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RICCI-FLAT GRAPHS WITH MAXIMUM DEGREE AT MOST 4*

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Abstract. A graph is called Ricci-flat if its Ricci curvatures vanish on all edges, here the definition of Ricci curvature on graphs was given by Lin-Lu-Yau [7]. The authors in [8] and [3] obtained a complete characterization for all Ricci-flat graphs with girth at least five. In this paper, we completely determined all Ricci-flat graphs with maximum degree at most 4.

Key words. Ricci curvature, Ricci-flat graph, Graph construction.

Mathematics Subject Classification. 05C75.

1. Introduction. There is an increasing interest in applying tools and ideas from continuous geometry to discrete setting such as graphs. One of the principal developments in this area concerns curvature for graphs. It is known that Ricci curvature plays a very important role on geometric analysis on Riemannian manifolds. As for graphs, the first definition of Ricci curvature was introduced by Fan Chung and Yau[12] in 1996, their definition provides a curvature at each vertex. In 2009, Ollivier[10] gave a notation of coarse Ricci curvature of Markov chains valid on arbitrary metric spaces, including graphs. His definition on graphs provides a curvature on each edge and depends on a so-called idleness parameter. For a more general definition of Ricci curvature, Lin and Yau [9] gave a generalization of lower Ricci curvature bound in the framework of graphs in term the notation of Bakry and Emery. In 2011, Lin-Lu-Yau [7] modified Ollivier's definition. The modified version is a more suitable definition for graphs. There have been many results about the Ricci curvature, one can refer to [1, 5, 6, 11].

By Lin-Lu-Yau's definition, a Ricci-flat graph is a graph where Ricci curvature vanishes on every edge. In Riemannian manifold, there are many works on constructing Calabi-Yau manifolds which is a class of manifolds with zero Ricci curvature. As an analog, we want to know what do Ricci-flat graphs look like? There have been works on classifying Ricci-flat graphs. At first, [8] and [3] classified all Ricci-flat graphs with girth at least five. Then the authors in [4] characterized all Ricci-flat graphs of girth four with vertex-disjoint 4-cycles, their results show that there are two such graphs. While the fact is there are infinitely many Ricci-flat graphs with girth three or four. In this paper, we will completely classify these Ricci-flat graphs with maximum vertex degree at most 4.

Throughout this paper, let $G = (V, E)$ represent an undirected connected graph with vertex set V and edge set E without multiple edges or self loops. For any vertices $x, y \in V$, let $d(x)$ denote the degree of vertex x, $d(x, y)$ denote the distance from x to y, i.e. the length of the shortest path from x to y. Denote $\Gamma(x)$ as the set of vertices that are adjacent to x, and $N(x) = \Gamma(x) \cup \{x\}$. Notation $x \sim y$ represent that two vertices x and y are adjacent and (x, y) represent the edge. Let C_3, C_4, C_5 represent any cycle of length $3, 4, 5$ respectively. Let $\mathcal{G}$ be the set of simple graphs

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with maximum degree at most 4 that contains at least one copy of C_3 or C_4. Since the Ricci-flat graphs with girth at least five have been completely determined [7], we will only need to find the Ricci-flat graphs from the class $\mathcal{G}$ for our purpose.

Definition 1. *A probability distribution over the vertex set V is a mapping $\mu : V \to [0,1]$ satisfying $\sum_{x \in V} \mu(x) = 1$. Suppose that two probability distributions μ_1 and μ_2 have finite support. A coupling between μ_1 and μ_2 is a mapping $A : V \times V \to [0,1]$ with finite support so that*

$$\sum_{y \in V} A(x,y) = \mu_1(x) \text{ and } \sum_{x \in V} A(x,y) = \mu_2(y).$$

The transportation distance between two probability distributions μ_1 and μ_2 is defined as follows:

$$W(\mu_1,\mu_2) = \inf_A \sum_{x,y \in V} A(x,y)d(x,y),$$

where the infimum is taken over all coupling A between μ_1 and μ_2.

A coupling function provides a lower bound for the transportation distance, the following definition can provide an upper bound for the transportation distance.

Definition 2. *Let $G = (V,E)$ be a locally finite graph. Let $f : V \to \mathbb{R}$. We say f is 1-Lipschitz if*

$$f(x) - f(y) \le d(x,y),$$

for each $x, y \in V$.

By the duality theorem of a linear optimization problem, the transportation distance can also be written as follows:

$$W(\mu_1,\mu_2) = \sup_f \sum_{x \in V} f(x)[\mu_1(x) - \mu_2(x)],$$

where the supremum is taken over all 1-Lipschitz functions f.

For any vertex $x \in V$ and any value $\alpha \in [0,1]$, the probability distribution μ_x^α is defined as:

$$\mu_x^\alpha(z) = \begin{cases} \alpha, & \text{if } z = x, \\ \frac{1-\alpha}{d(x)}, & \text{if } z \sim x, \\ 0, & \text{otherwise.} \end{cases}$$

For any $x, y \in V$, the α-Ricci curvature k_α is defined to be

$$k_\alpha(x,y) = 1 - \frac{W(\mu_x^\alpha, \mu_y^\alpha)}{d(x,y)}.$$

Then the Ollivier-Ricci curvature $k(x,y)$ is defined by Lin-Lu-Yau as

$$k(x,y) = \lim_{\alpha \to 1} \frac{k_\alpha(x,y)}{1-\alpha}.$$

In [2], the authors introduced the concept of idleness function and studied its several properties. In the α-Ollivier-Ricci curvature, for every edge (x,y) in $G =$

(V, E), the value α is called the *idleness*, and function $\alpha \to k_\alpha(x, y)$ is called the *Ollivier-Ricci idleness function*.

THEOREM 1. *Let $G = (V, E)$ be a locally finite graph. Let $x, y \in V$ such that $x \sim y$ and $d(x) \geq d(y)$. Then $\alpha \to k_\alpha(x, y)$ is a piece-wise linear function over $[0, 1]$ with at most 3 linear parts. Furthermore, $k_\alpha(x, y)$ is linear on $[0, \frac{1}{lcm(d(x),d(x))+1}]$ and is also linear on $[\frac{1}{\max(d(x),d(x))+1}, 1]$. Thus, if we have further condition $d(x) = d(y)$, then $k_\alpha(x, y)$ has at most two linear parts.*

By above theorem, to study the local structure of an edge (x, y) such that $k(x, y) = 0$, we only need to consider

$$k_\alpha(x, y) = 0, \quad \text{for } \alpha = \frac{1}{\max(d(x), d(x)) + 1},$$

equivalently,

$$W(\mu_x^\alpha, \mu_y^\alpha) = 1, \quad \text{for } \alpha = \frac{1}{\max(d(x), d(y)) + 1}.$$

Here are some helpful lemmas.

LEMMA 1 ([7]). *Suppose that an edge (x, y) in a graph G is not in any C_3, C_4 or C_5. Then $k(x, y) = \frac{2}{d(x)} + \frac{2}{d(y)} - 2$.*

COROLLARY 1 ([7]). *Suppose that x is a leaf-vertex(i.e. $d(x) = 1$). Let y be the only neighbor of x. Then $k(x, y) > 0$.*

LEMMA 2 ([7]). *Suppose that an edge (x, y) in a graph G is not in any C_3 or C_4. Then $k(x, y) \leq \frac{1}{d(x)} + \frac{2}{d(y)} - 1$.*

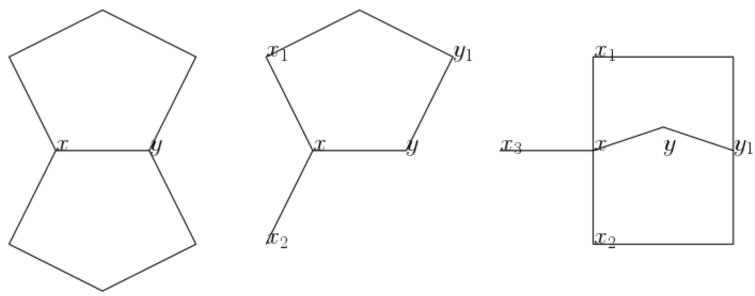
For any edge (x, y) in a Ricci-flat graph G, we require $k(x, y) = 0$, by Lemma 1, if $0 = k(x, y) \neq \frac{2}{d(x)} + \frac{2}{d(y)} - 2$, then (x, y) must be either in C_3 or C_4 or C_5. Similarly, by Lemma 2, if $0 = k(x, y) > \frac{1}{d(x)} + \frac{2}{d(y)} - 1$, then (x, y) must be either in C_3 or C_4; by Corollary 1, there is no leaf-vertex in a Ricci-flat graph.

The road-map constructing Ricci-flat graphs is as follows: in Section 2, we study the local structure of an Ricci-flat edge xy with $d_x, d_y \leq 4$, that is, the distance between each pair of vertices in $\Gamma(x) \times \Gamma(y)$. Results are shown in Lemma 4. When considering Ricci-flat graphs in class $\mathcal{G}$, we then ruled out several cases. Results are shown in Section 2.8. With all "good" structures that have been preserved, we construct Ricci-flat graphs in the following sections. In Section 3, we deal with Ricci-flat graphs which have vertex degree 3 in class $\mathcal{G}$. Results are shown in Theorem 3 and Theorem 4. Based on previous sections, in Section 4, we deal with Ricci-flat graphs in class $\mathcal{G}$ whose degrees on endpoints of every edge is $(2, 4), (4, 4)$. We distinguish these Ricci-flat graphs according to their girth length. Ricci-flat graph whose vertex degree is either 2 or 4 and girth is 3 are shown in Theorem 5. The rest Ricci-flat whose girth is 4 are then divided into three sets according to the connection between any two cycles of length 4. Ricci-flat graphs found in this part are shown in Theorems 6, 7, 8, 9 and 10.

2. Local structures in $\Gamma(x)\cup\Gamma(y)$. In this section, let (x, y) be an edge in any Ricci-flat graph, we will study the distance between any pair of vertices in $\Gamma(x)\cup\Gamma(y)$. Recall the local structures when edges are not contained in any C_3 or C_4.

LEMMA 3 ([7]). *Suppose that an edge (x, y) in a graph G is not in any C_3 or C_4. Without loss of generality, we assume $d(x) \geq d(y)$. If $k(x, y) = 0$, then one of the following statements holds:*

1. *$d(x) = d(y) = 2$. In this case, (x, y) is not in any C_5.*
2. *$d(x) = d(y) = 3$. In this case, (x, y) is shared by two C_5s.*
3. *$d(x) = 2, d(y) = 2$. In this case, let y_1 be the other neighbor of x other than y. Let x_1, x_2 be two neighbors of x other than y. Then $\{d(x_1, y_1), d(x_2, y_1)\} = \{2, 3\}$.*
4. *$d(x) = 4$ and $d(y) = 2$. In this case, let y_1 be the other neighbor of y other than x. Let x_1, x_2, x_3 be three neighbors of x other than y. Then at least two of x_1, x_2, x_3 have distance 2 from x.*



In the following, we consider the case when an edge is supported on C_3 or C_4 or both. For convenience, we label all vertices by nonnegative integers, where x and y are labeled by $0, 1$ respectively. Vertices in $\Gamma(x)$ are labeled by first $d(x)$ positive integers, vertices in $\Gamma(y)$ are labeled by the succeeding integers. Note in [4], the authors also analyzed the local structures of edge with Ricci curvature 0, the difference is that their conclusions are based on that the graph has girth 4 and the 4-cycles in the graph are mutually vertex-disjoint. Our results on these local structures can be applied to graphs without any restriction.

2.1. Local Characteristics of Ricci-flat edges. For edge with zero Ricci curvature in an arbitrary graph, we will prove the following results.

LEMMA 4. *Suppose (x, y) is an edge in a graph G with Ricci curvature $k(x, y) = 0$. Then one of the following statements holds.*

Type 1: $d(x) = 2, d(y) = 2$, then (x, y) is not in any C_3, C_4 or C_5.

Type 2: $d(x) = 3, d(y) = 2$, then (x, y) is in exactly one C_5. In other words, let x_1, x_2 be the neighbors of vertex x, y_1 be the neighbor of y, then $\{d(x_1, y_1), d(x_2, y_2)\} = \{2, 3\}$.

Type 3: $d(x) = 4, d(y) = 2$, then (x, y) is either in exactly one C_4 or is in at least two C_5s. In other words, let x_1, x_2, x_3 be the neighbors of vertex x, y_1 be the neighbor of y, if $d(x_1, y_1) = 1$, then $d(x_2, y_1) = d(x_3, y_1) = 3$. If $d(x_1, y_1) = 2$, then at least one of $d(x_2, y_1), d(x_3, y_1)$ is 2.

Type 4: $d(x) = 3, d(y) = 3$, then (x, y) either is in at least one C_4 or shares two C_5s. In other words, let x_1, x_2 be the neighbors of vertex x, y_1, y_2 be the neighbors of y. Then there are two main cases:

Case 1: *If* $d(x_1, y_1) = 1$, *then* $d(x_2, y_2) = 3$. *If further* $d(x_1, y_2) = 1, 2$, *then* $d(x_2, y_1) = 3$. *Similarly, if* $d(x_2, y_1) = 1, 2$, *then* $d(x_1, y_2) = 3$.
Case 2: *If* $d(x_1, y_1) = 2$, *then* $d(x_2, y_2) = 2$.

Type 5: $d(x) = 3, d(y) = 4$, *then* (x, y) *either shares one* C_3 *or shares one* C_4 *plus one* C_5. *In other words, let* x_1, x_2 *be the neighbors of vertex* x, y_1, y_2, y_3 *be the neighbors of* y. *Then there are three main cases:*
Case 1: *If* $x_1 = y_1$, *then* $d(x_2, y_2) = d(x_2, y_3) = 3$*;*
Case 2: *If* $d(x_1, y_1) = d(x_1, y_2) = 1$, *then* $d(x_2, y_3) = 3$ *and at least one of* $d(x_2, y_1), d(x_2, y_2)$ *is* 3*;*
If $d(x_1, y_1) = d(x_2, y_1) = 1$, *then* $\{d(x_1, y_2), d(x_1, y_3), d(x_2, y_2), d(x_2, y_3)\}$ *is in the set* $\{2, 2, 3, 3\}$ *or* $\{3, 3, 2, 2\}$*;*
Case 3: *If* $d(x_1, y_1) = 1$, *then at least one of* $d(x_2, y_2), d(x_2, y_3)$ *is* 2. *Further, there are two types.*
Type 5a: *If* $d(x_2, y_2) = d(x_2, y_3) = 2$, *then* $d(x_1, y_2) = d(x_1, y_3) = 3$.
Type 5b: *If* $d(x_2, y_2) = 2, d(x_2, y_3) = 3$, *then* $d(x_1, y_3) = 2$, $d(x_1, y_2) = 2$ *or* 3.

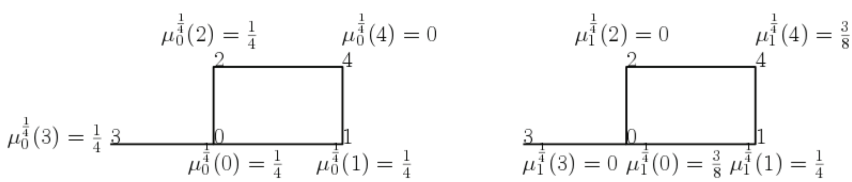
Type 6: $d(x) = d(y) = 4$, *then* (x, y) *either shares one* C_3 *plus one* C_5 *or shares two* C_4*s or shares one* C_4 *plus two* C_5*s. In other words, let* x_1, x_2, x_3 *be the neighbors of vertex* x, y_1, y_2, y_3 *be the neighbors of* y. *Then there are three main cases:*
Type 6a: $x_1 = y_1$ *and* $d(x_2, y_2) = 2$, *then* $d(x_3, y_3)$ *must be* 3.
Type 6b: $d(x_1, y_1) = 1$ *and* $d(x_2, y_2) = 1$, *then* $d(x_3, y_3)$ *must be* 3.
Type 6c: $d(x_1, y_1) = 1$, $d(x_2, y_2) = 2$ *and* $d(x_3, y_3) = 2$.

The reasons of above lemma will be given in the next few sections 2.2, 2.3, 2.4, 2.5, 2.6 and 2.7. Note that we do not use the fact that the maximum degree of the graph is at most 4.

2.2. $d(x) = 2, d(y) = 2$. By calculating, Ricci curvature κ on edges of cycle C_3, C_4, C_5 are $1.5, 1, 0.5$ respectively. Thus for the case $k(x, y) = 0$ with $d(x) = d(y) = 2$, (x, y) cannot appear in any C_3 or C_4 or C_5.

2.3. $d(x) = 3, d(y) = 2$. Since $\frac{2}{d(x)} + \frac{2}{d(y)} - 2 = \frac{2}{3} + \frac{2}{2} - 2 \neq 0$. Thus (x, y) must be either in C_3, or C_4 or C_5 by Lemma 1.

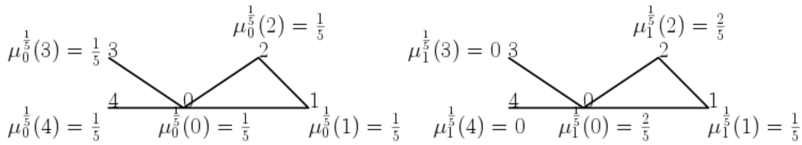
- If (x, y) in C_4, see the following graph, taking $A(3, 0) = A(3, 4) = \frac{1}{8}, A(2, 4) = \frac{1}{4}$ and other values $A(i, j) = 0$, we have $W(\mu_x^{\frac{1}{4}}, \mu_y^{\frac{1}{4}}) \leq A(3, 0) \times d(3, 0) + A(3, 4) \times d(3, 4) + A(2, 4) \times d(2, 4) = \frac{1}{8} \times 1 + \frac{1}{8} \times 3 + \frac{1}{4} \times 1 = 0.75 < 1$.



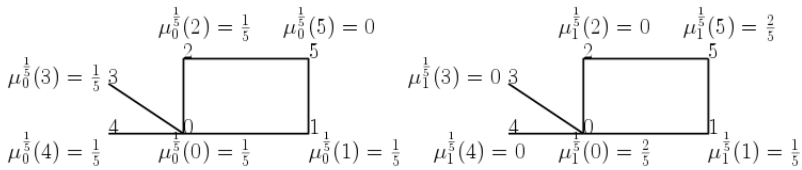
Thus (x, y) cannot appear in any C_3 or C_4.

2.4. $d(x) = 4, d(y) = 2$. Since $\frac{2}{d(x)} + \frac{2}{d(y)} - 2 = \frac{2}{4} + \frac{2}{2} - 2 \neq 0$. Thus (x, y) must be either in C_3, or C_4 or C_5 by Lemma 1.

- If (x, y) in C_3, see the following graph, $W(\mu_x^{\frac{1}{5}}, \mu_y^{\frac{1}{5}}) < 1$ by taking $A(3, 0) = A(4, 2) = \frac{1}{5}$ and other values $A(i, j) = 0$.



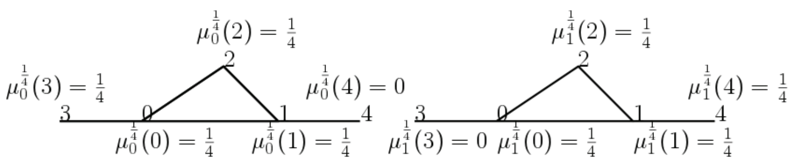
- If (x, y) is in C_4. See the following $W(\mu_x^{\frac{1}{5}}, \mu_y^{\frac{1}{5}}) = 1$. For $W(\mu_x^{\frac{1}{5}}, \mu_y^{\frac{1}{5}}) < 1$, we take $A(0,3) = A(5,2) = A(5,4) = \frac{1}{5}$ and other values $A(i,j) = 0$. For $W(\mu_x^{\frac{1}{5}}, \mu_y^{\frac{1}{5}}) \leq 1$, we take $f(0) = 2, f(1) = 2, f(2) = f(3) = f(4) = 1$ and $f(5) = 3$.



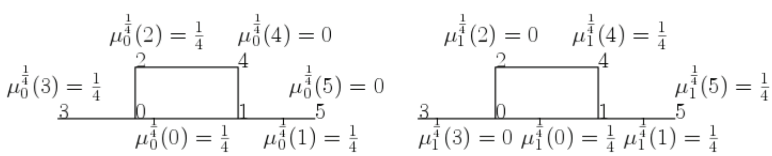
Note if $d(3,5) = 2$, $W(\mu_x^{\frac{1}{5}}, \mu_y^{\frac{1}{5}}) < 1$ by taking $A(4,0) = A(2,5) = A(3,5) = \frac{1}{5}$, and other values $A(i,j) = 0$. Similarly for $d(4,5)$. Thus if $(0,1)$ is in C_4 with $d(2,5) = 1$, then $d(3,5)$ and $d(4,5)$ must be 3.

2.5. $d(x) = 3, d(y) = 3$**.** Since $\frac{2}{d(x)} + \frac{2}{d(y)} - 2 = \frac{2}{3} + \frac{2}{3} - 2 \neq 0$. Thus (x, y) must be either in C_3, or C_4 or C_5 by Lemma 1.

- If (x, y) is in C_3, see the following graph, $W(\mu_x^{\frac{1}{4}}, \mu_y^{\frac{1}{4}}) < 1$ by taking $A(3,4) = \frac{1}{4}$ and other values $A(i,j) = 0$. Contradiction.



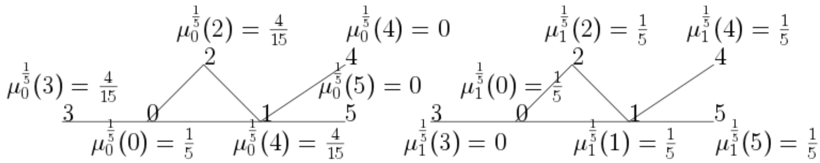
- If (x, y) is in C_4, see the following graph, $W(\mu_x^{\frac{1}{4}}, \mu_y^{\frac{1}{4}}) = 1$. For $W(\mu_x^{\frac{1}{4}}, \mu_y^{\frac{1}{4}}) \leq 1$, we can take $A(2,4) = A(3,5) = \frac{1}{4}$. For $W(\mu_x^{\frac{1}{4}}, \mu_y^{\frac{1}{4}}) \geq 1$, we can take $f(0) = 2, f(1) = 1, f(2) = f(3) = 3, f(4) = 2, f(5) = 0$.



Note if $d(2,4) = 1$, then $d(3,5)$ must be 3. If further $d(2,5) = 1$ or 2 then $d(3,4)$ must be 3, and we still have $W(\mu_x^{\frac{1}{4}}, \mu_y^{\frac{1}{4}}) = 1$. Similarly, if further $d(3,4) = 1$ or 2 then $d(2,5)$ must be 3, still $W(\mu_x^{\frac{1}{4}}, \mu_y^{\frac{1}{4}}) = 1$.

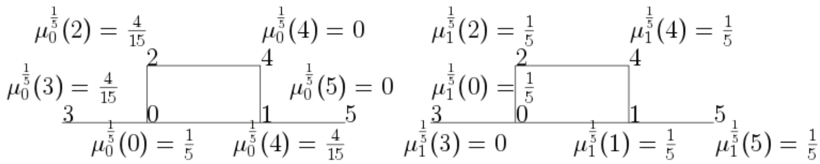
2.6. $d(x) = 3, d(y) = 4$**.** By Lemma 2, $\frac{1}{d(x)} + \frac{2}{d(y)} - 1 = \frac{1}{3} + \frac{2}{4} - 1 < 0$. Thus (x, y) must be either in C_3 or C_4.

- If (x, y) is in C_3, see the following graph. $W(\mu_x^{\frac{1}{5}}, \mu_y^{\frac{1}{5}}) = 1$. For $W(\mu_x^{\frac{1}{5}}, \mu_y^{\frac{1}{5}}) \leq 1$, we can take $A(1,4) = A(1,5) = \frac{1}{30}$, $A(3,4) = A(3,5) = \frac{2}{15}$, $A(2,4) = A(2,5) = \frac{1}{30}$ and other values $A(i,j) = 0$. For $W(\mu_x^{\frac{1}{5}}, \mu_y^{\frac{1}{5}}) \geq 1$, we can take $f(0) = 2, f(1) = 1, f(2) = 1, f(3) = 3, f(4) = f(5) = 0$.



If $d(3,5) = 2$, taking the same function $A(i,j)$, we have $W(\mu_x^{\frac{1}{5}}, \mu_y^{\frac{1}{5}}) < 1$. Thus $d(3,5), d(3,4)$ must be 3.

- If (x,y) is in C_4, see the following graph, $W(\mu_x^{\frac{1}{5}}, \mu_y^{\frac{1}{5}}) = 1.2666$ by taking $f(0) = 2, f(1) = 1, f(2) = f(3) = 3, f(4) = f(5) = f(6) = 0$.



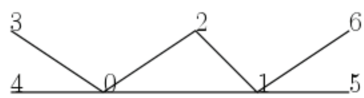
We calim (x,y) cannot share two C_4s, otherwise if both $d(2,4), d(3,5)$ are 1, that is, $(0,1)$ shares two C_4s, then $W(\mu_x^{\frac{1}{5}}, \mu_y^{\frac{1}{5}}) \leq 0.8666$ by taking $A(2,4) = A(3,5) = \frac{1}{5}$ and other values $A(i,j) = 0$.
Thus there are two main cases left:
  - (x,y) is contained in exactly two C_4s.
    Case a: $(0,1)$ is in two C_4s with $C_4 = 0-2-4-1-0$ and $C_4 = 0-2-5-1-0$.
    We claim that $d(3,6)$ cannot be 2 under this assumption. Otherwise, take $A(1,5) = A(2,5) = \frac{1}{15}, A(2,4) = \frac{3}{15}, A(3,5) = \frac{1}{15}, A(3,6) = \frac{2}{15}$, then $W(\mu_0^{\frac{1}{5}}, \mu_1^{\frac{1}{5}}) \leq \frac{1}{15} + \frac{1}{15} + \frac{3}{15} + d(3,5) \times \frac{1}{15} + d(3,6) \times \frac{2}{15} = \frac{9}{15} + \frac{d(3,5)}{15} \leq \frac{12}{15} < 1$, a contradiction.
    We claim whatever $d(2,6)$ is (2 or 3), $d(3,4), d(3,5)$ cannot be both 3, since we can take $f(0) = 2, f(1) = 1, f(2) = 1, f(3) = 3, f(4) = f(5) = f(6) = 0$, then $W(\mu_0^{\frac{1}{5}}, \mu_1^{\frac{1}{5}}) = f(0) \times 0 + f(1) \times \frac{1}{15} + (f(2) + f(3)) \times \frac{4}{15} - (f(4) + f(5) + f(6)) \times \frac{3}{15} = \frac{1}{15} + \frac{16}{15} > 1$. Thus one of $d(3,4), d(3,5)$ must be 2.
    By the symmetry of vertex 4 and vertex 5, wlog, let $d(3,5) = 2$. See the following graph, we have $W(\mu_1^{\frac{1}{5}}, \mu_1^{\frac{1}{5}}) = 1$ by taking $A(1,6) = A(2,5) = \frac{1}{15}, A(3,6) = A(3,5) = \frac{2}{15}, A(2,4) = \frac{3}{15}$, and $f(0) = 2, f(1) = 1, f(2) = 2, f(3) = 3, f(4) = f(5) = 1, f(6) = 0$.
    Case b: $(0,1)$ is in two C_4s with $C_4 = 0-1-4-2-0$, $C_4 = 0-1-4-3-0$.
    We analysis the values of $d(2,5), d(2,6), d(3,5), d(3,6)$. Currently we have $W(\mu_0^{\frac{1}{5}}, \mu_1^{\frac{1}{5}}) \geq 1.26666$. Thus one of $d(2,5), d(2,6), d(3,5), d(3,6)$ must be 2. Wlog, let $d(2,5) = 2$. We calim that $d(3,6) = 3$, otherwise let $d(2,6) = d(3,5) = 3$, take $A(1,5) = \frac{1}{15}, A(2,4) = \frac{2}{15}, A(3,4) = \frac{1}{15}, A(3,6) = \frac{3}{15}$ and others $A(i,j) = 0$, then $W(\mu_0^{\frac{1}{4}}, \mu_1^{\frac{1}{4}}) \leq 0.9333$, thus $d(3,6)$ must be 3. Similarly, if $d(2,6) = 2$, then $d(3,5)$ must be 3. Assume $d(2,6) = 3, d(3,5) = 2$, take the 1-Lipschitz function $f(0) = 2, f(1) = 1, f(2) = f(3) = 3, f(4) = 2, f(5) = 1, f(6) = 0$, then $W(\mu_0^{\frac{1}{4}}, \mu_1^{\frac{1}{4}}) \geq \frac{16}{15} > 1$. Thus $d(2,6) = 2$, $d(3,5) = 3$.
  - (x,y) is contained in exactly one C_4, then at least one of $d(3,5), d(3,6)$

is 2.
Wlog, let $d(3,5)=2$, we have $W(\mu_x^{\frac{1}{5}},\mu_y^{\frac{1}{5}})\geq 1.066666$ by taking $f(0)=2, f(1)=1, f(2)=f(3)=3, f(4)=2, f(6)=0, f(5)=1$. Now there are two cases to consider.

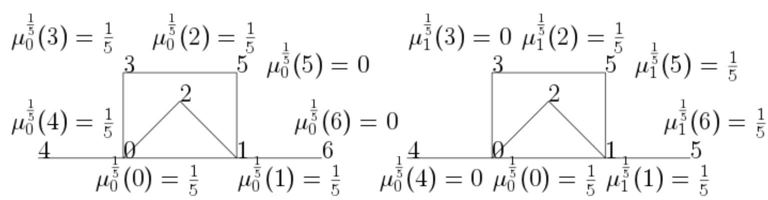
1. If further $d(3,6)=2$, then $d(2,5)=d(2,6)=3$. For $W(\mu_x^{\frac{1}{5}},\mu_y^{\frac{1}{5}})\leq 1$, we can take $A(1,5)=A(1,6)=A(2,5)=A(2,6)=\frac{1}{30}$, $A(3,5)=A(3,6)=\frac{2}{15}$, $A(2,4)=\frac{3}{15}$ and other values $A(i,j)=0$. For $W(\mu_x^{\frac{1}{5}},\mu_y^{\frac{1}{5}})\geq 1$, we can take $f(0)=2, f(1)=1, f(2)=3, f(3)=2, f(4)=2, f(5)=f(6)=0$. From the values of $A(i,j)$, $d(2,5), d(2,6)$ must be 3.
2. If further $d(3,6)=3$, then $d(2,6)=2$, $d(2,5)$ could be 2 or 3. When $d(2,6)=2$, taking $A(1,6)=A(2,6)=\frac{1}{15}$, $A(3,5)=\frac{2}{15}$, $A(2,4)=\frac{3}{15}$ and other values $A(i,j)=0$, we get $W(\mu_0^{\frac{1}{5}},\mu_1^{\frac{1}{5}})\leq \frac{3}{15}+2\times\frac{1}{15}+2\times\frac{3}{15}+\frac{1}{15}+3\times\frac{1}{15}=1$. Taking $f(0)=2, f(1)=1, f(2)=2, f(3)=3, f(4)=1, f(5)=1, f(6)=0$, we have $W(\mu_0^{\frac{1}{5}},\mu_1^{\frac{1}{5}})\geq 1$. From the values of $f(i)$, $d(2,6)$ must be 2.

2.7. $d(x)=4, d(y)=4$**.** By Lemma 2, $\frac{1}{d(x)}+\frac{2}{d(y)}-1=\frac{1}{4}+\frac{2}{4}-1<0$. Thus (x,y) must be either in C_3 or C_4.

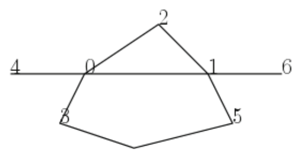
- If (x,y) is in C_3, see the following graph. $W(\mu_x^{\frac{1}{5}},\mu_y^{\frac{1}{5}})\geq 1.2$, since we can take $f(0)=2, f(1)=f(2)=1, f(3)=f(4)=3, f(5)=f(6)=0$.



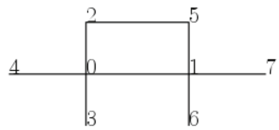
Note (x,y) cannot share both C_3 and C_4. For example, if $d(3,5)=1$, ever $d(4,6)=3$, we have $W(\mu_0^{\frac{1}{5}},\mu_1^{\frac{1}{5}})\leq 0.8$ by taking $A(3,5)=A(4,6)=\frac{1}{5}$. It implies that (x,y) cannot share two C_3s.



Then we need to solve $d(3,5)\times A(3,5)+d(3,6)\times A(3,6)+d(4,5)\times A(4,5)+d(4,6)\times A(4,6)$, by symmetric of these four vertices, there is only one solution: if $d(3,5)=2$, then $d(4,6)$ must be 3 and $A(3,5)=A(4,6)=\frac{1}{5}$, $d(3,6), d(4,5)$ could be 2 or 3. To see this, let $d(3,5)=2$, if $d(4,6)=2$, then $W(\mu_0^{\frac{1}{5}},\mu_1^{\frac{1}{5}})\leq 0.8$ by taking $A(3,5)=A(4,6)=\frac{1}{5}$ and other values $A(i,j)=0$. Thus we have when the edge $(0,1)$ is in the $C_3:=0-1-2-0$, then if $d(3,5)=2$, then $d(4,6)=3$.
If further $d(3,6)=3$, then $d(4,5)$ could be 2 or 3, since we can take $f(0)=2, f(1)=f(2)=1=f(5), f(3)=f(4)=3$ and $f(6)=0$ such that $W(\mu_0^{\frac{1}{5}},\mu_1^{\frac{1}{5}})\geq 1$ and take $A(3,5)=A(4,6)=\frac{1}{5}$ and other values $A(i,j)=0$ such that $W(\mu_0^{\frac{1}{5}},\mu_1^{\frac{1}{5}})\leq 1$.

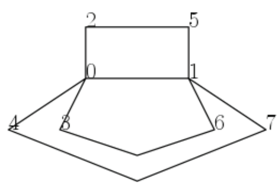


- When (x, y) is in one C_4, see the following graph. $W(\mu_x^{\frac{1}{5}}, \mu_y^{\frac{1}{5}}) = 1.4$.



Then we analysis the other pair of neighbors $(3, 6), (3, 7), (4, 6), (4, 7)$.

  – Assume $d(3, 6) = 1$, then $d(4, 7)$ must be 3. Otherwise $W(\mu_0^{\frac{1}{5}}, \mu_1^{\frac{1}{5}}) \leq 0.8$ by taking $A(2, 5) = A(3, 6) = A(4, 7) = \frac{1}{5}$. Let $d(4, 7) = 3$, then (x, y) shares two C_4s, we have $W(\mu_0^{\frac{1}{5}}, \mu_0^{\frac{1}{5}}) \leq 1$ by taking $A(2, 5) = A(3, 6) = A(4, 7) = \frac{1}{5}$; and $W(\mu_x^{\frac{1}{5}}, \mu_y^{\frac{1}{5}}) \geq 1$ by taking take $f(0) = 2, f(1) = 1, f(2) = f(3) = f(4) = 3, f(5) = f(6) = 2, f(7) = 0$.
    When $d(2, 6) = 1$ (and $d(3, 5) = 1$), then $d(4, 5), d(4, 6)$ could be 1. When $d(2, 7) = 1$ or $d(3, 7) = 1$, then $d(4, 5), d(4, 6)$ must be 3. When $d(2, 7) = 2$ (and $d(3, 7) = 2$), $d(4, 5), d(4, 6)$ could be 2 or 3.
  – Assume $d(3, 6) = 2$, then $d(4, 7)$ must be 2. Otherwise we can take $f(0) = 2, f(1) = 2, f(2) = f(3) = f(4) = 3, f(5) = 2, f(6) = 1, f(7) = 0$, then $W(\mu_0^{\frac{1}{5}}, \mu_1^{\frac{1}{5}}) \geq 1.2$. Let $d(3, 6) = 2 = d(4, 7) = 2$, then $W(\mu_0^{\frac{1}{5}}, \mu_1^{\frac{1}{5}}) = 1$ by taking $f(2) = f(3) = f(4) = 3, f(5) = 2, f(7) = 1, f(6) = 1$ and $A(2, 5) = A(3, 6) = A(4, 7) = \frac{1}{5}$.



If further $d(2, 6) = d(2, 7) = 1$, $d(3, 5) = d(4, 5) = d(4, 6) = d(3, 7) = 2$, then still $W(\mu_0^{\frac{1}{5}}, \mu_1^{\frac{1}{5}}) = 1$.

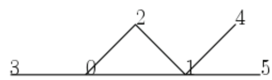
2.8. Further analysis. Above conclusions can be applied to Ricci-flat graphs without degree restriction. We now give several simple facts and further study these conclusions on Ricci-flat graphs with maximum degree at most 4.

LEMMA 5. *Let G be a Ricci-flat graph that does not contain edge with endpoints degree $\{2, 2\}$, then any edge in G is either contained in a C_3 or C_4 or C_5.*

LEMMA 6. *Let G be a Ricci-flat graph with maximum degree at most 4, then no edge in G shares C_3 and C_4.*

LEMMA 7. *Let G be any Ricci-flat graph with maximum degree at most 4 that contains an edge (x, y) with $d(x) = 3$ and $d(y) = 4$, then (x, y) is not contained in any C_3. Thus Case 1 in "Type 5" is excluded for our purpose.*

Proof. We suppose G contains and edge $(0,1)$ with $d(0)=3$, $d(1)=4$ such that $0\sim 2,3, 1\sim 2,3,4$. See the following graph. By previous analysis $d(3,5), d(3,4)$ must be 3. Observe that the edge $(0,2)$ is in a C_3 with $d(0)=3$, then $d(2)=4$. By Lemma 6, $2\not\sim 4$ or $2\not\sim 5$. Let $2\sim 6$ and $2\sim 7$. Since the edge $(0,2)$ is contained in a C_3 sharing vertex 1, thus $d(3,6), d(3,7)$ must be 3. Then the edge $(0,3)$ cannot be contained in any C_3 or C_4 or C_5, a contradiction to Lemma 5. A contradiction.

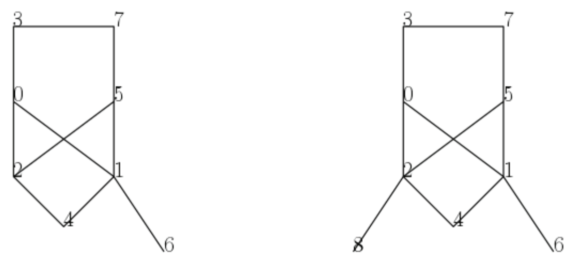


□

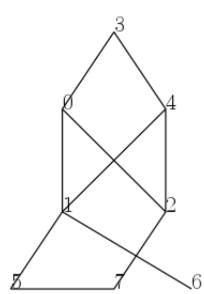
LEMMA 8. *Let G be a Ricci-flat graph with maximum degree at most 4, if there exists an edge (x,y) with $d(x)=3$ and $d(y)=4$, then (x,y) must be contained in exactly one C_4. Thus Case 2 in "Type 5" is excluded for our purpose.*

Proof. Let $x=0, y=1$ and $0\sim 2,3, 1\sim 4,5,6$, Let $(0,1)$ is in the 4-cycle $C_4 = 0-2-4-1-0$. There are two different situations for the edge $(0,1)$ to be contained in two C_4s.

- $(0,1)$ is in two C_4s with $C_4 = 0-2-4-1-0$ and $C_4 = 0-2-5-1-0$. By previous analysis, we have $d(3,6)=3$ and one of $d(3,4), d(3,5)$ is 2. Wlog, let $d(3,5)=2$.



  Consider the edge $(0,2)$, note $d(2)\neq 3$, since the edge $(0,2)$ does not satisfy "Type 4". Thus $d(2)$ must be 4. B Lemma 6, vertex 2 is not adjacent to vertex 3, 6 or 7, we need a new vertex $8\sim 2$. We have $d(3,8)=3$ with the same reason as $d(3,6)=3$. From above local structure, the edge $(0,3)$ is not "Type 2", then $d(3)\neq 2$. Since edge $(0,3)$ cannot be in any C_4, then $d(3)\neq 4$. Then it must be $d(3)=3$, and edge $(0,3)$ must share two separated C_5s which would lead to $d(3,6)=2$ or $d(3,8)=2$, a contradiction.
- $(0,1)$ is in two C_4s with $C_4 = 0-1-4-2-0$, $C_4 = 0-1-4-3-0$. By previous analysis, wlog, let $d(2,5)=d(2,6)=2, d(3,5)=d(3,6)=3$. Let vertex 7 connect 2 and 5.



  Now we have $d(2), d(4)\geq 3$, there are several different cases to consider.

If $d(2) = 3, d(4) = 3,$ then we have $W(\mu_2^{\frac{1}{5}}, \mu_4^{\frac{1}{5}}) \leq 0.75$ as the edge $(2,4)$ does not satisfies "Type 4".

If $d(2) = 3, d(4) = 4,$ consider the edge $(2,4)$, it is contained in two $C_4 := 2-4-1-0-2, 2-4-3-0-2$, which can be compared with the first situation of edge $(0,1)$, this case is excluded.

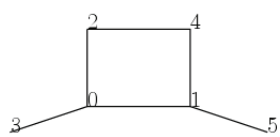
If $d(2) = 4, d(4) = 3,$ let $2 \sim 8$. For edge $(0,2)$, it is contained in two $C_4 := 0-2-4-3-0, 0-2-4-1-0$ and $d(1,7) = 2$, then $d(1,8)$ must be 2, $d(3,7), d(3,8) = 3$. Since $(0,3)$ is contained in two C_4s, then $d(3) \neq 2$. If $d(3) = 3$ with the third neighbor 9, then $d(9,1) = d(9,2) = 3$ which would result that the edge $(3,9)$ cannot be in any C_3, C_4 or C_5, a contradiction. Let $d(3) = 4$ with $3 \sim 9$ and $3 \sim 10$, compare the edge $(0,3)$ with the second situation of edge $(0,1)$, then either $d(1,9) = d(2,9) = 2, d(1,10) = d(2,10) = 3$ or $d(1,9) = d(2,9) = 3, d(1,10) = d(2,10) = 2$, wlog, we use the latter case, then the edge $(3,9)$ cannot be in any C_3, C_4 or C_5, a contradiction.

If $d(2) = 4, d(4) = 4,$ let $2 \sim 8$, $4 \sim 9$. Consider the edge $(0,2)$, since $d(1,7) = 2$, then $d(1,8) = 2$, $d(3,7) = d(3,8) = 3$. Note $d(3,5) = d(3,6) = 3$, then any neighbor of 3 must have distance 3 to both vertices 1 and 2. Then $d(3) \neq 3$ by considering edge $(3,0)$. Let $d(3) = 4$ with $3 \sim 10, 3 \sim 11$. Then for the edge $(0,3)$, we have both $d(1,10) = d(1,11) = 3$ and $d(2,10) = d(2,11) = 3$, a contradiction to "Type 5".

We exclude the two situations, thus $(0,1)$ must be contained exactly one C_4 in a Ricci-flat graph. The result follows. □

LEMMA 9. *Let G be a Ricci-flat graph with maximum degree at most 4, if there exists an edge (x,y) with endpoint degree $(d(x), d(y)) = (3,3)$, then (x,y) is not contained in any C_4. Thus Case 1 in "Type 4" is excluded for our purpose.*

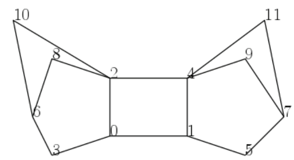
Proof. Let $x = 0, y = 1$, $0 \sim 2,3$, $1 \sim 4,5$. By contradiction, assume $(0,1)$ is contained in a $C_4 := 0-1-4-2-0$.



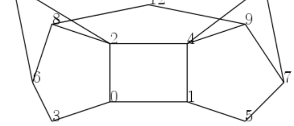
There are three different cases.

- Case 1: $d(2,4) = d(2,5) = 1$, then $d(3,4), d(3,5)$ must be 3 by "Type 4". If $d(2) = 3$, then the edge $(0,3)$ cannot be in any C_3 or C_4 or C_5 through vertex 1 or 2, contradicting to Lemma 5. Then $d(2) = 4$. For edge $(0,2)$, we have $(d(0), d(2)) = (3,4)$ and it is the first situation of Lemma 8, this case should be excluded.
- Case 2: $d(2,4) = 1, d(2,5) = 2$, then $d(3,4), d(3,5)$ must be 3. Let $2 \sim 6 \sim 5$. If $d(2) = 3$, since $d(1,4) = 1$, then $d(3,6)$ must be 3. Then the edge $(0,3)$ cannot be in any C_3 or C_4 or C_5 through vertex 1 or 2, contradicting to Lemma 5. Let $d(2) = 4$ with $2 \sim 7$. Consider the edge $(1,5)$, note since $d(0,6) = d(4,6) = 2$, then $d(5) \neq 2$. Since $(1,5)$ cannot be in any C_4, then $d(5) \neq 4$. Thus $d(5) = 3$ and $(1,5)$ shares two C_5s. However, since $d(3,5) = 3$, this case cannot happen either.
- Case 3: $d(2,4) = 1, d(2,5) = d(3,4) = d(3,5) = 3$. By the same reason as case 1, $d(2)$ must be 4. Similarly, $d(4) = 4$. Still the edge $(0,3)$ cannot be in

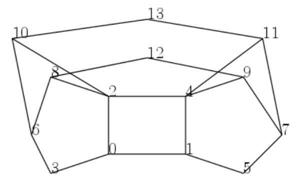
any C_3 or C_4, then $d(3) \neq 4$. Assume $d(3) = 3$ with new neighbors $6, 7$, note $(0, 3)$ must share two C_5s, then we need $d(1, 6) = 2$ or $d(1, 7) = 2$, both would lead to $d(3, 5) = 2$, a contradiction. Thus $d(3) = 2$ with $3 \sim 6$. Similarly, $d(5) = 2$ with $5 \sim 7$. Let the edge $(0, 3)$ in a $C_5 = 0-3-6-8-2-0$, where 8 is new neighbor of vertex 2, the edge $(1, 5)$ in a $C_5 = 1-5-7-9-4-1$, where 9 is new neighbor of vertex 4, let the fourth neighbor of 2 be 10, the fourth neighbor of 4 be 11. Since $d(2, 5) = d(3, 4) = 3$, then $d(0, 11) = d(1, 10) = 3$, for the edge $(0, 2)$, we need $d(3, 10) = 2$, thus $6 \sim 10$, similarly, for the edge $(1, 4)$, let $7 \sim 11$.



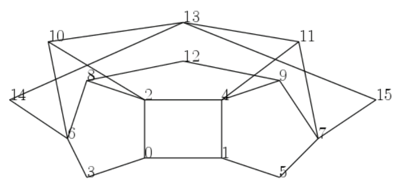
Consider the edge $(3, 6)$, since $d(0, 8) = d(0, 10) = 2$, then $d(6) \neq 3$. Thus $d(6) = 4$, similarly, $d(7) = 4$.
Consider the edge $(2, 8)$, since it is in a C_4 and C_5, then $d(8) \neq 2$. By symmetric, $d(9), d(10), d(11) \neq 2$. For the edge $(2, 4)$, wlog, either we need $d(8, 9) = d(10, 11) = 2$, either $d(8, 9) = 1, d(10, 11) = 3$. For the latter case, since edge $(2, 8)$ is in two separate C_4s, then $d(8)$ must be 4. Similarly, $d(9) = 4$. Let $12, 13$ be fourth neighbors of 8 and 9 respectively. Now consider the edge $6, 8$, note $d(6) = d(8) = 4$ and $d(10, 2) = 1$, both $d(3, 9), d(3, 12) \neq 1$, then one of them must be 2. If $d(3, 12) = 2$, then $0 \sim 12$ which cannot happen or $6 \sim 12$ which will contradict to Lemma 6; similarly it cannot be $d(3, 9) = 2$, a contradiction.
Thus for the edge $(2, 4)$, we must have that $d(8, 9) = d(10, 11) = 2$. For $d(8, 9) = 2$, assume $8 \sim 7$, then the edge $(7, 8)$ must be in a C_4, then either $9 \sim 6$ or $11 \sim 6$, wlog, let $9 \sim 6$. Then $d(8) \neq 3$ since both $d(2, 5), d(2, 11)$ are 3. Let $d(8) = 4$ with the fourth neighbor 12, then we need $d(12, 11) = 1$ for the edge $(8, 7)$. Consider the edge $(2, 8)$, since both $d(0, 7), d(0, 12)$ are 3, we need $d(4, 12) = 1$ which cannot happen. A contradiction.
For $d(8, 9) = 2$, let 12 be the common neighbor of 8 and 9. We claim $d(8) \neq 4$. Otherwise, let 13 be the fourth neighbor of 8, for the edge $(2, 8)$, since $d(6, 10) = 1$ and $d(12, 4) = 2$, then $d(13, 0)$ must be 2, however, this cannot happen. Thus $d(8) = 3$ with neighbors $2, 6, 12$. Similarly, $d(9) = d(10) = d(11) = 3$.



Note we need another new vertex to connect $10, 11$ such that $d(10, 11) = 2$. Note it cannot be 12 since then $(8, 2)$ is contained in two C_4s which contradicts to Lemma 8. Let this vertex be 13. Observe that the edge $(8, 12)$ cannot be in any C_4, thus $d(12) \neq 4, d(13) \neq 4$. When $d(12) = d(13) = 2$, we only need to consider edge $(3, 6)$ and $(3, 7)$. Let $6 \sim 14$, assume $14 \sim 7$. Consider the edge $(6, 8)$, we need $d(12, 14) = 2$, let $12 \sim 15 \sim 14$, then $d(4) \geq 3$. Similarly, we need $d(13, 14) = 2$ for the edge $(6, 10)$, thus the edge $(6, 14)$ cannot be in any C_4 which implies $d(14) \neq 3, 4$. A contradiction. Let vertex $14, 15$ be neighbors of $6, 7$ respectively. And $d(12) = d(13) = 3$.



Now consider the edge $(6, 10)$ since $d(14, 13)$ must be 2, we need new vertex 16 to connect them. Similarly, consider edge $(7, 11)$, we need $d(13, 15) = 2$. Then 15 must be adjacent to vertex 16 in order to keep $d(13) = 3$. Since $(13, 16)$ cannot be in a C_4, then $d(16) = 3$.
For the edge $(6, 8)$, we need $d(12, 14) = 2$, thus need new vertex 17 to connect them. Similarly, the edge $(7, 9)$, we need $d(12, 15) = 2$, since $(8, 12)$ cannot be in a C_4, then $d(12) \neq 4$, thus 15 must be adjacent to 17 in order to have $d(12, 15) = 2$.

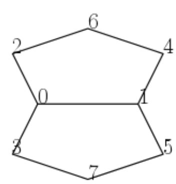


Observe that the edge $(6, 14), (7, 15)$ cannot be in any C_3 or C_4, thus $d(14), d(15)$ cannot be 3 or 4, a contradiction.

□

LEMMA 10. *Let G be a Ricci-flat graph with maximum degree at most 4, if there exists an edge (x, y) with $d(x) = d(y) = 3$, and (x, y) shares two C_5s. Then $d(z) = d(w)$ if both z, w are adjacent to x or y.*

Proof. Look at the following graph with $x = 0, y = 1$. We will show that $d(2) = d(3)$, then $d(4) = d(5)$ will follow immediately.

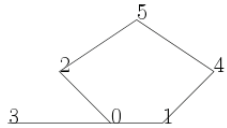


Suppose $d(2) = 2$, then $d(3, 6) = 3$ for the edge $(0, 2)$, thus the edge $(0, 3)$ cannot be in any C_4 or C_5 which lead to $d(3) = 2$. Suppose $d(2) = 3$ with the third neighbor 8. Since the edge $(0, 2)$ must share two C_5s, then the edge $(0, 3)$ cannot be in any C_4, thus $d(3) = 3$. Suppose $d(2) = 4$, then the edge $(0, 2)$ must be in a C_4 which can only pass through vertex 3, thus the edge $(0, 3)$ is also in a C_4, then $d(3)$ must be 4. □

THEOREM 2. *Let G be a Ricci-flat graph with maximum degree at most 4. If G contains an edge (x, y) with $d(x) = 3, d(y) = 2$. Then (x, y) shares two C_5s, and G is the Half-dodecaheral graph.*

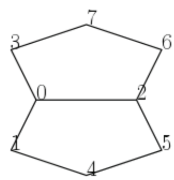
REMARK 1. *Note Half-dodecaheral graph is proved to be Ricci-flat in [7], however we cannot use their results directly since we make no assumption on the girth of G.*

Proof. Look at the following graph with $x = 0, y = 1$.



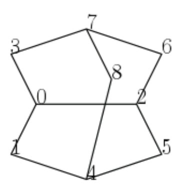
First note that $d(3,4)$ must be 3 by Lemma 4. If $d(2) = 2$, we must have $d(3,5) = 3$ for the edge $(0,2)$, then $(0,3)$ cannot be in any C_3, C_4 or C_5, a contradiction. If $d(2) = 4$, with new neighbors $6,7$, then the edge $(0,2)$ must be in a C_4. By Lemma 9, we get $d(3) = 4$. Let $C_4 := 2-0-3-6-2$, and $3 \sim 8,9$. Consider the edge $(0,3)$, we need one of $d(1,8), d(1,9)$ to be 2, which would contradict to $d(3,4) = 3$.

Thus $d(2) = 3$, let $2 \sim 6$. For the edge $(0,2)$, it must be in two C_5s, and $d(3) = d(1) = 2$, $d(5) = d(6)$. There are two cases: 1) $d(1,6) = d(3,5) = 2$ and $d(3,6) = 3$, then $6 \sim 4$ and need new vertex 7 such that $3 \sim 7 \sim 5$. Under this situation, the edge $(2,5)$ is contained in the $C_4 := 2-5-4-6-2$, then $d(5) = 4$, let $5 \sim 8$. By Item 5 of Lemma 4, since $d(0,8) = 3$, we need $d(6,8) = 2$, let $6 \sim 9 \sim 8$. Note $d(6) = d(5) = 4$ and the edge $(2,6)$ cannot be in two C_4s by Lemma8, thus $6 \not\sim 7$. Let $6 \sim 10$, we need $d(0,9) = 2$ or $d(0,10) = 2$, both cannot happen, a contradiction. Thus for the edge $(0,2)$ sharing two C_5s., it must be the second case 2)$d(1,5) = d(3,6) = 2$ with new vertex 7 achieving this and $d(3) = 2$.

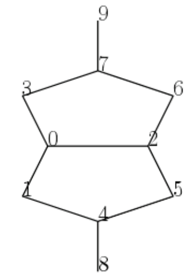


We focus on vertex 4. Note $d(4) \neq 2$ and $d(1,7) = d(3,4) = 3$, then $4 \not\sim 7$. Assume $d(4) = 4$, and assume z, w are third and fourth neighbors of 4, then one of $d(0,z), d(0,w)$ must be 2, which which implies 4 is adjacent to vertex 6. Then the edge $(2,5)$ is in a $C_4 := 4-6-2-5-4$, implies that $d(5) = 4$. Note we need new vertices as third and fourth neighbors of 5, let them be $9, 10$. However both $d(0,9), d(0,10)$ cannot be 2, a contradiction.

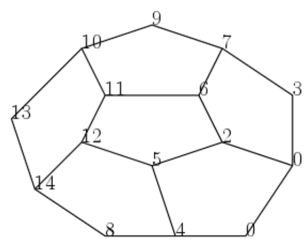
Thus $d(4) = 3$. Similarly, $d(7) = 3$. Let the third neighbor of 4 be z, by comparing edge $(0,1)$ and edge $(4,1)$, we obtained that $d(z) = 2$ just like $d(3) = 2$, thus $z \neq 6$. Let $4 \sim 8$. Actually there is an isomorphism f between the neighborhood $N(0) \cup N(1)$ and the neighborhood $N(1) \cup N(4)$, then $f(0) = 4, f(1) = 1, f(0) = 4, f(2) = 5, f(5) = 2, f(8) = 3$. Thus $d(5) = d(2) = 3, d(8) = d(3) = 2$. Assume $8 \sim 7$. Then $f(7) = 7$, then $d(7,5) = d(f(7), f(5)) = d(7,2) = 2$, note $5 \not\sim 3$ and $5 \not\sim 6$, thus we need new vertex to connect vertices $5, 7$, then $d(7) = 4$, a contradiction.



For $d(7) = 3$, let $7 \sim 9$. Since the distance from 3 to any neighbor of 9 cannot be 2, then $d(9)$ cannot be 3. If $d(9) = 4$, then the edge $(7,9)$ must be in C_4 which can only through vertex 6, then by considering the edge $(7,9)$, one of neighbors of 9 must has distance 2 to vertex 3, which cannot happen.



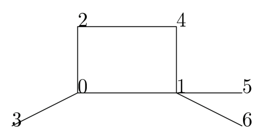
Thus $d(9) = 2$, similarly, $d((8) = 2$. The edge $(7, 9)$ must be in a C_5 through 6. If the $C_5 := 9-7-6-2-5-9$, then consider the edge $(1, 4)$, we need $d(8, 9) = 2$, then either $8 \sim 7$ or $9 \sim 4$, both cannot happen. Thus the C_5 cannot pass through vertex 2, we need new neighbor for 9 and 6, let them be $10, 11$ and $10 \sim 11$ respectively. Note $d(6) = d(5) = 3$. Then for the edge $(2, 6)$, it must be in two separate C_5s, let the other one be $C_5 : 6-2-5-12-11-6$ where 12 is a neighbor of 5. For the edge $(4, 5)$, we need the other $C_5 : 5-4-8-14-12-5$ where 14 is a neighbor of 8, then $14 \sim 12$. Now for the edge $(8, 14)$, any other new neighbor of 14 cannot have distance 1 or 2 to 4, thus $d(14)$ must be 3. Similarly $d(10) = 3$. Since there is no way for the edge $(6, 11)$ to be in any C_4, thus $d(11) = d(12) = 3$. Then for the edge $(11, 12)$, we need $d(10, 14) = 2$, let $10 \sim 13 \sim 14$. And $d(13) \neq 3, 4$.



∎

Lemma 11. *Let G be a Ricci-flat graph with maximum degree at most 4, if there exists an edge (x, y) with $d(x) = 3, d(y) = 4$, and (x, y) is contained in exactly one $C_4 := x - x_1 - y_1 - y - x$ and let x_2, y_2, y_3 be the neighbors of x and y that are not on the C_4, then $d(x_2) = 3$, $d(x_1) = 4$.*

Proof. Let $x = 0, y = 1$ and $0 \sim 1, 2, 3$, $1 \sim 4, 5, 6$ and the edge $(0, 1)$ be contained in exactly one $C_4 = 0-1-4-2-0$. See the following graph, where $x_1 = 2, x_2 = 3$.



By previous analysis, we know $d(3, 5) = 2$, let $3 \sim 7 \sim 5$. Since the edge $(0, 2)$ is in a C_4, then $d(2)$ must be 4 by Lemma 9, thus $d(x_1) = 4$. Consider the vertex 3, since $d(3, 5), d(3, 4), d(3, 6) \neq 1$, and the edge $(0, 2)$ is not in two C_4s, then the edge $(0, 3)$ cannot be in any C_4, thus $d(3) \neq 4$. Assume $d(3) = 2$, consider the edge $(0, 3)$, we must have $d(2, 7) = 3$. Let $8, 9$ the third and fourth neighbors of 2. Thus for the edge $(0, 2)$, one of $d(3, 8), d(3, 9)$ must be 2, this contradicts to the fact that $d(2, 7) = 3$. Thus $d(3) = 3$, then $d(x_2) = 3$. ∎

3. Ricci-flat graphs containing vertex with degree 3. Recall $\mathcal{G}$ is the collection of all simple graphs with maximum degree at most 4 that contains at least one copy of C_3 or C_4. In Theorem 2. We have determined the Ricci-flat graphs in class $\mathcal{G}$ that contains an edge with endpoint degree $\{2,3\}$. In this section, we continue with endpoint degree $(3,3)$ and $(3,4)$. To determine the former case (see Theorem 4), we will need the Theorem 3. So we start with the graphs that contain an edge with endpoint degrees $\{3,4\}$. There are two cases "Type 5a" and "Type 5b". Case "Type 5b" will be excluded for contracting a Ricci-flat graph in class $\mathcal{G}$. See Lemma 12. For "Type 5a", we have the following four Ricci-flat graphs.

THEOREM 3. *Let G be a Ricci-flat graph with maximum degree at most* 4, *let* (x,y) *be an edge in G with $d(x)=3, d(y)=4$ that satisfies "Type 5a". Then G is isomorphic to one of the following graphs.*

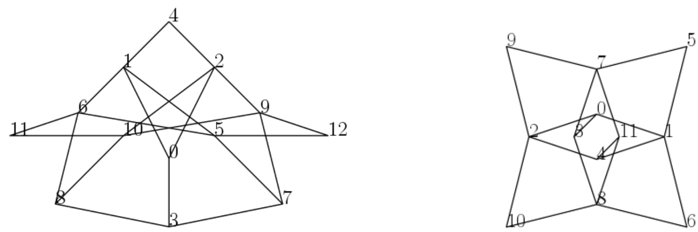


FIG. 1. *Ricci-flat graphs G_1, G_2*

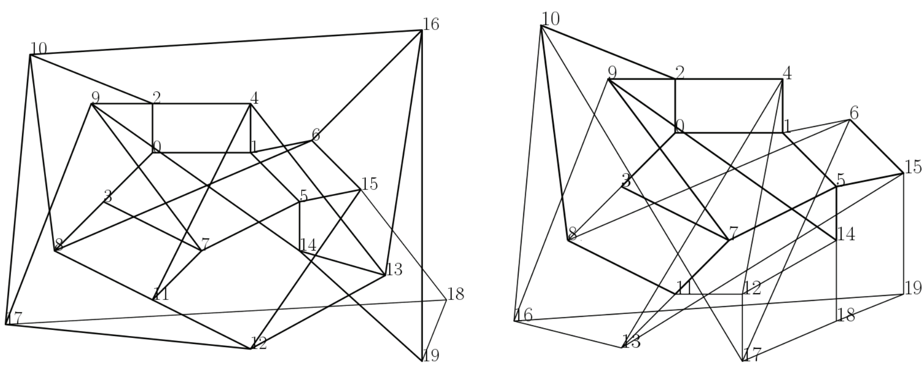
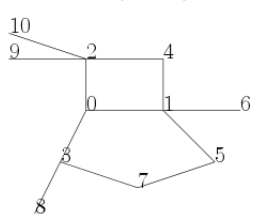


FIG. 2. *Ricci-flat graphs G_3, G_4*

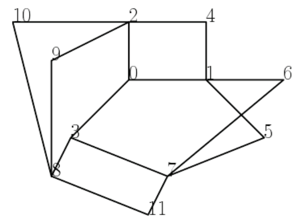
Proof. Let $x=0, y=1$ and $(0,1)$ be contained in exactly one $C_4: 0-2-4-1-10$, with $d(2,5)=d(2,6)=3$. By Lemma 11, $d(3)=3$ and $d(2)=4$. Let $3\sim 7,8$, $7\sim 5$, and $2\sim 9,10$. Then we have the following subgraph of G:



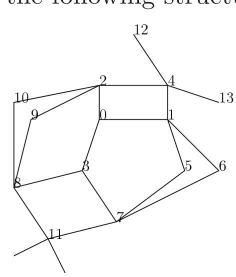
We focus on the degree of vertex 7. There are two cases:

1. Case 1a: $d(7) = 3$. Since $d(2,5), d(2,6) = 3$, then $d(1,9) = d(1,10) = 3$. Then for the edge $(0,2)$, we need $d(3,9) = d(3,10) = 2$. Then vertex 7 must be adjacent to one of $6, 9, 10$, otherwise we need 8 to be adjacent to all three then $d(8) = 4 \neq d(7)$, a contradiction to Lemma 10. If $7 \sim 6$, then $8 \sim 9$ and $8 \sim 10$. Consider the edge $(3,7)$, it should share two C_5s, thus one of $d(8,5), d(8,6)$ must be 2, wlog let $d(8,5) = 2$. Note $d(8) = d(7) = 3$, then it must be either $5 \sim 9$ or $5 \sim 10$, which would contradict to the fact that $d(2,5) = d(2,6) = 3$. A contradiction.
   Now let $7 \sim 9$. Since we require $d(3,6) = 2$, then $8 \sim 6$. Still for the edge $(0,2)$, we need $d(3,10) = 2$ which implies that $8 \sim 10$. For the edge $(3,7)$ to share two C_5s, there are two cases, either $d(8,9) = 2$ or $d(8,5) = 2$, by symmetry of the current graph, wlog, let $d(8,9) = 2$ which implies that $9 \sim 10$, a C_3 appears, thus $d(9) = d(10) = d(2) = 4$. For the edge $(8,10)$, it must be in a C_4 which can only passes through vertex 6, thus we need a new vertex 11 as common of $6, 10$. Similar analysis for the edge $(7,9)$, we need a new vertex 12 as common of $5, 9$.
   Consider $d(4)$, if $d(4) = 3$, then we need new vertex as its neighbor say it is 13. For the edge $(2,4)$, since$d(1,9) = d(1,10) = 3$, we need $d(13,9) = d(13,10) = 2$, then it must be $13 \sim 11$ and $13 \sim 12$, however this is not good for the edge $(2,9)$ and $(2,10)$ as for them we must have $d(4,11) = 3, d(4,12) = 3$. If $d(4) = 4$, we need two new vertices $13, 14$ as neighbors of 4. However, $d(10,13), d(10,14), d(9,13), d(9,14)$ must be 3 which is not good for the edge $(2,4)$. Thus $d(4) = 2$. Similarly, $d(11) = d(12) = 2$.
   Note $d(6), d(5) \neq 3$ by Lemma 9. Observe that the edge $(1,6)$ cannot be in any C_4, then it must be in the $C_5 := 1-6-5-1$. Then $6 \sim 5$. The resulting graph is G_1.
2. Case 1b: $d(7) = 4$. There are several cases with respect to the neighbors of vertex 7.
   - Case 1b1: $7 \sim 6$, $7 \sim 9$. Then $8 \sim 10$ for the edge $(0,2)$. Since the edge $(3,7)$ must be in a C_4, there are two distinct possible cases: $C_4 := 7-3-8-5-7$ or $C_4 := 7-3-8-9-7$.
     We will reject the former case. Otherwise, we need $d(8,9) = d(8,6) = 3$ for the edge $(3,7)$ as $d(0,5) = d(0,6) = 2$. Assume $8 \sim 4$. Then both the edges $(2,4), (5,8)$ are in two C_4s, $d(4) = d(5) = 4$. Let $4 \sim 11$, a new vertex, then $5 \not\sim 11$ for the edge $(1,5)$. Let $5 \sim 12$. However, we need $d(6,12) = 3$ for the edge $(1,5)$, $d(10,12) = 3$ for the edge $(8,5)$, $d(9,12) = 3$ for the edge $(7,5)$, then the edge $(5,12)$ cannot be in any C_3, C_4, then it must be in a C_5 which needs $11 \sim 12$, and $d(12) = 2$. Now we need $d(6,11) = 3$ for the edge $(1,4)$, $d(9,11) = 3$ for the edge $(2,4)$, $d(10,11) = 3$ for the edge $(4,8)$, then the edge $(4,11)$ cannot be in any C_3 and C_4. Thus $d(11) = 2$, a contradiction for the edge $(11,12)$. Thus vertex 8 is adjacent to a new vertex. Let $8 \sim 11$. For the edge $(3,8)$, since $d(0,11)$ cannot be 2, then we need $d(7,11) = 2$ which would be true if $11 \sim 6$ or $11 \sim 9$, while both cannot happen since otherwise $d(8,9) = d(8,6) = 2$. Thus $8 \not\sim 5$, similarly, $8 \not\sim 6$.
     Thus $(3,7)$ is in the $C_4 := 7-3-8-9-7$ which implies $8 \sim 9$. Note when $8 \sim 9, 10$, by similar arguments as above, 7 should not be adjacent to vertex 9. A contradiction. As a result, the edge $(3,7)$ cannot be in C_4, a contradiction.

- Case 1b2: Assume $7 \sim 6$, $7 \sim 11$. Note for the edge $(0,2)$, we need $8 \sim 9, 10$. Since $(3,7)$ must be in a C_4, and this C_4 can only be $7-3-8-11-7$. Thus $8 \sim 11$. For edges $(3,7),(3,8)$, we need $d(8,5) = d(8,6) = 3$ since $d(0,5) = d(0,6) = 2$ and $d(7,9) = d(7,10) = 3$ since $d(0,5) = d(0,10) = 2$.



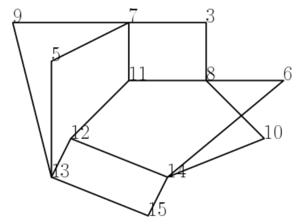
Now we consider $d(4)$. Observe that $d(5) \neq 2$, and any new neighbor of 5 has distance 3 to vertex 0, thus at least one of these new neighbors of 5 should have distance 1, 2 to vertex 4 by considering the edge $(1,5)$. Thus $d(4) \neq 2$. Assume $d(4) = 3$, and assume $4 \sim 11$. Then $d(11)$ cannot be just 3. Let $11 \sim 12$. Then $5 \sim 12$ for the edge $(1,5)$. However, this is not good for the edge $(7,11)$. Thus $4 \not\sim 11$. Let 12 be the third neighbor of 4, then $d(12,5) = d(12,6) = 2$ for the edge $(1,5)$ since $d(2,5) = d(2,6) = 3$. Let $5 \sim 13 \sim 12$. Note $d(5) = 3$, since both third and fourth neighbors of 5 have distance 3 to 0 and distance at least 2 to vertex 4 which is not good for the edge $(1,5)$ when $d(5) = 4$. Since $(1,5)$ must be in exactly one C_4, then $6 \not\sim 13$. For $d(12,6) = 2$, let $6 \sim 14 \sim 12$. Similarly, $d(6) = 3$. Note when $d(4) = 3$, $d(12) = 3$ by similar analysis in Lemma 11. Similarly $d(13) = d(14) = 3$. Now we need $d(12,9) = d(12,10) = 2$ for the edge $(2,4)$ since $d(1,9) = d(1,10) = 3$. Then wlog, let $9 \sim 13, 10 \sim 14$. Now we cannot add new neighbors to vertices $12, 13, 14$, thus cannot guarantee the edge $(5,7)$. A contradiction. Thus $d(4) \neq 3$.
Let $d(4) = 4$, similarly $d(11) = 4$. Still assume $4 \sim 11$, then the edge $(4,11)$ have to be in a C_4 that passes through new neighbors. Let $4 \sim 12 \sim 13 \sim 11$. Consider the edge $(1,4)$, since $d(5,11) = d(6,11) = 2$, then at least one of $d(5,12), d(6,12)$ is 2. Similarly we need at least one of $d(5,13), d(6,13)$ is 2 for the edge $(7,11)$. Wlog, let $d(5,12) = 2$ with $5 \sim 14 \sim 12$. Note $d(5) = 3$, since both third and fourth neighbors of 5 have distance 3 to 0 and distance at least 2 to vertex 4 which is not good for the edge $(1,5)$ when $d(5) = 4$. Thus we need $d(14,11) = 2$ for the edge $(5,7)$, let $14 \sim 13$. And $d(14) = 3$. However, the edge $(12,13)$ is sharing a $c_4 := 12-13-11-4-12$ and a $C_3 := 12-13-14-12$, a contradiction to Lemma 6. Thus $4 \not\sim 11$. Let $4 \sim 12, 13$. The Ricci-flat graph which contains an edge with endpoints degree $(3,4)$ and $d(7) = 4, 7 \sim 5, 7 \sim 6$ must have the following structure.



If $d(5) = 3$ and let z represent the third neighbor of vertex 5,the $d(z) = 4$. However, the edge $(5, 7)$ must be in exactly one C_4 that passes through edge $(5, z)$, similarly, the edge $(1, 5)$ should also be in exactly one C_4 that passes through edge $(5, z)$, the edge $(5, z)$ share two C_4s , a contradiction to "Type 5". Thus $d(5) = 4$. Similarly, $d(6) = d(9) = d(10) = 4$.
Let $d(5) = 4$, observe that we need $d(7, 9) = d(7, 10) = 3$ and $d(2, 5) = d(2, 6) = 3$, then $5, 6 \not\sim 9$ and $5, 6 \not\sim 10$. Since any new neighbor of 5 has distance 3 to 0, then we need that one of new neighbors of 5 has distance 1 to vertex 4 for the edge $(1, 5)$, thus $5 \sim 12$ or $5 \sim 13$, wlog, assume the former. Consider the edge $(1, 4)$, we need $d(6, 13) = 3$. Then for the $(1, 6)$ we need $6 \sim 12$ for the edge $(1, 5)$ and we need $d(5, 13) = 3$ for the edge $(1, 4)$. Let $5 \sim 14$. Note $6 \not\sim 14$ for the edge $(1, 5)$. Let $6 \sim 15$. Consider the edge $(5, 7)$, we have $d(6, 1) = d(6, 12) = 1$, $d(3, 1) = 2$, $d(3, 12) = d(3, 14) = 3$. Note $14 \not\sim 11$, otherwise $d(6, 12) = 1, d(1, 3) = 2$ and $d(14, 11) = 1$ would result $k(5, 7) \neq 0.8$. So we need $d(14, 11) = 2$. Similarly, $d(11, 15) = 2$ for the edge $(6, 7)$. Let $11 \sim 16 \sim 14$.
Now consider the edge $(4, 12)$ which is sharing two C_4s, thus $d(12) = 4$. Since $d(2, 6) = d(13, 6) = 3$, we need the fourth neighbor of 12 to have distance 1 to either 2 or 13. Assume $12 \sim 9$, then consider the edge $(2, 4)$, then we need $d(10, 13) = 3$. Observe that the edge $(4, 13)$ cannot be in any C_3, if it is in a C_4, then we need $9 \sim 13$. Consider the edge $(8, 9)$, we need $11 \sim 13$. Then for $d(11, 15) = 2$, we need $15 \sim 16$. However, this situation is not good for the edge $(7, 11)$. Similarly, $12\ /sim10$. Thus, we need the fourth neighbor of 12 to have distance 1 to vertex 13, which need new vertex $18 \sim 12, 13$. Now consider the edge $(2, 4)$, we need at least one of $d(12, 9), d(12, 10)$ is 2. Wlog, let $d(12, 9) = 2$, then $9 \sim 18$. Then consider the edge $(2, 9)$, let z be the fourth neighbor of 9, since $d(0, z) = d(0, 18) = 3$, $d(4, 18) = 2$, then we need $d(4, z) = 1$. Thus $9 \sim 13$. However, this would result the edge $(13, 18)$ shares a $C_3 := 13 - 18 - 9$ and a $C_4 := 13 - 18 - 12 - 4$. A contradiction.
Therefore, $d(5) \geq 5$. We should exclude "Case 1b2".

- Case 1b3: $7 \sim 9$, then $8 \sim 6$ and $8 \sim 10$. For edge $(3, 7)$ in a C_4, we need new vertex $7 \sim 11 \sim 8$. Note if $d(11) = 2$, then $d(5)$ cannot be $2, 3, 4$. Similarly, $d(4) \neq 2$. If $d(11) = 3$, note $11 \not\sim 5, 6, 9, 10$. Assume $4 \sim 11$, then we get the Ricci-flat graph G_2. Note G_2 is isomorphic to one of two graphs found in [4].
Next we claim there is no Ricci-flat graphs with $d(11) = 3$ and $11 \not\sim 4$:
Proof. Now we consider the case $d(11) = 3$ and $11 \sim 12$ a new vertex, then for the edge $(8, 11)$, since $d(7, 6) = d(7, 10) = 3$, then $d(12, 6)$ and $d(12, 10)$ must be 2. For the edge $(7, 11)$, since $d(8, 5) = d(8, 9) = 3$, then $d(12, 5)$ and $d(12, 9)$ must be 2. Observe that $(11, 12)$ cannot be in any C_4, thus $d(12) \neq 4$. $d(12)$ must be 3. Assume $d(5) = 3$. Then the edge $(5, 7)$ must be in a C_4 which also must pass through vertex 9. Let this be $C_4 := 5 - 7 - 9 - 13 - 5$, and since $d(1, 11) = 3$, we need $d(13, 11) = 2$, thus $13 \sim 12$. Then consider the edge $(1, 5)$ which must also be contained in a C_4 and this C_4 can only pass through vertex 13 and 6 which implies that $6 \sim 13$. Then the edge $(5, 13)$ are contained in two C_4s with endpoints degree $(3, 4)$, a contradiction. Thus For above graph, $d(5) = 4$. By symmetric of vertices 5 and 9, $d(9) = 4$, symmetric

of vertices 7 and 8, $d(6) = d(10) = 4$. Note when $d(5) = 4$, Note $(5, 7)$ cannot be in any C_3, we still need $5 \sim 13 \sim 9$ and $13 \sim 12$. Similarly, the edge $(6, 8)$ must be in a C_4 which also must pass through vertex 10. Let this be $C_4 := 6-8-10-z-6$, note the vertex z cannot be 13 since then $d(13) = 5$. Thus we need a new vertex say 14 as z. And we need $d(11, 14) = 2$, thus $14 \sim 12$. Consider the edge $(12, 13), (12, 14)$, if $d(13$ must be 4, by symmetry, $d(14)$ must be also 4, then we need $d(9, 14) = 2$ which implies $9 \sim 10$. However, $d(7, 10) = 2$, a contradiction. Thus $d(13) = 4$, the edge $(12, 13)$ must be in a C_4 that passes through vertex 14, then $d(14) = 4$, let $13 \sim 15 \sim 14$. Now the graph has the following subgraph which is isomorphic to subgraph in the "Case 1b2", thus we should exclude this case.



□

In the following part, we consider the last case $d(11) = 4$. By symmetry of vertex 4 and vertex 11, we have $d(4) = 4$. We will prove the Ricci-flat graphs exist and they are G_3 or G_4.

– For case $4 \sim 11 \sim 12$. Note $4 \not\sim 12$ when consider edge $(4, 11)$. Let 13 be the fourth neighbor of 4. Then the edge $(4, 11)$ must be in a C_4 which can only be $4-11-12-13-4$. Consider the edge $(1, 4)$, we have $d(5, 13), d(6, 13)$ must be 2. Consider the edge $(7, 11)$, we have $d(5, 12), d(9, 12)$ must be 2. Similarly, $d(9, 13), d(10, 13), d(10, 12), d(6, 12)$ are all 2.
  Consider the edge $(5, 7)$, since $d(5) \neq 2$, then $(5, 7)$ must be in a C_4 which must pass through vertex 9. Then we need a new vertex 14 as a common for 5 and 9. Note $d(5) \neq 3$ since $d(1) \neq 3$ by Lemma 11. Let $15 \sim 5$ be the fourth neighbor of 5, for the edge $(5, 7)$, since $d(1, 3) = d(1, 11) = 2$, then we need $d(15, 3) = 2$ or $d(15, 11) = 2$, obviously, it must be the latter case. Then $15 \sim 12$.
  For $d(5, 13) = 2$, note $13 \sim 15$ would result edge $(12, 13)$ sharing C_4 and $C_3 := 12-13-15$, which cannot happen. Thus $13 \sim 14$. Then consider the edge $(1, 5)$ which must be in a C_4 and can only pass through edge $(1, 5)$. If $6 \sim 14$, then the edge $(5, 14)$ share two C_4s but $d(15, 13) = 2$, a contradiction. Thus $6 \sim 15$.
  Now consider the edge $(6, 8)$ which must be in a C_4 that can only pass through edge $(8, 10)$. Thus we need a common vertex for 6 and 10. Let $6 \sim 16 \sim 10$. Now consider the edge $(1, 6)$, since $d(8, 0) = d(8, 4) = 2$ and $d(16, 0) = 3$, we need $d(16, 4) = 2$ which lead to $16 \sim 13$.
  Now consider the edge $(2, 9)$ which must be in a C_4 and can only pass through vertices $9-2-10-*-9$. Since all vertices in the current graph have degrees at least 3, we need new vertex 17 as the common neighbor of $9, 10$. Now consider the edge $(7, 9)$, we need

$d(11,17) = 2$, then it must be $17 \sim 12$. Then $d(12) = 4$. Consider the edge $(5,15)$, by Lemma 11 we have $d(15) \neq 3$ as $d(12) \neq 3$. Then $d(15) = 4$, similarly, $d(14) = 4, d(16) = 4, d(17) = 4$. Observe that vertices $14, 15, 16, 17$ are not adjacent to each other, otherwise there would be edges sharing C_3 and C_4. Let $15 \sim 18$ a new vertex. Observe that the edge $(5,15)$ can only be in one $C_4 := 5{-}15{-}6{-}1{-}5$ and must be "Type 6c", then we need $d(18,14) = 2$. Let $14 \sim 19 \sim 18$. Then consider the edge $(13,14)$ which must be in a C_4 that passes through vertices 16 and 19, let $16 \sim 19$. Consider the edge $(9,14)$, we need $d(17,19) = 2$. There are two case, either we need a common vertex for $17, 19$ or $17 \sim 18$. Then consider the edge $(12,15)$ which must be in a C_4 that passes through vertices 16 and 19,we need $17 \sim 18$. A new Ricci-flat graph is obtained:

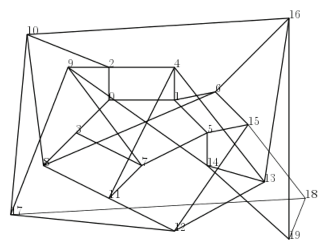


FIG. 3. *Ricci-flat graph G_3*

– For case $4 \not\sim 11$. Let $12, 13$ be the third and fourth neighbor of 11. If $d(5) = 3$ and let z represent the third neighbor of vertex 5. Then the edge $(5,7)$ must be in exactly one C_4 that passes through edge $(5,z)$, similarly, the edge $(1,5)$ should also be in exactly one C_4 that passes through edge $(5,z)$, the edge $(5,z)$ share two C_4s , a contradiction to "Type 5". Thus $d(5) = 4$. Similarly $d(6) = d(9) = d(10) = 4$.
  Consider the edge $(7,11)$. There are two cases: $d(5,12) = 1$ and $d(9,13) = 3$; (2) $d(5,12) = d(9,13) = 2$.
  * Case (1): When $5 \sim 12$, we claim $5 \not\sim 13$. Otherwise we need $d(9,12) = 2$ for the edge $(5,7)$. Then the C_4 that passes through edge $(7,9)$ must pass through vertex 5 then 12 or 13, which implies $9 \sim 12$ or $9 \sim 13$, a contradiction. Let $5 \sim 14$, a new vertex. We claim $9 \sim 12$. Also $d(1,3) = 2$ then $d(9,14) \neq 1$ for the edge $(5,7)$. Since $d(9) = 4$, then the edge $(7,9)$ must be in C_4 that can only pass through vertices 12. Thus $9 \sim 12$. Let $9 \sim 15$. Now for the edge $(5,7)$, consider the $C_4 : -5-7-9-12-5$, we need $d(14,11) = 2$, but if we use $C_4 := 5{-}7{-}11{-}12{-}5$, we need $d(9,14) = 2$. Similarly, for the edge $(7,9)$, either $d(15,11) = 2$ or $d(15,5) = 2$. For $d(15,11) = 2$, either $15 \sim 12$ then the edge $(9,12)$ shares a C_3 and C_4, or $9 \sim 13$, however, both cases cannot happen. Thus we need $d(15,5) = 2$, which implies $15 \sim 4$. Since the edge $(5,12)$ share two $C_4 := 5 - 12 - 11 - 7 - 5, 5 - 12 - 9 - 7 - 5$, then $d(12) \neq 3$. Observe vertex 12 cannot be adjacent to any existing vertices in the current subgraph. Let $12 \sim 16$,a new vertex. However, for the edge $(5,12)$, $d(1,9) =$

$d(1,11) = d(1,16) = 3$, we need $d(14,16) = 1$. Then for the edge $(9,12)$, since $d(2,11) = d(2,16) = d(2,5) = 3$, we need $d(15,16) = 1$, this would result that the edge $(14,16)$ shares a $C_3 := 14 - 15 - 16 - 14$ and a $C_4 := 14 - 16 - 12 - 5 - 14$. A contradiction. Thus the edge $(7,11)$ only shares exactly one C_4, similar analysis indicates each edges of $(8,11), (1,4)$, $(2,4)$ shares exactly one C_4.

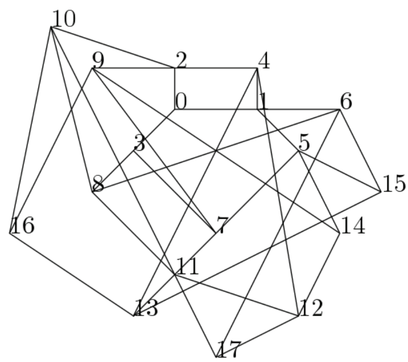
∗ Case (2) For $d(5,12) = 2$, let 14 be the common neighbor of $5,12$. Let 15 be the fourth neighbor of 5. The edge $(5,7)$ must be in a C_4 which can only pass through $5 - 7 - 9 - t - 5$ where t is either 14 or 15.

We assume $15 \sim 12$. Then by symmetry of vertices $14,15$, wlog, let $9 \sim 14$. Note $9 \not\sim 15$, otherwise for $d(9,13) = 2$, we need $14 \sim 13$(similar for $15 \sim 13$). Then we must have $6 \sim 15$ for the C_4 that passes through edge $(1,5)$. However, this would result the edge $(5,15)$ to share three C_4s, a contradiction. Thus $9 \sim 15$. Still consider C_4 that passes through edge $(1,5)$. Note $6 \not\sim 14$ for the edge $(5,14)$, thus $6 \sim 15$. Then consider the fourth neighbor of vertex 9, we need new vertex as the fourth neighbor of 9, let it be 16. For the edge $(7,9)$, we need $d(11,16) = 2$, then either $16 \sim 12$ or $16 \sim 13$. For $d(9,13) = 2$, either $14 \sim 13$ or $16 \sim 13$. Assume $14 \sim 13, 16 \not\sim 13$, then $16 \sim 12$, then the edge $(12,14)$ shares three C_4s, a contradiction. Assume $14 \sim 13$ and $16 \sim 13$. Now we have $d(14) = 4$, then we need $10 \sim 16$ for the C_4 that passes through edge $(2,9)$. Then we need $d(4,14) = 2$ for the edge $(2,9)$ which implies $4 \sim 12$ or $4 \sim 13$. Note $12 \not\sim 4$ for the edge $(12,14)$ and $4 \not\sim 13$ for the edge $(13,14)$, a contradiction. Assume $14 \not\sim 13$, then $16 \sim 13$. Now consider the edge $(5,14)$ which shares two C_4s, thus we need the fourth neighbor of 14 to have distance 3 to vertex 1. For the edge $(2,9)$,if $10 \sim 16$, then we need $d(4,14) = 2$. For these two requirements, we need $4 \sim 12$. Then we also need $4 \sim 13$ for the C_4 passing through edge $(11,12)$. Then consider the edge $(4,13)$, we need the fourth neighbor of 13 to have distance 2 to vertex 1. Then $13 \sim 15$ which is not good for the edge $(5,15)$ as it does not satisfy "Type 6b". The other case for the edge $(2,9)$ is $10 \sim 14$. Then consider the edge $(1,5)$, we need $d(4,14) = 2$ which implies $4 \sim 12$ then $4 \sim 13$. Then still consider the edge $(4,13)$, we need $13 \sim 15$, a contradiction. Thus $15 \not\sim 12$.

Then for the edge $(5,7)$. There are two cases: either $9 \sim 14$ or $9 \sim 15$.

· Assume $9 \sim 14$. We need $d(15,11) = 2$ for the edge $(5,7)$. Note $15 \not\sim 12$, we must have $15 \sim 13$. Consider the C_4 that passes through edge $(1,5)$, assume $6 \sim 14, 6 \not\sim 15$. Then consider the C_4 passing through edge $(5,15)$, we must have $9 \sim 15$, a contradiction for the edge $(5,14)$. Thus Thus under the assumption $9 \sim 14$, we must have $13 \sim 15$ and $6 \sim 15$. Note $9 \not\sim 15$ for the edge $(5,15)$. Let $9 \sim 16$. Consider the C_4 passing through vertices $2,9,10$, assume $10 \not\sim 16$, then

we need $10 \sim 14$, however, the edge $(9,16)$ cannot be in any C_3, C_4, a contradiction. Thus $10 \sim 16$. If $10 \sim 14$, then consider the edge $(10,14)$, we need $d(16,5) = 2$, which implies $16 \sim 15$. Then for the C_4 passing through vertices $8, 6, 10$, we need $16 \sim 6$ which is not good for the edges $(10,16), (6,16)$. Note $10 \not\sim 15$ for the edge $(5,15)$ Thus we need a new vertex as the fourth neighbor of vertex 10, let $10 \sim 17$. Still consider the C_4 passing through vertices $8, 6, 10$, either $6 \sim 16$ or $6 \sim 17$. If $6 \not\sim 17$, then $6 \sim 16$, consider the edge $(6,16)$, since both $d(1,9), d(15,9)$ are 3, we need the fourth neighbor of vertex 16 to have distance 1 from vertex 15. Thus $16 \sim 13$. Then for the C_4 that passes through edge $(10,17)$ we must have $17 \sim 13$, however the edge $(13,16)$ does not satisfy "Type 6b", a Contradiction. Thus, we need $6 \sim 17$.
Still for the edge $(8,10)$, we need $d(16,11) = 2$, for the edge $(8,11)$ we need either $d(6,12) = 2$ or $d(10,12) = 2$. Thus the we needs either $16 \sim 12$ or $17 \sim 12$. Since when $5 \sim 14 \sim 12$, we have $6 \sim 15 \sim 13$, thus for $9 \sim 14 \sim 12$, and $10 \sim 16$, we must have $16 \sim 13$.
For the edge $(1,5)$, we need $d(4,14) = 2$. Note vertices $4, 14$ cannot be both adjacent to 16 or 17 by considering the edges $(2,4), (1,6)$ respectively. Then two cases are left:
Case a: If $4 \sim 12$, then $4 \sim 13$ in order to make $(11,12)$ in a C_4.



Observe that the edge $(12,14)$ must be in a C_4 that passes through edge $(12,17)$, then we need a new vertex as the common of $14, 17$, let it be $18 \sim 14, 17$. Then we need $d(15,18) = 2$ for the edge $(5,14)$, let $15 \sim 19 \sim 18$. Observe that $d(18)$ cannot be 4 since the new edge cannot be in any C_3, C_4, C_5. Similarly, $d(19) = 3$. This is graph G_4.
If we need a new vertex 18 for $d(4,14) = 2$. Let $4 \sim 18 \sim 14$. Then $4 \not\sim 12$, otherwise $4 \sim 13$ which would make $d(4) = 5$, a contradiction. Thus we need a new vertex as the fourth neighbor of vertex 14. Let $4 \sim 19$. For the edge $(5,14)$, we need either $d(1,12) = 3, d(15,18) = 1$ or $d(1,18) = 2, d(15,12) = 2$. Note for the former case, we need $15 \sim 18$, then the edge $(5,15)$ does not satisfy "Type 6". Thus we need the latter case.

For $d(15,12) = 2$, note $12 \not\sim 13$ as the edge $(11,12)$ cannot satisfy "Type 6". We should also reject the case $12 \sim 19 \sim 15$. As the C_4 passing through edge $(4,19)$ must pass through edge $4,18$, then either $18 \sim 15$ or $18 \sim 12$ or 18 is adjacent to a new vertex. The first case was rejected, the second case cannot guarantee the edge $(12,14)$. For the third case, we have $d(2,15) = d(2,12) = 3, d(1,15) = 2, d(1,12) = 3$, then the edge $(4,19)$ does not satisfy "Type 6".

Thus we need a new vertex for $d(15,12) = 2$. Let $12 \sim 20 \sim 15$. Note vertex 16 is not adjacent to vertex 12 for the edge $(9,16)$, then for the edge $(8,11)$, if we need $d(10,12) = d(6,13) = 2$, then it must be $17 \sim 12$ and $17 \sim 13$, which would result the edge $(13,17)$ to share three C_4s. Then we need $d(10,13) = d(6,12) = 2$ which only implies $17 \sim 12$. Then $d(17) = 4$, let z represent the fourth neighbor of vertex 17. Consider the edge $(10,17)$, as $d(2,12) = d(2,z) = 3$ and $16 \not\sim 12$, we need $16 \sim z$. However, the edge $(10,16)$ would share two $C_4 := 10-16-9-2-10, 10-16-z-17-10$ but the third pair neighbors has $d(12,13) = 2$, a contradiction to "Type 6b".

· Next we consider the case when $9 \sim 15$. Since we need $d(9,13) = 2$, there are two cases: Case 1 $15 \sim 13$, we ignore this case since it can be compared when $5 \sim 14 \sim 12 \sim 11, 9 \sim 14$.

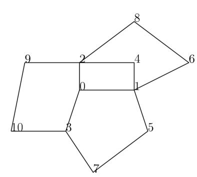
Case 2: $9 \sim 16 \sim 13$. Consider the edge $(2,9)$ which should be in a C_4 that passes through vertex 10, there are two cases: Case a: $10 \sim 15$. Then we need $d(4,16) = 2$. Obviously, $d(15) \neq 3$. Let $d(15) = 4$. For the edge $(1,5)$ which should be in a C_4 that passes through vertex 6, assume $6 \sim 15$, then consider the edge $(6,15)$, we need $d(5,z) = 3$ where z is the fourth neighbor of 6. Note $6 \not\sim 12, 13$ since the edge $(8,11)$ should be in exactly one C_4, $6 \not\sim 16$. Then we need a new vertex 17 as the fourth neighbor of 6. Consider the edge $(8,10)$, we need the fourth neighbor of 10 has distance 2 to vertex 7, since $10 \not\sim 12, 13$, then 10 must be adjacent to vertex 14. However, this is not good for the edge $(5,15)$. Thus under the condition $10 \sim 15$, we have $6 \not\sim 15$. Then for the edge $(1,5)$ to be in a C_4, we need $6 \sim 14$. Consider the edge $(8,6)$, note $10 \sim 14$ is not good for the edge $(5,14)$, then we need a new vertex for the edge $(8,6)$ to be in a C_4, let it be $10 \sim 17 \sim 6$. Consider the edge $(5,15)$, note $d(15) \neq 3$, since both $d(10,1) = d(9,1) = 3$. However, $15 \not\sim 12, 13, 16, 17$. Let $15 \sim 18$, then we need $14 \sim 18$, however, this is not good for the edge $(5,14)$. Now we consider the last one. Case b: $10 \sim 16$. For the edge $(1,5)$, assume $6 \sim 15$. Note $d(15) \neq 3$, since both $d(5,8) = d(9,8) = 3$. Then $d(15) = 4$, note $15 \not\sim 16$ for the edge $(9,15)$. Assume $15 \sim 13$, then consider the edge $(2,9)$, we need $d(4,15) = 2$, then $4 \sim 13$. Then for the edge $(11,13)$ to be in a C_4, we must have $4 \sim 12$. Consider the edge $(4,12)$, since both vertex 14 and the fourth neighbor of 12 have distance 3 to vertex 0, and $d(1,14)=2$, a contradiction. Thus $15\sim 17$ a new vertex and $4\sim 17$. Consider the edge $(6,15)$, we need $d(8,17)=2$, then $17\sim 10$, the edge $(2,4)$ is in two C_4, which is rejected before. Thus we have arrived another contradiction.

□

Next we will reject the other case "Type 5b" for the edge with endpoint degrees $\{3,4\}$.

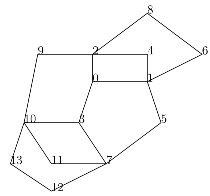
Lemma 12. *Let G be a graph with maximum degree at most 4, let (x,y) be an edge in G with $d(x)=3, d(y)=4$ that satisfies "Type 5b". Then G cannot be a Ricci-flat graph.*

Proof. Assume G contains an edge $(0,1)$ with $d(0)=3, d(1)=4$, $(0,1)$ is contained in exactly one $C_4: 0-2-4-1-10$, and $d(3,5)=2, d(3,6)=3, d(2,6)=2$. By contradiction, we assume G is Ricci-falt. As this is the last case of "Type 5", then for any other edge (x',y') in G with endpoint degrees $\{3,4\}$, there must be an isomorphism from the neighborhood $N(0)\cup N(1)$ to neighborhood $N(x')\cup N(y')$. Let $5\sim 7\sim 3$, $2\sim 8\sim 6$. Since $d(3,6)=3$, we need a new vertex as the third neighbor of 3. Note $2\not\sim 7$, otherwise $(0,2)$ would be in two C_4s. Thus we need a new vertex as the fourth neighbor of 2. Let $2\sim 9$. $3\sim 10$. Note for the edge $(0,2)$ since $(d(0),d(2))=(3,4)$, we also have either $d(3,9)=2, d(3,8)=3, d(1,8)=2, d(1,9)=2$ or 3 or $d(3,8)=2, d(3,9)=3, d(1,9)=2, d(1,8)=2$ or 3. The latter case is not preserved under isomorphism then we consider the former case. For $d(3,9)=2$, either $9\sim 10$ or $9\sim 7$, since the edge $(0,3)$ share two separate C_5s, we have $9\sim 10$.



Note $7\not\sim 6,8$, $10\not\sim 6,8$. If $d(7)=3$, then the edge $(3,7)$ shares two C_5s, which implies $d(10)=3$. Similarly, if $d(10)=3$, then $d(7)=3$. We will first exclude the following case:

- If $d(10)=4$, then the edge $(3,10)$ must be in a C_4 which must pass through vertex 7. Then $d(7)=4$. Assume $10\sim 5$ and $10\sim 11$, a new vertex. Then we need $d(7,11)=2$. Let $7\sim 12\sim 11$. Consider the fourth neighbor of 7, note $7\not\sim 9$. Otherwise the edge $(3,10)$ would be contained in two C_4, a contradiction. Then $7\sim 13$, a new vertex. Then consider the edge $(3,7)$, we have both $d(0,12), d(0,13)$ are 3, a contradiction. Thus for the $(3,10)$ to be in a C_4 that passes through vertex 10, let $10\sim 11\sim 7$, $7\sim 12$ and $10\sim 13$ and we still need $d(7,13)=2$, for the edge $(3,7)$, we need $d(10,12)=2$. Let $12\sim 13$.



Consider $d(5)$, if $d(5) = 2$, we need one of $d(1, 12) = 2$, $d(1, 11) = 2$ for the edge $(5, 7)$, note we also need one of $d(7, 6) = 2$, $d(7, 4) = 2$ for the edge $(1, 5)$. By symmetry of the current graph, there are three different cases for the two conditions. Case 1: Assume $12 \sim 4$ and $12 \sim 6$, then $d(12) = 4$, the edge $(7, 12)$ must be in a C_4 that passes through vertex 11, then either $11 \sim 6$ which is not good for the edge $(6, 12)$, or $11 \sim 4$ which is not good for the edge $(1, 4)$. Case 2: Assume $12 \sim 4$, $11 \sim 6$. Then still the edge $(7, 12)$ must be in a C_4 that passes through vertex 11, then $11 \sim 4$ which is not good for the edge $(1, 4)$. Case 3: $11 \sim 4$. If $d(4) = 3$, then $d(11) = 3$, the current graph is all good except the edge $(6, 8), (12, 13)$, however, there is no way to generate a Ricci-flat graph. Thus $d(4) = 4$. Since $4 \not\sim 6, 8, 12, 13$, we need a new vertex as its fourth neighbor, let it be 14. Note $11 \not\sim 14$, let $11 \sim 15$. Since we need the edge $(4, 11)$ to be in a C_4, it must be $14 \sim 15$. Then we need $d(6, 14) = d(8, 14) = d(12, 15) = d(13, 15) = 2$ for edges $(1, 4), (2, 4), (7, 11), (10, 11)$ respectively. Then $d(6) \neq 2$, however the edge $(1, 6)$ cannot be in any C_4, a contradiction. Thus $d(5) \neq 2$. If $d(5) = 3$, let z represent the third neighbor, then the edge $(5, 7)$ must be in exactly one C_4 passing through edge $(5, z)$, the edge $(1, 5)$ should also be in exactly one C_4 passing through ege $(5, z)$, then the edge $(5, z)$ is in two C_4s, a contradiction to Lemma 8. Thus $d(5) = 4$. Similarly $d(9) = 4$.

Consider the edge $(1, 5)$, if it is contained in a C_3, then $5 \sim 6$, then we need the fourth neighbors of vertex 5 and vertex 6 to have distance 3 from vertex 4 . Assume $5 \sim 12$, then $d(12) = 4$. For the edge $(5, 6)$, assume $d(8, 7) = 2$, however neither $8 \sim 12$, otherwise the $(5, 6)$ would share a C_3 and C_4, nor $8 \sim 11$ considering the edge $(5, 7)$. Similarly, $13 \not\sim 6, 4$. Assume $d(8, 12) = 2$, consider the case $8 \sim 13$. Note if the edge $(6, 8)$ is in a C_3, it must be $C_3 := 6 \sim z \sim 8$, where z is the a new vertex, however, we have $d(1, 2) = d(5, 13) = 2$, a contradiction. Thus edge $(6, 8)$ must be in a C_4. Let $6 \sim 14$, assume $14 \sim 13$. Then the edge $(12, 13)$ must be in a C_4 that passes through the fourth neighbor of 12,let it be 15. Note $15 \not\sim 8$ for the edge $(8, 13)$, then it must be $15 \sim 14$. Note we need the edge $(10, 13)$ to be in a C_4, however, there is no C_4 satisfying the current structure. Thus for the edge $(6, 8)$ to be in a C_4, it must be $14 \sim 15 \sim 8$. Similarly we need $12 \sim 16 \sim 17 \sim 13$ for edge $(12, 13)$ to be in a C_4. Then consider the edge $(8, 13)$, it must be in the $C_4 := 8 - 13 - 17 - 15 - 8$. Then there is no C_4 for the edge $(10, 13)$. Thus $8 \not\sim 13$. Hence, for $d(8, 12) = 2$, let $8 \sim 14 \sim 12$. For the edge $(6, 8)$ in a C_4, let $6 \sim 15 \sim 16 \sim 8$. Then consider the edge $(2, 8)$ which must be in a C_4, assume the C_4 passes through vertex 4, then $d(4) = 4$. Consider the edge $(1, 4)$ since each of the third and fourth neighbors of 4 has distance 3 to vertex 5, then one of them must have distance 1 to vertex 6, then the edge $(1, 6)$ shares a C_3 and C_4, a contradiction. Then the C_4 for edge $(2, 8)$ must pass through vertex 9. Assume $9 \sim 14$. Observe that since $4 \not\sim 14, 15$, then any neighbor of vertex 4 has distance 3 to vertex

5, in addition, we have $d(2,5) = 3$, then $d(4) \neq 3$. Let $d(4) = 4$ with the third and fourth neighbors w, l, then we need $d(8,w) = d(9,l) = 2$. However, $d(8,w) = 2$ cannot be true.

Thus the edge $(1,5),(5,7)$ must be in C_4s. Assume the C_4 passes through vertex 4, then $d(4) = 4$, let $5 \sim 14 \sim 4$. Then we need the fourth neighbor of vertex 5 to have distance 2 from vertex 6. Assume $5 \sim 13$, then $d(13,6) = 2$. Consider the edge $(5,13)$, since both $d(10,1) = 3, d(12,1) = 3$, then $d(13) \neq 3$. Let $d(13) = 4$, we need the fourth neighbor of 13 to have distance 1 to vertex 14. Note if $13 \sim 4$, we need $d(13,6) = 3$ for the edge $(1,4)$, a contradiction. Assume $13 \sim 8$, then we need $d(14,10) = 2$ for the edge $(5,13)$ and $d(14,11) = 2$ for the edge $(5,7)$. Thus $14 \sim 9$ for $d(14,10) = 2$, let $14 \sim 15 \sim 11$ for $d(14,11) = 2$. Then vertex 4 must be adjacent to a new vertex 16. Note $d(6,16) = d(8,16) = 3$ for the edges $(1,4),(2,4)$ respectively, then for the edge $(16,4)$ to be in any C_4, it must pass through vertex 9, that is $9 \sim 16$. However, it is not good for the edge $(2,9)$. Thus let $13 \sim 15 \sim 14$. For the edge $(5,7)$, we need $d(14,11) = 2$. Assume $11 \sim 15$, then we need $d(5,9) = 2$ for the edge $(10,13)$ which implies $14 \sim 9$. Consider the edge $(9,10)$ which must be in a C_4 that passes through the fourth neighbor of 9. Note $9 \not\sim 15$ for the edge $(9,15)$, if $9 \sim 12$, then $6 \not\sim 12$ or $6 \not\sim 13$ for the edge $(10,13)$, a contradiction to $d(13,6) = 2$. Thus for the edge $(9,10)$ to be in a C_4, we need $9 \sim 16 \sim 11$. However, the edge $(10,11)$ is contained in three separated C_4s, a contradiction. Thus there is no way to form a C_4 for the edge $(9,10)$. Then for $d(14,11) = 2$, we assume $11 \sim 4$. Then the edge $(4,11)$ must be in a C_4 that passes through vertex 14, let $14 \sim 16 \sim 11$, however, we cannot guarantee the edge $(4,14)$. Then for $d(14,11) = 2$, assume $14 \sim 9$. Note $11 \not\sim 4$ or $11 \not\sim 15$. Then still for the edge $(9,10)$ to be in a C_4, we need $9 \sim 16 \sim 11$. Then $d(11) = 4$, let z be the fourth neighbor of 11, we need $d(11,13)$ for the edge $(10,11)$, However, there is no way to form a C_4 for the edge $(10,13)$. A contradiction. Thus for $d(14,11) = 2$, we need a new vertex 16 with $14 \sim 16 \sim 11$. However, now the edge $(5,14)$ is in two C_4s, but the third pair of neighbor $(16,7)$ which are not on the C_4s, has distance $d(16,7) = 2$, a contradiction.

Thus the C_4 for the edge $(1,5)$ does not passes through vertex 4, then it must pass through vertex 6. Then the edges $(1,6)$ must be in a C_4, so do edges $(7,12),(2,8),(10,13)$. Let $5 \sim 14 \sim 6$. Then we need the fourth neighbor of vertex 5 to have distance 2 from vertex 4. Similarly, we need the fourth neighbor of vertex 5 to have distance 2 from vertex 11. Note $14 \sim 8$ otherwise, the edge $(6,14)$ would share a C_3 and C_4. Thus for the edge $(2,9)$ to be in a C_4, let $9 \sim 15 \sim 8$. Then we need the fourth neighbor of vertex 9 to have distance 2 from vertex 4. Thus $5 \not\sim 9$. Assume $5 \sim 8$, then for the edge $(5,7)$ to be in a C_4, it must be $C_4 := 5 - 7 - 12 - 14$. Note we need $d(8,11) = 2$, since $11 \not\sim 15$, then $11 \sim 6$. Consider the edge $(6,8)$, we need $d(11,15) = 2$. For the edge $(5,8)$, we need $d(7,15) = 2$, thus 15 must be adjacent to vertex 12 and $11 \sim 16 \sim 15$. Since for the edge $(1,6)$, we need $d(4,11) = 2$, then $4 \sim 16$. However, we cannot guarantee the edge $(6,11)$.

Thus $5 \not\sim 8$, similarly, $5 \not\sim 13$, $9 \not\sim 6$ and $9 \not\sim 12$. If $5 \sim 15$, we need a new vertex 16 for $d(4,15) = 2$, let $15 \sim 16 \sim 4$, since $d(15)$ is at most 4, then for $d(11,15) = 2$, we need $11 \sim 16$. Now observe that the C_4 for the edge $(5,15)$ must pass through vertex 14. Note $14 \sim 9$ is not good for the edge $(9,15)$,

then it must be $14 \sim 16$, also we need $d(7,9) = 2$, which implies $9 \sim 11$ or $9 \sim 12$, contradiction. Thus $5 \not\sim 15$. Similarly, $9 \not\sim 14$.
Let $5 \sim 16$. For $d(16,4) = 2$, assume $4 \sim 12 \sim 16$, since we also need $d(z,4) = d(w,4) = 2$ where z, w represent the neighbors of vertices $6, 8$ respectively, then the fourth neighbor of vertex 4 should be adjacent to vertices $4, z, w$, then the fourth neighbor of vertex 4 should have degree at most 1 in the current graph, which implies it must be a new vertex. Let $4 \sim 17$. Note $d(17) \geq 3$, then the edge $(4,17)$ must be in a C_4 that can only pass through vertex 12, then either $17 \sim 16$ or $17 \sim 13$. Assume $17 \sim 16$, then we need $d(2,13) = 2$ for the edge $(4,12)$, which implies $8 \sim 13$. Then $17 \sim 13$ for $d(13,4) = 2$, however the edge $(8,13)$ cannot be in any C_4, a contradiction. Assume $17 \sim 13$, then we need $d(2,16) = 2$ for the edge $(4,12)$, note $16 \sim 8$ implies $17 \sim 8$, thus we need $16 \sim 9$. Note we need $d(4,11) = 2$ for the edge $(7,12)$, then $11 \sim 17$. However, we cannot guarantee the edge $(10,11)$. Thus for $d(16,4) = 2$, we need a new vertex 17 such that $4 \sim 17 \sim 16$.
Now consider the fourth neighbor of vertex 6. Assume $6 \sim 12$, then $12 \sim 17$, however the edge $(5,7)$ cannot be in any C_4 through vertex 12. Assume $6 \sim 13$, then $13 \sim 17$, however, we cannot guarantee the edge $(6,13)$. Note $6 \not\sim 15$, otherwise the edge $(8,15)$ would share a C_3 and C_4. Assume $6 \sim 16$, then the edge $(6,16)$ is in two $C_4 := 6-16-5-1-6, 6-16-5-14-6$, thus $d(16) = 4$. However, $16 \not\sim 12$ for the edge $(5,16)$, thus we need $14 \sim 12$ for the edge $(5,7)$ in a C_4. Note for the edge $(6,16)$, we need 14 to have distance 2to the fourth neighbor of 16, but we also need the fourth neighbor of vertex 14 has distance 3 to vertex 16 considering edge $(5,14)$, a contradiction. Thus the fourth neighbor of vertex 6 should be a new vertex 18, and $18 \sim 17$.
Assume $8 \sim 19 \sim 17$ for the edge $(1,8)$. Then $d(17) = 4$, consider the edge $(4,17)$ which must be in a C_4 that passes through vertex 16. Now consider the fourth neighbor of vertex 4. Assume $4 \sim 12$, then $16 \sim 12$, then we need $d(2,13) = 2$ for the edge $(4,12)$ which implies $13 \sim 9$, a contradiction. Assume $4 \sim 13$, then $16 \sim 13$, then we need $d(1,12) = 2$ for the edge $(4,12)$ which is not possible. Note $4 \not\sim 14$ and $4 \not\sim 15$. Thus we need a new vertex $4 \sim 20 \sim 16$. Then consider the edge $(5,16)$ which must be in a C_4 that passes through vertex 14. If $14 \sim 20$, then we need the fourth neighbor of vertex 16 to have distance 2 to vertex 7, then $16 \not\sim 12$, we must have $14 \sim 12$ for the edge $(5,7)$ to be in a C_4. However, this is not good for the edge $(5,14)$. Assume $17 \sim 14$, then we need $16 \sim 12$ for the edge $(5,7)$ to be in a C_4. However, this is not good for the edge $(5,16)$. For all the other possible cases such that the edge $(5,16)$ is in a C_4(that is $d(14,16) = 2$), we must have $4 \sim 12$ consider the edge $(5,7)$ in a C_4. Then the edge $(5,14)$ shares two separated C_4s, we need the fourth neighbor of 14 to have distance 3 from vertex 16, a contradiction.
Assume $8 \sim 18$ for the edge $(1,8)$. The the edge $(6,8)$ shares a $C_3 := 6-8-18-6$. Thus we need $d(14,15) = 3$. Consider the edge $(6,14)$, note the new neighbor of vertex 14 should have distance 2 from vertex 8, then it must be adjacent to vertex 15 which would make $d(14,15) = 2$, a contradiction.

- We consider the case $d(7) = d(10) = 3$. Let $7 \sim 11, 10 \sim 12$, then $11 \sim 12$ for the edge $(3,7)$. Consider the edge $(3,7)$ and edge $(3,10)$, we have $d(5) = d(11)$ and $d(9) = d(12)$ by Lemma 10. There are two cases for $d(4)$.

Case 1: assume $d(4) = 3$. Note $4 \not\sim 11$, otherwise there is an isomorphism ϕ that maps the current subgraph to subgraph 3. In the following, we call the latter graph as " base graph" We must have $\phi(4) = 0, \phi(0) = 4$. There are two cases for ϕ depending on $\phi(1)$. If $\phi(1) = 1$, then follow the isomorphism we must have $\phi(10) = 12, \phi(9) = 9$. Since $10 \sim 9$ in base graph, we must have $12 \sim 9$, a contradiction. Similar analysis for case $\phi(1) = 2$. Thus the neighborhood of edge $(1, 4)$ is not preserved under isomorphism. Similarly, $4 \not\sim 12$. Note vertex 4 cannot be adjacent to any other existing vertices. Thus let $4 \sim 13$, a new vertex. Consider the isomorphism ϕ from current graph to base graph, we have $d(13) = d(3) = 3$. Let $13 \sim 14, 15$, then $d(14) = d(7) = 3, d(15) = d(10) = 3$. If $\phi(1) = 1$, then $\phi(5) = 6$ and $\phi(6) = 6$. Since $\phi(14) = 7$, then we must have $14 \sim 5$ as $7 \sim 5$ in the base graph. Similar analysis for $15 \sim 9$.
Consider the neighbors for vertices 14 and 15. Note $14 \not\sim 6$ or $14 \not\sim 8$ or or $14 \not\sim 9$ for edges $(1, 4), (2, 4), (13, 14)$ respectively. Assume $14 \sim 11$. Then $\phi(11) = 11$. Consider the edge $(13, 14)$, we must have $15 \sim 12$ and $\phi(12) = 12$. Then there is a $C_4 := 5 - 14 - 11 - 7 - 5$, thus $d(5) = d(11) = 4$.
Now we consider a new isomorphism ϕ between the neighborhood of edges $(7, 5)$ and $(0, 1)$. Let $\phi(5) = 1, \phi(7) = 0$, then ϕ is determined. We need the fourth neighbors of 5 to be mapped to 6 and the fourth neighbor of 11 to be mapped to 8. Consider vertices 6 and 8, note $6 \not\sim 5$ for edge $(1, 5)$, similarly, $8 \not\sim 9$. If $5 \sim 8$, then $11 \sim 6$. However, we cannot guarantee the edge $(1, 6)$. Thus $5 \not\sim 8, 11 \not\sim 6$. Similarly, $9 \not\sim 6, 12 \not\sim 8$, then all vertices $5, 6, 8, 9, 11, 12$ need new vertices, however, the edges $(1, 6), (2, 9)$ are not good under this situation as they cannot be in any C_3 or C_4 and $d(6) = d(8) = 2$ is not true. A similar analysis for case $\phi(5) = 2$. Thus $11 \not\sim 14$. Similarly, $12 \not\sim 15$.
Assume $14 \sim 12$. Still consider the isomorphism ϕ between the current graph to the base graph that satisfies $\phi(1) = 1, \phi(4) = 0$. Since $14 \sim 12$, then $\phi(12) = 11, \phi(11) = 12$, we must have $phi(10) \sim \phi(12)$, i.e $15 \sim 11$. Then we consider a new isomorphism ϕ between the neighborhood of edges $(7, 5)$ and $(0, 1)$. Similarly we cannot guarantee the edge $(1, 6)$.
Consider the last case: let $14 \sim 16$ and $15 \sim 17$. Under the isomorphism ϕ between the current graph to the base graph that satisfies $\phi(1) = 1, \phi(4) = 0$, we must have $\phi(16) = 11, \phi(17) = 12$, then $16 \sim 17$. If $d(5) = 3$, then the edge $(5, 7)$ must share two C_5s which needs $11 \sim 16$ and $12 \sim 17$. However, the edge $(1, 5)$ cannot be in any C_4, a contradiction. Thus we conclude that $d(4) = 4$.
Case 2: $d(4) = 4$. Note $4 \not\sim 11$, otherwise $4 \sim 12$ by symmetry, then $d(11) = d(12) = 4$ and they are adjacent to two new vertices by "Type 6a". Let $11 \sim 13, 12 \sim 14$. We need $d(2, 13) = d(2, 14) = 2$. Then the edge $(7, 11)$ must be in a C_4 passing through vertices 5 and 13. Let $5 \sim 13$, similarly, let $9 \sim 14$. Then we consider the isomorphism ϕ between the current graph to the base graph with $\phi(7) = 0$ and $\phi(11) = 1$. Then we have $\phi(3) = 3, \phi(12) = 5, \phi(4) = 6, \phi(2) = 12$, since $2 \sim 4$, then we have $12 \sim 6$, a contradiction. A similarly analysis for the isomorphism ϕ with $\phi(7) = 0$ and $\phi(11) = 2$.
As 4 cannot be adjacent to any other existing vertices, then let $4 \sim 13, 14$. Consider the edge $(1, 4)$, if it satisfies "Type 6b", then wlog, either $5 \sim 13$ or $6 \sim 13$. Case 2a: Assume $5 \sim 13$, then $d(6, 14) = 3$.
If further $d(5) = 3$, then $6 \not\sim 13$. Observe the edge $(1, 6)$, it cannot be in any

C_3 or C_4, thus $d(6) = 2$. Then we have $d(8) = 4$ considering the edge $(6, 8)$. Then under the isomorphism between neighborhoods of edges $(1, 5)$ and $(0, 1)$, we must have $13 \sim 8$ and need a new vertex 15 such that $11 \sim 15 \sim 13$. For the edge $(2, 8)$, we need the fourth neighbor of 8 to have distance 2 from vertex 9 by "Type 6b". Note $8 \not\sim 14$ or $9 \not\sim 14$ considering edge $(1, 4)$ and $(2, 4)$ respectively. Then the edge $(4, 14)$ can only be in a C_5 which implies $d(14) = 2$ and the fourth neighbor of 14 has distance 2 from vertices 2 and 13. Note $8 \not\sim 11, 12$ considering edge $(4, 14)$. Then we need a new vertex 16 as the fourth neighbor of vertex 8 such that $d(16, 9) = 2$. Then the edge $(2, 9)$ which cannot be in any C_3 or C_4, then $d(9) = 2$, however, we then cannot guarantee $d(16, 9) = 2$. A contradiction. Thus $d(5) = 4$. The edge $(5, 7)$ must be in a C_4 passing through vertex 11, implying $d(11) = 4$. Note if $13 \sim 11$, one cannot find isomorphism from the current graph to the base graph. Thus we need a new common vertex of 5 and 11. Note $5 \not\sim 12$ as under any isomorphism between neighborhood of $(0, 1)$ and $(7, 5)$, vertex 12 is mapped to vertex 4, vertex 3 is mapped to vertex 4, but the $d(3, 12) \neq d(3, 4)$. One can check vertex 9 or 8 or 14 is not the common for vertices 5 and 11. Thus we need a new vertex 15 as the common for vertices 5 and 11. Let $5 \sim 15 \sim 11$.
Consider two different maps between neighborhood of $(0, 1)$ and $(7, 5)$ specified by whether vertex 5 is mapped to vertex 1 or 2. For the latter isomorphism ϕ. See Figure 4 for illustration. Observe that 4 is not mapped to vertex 6, otherwise $4 \sim 11$ as 11 is mapped to vertex 1 and $1 \sim 6$, then $d(4) \geq 5$, a contradiction. Thus we need a new vertex as the image of vertex 6, let it be 16, then $11 \sim 16$, we also have $13 \sim 16$ as 13 is mapped to vertex 8 and $8 \sim 6$. Note vertices 12, 9 should share one more vertex by the isomorphism. Assume 14 is this common vertex. As vertices 9 and 12 cannot have other common vertex except 10, 14, then vertex 14 must be mapped to vertex 15. Then vertex 4 is mapped to vertex 14. Since $4 \sim 14$, then $14 \sim 15$ by isomorphism, again $15 \sim 4$ as 14, 15 are mapped to 15, 4. Then $d(4) \neq 5$, a contradiction. It is easy to check no existing vertices can be the common for vertices 9 and 12. Thus we need a new vertex 17 and $9 \sim 17 \sim 12$. Then $\phi(4) = 17$. Since $4 \sim 13$, then $17 \sim 8$ by ϕ and $\phi(17) = 15$. Consider vertex 8, note $d(8) \neq 3$ as there is no isomorphism between neighborhood of edges $(2, 8)$ and $(0, 1)$. Then $d(8) = 4$. Consider the edge $(8, 17)$, which is in the $C_4 = 8-17-9-2-8$. Note $12 \not\sim 8$, otherwise vertex 8 is the image of vertex 13, since which implies $12 \sim 13$ as 12, 13 are the images of vertex 2, 8 and $2 \sim 8$. This would result $12 \neq 5$, a contradiction. Similar analysis for $12 \not\sim 13$ or $12 \not\sim 14$ or $12 \not\sim 6$. Thus we need a new vertex as the fourth neighbor of vertex 12, let $12 \sim 18$. Then $\phi(18) = 3$ and $\phi(8) = 18$. Since $\phi(4) = 15$, then $15 \sim 18$. Now consider the edge $(11, 12)$ which satisfies "Type 6c". Then we need $d(16, 17) = 2$, then $d(\phi(16), \phi(17)) = d(15, 6) = 2$. Then consider the edge $(1, 5)$, we must have $6 \sim 13$ and $d(4, 15) = 2$ which implies $14 \sim 15$. Thus $\phi(14) = 14$, and $17 \sim 14, 18 \sim 16$.
Then consider the edge $(1, 6)$ which satisfies "Type 6c". We need a common vertex for 14 and the fourth neighbor of vertex 6. Similarly, for the edge $(2, 4)$ which satisfies "Type 6c", we need a common vertex for 14 and the fourth neighbor of vertex 9. Thus vertices 6 and 9 should share the fourth neighbor, otherwise $d(14) \geq 5$. Let $9 \sim 19 \sim 6$ and $14 \sim 19$. We also have $\phi(19) = 16$. Since $19 \sim 14$, we have $\phi(19) \sim \phi(14)$, i.e. $16 \sim 14$, then $d(14) \geq 5$. A

contradiction.

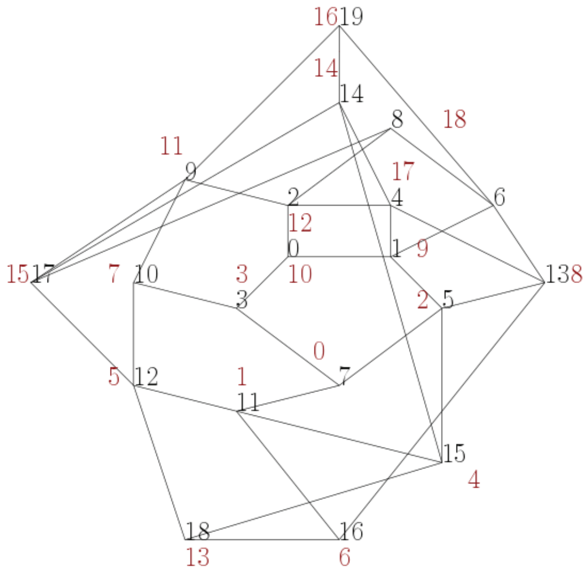
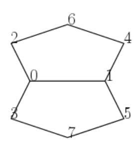


FIG. 4. *The red labels represent the image of black labels*

Case 2b: The case $6 \sim 13$ is included in above case. We omit the proof here. We have rejected all cases based on the structure stated in the theorem, thus there is no Ricci-flat graph of "Type 5b". □

THEOREM 4. *Let G be a Ricci-flat graph with maximum degree at most 4, if there exists an edge (x, y) with endpoint degree $(d(x), d(y)) = (3, 3)$, and (x, y) is contained in exactly two separate C_5s. Then G is one of the following: the dodecahedral graph, the Petersen graph, the half-dodecahedral graph and the Triplex graph.*

Proof. See the following subgraph of G, let $x = 0, y = 1$.



By Theorem 2, if $d(2) = d(3) = 2$ then $d(4), d(5)$ must be 3, and the graph is Half-dodecahedra, refer to Theorem 2. If $d(2) = d(3) = 4$ or $d(4) = d(5) = 4$, refer to Theorem 3. Thus we only need to consider the following case: $d(2) = d(3) = 3$, $d(4) = d(5) = 3$. Since $(0, 3)$ cannot be in any C_4, then 3 is not adjacent to 6 or 4, similarly, 2 is not adjacent to 7 or 5, 4 is not adjacent to 7 and 5 is not adjacent to 6. Thus vertices $2, 3, 4, 5$ need new neighbors, let $8, 9$ be the neighbors of vertices $2, 3$ respectively. If $d(6) = 2$ or $d(7) = 2$, then the edge $(2, 6)$ or $(3, 7)$ satisfy the condition in Theorem 2. Thus G is the half-dodecahreal graph. If $d(6) = 4$ or $d(7) = 4$, then the edge $(2, 6)$ or $(3, 7)$ satisfy the condition in Theorem 3. Thus G is determined.

Then we consider the case when $d(6) = 3, d(7) = 3$. Note all edges in the current structure are not in any C_4. Consider the edge $(2, 8)$ if it is in a C_4, then the C_4 must pass through the edge $(2, 6)$, a contradiction. Thus $d(8) = 3$. The edge $(2, 8)$ cannot be in any C_4, then the third neighbor of vertex 8 has degree 3. Similarly,

$d(9) = 3$ and its third neighbor also has degree 3. Following this process, there is no vertex with degree 4, then no edge in any C_4. Refer to [7], then G must be one of the dodecahedral graph, the Petersen graph, the Triplex graph. □

4. Ricci-flat graphs with vertex degrees $\{2, 4\}$. In previous sections, we have finished all cases when an edge has endpoint degrees $(2, 2), (2, 3), (3, 3), (3, 4)$. In this section, we consider the Ricci-flat graphs that contain edges with endpoint degrees $(2, 4), (4, 4)$, we first classify these that contain a copy of C_3.

4.1. Ricci-flat graphs that contain C_3.

THEOREM 5. *Let G be a Ricci-flat graph with maximum degree at most* 4 *and there exist an edge $e = (x, y)$ with $d(x) = d(y) = 4$ such that e is contained in a C_3. Then the graph is isomorphic to graphs G_5, G_6 and G_7.*

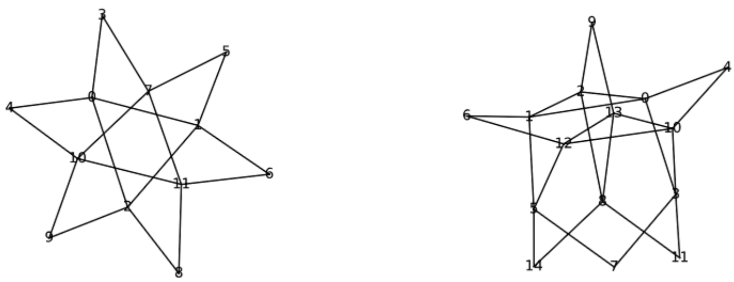


FIG. 5. *Ricci-flat graph G_5, G_6*

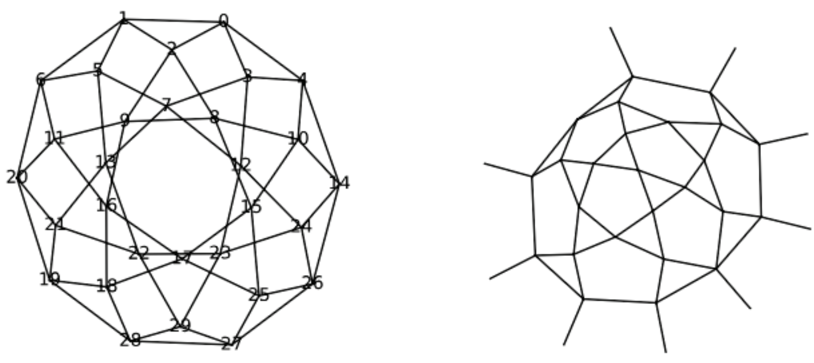
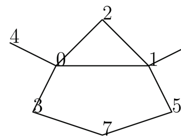


FIG. 6. *Ricci-flat graph G_7 and its local structure*

REMARK 2. *In Geometry, G_7 is a polyhedron called icosidodecahedron with* 20 *triangular faces,* 12 *pentagonal faces, and* 30 *vertices,* 60 *edges.*

Proof. Look at the follow subgraph with $x = 0, y = 1$. Since $(0, 2)$ is in C_3, then $d(2) = 4$. Note vertex 2 cannot be adjacent to any existing vertices, let $8, 9$ be the its new neighbors. Note we will exclude the case when there is an edge with endpoints degree $3, 4$.



- When $d(3) = 2$, then $d(7) = 4$. Then at least one of $d(7, 2) = 2, d(7, 4) = 2$ is true consider edge $(3, 7)$.
  - Assume $d(7, 2) = 2$, wlog, let $7 \sim 8$. Then $d(4, 9) = 3$ for the edge $(0, 2)$ and $d(6, 9) = 3$ for the edge $(1, 2)$. In this case, $7 \nsim 9$. Otherwise, we have $d(5, 8) = 3$ for the edge $(1, 2)$ and $d(8) = 4$, let $8 \sim 10, 11$, since $d(3, 10) = d(3, 11) = 3$, we need $d(5, 10) = 1$ or $d(5, 11) = 1$ which would result $d(5, 8) = 2$, a contradiction. Note the edge $(0, 4)$ cannot be in any C_3 or C_4, thus $d(4) = 2$, and the edge $(0, 4)$ is in a C_5. Assume $d(5) = 2$, then similarly, $d(6) = 2$. Let $4 \sim 10$, note for the edge $(0, 4)$ to be in a C_5, if $d(10, 1) = 2$ then $10 \sim 6$ which would result $d(4, 6) = 2$, a contradiction. Thus we need $d(10, 3) = d(10, 2) = 2$, then $10 \sim 7$ and $10 \sim 8$. Since vertices $8, 10$ are in a C_3, then $(8) = d(10) = 4$. Then the edge $(2, 8)$ must be in a C_4 that passes through vertices: $C_4 := 2-8-9-11-2$. Now consider the edge $(1, 6)$ since $d(6) = 2$, and 6 cannot be adjacent to any existing vertices, then let $6 \sim 12$, however, none of $d(12, 5), d(12, 2), d(12, 0)$ can be 2. A contradiction. Let $d(5) = 4$. For the edge $(0, 4)$ to be in a C_5, if we still assume $10 \sim 7$ and $10 \sim 8$. Then $(8) = d(10) = 4$. By same reasons as above, let $8 \sim 11 \sim 9$, $10 \sim 12$. Observe that the edge $(5, 7)$ cannot be in any C_3 or C_4, a contradiction. Thus for the edge $(0, 4)$ to be in a C_5, we must have $10 \sim 5$ and exactly one of $10 \sim 7$ and $10 \sim 8$. Assume $10 \sim 7$. Note $5 \nsim 6$. Let $5 \sim 11$, $10 \sim 12$. Then the edge $(1, 5)$ must be in the $C_4 := 11 - 5 - 1 - 6 - 11$. Consider the vertex 8, we claim $d(8) = 2$, otherwise $d(8) = 4$. then the edge $(7, 8)$ must be in a C_4 which cannot happen. Now consider the edge $(2, 9)$ which cannot be in any C_3, C_4 or C_5. Thus $d(5) \neq 4$. A contradiction. Thus $7 \nsim 8$.
  - So we need $d(7, 4) = 2$ for the edge $(0, 3)$. Let $4 \sim 10 \sim 7$. Note $7 \nsim 8, 9$. Observe the edge $(0, 4)$ cannot be in any C_3 or C_4, thus $d(4) = 2$. Assume $d(5) = 4$. For the edge $(0, 4)$, we assume $10 \sim 5$. For the edge $(1, 5)$, if it is contained in a C_3, then $5 \sim 6$. For vertices $7, 10$, let $7 \sim 11, 10 \sim 12$. Consider vertex 6, we have $d(6) = 4$, since 6 cannot be adjacent to any existing vertices, then let $6 \sim 13, 14$. Then for the edge $(5, 6)$ we need either $d(7, 13) = 2$ then $d(6, 11) = 2$ which is not good for the edge $(5, 7)$; or $d(10, 13) = 2$ then $d(6, 12) = 2$ which is not good for the edge $(5, 10)$. Thus vertex 5 needs a new vertex as its fourth neighbor. Let $5 \sim 11$. Observe that the edge $(1, 5)$ must be in a C_4, then $6 \sim 11$. Consider the edge $(1, 5)$, either $d(7, 2) = 2$ or $d(10, 2) = 2$, since $7 \nsim 8$, then we need $10 \sim 8$. For the edge on $C_3 := 5 - 10 - 7 - 5$, we need $d(11, 12) = d(11, 8) = d(12, 8) = 3$. Then $d(11) = 2$. Then the edge $(8, 10)$ cannot be in ant C_3 or C_4, we have $d(8) = 2$. Consider the edge $(2, 9)$, it cannot be in any C_3, C_4, then $d(9) = 2$, however, the second neighbor of 9 cannot have distance 2 to vertices 8 and 0, which is a contradiction for the edge $(2, 9)$ to be in two C_5s. Thus $d(5) \neq 4$.

    Let $d(5) = 2$, then similarly, $d(6) = 2$. Let $6 \sim 11$. Thus since at least two of $d(11, 5), d(11, 5), d(11, 0)$ are 2. Then it must be $11 \sim 7$. Now we have $d(10) = d(11) = 4$. Then for the edge $(0, 4)$, we must have $d(10, 2) = 2$ which implies $10 \sim 9$. Note the edges $(7, 10), (7, 11)$ cannot be in any C_4, they must be in C_3, thus $10 \sim 11$. Then for the edge $(1, 6)$, we must have $11 \sim 8$. Observe that $d(8), d(9)$ must be 2. The resulting graph is G_5.

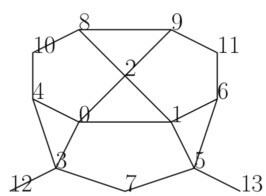
Now we let $d(3) = d(5) = 4$. For the edge $(0, 3)$, if it is contained in a C_4 then must pass through vertex 4. There are two cases:

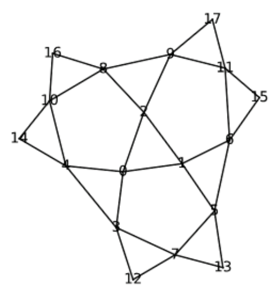
Case 1: $C_4 := 3 - 0 - 4 - 7 - 3$. Let $3 \sim 10, 11$, then for the edge $(0, 3)$, we need $d(10, 2) = d(11, 1) = 2$. For $d(10, 2) = 2$, wlog, let $10 \sim 8$. For $d(11, 1) = 2$, note for the edge $(0, 1)$, $11 \not\sim 6$. Thus let $11 \sim 5$. Note for the edge $(0, 1)$, $7 \not\sim 6$, for the edge $(3, 7)$, $7 \not\sim 8$, or the edge $(0, 2)$, $7 \not\sim 9$. Let $7 \sim 12$. Consider the edge $(0, 4)$, it is in the $C_4 := 0-4-7-3-0$ and a $C_5 := 0-4-7-5-1-0$, thus $d(4) = 4$. For the edge $(0, 4)$, we need that the new neighbors of 4 have distance 2 to vertices $1, 2$ respectively. Since $d(4, 6) = d(4, 9) = 3$, then we need that the new neighbors of 4 have distance 1 to vertices $8, 5$ respectively. Assume $4 \sim 10$ and $4 \sim 11$. Then vertices $10, 11$ are not adjacent to any existing vertices. Let $10 \sim 13, 11 \sim 14$. For the edge $(3, 10)$, we need $d(7, 13) = 2$. Note $13 \not\sim 12$ for the edge $(4, 7)$, thus $13 \sim 5$. Then consider the edge $(1, 5)$, it must be in a C_4 which passes through vertex 6. Then it must be $6 \sim 13$. Then for the edge $(5, 13)$, we need the fourth neighbor of vertex 13 has distance 2 to either vertex 7 or 11. By symmetry of $7, 11$, wlog, let $13 \sim 12$. Then consider the edge $(1, 2)$, we have both $d(5, 8) = 3, d(5, 9) = 3$, a contradiction. Now assume $4 \sim 13 \sim 5$ where 13 is a new vertex. Consider the edge $(1, 5)$ which must be in a C_4 that passes through vertex 6. Then either $6 \sim 11$ or $6 \sim 13$, however, both cases are not good for the edge $(0, 1)$. Thus $4 \not\sim 10$. Let $4 \sim 13 \sim 8$. Then assume $4 \sim 11$. By symmetry of $7, 11$, vertex 11 is not adjacent to $6, 8, 9, 10, 13$, if $11 \sim 12$, then consider the vertex 5, $5 \not\sim 13$ for the edge $(4, 7)$, $5 \not\sim 10$ for the edge $(3, 7)$. Let $5 \sim 14$. For the edge $(5, 7)$, we need either $d(4, 14) = 2$, then $13 \sim 14$ which is not good for the edge $(4, 11)$, or $d(1, 12) = 2$, then $6 \sim 12$. Then for the edge $(5, 11)$, we need $d(14, 12) = 2$, then $12 \sim 13$, however, this situation is not good for the edges $(4, 7)$ and $(4, 11)$. Thus we need new vertex as the fourth neighbor of 11, let $11 \sim 14$. Then consider the edge $(1, 5)$, it must be in a C_4 which passes through vertex 6 and new neighbor of 5, since $6 \not\sim 13$ for the edge $(0, 1)$, let $5 \sim 15 \sim 6$. Consider the edge $(1, 2)$, assume $d(5, 8) = 2$, then $8 \sim 15$. Consider the edge $(2, 8)$ which must be in a C_4 that passes through vertex 9. However the vertex $9 \not\sim 10, 15, 13$ for the edge $(0, 2)$ or $(1, 2)$, a contradiction. Thus for the edge $(1, 2)$, we need $d(5, 9) = 2$, then $9 \sim 15$. Then consider the edge $(1, 5)$, we need either $d(2, 7) = 2$ or $d(2, 11) = 2$, both cannot happen. A contradiction. For the requirement that new neighbors of 4 has distance 1 to vertices 5. We let $4 \sim 14 \sim 5$. Then consider the edge $(1, 2)$, assume $d(5, 8) = 2$, then either $8 \sim 11$ or $8 \sim 14$. Note $9 \not\sim 10, 13, 14, 11$ for the edge $(0, 2)$, then edge $(2, 8)$ cannot be in any C_4. Thus for the edge $(1, 2)$, we need $d(5, 9) = 2$, then either $9 \sim 11$ or $9 \sim 14$, a contradiction for the edge $(0, 2)$.

Case 2: $C_4 := 3-0-4-10-3$, where 10 is a new vertex. Let 11 be the fourth neighbor of 3. A similar analysis for the edge $(1, 5)$, it must be in a $C_4 := 1-5-12-6-1$. We need $d(2, 11) = 2$ for the edge $(0, 3)$ which lead to $d(8, 11) = 1$. For the edge $(0, 2)$, we have $d(3, 8) = 2$ thus $d(4, 9) = 3$. Then we consider a similar situation for the edge $(0, 2)$, we have $d(8) = 4$ and $(2, 8)$ is in a C_4 that pass

through vertex 9. Let $8 \sim 13 \sim 9$. By symmetry, let $5 \sim 14 \sim 8$.
Consider the edge $(0, 4)$, assume $d(4) = 4$, let $4 \sim z, w$. Then we need $d(1, z) = d(2, w) = 2$, which means 4 should be adjacent to these vertices: $7, 11, 13, 12, 14$. If $4 \sim 7$, by symmetry, $6 \sim 7$, then $d(4, 6) = 2$, a contradiction. Similar analysis, we would get contradictions for all other cases.
Thus $d(4) = 2$, then $d(10) = 2$. By symmetry, $d(6) = d(9) = 2, d(12) = d(13) = 4$. Consider the edge $(3, 10)$, let $10 \sim z, w$, we need $d(7, z) = d(11, w) = 2$. This fact also implies that the edges $(3, 7).(3, 11)$ cannot be in any C_3 or C_4, then $d(7) \neq 4, d(11) \neq 4$. Note for $d(7, z) = d(11, w) = 2$, $10 \not\sim 14$, otherwise by symmetry $7 \sim 13, 11 \sim 12$, a contradiction. Thus it must be $10 \sim 12, 13$, then consider the edge $(5, 12)$, we need the fourth neighbor of 12 has distance 2 to vertex 14, similarly, for the edge $(8, 13)$, we need the fourth neighbor of 13 has distance 2 to vertex 14. By symmetry $12 \sim 13$. The resulting graph is G_6 and it is Ricci-flat.

Case 3: Now we consider the case when the edge $(0, 3)$ is in a C_3, which must be $C_3 := 0 - 3 - 4 - 0$. Similarly, the edge $(1, 5)$ must be in a C_3 which must be $C_3 := 1 - 5 - 6 - 0$ and the edge $(2, 8)$ must be in a C_3 which must be $C_3 := 2 - 8 - 9 - 2$. Let $3 \sim 12, 5 \sim 13$. We must have $12 \sim 7$ and $13 \sim 7$ for the edge $(0, 3)$ and $(1, 5)$, etc.





Following this structure, we are actually constructing a simple graphs in which every vertex has degree 4 and every edge shares one C_3 and one C_5, then eventually the graph is isomorphic to G_7.

□

4.2. Ricci-flat graphs that contain C_4. Now all Ricci-flat graphs containing C_3 are determined. In the following section we will only consider the Ricci-flat graphs containing C_4s. Basically these can be classified into three cases:

- all C_4s are vertex-disjoint;
- exist two C_4s sharing an edge;
- no two C_4s sharing an edge, but exist two C_4s sharing a vertex.

We consider each case.

THEOREM 6. *Let G be a Ricci-flat graph with maximum vertex degree 4, assume G contains no edges with endpoint degree $(3, 4)$, and all C_4s of G are vertex-disjoint. Then G is isomorphic to G_8.*

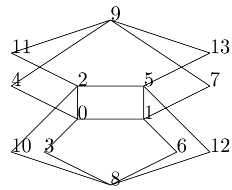
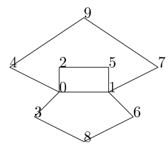


FIG. 7. *Ricci-flat graph G_8*

Proof. Let G be a Ricci-flat graph in class $\mathcal{G}$ with the following sub-structure, where $d(0) = d(1) = 4$ and the edge $(0, 1)$ is in the $C_4 := 0 - 1 - 5 - 2 - 0$.



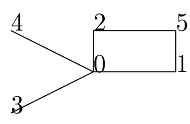
Since C_4s in G are vertex-disjoint, then $8 \neq 9$, and the edges $(0, 2), (0, 3), (0, 4)$ are not in any other C_4s, which implies $d(3), d(4)$ must be 2. Similarly, $d(5) = d(6) = 2$. Then $d(8) = d(9) = 4$. Consider vertex 2, it is not adjacent to 8 or 9. Let $2 \sim 10, 11$. Since the edge $(2, 5)$ is not in any C_4 or other C_4, let $5 \sim 12, 13$. Then wlog, we must have $d(10, 12) = d(11, 13) = 2$ and we must make connection with vertices $8, 9$. Then $10, 12$ must be adjacent to 8 or 9, wlog, let $10 \sim 8 \sim 12$. Then $11 \sim 9 \sim 13$. Now the graph is G_8, this is the unique Ricci-flat graph such that C_4s are vertex-disjoint. Note G_8 is isomorphic to the second graph found in [4]. □

In the following parts, we will see many infinite Ricci-flat graphs, we call these as *primitive graphs*, as from each, we can obtain finite Ricci-flat graphs which are obtained by a vertex-preserve projection from the primitive graph.

For the second and third cases, we first consider the case when a vertex in some C_4 has degree 2, see the following results:

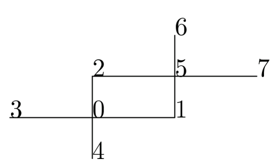
THEOREM 7. *Let G be a Ricci-flat graph with maximum degree at most 4. Let edge $(x, y) \in E(G)$ be in a C_4 with $d(x) = 4$ and $d(y) = 2$. Then G is isomorphic to graphs with the primitive graphs showing in the Figures 8, 11, 25, 26.*

Proof. See the following graph, let $x = 0, y = 1$, $C_4 := 0 - 1 - 5 - 2 - 0$, then $d(3, 5) = d(4, 5) = 3$ by "Type 3". Since edge $(1, 5)$ is in C_4, then $d(5)$ must be 4.



We focus on vertex 2, note we will exclude the case when an edge has endpoints degree 3, 4. Thus there are two cases:

Case 1 Assume $d(2) = 2$. Let $5 \sim 6, 7$. Note $d(0, 6), d(0, 7)$ must be 3.



Assume $d(6) = 3$, then the edge $(5, 6)$ must be contained in a C_4 passing through vertex 7 let it be $C_4 := 6-5-7-8-6$, and 9 be the third neighbor of 6, so we need one of $d(9, 1), d(9, 2)$ to be 2 which cannot happen.
Assume $d(6) = 4$, then the edge $(5, 6)$ is in a C_4 which must pass through vertex 7, still we need the neighbors of 6 have distance 2 from vertex 1 and vertex 2, which cannot happen.
Thus $d(6) = 2$. Similarly, $d(7) = 2$. We assume the edge $(5, 6)$ must be in a C_5 passing through vertex 7 and new vertices, let this $C_5 := 6-5-7-9-8-6$. However we need then $d(8, 2)$ or $d(8, 1)$ to be 2 which cannot happen. Thus the edge $(5, 6)$ must be in a C_4 passing through vertex 8, $C_4 := 6-5-7-8-6$. Now compare the $C_4 := 0-1-5-2-0$ and $C_4 := 5-7-8-6-5$, the situation is same as the initial stage, we have that $d(8)$ must be 4 and its neighbors must have degree 2. Then we can extend this process infinitely times to get an infinite graph that consists of a sequence of C_4s where the vertices degrees are $2, 4, 2, 4$ in the cycle order and each two consecutive C_4s sharing one vertex of degree 4. To get a finite graph, the only way is to combine the neighbors of two degree 4 vertices only if the distance of these two vertices is at least 3. For example, we could have $8 \sim 3, 4$, then get the smallest Ricci-flat graph of this type, we could also let $8 \sim 9, 10, 11 \sim 9, 10, 3, 4$. Thus there are infinitely many such Ricci-flat graphs.

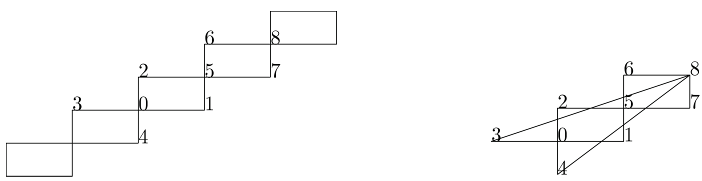
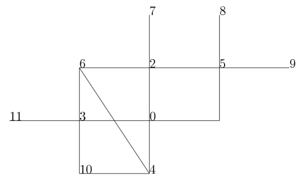


FIG. 8. *Ricci-flat graphs: the primitive graph and an example of finite type*

Case 2 Assume $d(2) = 4$. Let vertices $6, 7$ be neighbors of 2, vertices $8, 9$ be neighbors of 5. Note $d(0, 8) = d(0, 9) = 3$. We claim the edge $(0, 3), (0, 4)$ must be in C_4. Suppose $(0, 3)$ is in a C_5, then $d(3) = 2$, let $3 \sim 10 \sim 6$. We have $d(3, 6) = 2$, thus $4 \not\sim 7$ for the edge $(0, 2)$. Assume $d(4) = 2$, then we must have $d(4, 6) = 2$, note $4 \not\sim 10$, otherwise the edge $(0, 3)$ would be in a C_4. Let $4 \sim 11 \sim 6$. Consider the edge $(0, 4)$, by "Type 3" and the fact $d(11, 1) = 3$, we also need $d(11, 3) = 2$. Then $11 \sim 10$, a C_3 appears, a contradiction. Thus $d(4) = 4$ and the edge $(0, 4)$ is in a C_4 that must be $C_4 := 4-0-2-6-4$. Consider the edge $(0, 4)$, let z, w represent the third and fourth neighbor of vertex 4, then we need $d(z, 1) = 2$, which implies $4 \sim 5$, then $d(4, 5) = 2$, a contradiction. Thus the edge $(0, 3), (0, 4)$ must be in C_4.
Note $d(3), d(4)$ cannot be both 2. Wlog, let $d(3) = 4$. Now we assume neither the C_4 for edge $(0, 3)$ nor the C_4 for the edge $(0, 4)$ pass through edge $(0, 2)$. Let $3 \sim 1011, 12$ with $C_4 := 0-3-10-4-0$. Consider the edge $(0, 3)$, we need at least one of $d(1, 11), d(1, 12)$ is 2, however, both would result $d(3, 5) = 2$, a contradiction. Thus one of C_4s for edges $(0, 3), (0, 4)$ must pass through edge $(0, 2)$. Wlog, let $3 \sim 6$. Then we need $d(4, 7) = 3$ for the edge $(0, 2)$. In the following, there are two cases for the C_4 that passes through edge $(0, 4)$.

– If the edges $(0, 4)$ also passes through edge $(0, 2)$, then it must be $4 \sim 6$. Then $d(3, 7) = 3$. In this case, $d(3) = d(4) = 4$ considering the edge

$(0,3)$. Let $3 \sim 10, 11$, since $d(3,5) = d(4,5) = 3$, then $d(1,10) = d(1,11) = 3$ which implies $d(4,10) = 1$.



Consider vertex 4, note either $4 \sim 11$ or $4 \sim 12$ where 12 is a new vertex. Assume the second case, then consider the edge $(2,6)$, let $6 \sim t$, then we need $d(5,t) = 1$ or $d(7,t) = 1$. For $d(5,t) = 1$, wlog, let $6 \sim 8$, still consider the edge $(5,9)$ which must be in a C_4 that pass through vertex 8, let $9 \sim w \sim 8$, where t could be 11 or 12 or a new vertex. Then consider the edge $(2,7)$ which must be in a C_4 that passes through vertex 8, let $7 \sim 8$. Observe that we have $d(7,9) = d(7,3) = d(7,4) = 3$, then vertex 7 must be adjacent to new vertices which have distances 3 from vertices 0 and 5, then it is not good for the edge $(2,7)$.
Thus $4 \sim 11$. Note one of the edge $(5,8), (5,9)$ must be in a C_4 that pass through the edge $(2,5)$. Assume $6 \sim 8$, then consider the edge $(5,9)$ which must be in a C_4 that pass through vertex 8, since $d(6) = 4$ in the current graph, we need a new vertex as common for 8, 9. Observe that $d(8,10) = d(8,11) = 3$ for the edge $(3,6)$, then we need a new vertex 12 such that $9 \sim 12 \sim 8$. Then consider the edge $(6,8)$, we have both $d(3,12), d(4,12)$ are not 1, then at least one of them must be 2 which need $12 \sim 10$ or $12 \sim 11$, however, we would have $d(8,10) = 2$ or $d(8,11) = 2$, a contradiction.
Thus when $4 \sim 11$, we need $7 \sim 8$. Let a new vertex 12 as the fourth neighbor of 6. Consider the edge $(2,6)$, since $d(4,5) = d(3,5) = 3$, we need $d(7,12) = 1$. Consider the edge $(5,8)$, $d(8) \neq 2$, otherwise the edge $(5,9)$ cannot be in any C_4. Let $d(8) = 4$, consider the edge $(5,8)$, since vertex 1 has distance 3 to any new neighbor of vertex 8, then we need vertex 9 to have distance 1 from neighbor of vertex 8, that is, a common vertex for 8, 9. Note $d(10,12) = d(11,12) = 3$ for the edge $(3,6)$. Then $d(12) \neq 4$. Otherwise the new neighbor of vertex 12 must have distance 2 from 3 or 4, then it must be adjacent to vertex 10 or 11 which would make $d(10,12) = 2$ or $d(1,12) = 2$. Consider the vertices 10, 11, we have $d(10) = 4$ and the edge $(3,10)$ satisfies "Type 6b", such that the second $C_4 := 3 - 10 - w - 11 - 3$ where w is a new neighbor of vertex 10. A similar situation for the vertex 11 and the edge $(3,11)$. Thus there is at least new common vertex for vertices 10 and 11. Let z, t be the fourth neighbors of vertex 10 and 11 respectively. Now the C_4 for edge $(z,10)$ must pass through edge $(10,c)$ and the C_4 for edge $(11,t)$ must pass through edge $(11,c)$. Then either $z = t$ or they are two different vertices.
Consider the vertex 7, there are two cases: either $7 \not\sim 9$ or $7 \sim 9$. Next, we consider the different situations under each case:

Cases for the fourth neighbor of vertex 7.

* Assume $7 \sim 9$. Then $d(9) = 4$. By above analysis, there are four different situations:

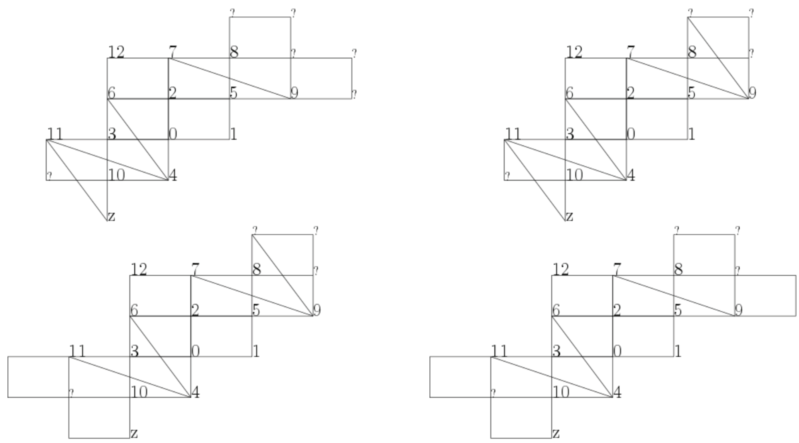


FIG. 9. *Structures of the Ricci-flat graphs when* $7 \sim 9$

* Assume $7 \not\sim 9$. Let $7 \sim 13$. Then we have $d(13) = 2$ considering the edge $(7, 13)$, and $d(9) = 2$ considering the edge $(5, 9)$. Still we need a common for vertices $8, 13$ and a common for vertices $8, 9$. A similar analysis for the vertices 10 and 11, the difference lies in whether the vertices $10, 11$ share two more common neighbors or just one more common neighbor, thus we have two different infinite structures. See Figure 10:

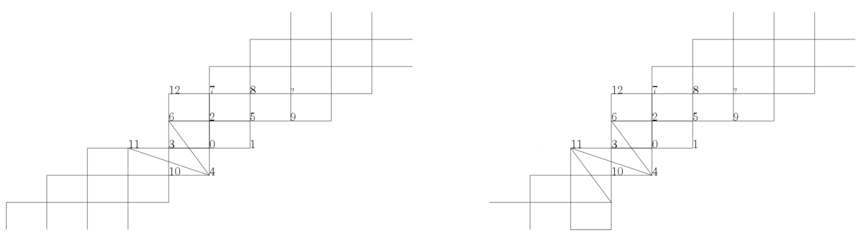


FIG. 10. *Structure of Ricci-flat graph(when* $7 \not\sim 9$*)*

As we can see, we can also combine above different situations ($7 \sim 9$ and $7 \not\sim 9$) into one graph. The construction is: we can add the "slash edges"(except $(4, 6), (4, 11)$) at any stage. To get a finite one, always merge the vertices on the left-upper corner region and vertices on the right-down conner region. See the following illustration:
For examples:

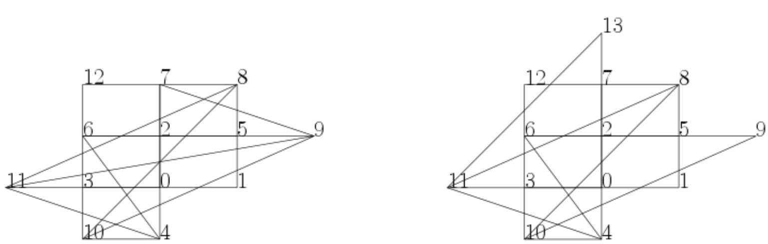


FIG. 12. *Examples of finite Ricci-flat graphs*

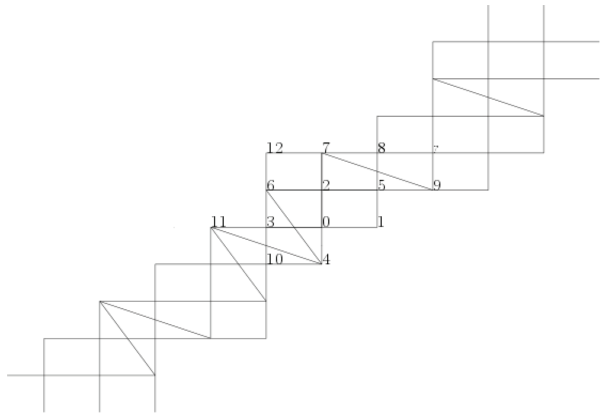


FIG. 11. *Ricci-flat graphs: the primitive graph*

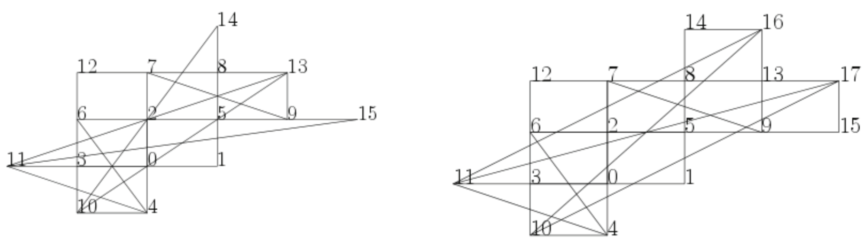
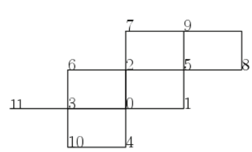


FIG. 13. *Examples of finite Ricci-flat graphs*

– We consider the other case for the C_4 that passes through edge $(0,4)$, that is, this C_4 passes through edge $(0,3)$ but not vertex 6. Let $3 \sim 10 \sim 4$ and $3 \sim 11$. Observe that $d(4) = 2$ considering the edge $(0,4)$. By symmetry, we consider the same situation for edges $(5,8), (5,9)$, wlog, let $d(8) = 2, d(9) = 4$. Then the edge $(5,8)$ is in a C_4 that passes through 9. Consider the edge $(5,9)$, since any new neighbor of vertex 9 has distance 3 from vertex 1, we need one of them to be adjacent to vertex 2 such that $(5,9)$ shares two C_4. Then either $9 \sim 6$ or $9 \sim 7$. Note we must have $9 \sim 7$ considering the edge $(2,7)$. Then $d(6,8) = 3$ for the edge $(2,5)$. Now consider the edge $(2,6)$ according to the following cases:
  * Assume $d(6) = 2$. Then $d(7) = 2$, $d(11) = 2$ considering the edge $(2,7), (3,11)$ respectively. Similarly, the fourth neighbor of vertex 9 has degree 2, and the fourth neighbor of vertex 10 has degree 2 as $d(4) = 2$.



Continue with the similar process, we will get the following structure which is an infinite Ricci-flat graph:

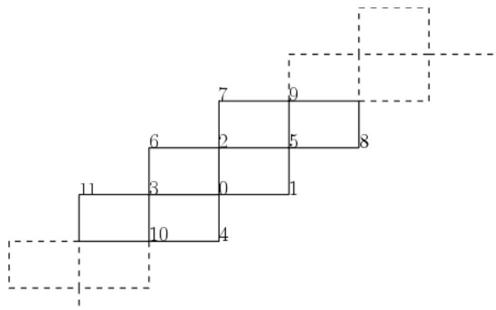


FIG. 14. *The primitive Ricci-flat graph*

To get a finite one we need to merge the vertices on the ends. For example, let vertex 10 be the common neighbor of vertices 8 and 9. Then we need $9 \sim 11$ for the edge $(3, 10)$ as both $d(6, 8), d(6, 9)$ are 3. We obtain a Ricci-flat graph:

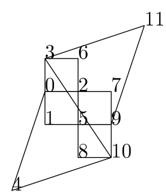


FIG. 15. *Ricci-flat graph*

Assume we need a new vertex 12 as the common of $8, 9$, then $d(12) = 4$, we need the edge $(9, 12)$ to share two C_4s, note $9 \not\sim 11$ as $d(11) = 2$. Let $9 \sim 13$, thus we need a common vertex for $12, 13$. Note if $10 \sim 13$, then $10 \sim 12$, consider the edge $(3, 10)$, we have both $d(6, 12), d(6, 13)$ are 3, then we need $11 \sim 12$.

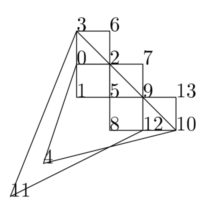


FIG. 16. *Ricci-flat graph*

Assume 10 is not the common vertex for $12, 13$. Then let $12 \sim 14 \sim 13$, then $d(14) = 4$, for the fourth neighbor of vertex 12, assume $12 \sim 11$, then we have $12 \sim 10$, or assume $12 \sim 10$, then $12 \sim 11$ which would make $d(12) = 5$. Thus let $12 \sim 15$, we need a common vertex for $14, 15$. Still assume $10 \sim 14, 14$. Then we need $14 \sim 11$ for the C_4 that passes through edge $(3, 10)$.

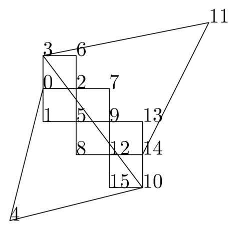


FIG. 17. *Ricci-flat graph*

Continue with the similar process, we will obtain the following Ricci-flat graphs.

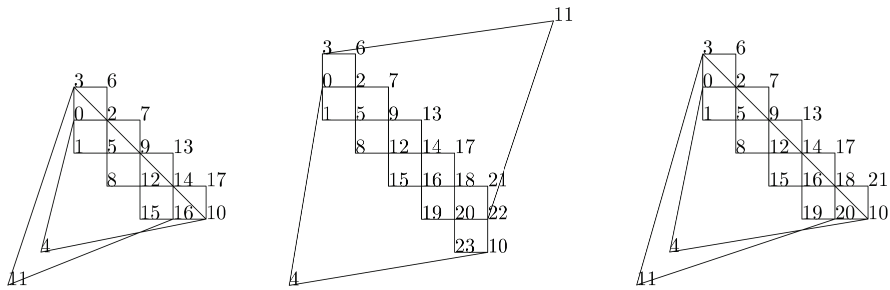


FIG. 18. *Ricci-flat graphs*

The idea behind these Ricci-flat graphs is simple, we can first have the infinite Ricci-flat graph(see Figure 14), then two ways to obtain a finite Ricci-flat:

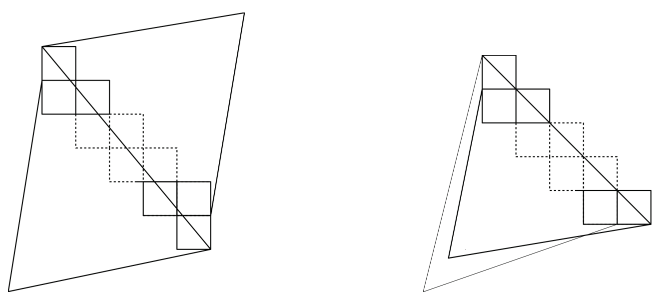
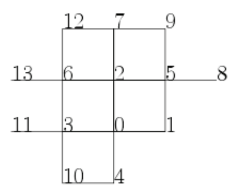


FIG. 19. *Two ways to generate finite types*

∗ Assume $d(6) = 4$. Then $d(7) = 4$. For the edge $(2, 6)$ in a C_4, there are two cases: either $6 \sim 12, 13$ and $7 \sim 12$, or $6 \sim 9, 12$. Consider the latter case: $6 \sim 9, 12$, then we need $d(7, 12) = d(5, 12) = 3$ for the edge $(2, 6)$. For the edge $(2, 7)$, note $7 \not\sim 11$, as both $d(5, w), d(6, w)$ cannot be 2 where w is the fourth neighbor of vertex 7. Thus 7 is adjacent to two new vertices, let $7 \sim 13, 14$, since the vertex 0 has distance 3 from both vertices $13, 14$, we need $5 \sim 13$ or $6 \sim 14$, a contradiction. Thus $6 \not\sim 9$.
Let $6 \sim 12, 13$ with two new vertices. Note we still need the "Type 6b" for the edge $(2, 6)$, otherwise suppose $d(5, 13) = 2$, then either

$9 \sim 12$ or $8 \sim 12$. The latter case implies the former case, since $d(8) = 2$ and we need a common vertex for $8, 9$. While the former case is not good for the edge $(5, 9)$ as the vertex 1 always has distance 3 from the fourth neighbor of vertex 9, thus we cannot have "Type 6c". Using "Type 6b", we need $12 \sim 7$.



· Assume $d(12) = 2$. Then both vertex 13 and the fourth neighbor of vertex 7 have degree 2. For the edge $(6, 13)$, we need $13 \sim 10$ or $13 \sim 11$. Note if $3 \sim 10$, then we need $d(11, 12) = 3$ for the edge $(3, 6)$. Then the C_4 for the edge $(3, 11)$ must pass through edge $3, 10$, then we need $d(13, 6) = 2$ for the edge $(3, 10)$, contradict to the fact $d(13, 6) = 1$. Let $13 \sim 11$. Continue with this process, we will get the following infinite Ricci-flat graph:

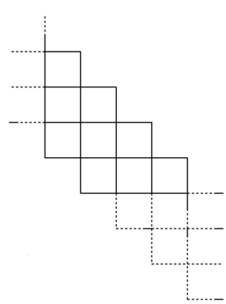


FIG. 20. *The primitive Ricci-flat graph*

To get a finite one, need to merge the vertices on the ends. Now consider the common vertex for $8, 9$, from the current vertices, it can be vertex 10 or 11. Assume $11 \sim 8, 9$, then we need $9 \sim 10$ however, we cannot guarantee the edge $(9, 10)$. Assume $10 \sim 8, 9$, then we need $9 \sim 11$ for the edge $(9, 10)$. Then for the edge $(3, 11)$, we need $11 \sim 13$, then we need $7 \sim 14 \sim 11$ for the edge $(9, 11)$. Another Ricci-flat graph:

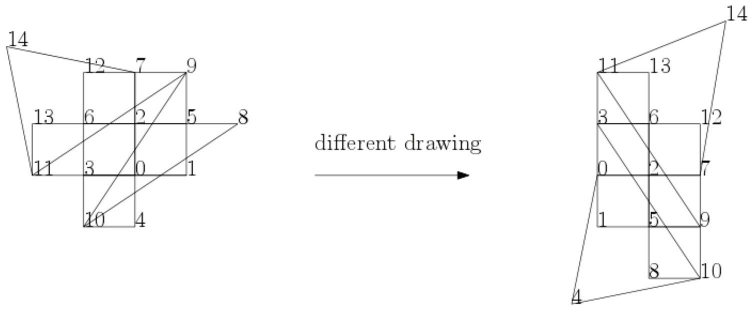


FIG. 21. *Ricci-flat graph*

If we need a new vertex 14 as common for vertex $8, 9$. We can have the following Ricci-flat graphs by similarly arguments:

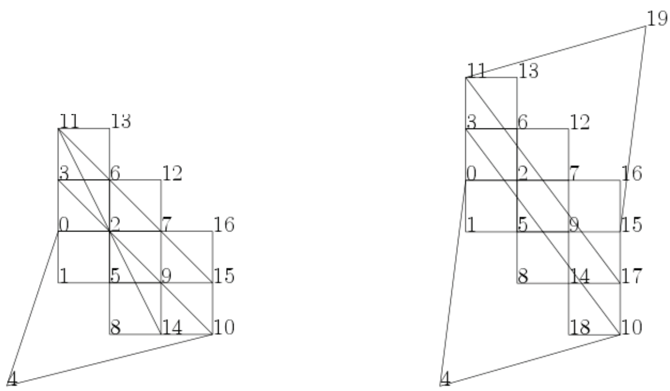


FIG. 22. *Ricci-flat graphs*

Now we can see, there are also two ways to generate finite Ricci-flat graphs:

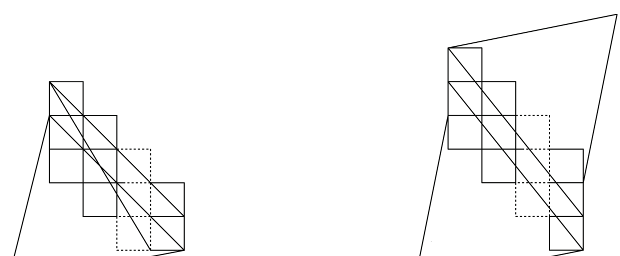


FIG. 23. *The primitive Ricci-flat graph*

· Assume $d(12) = 4$. Let $12 \sim 14$. Then assume $d(14) = 2$ or 4. For each case, the similar arguments are used, here we omit the details. We will have the following infinite and finite Ricci-flat graphs:

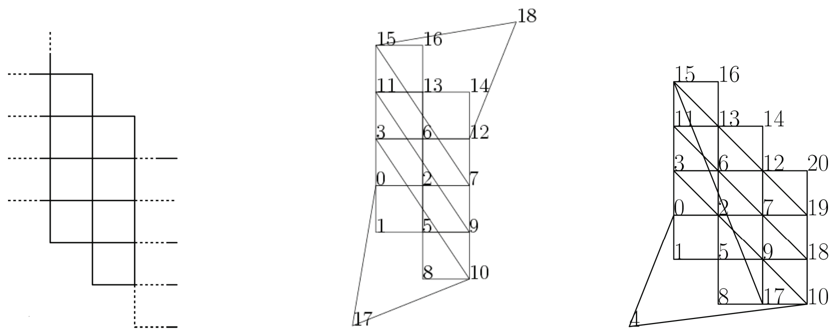


FIG. 24. *Ricci-flat graphs*

Thus there are infinitely many finite Ricci-flat graphs. To describe them, we first specify the drawing as showing above, define the number of C_4 in each column as the length of the graph, define the width as the number of columns in the drawing. For example above finite graphs have length 4 (if $d(14) = 4$, we could have a larger length), width 2 and width 3 respectively. We use an example with length 5 to illustrate the infinite Ricci-flat graphs and the first way to generate finite types:

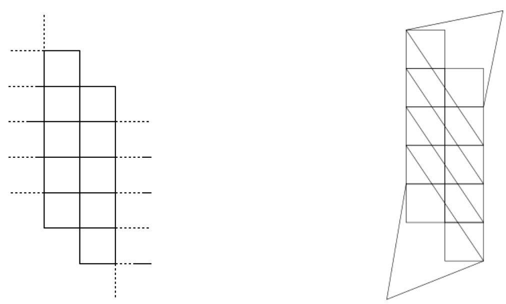


FIG. 25. *Infinite Ricci-flat graph of length* 5*, the first way to generate a finite type with width* 2*.*

The second way to generate finite Ricci-flat graph starts with width at least 3, see the following example.

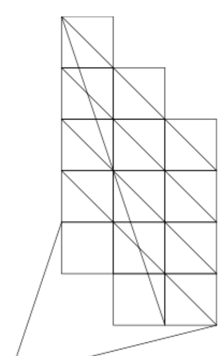


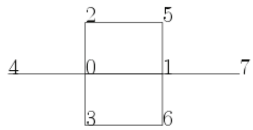
FIG. 26. *The second way to generate a finite type with width* 3*.*

□

The other situation for the second case is that all four vertices in any C_4 have degree 4, then G must be 4-regular, we have the following results.

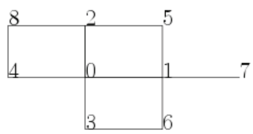
THEOREM 8. *Let G be a 4-regular Ricci-flat graph that contains two C_4s sharing one edge. Then G is isomorphic to to graphs with the primitive graphs showing in the Figures 28, 30.*

Proof. Start with the following structure where the edge $(0,1)$ shares two C_4s:



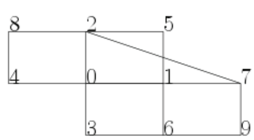
There are two cases for the edge $(0,2)$:

Case 1: $(0,2)$ shares two C_4, let the second one be $C_4 = 0-2-8-4-0$.



Note for the edge $(0,2)$, we need the fourth neighbor of 2 has distance 3 from vertex 3, thus $2 \not\sim 6$. there are two cases, either $2 \sim 7$ or $2 \sim 9$ where 9 is a new vertex.

- Assume $2 \sim 7$. Then $d(3,7) = d(3,5) = d(4,7) = d(4,5) = 3$. We will show that the edge $(1,6)$ does not satisfy "Type 6c". We first assure the C_4 for edge $(1,6)$ is shown in above subgraph. Suppose we also have $C_4 := 1-6-0-4-1$. Then according to 'Type 6c", we need at least one of $d(3,5), d(3,7)$ to be 2, a contradiction. Thus $6 \not\sim 4$. Then vertex 6 must be adjacent to at least one new vertex, let it be $6 \sim 9$. Wlog, we need $d(5,9) = 2$ for the edge $(1,6)$. Then we need a new vertex as common for vertices 5, 9, let $5 \sim 10 \sim 9$. Consider the edge $(1,5)$, since $d(4,5) = d(3,5) = 3$, as the vertex 0 has distance 3 from any new neighbor of vertex 5, then we need vertex 6 to be adjacent to the fourth neighbor of vertex 5, a contradiction. Thus the neighborhood for the edge $(1,6)$ must be "Type 6b". That is, the edge $(1,6)$ must share a C_4 that passes through either vertex 5 or vertex 7. Assume it passes through vertex 5. Then we need a new vertex 10 as the common for 5, 6. Let $5 \sim 10$, we need $d(0,10) = d(7,10) = 3$ for the edge $(1,5)$. Then let $7 \sim 11, 12$, we have the C_4s for the edge $(7,11)$ or edge $(7,12)$ must pass through edge $(7,2)$ then $(2,8)$, which implies $8 \sim 11, 12$. However, for the edge $(2,5)$, since both vertices 0 and 7 have distance 3 from the third and fourth neighbors of vertex 5, then we need vertex 8 to have distance 1 to them, that is, $5 \sim 11$ or $5 \sim 12$, however, both are not good for the edge $(2,7)$. A contradiction. Thus let $6 \sim 9 \sim 7$.



Let $7 \sim 10$. We need $d(5,10) = 3$ and $d(0,10) = 3$ for the edge $(1,7)$. Consider the edge $(2,7)$ which is in the $C_4 := 2-7-1-5-2$ and $2-7-1-0-2$, since $d(5,9) = d(5,10) = d(0,9) = d(0,10) = 3$, we need either $8 \sim 9$ or $8 \sim 10$ or both. For any of these three cases, we need vertex 5 to have distance 3 to new neighbors of vertex 8. Let s represent the new neighbor of vertex 5, we have $d(8,s) = 3$. Then the situation is not good for the edge $(2,5)$. A contradiction again. Thus $2 \not\sim 7$.

- Assume 2 is adjacent to a new vertex 9. Then we need $d(3,9) = d(4,7) = 3$. We will show that the edge $(0,6)$ does not satisfy "Type 6c". Otherwise, consider the C_4 which would be used for "Type 6c" for edge $(1,6)$, there are three cases:
  - Assume $C_4 := 1-6-4-0-1$. Then $d(3,7) = 3$ for the edge $(0,1)$. Let $6 \sim 10$, we need $d(3,5) = d(7,10) = 2$. Then the C_4 the edge $(1,7)$ must pass through edge $(1,5)$. Then we need a new vertex as the common for vertices $5,7$, let it be $5 \sim 11 \sim 7$. Note for $d(3,5) = 2$, $3 \not\sim 11$ as $d(3,7) = 3$, let $3 \sim 12 \sim 5$, then we have $d(6,12) = 2$, however, this will contradict to the edge $(1,5)$ which requires $d(6,12) = 3$.
  - Assume $C_4 := 1-6-3-5-1$. Then we need the fourth neighbor of vertex 5 to have distance 3 from vertex 7, then the C_4 the edge $(1,7)$ cannot pass through edge $(1,5)$, it must be pass through edge $(0,1)$ which implies $3 \sim 7$. Then consider the edge $(3,5)$, which now shares two $C_4s := 3-5-1-7-3, 3-5-2-0-3$, then we need the fourth neighbor of vertex 5 to have distance 3 from vertex 6. Since $d(3,9) = 2$, then $6 \not\sim 8,9$. Let $6 \sim 10,11$, then we have $d(5,10) = d(5,11) = 3$. Then we need $d(0,10) = 2$ for the edge $(1,6)$ which implies $4 \sim 10$, which is not good for the edge $(0,1)$. A contradiction.
  - Assume $C_4 := 1-6-3-7-1$. Then $d(4,6) = 3$ for the edge $(0,1)$. Consider the edge $(0,3)$, since $d(4,7) = d(4,6) = 3$, then we either need $d(2,6) = 2$(or $d(2,7) = 2$ by symmetry of vertices 6 and 7) or $d(2,z) = 1$ where z is the fourth neighbor of vertex 3. Consider $d(2,6) = 2$, then either $6 \sim 8$ or $6 \sim 9$. However $6 \sim 8$ would contradict $d(4,6) = 3$, $6 \sim 9$ would contradict $d(3,9) = 2$. Thus let $d(2,z) = 1$. Note $3 \sim 5$ would give us a same situation as previous item. Thus we need $3 \sim 8$. Then $d(4,9) = 3$ for the edge $(0,2)$. Consider the fourth neighbor of vertex 8, let $8 \sim 10$, since $d(2,6), d(2,7), d(4,6), d(4,7)$ are all 3, we need either $6 \sim 10$ or $7 \sim 10$, wlog, let $6 \sim 10$. Then consider the edge $(3,6)$ which shares two $C_4 := 3-6-1-0-3, 3-6-10-8-3$, then we need $d(7,11) = 3$ where 11 is the fourth neighbor of vertex 6, which would contradict to "Type 6c" for the edge $(1,6)$.

Since all possible cases lead to contradictions, we conclude that the edge $(1,6)$ must satisfy "Type 6b", that is, the edge $(1,6)$ must share a C_4 that pass through either vertex 5 or vertex 7. Assume it pass through vertex 5. Then we need a new vertex 10 as the common for $5,6$. Then we need vertex 7 to have distance 3 from the fourth neighbors of vertices 5 and 6 respectively, which would result that the edge $(1,7)$ cannot be in any C_4, a contradiction. Thus the C_4 for the edge $(1,6)$ must pass through vertex 7, let $6 \sim 10 \sim 7$. Similarly, let $6 \sim 11$. We obtain the following structure.

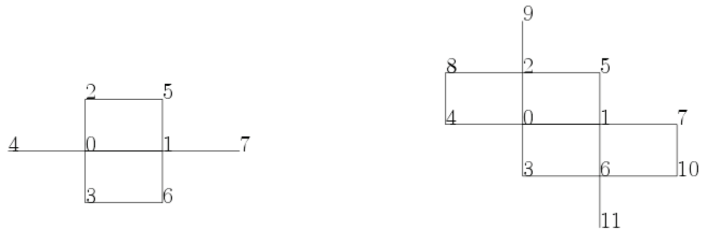
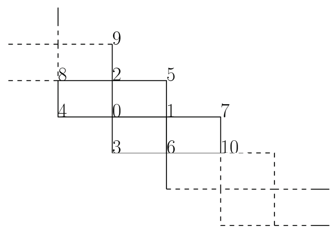


FIG. 27. *The left structure generate the unique right structure.*

Continue with the similar arguments, we will have the following structure:



The above is a Ricci-flat primitive graph, as we require that all vertices have degree 4, we consider two different situations according to the neighbors of vertex 4.

– Assume $4 \sim 5$. Consider the fourth neighbor of vertex 5, note $5 \not\sim 10$ as there is no way to generate a C_4 passing through edge $(5, 10)$ in above structure. The other possible case is $5 \sim 3$ or $5 \sim 12$, a new vertex. Consider $5 \sim 12$, then $d(9, 12) = 3$ for the edge $(2, 5)$. Note for the edge $(2, 8)$, which should share two C_4s, we need the fourth neighbor of vertex 8 to have distance 3 from vertex 5, thus the only possible way to form a C_4 passing through the edge $(5, 12)$ is through edge $(5, 1)$ then edge $(1, 7)$. Let $7 \sim 12$. Then consider the edge $(5, 12)$, since we need vertex 12 to have distance 3 to the fourth neighbor of vertex 4, then vertex 4 has distance 3 to the third and fourth neighbor of vertex 12, then for the edge $(5, 12)$, we need $12 \sim 8$. However, there is no way to generate a C_4 passing through vertices $2, 9, 8$. A contradiction. Thus, when $4 \sim 5$, we need $5 \sim 3$. Similarly, $3 \sim 7, 4 \sim 9$. Following the similar arguments, we have the following infinite Ricci-flat graph:

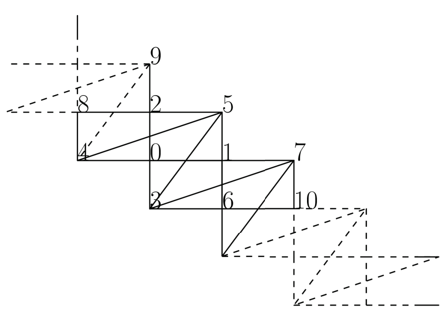


FIG. 28. *A primitive Ricci-flat graph*

To get a finite one, always merge the vertices on the left-upper corner region and vertices on the right-down corner region. For example, let $10, 11 \sim 8, 9$ or new vertex as common for vertices 8 and 9. See the following finite Ricci-flat graphs based on above structure:

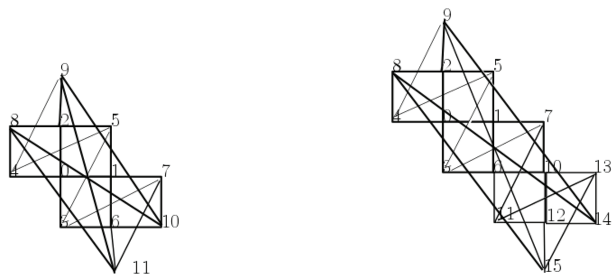


FIG. 29. *Two finite 4-regular Ricci-flat graphs*

– Now we consider the case when $4 \not\sim 5$, by symmetry $3 \not\sim 5$. Assume $3 \sim 8$, then $4 \sim 6$. Then $3 \not\sim 7$ and $4 \not\sim 9$. Let $3 \sim 10$, note $4 \not\sim 10$ for the edge $(0, 3)$. For the edge $(1, 7)$ to be in a C_4, which must pass through edge $(1, 6)$, let $6 \sim 12 \sim 7$. Similarly, let $9 \sim 13 \sim 8$. Note $12 \neq 13$, since then the edge $(3, 10)$ cannot be in any C_4. Consider the edge $(0, 3), (3, 6), (3, 8)$, we need $d(2, 10) = 3, d(10, 12) = 3, d(10, 13) = 3$ respectively, then the edge $(3, 10)$ cannot be in any C_4. A contradiction. Now consider the case when $3 \sim 10, 11$, then the edge $(3, 10)$ must be in a C_4 that pass through edge $(0, 4)$. Let $4 \sim 10, 4 \sim 12$. Continue with this process we can get an infinite lattice.

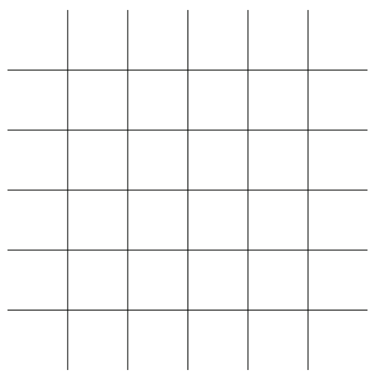


FIG. 30. *A primitive 4-regular Ricci-flat graph of "lattice type"*

We define the length as the number of C_4s in each row and the length as the number of C_4s in each column, then width and length of the lattice could be extended. Note the length and width should be at least 6 as for each edge there is a pair of neighbors with distance 3. To get a finite one, the vertices on parallel boundaries can be connected in the following ways.

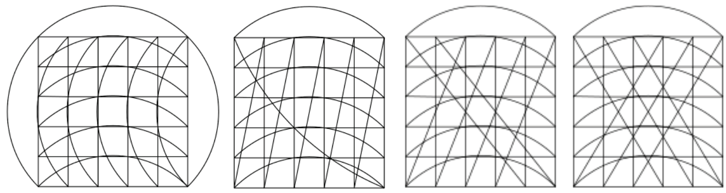
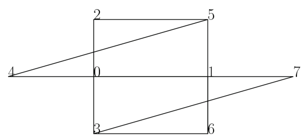


FIG. 31. *Finite 4-regular Ricci-flat graphs of "lattice type"*

Case 2: All edges $(0,2),(0,3),(1,5),(1,6)$ are "Type 6c". Then the C_4 for edge $(0,4)$ cannot pass through vertex 2 and its neighbor, then $4 \sim 5$ or $4 \sim 6$. Wlog, let $4 \sim 5$. Then $d(2,7) = 3$ for the edge $(0,1)$. Consider the C_4 for edge $(0,7)$, we have $7 \sim 3$. See the following graph.



Then $d(2,6) = 3, d(4,6) = 3$. Then consider the edge $(1,5)$, since we have $d(2,7) = d(2,6) = d(4,7) = d(4,6) = 3$, then we need the fourth neighbor of vertex 5 to have distance 1 to either vertex 6 or vertex 7. Let $5 \sim 9 \sim 6$. Then edge $(1,6)$ shares two $C_4 := 1-6-3-0-1, 1-6-9-5-1$, a contradiction. □

THEOREM 9. *Let G be a Ricci-flat graph that contains two C_4s sharing one edge and all vertices on any C_4 have degree 4. Assume G is not 4-regular, G is isomorphic to Figure 32.*

Proof. Start with the following structure where the edge $(0,1)$ shares two C_4s and $d(2) = d(5) = d(3) = d(6) = 4$. Note $d(4), d(7) \neq 4$, otherwise, we would get a 4-regular Ricci-flat graph, thus $d(4) = d(7) = 2$. For edge $(2,5)$ to be in two C_4's, let $2 \sim 8 \sim 9 \sim 5$. As vertices $2,5$ cannot be adjacent to any existing vertices, let $2 \sim 9, 5 \sim 11$, similarly, $d(10) = d(11) = 2$. Observe that $4 \not\sim 10$, otherwise $d(4) = d(10) = 4$. However vertex 4 cannot not be adjacent to new vertex, thus it has be adjacent to vertex 9 considering edge $(0,2)$. Similarly, $7 \sim 8, 10 \sim 6, 11 \sim 3$. Continue with this process, if vertex 8 is adjacent to a new vertex called a, then a must be adjacent to vertex 11, a contradiction to $d(11) = 2$, thus vertex 8 is adjacent to existing one and it can only vertex 3. Similarly, vertex $9 \sim 6$. Now the resulting graph is done and it is a Ricci-flat graph.

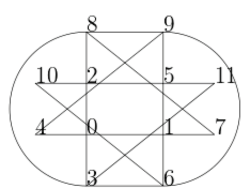


FIG. 32. *A Ricci-flat graph*

□

For the third case, let G be a 4-regular Ricci-flat graph in class $\mathcal{G}$ such that there exist two C_4s sharing one vertex and any two C_4s don't share an edge. Then G

contains a C_5, since every edge on the C_5 must be in a C_4, there are three distinct substructures:

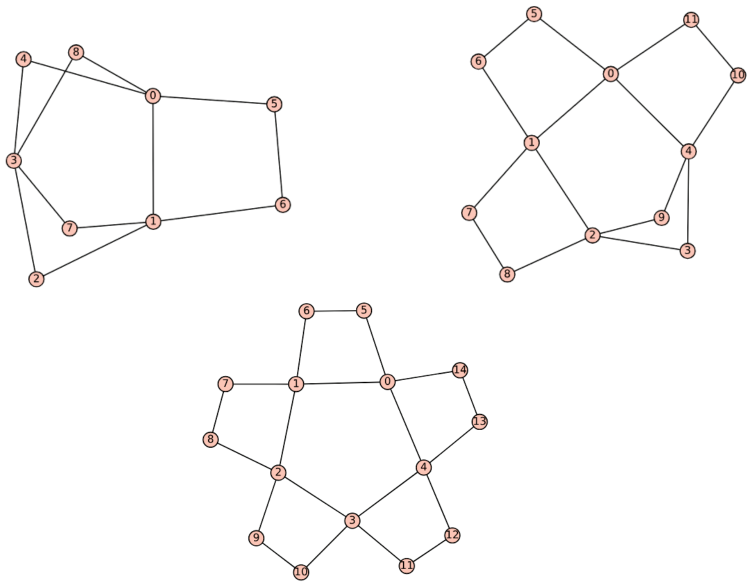


FIG. 33. *Type A, Type B, Type C*

Next, we consider each case.

LEMMA 13. *Let G be a 4-regular Ricci-flat graph, then G does not contain subgraph of "Type A."*

Proof. We study the graph based on this structure. Note all vertices cannot be adjacent to any other vertices in the current subgraph in order to avoid C_4s sharing on edge. Consider the edge $(0,4)$, let $4 \sim 9, 10$, wlog, then we need $d(1,9) = 2$ and $d(5,10) = 2$. For $d(1,9) = 2$, note $2 \not\sim 9$, otherwise the edge $(3,4)$ would share two C_4s. Similarly, $7 \not\sim 9$ considering the edge $(3,7)$. Then it must be $6 \sim 9$. Consider the vertex 8, $8 \sim 11, 12$ and $6 \sim 11$. Then $d(6) = 4$, consider the edge $(1,6)$, wlog, we need $d(2,9) = d(7,11) = 2$. Since $2, 7$ are not adjacent to any other vertices in the current subgraph, we need a common vertex 13 such that $2 \sim 13 \sim 9$. Note $7 \not\sim 13$, then let $7 \sim 14 \sim 11$. Let $2 \sim 15, 7 \sim 16$.

Consider the edge $(2,3)$, wlog, let $d(13,4) = 2$ and $d(15,8) = 2$. Assume $15 \sim 11$, then consider the edge $(6,11)$ in a C_4, assume $9 \sim 15$, then $10 \not\sim 15$ considering the edge $(9,15)$. It must be $10 \sim 13$ for the C_4 that passes through vertices $4, 9, 10$. Consider the edge $(2,13)$, since we have $d(1,10) = 3, d(3,10) = 2$, we also need $d(1,z) = 2$ where z is the fourth neighbor of vertex 13, However, this cannot happen in the current subgraph. Thus When $15 \sim 11$, $15 \not\sim 9$. Then for the edge $(6,11)$ in a C_4 that passes through vertex 9, it must be the case $9 \sim 14$. Then still we have $10 \sim 13$ for the C_4 that passes through vertices $4, 9, 10$. A contradiction again for the edge $(2,13)$. Thus $5 \not\sim 11$ for $d(15,8) = 2$, we must have $5 \sim 12$. Similarly, we have $15 \not\sim 9, 16 \not\sim 9, 11$, and $16 \sim 10$ for the edge $(3,7)$.

Now consider the edge $(1,2)$. Since $d(6,15) \neq 2$, we must have $d(0,15) = 2$ which implies $15 \sim 5$. Then for the edge $(1,7)$, we need $d(0,16) = 2$ which implies $16 \sim 5$.

Consider the C_4 that passes through vertices $5, 15, 16$, we need a common vertices for $15, 16$. Note $15 \not\sim 13, 9, 11$ and, $16 \not\sim 14, 9, 11$. Assume $15 \sim 10$, then consider the edge $(10, 15)$, we have $10 \not\sim 13$. Thus we must have $13 \sim 12$ for the C_4 that passes through vertices $2, 15, 13$. Consider the edge $(5, 6)$, either $d(15, 9) = 2$ or $d(15, 11) = 2$. If it is the former case, then $9 \sim 10$ or $9 \sim 12$, both would result appearances of C_3, a contradiction. Thus it must be $d(15, 11) = 2$ which implies $11 \sim 10$. Then for the edge $(6, 11)$ in a C_4 that passes through vertex 9, it must be the case $9 \sim 14$. It follows that $12 \sim 14$ for the C_4 that passes through vertex $12, 8, 11$. However, the edge $(9, 14)$ shares two C_4s, a contradiction.

We need a new vertex to be adjacent to $15, 16$. Let it be $15 \sim 17 \sim 16$. Then consider the edge $(2, 15)$, both $d(3, 5) = 3$ and $d(3, 17) = 3$, a contradiction.

Hence, there is no Ricci-flat graph based on "Type A" □

Lemma 14. *Let G be a 4-regular Ricci-flat graph, then G does not contain subgraph of "Type B".*

Proof. First note vertices $3, 9$ are not adjacent to any existing vertices in the current subgraph. Consider the edge $(0, 1)$, if $d(4, 7) = 2, d(11, 2) = 2$, then we must have $11 \sim 8, 7 \sim 10$, then the edge $(7, 8)$ would share two C_4s, a contradiction. Thus we need $d(11, 7) = 2, d(4, 2) = 2$ for the edge $(0, 1)$. Let $7 \sim 12 \sim 11$. Consider the edge $(0, 4)$, since $d(1, 9) = 3$, we need $d(5, 9) = 2$, note $5 \not\sim 12$ and any other existing vertices, let $5 \sim 14 \sim 9$. Similarly, we need $d(6, 9) = 2$ for the edge $(1, 2)$, since $6 \not\sim 14$ and any other existing vertices, let $6 \sim 15 \sim 9$. Now consider the C_4 that passes through vertices $9, 14, 15$, we need a vertex as the common for vertices $14, 15$. Consider the edge $(2, 9)$, note $14 \not\sim 7$, otherwise the cycle $C_5 := 14 - 7 - 1 - 6 - 5 - 14$ would be a "Type A". Thus $d(1, 14) = 3$, then we need $d(8, 14) = 2$, since $8 \not\sim 5$, otherwise, there would be a $C_5 := 5 - 0 - 1 - 2 - 8 - 5$ of "Type A', also $8 \not\sim 12$ and any existing vertices in the current subgraph, we need a new vertex 16 as the common of $14, 8$. Similarly, we need $d(10, 15) = 2$.

Assume $10 \sim 16 \sim 15$, then we need a common vertex for $8, 10$ to form a C_4 considering the $C_5 := 8 - 2 - 9 - 15 - 16 - 8$. Let $10 \sim 17 \sim 8$. Then consider the edge $(9, 14)$, since $d(2, 5) = 3$, we need $d(2, z) = 2$ where z is the fourth neighbor of vertex 14, then z should be adjacent to neighbor of 2. However all neighbors of 2 have degree 4, a contradiction. Thus for $d(10, 15) = 2$, let $10 \sim 17 \sim 15$. Then for the C_4 that passes through vertices $9, 14, 15$, since $15 \not\sim 16, 14 \not\sim 17$, we need a new vertex, let $14 \sim 13 \sim 15$.

For the edge $(5, 14)$ in a C_4 that passes through vertex 16, assume $5 \sim 17 \sim 16$. Then the edge $(5, 17)$would share two $C_4 := 5 - 17 - 16 - 14 - 5, 5 - 17 - -15 - 6 - 5$, a contradiction. Thus we need a new vertex as common for $5, 16$, let $5 \sim 18 \sim 16$. Similarly, for $d(6, 17) = 2$, note both $6, 17$ cannot be adjacent to 18 considering the edge $(5, 6)$. Thus let $6 \sim 19 \sim 17$.

Consider the C_4 that passes through vertices $6, 15, 9, 17$, then $19 \not\sim 13$, otherwise $(6, 15)$ would share two C_4s. Similarly, $18 \not\sim 13$.

Now consider the fourth neighbor of vertex 7:

- Assume $7 \sim 13$, then consider the edge $(1, 7)$, if $d(6, 13) = 2$, then $13 \sim 19$, a contradiction. Thus we need $d(0, 13) = 2$, then $13 \sim 11$. Consider the edge $(7, 8)$, note $d(16, 13) \neq 2$, otherwise 16 must be adjacent to one of neighbors of vertex 13, however, all neighbors have degree 4. Similar, let s represent the fourth neighbor of vertex 8, we have $d(z, 13) = 3$. Thus $d(8) \neq 4$. a contradiction. Actually, when $d(8) = 3$, this Ricci-flat graph is isomorphic to graph G_4 in Theorem 3.

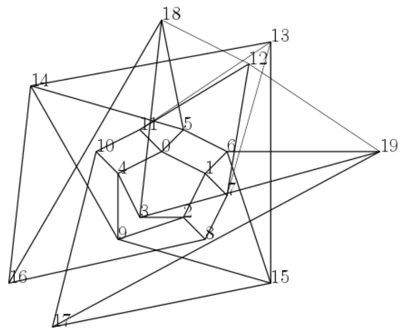


FIG. 34. *A Ricci-flat graph isomorphic to G_4*

- Assume $7 \sim 17$, consider the C_4 that passes through vertices $17, 10, 7, 12$, then $10 \sim 12$, a contradiction.
- Assume $7 \sim 10$, consider the C_4 that passes through vertices $7, 10, 17, 12$, then $17 \sim 12$. However the edge $(12, 17)$ would share two C_4s, with the second one $C_4 := 17 - 12 - 11 - 10 - 17$, a contradiction.
- Then $7 \sim 20$, a new vertex. For the edge $(1, 7)$, since $d(0, 20) \neq 2$, we need $d(6, 20) = 2$ which implies $20 \sim 19$. Consider the edge $(6, 19)$, we need $d(5, w) = 2$ where w is the fourth neighbor of 19, then $w \sim 18$. Assume $w = 12$, then there is a C_4 that passes through vertices $7, 12, 19, 20$. However, consider the edge $(7, 20)$, any new neighbor of vertex 20 has distance 3 from vertex 1, a contradiction.

We have considered all possible neighbors for vertex 7. Thus there is no 4-regular Ricci-flat graph of "Type B ". ∎

Now we consider all graphs in which every C_5 must be "Type C". We have the following result.

THEOREM 10. *Let G be a 4-regular Ricci-flat graph that contains a subgraph isomorphic to "Type C". And G is isomorphic to the graph showing in Figure 35.*

Proof. We start from such a subgraph of "Type C" in G by consider the neighborhood of each edge on the C_5, wlog, for the edge $(0, 1)$, there are two cases:

Case 1: $d(4, 7) = 2, d(14, 2) = 2$. For $d(4, 7) = 2$, we assume $13 \sim 7$. However the $C_5 := 7-1-0-14-13-7$ does not satisfy "Type C" Thus it must be $12 \sim 7$. Then consider the $C_5 := 12 - 4 - 0 - 1 - 7 - 12$, we need a C_4 that passes through edge $(12, 7)$ with new vertices added, let it be $12 \sim 16 \sim 15 \sim 7$. For $d(14, 2) = 2$, similarly, we need $14 \sim 9$.

Now consider the edge $(1, 7)$, there are also two cases:

Case 1a: $d(0, 15) = 2$, $d(6, 12) = 2$. Case 1a: By the "Type C", it must be $14 \sim 15$ and $6 \sim 11$. Consider the $C_5 := 1 - 6 - 11 - 12 - 7 - 1$, we need two new vertices to form a C_4 with $6, 11$, let them be $6 \sim 17 \sim 18 \sim 11$. Consider the $C_5 := 14-0-1-7-15-14$, to satisfy "Type C", the edge $(14, 15)$ must be in a C_4 that pass through vertex 9, there are two case: $9 \sim 17 \sim 15$ or $15, 9$ are adjacent to a new vertex 19. Assume the former. Consider the edge $(0, 14)$, there are two cases: Case 1a1: $d(1, 15) = 2$, $d(5, 9) = 2$; Case 1a2: $d(1, 9) = 2, d(5, 15) = 2$. For Case 1a1, we need $5 \sim 10$, for Case 1a2, we need $5 \sim 16$. Assume Case 1b1 is true, then for the edge $(5, 6)$, we need either $5 \sim 16$ or $5 \sim 19$ where 19 is a new

vertex such that $19 \sim 18$. However the latter case would contradict to the "Type C" for the $C_5 := 18 - 11 - 6 - 5 - 19 - 18$. Thus for the edge $(0, 14)$, Case 1a2 is always true, that is $5 \sim 16$. Assume $5 \not\sim 10$. Let $5 \sim 19$, a new vertex. Then we need $16 \sim 18$ and $19 \not\sim 18$. However, then $d(19, 17), d(19, 11) > 2$, a contradiction. Thus for the edge $(0, 14)$, we have $5 \sim 16$, and $5 \sim 10$.
Note $16 \sim 18$ by "Type C". By similar reasons, for the edge $(2, 8)$, we need $8 \sim 13, 8 \sim 18$. The for the edge $(8, 18)$ to be in a C_4, the only choice is passing through 13, let $18 \sim 19 \sim 13$. Then consider the edge $(4, 13)$, we must have $19 \sim 16$. Then for the edge $(5, 10)$ to be in a C_4, the only choice is $16 \sim 19$. Done.

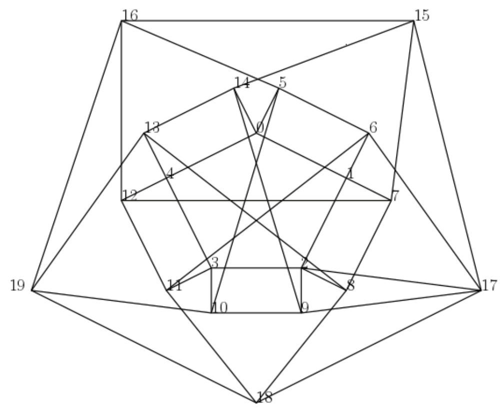
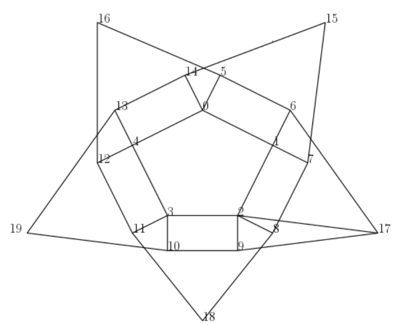


FIG. 35. *Ricci-flat graph*

Case 1b: $d(0, 12) = 2, d(6, 15) = 2$. This case is included by above case.

Case 2: $d(4, 2) = 2, d(14, 7) = 2$. Note we consider the same situation for all the other edges on the C_5. For $d(14, 7) = 2$, we need a new vertex 15. Similarly, we must have $6 \sim 16 \sim 9, 8 \sim 17 \sim 11, 10 \sim 18 \sim 13, 12 \sim 19 \sim 5$. Now, we conclude that except above Ricci-flat graph, any other 4-regular Ricci-flat graph in class $\mathcal{G}$ such that no edges share two C_4s must contain a subgraph that isomorphic to the following one:



Consider the $C_5 := 5 - 0 - 4 - 12 - 19 - 5$, then the edge $(12, 19)$ must be in a C_4. Note $12 \not\sim 13, 14, 5, 6, 10, 8$. Assume $12 \sim 9$. Then $19 \sim 16$ which lead to the edge $(5, 6)$ share two C_4, a contradiction. Assume $12 \sim 7$. Then $19 \sim 15$. Now consider the edge $(1, 7)$, since $d(0, 12) = 3, d(0, 15) = 2, d(2, 12) \geq 2, d(6, 12) \geq 2$, then we need $d(6, 12) = 2$, thus $6 \sim 11$. Similarly,

$5 \sim 10, 19 \sim 18, 14 \sim 9, 15 \sim 16, 16 \sim 17, 13 \sim 8, 17 \sim 18$. Done. The resulted graph is actually isomorphic to previous one.

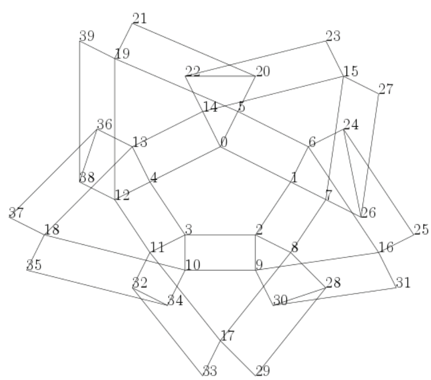
Now we consider another case for the circle $C_5 := 5{-}0{-}4{-}12{-}19{-}5$. That is, we need new vertices to form a C_4 for the edge $(5, 19)$ Let $5 \sim 20 \sim 21 \sim 19$. Note we should not have $15 \sim 19$ for this case, otherwise, it would result a same graph as previous one. Consider the edge $(0, 5)$, assume $d(4, 20) = 2, d(14, 19) = 2$, then for $d(4, 20) = 2$, either $20 \sim 13$ or $20 \sim 12$. however both assumptions do not guarantee the "Type C" for $C_5 := 20 - 13 - 4 - 0 - 5 - 20$ or $C_5 := 20 - 12 - 4 - 0 - 5 - 20$. Then it muse be the case $(4, 19) = 2, d(14, 20) = 2$ for the edge $(0, 5)$. For $d(14, 20) = 2$, note $14 \not\sim 21$ since the $C_5 := 14 - 21 - 20 - 5 - 0 - 14$ does not satisfy "Type C". Note $14 \not\sim 16$ or $14 \not\sim 17$ as the $C_5 := 14{-}0{-}5{-}6{-}16{-}14$ and $C_5 := 14{-}15{-}7{-}8{-}17{-}14$ cannot satisfy "Type C". It is easy to see that 14 is not adjacent to any other existing vertices in the current subgraph. Thus we need a new vertex for $d(14, 20) = 2$, let $14 \sim 22 \sim 20$. Then consider the edge $(0, 14)$, note $d(1, 22) \neq 2$, otherwise $22 \sim 6$ or $22 \sim 7$, both cases would generate C_5s that do not satisfy "Type C". Thus we need $d(1, 15) = 2$ and $d(5, 22) = 2$. The current subgraph meet this requirement. Now consider the edge $(4, 15)$ which must be in a C_4 that passes through vertex 22. Note $15 \not\sim 20$ for this requirement as $C_5 := 14 - 15 - 20 - 5 - 0 - 14$ cannot satisfy "Type C". Thus let $22 \sim 23 \sim 15$.

Next, we consider the fourth neighbor of vertex 6. Consider the edge $(5, 20)$, we need $d(6, w) = 2$ where w is the fourth neighbor of vertex 20, thus $20 \not\sim 16$, and $6 \not\sim 22$. Note $6 \not\sim 23$, otherwise the C_4 for the edge $(6, 23)$ must pass through the fourth neighbor of vertex 23 which is adjacent to vertex 16. Let $23 \sim 24 \sim 16$. Then consider the edge $(15, 7)$ on the $C_5 := 1{-}6{-}23{-}15{-}7{-}1$, note $7 \not\sim 22, 24$, since these two vertices are on the C_4s attached for this cycle. Note $7 \not\sim 21$ as we cannot guarantee the edge $(7, 21)$ any more. It is easy to see that 7 is not adjacent to any other existing vertices in the current subgraph. Thus let $7 \sim 25 \sim 26 \sim 15$. Consider the edge $(5, 19)$, we need the fourth neighbor of vertex 19 to have distance 2 form vertex 6. Consider the edge $(5, 20)$, we also need the fourth neighbor of vertex 20 to have distance 2 form vertex 6, since both the fourth neighbors should be adjacent to vertex 16 as the there has been 4 neighbors for vertex 23, then $d(16) \geq 5$, a contradiction. It is easy to see that 6 is not adjacent to any other existing vertices in the current subgraph.

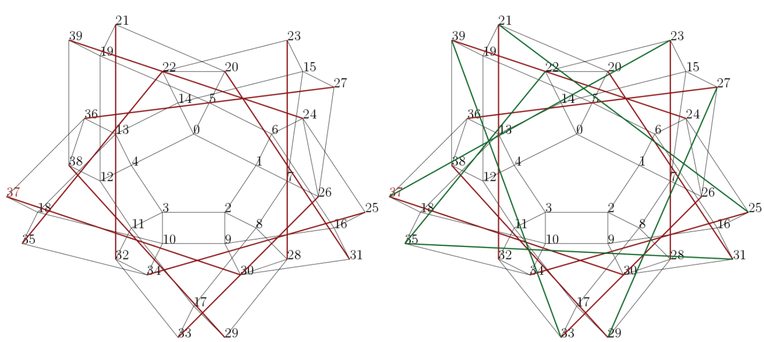
Thus we need a new vertex 24 as the fourth neighbor of vertex 6. Note vertex 7 is not adjacent to any existing vertices, we need a new vertex 26 as the fourth neighbor of it. Same reason as why $20 \sim 22$, we have $24 \sim 26$. For the edge $(6, 16)$ to be in a C_4 that passes through vertex 24, assume $24 \sim 23 \sim 16$. Then the $C_5 := 23 - 24 - 26 - 7 - 15 - 23$ cannot satisfy "Type C". Thus we need a new to form C_4 for the edge $(6, 16)$, let $24 \sim 25 \sim 16$. And we need a new neighbor for vertex 15 to from a $C_4 := 7 - 26 - 27 - 5$.

We now consider the fourth neighbor of vertex 8 and 9. Assume $8 \sim 20$, then for the edge $(8, 17)$ in a C_4 that pass through vertex 20, we must have $17 \sim 22$. However, for the edge $(8, 20)$, since $d(2, 5) = 3$, we need $d(2, 21) = 2$ and $d(5, 9) = 2$, which implies $9 \sim 21$ and $9 \sim 19$, then $d(9) \geq 5$, a contradiction. Similar analysis shows that $8 \not\sim 22$, $9 \not\sim 20, 9 \not\sim 22$. And it is easy to check vertices 8 and 9 are not adjacent to any other existing vertices in the

current subgraph. Thus let $8 \sim 28, 9 \sim 30$ and we also have $28 \sim 30$ by similar reasons for $20 \sim 22, 24 \sim 26$. Now we analysis the common vertex for $28, 17$. We consider the vertex with degree 2 in the current subgraph. Assume $28 \sim 21 \sim 17$. Then we need either $d(22, 17) = 2$ or $d(22, 28) = 2$ considering the edge $(20, 21)$, and $d(z, 28) = 2$ where z is the fourth neighbor of vertex 19. Since $d(28) = 3$ in the current subgraph and the vertices $19, 22$ cannot share a common vertex, we actually need $d(22, 17) = 2$. Note $17 \not\sim 20$ and $22 \not\sim 11$, thus we need a new vertex 29 such that $22 \sim 29 \sim 17$, then consider the edge $(8, 17)$, we need $d(7, 29) = 2$, then $29 \sim 26$. Since we need a C_4 for the edge $11, 17$ that passes through vertex 29 and a C_4 for the edge $24, 26$ that pa passes through vertex 29, then $d(29) \geq 5$, a contradiction. One can also check that $28, 17$ are not both adjacent to vertex $23, 27, 25$. Thus we need new vertex. Let $8 \sim 18 \sim 29 \sim 17$, similarly, $9 \sim 30 \sim 31 \sim 16$.
Similar argument will give us $11 \sim 32 \sim 33 \sim 17$, $10 \sim 34 \sim 35 \sim 18$, $32 \sim 34$, and $13 \sim 36 \sim 37 \sim 18, 12 \sim 28 \sim 39 \sim 19, 36 \sim 38$. See the following structure:



Now observe the edge $(5, 6)$, since $d(9, 16)$ cannot be 2, we must have $d(9, 14) = 2$ and $d(20, 6) = 2$. Note $24 \not\sim 21$, as it would generate the $C_5 := 19 - 21 - 24 - 6 - 5 - 19$ which does not satisfy "Type C". Then it must be $24 \sim 39$, similarly $20 \sim 31, 26 \sim 33, 28 \sim 23, 30 \sim 37, 34 \sim 25, 32 \sim 21, 38 \sim 29, 36 \sim 27, 22 \sim 35$. Consider the C_4 for the edge $(20, 22)$, we must have $31 \sim 35$, similarly, $33 \sim 39$, $23 \sim 37, 21 \sim 25, 27 \sim 29$.



Now all vertices have degree 4, however we cannot guarantee the edge $(20, 21)$, since $d(22, 32)$ cannot be 2. Thus there is no Ricci-flat graph for this case.

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