For an $\mathbb{I}_n$-modules $M$, we denote by $\mathrm{soc}_{\mathbb{I}_n}(M)$ its *socle*, i.e., the sum of all simple $\mathbb{I}_n$-submodules of $M$ if there are simple submodules, and $\mathrm{soc}_{\mathbb{I}_n}(M)=0$, otherwise.

THEOREM 2.5. *Every weight $\mathbb{I}_n$-module is a direct sum of simple weight $\mathbb{I}_n$-modules. In particular, the category* $\mathrm{W}(\mathbb{I}_\mathrm{n})$ *of weight $\mathbb{I}_n$-modules is a semisimple category.*

*Proof.* (a) *For all $\lambda=(\lambda_1,\dots,\lambda_n)\in K^n$, the generalized weight $\mathbb{I}_n$-module*

$$\mathbb{I}_n(\lambda) := \mathbb{I}_n/\mathbb{I}_n(H_1-\lambda_1,\dots,H_n-\lambda_n)$$

*has finite length*: By [9, Theorem 3.6.(2)] (or by Lemma 2.2.(2) and Eq. (14)), the statement (a) holds when $n=1$. Suppose that $n>1$. Then $\mathbb{I}_n=\otimes_{i=1}^n\mathbb{I}_i(\lambda_i)$, and the statement (a) follows from Lemma 2.2.(2), Eq. (14) and Theorem 2.3.

(b) *For all nonzero generalized weight $\mathbb{I}_n$-modules $M$,* $\mathrm{soc}_{\mathbb{I}_n}(M)\neq 0$*:* The $\mathbb{I}_n$-module $M$ contains a submodule, say, $N$ which is an epimorphic image of the $\mathbb{I}_n$-module $\mathbb{I}_n(\lambda)$ for some $\lambda\in K^n$. By the statement (a), the generalized weight $\mathbb{I}_n$-module $M$ contains a simple module which is necessarily weight, by Theorem 2.3.

Now, the theorem follows from the statements (b) and (c).

(c) *Every short exact sequence of $\mathbb{I}_n$-modules, $0\to M\to M'\xrightarrow{f}\overline{M}\to 0$, splits provided the $\mathbb{I}_n$-module $M'$ is weight, the $\mathbb{I}_n$-module $M$ is semisimple and the $\mathbb{I}_n$-module $\overline{M}$ is simple.*

*Proof of the theorem.* Let $M'$ be a nonzero weight $\mathbb{I}_n$-module. By the statement (b), $\mathrm{soc}_{\mathbb{I}_n}(M')\neq 0$. We have to show that

$$M'=\mathrm{soc}_{\mathbb{I}_n}(M').$$

Suppose that this is not the case, we seek a contradiction. Then

$$\mathrm{soc}_{\mathbb{I}_n}(M'/\mathrm{soc}_{\mathbb{I}_n}(M'))\neq 0.$$

Fix a nonzero, necessarily weight, $\mathbb{I}_n$-submodule, say $\overline{M}$, of $\mathrm{soc}_{\mathbb{I}_n}(M'/\mathrm{soc}_{\mathbb{I}_n}(M'))\neq 0$. Then the short exact sequence of $\mathbb{I}_n$-modules $0\to M\to M'\xrightarrow{f}\overline{M}\to 0$ is non-split. This contradicts to the statement (c).

*Proof of the statement (c).* In view of (9), we may assume that $\mathrm{Supp}(M),\mathrm{Supp}(\overline{M})\subseteq\mathcal{O}$ for an orbit $\mathcal{O}\in K^n/\mathbb{Z}^n$. Notice that

$$\mathrm{Supp}(M')=\mathrm{Supp}(M)\bigcup\mathrm{Supp}(\overline{M})\subseteq\mathcal{O}.$$

For each element $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathrm{Supp}(\overline{M})\subseteq\mathcal{O}$, we have the short exact sequence of $K_{\mathfrak{m}_\lambda}$-modules,

$$0\to M_\lambda\to M'_\lambda\xrightarrow{f_\lambda}\overline{M}_\lambda\to 0,$$

where $K_{\mathfrak{m}_\lambda}:=D_n/\mathfrak{m}_\lambda$ and $\mathfrak{m}_\lambda=(H_1-\lambda_1,\dots,H_n-\lambda_n)$. Notice that $\dim_K(\overline{M}_\lambda)=1$. Since the algebra $K_{\mathfrak{m}_\lambda}$ is a field, the short exact sequence above splits. Let $\overline{e}=\overline{e}_\lambda$ be a nonzero element of $\overline{M}_\lambda$ and $e=e_\lambda$ be an element of $M'_\lambda$ such that

$$f_\lambda(e)=\overline{e}.$$

To finish the proof it suffices to show that the $\mathbb{I}_n$-submodule of $M'$,

$$N := \mathbb{I}_n e,$$

is a simple $\mathbb{I}_n$-module: Indeed, in this case, $M \cap N = 0$ since $f(M) = 0$, $f(N) = \overline{M}$ and the $\mathbb{I}_n$-module $N$ is simple. Clearly, $M + N = M'$, and so

$$M' = M \oplus N,$$

as required.

There are two cases to consider: either $\overline{\mathcal{D}}_{\mathcal{O}} = \emptyset$ or $\overline{\mathcal{D}}_{\mathcal{O}} \neq \emptyset$.

(i) $\overline{\mathcal{D}}_{\mathcal{O}} = \emptyset$: In this case, $\overline{M} = \bigotimes_{i=1}^n \overline{M}_i$ where $\overline{M}_i = \mathbb{I}_1(i)/\mathbb{I}_1(i)(H_i - \lambda_i)$ and $\lambda_i \in K \backslash \mathbb{Z}$. For each number $i = 1, \ldots, n$, we have the $\mathbb{I}_1(i)$-module epimorphism

$$\overline{M}_i = \mathbb{I}_1(i)/\mathbb{I}_1(i)(H_i - \lambda_i) \rightarrow \mathbb{I}_1(i)e, \;\; 1 + \mathbb{I}_1(i)(H_i - \lambda_i) \mapsto e,$$

which is necessarily an isomorphism since the $\mathbb{I}_1(i)$-module $\overline{M}_i$ is simple. Since $\mathbb{I}_n = \mathbb{I}_1(1) \otimes \cdots \otimes \mathbb{I}_1(n)$, we have the $\mathbb{I}_n$-epimorphism

$$\overline{M} = \bigotimes_{i=1}^n \overline{M}_i = \bigotimes_{i=1}^n \mathbb{I}_1(i)/\mathbb{I}_1(i)(H_i - \lambda_i) \rightarrow \mathbb{I}_n e = N$$

which is necessarily an isomorphism since the $\mathbb{I}_n$-module $\overline{M}$ is simple.

(ii) $\overline{\mathcal{D}}_{\mathcal{O}} \neq \emptyset$: Without loss of generality we can assume that $\overline{\mathcal{D}}_{\mathcal{O}} = \{1, \ldots, m\}$ for some natural number $m$ such that $1 \leq m \leq n$. Then

$$\overline{M}_i = \begin{cases} K[x_i] & \text{if } i = 1, \ldots, m, \\ M(1, \lambda_i) & \text{otherwise.} \end{cases}$$

Notice that $\text{Supp}_{\mathbb{I}_1(i)}(K[x_i]) = \{1, 2, \ldots\}$. Choose $\lambda = (\lambda_1, \ldots, \lambda_n) \in \text{Supp}(\overline{M})$ such that $\lambda_i = 1$ for all $i = 1, \ldots, m$. Then

$$H_i e = \begin{cases} e & \text{if } i = 1, \ldots, m, \\ \lambda_i e & \text{otherwise.} \end{cases}$$

For each $i = 1, \ldots, m$,

$$0 = (H_i - 1)e = (\partial_i x_i - 1)e = (x_i \partial_i + [\partial_i, x_i] - 1)e = (x_i \partial_i + 1 - 1)e = x_i \partial_i e.$$

Since $\overline{M} = \bigotimes_{j=1}^n \overline{M}_j$ and $\overline{M}_i = K[x_i]$ for $i = 1, \ldots, m$, the map

$$x_i \cdot : \overline{M} \rightarrow \overline{M}, \;\; m \mapsto x_i m$$

is an injection. Therefore, $\partial_i e = 0$ for $i = 1, \ldots, m$. So, by Lemma 2.2.(1), we have an $\mathbb{I}_1(i)$-epimorphism

$$K[x_i] = \mathbb{I}_1(i)/\mathbb{I}_1(i)\partial_i \rightarrow \mathbb{I}_1(i)e, \;\; 1 + \mathbb{I}_1(i)\partial_i \mapsto e,$$

which is necessarily an isomorphism since the $\mathbb{I}_1(i)$-module $K[x_i]$ is simple. Using the same argument as in the case (i), we see that for all $j > m$, $\mathbb{I}_1(j)e \simeq M(1, \lambda_j)$ and the $\mathbb{I}_n$-epimorphism

$$\overline{M} = \bigotimes_{i=1}^n \overline{M}_i \rightarrow \mathbb{I}_n e = N$$

is an isomorphism since the $\mathbb{I}_n$-module $\overline{M}$ is simple. The proof of the theorem is complete. □

By Theorem 2.5, each weight $\mathbb{I}_n$-module $M$ is a unique direct sum

$$M = \bigoplus_{\mathcal{O}\in K^n/\mathbb{Z}^n} \bigoplus_{\mathcal{D}_{\mathcal{O}}\subseteq\mathbb{D}_{\mathcal{O}}} M(\mathcal{D}_{\mathcal{O}})^{\mu(\mathcal{D}_{\mathcal{O}})} \tag{18}$$

where $M(\mathcal{D}_{\mathcal{O}})^{\mu(\mathcal{D}_{\mathcal{O}})}$ is a direct sum of $\mu(\mathcal{D}_{\mathcal{O}})$ copies of the module $M(\mathcal{D}_{\mathcal{O}})$ and $\mu(\mathcal{D}_{\mathcal{O}})$ is the multiplicity of $M(\mathcal{D}_{\mathcal{O}})$ (which can be any set).

**3. Explicit description of indecomposable generalized weight $\mathbb{I}_n$-modules.** In this section, an explicit description of indecomposable generalized weight $\mathbb{I}_n$-modules is obtained (Theorem 3.6). One of the key steps is to show that each category $\mathrm{GW}(\mathbb{I}_n,\mathcal{O})$ is a direct sum of its subcategories $\mathrm{GW}(\mathbb{I}_n,\mathcal{D}_{\mathcal{O}})$ that are generated by the single simple weight $\mathbb{I}_n$-modules $M(\mathcal{D}_{\mathcal{O}})$, see (19). Using (19) and some results about representations of Artinian rings, a criterion is given for the category $\mathrm{GW}(\mathbb{I}_n,\mathcal{O})$ and its subcategories $\mathrm{GW}(\mathbb{I}_n,\mathcal{D}_{\mathcal{O}})$ to be of finite representation type, tame or wild. Explicit classes of indecomposable modules in $\mathrm{GW}(\mathbb{I}_n,\mathcal{O})$ are considered.

Let $A$ be an algebra, $\mathfrak{m}$ be a *co-finite ideal* of $A$ (i.e., $\dim_K(A/\mathfrak{m}) < \infty$). An $A$-module $M$ is called a *locally finite module* if for each $m \in M$, $\dim_K(Am) < \infty$. An $A$-module $M$ is called an $\mathfrak{m}$-*locally finite module* if for each $m \in M$, $\dim_K(Am) < \infty$ and $\mathrm{ann}_A(Am) \subseteq \mathfrak{m}^i$ for some $i \geq 1$ (If in addition $A$ is a commutative algebra then the last condition is equivalent to the condition that $\mathfrak{m}^i m = 0$ for some $i \geq 1$). We denote by $\mathrm{LF}_{\mathfrak{m}}(A)$ the category of all $\mathfrak{m}$-locally finite $A$-modules. The category $\mathrm{LF}_{\mathfrak{m}}(A)$ is closed under arbitrary direct sums, submodules and factor modules. An $\mathbb{I}_n$-module $M \in \mathrm{GW}(\mathbb{I}_n,\mathcal{O})$ (resp., $M \in \mathrm{W}(\mathbb{I}_n,\mathcal{O})$) is called *equidimensional* if $\dim_K(M^{\mathfrak{m}}) = \dim_K(M^{\mathfrak{n}})$ (resp., $\dim_K(M_{\mathfrak{m}}) = \dim_K(M_{\mathfrak{n}})$) for all $\mathfrak{m},\mathfrak{n} \in \mathrm{Supp}(M)$. If the common value of all $\dim_K(M^{\mathfrak{m}})$ (resp., $\dim_K(M_{\mathfrak{m}})$) is $d$, we say that $M$ is *$d$-equidimensional.* Let $\mathfrak{m}$ be a maximal ideal of the polynomial algebra $D_n$ and $\mathcal{I}(D_n,\mathfrak{m})$ be the set of all ideals $I$ of $D_n$ such that

$$\mathfrak{m} \supseteq I \supseteq \mathfrak{m}^i \ \text{ for some } \ i \geq 1.$$

For all ideals $I \in \mathcal{I}(D_n,\mathfrak{m})$, the factor algebra $D_n/I$ is a local, finite dimensional, commutative algebra with maximal ideal $\mathfrak{m}/I$.

**The category $\mathrm{GW}(\mathbb{I}_n,\mathcal{O})$ is a direct sum of subcategories $\mathrm{GW}(\mathbb{I}_n,\mathcal{D}_{\mathcal{O}})$.** Let $\mathrm{GW}(\mathbb{I}_n,\mathcal{D}_{\mathcal{O}})$ be the full subcategory of $\mathrm{GW}(\mathbb{I}_n,\mathcal{O})$ generated by the simple weight $\mathbb{I}_n$-module $M(\mathcal{D}_{\mathcal{O}})$. There are precisely $2^{|\mathbb{D}_{\mathcal{O}}|}$ such subcategories in the categories $\mathrm{GW}(\mathbb{I}_n,\mathcal{O})$. They are key objects in the description of all indecomposable generalized weight modules $M$ with $\mathrm{Supp}(M) \subseteq \mathcal{O}$ since

$$\mathrm{GW}(\mathbb{I}_n,\mathcal{O}) = \bigoplus_{\mathcal{D}_{\mathcal{O}}\subseteq\mathbb{D}_{\mathcal{O}}} \mathrm{GW}(\mathbb{I}_n,\mathcal{D}_{\mathcal{O}}) \quad \text{(Theorem 3.2)}. \tag{19}$$

Let $R$ be a ring, $M$ be a nonzero $R$-module and $\mathfrak{p} = \mathrm{ann}_R(M)$. The $R$-module $M$ is called a *prime* $R$-module (or a $\mathfrak{p}$-*prime* $R$-module) if $\mathfrak{p}$ is a prime ideal of $R$ and $\mathrm{ann}_R(N) = \mathfrak{p}$ for all nonzero $R$-submodules $N$ of $M$. The $R$-module $M$ is called an *absolutely prime* $R$-module (or an *absolutely* $\mathfrak{p}$-*prime* $R$-module) if $\mathfrak{p}$ is a prime ideal of $R$ and $\mathrm{ann}_R(N) = \mathfrak{p}$ for all nonzero subfactors of $M$, i.e., $N = M_2/M_1$ for some submodules $M_1$ and $M_2$ of $M$ such that $0 \subseteq M_1 \subset M_2 \subseteq M$.

Lemma 3.1 shows that all nonzero modules in each category $\mathrm{GW}(\mathbb{I}_{\rm n}, \mathcal{D}_{\mathcal{O}})$ are absolutely prime $\mathbb{I}_n$-modules (Corollary 3.4). Lemma 3.1 is one of the key steps in proving that the equality (19) holds (Theorem 3.2).

LEMMA 3.1. *Let* $0 \neq M \in \mathrm{GW}(\mathbb{I}_{\rm n}, \mathcal{D}_{\mathcal{O}})$. *Then*
(1) $\mathrm{Supp}(M) = \mathrm{Supp}(M(\mathcal{D}_{\mathcal{O}}))$.
(2) $\mathrm{ann}_{\mathbb{I}_n}(M) = \mathfrak{a}(\mathcal{D}_{\mathcal{O}})$ *where*

$$\mathfrak{a}(\mathcal{D}_{\mathcal{O}}) = \mathrm{ann}_{\mathbb{I}_n}(M(\mathcal{D}_{\mathcal{O}})) = \sum_{i \in \mathcal{N}_{\mathcal{O}}} \mathfrak{p}_i \in \mathrm{Spec}(\mathbb{I}_n).$$

*So, all nonzero modules in the category* $\mathrm{GW}(\mathbb{I}_{\rm n}, \mathcal{D}_{\mathcal{O}})$ *are absolutely* $\mathfrak{a}(\mathcal{D}_{\mathcal{O}})$-*prime* $\mathbb{I}_n$-*modules.*

*Proof.* 1. By the definition, the category of generalized weight $\mathbb{I}_n$-modules $\mathrm{GW}(\mathbb{I}_{\rm n}, \mathcal{D}_{\mathcal{O}})$ is generated by the simple weight $\mathbb{I}_n$-module $M(\mathcal{D}_{\mathcal{O}})$, and statement 1 follows, by (10).

2. The ideal

$$\mathfrak{a}(\mathcal{D}_{\mathcal{O}}) = \mathrm{ann}_{\mathbb{I}_n}(M(\mathcal{D}_{\mathcal{O}})) = \sum_{i \in \mathcal{N}_{\mathcal{O}}} \mathfrak{p}_i$$

is a prime ideal of $\mathbb{I}_n$. Let $N = \{m \in M \mid \mathfrak{a}m = 0\}$ where $\mathfrak{a} = \mathfrak{a}(\mathcal{D}_{\mathcal{O}})$. We have to show that $M = N$. Suppose that this is not the case, we seek a contradiction. Then

$$S := \mathrm{soc}_{\mathbb{I}_n}(M/N) \neq 0$$

since $0 \neq M/N \in \mathrm{GW}(\mathbb{I}_{\rm n}, \mathcal{D}_{\mathcal{O}})$. Now, $S = L/N$ for some submodule $L$ of $M$ such that $N \subsetneq L$. Recall that $\mathfrak{a}^2 = \mathfrak{a}$. So, $0 \neq \mathfrak{a}L = \mathfrak{a}^2 L \subseteq \mathfrak{a}N = 0$, a contradiction. □

THEOREM 3.2. *For all orbits* $\mathcal{O} \in K^n/\mathbb{Z}^n$, *the equality (19) holds.*

*Proof.* CLAIM. $\sum_{\mathcal{D}_{\mathcal{O}} \subseteq \mathbb{D}_{\mathcal{O}}} \mathrm{GW}(\mathbb{I}_{\rm n}, \mathcal{D}_{\mathcal{O}}) = \bigoplus_{\mathcal{D}_{\mathcal{O}} \subseteq \mathbb{D}_{\mathcal{O}}} \mathrm{GW}(\mathbb{I}_{\rm n}, \mathcal{D}_{\mathcal{O}})$.

Suppose that the Claim does not hold, i.e.,

$$M_1 + \cdots + M_s \neq M_1 \oplus \cdots \oplus M_s$$

for some modules $M_i \in \mathrm{GW}(\mathbb{I}_{\rm n}, \mathcal{D}_{\mathcal{O}_{\rm i}})$ such that the sets $\mathcal{D}_{\mathcal{O}_1}, \dots, \mathcal{D}_{\mathcal{O}_s}$ are distinct. We may assume that the number $s$ is the least possible. Then $s \geq 2$. By Lemma 2.4, the prime ideals

$$\mathfrak{a}_1 = \mathrm{ann}_{\mathbb{I}_1}(M_1), \dots, \mathfrak{a}_s = \mathrm{ann}_{\mathbb{I}_1}(M_s)$$

are *distinct*. Up to order, we may assume that the ideal $\mathfrak{a}_1$ is a minimal element (with respect to $\subseteq$) of the set $\{\mathfrak{a}_1, \dots, \mathfrak{a}_s\}$. By replacing the module $M_1$, by a (possibly) smaller nonzero submodule, say $M_1' \subseteq M_1$, we may assume that

$$M_1' \subseteq M_2 + \cdots + M_s.$$

By Lemma 3.1.(2), $\mathrm{ann}_{\mathbb{I}_1}(M_1') = \mathfrak{a}_1$. Let $\mathfrak{a} = \mathfrak{a}_2 \cdots \mathfrak{a}_s$. Then $\mathfrak{a}M_1' \subseteq \mathfrak{a}(M_2 + \cdots + M_s) = 0$, and so

$$\mathfrak{a}_2 \cdots \mathfrak{a}_s \subseteq \mathrm{ann}_{\mathbb{I}_1}(M_1') = \mathfrak{a}_1.$$

Recall that the ideal $\mathfrak{a}_1$ is a prime ideal (Lemma 3.1.(2)), hence

$$\mathfrak{a}_i \subseteq \mathfrak{a}_1$$

for some $i$ such that $2 \leq i \leq s$. This contradicts the minimality of the ideal $\mathfrak{a}_1$. Now, the theorem follows from Proposition 3.3. □

PROPOSITION 3.3. *Let $\mathcal{O} \in K^n/\mathbb{Z}^n$. Suppose that $\mathcal{D}_{\mathcal{O}}$ and $\mathcal{D}'_{\mathcal{O}}$ are distinct subsets of the set $\mathbb{D}_{\mathcal{O}}$. Let $M = M(\mathcal{D}_{\mathcal{O}})$ and $M' = M(\mathcal{D}'_{\mathcal{O}})$. Then* $\mathrm{Ext}^1_{\mathbb{I}_n}(M', M) = 0$.

*Proof.* By Lemma 3.1.(2), the ideals $\mathfrak{a} = \mathrm{ann}_{\mathbb{I}_n}(M)$ and $\mathfrak{a}' = \mathrm{ann}_{\mathbb{I}_n}(M')$ are *distinct* prime ideals of the algebra $\mathbb{I}_n$. Therefore, either $\mathfrak{a} \nsubseteq \mathfrak{a}'$ or otherwise $\mathfrak{a} \subset \mathfrak{a}'$ (a proper inclusion since $\mathfrak{a} \neq \mathfrak{a}'$). Let

$$0 \to M \to N \to M' \to 0$$

be a short exact sequence of $\mathbb{I}_n$-modules. To finish the proof it suffices to show that the short exact sequence splits.

(i) *Suppose that* $\mathfrak{a} \nsubseteq \mathfrak{a}'$: Then $\mathfrak{a}M' \neq 0$ (since otherwise $\mathfrak{a}M' = 0$, and so $\mathfrak{a} \subseteq \mathrm{ann}_{\mathbb{I}_n}(M') = \mathfrak{a}'$, a contradiction). In particular, $\mathfrak{a}M' = M'$ since the $\mathbb{I}_n$-module $M'$ is simple. Now,

$$N \supseteq \mathfrak{a}N = \mathfrak{a}(N/M) = \mathfrak{a}M' = M'.$$

Therefore, the $\mathbb{I}_n$-submodule $\mathfrak{a}N$ of $N$ is isomorphic to the simple $\mathbb{I}_n$-module $M'$. Hence,

$$N \supseteq M + \mathfrak{a}N = M \oplus \mathfrak{a}N \simeq M \oplus M',$$

i.e., $N = M \oplus \mathfrak{a}N \simeq M \oplus M'$, i.e., the short exact sequence splits.

(ii) *Suppose that* $\mathfrak{a} \subset \mathfrak{a}'$: Let $\mathcal{D} = \mathcal{D}_{\mathcal{O}}$ and $\mathcal{D}' = \mathcal{D}'_{\mathcal{O}}$. By Lemma 3.1.(2),

$$\mathfrak{a} = \sum_{i \in C\mathcal{D}} \mathfrak{p}_i \subset \mathfrak{a}' = \sum_{j \in C\mathcal{D}'} \mathfrak{p}_j,$$

and so $C\mathcal{D} \subset C\mathcal{D}'$ or, equivalently, $\mathcal{D} \supset \mathcal{D}'$. Up to order, let

$$\mathbb{D}_{\mathcal{O}} = \{1, \dots, m\}, \;\; \mathcal{D} = \{1, \dots, l\} \coprod \mathcal{D}' \;\text{ and }\; \mathcal{D}' = \{l+1, \dots, k\}$$

provided $\mathcal{D}' \neq \emptyset$. Notice that $1 \leq l \leq m$ and $k \leq m$,

$$\mathcal{O} = \mathbb{Z}^m \times (\lambda_{m+1} + \mathbb{Z}) \times \cdots \times (\lambda_n + \mathbb{Z})$$

where $\lambda_i \notin \mathbb{Z}$ for all $i$ such that $m+1 \leq i \leq n$.

Clearly,

$$\mathrm{Supp}(M) = \mathbb{N}_+^k \times \mathbb{Z}^{m-k} \times \prod_{i=m+1}^{n} (\lambda_i + \mathbb{Z}),$$

$$\mathrm{Supp}(M') = \mathbb{Z}^l \times \mathbb{N}_+^{k-l} \times \mathbb{Z}^{m-k} \times \prod_{i=m+1}^{n} (\lambda_i + \mathbb{Z})$$

since

$$M = K[x_1, \dots, x_k] \otimes \bigotimes_{j=k+1}^{m} \mathbb{I}_1(j)/\mathbb{I}_1(j)H_j \otimes \bigotimes_{i=m+1}^{n} \mathbb{I}_1(i)/\mathbb{I}_1(i)(H_i - \lambda_i),$$

$$M' = \bigotimes_{s=1}^{l} \mathbb{I}_1(s)/\mathbb{I}_1(s)H_s \otimes K[x_{l+1}, \dots, x_k] \otimes \bigotimes_{j=k+1}^{m} \mathbb{I}_1(j)/\mathbb{I}_1(j)H_j \otimes \bigotimes_{i=m+1}^{n} \mathbb{I}_1(i)/\mathbb{I}_1(i)(H_i - \lambda_i).$$

Let $\lambda = (1, \dots, 1, \lambda_{m+1}, \dots, \lambda_n)$. Then $\dim_K(M^\lambda) = \dim_K(M'^\lambda) = 1$ and $\dim_K(N^\lambda) = 2$ since there is a short exact sequence

$$0 \to M^\lambda \to N^\lambda \to M'^\lambda \to 0$$

of $K$-modules. Fix an element $v \in N^\lambda \setminus M^\lambda$. Then $v \neq 0$. Let $\theta := (1-e)v$ where

$$e = e_{00}(1) = 1 - \int_1 \partial_1.$$

Then $\theta \equiv v \mod M$ since $eM' = 0$. In particular, $\theta \neq 0$.

(iii) $\int_1 \partial_1 \theta = \theta$: Notice that $\int_1 \partial_1 = 1 - e$ is an idempotent and the result follows:

$$\int_1 \partial_1 \theta = (1-e)\theta = (1-e)(1-e)v = (1-e)v = \theta.$$

(iv) $\partial_1 \theta \neq 0$: Since $\theta \neq 0$, the statement (iv) follows from the statement (iii).

(v) $H_1 \partial_1 \theta = 0$: Since $(H_1 - 1)M'^\lambda = 0$, we must have $(H_1 - 1)N^\lambda \subseteq M^\lambda$. Now,

$$H_1 \partial_1 \theta = \partial_1 (H_1 - 1)\theta \in \partial_1 (H_1 - 1) N^\lambda \subseteq \partial_1 M^\lambda = 0.$$

The $\mathbb{I}_1(1)$-module $\mathbb{I}_1(1)/\mathbb{I}_1(1)H_1$ is a simple weight module. By the statement (v), the $\mathbb{I}_1(1)$-submodule

$$L_1 = \mathbb{I}_1(1)\partial_1 \theta$$

of $N$ is isomorphic to the $\mathbb{I}_1(1)$-module $\mathbb{I}_1(1)/\mathbb{I}_1(1)H_1$. Recall that $\mathrm{End}_{\mathbb{I}_1(1)}(L_1) = K$ (Lemma 2.1.(3)) and $\mathbb{I}_n = \mathbb{I}_1 \otimes \mathbb{I}_{n-1}$. By [4], the $\mathbb{I}_n$-submodule $\mathbb{I}_n \partial_1 \theta$ of $N$ is isomorphic to the tensor product $L_1 \otimes L$ of the $\mathbb{I}_1$-module $L_1$ and an $\mathbb{I}_{n-1}$-module $L$.

(vi) $N = M \oplus L_1 \otimes L$: Since the map $\partial_1 \cdot : L_1 \otimes L \to L_1 \otimes L$, $u \mapsto \partial_1 u$ is a bijection (since the map $\partial_1 \cdot : L_1 \to L_1$, $w \mapsto \partial_1 w$ is so) and the map $\partial_1 \cdot : M \to M$, $p \mapsto \partial_1 p$ has nonzero kernel (since $\partial_1 M^\lambda = 0$ and $M^\lambda \neq 0$), the simple $\mathbb{I}_n$-module $M$ is not a submodule of $L_1 \otimes L$. Hence, $M \cap L_1 \otimes L = 0$, and so $M \oplus L_1 \otimes L \subseteq N$, and the statement (vi) follows since the length of the $\mathbb{I}_n$-module $N$ is 2.

Now, the proposition follows from the statement (vi). □

Corollary 3.4.

(1) *Every module $M \in \mathrm{GW}(\mathbb{I}_n, \mathcal{O})$ is a unique direct sum of absolutely prime generalized weight $\mathbb{I}_n$-modules, and this direct sum is $M = \bigoplus_{\mathcal{D}_\mathcal{O} \subseteq \mathbb{D}_\mathcal{O}} M_{\mathcal{D}_\mathcal{O}}$ where $M_{\mathcal{D}_\mathcal{O}} \in \mathrm{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$.*

(2) *Every generalized weight module is a unique direct sum of absolutely prime (generalized weight) $\mathbb{I}_n$-modules.*

*Proof.* 1. Statement 1 follows from Lemma 2.4 and Theorem 3.2.
2. Statement 2 follows from statement 1 and (9). □

The next proposition shows that there are plenty of indecomposable generalized weight $\mathbb{I}_n$-modules with support from a single orbit.

PROPOSITION 3.5. *Let $\mathcal{O} \in \mathcal{M}_n/G$ and $\mathfrak{m} \in \mathcal{O}$.*

(1) *If $\mathbb{D}_\mathcal{O} = \emptyset$ then*

$$\{V(I) \mid I \in \mathcal{I}(D_n, \mathfrak{m})\} \subseteq \mathrm{GW}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O})$$

*where $V(I) := B_n \otimes_{D_n} D_n/I$, and each $\mathbb{I}_n$-module $V(I)$ is an indecomposable, equidimensional, generalized weight $\mathbb{I}_n$-module of length $d := \dim_K(D_n/I) < \infty$ which is isomorphic to the $\mathbb{I}_n$-module*

$$B_n/B_nI = \bigoplus_{\alpha\in\mathbb{Z}^n} \partial^\alpha D_n/I$$

*(where $\partial^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$) and $\dim_K(V(I)^\mathfrak{n}) = d$ for all $\mathfrak{n} \in \mathcal{O}$, and $\mathrm{ann}_{\mathbb{I}_n}(V(I)) = \mathfrak{a}_n$.*

(2) *Suppose that $\mathbb{D}_\mathcal{O} = \{1, \dots, l\}$ (up to order) for some $l$ such that $1 \leq l \leq n$.*

(a) *Suppose that $\mathcal{D}_\mathcal{O} = \emptyset$. Let $\mathfrak{m} = (H_1, \dots, H_l, H_{l+1} - \lambda_{l+1}, \dots, H_n - \lambda_n) \in \mathrm{Supp}(M(\mathcal{D}_\mathcal{O}))$. Then*

$$\{V(I) \mid I \in \mathcal{I}(D_n, \mathfrak{m})\} \subseteq \mathrm{GW}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O})$$

*where $V(I) := B_n \otimes_{D_n} D_n/I$, and each $\mathbb{I}_n$-module $V(I)$ is an indecomposable, equidimensional, generalized weight $\mathbb{I}_n$-module of length $d = \dim_K(D_n/I) < \infty$ which is isomorphic to the $\mathbb{I}_n$-module*

$$B_n/B_nI = \bigoplus_{\alpha\in\mathbb{Z}^n} \partial^\alpha D_n/I$$

*and $\dim_K(V(I)^\mathfrak{n}) = d$ for all $\mathfrak{n} \in \mathrm{Supp}(V(I)) = \mathrm{Supp}(M(\mathcal{D}_\mathcal{O}))$, and $\mathrm{ann}_{\mathbb{I}_n}(V(I)) = \mathfrak{a}_n$.*

(b) *If $\mathcal{D}_\mathcal{O} \neq \emptyset$ then $\mathcal{D}_\mathcal{O} = \{1, \dots, m\}$, up to order, for some $m$ such that $1 \leq m \leq l$. Let $k = n - m$, $D_k = K[H_{m+1}, \dots, H_n]$ and $B_k = \otimes_{i=m+1}^n B_1(i)$. Then*

$$\{V(I) \mid I \in \mathcal{I}(D_k, \mathfrak{m}')\} \subseteq \mathrm{GW}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O})$$

*where $V(I) := P_m \otimes (B_k \otimes_{D_k} D_k/I)$,*

$$\mathfrak{m}' = (H_{m+1}, \dots, H_l, H_{l+1} - \lambda_{l+1}, \dots, H_n - \lambda_n) \in S(\mathcal{D}_\mathcal{O}),$$

*and each $\mathbb{I}_n$-module $V(I)$ is an indecomposable, equidimensional, generalized weight $\mathbb{I}_n$-module of length $d := \dim_K(D_k/I) < \infty$ which is isomorphic to the $\mathbb{I}_n$-module*

$$P_m \otimes B_k/B_kI \simeq \bigoplus_{\alpha\in\mathbb{Z}^k} P_m \otimes \partial^\alpha D_k/I$$

*(where $\partial^\alpha = \partial_{m+1}^{\alpha_1}\cdots\partial_n^{\alpha_k}$) and $\dim_K(V(I)^\mathfrak{n}) = d$ for all $\mathfrak{n} \in \mathrm{Supp}(V(I)) = \mathrm{Supp}(M(\mathcal{D}_\mathcal{O}))$, and $\mathrm{ann}_{\mathbb{I}_n}(V(I)) = \mathfrak{a}_n(\mathcal{D}_\mathcal{O})$.*

*Proof.* 1. Let $I \in \mathcal{I}(D_n, \mathfrak{m})$. As $B_n = \bigoplus_{\alpha\in\mathbb{Z}^n} \partial^\alpha D_n$, we have

$$V(I) = \bigoplus_{\alpha\in\mathbb{Z}^n} \partial^\alpha D_n \otimes_{D_n} D_n/I \simeq \bigoplus_{\alpha\in\mathbb{Z}^n} \partial^\alpha \otimes D_n/I = \bigoplus_{\alpha\in\mathbb{Z}^n} \partial^\alpha D_n/I.$$

So, the $\mathbb{I}_n$-module $V(I)$ is an equidimensional, generalized weight module with $\mathrm{Supp}(V(I)) = \mathcal{O}$ and

$$\dim_K(V(I)^{\mathfrak{n}}) = d = \dim_K(D_n/I) < \infty.$$

By Theorem 2.3, the simple (generalized) weight $\mathbb{I}_n$-module $M(\mathcal{D}_\mathcal{O})$ from $\mathrm{GW}(\mathbb{I}_\mathrm{n}, \mathcal{O})$ has support $\mathcal{O}$ and is 1-equidimensional. Hence, the length of $V(I)$ equals $d$.

It remains to show that the $\mathbb{I}_n$-module $V(I)$ is an indecomposable. The functor

$$B_n \otimes_{D_n} - : D_n\text{-Mod} \to B_n\text{-Mod}, \quad N \mapsto B_n \otimes_{D_n} N$$

is exact. The commutative algebra $D_n/I$ is a local, commutative, finite dimensional algebra with maximal ideal $\mathfrak{m}/I$. Since $(D_n/I)/(\mathfrak{m}/I) \simeq D/\mathfrak{m}$ is a field, the $D_n$-module $D_n/I$ is indecomposable. Hence, so is the induced module

$$V(I) = B_n \otimes_{D_n} D_n/I.$$

Clearly, $\mathfrak{a}_n \subseteq \mathrm{ann}_{\mathbb{I}_n}(V(I))$. Since $\mathfrak{a}_n$ is a maximal ideal of $\mathbb{I}_n$ and $V(I) \neq 0$, we must have $\mathrm{ann}_{\mathbb{I}_n}(V(I)) = \mathfrak{a}_n$.

2(a). Repeat the arguments of statement 1.

2(b). The functor

$$P_m \otimes - : \mathbb{I}_k\text{-Mod} \to \mathbb{I}_n\text{-Mod}, \quad L \mapsto P_m \otimes L$$

is an exact functor. Now, statement 2 follows from statement 2(a). □

**Explicit description of modules in** $\mathrm{GW}(\mathbb{I}_\mathrm{n}, \mathcal{O})$**.** In view of Theorem 3.2, Theorem 3.6 below is an explicit description of generalized weight $\mathbb{I}_n$-modules.

THEOREM 3.6. *Let* $\mathcal{O} \in \mathcal{M}_n/G$.

(1) *Suppose that* $\mathcal{D}_\mathcal{O} = \emptyset$, $\mathfrak{m} \in \mathcal{O}$ *if* $\mathbb{D}_\mathcal{O} = \emptyset$ *and* $\mathfrak{m} = (H_1, \dots, H_l, H_{l+1} - \lambda_{l+1}, \dots, H_n - \lambda_n) \in S(\mathcal{D}_\mathcal{O})$ *if* $\mathbb{D}_\mathcal{O} = \{1, \dots, l\} \neq \emptyset$, *up to order. Then the functor*

$$\mathrm{GW}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O}) \to \mathrm{LF}_\mathfrak{m}(D_n), \quad M \mapsto M^\mathfrak{m}$$

*is an equivalence of categories with the inverse* $N \mapsto B_n \otimes_{D_n} N$, *the induced functor.*

(2) *Suppose that* $\mathcal{D}_\mathcal{O} \neq \emptyset$ *and, up to order,* $\mathcal{D}_\mathcal{O} = \{1, \dots, m\}$, $\mathbb{D}_\mathcal{O} = \{1, \dots, l\}$ *for some* $m$ *such that* $1 \leq m \leq l \leq n$. *Then* $\mathbb{I}_n = \mathbb{I}_m \otimes \mathbb{I}_{n-m}$ *and* $\mathcal{O} = \mathbb{Z}^m \times \mathcal{O}'$ *where* $\mathcal{O}' = \mathbb{Z}^{l-m} \times \mathcal{O}_1 \times \cdots \times \mathcal{O}_{n-l}$ *and* $\mathcal{O}_i \neq \mathbb{Z}$ *for all* $i = 1, \dots, n-l$, *and*

  (a) $\mathrm{GW}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O}) = P_m \otimes \mathrm{GW}(\mathbb{I}_{n-m}, \mathcal{D}'_{\mathcal{O}'}) := \{P_m \otimes M \mid M \in \mathrm{GW}(\mathbb{I}_{n-m}, \mathcal{D}'_{\mathcal{O}'})\}$ *with* $\mathcal{D}'_{\mathcal{O}'} = \emptyset$.

  (b) *Fix* $\mathfrak{m} \in \mathcal{O}$ *such that* $\mathfrak{m} = (H_1 - 1, \dots, H_m - 1, H_{m+1}, \dots, H_l, H_{l+1} - \lambda_{l+1}, \dots, H_n - \lambda_n)$. *Then* $\mathfrak{m}' = (H_{m+1}, \dots, H_l, H_{l+1} - \lambda_{l+1}, \dots, H_n - \lambda_n) \in \mathcal{O}'$ *and the functor*

$$\mathrm{GW}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O}) \to \mathrm{LF}_{\mathfrak{m}'}(D_{n-m}), \quad P_m \otimes M \mapsto M^{\mathfrak{m}'}$$

  *is an equivalence of categories with the inverse* $N \mapsto P_m \otimes (B_{n-m} \otimes_{D_{n-m}} N)$ *where* $D_{n-m} = K[H_{m+1}, \dots, H_n]$ *and* $D_0 := K$.

*Proof.* 1. Let $M \in \mathrm{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$.

(i) *$M$ is a sum of modules $V(I)$ where $I \in \mathcal{I}(D_n, \mathfrak{m})$.* The statement follows from Proposition 3.5(1).

(ii) $\mathrm{ann}_{\mathbb{I}_n}(M) = \mathfrak{a}_n$, by Lemma 3.1 (since $\mathrm{ann}_{\mathbb{I}_n}(V(I)) = \mathfrak{a}_n$).

The statement (ii) means that $M$ is a $\mathbb{Z}^n$-graded $B_n$-module. By Proposition 3.5.(1), the functor

$$\mathrm{GW}(\mathbb{I}_n, \mathcal{O}) \to \mathrm{LF}_{\mathfrak{m}}(D_n), \quad M \mapsto M^{\mathfrak{m}}$$

is an equivalence of categories with the inverse $N \mapsto B_n \otimes_{D_n} N$.

2. Recall that $\mathfrak{m} = (H_1 - 1, \dots, H_m - 1, H_{m+1}, \dots, H_l, H_{l+1} - \lambda_{l+1}, \dots, H_n - \lambda_n)$. Let $M \in \mathrm{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$.

(i) *$M$ is a sum of modules $V(I)$ where $I \in \mathcal{I}(D_k, \mathfrak{m}')$.* The statement follows from Proposition 3.5(2). Since, for all $I \in \mathcal{I}(D_k, \mathfrak{m}')$,

$$V(I)^{\mathfrak{m}} = (P_m \otimes (B_k \otimes_{D_k} D_k/I))^{\mathfrak{m}} \simeq D_k/I,$$

the statement 2(a) follows from Proposition 3.5(2) and statement 1. Now, the statement 2(b) follows from the statement 2(a) and statement 1. □

Corollary 3.7. *Let $\mathcal{O} \in \mathcal{M}_n/G$. All modules in $\mathrm{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$ and $\mathrm{W}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$ are equidimensional and the length of the module is equal to the dimension of any (generalized) weight component. In particular, all indecomposable generalized weight $\mathbb{I}_n$-modules are equidimensional.*

*Proof.* The corollary follows from Theorem 3.6. □

The next corollary is a criterion for a generalized weight $\mathbb{I}_n$-modules to be finitely generated.

Corollary 3.8. *Let $M$ be a generalized weight $\mathbb{I}_n$-module. The $\mathbb{I}_n$-module $M$ is finitely generated iff its support is a subset of a union of finitely many orbits in $\mathcal{M}_n/G$ and there is $s \in \mathbb{N}_+$ such that the dimensions of all generalized weight components are bounded by $s$.*

*Proof.* The corollary follows from Theorem 3.6 and Proposition 3.5. □

**Criterion for the category $\mathrm{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$ to be of finite representation type, tame or wild.** Let $A$ be an algebra and $M$ be an $A$-module. We denote by $[M]$ the isomorphism class of the $A$-module $M$ and $A - \mathrm{Mod}/\simeq$ is the set of all the isomorphism classes of $A$-modules. In particular, $\mathrm{LF}_{\mathfrak{m}}(D_n)/\simeq$ is the set of isomorphism classes of $D_n$-modules in $\mathrm{LF}_{\mathfrak{m}}(D_n)$. A category of modules is called a *category of finite representation type* if it contains only finitely many indecomposable modules up to isomorphism. The reader can find the definition of tame and wild category in [21]. Notice that every category of finite representation type is tame but not vice versa.

Theorem 3.9. *Let $\mathcal{O} \in \mathcal{M}_n/G$.*

(1) *The category $\mathrm{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$ is of finite representation type iff $\mathcal{O} = \mathbb{Z}^n$ and $\mathcal{D}_{\mathcal{O}} = \{1, \dots, n\}$, and in this case the simple $\mathbb{I}_n$-module $P_n = K[x_1, \dots, x_n]$ is the unique indecomposable $\mathbb{I}_n$-module in the category $\mathrm{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$.*

(2) *The category* $\mathrm{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$ *is tame iff* $|\mathcal{D}_{\mathcal{O}}| = n-1$, *and in this case, up to order,* $\mathcal{O} = \mathbb{Z}^{n-1} \times (\lambda_n + \mathbb{Z})$ *for some fixed* $\lambda_n = \lambda_n(\mathcal{O}) \in K$, $\mathcal{D}_{\mathcal{O}} = \{1, \dots, n-1\}$ *and*

$$\{P_{n-1} \otimes M(i, \lambda) \,|\, i \in \mathbb{N}_+ \text{ where } \lambda = \lambda_n(\mathcal{O}) \text{ if } \lambda_n \notin \mathbb{Z} \text{ and } \lambda = 0 \text{ if } \lambda_n \in \mathbb{Z}\}$$

*is the set of all indecomposable, pairwise non-isomorphic modules in* $\mathrm{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$.

(3) *The category* $\mathrm{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$ *is wild iff* $n \geq 2$ *and* $m := |\mathcal{D}_{\mathcal{O}}| < n-1$, *and in this case, up to order,* $\mathcal{D}_{\mathcal{O}} = \{1, \dots, m\}$ *where* $1 \leq m < n-1$, *and*

$$\{P_m \otimes (B_{n-m} \otimes_{D_{n-m}} N) \,|\, [N] \in \mathrm{LF}_{\mathfrak{m}'}(D_{n-m})/\simeq \text{ and } N \text{ is an indecomposable } D_{n-m}-\text{module}\}$$

*is a set of indecomposable, pairwise non-isomorphic modules in* $\mathrm{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$ *where* $D_{n-m} = K[H_{m+1}, \dots, H_n]$ *and* $D_0 = K$.

*Proof.* If $n \geq 2$ and $|\mathcal{D}_{\mathcal{O}}| < n-1$ then by Theorem 3.6 and [20] (see, also [31]), the category $\mathrm{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$ is wild. If either $n = 1$ or $n \geq 2$ and $|\mathcal{D}_{\mathcal{O}}| \geq n-1$ then by Theorem 3.6 and [20] (see, also [31]), the category $\mathrm{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$ is tame. Clearly, the category $\mathrm{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$ is of finite representation type iff $|\mathcal{D}_{\mathcal{O}}| = n$. □

**Criterion for the category $\mathrm{GW}(\mathbb{I}_n, \mathcal{O})$ to be of finite representation type, tame or wild.**

COROLLARY 3.10. *Let* $\mathcal{O} \in \mathcal{M}/G$.
(1) *The category* $\mathrm{GW}(\mathbb{I}_n, \mathcal{O})$ *is tame iff* $n = 1$.
(2) *The category* $\mathrm{GW}(\mathbb{I}_n, \mathcal{O})$ *is wild iff* $n \geq 2$.
(3) *None of the categories* $\mathrm{GW}(\mathbb{I}_n, \mathcal{O})$ *is of finite representation type.*

*Proof.* The corollary follows from Theorem 3.9 and Eq. (19). □

**Explicit classes of indecomposable $\mathbb{I}_n$-modules in $\mathrm{GW}(\mathbb{I}_n, \mathcal{O})$.** By Theorem 3.6, the problem of classifying indecomposable generalized weight $\mathbb{I}_n$-modules in $\mathrm{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$ is equivalent to the problem of classifying indecomposable modules in $\mathrm{LF}_{\mathfrak{m}}(D_{n'})$ for some $n' \leq n$. The set $\mathrm{ind.LF}_{\mathfrak{m}}(D_n)$ of isomorphism classes of indecomposable modules in $\mathrm{LF}_{\mathfrak{m}}(D_n)$ is the union

$$\mathrm{ind.LF}_{\mathfrak{m}}(D_n) = \bigcup_{i \geq 1} \mathrm{ind}(D_n, \mathfrak{m}^i)$$

where the set $\mathrm{ind}(D_n, \mathfrak{m}^i)$ contains the isomorphism classes of all the indecomposable $D_n$-modules $M$ with $\mathfrak{m}^i M = 0$. Clearly,

$$\mathrm{ind}(D_n, \mathfrak{m}) = \{D_n/\mathfrak{m}\} \subseteq \mathrm{ind}(D_n, \mathfrak{m}^2) \subseteq \dots.$$

By Theorem of Drozd, see [20],

- $\mathrm{ind}(D_n, \mathfrak{m}^i)$ *is tame iff either* $n = 1$ *or* $n = 2$ *and* $m = 1, 2$.

**Description of the set $\mathrm{ind}(D_2, \mathfrak{m}^2)$.** Let

$$L = D_2/\mathfrak{m}^2, \ \mathfrak{m} = (h_1, h_2) \text{ where } h_1 = H_1 - \lambda_1 \text{ and } h_2 = H_2 - \lambda_2$$

for some $\lambda_1, \lambda_2 \in K$ and $M \in \mathrm{ind}(D_2, \mathfrak{m}^2)$. Then $M = M_1 \oplus M_2$ where $M_2 = \mathfrak{m}M$ is a $L$-module and $M_1$ is any (fixed) complement subspace of the vector space $M_2$.

Clearly, $M = 0$ iff $M_2 = M$ iff $M_1 = 0$. The $L$-module structure on $M$ is uniquely determined by the linear maps

$$M_1 \overset{h_1}{\underset{h_2}{\rightrightarrows}} M_2\,, \quad m_1 \mapsto h_1 m_1, \quad m_1 \mapsto h_2 m_1 \quad \text{(where } m_1 \in M_1\text{)}.$$

So, the problem of describing the set $\mathrm{ind}(D_2, \mathfrak{m}^2)$ is 'almost' equivalent to the problem of classifying indecomposable finite dimensional representations of the *Kronecker quiver*:

$$1 \overset{h_1}{\underset{h_2}{\rightrightarrows}} 2\,.$$

More precisely, every indecomposable finite dimensional representation of the Kronecker quiver $(M_1, M_2)$ such that $M_1 \neq 0$ belongs to $\mathrm{ind}(D_2, \mathfrak{m}^2)$, and vice versa. Up to isomorphism, there are the following 5 series of indecomposable modules (in bracket bases of the vector spaces $M_1$ and $M_2$ are given):

(1) $K = D_2/\mathfrak{m}$.
(2) For each $n \geq 1$, $M_1 = \langle e_1, \ldots, e_n \rangle$, $M_2 = \langle e'_1, \ldots, e'_{n+1} \rangle$, $h_1 e_i = e'_i$ and $h_2 e_i = e'_{i+1}$ for $i = 1, \ldots, n$.
(3) For each $n \geq 1$, $M_1 = \langle e_1, \ldots, e_{n+1} \rangle$, $M_2 = \langle e'_1, \ldots, e'_n \rangle$, $h_1 e_i = e'_i$ and $h_2 e_{i+1} = e'_i$ for $i = 1, \ldots, n$, $h_1 e_{n+1} = 0$ and $h_2 e_1 = 0$.
(4) For each $n \geq 1$, $M_1 = \langle e_1, \ldots, e_n \rangle$, $M_2 = \langle e'_1, \ldots, e'_n \rangle$, $h_1 e_i = e'_i$ and $(h_2 - \lambda) e_i = e'_{i-1}$ for $i = 1, \ldots, n$ where $e'_0 = 0$ and $\lambda \in K$.
(5) For each $n \geq 1$, $M_1 = \langle e_1, \ldots, e_n \rangle$, $M_2 = \langle e'_1, \ldots, e'_n \rangle$, $h_1 e_i = e'_{i-1}$ and $h_2 e_i = e'_i$ for $i = 1, \ldots, n$ where $e'_0 = 0$.

The set $\mathrm{ind.LF}_{\mathfrak{m}}(D_n)$ is a disjoint union of subsets,

$$\mathrm{ind.LF}_{\mathfrak{m}}(D_n) = \bigsqcup_{I \in \mathcal{I}(D_n, \mathfrak{m})} \mathrm{ind.LF}_{\mathfrak{m}}(D_n, I) \tag{20}$$

where the set $\mathrm{ind.LF}_{\mathfrak{m}}(D_n, I)$ contains all the indecomposable modules $M \in \mathrm{ind.LF}_{\mathfrak{m}}(D_n)$ with $\mathrm{ann}_{D_n}(M) = I$. By Theorem of Drozd, see [20],

- $\mathrm{ind}(D_n, I)$ *is tame iff either* $n = 1$ *or* $n = 2$ *and* $I$ *contains a product* $h'_1 h'_2$ *of elements* $h'_1, h'_2 \in \mathfrak{m}$ *such that their images in the* $K$*-vector space* $\mathfrak{m}/\mathfrak{m}^2$ *are* $K$*-linearly independent (equivalently, are a basis).*

Let $\Gamma = D_2/(h_1 h_2) = K[h_1, h_2]/(h_1 h_2)$. In the second case (i.e., $n = 2$), the elements $h'_1$ and $h'_2$ are $K$-algebra generators for the algebra $\Gamma$. So, up to change of algebra generators, we can assume that $h_1 h_2 \in I$. Then

$$\mathrm{ind.LF}_{\mathfrak{m}}(D_2, I) = \{ M \in \mathrm{ind}_f(\Gamma) \mid IM = 0 \}$$

where $\mathrm{ind}_f(\Gamma)$ is the set of isomorphism classes of indecomposable finite dimensional left $\Gamma$-modules.

**Description of** $\mathrm{ind}_f(\Gamma)$**.** Let $W = \langle h_1, h_2\rangle$ be a free (noncommutative) semigroup. Each element (word) $w \in W$ is a unique product $w_1 \cdots w_l$ where $w_i \in \{h_1, h_2\}$ and $l = 1, 2, \ldots$. The number $l = l(w)$ is called the *length* of the word $w$ and

$$W = \sqcup_{l\geq 1} W_l,$$

a disjoint union, where $W_l$ is the set of all words of length $l$. The cyclic group of order $l$,

$$C_l = \langle \tau_l\rangle = \{\tau_l^i \mid i = 0, \ldots, l-1\}, \text{ where } \tau_l = (12\ldots l),$$

acts on the set $W_l$ by the rule $\tau_l(w_1 \cdots w_l) = w_{\tau_l(1)} \ldots w_{\tau_l(l)}$. Let $W_l/C_l$ be the set of orbits. We say that two elements $w$ and $w'$ of $W$ are *equivalent*, $w \sim w'$, if they belong to the same orbit ($w \sim w'$ iff $l(w) = l(w')$ and $w = \tau_l^i(w')$ for some $i$ where $l = l(w)$). An orbit

$$\mathbb{O} \in W_l/C_l$$

is called a *periodic* orbit if it contains an element $w$ such that $w = \theta^i$ for some $\theta \in W$ and $i \geq 2$. We denote by $\mathcal{N}$ the set of all *non-periodic orbits*. The simple module $K = \Gamma/(h_1, h_2)$ belongs to $\mathrm{ind}_f(\Gamma)$. The set of *non-simple* indecomposable finite dimensional $\Gamma$-modules consists of two sets of modules: the modules of the first and second type, see [24]:

$$\mathrm{ind}_f(\Gamma) \setminus \{K\} = \mathrm{ind}_1(\Gamma) \bigsqcup \mathrm{ind}_2(\Gamma) \qquad (21)$$

where

(1) $\mathrm{ind}_1(\Gamma) = \{M_w \mid w \in W\}$ and $M_w = \langle e_1, e_2, \ldots, e_{l+1}\rangle$ where $l = l(w)$, $w = w_1 \ldots w_l$ and $w_i \in \{h_1, h_2\}$,

$$h_1 e_i = \begin{cases} e_{i+1} & \text{if } w_i = h_1, \\ 0 & \text{otherwise,} \end{cases} \qquad h_2 e_i = \begin{cases} e_{i-1} & \text{if } w_{i-1} = h_2, i \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

(2) $\mathrm{ind}_2(\Gamma) = \{N(\mathbb{O}, n, \lambda) \mid \mathbb{O} \in \mathcal{N}, n \in \mathbb{N}_+, \lambda \in K^*\}$. Let $w = w_1 \cdots w_l \in \mathbb{O}$ where $l = l(w)$. Then

$$N(\mathbb{O}, n, \lambda) = \bigoplus_{i\in\mathbb{Z}/l\mathbb{Z}} N_i$$

is a direct sum of $n$-dimensional vector spaces $N_i = K^n$ and the action of the elements $h_1$ and $h_2$ is given below. Schematically, it can be represented by the following diagram

$$N_1 \xrightarrow{\text{id}} N_2 \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} N_l, \qquad N_l \xrightarrow{J_n(\lambda)} N_1$$

$$h_1|_{N_i} : N_i \to N_{i+1}, \quad h_1|_{N_i} = \begin{cases} \text{id} & \text{if } i \neq l, w_i = h_1, \\ 0 & \text{if } i \neq l, w_i = h_2, \\ J_n(\lambda) & \text{if } i = l, w_l = h_1, \\ 0 & \text{if } i = l, w_l = h_2, \end{cases}$$

$$h_2|_{N_i} : N_i \to N_{i-1}, \quad h_2|_{N_i} = \begin{cases} 0 & \text{if } i \neq l, w_i = h_1, \\ \text{id} & \text{if } i \neq l, w_i = h_2, \\ 0 & \text{if } i = l, w_l = h_1, \\ J_n(\lambda) & \text{if } i = l, w_l = h_2, \end{cases}$$

where $J_n(\lambda)$ is the $n \times n$ Jordan block with diagonal elements $\lambda$.
Up to isomorphism, the module $N(\mathbb{O}, n, \lambda)$ does not depend on the choice of the representative $w$ of the orbit $\mathbb{O}$.

**Description of** $\text{ind}_f(A)$ **where** $A = K[h_1, h_2]/(h_1^2, h_2^2)$**.** The field $K$ is an algebraically closed field of characteristic zero. Let $i = \sqrt{-1}$, $h_1' = h_1 + ih_2$ and $h_2' = h_1 - ih_2$. Then $h_1'h_2' = h_1^2 + h_2^2 \in (h_1^2, h_2^2)$, and so $A$ is tame, by Theorem of Drozd, see [20]. Let $C = K[h_1, h_2]/(h_1^2, h_1h_2, h_2^2)$.

LEMMA 3.11. $\text{ind}_f(A) = \text{ind}_f(C) \cup \{_AA\}$.

*Proof.* (i) $_AA$ is *indecomposable* (since $A$ is local).

(ii) $_AA$ is *an injective module*: straightforward.

(iii) *Any finite dimensional $A$-module $M$ such that $\mathfrak{m}^2M \neq 0$ contain $_AA$ where* $\mathfrak{m} = (h_1, h_2)$: Since $\mathfrak{m}^2M \neq 0$, we can find a nonzero element $a \in M$ such that $\mathfrak{m}^2a \neq 0$. Then $Aa \simeq_A A$, as required (since $\mathfrak{m}^2 = (h_1h_2)$).

(iv) $\text{ind}_f(A) = \text{ind}_f(C) \cup \{_AA\}$: Let $M \in \text{ind}_f(A)$. If $\mathfrak{m}^2M \neq 0$ then $M \simeq_A A$, by the statement (iii). If $\mathfrak{m}^2M = 0$ then $M \in \text{ind}_f(C)$ since $C = A/\mathfrak{m}^2$. □

**4. Generalized weight right $\mathbb{I}_n$-modules.** In this section, a classification of simple (generalized) weight right $\mathbb{I}_n$-modules is given (Theorem 4.2). The category of weight right $\mathbb{I}_n$-modules is a semisimple category (Theorem 4.3). An explicit description of generalized weight $\mathbb{I}_n$-modules is given (Theorem 4.4).

The algebra $\mathbb{I}_n$ admits an involution $*$ given by the rule, see [12]: For $i = 1, \ldots, n$,

$$\partial_i^* = \int_i, \quad \int_i^* = \partial_i \text{ and } H_i^* = H_i.$$

Recall that an involution $*$ on $\mathbb{I}_n$ is a $K$-algebra *anti-isomorphism* of $\mathbb{I}_n$ ($(ab)^* = b^*a^*$) such that $a^{**} = a$ for all elements $a \in \mathbb{I}_n$. Clearly, the involution $*$ above acts as the identity map on the algebra $D_n$.

Every left $\mathbb{I}_n$-module $M$ can be seen as a right $\mathbb{I}_n$-module $M^*$ where $M^* = M$, equality of vector spaces, and the right $\mathbb{I}_n$-module structure on $M$ is given by the rule: For all $m \in M$ and $a \in \mathbb{I}_n$, $ma := a^*m$. Similarly, every right $\mathbb{I}_n$-module $N$ can be seen as a left $\mathbb{I}_n$-module $N^*$ where $N^* = N$, equality of vector spaces, and, for all $n \in N$ and $a \in \mathbb{I}_n$, $an := na^*$. The functor

$$\mathbb{I}_n\text{-Mod} \to \text{Mod-}\mathbb{I}_n, \quad M \mapsto M^*$$

is an equivalence of categories with the inverse $N \mapsto N^*$. Clearly, $M^{**} = M$ and $N^{**} = N$.

EXAMPLE 4.1. *Recall that the polynomial algebra $P_n$ is a left $\mathbb{I}_n$-module isomorphic to the factor module $\mathbb{I}_n/\mathbb{I}_n(\partial_1, \ldots, \partial_n) \simeq K[\int_1, \ldots, \int_n]\bar{1}$ where* $\bar{1} = 1 +$

$\mathbb{I}_n(\partial_1, \ldots, \partial_n)$ *(since for all* $\alpha \in \mathbb{N}^n$, $(\alpha!)^{-1} \int^\alpha 1 = x^\alpha$*). Hence,*

$$\begin{aligned}(P_n^*)_{\mathbb{I}_n} &\simeq \mathbb{I}_n/(\partial_1^*, \ldots, \partial_n^*)\mathbb{I}_n \simeq \mathbb{I}_n/(\int_1, \ldots, \int_n)\mathbb{I}_n \\ &\simeq \tilde{1}K[\int_1^*, \ldots, \int_n^*] \simeq \tilde{1}K[\partial_1, \ldots, \partial_n] = \tilde{1}\mathcal{D}_n \simeq \mathcal{D}_n \quad (22)\end{aligned}$$

*where* $\tilde{1} = 1 + (\int_1, \ldots, \int_n)\mathbb{I}_n$ *and* $\mathcal{D}_n = K[\partial_1, \ldots, \partial_n]$ *is a polynomial algebra. The algebra* $\mathcal{D}_n$ *is a maximal commutative subalgebra of* $\mathbb{I}_n$*. Let* $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ *be the standard basis of the free abelian group* $\mathbb{Z}^n = \oplus_{i=1}^n \mathbb{Z}e_i$*. The right* $\mathbb{I}_n$*-module* $\mathcal{D}_n = (P_n^*)_{\mathbb{I}_n}$ *is simple (since* ${}_{\mathbb{I}_n}P_n$ *is simple) and* $\{\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$ *is a* $K$*-basis of* $\mathcal{D}_n$*. The right action of the generators* $H_i, \partial_i, \int_i$ *(*$i = 1, \ldots, n$*) of the algebra* $\mathbb{I}_n$ *on* $\partial^\alpha$ *are given below:*

$$\partial^\alpha H_i = \partial^\alpha(\alpha_i + 1), \quad \partial^\alpha \partial_i = \partial^{\alpha+e_i} \text{ and } \partial^\alpha \int_i = \begin{cases} \partial^{\alpha-e_i} & \text{if } \alpha_i \geq 1, \\ 0 & \text{if } \alpha_i = 0. \end{cases}$$

The definition of generalized weight right $\mathbb{I}_n$-modules is given in the same way as their left counterparts. We add the subscript 'r' to all the notation introduced for generalized weight left modules to indicate that we deal with *right* modules.

Since the involution $*$ acts as the identity map on the polynomial algebra $D_n = K[H_1, \ldots, H_n]$, we have, for each orbit $\mathcal{O} \in \mathcal{M}_n/G$,

$$\mathrm{W}(\mathbb{I}_\mathrm{n}, \mathcal{O})^* = \mathrm{W}_\mathrm{r}(\mathbb{I}_\mathrm{n}, \mathcal{O}), \quad \mathrm{GW}(\mathbb{I}_\mathrm{n}, \mathcal{O})^* = \mathrm{GW}_\mathrm{r}(\mathbb{I}_\mathrm{n}, \mathcal{O}), \quad (23)$$

$$\mathrm{W}_\mathrm{r}(\mathbb{I}_\mathrm{n}, \mathcal{O})^* = \mathrm{W}(\mathbb{I}_\mathrm{n}, \mathcal{O}), \quad \mathrm{GW}_\mathrm{r}(\mathbb{I}_\mathrm{n}, \mathcal{O})^* = \mathrm{GW}(\mathbb{I}_\mathrm{n}, \mathcal{O}), \quad (24)$$

$$\mathrm{W}_\mathrm{r}(\mathbb{I}_\mathrm{n}) = \bigoplus_{\mathcal{O} \in \mathcal{M}_n/G} \mathrm{W}_\mathrm{r}(\mathbb{I}_\mathrm{n}, \mathcal{O}) \text{ and } \mathrm{GW}_\mathrm{r}(\mathbb{I}_\mathrm{n}) = \bigoplus_{\mathcal{O} \in \mathcal{M}_n/G} \mathrm{GW}_\mathrm{r}(\mathbb{I}_\mathrm{n}, \mathcal{O}). \quad (25)$$

So, for each $M \in \mathrm{GW}(\mathbb{I}_\mathrm{n}, \mathcal{O})$, $\mathrm{Supp}(M^*) = \mathrm{Supp}(M)$ and $(M^*)^\mathfrak{m} = M^\mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{Supp}(M)$.

For each $\mathcal{D}_\mathcal{O}, M(\mathcal{D}_\mathcal{O})_r := M(\mathcal{D}_\mathcal{O})^*$ is a simple right $\mathbb{I}_n$-module with

$$\mathrm{Supp}(M(\mathcal{D}_\mathcal{O})_r) = \mathrm{Supp}(M(\mathcal{D}_\mathcal{O})) = S(\mathcal{D}_\mathcal{O}),$$

see (17). Then $\mathrm{GW}_\mathrm{r}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O}) := \mathrm{GW}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O})^*$ is the full subcategory of $\mathrm{GW}_\mathrm{r}(\mathbb{I}_\mathrm{n}, \mathcal{O})$ generated by the simple right weight $\mathbb{I}_n$-module $M(\mathcal{D}_\mathcal{O})_r$. There are precisely $2^{|\mathbb{D}_\mathcal{O}|}$ such subcategories in the category $\mathrm{GW}_\mathrm{r}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O})$, and

$$\mathrm{GW}_\mathrm{r}(\mathbb{I}_\mathrm{n}, \mathcal{O}) = \bigoplus_{\mathcal{D}_\mathcal{O} \subseteq \mathbb{D}_\mathcal{O}} \mathrm{GW}_\mathrm{r}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O}), \quad (26)$$

by Theorem 3.2 (apply $^*$ to (19)).

**Description of simple weight modules.** We denote by $\widehat{\mathbb{I}}_n(\text{weight})_r$ (resp., $\widehat{\mathbb{I}}_n(\text{gen. weight})_r$) the set of isomorphism classes of simple weight right (resp., generalized weight right) $\mathbb{I}_n$-modules. The next theorem classifies (up to isomorphism) all the simple weight right $\mathbb{I}_n$-modules.

THEOREM 4.2.
(1)

$$\widehat{\mathbb{I}}_n(\textit{gen. weight})_r = \widehat{\mathbb{I}}_n(\textit{weight})_r = \widehat{\mathbb{I}}_1(\textit{weight})_r^{\otimes n},$$

*i.e., any simple generalized weight right $\mathbb{I}_n$-module is a simple weight right $\mathbb{I}_n$-module, and vice versa; any simple weight right $\mathbb{I}_n$-module $M$ is isomorphic to the tensor product $M_1 \otimes \cdots \otimes M_n$ of simple weight right $\mathbb{I}_1$-modules and two such modules are isomorphic over $\mathbb{I}_n$,*

$$M_1 \otimes \cdots \otimes M_n \simeq M_1' \otimes \cdots \otimes M_n',$$

*iff for each $i = 1, \ldots, n$, the $\mathbb{I}_1$-modules $M_i$ and $M_i'$ are isomorphic. Furthermore,*

$$\widehat{\mathbb{I}}_1(\textit{weight})_r = \{P_1^* \simeq \mathbb{I}_1/\int \mathbb{I}_1 \simeq K[\partial],\ M(1,\lambda)^* \simeq B_1/(H-\lambda)B_1 \,|\, \lambda \in K\},$$

$\mathrm{Supp}(K[\partial]) = \mathbb{N}_+$ *and* $\mathrm{Supp}(M(1,\lambda)^*) = \lambda + \mathbb{Z}$.

(2) *For each simple weight right $\mathbb{I}_n$-module $M = \bigotimes_{i=1}^n M_i$, $\mathrm{Supp}(M) = \prod_{i=1}^n \mathrm{Supp}(M_i)$.*

*Proof.* The theorem follows at once from Theorem 2.3, (23), (24) and (25). □

THEOREM 4.3. *Every weight right $\mathbb{I}_n$-module is a direct sum of simple weight right $\mathbb{I}_n$-modules. In particular, the category $\mathrm{W_r}(\mathbb{I}_n)$ of weight right $\mathbb{I}_n$-modules is a semisimple category.*

*Proof.* The theorem follows from Theorem 2.5, (23), (24) and (25). □

**Explicit description of modules in $\mathrm{GW_r}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$.**

THEOREM 4.4. *Let $\mathcal{O} \in \mathcal{M}_n/G$.*

(1) *Suppose that $\mathcal{D}_\mathcal{O} = \emptyset$, $\mathfrak{m} \in \mathcal{O}$ if $\mathbb{D}_\mathcal{O} = \emptyset$ and $\mathfrak{m} = (H_1, \ldots, H_l, H_{l+1} - \lambda_{l+1}, \ldots, H_n - \lambda_n) \in S(\mathcal{D}_\mathcal{O})$ if $\mathbb{D}_\mathcal{O} = \{1, \ldots, l\} \neq \emptyset$, up to order. Then the functor*

$$\mathrm{GW_r}(\mathbb{I}_n, \mathcal{D}_\mathcal{O}) \to \mathrm{LF}_\mathfrak{m}(D_n), \quad M \mapsto M^\mathfrak{m}$$

*is an equivalence of categories with the inverse $N \mapsto N \otimes_{D_n} B_n$, the induced functor.*

(2) *Suppose that $\mathcal{D}_\mathcal{O} \neq \emptyset$ and, up to order, $\mathcal{D}_\mathcal{O} = \{1, \ldots, m\}$, $\mathbb{D}_\mathcal{O} = \{1, \ldots, l\}$ for some $m$ such that $1 \leq m \leq l \leq n$. Then $\mathbb{I}_n = \mathbb{I}_m \otimes \mathbb{I}_{n-m}$ and $\mathcal{O} = \mathbb{Z}^m \times \mathcal{O}'$ where $\mathcal{O}' = \mathbb{Z}^{l-m} \times \mathcal{O}_1 \times \cdots \times \mathcal{O}_{n-l}$ and $\mathcal{O}_i \neq \mathbb{Z}$ for all $i = 1, \ldots, n-l$, and*
   (a) $\mathrm{GW_r}(\mathbb{I}_n, \mathcal{D}_\mathcal{O}) = P_m^* \otimes \mathrm{GW_r}(\mathbb{I}_{n-m}, \mathcal{D}'_{\mathcal{O}'}) := \{P_m^* \otimes M^* \,|\, M \in \mathrm{GW}(\mathbb{I}_{n-m}, \mathcal{D}'_{\mathcal{O}'})\}$ *with* $\mathcal{D}'_{\mathcal{O}'} = \emptyset$.
   (b) *Fix $\mathfrak{m} \in \mathcal{O}$ such that $\mathfrak{m} = (H_1 - 1, \ldots, H_m - 1, H_{m+1}, \ldots, H_l, H_{l+1} - \lambda_{l+1}, \ldots, H_n - \lambda_n)$. Then $\mathfrak{m}' = (H_{m+1}, \ldots, H_l, H_{l+1} - \lambda_{l+1}, \ldots, H_n - \lambda_n) \in \mathcal{O}'$ and the functor*

$$\mathrm{GW_r}(\mathbb{I}_n, \mathcal{D}_\mathcal{O}) \to \mathrm{LF}_{\mathfrak{m}'}(D_{n-m}), \quad P_m^* \otimes M^* \mapsto M^{\mathfrak{m}'}$$

*is an equivalence of categories with the inverse* $N \mapsto P_m^* \otimes (N \otimes_{D_{n-m}} B_{n-m})$ *where* $D_{n-m} = K[H_{m+1}, \dots, H_n]$ *and* $D_0 := K$.

*Proof.* The theorem follows from Theorem 3.6 by applying $^*$. □

**Criterion for the category** $\mathrm{GW_r}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O})$ **to be of finite representation type, tame or wild.** Theorem 4.5 is such a criterion.

THEOREM 4.5. *Let* $\mathcal{O} \in \mathcal{M}_n/G$.

(1) *The category* $\mathrm{GW_r}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O})$ *is of finite representation type iff* $\mathcal{O} = \mathbb{Z}^n$ *and* $\mathcal{D}_\mathcal{O} = \{1, \dots, n\}$, *and in this case the simple right* $\mathbb{I}_n$*-module* $P_n^*$ *is the unique indecomposable right* $\mathbb{I}_n$*-module in the category* $\mathrm{GW_r}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O})$.
(2) *The category* $\mathrm{GW_r}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O})$ *is tame iff* $|\mathcal{D}_\mathcal{O}| = n-1$, *and in this case, up to order,* $\mathcal{O} = \mathbb{Z}^{n-1} \times (\lambda_n + \mathbb{Z})$ *for some fixed* $\lambda_n = \lambda_n(\mathcal{O}) \in K$, $\mathcal{D}_\mathcal{O} = \{1, \dots, n-1\}$ *and*
$$\{P_{n-1}^* \otimes M(i,\lambda)^* \,|\, i \in \mathbb{N}_+, \text{ where } \lambda = \lambda_n \text{ if } \lambda_n \notin \mathbb{Z} \text{ and } \lambda = 0 \text{ if } \lambda_n \in \mathbb{Z}\}$$
*is the set of all indecomposable, pairwise non-isomorphic modules in* $\mathrm{GW_r}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O})$.
(3) *The category* $\mathrm{GW_r}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O})$ *is wild iff* $n \geq 2$ *and* $m := |\mathcal{D}_\mathcal{O}| < n-1$, *and in this case, up to order,* $\mathcal{D}_\mathcal{O} = \{1, \dots, m\}$ *where* $1 \leq m < n-1$, *and*
$$\{P_m^* \otimes (N \otimes_{D_{n-m}} B_{n-m}) \,|\, [N] \in \mathrm{LF}_{\mathfrak{m}'}(D_{n-m})/\simeq \text{ and } N \text{ is an indecomposable } D_{n-m} - \text{module}\}$$
*is a set of indecomposable, pairwise non-isomorphic modules in* $\mathrm{GW_r}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O})$ *where* $D_{n-m} = K[H_{m+1}, \dots, H_n]$ *and* $D_0 = K$.

*Proof.* The theorem follows from Theorem 3.9 by applying $^*$. □

COROLLARY 4.6. *Let* $\mathcal{O} \in \mathcal{M}_n/G$. *All modules in* $\mathrm{GW_r}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O})$ *and* $\mathrm{W_r}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O})$ *are equidimensional and the length of the module is equal the dimension of any of (generalized) weight component. In particular, all indecomposable right generalized weight* $\mathbb{I}_n$*-modules are equidimensional.*

*Proof.* The corollary follows from Corollary 3.7. □

The next corollary is a criterion for a generalized weight right $\mathbb{I}_n$-modules to be finitely generated.

COROLLARY 4.7. *Let* $M$ *be a generalized weight right* $\mathbb{I}_n$*-module. The* $\mathbb{I}_n$*-module* $M$ *is finitely generated iff its support is a subset of a union of finitely many orbits in* $\mathcal{M}_n/G$ *and there is* $s \in \mathbb{N}_+$ *such that the dimensions of all generalized weight components are bounded by* $s$.

*Proof.* The corollary follows from Corollary 3.8. □

COROLLARY 4.8.

(1) *Every module* $M \in \mathrm{GW_r}(\mathbb{I}_\mathrm{n}, \mathcal{O})$ *is a unique direct sum of absolutely prime generalized weight right* $\mathbb{I}_n$*-modules, and this direct sum is* $M = \bigoplus_{\mathcal{D}_\mathcal{O} \subseteq \mathbb{D}_\mathcal{O}} M_{\mathcal{D}_\mathcal{O}}$ *where* $M_{\mathcal{D}_\mathcal{O}} \in \mathrm{GW_r}(\mathbb{I}_\mathrm{n}, \mathcal{D}_\mathcal{O})$.
(2) *Every generalized weight right* $\mathbb{I}_n$*-module is a unique sum of absolutely prime (generalized weight) right* $\mathbb{I}_n$*-modules.*

*Proof.* The corollary follows from Corollary 3.4 by applying $^*$. □

**Criterion for the category $\mathrm{GW_r}(\mathbb{I}_\mathrm{n}, \mathcal{O})$ to be of finite representation type, tame or wild.** Corollary 4.9 is such a criterion.

COROLLARY 4.9. *Let $\mathcal{O} \in \mathcal{M}/G$.*
(1) *The category $\mathrm{GW_r}(\mathbb{I}_\mathrm{n}, \mathcal{O})$ is tame iff $n = 1$.*
(2) *The category $\mathrm{GW_r}(\mathbb{I}_\mathrm{n}, \mathcal{O})$ is wild iff $n \geq 2$.*
(3) *None of the categories $\mathrm{GW_r}(\mathbb{I}_\mathrm{n}, \mathcal{O})$ is of finite representation type.*

*Proof.* The corollary follow from Corollary 3.10 by applying $^*$. □

Using the involution, we can consider right analogues of indecomposable $\mathbb{I}_n$-modules considered at the end of Section 2. We leave this to the interested reader.

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REFERENCES

[1] V. V. BAVULA, *Finite-dimensionality of* $\mathrm{Ext}_n$ *and* $\mathrm{Tor}_n$ *of simple modules over a class of algebras*, Funct. Anal. Appl., 25:3 (1991), pp. 229–230.
[2] V. V. BAVULA, *Generalized Weyl algebras and their representations*, Algebra i Analiz, 4:1 (1992), pp. 75–97; English transl. in St. Petersburg Math. J., 4:1 (1993), pp. 71–92.
[3] V. V. BAVULA, *Simple $D[X, Y; \sigma, a]$-modules*, Ukraïn. Mat. Zh., 44:12 (1992), pp. 1628–1644; English transl. in Ukrainian Math. J., 44:12 (1992), pp. 1500–1511.
[4] V. V. BAVULA, *Each Schurian algebra is tensor-simple*, Comm. Algebra, 23:4 (1995), ppp. 1363–1367.
[5] V. V. BAVULA AND V. BEKKERT, *Indecomposable representations of generalized Weyl algebras*, Comm. Algebra, 28:11 (2000), pp. 5067–5100.
[6] V. V. BAVULA, V. BEKKERT AND V. FUTORNY, *Indecomposable generalized weight modules over the algebra of polynomial integro-differential operators*, Proc. Am. Math. Soc., 146:6 (2018), pp. 2373–2380.
[7] V. V. BAVULA, *The algebra of integro-differential operators on a polynomial algebra*, J. Lond. Math. Soc. (2), 83:2 (2011), pp. 517–543.
[8] V. V. BAVULA, *The group of automorphisms of the algebra of polynomial integro-differential operators*, J. Algebra, 348 (2011), pp. 233–263.
[9] V. V. BAVULA, *The algebra of integro-differential operators on an affine line and its modules*, J. Pure Appl. Algebra, 217:3 (2013), pp. 495–529.
[10] V. V. BAVULA, *The algebra of polynomial integro-differential operators is a holonomic bimodule over the subalgebra of polynomial differential operators*, Algebr. Represent. Theory, 17:1 (2014), pp. 275–288.
[11] V. V. BAVULA AND T. LU, *The quantum Euclidean algebra and its prime spectrum*, Israel J. Math., 219:2 (2017), pp. 929–958.
[12] V. V. BAVULA, *The global dimension of the algebras of polynomial integro-differential operators $\mathbb{I}_n$ and the Jacobian algebras $\mathbb{A}_n$*, J. Algebra Appl., 19:2 (2020), 2050030 (28 pages).
[13] V. V. BAVULA AND T. LU, *Torsion simple modules over the quantum spatial ageing algebra*, Comm. Algebra, 45:10 (2017), pp. 4166–4189.
[14] V. V. BAVULA AND T. LU, *Prime ideals of the enveloping algebra of the Euclidean algebra and a classification of its simple weight modules*, J. Math. Phys., 58:1 (2017), 011701, 33p.
[15] V. V. BAVULA AND T. LU, *Classification of simple weight modules over the Schrödinger algebra*, Canad. Math. Bull., 61:1 (2018), pp. 16–39.
[16] V. V. BAVULA AND T. LU, *The universal enveloping algebra $U(\mathrm{sl}_2 \ltimes V_2)$, its prime spectrum and a classification of its simple weight modules*, J. Lie Theory, 28:2 (2018), pp. 525–560.
[17] V. V. BAVULA AND T. LU, *The prime spectrum of the algebra $\mathbb{K}_q[X, Y] \rtimes U_q(\mathrm{sl}_2)$ and a classification of simple weight modules*, J. Noncomm. Geometry, 12:3 (2018), pp. 889–946.

[18] V. Bekkert, G. Benkart and V. Futorny, *Weight modules for Weyl algebras*, Kac-Moody Lie algebras and related topics, pp. 17–42, Contemp. Math., 343, Amer. Math. Soc., Providence, RI, 2004.

[19] R. E. Block, *The irreducible representations of the Lie algebra* $\mathfrak{sl}(2)$ *and of the Weyl algebra*, Adv. Math., 39 (1981), pp. 69–110.

[20] Yu. A. Drozd, *Representations of commutative algebras*, Funct. Anal. Appl., 6 (1972), pp. 286–288.

[21] Yu. A. Drozd, *Tame and wild matrix problems*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), pp. 242–258, Lecture Notes in Math., 832, Springer, Berlin-New York, 1980.

[22] V. Futorny, D. Grantcharov and V. Mazorchuk, *Weight modules over infinite dimensional Weyl algebras*, Proc. Am. Math. Soc., 142:9 (2014), pp. 3049–3057.

[23] V. Futorny and U. Iyer, *Representations of* $D_q(k[x])$, Israel J. Math., 212:1 (2016), pp. 473–506.

[24] I. M. Gelfand and V. A. Ponomarev, *Indecomposable representations of the Lorenz group*, Russian Math. Surveys, 23 (1968), pp. 1–58.

[25] L. Guo, G. Regensburger and M. Rosenkranz, *On integro-differential algebra*, J. Pure Appl. Algebra, 218 (2014), pp. 456–471.

[26] J. Hartwig, *Locally finite simple weight modules over twisted generalized Weyl algebras*, J. Algebra, 303:1 (2006), pp. 42–76.

[27] J. Hartwig, *Pseudo-unitarizable weight modules over generalized Weyl algebras*, J. Pure Appl. Algebra, 215:10 (2011), pp. 2352–2377.

[28] V. Mazorchuk and L. Turowska, *Simple weight modules over twisted generalized Weyl algebras*, Commun. Algebra, 27:6 (1999), pp. 2613–2625.

[29] R. Lu, V. Mazorchuk and K. Zhao, *Simple weight modules over weak generalized Weyl algebras*, J. Pure Appl. Algebra, 219:8 (2015), pp. 3427–3444.

[30] J. C. McConnell and J. C. Robson, *Homomorphisms and extensions of modules over certain differential polynomial rings*, J. Algebra, 26 (1973), pp. 319–342.

[31] C. M. Ringel, *The representation type of local algebras*, Proceedings of the International Conference on Representations of Algebras (Carleton Univ., Ottawa, Ont., 1974), Paper No. 22, 24 pp. Carleton Math. Lecture Notes, No. 9, Carleton Univ., Ottawa, Ont., 1974.

[32] I. Shipman, *Generalized Weyl algebras: category O and graded Morita equivalence*, J. Algebra, 323:9 (2010), pp. 2449–2468.