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GENERATING FUNCTIONS FOR OHNO TYPE SUMS OF FINITE AND SYMMETRIC MULTIPLE ZETA-STAR VALUES*

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Abstract. Ohno's relation states that a certain sum, which we call an Ohno type sum, of multiple zeta values remains unchanged if we replace the base index by its dual index. In view of Oyama's theorem concerning Ohno type sums of finite and symmetric multiple zeta values, Kaneko looked at Ohno type sums of finite and symmetric multiple zeta-star values and made a conjecture on the generating function for a specific index of depth three. In this paper, we confirm this conjecture and further give a formula for arbitrary indices of depth three.

Key words. Multiple zeta(-star) values, Finite multiple zeta(-star) values, Symmetric multiple zeta(-star) values, Ohno's relation, Oyama's relation.

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1. Introduction.

1.1. Finite and symmetric multiple zeta(-star) values.

For positive integers $k_1, \dots, k_r$ with $k_r \geq 2$, the multiple zeta values and the multiple zeta-star values are defined by

$$\zeta(k_1, \dots, k_r) = \sum_{1 \leq n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \in \mathbb{R},$$
$$\zeta^\star(k_1, \dots, k_r) = \sum_{1 \leq n_1 \leq \cdots \leq n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \in \mathbb{R}.$$

Kaneko and Zagier [7] introduced the finite multiple zeta(-star) values and the symmetric multiple zeta(-star) values. Set $\mathcal{A} := \prod_p \mathbb{F}_p / \bigoplus_p \mathbb{F}_p$, where p runs over all primes. For positive integers $k_1, \dots, k_r$, the finite multiple zeta(-star) values are defined by

$$\zeta_{\mathcal{A}}(k_1, \dots, k_r) = \left(\sum_{1 \leq m_1 < \cdots < m_r < p} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \bmod p \right)_p \in \mathcal{A},$$
$$\zeta_{\mathcal{A}}^\star(k_1, \dots, k_r) = \left(\sum_{1 \leq m_1 \leq \cdots \leq m_r < p} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \bmod p \right)_p \in \mathcal{A}.$$

Let $\mathcal{Z}$ be the $\mathbb{Q}$-linear subspace of $\mathbb{R}$ spanned by 1 and all multiple zeta values. For

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positive integers $k_1, \dots, k_r$, we define the symmetric multiple zeta(-star) values by

$$\zeta_{\mathcal{S}}(k_1,\dots,k_r)=\sum_{i=0}^{r}(-1)^{k_{i+1}+\dots+k_r}\zeta(k_1,\dots,k_i)\zeta(k_r,\dots,k_{i+1}) \bmod \zeta(2)\in\mathcal{Z}/\zeta(2)\mathcal{Z},$$

$$\zeta^{\star}_{\mathcal{S}}(k_1,\dots,k_r)=\sum_{i=0}^{r}(-1)^{k_{i+1}+\dots+k_r}\zeta^{\star}(k_1,\dots,k_i)\zeta^{\star}(k_r,\dots,k_{i+1}) \bmod \zeta(2)\in\mathcal{Z}/\zeta(2)\mathcal{Z},$$

where we understand $\zeta(\emptyset)=\zeta^{\star}(\emptyset)=1$. Here, the symbols ζ and $\zeta^{\star}$ on the right-hand side mean the regularized values coming from harmonic regularization, i.e. real values obtained by taking constant terms of harmonic regularization as explained in [4].

Denoting by $\mathcal{Z}_{\mathcal{A}}$ the $\mathbb{Q}$-vector subspace of $\mathcal{A}$ spanned by 1 and all finite multiple zeta values, Kaneko and Zagier conjectured that there is an isomorphism between $\mathcal{Z}_{\mathcal{A}}$ and $\mathcal{Z}/\zeta(2)\mathcal{Z}$ as $\mathbb{Q}$-algebras such that $\zeta_{\mathcal{A}}(k_1,\dots,k_r)$ and $\zeta_{\mathcal{S}}(k_1,\dots,k_r)$ correspond to each other (for details, see [6] and [7]). In the sequel, the letter $\mathcal{F}$ stands for either $\mathcal{A}$ or $\mathcal{S}$.

1.2. Main results. For a sequence $\boldsymbol{k}=(k_1,\dots,k_r)$, we call $|\boldsymbol{k}|=k_1+\dots+k_r$ its weight and r its depth. For two sequences $\boldsymbol{k}$ and $\boldsymbol{l}$ of the same depth, we denote by $\boldsymbol{k}\oplus\boldsymbol{l}$ the sequence obtained by componentwise addition. We call a (possibly empty) sequence of positive integers an index. Throughout this paper, we always assume that $\boldsymbol{e}$ runs over sequences of nonnegative integers having suitable depth.

Ohno obtained the following remarkable result:

THEOREM 1.1 (Ohno's relation; Ohno [9]). *For a nonempty index $\boldsymbol{k}$ whose last component is greater than 1 and a nonnegative integer m, we have*

$$\sum_{|\boldsymbol{e}|=m}\zeta(\boldsymbol{k}\oplus\boldsymbol{e})=\sum_{|\boldsymbol{e}|=m}\zeta(\boldsymbol{k}^{\dagger}\oplus\boldsymbol{e}),$$

where $\boldsymbol{k}^{\dagger}$ is the dual index of $\boldsymbol{k}$ (see [9] for a precise definition).

DEFINITION 1.2 (Hoffman's dual index). For a nonempty index $\boldsymbol{k}=(k_1,\dots,k_r)$, we define Hoffman's dual index of $\boldsymbol{k}$ by

$$\boldsymbol{k}^{\vee}=(\underbrace{1,\dots,1}_{k_1}+\underbrace{1,\dots,1}_{k_2}+1,\dots,1+\underbrace{1,\dots,1}_{k_r}).$$

In contrast to Theorem 1.1, Oyama proved the following:

THEOREM 1.3 (Oyama [10]). *For a nonempty index $\boldsymbol{k}$ and a nonnegative integer m, we have*

$$\sum_{|\boldsymbol{e}|=m}\zeta_{\mathcal{F}}(\boldsymbol{k}\oplus\boldsymbol{e})=\sum_{|\boldsymbol{e}|=m}\zeta_{\mathcal{F}}((\boldsymbol{k}^{\vee}\oplus\boldsymbol{e})^{\vee}).$$

For a positive integer k, let

$$\mathfrak{z}_{\mathcal{A}}(k)=(B_{p-k}/k \bmod p)_p\in\mathcal{A},\qquad \mathfrak{z}_{\mathcal{S}}(k)=\zeta(k)\bmod\zeta(2)\in\mathcal{Z}/\zeta(2)\mathcal{Z},$$

where B_n is the n-th Bernoulli number; here for each k there are only finitely many p for which B_{p-k} is undefined, and so $\mathfrak{z}_{\mathcal{A}}(k)$ is well-defined as an element of $\mathcal{A}$. In light of this theorem, Kaneko looked at the generating function

$$O_{\mathcal{F}}(\boldsymbol{k})=\sum_{\boldsymbol{e}}\zeta^{\star}_{\mathcal{F}}(\boldsymbol{k}\oplus\boldsymbol{e})X^{|\boldsymbol{k}\oplus\boldsymbol{e}|}+\sum_{\boldsymbol{e}}(-1)^{|\boldsymbol{e}|}\zeta^{\star}_{\mathcal{F}}(\boldsymbol{k}^{\vee}\oplus\boldsymbol{e})X^{|\boldsymbol{k}^{\vee}\oplus\boldsymbol{e}|}$$

and gave the following conjecture:

$$O_{\mathcal{F}}(2,1,2) = (3\mathfrak{Z}_{\mathcal{F}}(3)X^3 + 5\mathfrak{Z}_{\mathcal{F}}(5)X^5 + 7\mathfrak{Z}_{\mathcal{F}}(7)X^7 + \cdots)^2.$$

In this paper, we prove a theorem that generalizes this conjecture. For positive integers k and i, let

$$F_{k,i}(X) = \sum_{n\geq k+i} \left((-1)^k \binom{n-1}{k-1} - (-1)^i \binom{n-1}{i-1} \right) \mathfrak{Z}_{\mathcal{F}}(n) X^n.$$

THEOREM 1.4 (Main theorem). *For positive integers k_1, k_2, k_3, we have*

$$O_{\mathcal{F}}(k_1,k_2,k_3) = \begin{cases} -\sum\limits_{\substack{i+j=k_2+1 \\ i,j\geq 2}} F_{k_1,i}(X)F_{k_3,j}(X) & (k_2 \geq 2), \\ F_{k_1,1}(X)F_{k_3,1}(X) & (k_2 = 1). \end{cases}$$

REMARK 1.5. Theorem 1.4 implies that $O_{\mathcal{F}}(k_1, 2, k_3) = 0$. We also note that the theorem implies Kaneko's conjecture when $k_1 = 2, k_2 = 1, k_3 = 2$, since $F_{2,1}(X) = \sum_{n\geq 3} n\mathfrak{Z}_{\mathcal{F}}(n)X^n$ and $\mathfrak{Z}_{\mathcal{F}}(2n) = 0$ for all positive integers n.

REMARK 1.6. By the duality formula (Theorem 2.6), the second sum in $O_{\mathcal{F}}(\boldsymbol{k})$ is equal to $-\sum_{\boldsymbol{e}}(-1)^{|\boldsymbol{e}|}\zeta^{\star}_{\mathcal{F}}((\boldsymbol{k}^{\vee} \oplus \boldsymbol{e})^{\vee})X^{|\boldsymbol{k}^{\vee}\oplus\boldsymbol{e}|}$, which more closely resembles the right-hand side of Theorem 1.3.

REMARK 1.7. We note that $O_{\mathcal{F}}(k) = 0$ and $O_{\mathcal{F}}(k_1, k_2) = 0$ hold for all positive integers k, k_1, k_2 (see Section 2).

REMARK 1.8. Hirose-Imatomi-Murahara-Saito [1] shows that $\sum_{|\boldsymbol{e}|=m} \zeta^{\star}_{\mathcal{F}}(\boldsymbol{k}^{\vee} \oplus \boldsymbol{e})$ can be represented as a $\mathbb{Z}$-linear combination of $\zeta^{\star}_{\mathcal{F}}(\boldsymbol{k} \oplus \boldsymbol{e})$'s.

We will give proofs of $O_{\mathcal{F}}(k) = 0$ and $O_{\mathcal{F}}(k_1, k_2) = 0$ in Section 2 and of our main theorem (Theorem 1.4) in Section 3.

2. Proofs of $O_{\mathcal{F}}(k) = 0$ and $O_{\mathcal{F}}(k_1, k_2) = 0$. For an index $\boldsymbol{k}$ and a positive integer k, we let

$$\tilde{\zeta}_{\mathcal{F}}(\boldsymbol{k}) = \zeta_{\mathcal{F}}(\boldsymbol{k})X^{|\boldsymbol{k}|}, \quad \tilde{\zeta}^{\star}_{\mathcal{F}}(\boldsymbol{k}) = \zeta^{\star}_{\mathcal{F}}(\boldsymbol{k})X^{|\boldsymbol{k}|}, \text{ and } \quad \tilde{\mathfrak{Z}}_{\mathcal{F}}(k) = \mathfrak{Z}_{\mathcal{F}}(k)X^k.$$

2.1. Proof of $O_{\mathcal{F}}(k) = 0$.

PROPOSITION 2.1. *For a positive integer k, we have*

$$\tilde{\zeta}_{\mathcal{F}}(k) = \tilde{\zeta}^{\star}_{\mathcal{F}}(k) = 0.$$

Proof. See Kaneko [6]. □

PROPOSITION 2.2 (Hoffman [3], Murahara [8]). *For a nonempty index $(k_1, \ldots, k_r)$, we have*

$$\sum_{\sigma\in S_r} \tilde{\zeta}_{\mathcal{F}}(k_{\sigma(1)}, \ldots, k_{\sigma(r)}) = \sum_{\sigma\in S_r} \tilde{\zeta}^{\star}_{\mathcal{F}}(k_{\sigma(1)}, \ldots, k_{\sigma(r)}) = 0,$$

where S_r is the symmetric group of degree r.

For a nonnegative integer m, we let $\{1\}^m$ denote the all-one sequence of length m.

PROPOSITION 2.3. *For a positive integer k, we have*

$$O_{\mathcal{F}}(k) = 0.$$

Proof. By Propositions 2.1 and 2.2, we have

$$\begin{aligned} O_{\mathcal{F}}(k) &= \sum_{m\geq 0} \tilde{\zeta}^{\star}_{\mathcal{F}}(k+m) + \sum_{\boldsymbol{e}} (-1)^{|\boldsymbol{e}|} \tilde{\zeta}^{\star}_{\mathcal{F}}((\{1\}^k) \oplus \boldsymbol{e}) \\ &= 0 + 0 = 0. \quad \square \end{aligned}$$

2.2. Proof of $O_{\mathcal{F}}(k_1, k_2) = 0$.

PROPOSITION 2.4. *For positive integers k_1, k_2, we have*

$$\tilde{\zeta}_{\mathcal{F}}(k_1, k_2) = \tilde{\zeta}^{\star}_{\mathcal{F}}(k_1, k_2) = (-1)^{k_1+1} \binom{k_1+k_2}{k_1} \tilde{\mathfrak{Z}}_{\mathcal{F}}(k_1+k_2).$$

Proof. See Kaneko [6]. □

The next formula is well known (see [12], for example).

PROPOSITION 2.5. *For a nonempty index $(k_1, \dots, k_r)$, we have*

$$\sum_{i=0}^{r} (-1)^i \tilde{\zeta}^{\star}_{\mathcal{F}}(k_1, \dots, k_i) \tilde{\zeta}_{\mathcal{F}}(k_{r,}, \dots, k_{i+1}) = 0.$$

Here, we understand $\tilde{\zeta}_{\mathcal{F}}(\emptyset) = \tilde{\zeta}^{\star}_{\mathcal{F}}(\emptyset) = 1$.

THEOREM 2.6 (Duality formula; Hoffman [3], Jarossay [5]). *For a nonempty index $\boldsymbol{k}$, we have*

$$\tilde{\zeta}^{\star}_{\mathcal{F}}(\boldsymbol{k}) = -\tilde{\zeta}^{\star}_{\mathcal{F}}(\boldsymbol{k}^{\vee}).$$

PROPOSITION 2.7. *For positive integers k_1, k_2, we have*

$$O_{\mathcal{F}}(k_1, k_2) = 0.$$

Proof. By Propositions 2.2 and 2.5, we have

$$\begin{aligned} &\sum_{\boldsymbol{e}} (-1)^{|\boldsymbol{e}|} \tilde{\zeta}^{\star}_{\mathcal{F}}((\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}) \oplus \boldsymbol{e}) \\ &= \sum_{\boldsymbol{e}} (-1)^{k_1+k_2+|\boldsymbol{e}|} \tilde{\zeta}_{\mathcal{F}}((\{1\}^{k_2-1}, 2, \{1\}^{k_1-1}) \oplus \boldsymbol{e}). \end{aligned}$$

By Theorem 1.3, we have

$$\begin{aligned}&\sum_{\boldsymbol{e}}(-1)^{k_1+k_2+|\boldsymbol{e}|}\tilde{\zeta}_{\mathcal{F}}((\{1\}^{k_2-1},2,\{1\}^{k_1-1})\oplus\boldsymbol{e})\\&=\sum_{\boldsymbol{e}}(-1)^{k_1+k_2+|\boldsymbol{e}|}\tilde{\zeta}_{\mathcal{F}}(((k_2,k_1)\oplus\boldsymbol{e})^{\vee}).\end{aligned}$$

By Propositions 2.2, 2.5, and Theorem 2.6, we have

$$\begin{aligned}&\sum_{\boldsymbol{e}}(-1)^{k_1+k_2+|\boldsymbol{e}|}\tilde{\zeta}_{\mathcal{F}}(((k_2,k_1)\oplus\boldsymbol{e})^{\vee})\\&=\sum_{\boldsymbol{e}}\tilde{\zeta}^{\star}_{\mathcal{F}}(((k_1,k_2)\oplus\boldsymbol{e})^{\vee})\\&=-\sum_{\boldsymbol{e}}\tilde{\zeta}^{\star}_{\mathcal{F}}((k_1,k_2)\oplus\boldsymbol{e}).\end{aligned}$$

Then we have

$$\begin{aligned}&O_{\mathcal{F}}(k_1,k_2)\\&=\sum_{\boldsymbol{e}}\tilde{\zeta}^{\star}_{\mathcal{F}}((k_1,k_2)\oplus\boldsymbol{e})+\sum_{\boldsymbol{e}}(-1)^{|\boldsymbol{e}|}\tilde{\zeta}^{\star}_{\mathcal{F}}((\{1\}^{k_1-1},2,\{1\}^{k_2-1})\oplus\boldsymbol{e})\\&=0.\quad\square\end{aligned}$$

3. Proof of Theorem 1.4.

3.1. Properties of $\tilde{\zeta}_{\mathcal{F}}$ and $\tilde{\zeta}^{\star}_{\mathcal{F}}$. To prove our main theorem (Theorem 1.4), we need Lemmas 3.6, 3.7, and 3.11. The following known results will be used to prove these lemmas.

PROPOSITION 3.1 (Reversal formula). *For an index $(k_1,\dots,k_r)$, we have*

$$\zeta_{\mathcal{F}}(k_1,\dots,k_r)=(-1)^{k_1+\cdots+k_r}\zeta_{\mathcal{F}}(k_r,\dots,k_1).$$

Proof. For $\mathcal{F}=\mathcal{A}$, the formula follows from the change of variables $m_i\mapsto p-m_{r-i+1}$ in the definition of $\zeta_{\mathcal{A}}(k_1,\dots,k_r)$. For $\mathcal{F}=\mathcal{S}$, the formula is immediate from the definition of $\zeta_{\mathcal{S}}(k_1,\dots,k_r)$. $\square$

PROPOSITION 3.2. *For nonnegative integers a and b, we have*

$$\tilde{\zeta}_{\mathcal{F}}(\{1\}^a,2,\{1\}^b)=\tilde{\zeta}^{\star}_{\mathcal{F}}(\{1\}^a,2,\{1\}^b)=(-1)^{a+1}\binom{a+b+2}{a+1}\mathfrak{Z}_{\mathcal{F}}(a+b+2).$$

Proof. By Propositions 2.2, 2.5, and 3.1, we have

$$\tilde{\zeta}_{\mathcal{F}}(\{1\}^a,2,\{1\}^b)=\tilde{\zeta}^{\star}_{\mathcal{F}}(\{1\}^a,2,\{1\}^b).$$

By Proposition 2.4 and Theorem 2.6, we have

$$\begin{aligned}\tilde{\zeta}^{\star}_{\mathcal{F}}(\{1\}^a,2,\{1\}^b)&=-\tilde{\zeta}^{\star}_{\mathcal{F}}(a+1,b+1)\\&=(-1)^{a+1}\binom{a+b+2}{a+1}\mathfrak{Z}_{\mathcal{F}}(a+b+2).\end{aligned}$$

This finishes the proof. □

Corollary 3.3. *For nonnegative integers a and b, we have*

$$\tilde{\zeta}_{\mathcal{F}}^{\star}(\{1\}^a, 2, \{1\}^b) = (-1)^{a+b+1}\tilde{\zeta}_{\mathcal{F}}^{\star}(\{1\}^a, 2, \{1\}^b).$$

Theorem 3.4 (Sum formula; Saito-Wakabayashi [11], Murahara [8]). *For nonnegative integers i and j, we have*

$$\sum_{\substack{k_1,\dots,k_{i+j+1}\geq 1\\ k_{i+1}\geq 2}} \tilde{\zeta}_{\mathcal{F}}(k_1,\dots,k_{i+j+1}) = F_{i+1,j+1}(X),$$
$$\sum_{\substack{k_1,\dots,k_{i+j+1}\geq 1\\ k_{i+1}\geq 2}} \tilde{\zeta}_{\mathcal{F}}^{\star}(k_1,\dots,k_{i+j+1}) = (-1)^{i+j+1}F_{i+1,j+1}(X).$$

We denote by $\mathcal{I}$ the space of formal $\mathbb{Q}$-linear combinations of indices. We define a $\mathbb{Q}$-bilinear product $\widetilde{ш}$ on $\mathcal{I}$ inductively by setting

$$\boldsymbol{k}\,\widetilde{ш}\,\emptyset = \emptyset\,\widetilde{ш}\,\boldsymbol{k} = \boldsymbol{k},$$
$$(k_1,\boldsymbol{k})\,\widetilde{ш}\,(l_1,\boldsymbol{l}) = (k_1,\boldsymbol{k}\,\widetilde{ш}\,(l_1,\boldsymbol{l})) + (l_1,(k_1,\boldsymbol{k})\,\widetilde{ш}\,\boldsymbol{l})$$

for all indices $\boldsymbol{k}$, $\boldsymbol{l}$ and all positive integers k_1, l_1. We $\mathbb{Q}$-linearly extend $\zeta_{\mathcal{F}}$, $\zeta_{\mathcal{F}}^{\star}$, $\tilde{\zeta}_{\mathcal{F}}$, and $\tilde{\zeta}_{\mathcal{F}}^{\star}$ to $\mathcal{I}$.

Theorem 3.5 (Hirose-Imatomi-Murahara-Saito [1, Lemma 2.5]). *For a nonempty index $\boldsymbol{k}$ and a nonnegative integer m, we have*

$$\tilde{\zeta}_{\mathcal{F}}^{\star}(\boldsymbol{k}\,\widetilde{ш}\,(\{1\}^m)) = \sum_{|\boldsymbol{e}|=m} \tilde{\zeta}_{\mathcal{F}}^{\star}(\boldsymbol{k}\oplus\boldsymbol{e}).$$

3.2. Calculation of $\sum_{\boldsymbol{e}}\tilde{\zeta}_{\mathcal{F}}^{\star}((k_1,k_2,k_3)\oplus\boldsymbol{e})$. We use Hoffman's algebraic setup with a slightly different convention (see [2]). Let $\mathfrak{H} = \mathbb{Q}\langle x, y\rangle$ be the noncommutative polynomial ring in two indeterminates x and y. We define a $\mathbb{Q}$-linear map $p\colon y\mathfrak{H}y \to \mathcal{I}$ by $p(yx^{k_1-1}y\cdots x^{k_r-1}y) = (k_1,\dots,k_r)$. For positive integers l_1, l_2, l_3, and a nonegative integer m, we define a polynomial $P_m(l_1,l_2,l_3)$ in $\mathfrak{H}$ by

$$P_m(l_1,l_2,l_3) = (-1)^m \sum_{\substack{e_1+e_2+e_3=m\\ e_1,e_2,e_3\geq 0}} y^{l_1+e_1}xy^{l_2+e_2-1}xy^{l_3+e_3}.$$

Lemma 3.6. *For positive integers k_1, k_2, k_3, we have*

$$\begin{aligned}&\sum_{\boldsymbol{e}}\tilde{\zeta}_{\mathcal{F}}^{\star}((k_1,k_2,k_3)\oplus\boldsymbol{e})\\ &= F_{k_1,1}(X)F_{k_3,1}(X) - \sum_{\substack{i+j\leq k_2+1\\ i,j\geq 2}}\ \sum_{\substack{n_1\geq k_1+i-1\\ n_3\geq k_3+j-1}} (-1)^i\binom{n_1}{i-1}\mathfrak{Z}_{\mathcal{F}}(n_1)\times(-1)^j\binom{n_3}{j-1}\mathfrak{Z}_{\mathcal{F}}(n_3)\\ &\quad + (-1)^{k_1+k_2+k_3}\sum_{m\geq 0}\tilde{\zeta}_{\mathcal{F}}(p(P_m(k_3,k_2,k_1))).\end{aligned}$$

Proof. We prove this lemma only for $k_2 \geq 2$. The case $k_2 = 1$ can be proved similarly. Put $\boldsymbol{e} = (e_1, e_2, e_3)$. By Theorem 2.6, we have

$$\begin{aligned}
&\sum_{\boldsymbol{e}} \tilde{\zeta}^{\star}_{\mathcal{F}}((k_1, k_2, k_3) \oplus \boldsymbol{e}) \\
&= -\sum_{\boldsymbol{e}} \tilde{\zeta}^{\star}_{\mathcal{F}}(((k_1, k_2, k_3) \oplus \boldsymbol{e})^{\vee}) \\
&= -\sum_{\boldsymbol{e}} \tilde{\zeta}^{\star}_{\mathcal{F}}(\{1\}^{k_1+e_1-1}, 2, \{1\}^{k_2+e_2-2}, 2, \{1\}^{k_3+e_3-1}).
\end{aligned}$$

By Propositions 2.2, 2.5, and 3.2, we have

$$\begin{aligned}
&-\sum_{\boldsymbol{e}} \tilde{\zeta}^{\star}_{\mathcal{F}}(\{1\}^{k_1+e_1-1}, 2, \{1\}^{k_2+e_2-2}, 2, \{1\}^{k_3+e_3-1}) \\
&= \sum_{\boldsymbol{e}} \Bigg(\sum_{\substack{i+j=k_2+e_2+2 \\ i,j \geq 2}} (-1)^{k_3+e_3+j} \tilde{\zeta}^{\star}_{\mathcal{F}}(\{1\}^{k_1+e_1-1}, 2, \{1\}^{i-2}) \tilde{\zeta}_{\mathcal{F}}(\{1\}^{k_3+e_3-1}, 2, \{1\}^{j-2}) \\
&\qquad + (-1)^{k_1+k_2+k_3+e_1+e_2+e_3} \tilde{\zeta}_{\mathcal{F}}(\{1\}^{k_3+e_3-1}, 2, \{1\}^{k_2+e_2-2}, 2, \{1\}^{k_1+e_1-1}) \Bigg) \\
&= \sum_{\boldsymbol{e}} \sum_{\substack{i+j=k_2+e_2+2 \\ i,j \geq 2}} (-1)^{k_1+e_1+j} \binom{k_1+e_1+i-1}{k_1+e_1} \tilde{\mathfrak{Z}}_{\mathcal{F}}(k_1+e_1+i-1) \\
&\qquad\qquad \times \binom{k_3+e_3+j-1}{k_3+e_3} \tilde{\mathfrak{Z}}_{\mathcal{F}}(k_3+e_3+j-1) \\
&\quad + (-1)^{k_1+k_2+k_3} \sum_{m \geq 0} \tilde{\zeta}_{\mathcal{F}}(p(P_m(k_3, k_2, k_1))) \\
&= \sum_{\substack{i+j \geq k_2+2 \\ i,j \geq 2}} \sum_{\substack{n_1 \geq k_1+i-1 \\ n_3 \geq k_3+j-1}} (-1)^{n_1+i+j+1} \binom{n_1}{i-1} \tilde{\mathfrak{Z}}_{\mathcal{F}}(n_1) \times \binom{n_3}{j-1} \tilde{\mathfrak{Z}}_{\mathcal{F}}(n_3) \\
&\quad + (-1)^{k_1+k_2+k_3} \sum_{m \geq 0} \tilde{\zeta}_{\mathcal{F}}(p(P_m(k_3, k_2, k_1))).
\end{aligned}$$

Since $(-1)^{a+1} \mathfrak{Z}_{\mathcal{F}}(a) = \mathfrak{Z}_{\mathcal{F}}(a)$, we have

$$\begin{aligned}
&\sum_{\boldsymbol{e}} \tilde{\zeta}^{\star}_{\mathcal{F}}((k_1, k_2, k_3) \oplus \boldsymbol{e}) \\
&= \sum_{\substack{i+j \geq k_2+2 \\ i,j \geq 2}} \sum_{\substack{n_1 \geq k_1+i-1 \\ n_3 \geq k_3+j-1}} (-1)^{i} \binom{n_1}{i-1} \tilde{\mathfrak{Z}}_{\mathcal{F}}(n_1) \times (-1)^{j} \binom{n_3}{j-1} \tilde{\mathfrak{Z}}_{\mathcal{F}}(n_3) \\
&\quad + (-1)^{k_1+k_2+k_3} \sum_{m \geq 0} \tilde{\zeta}_{\mathcal{F}}(p(P_m(k_3, k_2, k_1))) \\
&= \Bigg(\sum_{i,j \geq 2} - \sum_{\substack{i+j \leq k_2+1 \\ i,j \geq 2}} \Bigg) \sum_{\substack{n_1 \geq k_1+i-1 \\ n_3 \geq k_3+j-1}} (-1)^{i} \binom{n_1}{i-1} \tilde{\mathfrak{Z}}_{\mathcal{F}}(n_1) \times (-1)^{j} \binom{n_3}{j-1} \tilde{\mathfrak{Z}}_{\mathcal{F}}(n_3) \\
&\quad + (-1)^{k_1+k_2+k_3} \sum_{m \geq 0} \tilde{\zeta}_{\mathcal{F}}(p(P_m(k_3, k_2, k_1))).
\end{aligned}$$

Here, we note that

$$\begin{aligned}
&\sum_{i,j\geq 2}\sum_{\substack{n_1\geq k_1+i-1\\ n_3\geq k_3+j-1}}(-1)^i\binom{n_1}{i-1}\tilde{\mathfrak{Z}}_{\mathcal{F}}(n_1)\times(-1)^j\binom{n_3}{j-1}\tilde{\mathfrak{Z}}_{\mathcal{F}}(n_3)\\
&=\sum_{\substack{n_1\geq k_1+1\\ n_3\geq k_3+1}}\sum_{\substack{2\leq i\leq n_1-k_1+1\\ 2\leq j\leq n_3-k_3+1}}(-1)^i\binom{n_1}{i-1}\tilde{\mathfrak{Z}}_{\mathcal{F}}(n_1)\times(-1)^j\binom{n_3}{j-1}\tilde{\mathfrak{Z}}_{\mathcal{F}}(n_3)\\
&=\sum_{\substack{n_1\geq k_1+1\\ n_3\geq k_3+1}}\sum_{\substack{2\leq i\leq n_1-k_1+1\\ 2\leq j\leq n_3-k_3+1}}\left((-1)^i\left(\binom{n_1-1}{i-1}+\binom{n_1-1}{i-2}\right)\tilde{\mathfrak{Z}}_{\mathcal{F}}(n_1)\right)\\
&\qquad\qquad\times\left((-1)^j\left(\binom{n_3-1}{j-1}+\binom{n_3-1}{j-2}\right)\tilde{\mathfrak{Z}}_{\mathcal{F}}(n_3)\right)\\
&=\sum_{\substack{n_1\geq k_1+1\\ n_3\geq k_3+1}}\left((-1)^{k_1}\binom{n_1-1}{k_1-1}+1\right)\tilde{\mathfrak{Z}}_{\mathcal{F}}(n_1)\times\left((-1)^{k_3}\binom{n_3-1}{k_3-1}+1\right)\tilde{\mathfrak{Z}}_{\mathcal{F}}(n_3)\\
&=F_{k_1,1}(X)F_{k_3,1}(X).
\end{aligned}$$

Thus we get the result. □

3.3. Calculation of $\sum_{\boldsymbol{e}}(-1)^{|\boldsymbol{e}|}\tilde{\zeta}^{\star}_{\mathcal{F}}((k_1,k_2,k_3)^{\vee}\oplus\boldsymbol{e})$. For positive integers l_1,l_2,l_3, and a nonnegative integer m, we define a polynomial $Q_m(l_1,l_2,l_3)$ in $\mathfrak{H}$ by

$$\begin{aligned}
Q_m(l_1,l_2,l_3)=\sum_{\substack{e_1+e_2+e_3=m\\ e_1,e_2,e_3\geq 0}}&\binom{l_1+e_1-1}{e_1}\binom{l_2+e_2-2}{e_2}\binom{l_3+e_3-1}{e_3}\\
&\times y^{l_1+e_1}xy^{l_2+e_2-1}xy^{l_3+e_3}.
\end{aligned}$$

Here, when $l_2=1$, we understand

$$\binom{l_2+e_2-2}{e_2}=\binom{e_2-1}{e_2}=\begin{cases}1 & (e_2=0),\\ 0 & (e_2\geq 1).\end{cases}$$

LEMMA 3.7. *For positive integers k_1,k_2,k_3, we have*

$$\begin{aligned}
&\sum_{\boldsymbol{e}}(-1)^{|\boldsymbol{e}|}\zeta^{\star}_{\mathcal{F}}((k_1,k_2,k_3)^{\vee}\oplus\boldsymbol{e})\\
&=-\sum_{\substack{i+j=k_2+2\\ i,j\geq 2}}F_{k_1,i-1}(X)F_{k_3,j-1}(X)-(-1)^{k_1+k_2+k_3}\sum_{m\geq 0}\tilde{\zeta}_{\mathcal{F}}(p(Q_m(k_3,k_2,k_1))).
\end{aligned}$$

Proof. We prove this lemma only for $k_2\geq 2$. The case $k_2=1$ can be proved similarly. By Theorem 3.5, we have

$$\begin{aligned}
&\sum_{\boldsymbol{e}}(-1)^{|\boldsymbol{e}|}\tilde{\zeta}^{\star}_{\mathcal{F}}((k_1,k_2,k_3)^{\vee}\oplus\boldsymbol{e})\\
&=\sum_{m\geq 0}(-1)^m\tilde{\zeta}^{\star}_{\mathcal{F}}((k_1,k_2,k_3)^{\vee}\,\widetilde{ш}\,(\{1\}^m))\\
&=\sum_{m\geq 0}(-1)^m\tilde{\zeta}^{\star}_{\mathcal{F}}((\{1\}^{k_1-1},2,\{1\}^{k_2-2},2,\{1\}^{k_3-1})\,\widetilde{ш}\,(\{1\}^m)).
\end{aligned}$$

By Propositions 2.2 and 2.5, we have

$$\begin{aligned}
&\sum_{m\geq 0}(-1)^m\tilde{\zeta}^{\star}_{\mathcal{F}}((\{1\}^{k_1-1},2,\{1\}^{k_2-2},2,\{1\}^{k_3-1})\,\widetilde{ш}\,(\{1\}^m))\\
&=-\sum_{\substack{m_1\geq 0\\ m_3\geq 0}}\sum_{\substack{i+j=k_2+2\\ i,j\geq 2}}(-1)^{k_3+m_1+j}\tilde{\zeta}^{\star}_{\mathcal{F}}((\{1\}^{k_1-1},2,\{1\}^{i-2})\,\widetilde{ш}\,(\{1\}^{m_1}))\\
&\qquad\qquad\times\tilde{\zeta}_{\mathcal{F}}((\{1\}^{k_3-1},2,\{1\}^{j-2})\,\widetilde{ш}\,(\{1\}^{m_3}))\\
&\quad-(-1)^{k_1+k_2+k_3}\sum_{m\geq 0}\tilde{\zeta}_{\mathcal{F}}((\{1\}^{k_3-1},2,\{1\}^{k_2-2},2,\{1\}^{k_1-1})\,\widetilde{ш}\,(\{1\}^m)).
\end{aligned}$$

By Proposition 3.2, Corollary 3.3, and Theorems 3.4 and 3.5, we have

$$\begin{aligned}
&\sum_{\substack{m_1\geq 0\\ m_3\geq 0}}\sum_{\substack{i+j=k_2+2\\ i,j\geq 2}}(-1)^{k_3+m_1+j}\tilde{\zeta}^{\star}_{\mathcal{F}}((\{1\}^{k_1-1},2,\{1\}^{i-2})\,\widetilde{ш}\,(\{1\}^{m_1}))\\
&\qquad\qquad\times\tilde{\zeta}_{\mathcal{F}}((\{1\}^{k_3-1},2,\{1\}^{j-2})\,\widetilde{ш}\,(\{1\}^{m_3}))\\
&=(-1)^{k_1+k_2+k_3}\sum_{\substack{m_1\geq 0\\ m_3\geq 0}}\sum_{\substack{i+j=k_2+2\\ i,j\geq 2}}\tilde{\zeta}^{\star}_{\mathcal{F}}((\{1\}^{k_1-1},2,\{1\}^{i-2})\,\widetilde{ш}\,(\{1\}^{m_1}))\\
&\qquad\qquad\times\tilde{\zeta}^{\star}_{\mathcal{F}}((\{1\}^{k_3-1},2,\{1\}^{j-2})\,\widetilde{ш}\,(\{1\}^{m_3}))\\
&=(-1)^{k_1+k_2+k_3}\sum_{\substack{m_1\geq 0\\ m_3\geq 0}}\sum_{\substack{i+j=k_2+2\\ i,j\geq 2}}\sum_{|\boldsymbol{e}_1|=m_1}\tilde{\zeta}^{\star}_{\mathcal{F}}((\{1\}^{k_1-1},2,\{1\}^{i-2})\oplus\boldsymbol{e}_1)\\
&\qquad\qquad\times\sum_{|\boldsymbol{e}_3|=m_3}\tilde{\zeta}^{\star}_{\mathcal{F}}((\{1\}^{k_3-1},2,\{1\}^{j-2})\oplus\boldsymbol{e}_3)\\
&=\sum_{\substack{i+j=k_2+2\\ i,j\geq 2}}F_{k_1,i-1}(X)F_{k_3,j-1}(X).
\end{aligned}$$

Since

$$(\{1\}^{k_3-1},2,\{1\}^{k_2-2},2,\{1\}^{k_1-1})\,\widetilde{ш}\,(\{1\}^m)=p(Q_m(k_3,k_2,k_1)),$$

we have the desired result. □

3.4. The equality $\tilde{\zeta}_{\mathcal{F}}(p(P_m(l_1,l_2,l_3)-Q_m(l_1,l_2,l_3)))=0$. Recall $\mathfrak{H}=\mathbb{Q}\langle x,y\rangle$. We define the shuffle product as the $\mathbb{Q}$-bilinear product $ш:\mathfrak{H}\times\mathfrak{H}\to\mathfrak{H}$ given by

$$\begin{aligned}
1\,ш\,w=w\,ш\,1&=w,\\
wu\,ш\,w'u'&=(w\,ш\,w'u')u+(wu\,ш\,w')u',
\end{aligned}$$

where $w,w'\in\mathfrak{H}$ and $u,u'\in\{x,y\}$. For $u_1,\dots,u_r\in\{x,y\}$, let $\overline{u_1\cdots u_r}=u_r\cdots u_1$. We denote by $|w|$ the degree of a word w, e.g., $|yx|=2$.

THEOREM 3.8 (Kaneko-Zagier [7]). *For words $w\in y\mathfrak{H}$ and $w'\in\mathbb{Q}\oplus y\mathfrak{H}$, we have*

$$\zeta_{\mathcal{F}}(p((w\,ш\,w')y))=(-1)^{|w'|}\zeta_{\mathcal{F}}(p(wy\overline{w'})).$$

Proof. See [6, Theorems 8.1 and 9.6]. See also [13, Theorem 6.3.4]. □

Lemma 3.9. *For $w \in y\mathfrak{H}$ and $w' \in \mathbb{Q} \oplus y\mathfrak{H}$, we have*

$$\zeta_{\mathcal{F}}(p(wy ш w'y)) = 0.$$

Proof. We may assume that w and w' are words. By Theorem 3.8, we have

$$\begin{aligned}\zeta_{\mathcal{F}}(p(wy ш w'y)) &= \zeta_{\mathcal{F}}(p((w ш w'y)y + (wy ш w')y)) \\ &= (-1)^{|w'y|}\zeta_{\mathcal{F}}(p(wy^2\overline{w'})) + (-1)^{|w'|}\zeta_{\mathcal{F}}(p(wy^2\overline{w'})) \\ &= 0. \qquad \square\end{aligned}$$

Proposition 3.10. *For positive integers l_1, l_2, l_3 and a nonnegative integer m, we have*

$$Q_m(l_1,l_2,l_3) = \sum_{i=0}^{m} \sum_{\substack{e_1+e_2+e_3=i \\ e_1,e_2,e_3\geq 0}} (-1)^i y^{l_1+e_1}xy^{l_2+e_2-1}xy^{l_3+e_3} ш y^{m-i}.$$

Proof. Fix nonnegative integers a_1, a_2, a_3 with $a_1 + a_2 + a_3 = m$. Then the coefficient of $y^{l_1+a_1}xy^{l_2+a_2-1}xy^{l_3+a_3}$ on the right-hand side is

$$\begin{aligned}&\sum_{j=0}^{a_1}(-1)^{a_1-j}\binom{l_1+a_1}{j} \times \sum_{j=0}^{a_2}(-1)^{a_2-j}\binom{l_2+a_2-1}{j} \times \sum_{j=0}^{a_3}(-1)^{a_3-j}\binom{l_3+a_3}{j} \\ &= \sum_{j=0}^{a_1}(-1)^{a_1-j}\left(\binom{l_1+a_1-1}{j} + \binom{l_1+a_1-1}{j-1}\right) \\ &\quad \times \sum_{j=0}^{a_2}(-1)^{a_2-j}\left(\binom{l_2+a_2-2}{j} + \binom{l_2+a_2-2}{j-1}\right) \\ &\quad \times \sum_{j=0}^{a_3}(-1)^{a_3-j}\left(\binom{l_3+a_3-1}{j} + \binom{l_3+a_3-1}{j-1}\right) \\ &= \binom{l_1+a_1-1}{a_1}\binom{l_2+a_2-2}{a_2}\binom{l_3+a_3-1}{a_3}.\end{aligned}$$

Here, we understand $\binom{n}{-1} = 0$ for all integers n. This finishes the proof. $\square$

Lemma 3.11. *For positive integers l_1, l_2, l_3 and a nonnegative integer m, we have*

$$\tilde{\zeta}_{\mathcal{F}}(p(P_m(l_1,l_2,l_3) - Q_m(l_1,l_2,l_3))) = 0.$$

Proof. By Proposition 3.10, we have

$$P_m(l_1,l_2,l_3) - Q_m(l_1,l_2,l_3) = -\sum_{i=0}^{m-1} \sum_{\substack{e_1+e_2+e_3=i \\ e_1,e_2,e_3\geq 0}} (-1)^i y^{l_1+e_1}xy^{l_2+e_2-1}xy^{l_3+e_3} ш y^{m-i}.$$

Thus, by Lemma 3.9, we have the desired result. $\square$

3.5. Proof of Theorem 1.4. Now we prove our main theorem.

Proof of Theorem 1.4. By Lemmas 3.6, 3.7, and 3.11 we have

$$\begin{aligned}
&O_{\mathcal{F}}(k_1,k_2,k_3)\\
&=F_{k_1,1}(X)F_{k_3,1}(X)-\sum_{\substack{i+j\le k_2+1\\ i,j\ge 2}}\sum_{\substack{n_1\ge k_1+i-1\\ n_3\ge k_3+j-1}}(-1)^i\binom{n_1}{i-1}\tilde{\mathfrak{z}}_{\mathcal{F}}(n_1)\times(-1)^j\binom{n_3}{j-1}\tilde{\mathfrak{z}}_{\mathcal{F}}(n_3)\\
&\quad-\sum_{\substack{i+j=k_2+2\\ i,j\ge 2}}F_{k_1,i-1}(X)F_{k_3,j-1}(X).
\end{aligned}$$

For positive integers k and s, let

$$U_{k,s}=\sum_{n\ge s}(-1)^k\binom{n-1}{k-1}\tilde{\mathfrak{z}}_{\mathcal{F}}(n).$$

Since $(-1)^{s+1}\mathfrak{z}_{\mathcal{F}}(s)=\mathfrak{z}_{\mathcal{F}}(s)$, we note that

$$\begin{aligned}
U_{k,s}-U_{k,s-1}&=(-1)^{k-1}\binom{s-2}{k-1}\tilde{\mathfrak{z}}_{\mathcal{F}}(s-1)=(-1)^{s-k-1}\binom{s-2}{s-k-1}\tilde{\mathfrak{z}}_{\mathcal{F}}(s-1)\\
&=U_{s-k,s}-U_{s-k,s-1}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{\substack{i+j\le k_2+1\\ i,j\ge 2}}\sum_{\substack{n_1\ge k_1+i-1\\ n_3\ge k_3+j-1}}(-1)^i\binom{n_1}{i-1}\tilde{\mathfrak{z}}_{\mathcal{F}}(n_1)\times(-1)^j\binom{n_3}{j-1}\tilde{\mathfrak{z}}_{\mathcal{F}}(n_3)\\
&=\sum_{\substack{i+j\le k_2+1\\ i,j\ge 2}}\sum_{\substack{n_1\ge k_1+i-1\\ n_3\ge k_3+j-1}}(-1)^{i-1}\left(\binom{n_1-1}{i-2}+\binom{n_1-1}{i-1}\right)\tilde{\mathfrak{z}}_{\mathcal{F}}(n_1)\\
&\qquad\times(-1)^{j-1}\left(\binom{n_3-1}{j-2}+\binom{n_3-1}{j-1}\right)\tilde{\mathfrak{z}}_{\mathcal{F}}(n_3)\\
&=\sum_{\substack{i+j\le k_2+1\\ i,j\ge 2}}(U_{i-1,k_1+i-1}-U_{i,k_1+i-1})(U_{j-1,k_3+j-1}-U_{j,k_3+j-1})\\
&=\sum_{\substack{i+j\le k_2+1\\ i,j\ge 2}}(U_{i-1,k_1+i-1}-U_{i,k_1+i}+U_{i,k_1+i}-U_{i,k_1+i-1})\\
&\qquad\times(U_{j-1,k_3+j-1}-U_{j,k_3+j}+U_{j,k_3+j}-U_{j,k_3+j-1})\\
&=\sum_{\substack{i+j\le k_2+1\\ i,j\ge 2}}(U_{i-1,k_1+i-1}-U_{i,k_1+i}+U_{k_1,k_1+i}-U_{k_1,k_1+i-1})\\
&\qquad\times(U_{j-1,k_3+j-1}-U_{j,k_3+j}+U_{k_3,k_3+j}-U_{k_3,k_3+j-1})\\
&=\sum_{\substack{i+j\le k_2+1\\ i,j\ge 2}}(F_{k_1,i}(X)-F_{k_1,i-1}(X))(F_{k_3,j}(X)-F_{k_3,j-1}(X)).
\end{aligned}$$

Thus we get

$$\begin{aligned}
&O_{\mathcal{F}}(k_1,k_2,k_3)\\
&=F_{k_1,1}(X)F_{k_3,1}(X)-\sum_{\substack{i+j\le k_2+1\\ i,j\ge 2}}(F_{k_1,i}(X)-F_{k_1,i-1}(X))(F_{k_3,j}(X)-F_{k_3,j-1}(X))\\
&\quad-\sum_{\substack{i+j=k_2+2\\ i,j\ge 2}}F_{k_1,i-1}(X)F_{k_3,j-1}(X).
\end{aligned}$$

When $k_2\ge 2$, we note that

$$\begin{aligned}
&\sum_{\substack{i+j\le k_2+1\\ i,j\ge 2}}(F_{k_1,i}(X)-F_{k_1,i-1}(X))(F_{k_3,j}(X)-F_{k_3,j-1}(X))\\
&=F_{k_1,1}(X)F_{k_3,1}(X)-\sum_{\substack{i+j=k_2+2\\ i,j\ge 2}}F_{k_1,i-1}(X)F_{k_3,j-1}(X)+\sum_{\substack{i+j=k_2+1\\ i,j\ge 2}}F_{k_1,i}(X)F_{k_3,j}(X).
\end{aligned}$$

Hence we find

$$O_{\mathcal{F}}(k_1,k_2,k_3)=\begin{cases}-\displaystyle\sum_{\substack{i+j=k_2+1\\ i,j\ge 2}}F_{k_1,i}(X)F_{k_3,j}(X) & (k_2\ge 2),\\ F_{k_1,1}(X)F_{k_3,1}(X) & (k_2=1).\end{cases}$$

This finishes the proof. □

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REFERENCES

[1] M. Hirose, K. Imatomi, H. Murahara, and S. Saito, *Ohno-type relations for classical and finite multiple zeta-star values*, Kyushu J. Math., 75 (2021), pp. 115–124.

[2] M. E. Hoffman, *The algebra of multiple harmonic series*, J. Algebra, 194 (1997), pp. 477–495.

[3] M. E. Hoffman, *Quasi-symmetric functions and mod p multiple harmonic sums*, Kyushu J. Math., 69 (2015), pp. 345–366.

[4] K. Ihara, M. Kaneko, and D. Zagier, *Derivation and double shuffle relations for multiple zeta values*, Compositio Math., 142 (2006), pp. 307–338.

[5] D. Jarossay, *Double mélange des multizêtas finis et multizêtas symétrisés*, C. R. Math. Acad. Sci. Paris, 352 (2014), pp. 767–771.

[6] M. Kaneko, *An introduction to classical and finite multiple zeta values*, Publications mathématiques de Besançon. Algèbre et théorie des nombres (2019), pp. 103–129.

[7] M. Kaneko and D. Zagier, *Finite multiple zeta values*, in preparation.

[8] H. Murahara, *A note on finite real multiple zeta values*, Kyushu J. Math., 70 (2016), pp. 345–366.

[9] Y. Ohno, *A generalization of the duality and sum formulas on the multiple zeta values*, J. Number Theory, 74 (1999), pp. 39–43.

[10] K. Oyama, *Ohno-type relation for finite multiple zeta values*, Kyushu J. Math., 72 (2018), pp. 277–285.

[11] S. Saito and N. Wakabayashi, *Sum formula for finite multiple zeta values*, J. Math. Soc. Japan, 67 (2015), pp. 1069–1076.

[12] K. Sakugawa and S. Seki, *On functional equations of finite multiple polylogarithms*, J. Algebra, 469 (2017), pp. 323–357.

[13] J. Zhao, *Multiple zeta functions, multiple polylogarithms and their special values*, vol. 12, Hackensack, NJ: World Scientific (2016).