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GAUSS-KRONECKER CURVATURE AND EQUISINGULARITY AT INFINITY OF DEFINABLE FAMILIES*

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Abstract. Assume given a polynomially bounded o-minimal structure expanding the real numbers. Let $(T_s)_{s\in\mathbb{R}}$ be a definable family of C^2-hypersurfaces of $\mathbb{R}^n$. Upon defining the notion of generalized critical value for such a family, we show that the functions $s \to |K|(s)$ and $s \to K(s)$, respectively the total absolute Gauss-Kronecker and total Gauss-Kronecker curvature of T_s, are continuous in any neighbourhood of any value which is not generalized critical. In particular this provides a necessary criterion of equisingularity for the family of the levels of a real polynomial.

Key words. Gauss-Kronecker curvature, total curvatures, generalized critical values, definable families.

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1. Introduction. One of the main goals of equisingularity theory (of families of subsets, functions, mappings) is to find relations between numerical data and regularity conditions. In the local complex analytic case, this subject has been widely studied since the end of the 60's and many interesting results, some of them now classical, have been established. For example, Hironaka [Hir] proved that the multiplicity is constant along the strata of a Whitney stratification of a complex analytic set. In [Tei1] Teissier proved that a μ^*-constant family of hypersurfaces with isolated singularities is Whitney equisingular. The reverse implication was proved later by Briançon and Speder [BrSp]. These results were extended to the case of *ICIS* by Gaffney [Gaf1]. Maybe the most important result of local complex analytic equisingularity theory is Teissier's polar equimultiplicity theorem [Tei2], which states that Whitney regularity is equivalent to constancy of polar multiplicities. Teissier's results were refined and extended by Gaffney [Gaf2] to obtain sufficient conditions for equisingularity of a family of mappings.

When one considers global equisingularity problems, the first natural family to study is the family of fibres of a polynomial mapping. Following [Tho], a polynomial function from $\mathbb{K}^n$ to $\mathbb{K}$, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, is a smooth locally trivial fibration above the connected components of the complement of a (minimal) finite subset $B(f)$ of $\mathbb{K}$, called the set of bifurcation values of f. In the complex plane case, Hà and Lê [HL] gave the following numerical criterion to characterize bifurcation values: *A value c does not lie in $B(f)$ if and only if the Euler characteristic of the fibres of f is constant in a neighborhood of c.* This result was generalized by Parusiński [Par] to the case of complex polynomials with isolated singularities at infinity, and then by Siersma and Tibăr [SiTi1] to the case of complex polynomials with isolated $\mathcal{W}$-singularities at infinity. In [Tib1] Tibăr studies the more general situation of a 1-parameter family of complex hypersurfaces, and proves a global version of the results of Teissier and Briançon and Speder mentioned above: Considering a family of complex affine hypersurfaces

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$X_\tau = \{x \in \mathbb{C}^n \ : \ F(\tau, x) = 0\}$ given by a polynomial function $F : \mathbb{C} \times \mathbb{C}^n \mapsto \mathbb{C}$, he defines the notion of t-equisingularity at infinity and proves, under some additional conditions, that t-equisingularity at infinity is controlled by the constancy of a finite sequence of numbers, called the generic polar intersection multiplicities. As a consequence, if the family consists of non-singular affine hypersurfaces, then the constancy of the generic polar intersection multiplicities at τ_0 implies that the family is C^∞ trivial at τ_0.

In the real semi-algebraic/sub-analytic setting (or more generally in the definable setting), it is hopeless to expect that *constancy of numerical data is equivalent to regularity conditions.* First, because of lack of connectivity, one cannot define invariants like the μ^*-sequence, polar multiplicities or generic polar intersection multiplicities. However, using arguments from differential topology and integral geometry, one sees that these invariants admit geometric characterizations that still make sense in the real case. For instance, the multiplicity of a complex analytic germ is equal to its density [Dra] and the μ^*-sequence, the polar multiplicities and the generic polar intersection multiplicities are related to curvature integrals (see [La, Loe, Dut1, SiTi2]). Unfortunately, in the real situation, these geometric quantities do not belong to discrete sets and therefore, one cannot expect results relating their constancy to regularity conditions. It is more reasonable to study properties like continuity or Lipschitz continuity in the parameters of the family. The first result in this direction is due to Comte [Com], who established a real version of Hironaka's theorem, proving that the density is continuous along the strata of a (w)-stratification of a sub-analytic set. This result was generalized and strengthened by Valette [Val]: continuity of the density holds for (b)-regular stratifications and the density is Lipschitz continuous along the strata of (w)-stratifications. Later Comte and Merle [ComMe] established a real version of Teissier's theorem [Tei2]. Using tools from integral geometry and geometric measure theory, they associated with each sub-analytic germ a sequence of numbers, called the local Lipschitz-Killing invariants, and showed that they are continuous along the strata of a (w)-stratification of a sub-analytic set. Recently, Nguyen and Valette [NgVa] extended this continuity result to (b)-stratifications and moreover proved that these invariants are Lipschitz continuous along the strata of a (w)-stratification (see also the first author's work [Dut2] for relations with the densities of polar images).

In the global real context, it is still true that the bifurcation set of a definable function from $\mathbb{R}^n$ to $\mathbb{R}$ is a finite set of points (see [NeZa, LoZa, Tib2, d'Ac1]). In [TiZa] Tibăr and Zaharia provided necessary and sufficient conditions for a real plane polynomial function to be locally trivial over the neighborhood of a regular value (see [JoTi] for a generalization to a family of real curves). Unlike the complex case, their criterion is not only numerical but involves topological conditions at infinity. Later in [CosPe], Coste and de la Puente proved an equivalent version of Tibăr-Zaharia's results in terms of polar curves. Due to the links between polar curves and the Gauss-Kronecker curvature of the levels of a function provided by exchange formulas, it seems natural to study the variations of the total curvature of the levels (i.e. the integral of the Gauss-Kronecker curvature on the level) of a definable function, and to seek how bifurcation values interfere in these variations.

That is what the second author did in two papers. In [Gra1] he considers a definable function $f : \mathbb{R}^n \mapsto \mathbb{R}$ of class at least C^2, and proved that the following functions:

$$t \mapsto \int_{f^{-1}(t)} \kappa(\mathbf{x})\,\mathrm{d}\mathbf{x}\,, \quad \text{and } t \mapsto \int_{f^{-1}(t)} |\kappa|(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

where κ is the Gauss-Kronecker curvature, admit at most finitely many discontinuities. Any connected C^2 hypersurface of $\mathbb{R}^n$ comes naturally equipped with the induced Euclidean metric. Therefore, integrating with respect to the volume form of such a hypersurface coincides with integrating with respect to the $(n-1)$-dimensional Hausdorff measure over $\mathbb{R}^n$. In [Gra2] was proved that if the function $t \mapsto \int_{f^{-1}(t)} |\kappa| \, \mathrm{d}\mathbf{x}$ is continuous at a regular value c which satisfies an extra condition, then c is not a bifurcation value of f. He explained that for a real polynomial function with isolated singularities at infinity this extra condition is always satisfied, so this result can be interpreted as a real version of Parusiński's result mentioned above.

The aim of the present paper is to provide a kind of reverse implication of the latter mentioned result.

We will work in the more general situation of a one parameter family of hypersurfaces. More precisely, we consider a definable function over an a priori given polynomially bounded o-minimal structure $F : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ of class C^{2+m} with non-negative integer m. Assuming that 0 is a regular value of F, the 0-level $M = F^{-1}(0)$ is thus a definable hypersurface in $\mathbb{R}^{n+1}$ of class C^{2+m}. We use the coordinates $(\mathbf{x}, t)$ in $\mathbb{R}^n \times \mathbb{R}$ and we write $\mathbf{t}_M : M \mapsto \mathbb{R}$, $(\mathbf{x}, t) \mapsto t$ for the projection on the t-axis.

For a value c in $\mathbb{R}$, let $M_c = \mathbf{t}_M^{-1}(c)$ and $T_c = \pi_M(M_c) \subset \mathbb{R}^n$, where π_M is the projection from M to $\mathbb{R}^n$. If c is a regular value, then the hypersurface T_c is oriented by $\partial_{\mathbf{x}} F(\mathbf{x}, c)$. Therefore, we consider the Gauss-Kronecker curvature κ_c of T_c and define two functions:

$$c \mapsto K(c) = \int_{T_c} \kappa_c(\mathbf{x}) \, \mathrm{d}\mathbf{x}\,, \quad \text{and} \quad c \mapsto |K|(c) = \int_{T_c} |\kappa_c|(\mathbf{x}) \, \mathrm{d}\mathbf{x}.$$

By a straightforward adaptation of the methods of [Gra1], we show that these two functions have finitely many discontinuities (Theorem 6.1) and in Theorem 8.1, we give a criterion on the regular value c of $\mathbf{t}_M$ for the function $t \mapsto |K|(t)$ to be continuous at c. Namely, we prove

THEOREM 8.1. *Let c be a regular value taken by $\mathbf{t}_M$ at which it is horizontally spherical at infinity. Then the total absolute curvature function $t \mapsto |K|(t)$ is continuous at c. Consequently the total curvature function $t \mapsto K(t)$ is continuous at c.*

The notion of *horizontally sphericalness at infinity* is a regularity condition at infinity: A regular value c of $\mathbf{t}_M$ is horizontally spherical at infinity if for any sequence $(\mathbf{p}_k)_{k \in \mathbb{N}}$ of M converging at infinity to $(\mathbf{u}, c)$, $\mathbf{u}$ is orthogonal to the limit of the unitary gradients $\frac{\nabla \mathbf{t}_M}{|\nabla \mathbf{t}_M|}(\mathbf{p}_k)$. A key ingredient of the proof of our main result, Theorem 8.1 is Lemma 8.2 stating, informally, that *under these hypotheses there is no accumulation of curvature at infinity nearby the level c.*

We also prove that (t)-equisingularity at infinity implies horizontal sphericalness (Corollary 5.2). Therefore Theorem 8.1 shows that (t)-equisingularity at infinity implies continuity of the function $t \mapsto |K|(t)$. This can be considered as a first step towards a real version of Tibăr's result [Tib1] mentioned above.

To be complete, we show here more than Theorem 8.1. Its conclusion also holds true in any connected component of the pencil of levels over a small interval of regular values $]c - \varepsilon, c + \varepsilon[$ (see Theorem 8.3). In other words the connected components of the pencil of levels cannot compensate altogether the a priori possible discontinuities of some.

The paper is organized as follows. Section 2 contains material on compactifications, o-minimal structures and Thom's (a_f) condition. In Section 3, we recall some facts about conormal geometry so that we can introduce the notion of t-equisingularity. Sections 4, 5 and 7 contain definitions and new results on regularity at infinity of definable C^{2+m}-families of hypersurfaces. In Section 6, we generalize the results of [Gra1] to our situation. Section 8 contains the proof of the main result. Section 9 deals with the particular case of the levels of a function. Last, some examples can be found in Section 10.

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2. Miscellaneous material. Let $\mathbb{R}^n$ be the Euclidean space of positive dimension n.

Let $\langle -, - \rangle$ be the associated scalar product. For any point $\mathbf{x}$ of $\mathbb{R}^n$, let $|\mathbf{x}|$ be the norm $\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ of $\mathbf{x}$.

Let $\mathbf{S}_R^{q-1}$ be the Euclidean sphere of $\mathbb{R}^q$ of center the origin and positive radius R.

Let $\mathbf{B}_R^q$ be the closed Euclidean ball of $\mathbb{R}^q$ of center the origin and positive radius R. When q is understood we will only write $\mathbf{B}_R$.

Let $\mathrm{clos}(-)$ denote the operation "taking the closure of" in $\mathbb{R}^n$. Each Euclidean space $\mathbb{R}^n$ embeds semi-algebraically in the closed unit-ball $\mathbf{B}_1^n$, as its interior via the mapping $\mathbf{x} \mapsto \frac{\mathbf{x}}{\sqrt{1+|\mathbf{x}|^2}}$. We may then speak of $\mathbf{B}_1^n$ as the *spherical compactification of* $\mathbb{R}^n$ (see the next section).

Let $\mathcal{M}$ be an o-minimal structure expanding the real field $\mathbb{R}$. Assume it is polynomially bounded and let $\mathbb{F}_\mathcal{M}$ be the field of its exponents ([vdDM, vdD]). Any subset of any $\mathbb{R}^p$ definable in $\mathcal{M}$ will be called below *definable.*

Let X be a subset of $\mathbb{R}^n$. A mapping $f : X \mapsto \mathbb{R}^p$ is definable if its graph is definable in $\mathbb{R}^{n+p}$.

We would like to remind the following fact (see [d'Ac1]): *Let $\gamma : [1, +\infty[\mapsto \mathbb{R}^n$ be a C^1 definable arc such that $\gamma(t) \mapsto \infty$ as t goes to $+\infty$. Then there exists a unit vector $\mathbf{u}$ of $\mathbf{S}^{n-1}$ such that*

$$\lim_\infty \frac{\gamma}{|\gamma|} = \mathbf{u} = \lim_\infty \frac{\gamma'}{|\gamma'|}.$$

Let $f : (\mathbb{R}_{\geq 1}, +\infty) \to \mathbb{R}$ be the germ at $+\infty$ of a continuous definable function. We write $f \sim t^e$ for an exponent e in $\mathbb{F}_\mathcal{M} \cup \{-\infty\}$, with the convention that $t^{-\infty} = 0$ for large t, to mean

$$f \sim t^e \iff \lim_{t \to +\infty} \frac{f(t)}{t^e} \in \mathbb{R}^* .$$

Note that there always exists such an exponent e.

Let $\mathbf{G}(p, n)$ be the Grassmann manifold of p-vector subspaces of $\mathbb{R}^n$. We denote $\mathbf{G}^\vee(p, n)$ the space of p-vector subspaces of the space $L(\mathbb{R}^n, \mathbb{R})$ of linear forms over $\mathbb{R}^n$, and sometimes we will call it the *dual* of $\mathbf{G}(n-p, n)$.

We recall *Thom's condition* (or relative Whitney's condition (a)).

Let X, Y be two connected C^1 submanifolds of a definable compactification of $\mathbb{R}^n$, such that Y is contained in $\mathrm{clos}(X) \setminus X$. Let $g : (X \sqcup Y) \mapsto \mathbb{R}$ be a C^1 mapping, for $X \sqcup Y$ the disjoint union of X and Y. Let $\mathbf{y}$ be a point of Y.

The function g satisfies *Thom* (a_g)*-condition at* $\mathbf{y}$ if the following two conditions hold:

(i) For any sequence $(\mathbf{x}_k)_k$ of points of X converging to $\mathbf{y}$ such that the sequence $(T_{\mathbf{x}_k}X)_k$ converges to T in the appropriate Grassmann bundle, then $T_\mathbf{y}Y$ is contained in T;
(ii) For any sequence $(\mathbf{x}_k)_k$ of points of X converging to $\mathbf{y}$, such that the sequence $(T_{\mathbf{x}_k}X)_k$ converges to T which contains $T_\mathbf{y}Y$ and the sequence $(\ker \mathrm{d}_{\mathbf{x}_k} g)_k$ converges to K in the appropriate Grassmann bundles, then $\ker \mathrm{d}_\mathbf{y} g$ is contained in K.

In practice we want to stratify g with Thom's condition asking that the stratum Y is contained in some specified level of g.

3. Compactification and conormal geometry and t-equisingularity at infinity. Let $\mathbf{0}$ be the origin of $\mathbb{R}^n$.

As already seen in the previous section, we can compactify $\mathbb{R}^n$ as the closed unit ball $\mathbf{B}_1^n$. An alternative presentation to the spherical compactification is the spherical blowing-up $\mathbf{bl}_\infty$ of $\mathbb{R}^n$ at infinity, that is the mapping given by

$$\begin{array}{rcrcl} \mathbf{bl}_\infty & : & \mathbf{S}^{n-1}\times]0,+\infty[& \mapsto & \mathbb{R}^n \setminus \{\mathbf{0}\} \\ & & (\mathbf{u}, r) & \mapsto & \frac{\mathbf{u}}{r} \end{array}.$$

It is a Nash diffeomorphism and a re-parametrization of $\mathbb{R}^n \setminus \{\mathbf{0}\}$ embedded in $\mathbf{B}_1^n$. It is more convenient to look at it this way since it is a good real avatar of the projective compactification $\mathbf{P}\mathbb{R}^n$ (which in our definable context is not as relevant as in the algebraic case).

We denote by $\mathbf{S}_\infty^{n-1} := \mathbf{S}^{n-1} \times 0$ the sphere at infinity. Let us denote and identify

$$\overline{\mathbb{R}^n} := \mathbb{R}^n \sqcup \mathbf{S}_\infty^{n-1} = (\mathbf{S}^{n-1} \times [0, +\infty[) \sqcup \mathbf{0},$$

the spherical compactification of $\mathbb{R}^n$ at infinity, with boundary $\partial\overline{\mathbb{R}^n} := \mathbf{S}_\infty^{n-1}$, the sphere at infinity.

Let $\overline{Z}$ be the closure of the subset Z of $\mathbb{R}^n$ taken into $\overline{\mathbb{R}^n}$. The *tangent link of Z at ∞* is defined as

$$Z^\infty := \overline{Z} \cap \mathbf{S}_\infty^{n-1}.$$

The *tangent cone of Z at infinity* $C_\infty(Z)$ is defined as the (non-negative) cone over Z^∞. Whence Z^∞ is not empty (equivalently Z is not bounded) we also observe that

$$Z^\infty := \mathrm{clos}\left\{\mathbf{u} \in \mathbf{S}^{n-1} \,:\, \exists Z \ni (x_k)_k \to \infty \text{ such that } \frac{x_k}{|x_k|} \to \mathbf{u}\right\}.$$

If a subset Z of $\mathbb{R}^n$ is definable then $\overline{Z}$ is definable in $\overline{\mathbb{R}^n}$. Thus whenever a subset Z of $\mathbb{R}^n$ is definable, *its tangent link at infinity* Z^∞ is definable and of dimension at most $\dim Z - 1$.

Although heavy to define it is convenient to use the formalism of conormal geometry. We are especially interested in conormal geometry at infinity.

Now let $Z = \{(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R} \,:\, G(\mathbf{x}, t) = 0\}$, where $G : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ is a definable function of class at least C^2, and let $\overline{Z}$ be its closure in $\overline{\mathbb{R}^n} \times \mathbb{R}$.

We assume that 0 is a regular value of G and we consider Z as a definable family $\{Z_t\}_{t\in\mathbb{R}}$ of hypersurfaces in $\mathbb{R}^n$. Let $g : Z \to \mathbb{R}$ be a definable function which we assume to be C^1. For any regular point $(\mathbf{x}, t)$ of the function g, let $T_{(\mathbf{x},t)}g$ be the subspace of $T_{(\mathbf{x},t)}Z$ tangent at $(\mathbf{x}, t)$ to the level of g through $(\mathbf{x}, t)$. Let us define the following subset of $\overline{\mathbb{R}^n} \times \mathbb{R} \times \mathbf{G}^\vee(1, n+1)$:

$$\mathscr{X}_g^\vee := \mathrm{clos}\{(\mathbf{x}, t, \xi) \in Z \setminus \mathrm{crit}(g) \times \mathbf{G}^\vee(1, n+1) \,:\, \xi(T_{(\mathbf{x},t)}g) = 0\},$$

where $\mathbf{G}^{\vee}(1, n+1)$ is the dual of $\mathbf{G}(n, n+1)$.

DEFINITION 3.1. *The* relative conormal space of g *is the space* $\mathscr{X}_g^{\vee}$.

Let $\pi_{n+1} : \overline{\mathbb{R}^n} \times \mathbb{R} \times \mathbf{G}^{\vee}(1, n+1) \mapsto \overline{\mathbb{R}^n} \times \mathbb{R}$ be the projection given as $\pi_{n+1}(\mathbf{x}, t, \xi) = (\mathbf{x}, t)$.

DEFINITION 3.2. *The* relative conormal space of g at infinity *is the space* $\mathscr{X}_g^{\infty}$ *defined as*

$$\mathscr{X}_g^{\infty} := \pi_{n+1}^{-1}(Z^{\infty}) \cap \mathscr{X}_g^{\vee},$$

where $Z^{\infty} = \overline{Z} \cap (\mathbf{S}_{\infty}^{n-1} \times \mathbb{R})$.

For any $\mathbf{p} \in Z^{\infty}$, *let* $(\mathscr{X}_g^{\infty})_{\mathbf{p}}$ *be the fibre of* $\mathscr{X}_g^{\vee}$ *above* p, *that is* $(\mathscr{X}_g^{\infty})_{\mathbf{p}} = \pi_{n+1}^{-1}(\mathbf{p}) \cap \mathscr{X}_g^{\vee}$.

We introduce now the notion of t-equisingularity [Tib1] adapted to the context of Section 4.

Let $\mathbf{r}_Z : Z \mapsto \mathbb{R}$ be defined as $(\mathbf{x}, t) \mapsto |\mathbf{x}|$. It is continuous definable and C^1 outside $\mathbf{0} \times \mathbb{R} \cap Z$.

The *space of characteristic covectors* $\mathscr{C}$ *of* Z *at infinity* is the subset of $\overline{\mathbb{R}^n} \times \mathbb{R}^n \times \mathbf{G}^{\vee}(1, n+1)$ defined as

$$\mathscr{C}(Z) := \mathscr{X}_{\mathbf{r}_Z}^{\infty}.$$

It is closed and definable.

Let $\tau : \overline{\mathbb{R}^n} \times \mathbb{R}$ be defined as $(\mathbf{x}, t) \mapsto t$.

The following notion is due to Tibăr [Tib1]:

DEFINITION 3.3. *Let* $\mathbf{p}$ *in* Z^{∞}.

(i) The family $\{Z_t\}_{t \in \mathbb{R}}$ *is* t*-equisingular at* $\mathbf{p}$ *if*

$$\mathscr{C}(Z)_{\mathbf{p}} \cap (\mathscr{X}_{\tau}^{\infty})_{\mathbf{p}} = \emptyset.$$

(ii) The family $\{Z_t\}_{t \in \mathbb{R}}$ *is* t*-equisingular at infinity at* c *if it is* t*-equisingular at* $\mathbf{p}$ *for all* $\mathbf{p}$ *in* $Z^{\infty} \cap \tau^{-1}(c)$.

The definition above is slightly different from those given in [SiTi1, Tib1, DiRuTi], since there it is given via the projective compactification of $\mathbb{R}^n$. Anyhow they are equivalent.

4. Regularities at infinity for definable families of hypersurfaces. We present here two regularity conditions at infinity for the function restriction of a coordinate projection along a definable one parameter family of hypersurfaces. In the next section, we will compare altogether these regularity conditions with t-equisingularity, introduced in the previous section.

Let $F : \mathbb{R}_{\mathbf{x}}^n \times \mathbb{R}_t \mapsto \mathbb{R}$ be a C^{2+m} definable function, for some non-negative integer m.

Assuming that $\mathbb{R}^n \times \mathbb{R}$ is equipped with the canonical Euclidean structure, let ∇F be the gradient field of F. Without further hypotheses, the real number 0 may be a critical value of F, and ∇F may be vanishing on the zero level of F.

WORKING HYPOTHESES. *(i)* 0 *is a regular value of* F.
(ii) There is no connected component of $F^{-1}(0)$ *contained in an affine hyperplane of the form* $\mathbb{R}^n \times c$.

Let M be the zero locus $F^{-1}(0)$ of the function F, which is a closed definable subset of $\mathbb{R}^{n+1}$ and a C^{2+m} hypersurface. Let $\overline{M}$ be its closure in $\overline{\mathbb{R}^n} \times \mathbb{R}$.

We define two mappings $(\pi, \mathbf{t}) : \overline{\mathbb{R}^n} \times \mathbb{R} \mapsto \overline{\mathbb{R}^n} \times \mathbb{R}$ obtained respectively as the projections over $\overline{\mathbb{R}^n}$ and over $\mathbb{R}$, and both are semi-algebraic.

Let $\mathbf{t}_M$ be $\mathbf{t}|_M$ the restriction of $\mathbf{t}$ to M and let π_M be the restriction of π to M, both are C^{2+m} and definable mappings. Let us write $M_c := \mathbf{t}_M^{-1}(c)$ and $T_c := \pi_M(M_c)$ subset of $\mathbb{R}^n$. Hypothesis (ii) above is equivalent to require that the critical locus $\mathrm{crit}(\mathbf{t}_M)$ has positive codimension in M.

Definition 4.1. *Let c be a value taken by $\mathbf{t}_M$. The function $\mathbf{t}_M$ is said locally C^k trivial at c if there exists a positive real number ε such that $\mathbf{t}_M^{-1}(]c-\varepsilon, c+\varepsilon[)$ is a trivial C^k-bundle with fibre M_c.*

Mimicking what was done for level hypersurfaces of functions [LoZa, TiZa, d'Ac1, d'AcGr1, d'AcGr2, Gra2], sufficient conditions about the gradient of $\mathbf{t}_M$ guarantee trivialization (see below). Since M is definable and each of its connected component is orientable, let ν_M be a C^{1+m} globally definable unitary field normal to M. Since 0 is not a critical value of F, we choose

$$\nu_M := \frac{\nabla F}{|\nabla F|} = \nu_M^{\mathbf{x}} + \nu_M^t \partial_t$$

where $\nu_M^{\mathbf{x}}$ is the component of ν_M in $\mathbb{R}^n \times 0$, and writing $\nabla F = \partial_{\mathbf{x}} F + \partial_t F \partial_t$, where $\partial_{\mathbf{x}} F$ lies in $\mathbb{R}^n \times 0$.

Let $\mathbf{p} = (\mathbf{x}, t)$ be a point of M. We have

$$T_{\mathbf{p}} M = \{(\mathbf{u}, w) \in \mathbb{R}^n \times \mathbb{R} \,:\, \langle \partial_{\mathbf{x}} F, \mathbf{u} \rangle + \partial_t F \cdot w = 0\}.$$

It is easy to prove the following relation:

$$\nabla \mathbf{t}_M = -\frac{\partial_t F}{|\nabla F|^2} \partial_{\mathbf{x}} F + \frac{|\partial_{\mathbf{x}} F|^2}{|\nabla F|^2} \partial_t = -\nu_M^t \nu_M^{\mathbf{x}} + |\nu_M^{\mathbf{x}}|^2 \partial_t \tag{4.1}$$

and thus

$$|\nabla \mathbf{t}_M| = \frac{|\partial_{\mathbf{x}} F|}{|\nabla F|} = |\nu_M^{\mathbf{x}}|.$$

The critical locus of $\mathbf{t}_M$ is

$$\mathrm{crit}(\mathbf{t}_M) = \{\mathbf{p} \in M \,:\, \partial_{\mathbf{x}} F(\mathbf{p}) = 0\}.$$

Since M is a C^{2+m} orientable hypersurface, the function $\mathbf{t}_M$ is C^{2+m} as well. Since it is definable, the set of its critical values $K_0(\mathbf{t}_M) := \mathbf{t}_M(\mathrm{crit}(\mathbf{t}_M))$ is finite.

Let $\nu_{\mathbf{t}_M} : M \setminus \mathrm{crit}(\mathbf{t}_M) \mapsto \mathbf{S}^n$ be the unitary gradient of $\nabla \mathbf{t}_M$,

$$\nu_{\mathbf{t}_M} := \frac{\nabla \mathbf{t}_M}{|\nabla \mathbf{t}_M|}.$$

The Local Conical Structure Theorem ensures the existence of a positive number S_M such that for any $S > S_M$ the hypersurface M is transverse with $\mathbf{S}_S^n$, the Euclidean sphere of radius S. As a consequence of this fact we also have:

LEMMA 4.2. *For any $A > max_{c \in K_0(\mathbf{t}_M)}|c|$, there exists R_A such that for any $R > R_A$ the definable C^{2+m} hypersurface $M \cap (\mathbb{R}^n \times]-A, A[)$ is transverse to the cylinder $\mathbf{S}_R^{n-1} \times \mathbb{R}$.*

Proof. Let $A \gg 1$ be given. Let us define the following subset

$$\Sigma := \{(\mathbf{x}, t) \in M \,:\, \mathbf{x} \wedge \nabla F(\mathbf{x}, t) = 0\}.$$

Note that $\Sigma \setminus \mathbf{0} \times \mathbb{R}$ is contained in $M \cap \{\partial_t F = 0\}$ and that Σ is a closed definable subset of M.

Let us assume that the statement of the lemma is not true. Thus there exists a C^1 definable path $\gamma : [1, +\infty[\mapsto \Sigma \cap \mathbb{R}^n \times]-A, A[$ such that $\gamma(s) \to (\mathbf{u}, c)$ in M^∞ as s goes to $+\infty$, with $|c| < A$.

We can parameterize γ in such a way that $|\gamma(s)| = s$, which gives the following

$$\gamma(s) = (s\mathbf{u}, 0) + s^e \mathbf{v}(s)$$

for a C^1 and definable mapping $s \mapsto s^e \mathbf{v}(s) \in \mathbb{R}^n \times \mathbb{R}$ such that $\lim_\infty \mathbf{v} \neq (0, 0)$ and $e < 1$. We also have that $s \mapsto \nu_M(s) = \frac{\nabla F}{|\nabla F|}(\gamma(s))$ goes to ν in $\mathbf{S}^n$ as s goes to $+\infty$. Note that $\frac{\gamma'(s)}{|\gamma'(s)|}$ goes to $(\mathbf{u}, 0)$ as s goes to $+\infty$. Since

$$\gamma(s) \wedge \nu_M(s) = 0 \;\text{ and }\; \langle \gamma'(s), \nu_M(s) \rangle = 0\,,$$

we deduce that

$$\mathbf{u} \wedge \nu = 0 \;\text{ and }\; \langle \mathbf{u}, \nu \rangle = 0,$$

which is absurd. □

We can introduce now the Malgrange regularity condition at infinity.

DEFINITION 4.3. *Let $c \in \mathbb{R}$ be a value.*

(i) The function $\mathbf{t}_M$ satisfies the Malgrange condition at c if there exist positive constants R, ε, A_c such that

$$|\mathbf{x}| > R,\ |t - c| < \varepsilon \Longrightarrow |\mathbf{x}| \cdot |\nabla \mathbf{t}_M(\mathbf{x}, t)| \geq A_c \tag{4.2}$$

which is equivalent to

$$|\mathbf{x}| > R,\ |t - c| < \varepsilon \Longrightarrow |\mathbf{x}| \cdot |\partial_{\mathbf{x}} F| \geq A_c |\nabla F|. \tag{4.3}$$

(ii) A value c which is not satisfying the Malgrange condition is called an asymptotic critical value *(ACV for short). Let $K_\infty(\mathbf{t}_M)$ be the set of ACV of $\mathbf{t}_M$.*

Similarly to the case of real or complex polynomial families [Par, Tib1, Tib2, TiZa] we find

THEOREM 4.4 (see also [LoZa, Kur, d'Ac1, d'AcGr1]). *(i) There exists a finite subset $B(\mathbf{t}_M)$ of $\mathbb{R}$ such that the function $\mathbf{t}_M$ is a locally C^{1+m} trivial at any value c not lying in $B(\mathbf{t}_M)$.*

(ii) $B(\mathbf{t}_M) \subset K_0(\mathbf{t}_M) \cup K_\infty(\mathbf{t}_M)$.

(iii) $K_\infty(\mathbf{t}_M)$ is finite.

(iv) If c is a regular value taken by $\mathbf{t}_M$ and does not lie in $K_\infty(\mathbf{t}_M)$, then the local trivialization can be realized by a vector field colinear to $\nu_{\mathbf{t}_M}$.

Proof. We are going to sketch the proofs of (iii) following [d'Ac1] and (iv) following [d'Ac1, d'Ac2, d'AcGr1]. Both (i) and (ii) can be deduced from these two points.

For simplicity we write f for $\mathbf{t}_M$.

Since $\mathcal{M}$ is polynomially bounded, there exists a definable function $\rho : [1, +\infty[\mapsto \mathbb{R}_+$ such that (see [d'Ac1, Lemma 3.3]):

$$(1)\ \lim_{R\to+\infty} R^{-1}\rho(R) = +\infty \quad \text{and} \qquad (2)\ K_\infty(f) = K_\infty^\rho(f)$$

where

$$\begin{aligned} K_\infty^\rho(f) :=& \{c \in \mathbb{R} : \exists(\mathbf{x}_k, t_k) \in M\, ,\, \mathbf{x}_k \to \infty \\ & \text{and } t_k \to c \text{ such that } \rho(|\mathbf{x}_k|)\cdot|\nabla f(\mathbf{x}_k, t_k)| \to 0\}. \end{aligned}$$

In particular for R large enough there exists an exponent α of $(\mathbb{F}_\mathcal{M})_{>1}$ such that $\rho(R) \geq R^\alpha$.

Following the steps of [d'Ac1, Theorem 3.4], we show that $K_\infty^\rho(f)$ is finite.

Assume that there exists $c' > c$ such that for each $c \leq t \leq c'$ the level $\{f = t\}$ is neither empty nor a critical level. Let us consider the following subset

$$\Delta := \{(s, w) \in \mathbb{R}^2\ :\ \exists(\mathbf{x}_k) \in M\ :\ \mathbf{x}_k \to \infty,\ f(\mathbf{x}_k) \to w \text{ and } |\mathbf{x}_k|\cdot|f(\mathbf{x}_k)| \to s\}.$$

This subset is definable. Let $\theta : [c, c'] \mapsto [0, +\infty[$ be the function defined as follows

$$\theta(t) := \inf\{\Delta \cap \{w = t\}\}.$$

It is definable. We wish to show that it vanishes only finitely many times on $[c, c']$. Assume that θ is identically 0 over $[c, c']$ (up to work with a smaller c'). Under these hypotheses the definable subset

$$\Sigma := \{\rho(|\mathbf{x}|)\cdot|\nabla f(\mathbf{x}, t)| < f(\mathbf{x}, t) - c\}$$

is not empty outside of a compact subset of $M \cap \mathbf{t}_M^{-1}[c-\varepsilon, c+\varepsilon]$ (see [d'Ac1, p. 40]). By definition of Σ, there exists a C^1 definable arc going to infinity $\gamma : [1, +\infty[\mapsto \Sigma$ such that

$$\lim_{+\infty} f \circ \gamma = c.$$

Let $h = f\circ\gamma - c$, and let us parameterize γ such that $|\mathbf{x}(\gamma(R))| = R$, so that $|\gamma'|$ goes to 1 at infinity. We find

$$0 < -h' \leq \frac{h}{|\rho|}|\gamma'| < 2\frac{h}{|\rho|}.$$

Let R_0 be large enough and let $u(R) = 2h(R_0) - h(R)$ once $R \geq R_0$, and let $a > 1$ be such that

$$\lim_{+\infty} R^{-a}\rho(R) = +\infty.$$

We deduce for $R \geq R_0$ (up to taking a larger R_0)

$$0 < u'(R) < 2\frac{u(R)}{|\rho(R)|} < \frac{u}{R^a}.$$

Applying the Gronwall Lemma provides for $R \geq R_0$

$$u(R) \leq h(R_0) \cdot \lambda(R_0) \quad \text{with} \quad \lambda(R_0) := \exp\left(\frac{1}{(a-1)R_0^{a-1}}\right).$$

We know that $\lim_{+\infty} u = 2h(R_0)$ but we can choose R_0 a priori such that $\lambda(R_0) < 2$, concluding that the function θ cannot vanish identically over $[c, c']$.

Point (iv) is of importance for the rest of the paper so we sketch its proof as a variation of the proof of [d'AcGr1, Theorem 3.5]. Let

$$\chi := \frac{1}{|\nabla \mathbf{t}_M|^2} \nabla \mathbf{t}_M \,.$$

Any trajectory γ of χ is parameterized by the levels of $\mathbf{t}_M$: starting at a point of M_c we find

$$\mathbf{t}_M(\gamma(s)) = c + s \,.$$

Since the Malgrange condition is not affected by a change of origin of $\mathbb{R}^n$, we can assume that for every small enough positive real number ε there exists a constant A_ε so that

$$|\mathbf{x}| \cdot |\nabla \mathbf{t}_M(\mathbf{p})| \geq A_\varepsilon \quad \text{for all} \quad \mathbf{p} = (\mathbf{x}, t) \in \mathbf{t}_M^{-1}[c - \varepsilon, c + \varepsilon] \,. \tag{4.4}$$

For $|s| \leq \varepsilon$ we deduce

$$|\gamma(s)| \leq |\gamma(0)| + \int_0^{|s|} \frac{1}{|\nabla \mathbf{t}_M(\gamma(z))|} dz \,.$$

Combining this latter inequality with the Gronwall Lemma provides

$$|\gamma(s)| \leq |\gamma(0)| \cdot \exp\left(\frac{|s|}{A_\varepsilon}\right) .$$

Since χ is C^{1+m} in $M \setminus \operatorname{crit}(\mathbf{t}_M)$, the function $\mathbf{t}_M$ is C^{1+m}-trivial at c by the flow of χ with initial conditions along M_c. □

Definition 4.5. *The set* of generalized critical values *is defined as*

$$K(\mathbf{t}_M) := K_0(\mathbf{t}_M) \cup K_\infty(\mathbf{t}_M).$$

The Malgrange condition at a regular value c encodes the geometry at infinity of the pencil of nearby fibres. Indeed we have the following

Lemma 4.6. *Let c be a value taken by $\mathbf{t}_M$ which is not a generalized critical value. Let $(\mathbf{p}_k)_k$ be a sequence of points of M converging to $(\mathbf{v}, c)$ in $\mathbf{S}^{n-1}_\infty \times \{c\} \cap \overline{M}$ while $\mathbf{t}_M(\mathbf{p}_k)$ goes to c. Assume that $\nu_{\mathbf{t}_M}(\mathbf{p}_k)$ converges in $\overline{M}$ to ν in $\mathbf{S}^n$. Then ν is orthogonal to $\mathbf{v}$.*

Proof. These limits can be achieved along a C^1 definable path $]0,1[\mapsto M$, $s \mapsto \mathbf{p}(s) = (\mathbf{x}(s), t(s))$ as s goes to 0 with $\lim_0 \mathbf{p} = (\mathbf{v}, c) \in M^\infty$. We choose the parameterization of $\mathbf{p}$ so that $|\mathbf{x}(s)| = s^{-1}$. Let $\mathbf{t}(s) := \mathbf{t}_M(\mathbf{p}(s))$ and so on. Let us write

$$\mathbf{t}(s) = c + As^a + o(s^a)$$
$$\mathbf{x}(s) = s^{-1}\mathbf{v} + o(s^{-1})$$
$$\nu_{\mathbf{t}_M}(s) = \nu + s^d\nu_1 + o(s^d)$$
$$\nu_M^{\mathbf{x}}(s) = s^e\nu^{\mathbf{x}} + o(s^e)$$
$$\nu_M^t(s) = s^f\nu^t + o(s^f)$$

where $a, d, e, f \in (\mathbb{F}_{\mathcal{M}})_{\geq 0} \cup \{+\infty\}$ with a, d positive exponents, $\min(e, f) = 0$, and $A \in \mathbb{R}$, $\nu_1 \in \mathbb{R}^n \times \mathbb{R}$, $\nu^{\mathbf{x}} \in \mathbb{R}^n \times 0$, $\nu^t \in \mathbb{R}$ are non-zero vectors whence the corresponding exponent is not ∞ and

$$\nu = -\nu^t \frac{\nu^{\mathbf{x}}}{|\nu^{\mathbf{x}}|} + |\nu^{\mathbf{x}}|\partial_t.$$

We deduce that there exists a continuous definable function $\varphi : (\mathbb{R}_{\geq 0}, 0) \mapsto \mathbb{R}$ with $\varphi(0) = Aa$, such that

$$\mathbf{t}'(s) = s^{a-1}\varphi(s) \quad \text{and} \quad \mathbf{x}'(s) = -s^{-2}\mathbf{v} + o(s^{-2}).$$

Using the Malgrange condition provides

$$1 \geq |\nu_M^{\mathbf{x}}(s)| = |\nabla \mathbf{t}_M(s)| \geq A_c\, s$$

for some positive constant A_c. Thus $e \leq 1$. Since

$$0 = \langle \nu_M^{\mathbf{x}} + \nu_M^t\partial_t, \mathbf{x}' + \mathbf{t}'\partial_t\rangle = \langle \nu_M^{\mathbf{x}}, \mathbf{x}'\rangle + \nu_M^t\mathbf{t}'$$

we deduce that

$$\nu_M^t\, \mathbf{t}' = -\langle \nu_M^{\mathbf{x}}, \mathbf{x}'\rangle = s^{e-2}[-\langle \nu^{\mathbf{x}}, \mathbf{v}\rangle + o(1)].$$

From this last equation we deduce that there exists an exponent $e' \geq e$ such that

$$s^{a-1+f} \sim |\nu_M^t\, \mathbf{t}'| = |\langle \nu_M^{\mathbf{x}}, \mathbf{x}'\rangle| \sim s^{e'-2},$$

so that $\langle \nu^{\mathbf{x}}, \mathbf{v}\rangle = 0$. □

To conclude this section we introduce a final regularity condition.

DEFINITION 4.7. *Let c be a regular value of $\mathbf{t}_M$ taken by $\mathbf{t}_M$.*
The function $\mathbf{t}_M$ is horizontally spherical at c at infinity *if for any sequence $(\mathbf{p}_k)_k$ of M converging to $(\mathbf{u}, c) \in M^\infty$, then*

$$\left\langle \lim_\infty \frac{\nu_M^{\mathbf{x}}}{|\nu_M^{\mathbf{x}}|}, \mathbf{u} \right\rangle = 0, \tag{4.5}$$

where $\lim_\infty \frac{\nu_M^{\mathbf{x}}}{|\nu_M^{\mathbf{x}}|}$ means the closed set of all the possible accumulation values, as k goes to infinity, of the unitary vector field of $\frac{\nu_M^{\mathbf{x}}}{|\nu_M^{\mathbf{x}}|}$ along the sequence $(\mathbf{p}_k)_k$.

Note that the following holds true:

LEMMA 4.8. *The condition of Equation 4.5 is equivalent to*

$$\langle \lim_\infty \nu_{\mathbf{t}_M}, \mathbf{u}\rangle = 0$$

along any sequence $(\mathbf{p}_k)_k$ *of* M *converging to a point* $(\mathbf{u}, c)$ *in* M^∞*.*

Indeed, similarly to what has been done for definable functions, we have the following:

PROPOSITION 4.9. *Let* c *be a regular value taken by* $\mathbf{t}_M$*. The function* $\mathbf{t}_M$ *is horizontally spherical at* c *at infinity if and only if there exists an exponent* e_c *in* $\mathbb{F}_{\mathcal{M}} \cap]-\infty, 1[$ *and a positive constant* E_c*, such that there exist positive real numbers* ε *and* R *such that*

$$(\mathbf{x}, t) \in \mathbf{t}_M^{-1}([c-\varepsilon, c+\varepsilon]) \setminus \mathbf{B}_R \Longrightarrow |\mathbf{x}| \cdot |\nabla \mathbf{t}_M| \geq E_c |\mathbf{t}_M(\mathbf{x}, t) - c|^{e_c}. \tag{4.6}$$

Proof. In this definable and polynomially bounded context, we can show (as in [d'AcGr2]) that a Bochnak-Łojasiewicz inequality type at the value c not in $K_0(\mathbf{t}_M)$ at infinity holds: there exists a positive constant L_c such that there exist positive real numbers ε and R such that

$$(\mathbf{x}, t) \in \mathbf{t}_M^{-1}([c-\varepsilon, c+\varepsilon]) \setminus \mathbf{B}_R \Longrightarrow |\mathbf{x}| \cdot |\nabla \mathbf{t}_M| \geq L_c |\mathbf{t}_M(\mathbf{x}, t) - c|. \tag{4.7}$$

1) Assume $\mathbf{t}_M$ is horizontally spherical at c at infinity.
Let $\mathbf{p} :]0, 1[\mapsto M$ be any continuous definable path such that it goes to $(\mathbf{u}, c) \in M^\infty$ as s goes to 0. Writing $\mathbf{p} = (\mathbf{x}, \mathbf{t})$ and parameterizing as $|\mathbf{x}(s)| = s^{-1}$, we have

$$\mathbf{p}(s) = (s^{-1}\mathbf{u} + o(s^{-1}), c + As^a + o(s^a))$$

for $A \neq 0$ and $a \in (\mathbb{F}_{\mathcal{M}})_{>0} \cup \{+\infty\}$. The numbers A and a depend on the choice of the path $s \mapsto \mathbf{p}(s)$. We obtain that along $\mathbf{p}$ there exists $a' \leq a$ such that

$$|\mathbf{x}| \cdot |\nabla \mathbf{t}_M| \sim s^{a'}.$$

Note that

$$a' < a \iff \lim_0 \left\langle \nu_{\mathbf{t}_M}, \frac{\mathbf{p}}{|\mathbf{p}|} \right\rangle = 0.$$

In particular the latter equivalence shows that

$$\mathbf{t}_M(\mathbf{x}, t) \to c \text{ as } \mathbf{x} \to +\infty \implies \frac{|\mathbf{t}_M(\mathbf{x}, t) - c|}{|\mathbf{x}| \cdot |\nabla \mathbf{t}_M(\mathbf{x}, t)|} \to 0 \text{ as } \mathbf{x} \to +\infty\,.$$

Let ε_0 be a small enough positive number such that $[c - \varepsilon_0, c + \varepsilon_0]$ contains only a single asymptotic critical value: c. Let R_0 be a positive large enough number. Let V_{ε_0, R_0} be the definable subset defined as

$$V_{\varepsilon_0, R_0} := \{(\mathbf{x}, t) \in M \,:\, |\mathbf{t} - c| \leq \varepsilon_0\,, |\mathbf{x}| \geq R_0\}.$$

Let $\mu_0 : [R_0, +\infty[\to \mathbb{R}$ be defined as

$$\mu_0(R) := \min\{|\mathbf{x}| \cdot |\nabla \mathbf{t}_M(\mathbf{x}, t)| \;\text{ for }\; (\mathbf{x}, t) \in V_{\varepsilon_0, R_0} \text{ and } |\mathbf{x}| = R\}.$$

The function μ_0 is definable and tends to 0 as R goes to infinity since c is an ACV. If R_0 is large enough, we can write

$$\mu_0(R) = A_0 R^{-a_0}(1 + o(1)) \text{ with } A_0 > 0,\, a_0 \in (\mathbb{F}_{\mathcal{M}})_{>0}.$$

Let V_0 be the closure of V_{ε_0,R_0} in $\overline{M}$, thus V_0 is compact in $\overline{\mathbb{R}^n}\times\mathbb{R}$. Let W_0 be the part at infinity of V_0, that is

$$W_0 := V_0 \cap (\mathbf{S}_\infty^{n-1}\times\mathbb{R}).$$

The function

$$\psi_0 : V_0 \setminus W_0 \ni (\mathbf{x},t) \to \frac{|\mathbf{t}_M(\mathbf{x},t)-c|}{|\mathbf{x}|\cdot|\nabla\mathbf{t}_M(\mathbf{x},t)|}$$

extends continuously and definably over V_0 taking the value 0 along W_0, by hypothesis of horizontal sphericalness. In the same way, the function

$$\rho_0 : V_0 \setminus W_0 \ni (\mathbf{x},t) \to |\mathbf{x}|^{-1}$$

also extends continuously and definably over V_0 taking the value 0 along W_0. Furthermore we see that

$$\rho_0 = 0 \Longrightarrow \psi_0 = 0\,.$$

Thus by a Łojasiewicz argument, there exist a positive exponent b and a positive constant B such that in V_0 the following inequality holds true:

$$\psi_0 \le B\rho_0^b \Longleftrightarrow \psi_0 \le B|\mathbf{x}|^{-b}.$$

Let μ_1 be the function defined as follows:

$$\mu_1 : V_0 \setminus W_0 \ni (\mathbf{x},t) \to \mu_0(|\mathbf{x}|).$$

The function μ_1 is definable, continuous and extends continuously to V_0 taking the value 0 along W_0. Therefore we deduce that in $V_0 \setminus W_0$ we have

$$|\mathbf{t}_M(\mathbf{x},t)-c| \le C_0\cdot\mu_1^{\frac{b}{a_0}}\cdot|\mathbf{x}|\cdot|\nabla\mathbf{t}_M(\mathbf{x},t)| \le C_0\cdot(|\mathbf{x}|\cdot|\nabla\mathbf{t}_M(\mathbf{x},t)|)^{\frac{b+a_0}{a_0}}$$

where C_0 is a positive constant. This latter inequality provides the announced result.

2) Assume the inequality holds.

Let $\mathbf{p} :]0,1[\mapsto M$ be a definable continuous path such that $\lim_0 \mathbf{p} = (\mathbf{u},c) \in M^\infty$. Writing $\mathbf{p} = (\mathbf{x},\mathbf{t})$ and parameterizing as $|\mathbf{x}(s)| = s^{-1}$, we have that

$$\begin{aligned}
\mathbf{t}(s) &= c + As^a + o(s^a)\\
\mathbf{p}(s) &= (s^{-1}\mathbf{u} + o(s^{-1}),\mathbf{t}(s)) \in \mathbb{R}^n\times\mathbb{R}\\
\nu_M^{\mathbf{x}}(s) &= s^b\nu + o(s^b) \in \mathbb{R}^n\times 0\\
\nu_M^t(s) &= s^d(\lambda v) + o(s^d) \in \mathbb{R}^n\times 0
\end{aligned}$$

with $A\neq 0$ and $a\in(\mathbb{F}_\mathcal{M})_{>0}\cup\{\infty\}$ while $b,d\in(\mathbb{F}_\mathcal{M})_{\ge 0}$, $\min(b,d)=0$, with $\lambda\neq 0$ and $\nu\in\mathbb{R}^n\setminus 0$.

Since the path $\mathbf{p}$ lies on M, we know that

$$\langle\nu_M,\mathbf{p}'\rangle = \langle\nu_M^{\mathbf{x}},\mathbf{x}'\rangle + \nu_M^t\mathbf{t}' = 0$$

from which we deduce

$$b-2 \le d+a-1\,. \tag{4.8}$$

We want to show that ν is orthogonal to $\mathbf{u}$, in other words $b < d + a + 1$.

We have the following estimates

$$|\nabla \mathbf{t}_M|(s) = |\nu_M^{\mathbf{x}}|(s) \sim s^b$$
$$|\mathbf{x}| \cdot |\nabla \mathbf{t}_M|(s) \sim s^{b-1}.$$

Using Inequality (4.6), we get

$$b - 1 \leq e_c \cdot a < a \text{ and } b < a + 1. \tag{4.9}$$

Since d is non negative, this yields the orthogonality of $\mathbf{u}$ and ν. □

5. Comparing regularity conditions and triviality.

We are working within the context of Section 4.

We have previously introduced three regularity conditions at infinity for the function $\mathbf{t}_M$. We are going to compare them here.

The hypersurface $M \subset \mathbb{R}^n \times \mathbb{R}$ is the definable family of the hypersurfaces $\{T_t\}_{t \in \mathbb{R}}$ of $\mathbb{R}^n$ and $\overline{M}$ is its closure in $\overline{\mathbb{R}^n} \times \mathbb{R}$. Let M^∞ be the intersection of $\overline{M}$ with the boundary at infinity $\mathbf{S}_\infty^{n-1} \times \mathbb{R}$. By Lemma 4.2, the definable function $\mathbf{r}_M : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$, defined as $(\mathbf{x}, t) \mapsto |\mathbf{x}|$, is transverse to $M \cap \mathbb{R}^n \times \mathbb{R} \times]-A, A[$ for some positive given A whenever $\mathbf{x}$ is large enough.

In Section 3 was defined the *space of characteristic covectors of M at infinity*

$$\mathscr{C}(M) := \mathscr{X}_{\mathbf{r}_M}^\infty,$$

which is a closed definable subset of $\mathbf{S}_\infty^{n-1} \times \mathbb{R} \times \mathbf{G}^\vee(1, n+1)$.

From Definition 3.3, we also know that: (i) the family $\{T_t\}_{t \in \mathbb{R}}$ is t-equisingular at $\mathbf{p} \in M^\infty$ if

$$\mathscr{C}(M)_{\mathbf{p}} \cap (\mathscr{X}_\tau^\infty)_{\mathbf{p}} = \emptyset, \tag{5.1}$$

where $\tau : \overline{\mathbb{R}^n} \times \mathbb{R} \mapsto \mathbb{R}$ is the projection on the last factor and, (ii) the family $\{T_t\}_{t \in \mathbb{R}}$ is t-equisingular at infinity at c if it is t-equisingular at $\mathbf{p} \in M^\infty$ for all $\mathbf{p} \in M^\infty \cap \tau^{-1}(c)$.

Let $\mathbf{p} = (\mathbf{u}, c) \in M^\infty$. The family $\{T_t\}_{t \in \mathbb{R}}$ is t-equisingular at $\mathbf{p}$ if for any sequence $\mathbf{p}_k = (\mathbf{u}_k, t_k)$ converging to $\mathbf{p}$ such that the sequence of T_k', the tangent space to the level of $\mathbf{r}_M$ through $\mathbf{p}_k$, converges to T', then the latter is not contained in $\mathbb{R}^n \times 0$. This definition is more geometric than the Malgrange condition, which is of interest since we have the following:

Proposition 5.1 (see [DiRuTi] for functions). *If the family $\{T_t\}_{t \in \mathbb{R}}$ is t-equisingular at infinity at c then the function $\mathbf{t}_M$ satisfies the Malgrange condition at c.*

Proof. Let $\beta : \mathbf{S}_\infty^{n-1} \times]0 + \infty[\times \mathbb{R} \mapsto (\mathbb{R}^n \setminus \mathbf{0}) \times \mathbb{R}$, be defined as $(\mathbf{u}, s, t) \mapsto \left(\frac{\mathbf{u}}{s}, t\right)$. Let $\mathbf{p} = (\mathbf{u}, c) \in M^\infty$, and consider a definable path

$$\gamma :]0, 1] \mapsto Z := \beta^{-1}(M), \quad s \mapsto (\mathbf{u}(s), s, t(s)),$$

with $\lim_0 \gamma = \mathbf{p} = (\mathbf{u}, c) \in M^\infty$. Thus $\mathbf{r}_Z(\gamma(s)) = s$. Along $\beta(\gamma)$ we also find

$$|\nabla \mathbf{t}_M| = |\nu_M^{\mathbf{x}}| \sim s^f, \quad \text{for } f \in F_{\mathcal{M}}.$$

Note that $(\mathscr{X}_\tau^\infty)_{\mathbf{p}}$ consists only of the "single" 1-form dt. The hypothesis implies the existence of a definable unit vector path

$$s \mapsto \xi(s) = (\mathbf{v}(s), 0, a(s)) \in T_{\gamma(s)} \mathbf{r}_Z = T_{\gamma(s)} Z \cap (T_{\mathbf{u}(s)} \mathbf{S}^{n-1} \times 0 \times \mathbb{R})$$

with $a(s) \to a_0 \neq 0$. Since $D\beta \cdot \xi$ is tangent to M along $\beta(\gamma)$, we obtain

$$\langle \nu_M^{\mathbf{x}}, \mathbf{v}(s)\rangle + s \cdot a(s)\nu_M^t = 0.$$

When $f > 0$, we get $\nu_M^t \to \pm 1$, from which we deduce

$$|\langle \nu_M^{\mathbf{x}}, \mathbf{v}(s)\rangle| \sim s.$$

Since $|\mathbf{v}| \leq 1$, we must have $f \leq 1$. Malgrange Condition is satisfied along any definable arc accumulating at a point of the compact set $M^\infty \cap \overline{\mathbb{R}}^n \times c$, therefore the announced result is proved. □

Since we just have seen that t-equisingularity at infinity implies the Malgrange condition, we need to check if there is a relation between these and sphericalness at infinity. To this end an obvious corollary of Proposition 4.9 is the following:

Corollary 5.2. *Let c not be a generalized critical value. Then $\mathbf{t}_M$ is horizontally spherical at c at infinity. In other words t-equisingularity at infinity at c implies horizontal sphericalness at c at infinity.*

Proof. It is just reformulating the fact that Malgrange at c is equivalent to have $e_c \leq 0$ in Equation (4.6). □

We can now state the last result of this section about local triviality:

Theorem 5.3. *Let c be a regular value at which $\mathbf{t}_M$ is horizontally spherical at infinity. Then $\mathbf{t}_M$ is C^{1+m} locally trivial at c.*

Proof. Once we have moved the origin of $\mathbb{R}^n \times 0$ so that its value is not c, we just have to integrate the field $\chi = \frac{1}{|\nabla \mathbf{t}_M|}\nu_{\mathbf{t}_M}$ as before. Inequality (4.6) now holds in $\mathbf{t}_M^{-1}[c-\varepsilon, c+\varepsilon] \setminus \mathbf{B}_{R_0}$ for a large positive R_0. As in [d'AcGr1, d'AcGr2] combining it with Gronwall Lemma will show that any trajectory of χ parameterized over $[0, \varepsilon]$ with initial point in $M_c \cap \mathbf{B}_R$ stays in $\mathbf{B}_{KR}$ for some constant K depending only on c and ε. □

As a final remark, there are polynomial examples in [d'AcGr1] with regular values which are ACV, but with exponent $e_c < 1$.

6. Curvature and absolute curvature of families of definable hypersurfaces. Some of the material presented here can also be found in [Gra1] (or adapted from it).

Let H be a definable and oriented hypersurface of $\mathbb{R}^n$ of class C^{1+m} with $m \geq 1$.

Assume that H is connected and let $\nu_H : H \mapsto \mathbf{S}^{n-1}$ be an orientation. The unitary normal mapping ν_H is definable and C^m.

Assume that the maximal rank of $\mathrm{d}_{\mathbf{x}}\nu_H$ when $\mathbf{x}$ ranges H is $n-1$.

There exist finitely many definable disjoint connected open subsets $(U_i)_{i\in I}$ of $\mathbf{S}^{n-1}$ such that

$$\mathrm{clos}(\nu_H(H)) = \cup_{i\in I}\mathrm{clos}(U_i)$$

and for each $i \in I$, the mapping ν_H induces a definable finite covering

$$\nu_H : H_i \mapsto U_i$$

where $H_i := \nu_H^{-1}(U_i)$ and such that

$$\dim \nu_H(H \setminus (\cup_{i\in I} H_i)) \leq n-2.$$

Denoting κ_H the determinant of $\mathrm{d}\nu_H$, that is the Gauss-Kronecker curvature of H at the considered point, the *total Gauss curvature* $K(H)$ *of* H is defined (if it exists, and it does as we see below) as

$$K(H) := \int_H \kappa_H(\mathbf{x})\mathrm{d}\mathbf{x}.$$

An application of the formula of change of variables gives

$$\int_H \kappa_H(\mathbf{x})\mathrm{d}\mathbf{x} = \sum_{i\in I}(-1)^{d_i}\mathrm{vol}_{n-1}(U_i)$$

for $(-1)^{d_i}$ the degree of the covering mapping $\nu_H|_{H_i} : H_i \mapsto U_i$ for each i.

We introduce another average of curvature, namely the *total absolute curvature* $|K|(H)$ *of* H defined as

$$|K|(H) := \int_H |\kappa_H(\mathbf{x})|\mathrm{d}\mathbf{x}.$$

Another application of the formula of change of variables yields

$$\int_H |\kappa_H(x)|\mathrm{d}\mathbf{x} = \sum_{i\in I} e_i \cdot \mathrm{vol}_{n-1}(U_i)$$

where e_i is the number of sheets of the covering $\nu_H|_{H_i} : H_i \mapsto U_i$.

The hypothesis on the rank of $\mathrm{d}\nu_H$ guarantees that e_i is positive. Otherwise both curvatures are 0.

Returning to the notations and hypotheses of Section 4, the hypersurface M can also be seen as a definable family of hypersurfaces $\mathcal{F}_{\mathbf{t}_M} := (T_c)_{c\in Im(\mathbf{t}_M)}$ of $\mathbb{R}^n$ where $Im(\mathbf{t}_M)$ is the image of the function $\mathbf{t}_M$. We can define the following mapping:

$$\begin{array}{cccc} \mathbf{N}: & M \setminus \mathrm{crit}(\mathbf{t}_M) & \mapsto & \mathbf{S}^{n-1} \\ & (\mathbf{x}, t) & \mapsto & \frac{\nu_M^{\mathbf{x}}}{|\nu_M^{\mathbf{x}}|} \end{array}.$$

The mapping $\mathbf{N}$ is called *the Gauss mapping of the family* $\mathcal{F}_{\mathbf{t}_M}$. It is definable and C^{1+m}. The restriction of $\mathbf{N}|_{T_c}$ is denoted $\mathbf{N}_c$, so that the family of mappings $(\mathbf{N}_c)_{c\in Im(\mathbf{t}_M)\setminus K_0(\mathbf{t}_M)}$ is definable. Let κ_c be the Gauss-Kronecker curvature of T_c. Thus we can define two functions

$$\begin{array}{cccc} K: & Im(\mathbf{t}_M) \setminus K_0(\mathbf{t}_M) & \mapsto & \mathbb{R} \\ & c & \mapsto & K(c) := \int_{T_c} \kappa_c(\mathbf{x})\mathrm{d}\mathbf{x}, \\ |K|: & Im(\mathbf{t}_M) \setminus K_0(\mathbf{t}_M) & \mapsto & \mathbb{R} \\ & c & \mapsto & |K|(c) := \int_{T_c} |\kappa_c|(\mathbf{x})\mathrm{d}\mathbf{x}. \end{array}$$

The introductory material of this section guarantees that both functions are well defined. The paper [Gra1] has dealt with the case where M is a graph. We can state now the result of this section:

Theorem 6.1. *(i) There are finitely many values in* $Im(\mathbf{t}_M) \setminus K_0(\mathbf{t}_M)$ *at which the function* $t \mapsto K(t)$ *is not continuous*

(ii) There are finitely many values in $Im(\mathbf{t}_M) \setminus K_0(\mathbf{t}_M)$ *at which the function* $t \mapsto |K|(t)$ *is not continuous*

(iii) If $|K|$ is continuous at c, so is K.

Sketch of Proof. It is a very similar proof to that of [Gra1, Sections 4,5,6].

Let us consider the following definable and C^{1+m} mapping

$$\begin{array}{rccl} \Psi: & M & \mapsto & \mathbf{S}^{n-1} \times \mathbb{R} \\ & \mathbf{p} & \mapsto & (\mathbf{N}(\mathbf{p}), \mathbf{t}_M(\mathbf{p})). \end{array}$$

It is a local diffeomorphism at any point of $M \setminus \mathrm{crit}(\Psi)$. Let $\Delta := \Psi(\mathrm{crit}(\Psi))$ which is definable, closed and of dimension lower than or equal to $n-1$. Let $\mathcal{U} := (\mathbf{S}^{n-1} \times \mathbb{R}) \setminus \Delta$.

There exists an integer number p_M such that for any $(\mathbf{u}, t) \in \mathcal{U}$ the fibre $\Psi^{-1}(\mathbf{u}, t)$ has at most p_M points. For any point $(\mathbf{u}, t)$ in $\mathcal{U}$ the degree $\delta(\mathbf{u}, t)$ of Ψ at $(\mathbf{u}, t)$ may range from $-p_M$ to p_M. In particular the function $(\mathbf{u}, t) \mapsto \delta(\mathbf{u}, t)$ is definable and

$$\delta(\mathbf{u}, t) = \deg_{\mathbf{u}} \mathbf{N}_t.$$

We define the following subsets

$$\begin{aligned} \mathcal{U}_k &:= \{(\mathbf{u}, t) \in \mathcal{U} \,:\, \#\Psi^{-1}(\mathbf{u}, t) = k\} \\ U_t &:= \{\mathbf{u} \in \mathbf{S}^{n-1} \,:\, (\mathbf{u}, t) \in \mathcal{U}\} = \mathbf{N}_t(T_t \setminus \mathrm{crit}(\mathbf{N}_t)) \\ U_{t,k} &:= \{\mathbf{u} \in \mathbf{S}^{n-1} \,:\, (\mathbf{u}, t) \in \mathcal{U}_k\}. \end{aligned}$$

The subsets U_t and $U_{t,k}$ are open, and we obtain finitely many definable families $(U_t)_{t \in Im(\mathbf{t}_M) \setminus K_0(\mathbf{t}_M)}$ and $(U_{t,k})_{t \in Im(\mathbf{t}_M) \setminus K_0(\mathbf{t}_M)}$.

Note that $U_t = \cup_k U_{t,k}$ and since the function $\mathbf{u} \mapsto \deg_{\mathbf{u}} \mathbf{N}_t$ is definable, it is constant on each connected component of U_t.

Let c be a regular value of $\mathbf{t}_M$. Since Hausdorff limits of closed definable subsets of a given compact space exist, we can set

$$\begin{array}{lcl} \mathscr{V}_c^+ := \lim_{t \to c, t > c} \mathrm{clos}(U_t) & \text{and} & \mathscr{V}_{c,k}^+ := \lim_{t \to c, t > c} \mathrm{clos}(U_{t,k}) \\ \mathscr{V}_c^- := \lim_{t \to c, t < c} \mathrm{clos}(U_t) & \text{and} & \mathscr{V}_{c,k}^- := \lim_{t \to c, t < c} \mathrm{clos}(U_{t,k}). \end{array}$$

Let $V_1, \ldots, V_{d_c}$ be the connected components of U_c. For each $i = 1, \ldots, s$, let k_i be the integer number such that $V_i \subset U_{c,k_i}$. For each $i = 1, \ldots, d_c$ there exists $l_i^+ \geq k_i$ and $l_i^- \geq k_i$ such that

$$V_i \subset \mathscr{V}_{c,l_i^-}^- \ \text{ and } \ V_i \subset \mathscr{V}_{c,l_i^+}^+.$$

In particular we deduce that for each i

$$\mathrm{vol}_{n-1}(V_i) \leq \min\{\, \mathrm{vol}_{n-1}(\mathscr{V}_{c,l_i^-}^-)\,,\, \mathrm{vol}_{n-1}(\mathscr{V}_{c,l_i^+}^+)\,\}\,.$$

Let δ_i be the degree of $\mathbf{N}_c$ at any point of V_i. We find

$$K(c) = \sum_{i=1}^{d_c} \delta_i \cdot \mathrm{vol}_{n-1}(V_i) \ \text{ and } \ |K|(s) = \sum_{i=1}^{d_c} k_i \cdot \mathrm{vol}_{n-1}(V_i).$$

From the previous arguments we get that each following limit exists

$$\begin{aligned} K_c^+ := \lim_{t \to c, t > c} K(t)\,,\ K_c^- := \lim_{t \to c, t < c} K(t)\,, \\ |K|_c^+ := \lim_{t \to c, t > c} |K|(t)\,,\ |K|_c^- := \lim_{t \to c, t < c} |K|(t)\,, \end{aligned}$$

and we obviously get

$$|K|(c) \leq \min(|K|_c^-\,,\,|K|_c^+).$$

The rest of the proof follows from the following arguments: Assume that each U_t has d_t connected $V_{t,1}, \ldots, V_{t,d_t}$. Each such connected component $V_{t,i}$ lies in $U_{t,k_i(t)}$ with $k_i(t) \leq k_j(t)$ if and only if $i \leq j$. Moreover the degree of $\mathbf{N}_t$ at any point of $V_{t,i}$ is constant and equal to $\delta_i(t)$. These comes from properties of Ψ and $\mathcal{U}$. From here we deduce that there exists a finite subset Z of $\mathbb{R}$ such that for any J connected component of $(\mathbb{R} \setminus Z) \cap Im(\mathbf{t}_M)$, the numbers $d_t, k_i(t), \delta_i(t)$ are independent of t in J. Moreover each function $t \mapsto \mathrm{vol}_{n-1}(V_{t,i})$ is continuous over J. □

7. More on regularity at infinity. Let $\mathbf{N} : M \setminus \mathrm{crit}(\mathbf{t}_M) \mapsto \mathbf{S}^{n-1}$ be the Gauss mapping of the family of the regular levels of $\mathbf{t}_M$. Similarly to the conormal geometry at infinity (in $\mathbb{R}^n \times \mathbb{R}$) of the function $\mathbf{t}_M$, we are interested in the limits of $\mathbf{N}$ at infinity (in $\mathbb{R}^n$).

Let $\Gamma(\mathbf{N})$, contained in $M \times \mathbf{S}^{n-1}$, be the graph of $\mathbf{N}$, let $\overline{\Gamma(\mathbf{N})}$ be its closure in $\overline{\mathbb{R}^n} \times \mathbb{R} \times \mathbf{S}^{n-1}$ and $\overline{\mathbf{N}} : \overline{\Gamma(\mathbf{N})} \mapsto \mathbf{S}^{n-1}$ be the projection onto $\mathbf{S}^{n-1}$, so that we can think of it as the extension by continuity of $\mathbf{N}$ to $\overline{\Gamma(\mathbf{N})}$.

The closed definable subset $T_{c,+}^\infty$ is defined as

$$T_{c,+}^\infty := \{\mathbf{u} \in \mathbf{S}_\infty^{n-1} \ : \ \exists(\mathbf{p}_k)_k \in M \ \text{ such that } \ \lim_\infty \mathbf{p}_k = (\mathbf{u}, c)\}.$$

Let $V_c^\infty := \overline{\mathbf{N}}(\pi^{-1}(T_{c,+}^\infty \times \{c\}))$, in other words it is the definable closed subset

$$V_c^\infty = \left\{\mathbf{v} \in \mathbf{S}^{n-1} \ : \ \exists((\mathbf{x}_k, \tau_k))_k \in M \text{ such that } \mathbf{x}_k \to \infty,\ \tau_k \to c,\ \mathbf{N}(\mathbf{x}_k, \tau_k) \to \mathbf{v}\right\},$$

corresponding to all the limits at infinity of normals to the hypersurfaces $(T_t)_t$ as t tends to c.

For each $\mathbf{u} \in \mathbf{S}_\infty^{n-1}$, let $V_{c,\mathbf{u}}^\infty := V_c^\infty \cap \overline{\mathbf{N}}(\pi^{-1}(\mathbf{u}) \times \{c\})$, that is

$$V_{c,\mathbf{u}}^\infty = \Big\{\mathbf{v} \in \mathbf{S}^{n-1} \ : \ \exists((\mathbf{x}_k, \tau_k))_k \in M \text{ such that } \mathbf{x}_k \to \infty,\ \tau_k \to c,\ \frac{\mathbf{x}_k}{|\mathbf{x}_k|} \to \mathbf{u},\ \mathbf{N}(\mathbf{x}_k, \tau_k) \to \mathbf{v}\Big\}.$$

Note that whenever $\mathbf{u}$ does not belong to $T_{c,+}^\infty$ we find that $V_{c,\mathbf{u}}^\infty$ is empty.

A very rigid consequence of $\mathbf{t}_M$ being horizontally spherical at c at infinity is the following:

Lemma 7.1. *Let c be a regular value taken by $\mathbf{t}_M$ at which it is horizontally spherical at infinity. Then each $\mathbf{u}$ in $T_{c,+}^\infty$ and each $\mathbf{v}$ in $V_{c,\mathbf{u}}^\infty$ are orthogonal.*

Proof. Obvious from the definition of the horizontal sphericalness. □

Let c be a regular value taken by $\mathbf{t}_M$ at which it is horizontally spherical at infinity. Let ε be a positive real number such that for each $t \in [c-\varepsilon, c+\varepsilon]$ the function $\mathbf{t}_M$ is horizontally spherical at t at infinity. Let $T_{c,\varepsilon} := \mathbf{t}_M^{-1}([c+\varepsilon, c-\varepsilon])$.

We find that for each for η in $]e_c, 1[$, there exists a positive real number R such that for every $(\mathbf{x}, t)$ belonging to $T_{c,\varepsilon} \setminus \mathrm{clos}(\mathbf{B}_R)$, we have

$$|\mathbf{x}| \cdot |\nabla \mathbf{t}_M(\mathbf{x}, t)| \geq |t - c|^\eta. \tag{7.1}$$

Let ξ be the following definable vector field

$$\xi := \frac{1}{|\nabla \mathbf{t}_M|} \frac{\nu^{\mathbf{S}}_{\mathbf{t}_M}}{|\nu^{\mathbf{S}}_{\mathbf{t}_M}|}, \quad \text{for } |(\mathbf{x},t)| \geq R \gg 1, \quad \text{and } |t-c| \leq \varepsilon.$$

It is definable and C^{1+m}, non vanishing, tangent to the Euclidean spheres. The flow of the differential equation

$$\dot{\mathbf{p}}(t) = \xi(\mathbf{p}(t)) \;\text{ and }\; \xi(0) \in T_c \times \{c\} \setminus \mathbf{B}_R$$

induces a C^{1+m} diffeomorphism $(T_c \times \{c\} \setminus \mathbf{B}_R) \times [-\varepsilon, \varepsilon] \mapsto T_{c,\varepsilon} \setminus \mathbf{B}_R$.

Using Inequality (7.1) we deduce that the length $l(\mathbf{p}_0, \mathbf{p}_t)$ of the trajectory of ξ between the point $\mathbf{p}_0$ of $T_c \times \{c\} \setminus \mathbf{B}_R$ and $\mathbf{p}_t$, point reached after time t, is bounded as

$$l(\mathbf{p}_0, \mathbf{p}_t) \leq |\mathbf{p}_0| \left(\frac{t^{1-\eta}}{1-\eta} \right). \tag{7.2}$$

Inequality (7.2) implies that the angle $\theta(t)$ between the vector $\mathbf{p}_t$ and $\mathbf{p}_0$ tends to 0 as t goes to 0. This proves the following:

LEMMA 7.2. *Let c be a regular value taken by the function $\mathbf{t}_M$ at which it is horizontally spherical at infinity. Then $T_c^\infty = T_{c,+}^\infty$, thus $T_{c,+}^\infty$ is of dimension at most $n-2$.*

8. Main result. Our main result Theorem 8.1 presented in this section is a consequence of results of equisingularity theory and of our context.

THEOREM 8.1. *Let $F : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ be a C^{2+m} definable function over a polynomially bounded o-minimal structure, for a non negative integer number m. Assuming that 0 is regular value of F, let M be the level $\{F = 0\}$. Let $\mathbf{t}_M$ be the projection of M onto $\mathbb{R}$.*

Let c be a regular value taken by $\mathbf{t}_M$ at which it is horizontally spherical at infinity. Then the total absolute curvature function $t \mapsto |K|(t)$ is continuous at c. Consequently the total curvature function $t \mapsto K(t)$ is continuous at c.

It is a straightforward consequence of the following

LEMMA 8.2. *Under the hypotheses of Theorem 8.1, we find*

$$\dim V_c^\infty \leq n-2.$$

Let us show the main result.

Proof of the main result. Let $|K| : t \mapsto |K|(t)$ be the total absolute curvature function of the family of hypersurfaces $(T_t)_t$. By Lemma 8.2 we find that V_c^∞ has $(n-1)$-dimensional volume zero. Following [Gra1, Proposition 6.8], we deduce there is no accumulation of curvature at infinity at c. In other words the function $|K|$ is continuous at c, and so is K by point (iii) of Theorem 6.1. □

Before going into the proof of Lemma 8.2, we observe that it states that there is no accumulation of curvature at infinity nearby the level c, or equivalently there are no half-branch at infinity of the generic polar curve along which the function $\mathbf{t}_M$ tends to c (see [Tib1, Gra2] for local triviality results with a similar flavor).

Proof of Lemma 8.2. Let $\mathbf{v}$ be a limit of normal direction lying in V_c^∞. By the Curve Selection Lemma we can find a definable continuous path, going to infinity, along which this limit is reached: there exists such a path γ such that

$$\mathbf{N}\circ\gamma \to \mathbf{v} \ \text{ and } \ \mathbf{t}_M\circ\gamma \to c.$$

In particular there exists a positive exponent α, in the field of exponents $\mathbb{F}_{\mathcal{M}}$ of the structure $\mathcal{M}$ such that

$$\frac{\mathbf{t}_M\circ\gamma - c}{|\gamma|^\alpha} \to a \in \mathbb{R}^*.$$

In other words there exists a positive exponent e such that the germ at infinity of γ lies in

$$\mathcal{H}_e := \{\mathbf{p}\in T_{c,\varepsilon}\setminus \mathbf{B}_R \,:\, |\mathbf{t}_M(\mathbf{p}) - c| \le |\mathbf{p}|^{-e}\}.$$

If the exponent e belongs to $F_{\mathcal{M}}$, then $\mathcal{H}_e$ is definable and so is its closure $\overline{\mathcal{H}_e}$ in $\mathbb{R}^n\times\mathbb{R}$. Let us define

$$V_c^{\infty,e} := \left\{\mathbf{v}\in\mathbf{S}^{n-1} \,:\, \exists\,(\mathbf{p}_k)_k \in \mathcal{H}_e \text{ such that } \mathbf{N}(\mathbf{p}_k)\to\mathbf{v}\right\}$$

which is a closed definable subset of $\mathbf{S}^{n-1}$ contained in V_c^∞ whenever e lies in $\mathbb{F}_{\mathcal{M}}$.

Let $\mathcal{H}_e^\infty$ be the intersection $\overline{\mathcal{H}_e}\cap\mathbf{S}_\infty^{n-1}\times\{c\}$. The function $\mathbf{t}_M$ extends continuously and definably to $\overline{\mathcal{H}_e}$ taking the value c along $\mathcal{H}_e^\infty$. Let $\mathbf{t}_e$ be the restriction of this extension to $\overline{\mathcal{H}_e}\setminus\mathbf{B}_R$.

According to [Bek, Loi], we can stratify the pair $(\mathbf{t}_e, \overline{\mathcal{H}_e}\setminus\mathbf{B}_R)$ with Thom's condition. Furthermore we can require that $X := \mathcal{H}_e\setminus\mathbf{B}_R$ and $Y := \mathcal{H}_e^\infty$ are union of strata.

Suppose first that X and Y are strata. The dimension of Y is $d\le n-2$ since $\mathcal{H}_e^\infty$ is contained in $T_{c,+}^\infty$, thus of dimension lower than or equal to $n-2$ by Lemma 7.2. Let $\mathbf{p} = (\mathbf{u}, c)$ be a point of Y and let $T := T_\mathbf{p}Y$ which is contained in $\mathbb{R}^n\times 0$. Note that T and $\mathbf{u}$ are orthogonal.

Let $\mathbf{v}$ be a limit of the normal $\mathbf{N}$ at infinity at $\mathbf{u}$ taken into $\mathcal{H}_e$ along a path γ. We will show that $\mathbf{v}$ and $T\oplus\mathbb{R}\mathbf{u}$ are orthogonal. We recall that $\nu_M = \nu_M^{\mathbf{x}} + \nu_M^t\partial_t$. Let ν be the limit of ν_M along γ as γ goes to infinity and let η be the limit of $\nu_{\mathbf{t}_M}$. Writing ν as $(\nu^{\mathbf{x}}, \nu^t)$ in $\mathbb{R}^n\times\mathbb{R}$, we have

$$\mathbf{v} = \frac{\nu^{\mathbf{x}}}{|\nu^{\mathbf{x}}|} \ \text{ and } \ \eta = -\nu^t\mathbf{v} + |\nu^{\mathbf{x}}|^2\partial_t.$$

Thom's condition implies that η and T are orthogonal. Moreover, by horizontal spherical-ness at infinity, η and $\mathbf{u}$ are also orthogonal, therefore η and $T\oplus\mathbb{R}\mathbf{u}$ are orthogonal too. Hence, if $\nu^t\neq 0$, then $\mathbf{v}$ is orthogonal to $T\oplus\mathbb{R}\mathbf{u}$ since $T\oplus\mathbb{R}\mathbf{u}$ is contained in $\mathbb{R}^n\times 0$. If $\nu^t = 0$ then $\mathbf{v} = \nu$. Using the arguments of the proof of Lemma 4.6, we see that $\mathbf{u}$ and ν are orthogonal. By Whitney's condition (a), we know that $T_\mathbf{p}Y$ is a subspace of $\lim_\infty T_\gamma M$ and so $T_\mathbf{p}Y$ and ν are orthogonal. Hence we conclude that $\mathbf{v} = \nu$ is orthogonal to $T_\mathbf{p}Y\oplus\mathbb{R}\mathbf{u}$.

Let $V_{c,\mathbf{u}}^{\infty,e} := V_c^{\infty,e}\cap V_{c,\mathbf{u}}^\infty$. We have proved that $\dim V_{c,\mathbf{u}}^{\infty,e} \le (n-1)-(d+1) = n-d-2$, and thus $\dim V_c^{\infty,e}\le n-2$.

In the general case the only thing to check is that whenever X contains a (definable) stratum S of dimension s at most $n-1$, then its contribution to V_c^∞ is at most of dimension $n-2$. This is so since the graph of $\mathbf{N}|S$ is of dimension s, so that its limits at infinity

$$\left\{\mathbf{v} \in \mathbf{S}^{n-1} \,:\, \exists S \ni (\mathbf{x}_k, \tau_k)_k \text{ such that } \tau_k \to c\,,\, \mathbf{N}(\mathbf{x}_k, \tau_k) \to \mathbf{v}\right\} \subset V_c^\infty$$

have dimension at most $s-1 \leq n-2$.

We conclude that $V_c^{\infty,e}$ has dimension lower than or equal to $n-2$ for any exponent e of $\mathbb{F}_\mathcal{M}$.

Since any limit $\mathbf{v}$ of V_c^∞ belongs to some $V_c^{\infty,e}$ for some e in $\mathbb{F}_\mathcal{M}$, and since the family $(V_c^{\infty,e})_{e\in(\mathbb{F}_\mathcal{M})>0}$ is increasing as e goes to 0, we get that V_c^∞ is the Hausdorff limit at $e=0$ of $V_c^{\infty,e}$, thus has dimension lower than or equal to $n-2$. □

We conclude with an interesting observation. For this purpose we need a few more preparations. Let c be regular value taken by $\mathbf{t}_M$. Let ε be a positive number such that $[c-\varepsilon, c+\varepsilon]$ consists only of regular values. Let Z be a connected component of $\mathbf{t}_M^{-1}(]c-\varepsilon, c+\varepsilon[)$. Let us consider now $\mathbf{t}_Z$ the restriction of $\mathbf{t}_M$ to Z. Let $Z_t \times \{t\} := \mathbf{t}_Z^{-1}(t) = M_t \cap Z$. Let $K_Z(t) := \int_{Z_t} \kappa(\mathbf{x})\mathrm{d}\mathbf{x}$ and $|K|_Z(t) := \int_{Z_t} |\kappa|(\mathbf{x})\mathrm{d}\mathbf{x}$ for t in $]c-\varepsilon, c+\varepsilon[$. Then we actually have showed the following:

Theorem 8.3. *Under the above hypotheses, assume furthermore that $\mathbf{t}_Z$ is horizontally spherical at infinity at c. Then the functions K_Z and $|K|_Z$ are continuous at c.*

To rephrase informally Theorem 8.3, the continuity of $t \mapsto |K|(t)$ nearby the value c at which the function $\mathbf{t}_M$ is horizontally spherical at infinity, is equivalent to the continuity nearby c of each function $t \mapsto |K|_Z(t)$ for each connected component Z of $\mathbf{t}_M^{-1}(]c-\varepsilon, c+\varepsilon[)$.

9. The special case of functions. We treat here briefly the case of functions which is a special case of the context presented here. The continuity of curvatures is the same property but the regularity conditions are a little bit different.

Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a C^{2+m}, with non-negative m, definable function. We denote the level $f^{-1}(t)$ by F_t and its closure in the spherical compactification by $\overline{F_t}$. Its intersection with the sphere at infinity $\mathbf{S}_\infty^{n-1}$ will be denoted F_t^∞. Let ν_f be the unitary gradient field $\frac{\nabla f}{|\nabla f|}$.

The function f satisfies the *Malgrange condition* at c if there are positive constants R, ε, A such that

$$|\mathbf{x}| > R,\ |f(\mathbf{x}) - c| < \varepsilon \implies |\mathbf{x}| \cdot |\nabla f(\mathbf{x})| \geq A.$$

We would like to introduce what the analogue of horizontal spherical-ness in this context would be. The function f is *spherical at the regular value c at infinity* if along any sequence of points $(\mathbf{x}_k)_k$ of $\mathbb{R}^n$ such that $|\mathbf{x}_k|$ goes to ∞ and $f(\mathbf{x}_k)$ goes to c, we have

$$\left\langle \lim_\infty \nu_f(\mathbf{x}_k)\ ,\ \lim_\infty \frac{\mathbf{x}_k}{|\mathbf{x}_k|} \right\rangle = 0,$$

whenever each limit exists.

This condition is equivalent to the following result already proved in [d'AcGr1, d'AcGr2] which justified the introduction for families of the notion of horizontal spherical-ness at infinity.

THEOREM 9.1 ([d'AcGr1, d'AcGr2]). *Let c be a regular value of f taken by f. The function f is spherical at infinity at c if and only if there exists an exponent e_c in $\mathbb{F}_{\mathcal{M}} \cap]-\infty, 1[$ and a positive constant E_c such that*

$$|\mathbf{x}| \gg 1,\ |f(\mathbf{x} - c| \ll 1 \implies |\mathbf{x}| \cdot |\nabla f(\mathbf{x})| \geq E_c |f(\mathbf{x}) - c|^{e_c}.$$

It is well known that t-regularity is equivalent to Malgrange [DiRuTi] (their proof goes through the definable context) and that Malgrange is equivalent to requiring having $e_c \leq 0$, thus spherical-ness at infinity.

Let $K(t)$ be the total Gauss-Kronecker curvature of F_t and $|K|(t)$ be the total absolute Gauss-Kronecker curvature of F_t. In the context of functions what we have proved is the following:

THEOREM 9.2. *Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a definable C^{2+m} function for some non-negative integer m. Let c be a regular value at which the function is spherical at infinity.*

(1) Then the function $t \mapsto |K|(t)|$ is continuous at c, and thus so is $t \mapsto K(t)$.

(2) As for Theorem 8.3, for any connected component Z of $f^{-1}]c - \varepsilon, c + \varepsilon[$ for positive ε small enough, the function $t \mapsto |K|_Z(t)$ is continuous at c, and thus so is $t \mapsto K_Z(t)$.

Let us end with a last result on equisingularity of the family of fibres of a function.

COROLLARY 9.3. *Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a definable C^{2+m} function for some non-negative integer m. Let c be a regular value at which the function is spherical at infinity.*

If n is odd then the following function is continuous at c

$$t \mapsto \int_{\mathbf{G}(n-1,n)} \chi\left(f^{-1}(t) \cap H\right) dH.$$

If n is even then the following function is continuous at c

$$t \mapsto \int_{\mathbf{G}(n-1,n)} \left[\chi\left(\{f \geq t\} \cap H\right) - \chi\left(\{f \leq t\} \cap H\right)\right] dH.$$

Proof. By Theorem 9.2, we know that the function $t \mapsto K(t)$ is continuous at c. Then we apply Theorem 4.5 in [Dut1]. If n is odd, the result is clear because the function $t \mapsto \chi\left(f^{-1}(t)\right)$ is constant in a neighborhood of c. If n is even, it is enough to prove that the functions $t \mapsto \chi\left(\{f \geq t\}\right)$ and $t \mapsto \chi\left(\{f \leq t\}\right)$ are constant in a neighborhood of c. By the Mayer-Vietoris sequence, if $t > c$ then we have

$$\chi\left(\{f \geq c\}\right) = \chi\left(\{f \geq t\}\right) + \chi\left(\{c \leq f \leq t\}\right) - \chi\left(f^{-1}(t)\right).$$

So if t is close enough to c then $\chi\left(\{f \geq c\}\right) = \chi\left(\{f \geq t\}\right)$, for f is a fibration over $[c, t]$. Similarly we can show that $\chi\left(\{f \leq c\}\right) = \chi\left(\{f \leq t\}\right)$ for $t > c$ close enough to c. The same argument works for $t < c$. $\square$

10. Some examples. (1) The first obvious situation is to apply our result to graph of a function, completing the results of [Gra1, Gra2].

(2) Let $(\mathbf{x}, t) = (x, y, t) \in \mathbb{R}^3$ and consider the polynomial function

$$F(\mathbf{x}, t) := x + y - t(x^2 + y^2).$$

The hypersurface $M = F^{-1}(0)$ (closure in $\mathbb{R}^3$ of the graph of a rational function) is a regular affine surface. Observe that $\mathbf{t}_M$ has no critical point. For any value $c \neq 0$, the level $\mathbf{t}_M^{-1}(c) = T_c \times c$ is a diffeomorphic to $\mathbf{S}^1$. Thus for every $c \neq 0$, we get

$$K(c) := \int_{T_c} \kappa_c(\mathbf{x}) \mathrm{d}\mathbf{x} = 2\pi,$$

where $\kappa_c(\mathbf{x})$ is the Gauss-Kronecker curvature of T_c at $\mathbf{x}$. Flowing (by a gradient flow of $\nabla \mathbf{t}_M$) a non-zero level onto a non-zero neighbouring one, we also check that

$$c \mapsto |K|(c) := \int_{T_c} |\kappa_c|(\mathbf{x}) \mathrm{d}\mathbf{x} (\geq 2\pi)$$

is continuous on $\mathbb{R} \setminus 0$.

Along the path $\gamma : s \mapsto (s, s, s^{-1})$, as $s \to +\infty$, we find out

$$\frac{\gamma}{|\gamma|} \to \mathbf{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \quad \text{while} \quad \frac{\nu_M^{\mathbf{x}}}{|\nu_M^{\mathbf{x}}|} = -\mathbf{u},$$

therefore $\mathbf{t}_M$ is not spherical at infinity at 0. Since the 0-level is the line $\{x + y = t = 0\}$, the value 0 is a bifurcation value of $\mathbf{t}_M$. Note that $|K|(0) = 0 = K(0)$.

(3) Let $g : \mathbb{R}^p \to \mathbb{R}$ be a C^{2+m} definable function and let $M = g^{-1}(0)$. We assume that 0 is a regular value of g. For a unit vector $\mathbf{v} \in \mathbf{S}^{p-1}$, let $\mathbf{v}^*$ be the function $\mathbf{y} \mapsto \langle \mathbf{v}, \mathbf{y} \rangle$, and let $\mathbf{v}_M^*$ be its restriction to M.

LEMMA 10.1. *There exists a closed definable subset $\Sigma \subset \mathbf{S}^{p-1}$, of positive codimension, such that if $\mathbf{v} \notin \Sigma$, then $K_\infty(\mathbf{v}_M^*) = \emptyset$.*

Proof. Let Σ be the subset of unit vectors $\mathbf{u} \in \mathbf{S}^{p-1}$ such that $\frac{\nabla g}{|\nabla g|}$ tends to $\mathbf{u}$ or $-\mathbf{u}$ along a sequence $(\mathbf{y}_k)_k$ in M, going to infinity. The subset Σ is a closed definable subset of positive codimension [Dut1, Proposition 3.1].

Assume that $K_\infty(\mathbf{v}_M^*) \neq \emptyset$. We can further assume that $\mathbf{v} = (0, \ldots, 0, 1)$. Thus there exists a sequence of points $(\mathbf{y}_k)_k$ in M going to infinity such that

$$|\mathbf{y}_k| \cdot \frac{|\partial_{\mathbf{y}'} g|}{|\nabla g|}(\mathbf{y}_k) \to 0,$$

where $\mathbf{y} = (\mathbf{y}', y_n)$. Since $|\partial_{\mathbf{y}'} g| = o(|\nabla g|)$ along $(\mathbf{y}_k)_k$, we deduce that

$$\frac{\nabla g}{|\nabla g|}(\mathbf{y}_k) \to \pm \mathbf{v},$$

in other words $\mathbf{v} \in \Sigma$. □

Let $p = n + 1$ and choose linear coordinates $(\mathbf{x}, t)$ in $\mathbb{R}^{n+1}$, so that $\mathbf{v} := \partial_t \notin \Sigma$ with g, M as above. Since $\mathbf{t}_M^{-1}(c) = T_c \times c = M \cap \{\mathbf{v}^* = c\}$, the following is a corollary of the previous lemma and Theorem 8.1.

COROLLARY 10.2. *Let $\kappa_c(\mathbf{x})$ be the Gauss-Kronecker curvature of the hypersurface $T_c \subset \mathbb{R}^n$ at $\mathbf{x} \notin \mathrm{crit}(\mathbf{t}_M)$. The functions*

$$c \mapsto \int_{T_c} |\kappa_c(\mathbf{x})| \mathrm{d}\mathbf{x} \text{ and } c \mapsto \int_{T_c} \kappa_c(\mathbf{x}) \mathrm{d}\mathbf{x},$$

are continuous on $\mathbb{R} \setminus K_0(\mathbf{t}_M)$.

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