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A NEW PROOF FOR GLOBAL RIGIDITY OF VERTEX SCALING ON POLYHEDRAL SURFACES*

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Abstract. The vertex scaling for piecewise linear metrics on polyhedral surfaces was introduced by Luo [17], who proved the local rigidity by establishing a variational principle and conjectured the global rigidity. Luo's conjecture was solved by Bobenko-Pinkall-Springborn [3], who also introduced the vertex scaling for piecewise hyperbolic metrics and proved its global rigidity. Bobenko-Pinkall-Spingborn's proof is based on their observation of the connection between vertex scaling, the geometry of polyhedra in 3-dimensional hyperbolic space and the concavity of the volume of ideal and hyper-ideal tetrahedra. In this paper, we give an elementary and short variational proof of the global rigidity of vertex scaling without involving 3-dimensional hyperbolic geometry. The method is based on continuity of eigenvalues and the extension of convex functions.

Key words. Rigidity, Vertex scaling, Piecewise linear metric, Piecewise hyperbolic metric.

Mathematics Subject Classification. 52C25, 52C26.

1. Introduction. The most important two discrete conformal metrics on polyhedral surfaces are circle packing metrics and vertex scaling of polyhedral metrics on surfaces. There are lots of important works on circle packing metrics, please refer to [1, 2, 4, 5, 6, 15, 18, 19, 22, 25, 30] and others. In this paper, we focus on vertex scaling of polyhedral metrics on surfaces, which is an analogue of the conformal transformation in Riemannian geometry. The vertex scaling of piecewise linear metrics (PL metrics for short in the following) on polyhedral surfaces was introduced physically by Rǒcek-Williams [24] and mathematically by Luo [17] independently. Luo proved the local rigidity of vertex scaling for PL metrics by establishing a variational principle and conjectured the global rigidity in [17], where Luo also introduced the corresponding combinatorial Yamabe flow and studied its properties. Luo's conjecture was solved affirmatively by Bobenko-Pinkall-Springborn in their important work [3] by establishing the connection of vertex scaling and the geometry of ideal tetrahedra in 3-dimensional hyperbolic space and using Rivin's result on the concavity of the volume of ideal tetrahedra [23]. Bobenko-Pinkall-Springborn [3] further introduced the vertex scaling for piecewise hyperbolic metrics (PH metrics for short in the following) and proved its global rigidity by connecting the hyperbolic vertex scaling to the geometry of hyper-ideal tetrahedra in 3-dimensional hyperbolic space and using Leibon's result on the concavity of the volume of hyper-ideal tetrahedra [16]. Based on Bobenko-Pinkall-Springborn's observations, the important discrete uniformization theorems for vertex scaling on closed surfaces were recently established in [11, 12, 21]. This paper aims at giving an elementary, direct and short variational proof for the global rigidity of vertex scaling of PL and PH metrics on surfaces without involving 3-dimensional hyperbolic geometry.

Suppose M is a closed surface with a triangulation $\mathcal{T} = (V, E, F)$, where V, E, F represent the sets of vertices, edges and faces respectively. A discrete metric is a map $l : E \to (0, +\infty)$ such that the triangle inequalities are satisfied for l_{ij}, l_{ik}, l_{jk}

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on any triangle $\triangle v_iv_jv_k \in F$, where $l_{ij} := l(v_iv_j)$ with $v_iv_j \in E$. In this case, we can attach a Euclidean metric to each triangle $\triangle v_iv_jv_k \in F$, which gives rise to a Euclidean triangle, still denoted by $\triangle v_iv_jv_k$. By gluing the Euclidean triangles isometrically along the edges, we have a PL metric on the triangulated surface $(M, \mathcal{T})$. If we replace the Euclidean metric by hyperbolic metric, then we obtain a PH metric on the triangulated surface $(M, \mathcal{T})$. For PL and PH metrics on $(M, \mathcal{T})$, there may exists cone singularities at the vertices, which could be described by combinatorial curvature. The combinatorial curvature K_i at the vertex v_i is 2π less the summation of inner angles of triangles at v_i.

DEFINITION 1.1 ([3, 17, 24]). Suppose $(M, \mathcal{T})$ is a triangulated surface and $u : V \to \mathbb{R}$ is a function defined on the vertices.

(1) If $l : E \to (0, +\infty)$ and $\widetilde{l} : E \to (0, +\infty)$ are two PL metrics on $(M, \mathcal{T})$ with

$$l_{ij} = \widetilde{l}_{ij} e^{\frac{u_i+u_j}{2}} \tag{1}$$

for any edge $v_iv_j \in E$, we say l is a Euclidean vertex scaling of $\widetilde{l}$.

(2) If $l : E \to (0, +\infty)$ and $\widetilde{l} : E \to (0, +\infty)$ are two PH metrics on $(M, \mathcal{T})$ with

$$\sinh \frac{l_{ij}}{2} = \sinh \frac{\widetilde{l}_{ij}}{2} e^{\frac{u_i+u_j}{2}} \tag{2}$$

for any edge $v_iv_j \in E$, we say l is a hyperbolic vertex scaling of $\widetilde{l}$.

The function $u : V \to \mathbb{R}$ is called a discrete conformal factor.

Bobenko-Pinkall-Spingborn proved the following global rigidity of vertex scaling of polyhedral metrics on surfaces in their important work [3].

THEOREM 1.2 ([3]). *Suppose $(M, \mathcal{T})$ is a closed triangulated surface. Then the discrete conformal factor is uniquely determined by the discrete curvature (up to a vector $t(1, 1, \cdots, 1), t \in \mathbb{R}$, in the PL case).*

Let us recall Bobenko-Pinkall-Spingborn's strategy to prove Theorem 1.2. In the PL case, they considered the Legendre transform of the volume of ideal tetrahedra in 3-dimensional hyperbolic space, which has an explicit formula in dihedral angles obtained by Milnor [20]. Based on Rivin's result [23] that the volume of ideal tetrahedra is a concave function of the dihedral angles and could be extended, they extended the definition of Legendre transform of the volume to to be a globally defined convex function. By modifying the Legendre transform of the volume by a linear function, they showed that this is a globally defined convex extension of Luo's action function which is locally convex. Then the global rigidity follows from the convexity of the extended function. In the PH case, the global rigidity is proved similarly with the volume of ideal tetrahedra replaced by the volume of hyper-ideal tetrahedra with one hyper-ideal vertex and three ideal vertices. The explicit formula of such hyper-ideal tetrahedra in terms of dihedral angles was obtained by Leibon in [16], where the concavity of the volume was also proved. Bobenko-Pinkall-Spingborn's approach established the connection of vertex scaling on polyhedral surfaces and the geometry of hyperbolic polyhedra in 3-dimensional hyperbolic space. In this approach they could define the vertex scaling for hyperbolic and spherical polyhedral metrics on surfaces and give the explicit formula of the action functional introduced by Luo, which has lots of applications.

In this paper, we give an elementary, direct and short variational proof for the global rigidity of vertex scaling of PL and PH metrics on polyhedral surfaces, which does not involve the volume of ideal and hyper-ideal tetrahedra in 3-dimensional hyperbolic space and their concavity with respect to dihedral angles. The main idea comes from [3, 6, 18, 28]. The first step is to give a characterization of the admissible space of conformal factors for any given initial discrete metric on a single triangle, which is proved to be simply connected with analytic boundaries by solving a quadratic inequality. The second step is to prove the Jacobian of the combinatorial curvature with respect to the discrete conformal factor is symmetric and positive definite, which could be reduced to the case that the Jacobian of the inner angles with respect to the conformal factors in a triangle is symmetric and negative definite. The symmetry could be proved by direct calculations. For the negative definiteness, we introduce a parameterized admissible space of conformal factors, which is the union of admissible spaces of conformal factors on a single triangle with different initial discrete metrics. This space is proved to be connected, from which the negativity of the Jacobian of inner angles with respect to the conformal factors in a triangle follows easily by the continuity of eigenvalues and calculating at a good point in the parameterized admissible space. The first step and second step enable us to define a locally convex function on the admissible space of conformal factors for a triangle with fixed initial metric. The third step is to extend the locally convex function to be a globally defined convex function, from which the global rigidity follows. This step is accomplished using Luo's extension theorem for continuous closed 1-forms [18], which is a development of Bobenko-Pinkall-Spingborn's extension. We will give the details of the proof of global rigidity for hyperbolic vertex scaling and just sketch the proof for Euclidean vertex scaling.

The paper is organized as follows. In Section 2, we characterize the admissible space of conformal factors for PH metrics on a single triangle. In Section 3, we prove the Jacobian matrix of the inner angles in terms of hyperbolic discrete conformal factors in a triangle is symmetric and negative definite, which enables us to define a locally convex function on the admissible space of conformal factors. In Section 4, we extend the locally convex function to be a globally defined convex function, from which the global rigidity of hyperbolic vertex scaling follows. In Section 5, we sketch the proof for global rigidity of vertex scaling for PL metrics.

2. Admissible space of discrete conformal factors for discrete hyperbolic metrics on a triangle. Suppose $\triangle v_iv_jv_k \in F$ is a triangle and $\widetilde{l}_{ij}, \widetilde{l}_{ik}, \widetilde{l}_{jk}$ is a discrete hyperbolic metric on $\triangle v_iv_jv_k$. The admissible space $\Omega^H_{ijk}(\widetilde{l})$ of discrete conformal factors for the triangle $\triangle v_iv_jv_k$ with discrete hyperbolic metric $\widetilde{l}_{ij}, \widetilde{l}_{ik}, \widetilde{l}_{jk}$ is defined to be the set of discrete conformal factors $(u_i, u_j, u_k) \in \mathbb{R}^3$ such that the triangle with edge lengths given by formula (2) exists in 2-dimensional hyperbolic space $\mathbb{H}^2$, i.e.

$$\Omega^H_{ijk}(\widetilde{l}) = \{(u_i, u_j, u_k) \in \mathbb{R}^3 | l_i + l_j > l_k, l_i + l_k > l_j, l_j + l_k > l_i\}.$$

Here and in the following, we use l_i to denote l_{jk} for simplicity. The parameterized admissible space of conformal factors for the triangle $\triangle v_iv_jv_k$ is defined to be

$$\Omega^H_{ijk} = \{(\widetilde{l}_i, \widetilde{l}_j, \widetilde{l}_k, u_i, u_j, u_k) \in \mathbb{R}^3_{>0} \times \mathbb{R}^3 | l_i + l_j > l_k, l_i + l_k > l_j, l_j + l_k > l_i\},$$

which could be taken as the union of the admissible space $\Omega^H_{ijk}(\widetilde{l})$ according to the parameters given by the initial discrete metrics $(\widetilde{l}_i, \widetilde{l}_j, \widetilde{l}_k)$.

By formula (2), if the edge lengths l_i, l_j, l_k satisfy the triangle inequalities, there are some restrictions on the discrete conformal factors.

Lemma 2.1. *Suppose the triangle $\triangle v_iv_jv_k$ is a triangle with discrete hyperbolic metric $(\widetilde{l}_i, \widetilde{l}_j, \widetilde{l}_k)$, l_i, l_j, l_k are the edge lengths defined by formula (2), then l_i, l_j, l_k satisfy the triangle inequalities if and only if*

$$Q := -S_i^4\xi_i^2 - S_j^4\xi_j^2 - S_k^4\xi_k^2 + 2S_i^2S_j^2\xi_i\xi_j + 2S_i^2S_k^2\xi_i\xi_k + 2S_j^2S_k^2\xi_j\xi_k + 4S_i^2S_j^2S_k^2 > 0,$$

where $S_i = \sinh\frac{\widetilde{l}_i}{2}$, $\xi_i = e^{-u_i}$.

Proof. l_i, l_j, l_k satisfy the triangle inequalities, i.e. $l_i + l_j > l_k, l_i + l_k > l_j, l_j + l_k > l_i$, is equivalent to

$$\sinh\frac{l_i+l_j+l_k}{2}\sinh\frac{l_i+l_j-l_k}{2}\sinh\frac{l_i+l_k-l_j}{2}\sinh\frac{l_j+l_k-l_i}{2} > 0.$$

By direct calculations, we have

$$\begin{aligned}
&\sinh\frac{l_i+l_j+l_k}{2}\sinh\frac{l_i+l_j-l_k}{2}\sinh\frac{l_i+l_k-l_j}{2}\sinh\frac{l_j+l_k-l_i}{2}\\
=&\frac{1}{4}(\cosh(l_i+l_j)-\cosh l_k)(\cosh l_k - \cosh(l_i-l_j))\\
=&\frac{1}{4}(-\cosh^2 l_i - \cosh^2 l_j - \cosh^2 l_k + 2\cosh l_i\cosh l_j\cosh l_k + 1)\\
=&-\sinh^4\frac{l_i}{2} - \sinh^4\frac{l_j}{2} - \sinh^4\frac{l_k}{2} + 2\sinh^2\frac{l_i}{2}\sinh^2\frac{l_j}{2}\\
&+2\sinh^2\frac{l_i}{2}\sinh^2\frac{l_k}{2} + 2\sinh^2\frac{l_j}{2}\sinh^2\frac{l_k}{2} + 4\sinh^2\frac{l_i}{2}\sinh^2\frac{l_j}{2}\sinh^2\frac{l_k}{2}\\
=&\xi_i^{-2}\xi_j^{-2}\xi_k^{-2}(-S_i^4\xi_i^2 - S_j^4\xi_j^2 - S_k^4\xi_k^2 + 2S_i^2S_j^2\xi_i\xi_j + 2S_i^2S_k^2\xi_i\xi_k + 2S_j^2S_k^2\xi_j\xi_k + 4S_i^2S_j^2S_k^2),
\end{aligned}$$

where the formula (2) is used in the last equality. □

Set

$$\begin{aligned}
h_i &= -S_i^4\xi_i + S_i^2S_j^2\xi_j + S_i^2S_k^2\xi_k,\\
h_j &= -S_j^4\xi_j + S_i^2S_j^2\xi_i + S_j^2S_k^2\xi_k,\\
h_k &= -S_k^4\xi_k + S_i^2S_k^2\xi_i + S_j^2S_k^2\xi_j,
\end{aligned}\tag{3}$$

then we have

$$Q = \xi_ih_i + \xi_jh_j + \xi_kh_k + 4S_i^2S_j^2S_k^2.$$

Lemma 2.1 implies that $(u_i, u_j, u_k) \in \mathbb{R}^3$ is a degenerate hyperbolic discrete conformal factor for a triangle $\triangle v_iv_jv_k$ if and only if

$$Q = \xi_ih_i + \xi_jh_j + \xi_kh_k + 4S_i^2S_j^2S_k^2 \leq 0.$$

Note that $4S_i^2S_j^2S_k^2 > 0$ for any $(\widetilde{l}_i, \widetilde{l}_j, \widetilde{l}_k) \in \mathbb{R}^3_{>0}$. If $(u_i, u_j, u_k) \in \mathbb{R}^3$ is a degenerate hyperbolic discrete conformal factor, we have

$$\xi_ih_i + \xi_jh_j + \xi_kh_k < 0,$$

which implies that at least one of h_i, h_j, h_k is negative. We further have the following result on the signs of h_i, h_j and h_k.

LEMMA 2.2. *Suppose $(u_i, u_j, u_k) \in \mathbb{R}^3$ is a degenerate hyperbolic discrete conformal factor for a triangle $\triangle v_iv_jv_k$, then one of h_i, h_j, h_k is negative and the others are positive.*

Proof. We claim that there exists no subset $\{r, s\} \subset \{i, j, k\}$ such that $h_r \leq 0$ and $h_s \leq 0$, from which the conclusion of the lemma follows. Otherwise, without loss of generality, we assume $h_i \leq 0, h_j \leq 0$, which is equivalent to $S_i^2\xi_i \geq S_j^2\xi_j + S_k^2\xi_k, S_j^2\xi_j \geq S_i^2\xi_i + S_k^2\xi_k$. This is impossible. $\square$

There is a nice geometric explanation of the result in Lemma 2.2 in the Euclidean case in terms of circumcircle center. Please refer to Remark 4.

THEOREM 2.3. *Given any initial nondegenerate hyperbolic discrete metric $\widetilde{l} = (\widetilde{l}_i, \widetilde{l}_j, \widetilde{l}_k)$ on a triangle $\triangle v_iv_jv_k$, the admissible space $\Omega^H_{ijk}(\widetilde{l})$ of hyperbolic discrete conformal factors $(u_i, u_j, u_k) \in \mathbb{R}^3$ for the triangle $\triangle v_iv_jv_k$ is nonempty and simply connected. Furthermore, the set of degenerate hyperbolic discrete conformal factors is a disjoint union $\bigcup_{\alpha\in\Lambda} V_\alpha$, where $\Lambda = \{i, j, k\}$ and V_α is a closed region in $\mathbb{R}^3$ bounded by an analytic graph on $\mathbb{R}^2$.*

Proof. Suppose $(u_i, u_j, u_k) \in \mathbb{R}^3$ is a degenerate hyperbolic discrete conformal factor for the triangle $\triangle v_iv_jv_k$, which is equivalent to $Q \leq 0$. Then by Lemma 2.2, one of h_i, h_j, h_k is negative and the others are positive. Without loss of generality, we assume $h_i < 0,\ h_j > 0,\ h_k > 0$. Note that $Q \leq 0$ is equivalent to the following quadratic inequality of ξ_i

$$A_i\xi_i^2 + B_i\xi_i + C_i \geq 0, \tag{4}$$

where

$$\begin{aligned} A_i &= S_i^4 > 0,\\ B_i &= -2S_i^2(S_j^2\xi_j + S_k^2\xi_k) < 0,\\ C_i &= S_j^4\xi_j^2 + S_k^4\xi_k^2 - 2S_j^2S_k^2\xi_j\xi_k - 4S_i^2S_j^2S_k^2. \end{aligned} \tag{5}$$

By direct calculations, $\Delta_i = B_i^2 - 4A_iC_i$ is given by

$$\Delta_i = 16S_i^4S_j^2S_k^2\xi_j\xi_k + 16S_i^6S_j^2S_k^2 > 0. \tag{6}$$

Combining formula (4), (5) with (6), we have

$$\xi_i \geq \frac{-B_i + \sqrt{\Delta_i}}{2A_i} \quad \text{or} \quad \xi_i \leq \frac{-B_i - \sqrt{\Delta_i}}{2A_i}.$$

Note that $-2h_i = 2A_i\xi_i + B_i$, so $h_i < 0$ is equivalent to $\xi_i > \frac{-B_i}{2A_i}$, which implies $\xi_i \geq \frac{-B_i+\sqrt{\Delta_i}}{2A_i}$. Therefore, $\mathbb{R}^3 \setminus \Omega^H_{ijk}(\widetilde{l}) \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$, where

$$V_i = \left\{(u_i, u_j, u_k) \in \mathbb{R}^3 | \xi_i \geq \frac{-B_i + \sqrt{\Delta_i}}{2A_i}\right\}$$

and V_j, V_k are defined similarly.

On the other hand, for any $(u_i, u_j, u_k) \in V_i$, we have $A_i\xi_i^2 + B_i\xi_i + C_i \geq 0$, which implies $Q \leq 0$, thus $V_i \subseteq \mathbb{R}^3 \setminus \Omega^H_{ijk}(\widetilde{l})$. Similar arguments imply $V_j, V_k \subseteq \mathbb{R}^3 \setminus \Omega^H_{ijk}(\widetilde{l})$. Therefore, $\Omega^H_{ijk}(\widetilde{l}) = \mathbb{R}^3 \setminus \bigcup_{\alpha\in\Lambda} V_\alpha$, where $\Lambda = \{i, j, k\}$.

For any $(u_i, u_j, u_k) \in V_i$, we have $\xi_i > \frac{-B_i}{2A_i}$, which is equivalent to $h_i < 0$. Similarly, for $(u_i, u_j, u_k) \in V_j$, we have $h_j < 0$ and for $(u_i, u_j, u_k) \in V_k$, we have $h_k < 0$. Then Lemma 2.2 implies $V_i \bigcap V_j = \varnothing$, $V_i \bigcap V_k = \varnothing$, $V_j \bigcap V_k = \varnothing$.

Note that V_i is bounded by an analytic graph on $\mathbb{R}^2$. In fact,

$$V_i = \{(u_i, u_j, u_k) \in \mathbb{R}^3 | u_i \le \log \frac{2A_i}{-B_i + \sqrt{\Delta_i}}\}.$$

This implies $\Omega^H_{ijk}(\widetilde{l}) = \mathbb{R}^3 \setminus \bigcup_{\alpha \in \Lambda} V_\alpha$ is homotopy equivalent to $\mathbb{R}^3$. Therefore, $\Omega^H_{ijk}(\widetilde{l})$ is simply connected. □

Remark 1. The ideal of the proof of Theorem 2.3 comes from [27], where the first author introduced the method of homotopy deformation to prove the admissible space of sphere packing metrics for a single tetrahedron is simply connected. This method is then developed [28] to prove the admissible space of inversive distance packing metrics for a single triangle is simply connected and used to proved the admissible space of Thurston's sphere packing metrics on a tetrahedron is simply connected [13, 14]. This method has lots of other applications in characterizing admissible spaces of discrete metrics, see [29].

Note that Q is a continuous function of $(\widetilde{l}_i, \widetilde{l}_j, \widetilde{l}_k, u_i, u_j, u_k) \in \mathbb{R}^3_{>0} \times \mathbb{R}^3$ and the space of hyperbolic discrete metrics $(\widetilde{l}_i, \widetilde{l}_j, \widetilde{l}_k)$ satisfying the triangle inequalities is connected. As a direct corollary of Lemma 2.1 and Theorem 2.3, we have the following result on the parameterized admissible space Ω^H_{ijk}.

Corollary 2.4. *Suppose $\triangle v_i v_j v_k \in F$. Then the parameterized admissible space Ω^H_{ijk} is connected.*

Denote $\alpha_i, \alpha_j, \alpha_k$ as the inner angles in the triangle $\triangle v_i v_j v_k$ so that α_i is opposite to the edge of length l_i. We further have the following property of the inner angles on the admissible space $\Omega^H_{ijk}(\widetilde{l})$.

Lemma 2.5. *The inner angles $\alpha_i, \alpha_j, \alpha_k$ defined for $(u_i, u_j, u_k) \in \Omega^H_{ijk}(\widetilde{l})$ could be extended to be continuous functions $\widetilde{\alpha}_i, \widetilde{\alpha}_j, \widetilde{\alpha}_k$ defined on $\mathbb{R}^3$.*

Proof. By Theorem 2.3, $\Omega^H_{ijk}(\widetilde{l}) = \mathbb{R}^3 \setminus \bigcup_{\alpha \in \Lambda} V_\alpha$, where $\Lambda = \{i, j, k\}$ and $V_i = \{(u_i, u_j, u_k) \in \mathbb{R}^3 | \xi_i \ge \frac{-B_i + \sqrt{\triangle}_i}{2A_i}\}$. Then $\partial V_i = \{(u_i, u_j, u_k) \in \mathbb{R}^3 | \xi_i = \frac{-B_i + \sqrt{\triangle}_i}{2A_i}\}$. Suppose $(u_i, u_j, u_k) \in \Omega^H_{ijk}(\widetilde{l})$ tends a point $(\overline{u}_i, \overline{u}_j, \overline{u}_k) \in \partial V_i$. By the proof of Lemma 2.1, we have

$$\begin{aligned}
4\xi_i^{-2}\xi_j^{-2}\xi_k^{-2}Q =& 4 \sinh \frac{l_i + l_j + l_k}{2} \sinh \frac{l_i + l_j - l_k}{2} \sinh \frac{l_i + l_k - l_j}{2} \sinh \frac{l_j + l_k - l_i}{2} \\
=& (\cosh(l_i + l_j) - \cosh l_k)(\cosh l_k - \cosh(l_i - l_j)) \\
=& \sinh^2 l_i \sinh^2 l_j - (\cosh l_i \cosh l_j - \cosh l_k)^2 \\
=& \sinh^2 l_i \sinh^2 l_j - \sinh^2 l_i \sinh^2 l_j \cos^2 \alpha_k \\
=& \sinh^2 l_i \sinh^2 l_j \sin^2 \alpha_k.
\end{aligned}$$

As $(u_i, u_j, u_k) \in \Omega^H_{ijk}(\widetilde{l})$ tends to $(\overline{u}_i, \overline{u}_j, \overline{u}_k) \in \partial V_i$, we have $Q \to 0$, which implies $\alpha_k \to 0$ or π. Similarly, we have $\alpha_i, \alpha_j \to 0$ or π.

By formula (8), we have

$$
\begin{aligned}
\frac{\partial \alpha_j}{\partial u_i} =& \frac{\cosh l_i + \cosh l_j - \cosh l_k - 1}{A(\cosh l_k + 1)} \\
=& \frac{\sinh^2 \frac{l_i}{2} + \sinh^2 \frac{l_j}{2} - \sinh^2 \frac{l_k}{2}}{A(\sinh^2 \frac{l_k}{2} + 1)} \\
=& \frac{\xi_i^{-1}\xi_j^{-1}\xi_k^{-1}}{A(S_k^2\xi_i^{-1}\xi_j^{-1} + 1)}(S_i^2\xi_i + S_j^2\xi_j - S_k^2\xi_k) \\
=& \frac{\xi_i^{-1}\xi_j^{-1}\xi_k^{-1}h_k}{AS_k^2(S_k^2\xi_i^{-1}\xi_j^{-1} + 1)},
\end{aligned} \tag{7}
$$

where $A = \sinh l_j \sinh l_k \sin \alpha_i$, $S_i = \sinh \frac{\widetilde{l}_i}{2}$, $\xi_i = e^{-u_i}$ and h_k is defined by formula (3). Note that for $(\overline{u}_i, \overline{u}_j, \overline{u}_k) \in \partial V_i$, by Lemma 2.2 and the proof of Theorem 2.3, we have $h_i < 0$, $h_j > 0$, $h_k > 0$ at $(\overline{u}_i, \overline{u}_j, \overline{u}_k)$. By formula (7), we have $\frac{\partial \alpha_j}{\partial u_i} > 0$ for $(u_i, u_j, u_k) \in \Omega_{ijk}^H(\widetilde{l})$ around $(\overline{u}_i, \overline{u}_j, \overline{u}_k) \in \partial V_i$. This implies $\alpha_j \to 0$ as $(u_i, u_j, u_k) \to (\overline{u}_i, \overline{u}_j, \overline{u}_k) \in \partial V_i$. Otherwise, we have $\alpha_j \to \pi$ as $(u_i, u_j, u_k) \to (\overline{u}_i, \overline{u}_j, \overline{u}_k) \in \partial V_i$ and then $\frac{\partial \alpha_j}{\partial u_i} > 0$ implies $\alpha_j > \pi$ for some $(u_i, u_j, u_k) \in \Omega_{ijk}^H(\widetilde{l})$ around $(\overline{u}_i, \overline{u}_j, \overline{u}_k) \in \partial V_i$, which is impossible for hyperbolic triangles. Similarly, we have $\alpha_k \to 0$ as $(u_i, u_j, u_k) \to (\overline{u}_i, \overline{u}_j, \overline{u}_k) \in \partial V_i$.

Furthermore, we have the following formula for the area S of the hyperbolic triangle in terms of the edge lengths ([26] page 66)

$$
\begin{aligned}
\tan^2 \frac{S}{4} =& \tanh \frac{p}{2} \tanh \frac{p - l_i}{2} \tanh \frac{p - l_j}{2} \tanh \frac{p - l_k}{2} \\
=& \frac{\xi_i^{-2}\xi_j^{-2}\xi_k^{-2}Q}{64 \cosh^2 \frac{p}{2} \cosh^2 \frac{p-l_i}{2} \cosh^2 \frac{p-l_j}{2} \cosh^2 \frac{p-l_k}{2}},
\end{aligned}
$$

where $p = \frac{1}{2}(l_i + l_j + l_k)$. Note that $Q \to 0$ as $(u_i, u_j, u_k) \to (\overline{u}_i, \overline{u}_j, \overline{u}_k) \in \partial V_i$, we have $S \to 0$. Then we have $\alpha_i \to \pi$ as $(u_i, u_j, u_k) \to (\overline{u}_i, \overline{u}_j, \overline{u}_k) \in \partial V_i$ by $S = \pi - \alpha_i - \alpha_j - \alpha_k$ and $\alpha_j, \alpha_k \to 0$. The case for the boundary ∂V_j and ∂V_k could be discussed similarly.

Therefore, we can extend $\alpha_i, \alpha_j, \alpha_k$ defined on $\Omega_{ijk}^H(\widetilde{l})$ to be continuous functions defined on $\mathbb{R}^3$ by setting

$$
\widetilde{\alpha}_i(u_i, u_j, u_k) = \begin{cases} \alpha_i, & \text{if } (u_i, u_j, u_k) \in \Omega_{ijk}^H, \\ \pi, & \text{if } (u_i, u_j, u_k) \in V_i, \\ 0, & \text{if } (u_i, u_j, u_k) \in V_j \text{ or } V_k. \end{cases}
$$

This completes the proof of the lemma. □

Remark 2. By the proof of Lemma 2.5, we have $\frac{\partial \alpha_j}{\partial u_i} \to +\infty$ and $\frac{\partial \alpha_k}{\partial u_i} \to +\infty$ as $(u_i, u_j, u_k) \to (\overline{u}_i, \overline{u}_j, \overline{u}_k) \in \partial V_i$. Recall the following formula obtained by Glickenstein-Thomas ([10] Proposition 9)

$$
\frac{\partial S}{\partial u_i} = \frac{\partial \alpha_j}{\partial u_i}(\cosh l_k - 1) + \frac{\partial \alpha_k}{\partial u_i}(\cosh l_j - 1),
$$

where S is the area of $\triangle v_i v_j v_k$, which could also be proved using Lemma 3.1 directly. For hyperbolic vertex scaling, we have $\frac{\partial S}{\partial u_i} \to +\infty$, which implies

$$\frac{\partial \alpha_i}{\partial u_i} = -\frac{\partial S}{\partial u_i} - \frac{\partial \alpha_j}{\partial u_i} - \frac{\partial \alpha_k}{\partial u_i} \to -\infty$$

as $(u_i, u_j, u_k) \to (\overline{u}_i, \overline{u}_j, \overline{u}_k) \in \partial V_i$.

3. Negative definiteness of Jacobian matrix.

LEMMA 3.1. *For any triangle $\triangle v_i v_j v_k$, let l_i, l_j, l_k be edge lengths of a hyperbolic triangle and $\alpha_i, \alpha_j, \alpha_k$ be the opposite angles so that α_i is facing the edge of length l_i, then*

$$\frac{\partial \alpha_i}{\partial u_j} = \frac{\partial \alpha_j}{\partial u_i} = \frac{\cosh l_i + \cosh l_j - \cosh l_k - 1}{A(\cosh l_k + 1)}, \tag{8}$$

$$\frac{\partial \alpha_i}{\partial u_i} = \frac{\cosh^2 l_j + \cosh^2 l_k - 2\cosh l_i \cosh l_j \cosh l_k + (1 - \cosh l_i)(\cosh l_j + \cosh l_k)}{A(1 + \cosh l_j)(1 + \cosh l_k)},$$

where $A = \sinh l_j \sinh l_k \sin \alpha_i$.

Proof. By the derivative cosine law (see Lemma A1 in [5] for example), we have

$$\frac{\partial \alpha_i}{\partial l_i} = \frac{\sinh l_i}{A}, \ \frac{\partial \alpha_i}{\partial l_j} = \frac{-\sinh l_i \cos \alpha_k}{A}, \ \frac{\partial \alpha_i}{\partial l_k} = \frac{-\sinh l_i \cos \alpha_j}{A},$$

where $A = \sinh l_j \sinh l_k \sin \alpha_i$. By formula (2), we have

$$\frac{\partial l_i}{\partial u_i} = 0, \ \frac{\partial l_i}{\partial u_j} = \frac{\partial l_i}{\partial u_k} = \tanh \frac{l_i}{2}.$$

Then according to the chain rules, we have

$$\begin{aligned}
\frac{\partial \alpha_i}{\partial u_j} &= \frac{\partial \alpha_i}{\partial l_i}\frac{\partial l_i}{\partial u_j} + \frac{\partial \alpha_i}{\partial l_j}\frac{\partial l_j}{\partial u_j} + \frac{\partial \alpha_i}{\partial l_k}\frac{\partial l_k}{\partial u_j} \\
&= \frac{\sinh l_i}{A}\tanh\frac{l_i}{2} - \frac{\sinh l_i \cos \alpha_j}{A}\tanh\frac{l_k}{2} \\
&= \frac{\sinh l_i}{A}\frac{\sinh l_i}{1 + \cosh l_i} - \frac{\sinh l_i \cos \alpha_j}{A}\frac{\sinh l_k}{1 + \cosh l_k} \\
&= \frac{\cosh l_i + \cosh l_j - \cosh l_k - 1}{A(\cosh l_k + 1)},
\end{aligned}$$

which implies $\frac{\partial \alpha_i}{\partial u_j} = \frac{\partial \alpha_j}{\partial u_i}$. Similarly, we have

$$\begin{aligned}
\frac{\partial \alpha_i}{\partial u_i} &= \frac{\partial \alpha_i}{\partial l_i}\frac{\partial l_i}{\partial u_i} + \frac{\partial \alpha_i}{\partial l_j}\frac{\partial l_j}{\partial u_i} + \frac{\partial \alpha_i}{\partial l_k}\frac{\partial l_k}{\partial u_i} \\
&= \frac{\cosh^2 l_j + \cosh^2 l_k - 2\cosh l_i \cosh l_j \cosh l_k + (1 - \cosh l_i)(\cosh l_j + \cosh l_k)}{A(1 + \cosh l_j)(1 + \cosh l_k)}.
\end{aligned}$$

∎

Lemma 3.1 shows that the matrix

$$\Lambda^H_{ijk} = \frac{\partial(\alpha_i, \alpha_j, \alpha_k)}{\partial(u_i, u_j, u_k)} = \begin{pmatrix} \frac{\partial\alpha_i}{\partial u_i} & \frac{\partial\alpha_i}{\partial u_j} & \frac{\partial\alpha_i}{\partial u_k} \\ \frac{\partial\alpha_j}{\partial u_i} & \frac{\partial\alpha_j}{\partial u_j} & \frac{\partial\alpha_j}{\partial u_k} \\ \frac{\partial\alpha_k}{\partial u_i} & \frac{\partial\alpha_k}{\partial u_j} & \frac{\partial\alpha_k}{\partial u_k} \end{pmatrix}$$

is symmetric on Ω^H_{ijk}. Furthermore, one has the following result for the matrix Λ^H_{ijk}.

THEOREM 3.2 ([3]). *The matrix Λ^H_{ijk} is symmetric, negative definite on Ω^H_{ijk}.*

Proof. By the chain rules, we have

$$\Lambda^H_{ijk} = \frac{\partial(\alpha_i, \alpha_j, \alpha_k)}{\partial(u_i, u_j, u_k)} = \frac{\partial(\alpha_i, \alpha_j, \alpha_k)}{\partial(l_i, l_j, l_k)} \cdot \frac{\partial(l_i, l_j, l_k)}{\partial(u_i, u_j, u_k)}.$$

By the calculations in the proof of Lemma 3.1, we have

$$\frac{\partial(\alpha_i, \alpha_j, \alpha_k)}{\partial(l_i, l_j, l_k)} = -\frac{1}{A} \begin{pmatrix} \sinh l_i & 0 & 0 \\ 0 & \sinh l_j & 0 \\ 0 & 0 & \sinh l_k \end{pmatrix} \begin{pmatrix} -1 & \cos\alpha_k & \cos\alpha_j \\ \cos\alpha_k & -1 & \cos\alpha_i \\ \cos\alpha_j & \cos\alpha_i & -1 \end{pmatrix} \tag{9}$$

and

$$\frac{\partial(l_i, l_j, l_k)}{\partial(u_i, u_j, u_k)} = \begin{pmatrix} \tanh\frac{l_i}{2} & 0 & 0 \\ 0 & \tanh\frac{l_j}{2} & 0 \\ 0 & 0 & \tanh\frac{l_k}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{10}$$

Denote the last matrices in (9) and (10) as Φ and $\mathcal{R}$ respectively. By direct calculations, we have

$$\begin{aligned} \det\Phi =& -1 + \cos\alpha_i^2 + \cos\alpha_j^2 + \cos\alpha_k^2 + 2\cos\alpha_i\cos\alpha_j\cos\alpha_k \\ =& 4\cos\frac{\alpha_i+\alpha_j-\alpha_k}{2}\cos\frac{\alpha_i-\alpha_j+\alpha_k}{2}\cos\frac{\alpha_i+\alpha_j+\alpha_k}{2}\cos\frac{\alpha_i-\alpha_j-\alpha_k}{2} > 0, \\ \det\mathcal{R} =& 2 > 0 \end{aligned}$$

for any $(\widetilde{l}_i, \widetilde{l}_j, \widetilde{l}_k, u_i, u_j, u_k) \in \Omega^H_{ijk}$, which implies $\det\Lambda^H_{ijk} < 0$ and then the Jacobian matrix Λ^H_{ijk} is non-singular. Therefore, the eigenvalues of Λ^H_{ijk} are non-zero. Combining with the continuity of the eigenvalues and the connectivity of the parameterized admissible space Ω^H_{ijk} in Corollary 2.4, the eigenvalues of Λ^H_{ijk} never change signs. So we just need to calculate at one point in Ω^H_{ijk} to prove that the eigenvalues of Λ^H_{ijk} are negative and then Λ^H_{ijk} is negative definite. Note that $p = (1,1,1,0,0,0) \in \Omega^H_{ijk}$. By Lemma 3.1, we have

$$\Lambda^H_{ijk}(p) = \frac{-(\cosh 1 - 1)}{A(1+\cosh 1)} \begin{pmatrix} 2\cosh 1 & -1 & -1 \\ -1 & 2\cosh 1 & -1 \\ -1 & -1 & 2\cosh 1 \end{pmatrix},$$

which is negative definite. Therefore, the eigenvalues of the Jacobian matrix Λ^H_{ijk} at $p = (1,1,1,0,0,0)$ are negative. This completes the proof of the theorem. □

REMARK 3. Theorem 3.2 was first obtained by Bobenko-Pinkall-Springborn in their important work [3] by taking the Jacobian matrix Λ^H_{ijk} as the Hessian matrix of Legendre transform of the volume of hyper-ideal tetrahedra in 3-dimensional hyperbolic space with prescribed metric. The negativity of Λ^H_{ijk} follows from Leibon's concavity of the volume of hyper-ideal tetrahedra with one hyper-ideal vertex and three ideal vertices [16], which depends on the explicit form of the volume formula in terms of dihedral angles. The proof of Theorem 3.2 presented here involves only the cosine law and the continuity of the eigenvalues.

4. Proof of the global rigidity of hyperbolic vertex scaling. By Theorem 2.3 and Theorem 3.2, the following function

$$F_{ijk}(u_i,u_j,u_k)=\int_{(\overline{u}_i,\overline{u}_j,\overline{u}_k)}^{(u_i,u_j,u_k)}\alpha_i du_i+\alpha_j du_j+\alpha_k du_k$$

is a well-defined locally strictly concave function of $(u_i,u_j,u_k)\in\Omega^H_{ijk}(\widetilde{l})$. We need to extend F_{ijk} to be a globally defined concave function on $\mathbb{R}^3$. Recall the following definition of closed continuous 1-form and extension of locally convex function of Luo [18], which is a development of Bobenko-Pinkall-Spingborn's extension in [3].

Definition 4.1 ([18], Definition 2.3). A differential 1-form $w=\sum_{i=1}^n a_i(x)dx^i$ in an open set $U\subset\mathbb{R}^n$ is said to be continuous if each $a_i(x)$ is continuous on U. A continuous differential 1-form w is called closed if $\int_{\partial\tau}w=0$ for each triangle $\tau\subset U$.

Theorem 4.2 ([18], Corollary 2.6). *Suppose $X\subset\mathbb{R}^n$ is an open convex set and $A\subset X$ is an open subset of X bounded by a real analytic codimension-1 submanifold in X. If $w=\sum_{i=1}^n a_i(x)dx_i$ is a continuous closed 1-form on A so that $F(x)=\int_a^x w$ is locally convex on A and each a_i can be extended continuous to X by constant functions to a function $\widetilde{a}_i$ on X, then $\widetilde{F}(x)=\int_a^x\sum_{i=1}^n\widetilde{a}_i(x)dx_i$ is a C^1-smooth convex function on X extending F.*

By Lemma 2.5 and Theorem 4.2, $F_{ijk}(u_i,u_j,u_k)$ defined on $\Omega^H_{ijk}(\widetilde{l})$ could be extended to be the following function

$$\widetilde{F}_{ijk}(u_i,u_j,u_k)=\int_{(\overline{u}_i,\overline{u}_j,\overline{u}_k)}^{(u_i,u_j,u_k)}\widetilde{\alpha}_i du_i+\widetilde{\alpha_j} du_j+\widetilde{\alpha}_k du_k,$$

which is a C^1-smooth concave function defined on $\mathbb{R}^3$ with $\nabla_u\widetilde{F}_{ijk}=(\widetilde{\alpha}_i,\widetilde{\alpha_j},\widetilde{\alpha}_k)^T$. Set

$$\widetilde{F}(u_1,\cdots,u_{|V|})=-\sum_{\triangle v_iv_jv_k\in F}\widetilde{F}_{ijk}(u_i,u_j,u_k)+\int_{\overline{u}}^{u}2\pi\sum_{i=1}^{|V|}du_i,$$

where $|V|$ is the number of vertices. Then $\widetilde{F}(u_1,\cdots,u_{|V|})$ is a C^1 smooth convex function on $\mathbb{R}^V$ with

$$\nabla_{u_i}\widetilde{F}(u_1,\cdots,u_{|V|})=-\sum_{\triangle ijk\in F}\widetilde{\alpha}_i+2\pi=\widetilde{K}_i,$$

where $\widetilde{K}_i=2\pi-\sum_{\triangle v_iv_jv_k\in F}\widetilde{\alpha}_i$ is an extension of K_i. Then the global rigidity of hyperbolic vertex scaling follows from the convexity of $\widetilde{F}$ on $\mathbb{R}^V$ and the locally strict convexity of $\widetilde{F}$ on $\cap_{\triangle v_iv_jv_k\in F}\Omega^H_{ijk}(\widetilde{l})$. This completes the proof of Theorem 1.2 in the hyperbolic case.

5. Rigidity for vertex scaling of PL metrics. As the main steps for the proof of global rigidity of Euclidean vertex scaling is paralleling to the hyperbolic case, we just list the main steps here.

Given any initial discrete Euclidean metric $\widetilde{l}_{ij}, \widetilde{l}_{ik}, \widetilde{l}_{jk}$ on the triangle $\triangle v_iv_jv_k$, the admissible space $\Omega^E_{ijk}(\widetilde{l})$ of Euclidean conformal factors is defined to be

$$\Omega^E_{ijk}(\widetilde{l}) = \{(u_i, u_j, u_k) \in \mathbb{R}^3 | l_i + l_j > l_k, l_i + l_k > l_j, l_j + l_k > l_i\},$$

where the edge lengths are given by formula (1) and we use l_i to denote l_{jk} for simplicity. The Euclidean parameterized admissible space of conformal factors for the triangle $\triangle v_iv_jv_k$ is defined to be

$$\Omega^E_{ijk} = \{(\widetilde{l}_i, \widetilde{l}_j, \widetilde{l}_k, u_i, u_j, u_k) \in \mathbb{R}^3_{>0} \times \mathbb{R}^3 | l_i + l_j > l_k, l_i + l_k > l_j, l_j + l_k > l_i\}.$$

LEMMA 5.1. *Suppose the triangle $\triangle v_iv_jv_k$ is a topological triangle, l_i, l_j, l_k are the edge lengths defined by (1), then the triangle inequalities are satisfied if and only if*

$$Q := -\widetilde{l}_i^4\xi_i^2 - \widetilde{l}_j^4\xi_j^2 - \widetilde{l}_k^4\xi_k^2 + 2\widetilde{l}_i^2\widetilde{l}_j^2\xi_i\xi_j + 2\widetilde{l}_i^2\widetilde{l}_k^2\xi_i\xi_k + 2\widetilde{l}_j^2\widetilde{l}_k^2\xi_j\xi_k > 0,$$

where $\xi_i = e^{-u_i}$.

Set

$$\begin{aligned} h_i &= -\widetilde{l}_i^4\xi_i + \widetilde{l}_i^2\widetilde{l}_j^2\xi_j + \widetilde{l}_i^2\widetilde{l}_k^2\xi_k, \\ h_j &= -\widetilde{l}_j^4\xi_j + \widetilde{l}_i^2\widetilde{l}_j^2\xi_i + \widetilde{l}_j^2\widetilde{l}_k^2\xi_k, \\ h_k &= -\widetilde{l}_k^4\xi_k + \widetilde{l}_i^2\widetilde{l}_k^2\xi_i + \widetilde{l}_j^2\widetilde{l}_k^2\xi_j. \end{aligned} \tag{11}$$

Then we have $Q = \xi_ih_i + \xi_jh_j + \xi_kh_k$. $(u_i, u_j, u_k) \in \mathbb{R}^3$ is a degenerate Euclidean discrete conformal factor if and only if $Q = \xi_ih_i + \xi_jh_j + \xi_kh_k \leq 0$, which implies at least one of h_i, h_j, h_k is nonpositive. Similar to Lemma 2.2 in the hyperbolic case, we have the following result on the signs of h_i, h_j, h_k in the Euclidean case.

LEMMA 5.2. *Suppose $(u_i, u_j, u_k) \in \mathbb{R}^3$ is a degenerate Euclidean discrete conformal factor for a triangle $\triangle v_iv_jv_k$, then one of h_i, h_j, h_k is negative and the others are positive.*

REMARK 4. Lemma 5.2 has the following interesting geometrical explanation. For a Euclidean triangle $\triangle v_iv_jv_k$ with a nondegenerate discrete conformal factor (u_i, u_j, u_k), there exists a geometric center C_{ijk} ([9] Proposition 4) of the triangle $\triangle v_iv_jv_k$ with the same Euclidean distance from C_{ijk} to each vertex of the triangle, which is in fact the circumcircle center for vertex scaling of PL metrics. h_i in formula (11) is positive multiplication of the signed distance $h_{jk,i}$ from C_{ijk} to the edge $\{jk\}$, which is defined to be positive if C_{ijk} is on the same side of the line determined by $\{jk\}$ as the triangle $\triangle v_iv_jv_k$ and negative otherwise (or zero if C_{ijk} is on the edge). By direct calculations, we have the following relationship for h_i and $h_{jk,i}$

$$h_{jk,i} = \frac{\xi_i^{-1}\xi_j^{-\frac{3}{2}}\xi_k^{-\frac{3}{2}}}{8S\widetilde{l}_i}h_i,$$

where S is the area of the Euclidean triangle $\triangle v_iv_jv_k$. See [8] for more general cases. For degenerate conformal factors for Euclidean vertex scaling, Lemma 5.2 implies that the circumcircle center lies in some special regions in the plane relative to the triangle $\triangle v_iv_jv_k$.

THEOREM 5.3 ([17]). *Given any initial nondegenerate Euclidean discrete metric $\widetilde{l}=(\widetilde{l}_i,\widetilde{l}_j,\widetilde{l}_k)$ on a triangle $\triangle v_iv_jv_k$, the admissible space $\Omega^E_{ijk}(\widetilde{l})$ of Euclidean discrete conformal factors $(u_i,u_j,u_k)\in\mathbb{R}^3$ for the triangle $\triangle v_iv_jv_k$ is nonempty and simply connected. Furthermore, the set of degenerate Euclidean discrete conformal factors is a disjoint union $\bigcup_{\alpha\in\Lambda}V_\alpha$, where $\Lambda=\{i,j,k\}$ and V_α is bounded by an analytic graph on $\mathbb{R}^2$ with*

$$V_i=\{(u_i,u_j,u_k)\in\mathbb{R}^3|u_i\leq-\ln(\widetilde{l}_j^2e^{-u_j}+\widetilde{l}_k^2e^{-u_k})+2\ln\widetilde{l}_i\}.$$

As a corollary, Ω^E_{ijk} is connected.

Following the hyperbolic case, as an application of Theorem 5.3, we have the following result, which was obtained by Luo [17] by direct calculations.

THEOREM 5.4 ([17]). *The matrix $\Lambda^E_{ijk}=[\frac{\partial\alpha_r}{\partial u_s}]_{3\times3}$ is symmetric, semi-negative definite on $\Omega^E_{ijk}(\widetilde{l})$ with null space $\{(t,t,t)\in\mathbb{R}^3|t\in\mathbb{R}\}$.*

REMARK 5. In fact, by the derivative cosine law (see [5] for example), we have $\frac{\partial\alpha_i}{\partial l_i}=\frac{l_i}{2S}$, $\frac{\partial\alpha_i}{\partial l_j}=-\frac{l_i\cos\alpha_k}{2S}$, $\frac{\partial\alpha_i}{\partial l_k}=-\frac{l_i\cos\alpha_j}{2S}$, where S is the area of the Euclidean triangle $\triangle v_iv_jv_k$. According to formula (1), we have $\frac{\partial l_i}{\partial u_i}=0$, $\frac{\partial l_i}{\partial u_j}=\frac{\partial l_i}{\partial u_k}=\frac{l_i}{2}$. By direct calculations with the chain rules, we have $\frac{\partial\alpha_i}{\partial u_j}=\frac{l_il_j\cos\alpha_k}{4S}$, $\frac{\partial\alpha_i}{\partial u_i}=-\frac{l_i^2}{4S}$, which implies $\Lambda^E_{ijk}=[\frac{\partial\alpha_r}{\partial u_s}]_{3\times3}$ is symmetric. Luo [17] proved the semi-negative definiteness of Λ^E_{ijk} by direct calculations for any nondegenerate Euclidean conformal factor. If we use the connectivity of Ω^E_{ijk}, we just need to check the signs of the eigenvalues of Λ^E_{ijk} at the point $p=(1,1,1,0,0,0)\in\Omega^E_{ijk}$. By direct calculations, we have

$$\Lambda^E_{ijk}(p)=\frac{-\sqrt{3}}{6}\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix},$$

which has two negative eigenvalues and one zero eigenvalue. This also implies semi-negative definiteness of Λ^E_{ijk}.

LEMMA 5.5 ([3]). *Suppose $(u_i,u_j,u_k)\in\mathbb{R}^3$ is a nondegenerate Euclidean discrete conformal factor for a triangle $\triangle v_iv_jv_k$, denote α_i as the angle at vertex v_i. Then $\alpha_i,\alpha_j,\alpha_k$ defined for $(u_i,u_j,u_k)\in\Omega^E_{ijk}(\widetilde{l})$ could be extended by constants to be continuous functions $\widetilde{\alpha}_i,\widetilde{\alpha}_j,\widetilde{\alpha}_k$ defined on $\mathbb{R}^3$.*

By Theorem 5.3 and Theorem 5.4, the following function

$$F_{ijk}(u_i,u_j,u_k)=\int_{(\overline{u}_i,\overline{u}_j,\overline{u}_k)}^{(u_i,u_j,u_k)}\alpha_idu_i+\alpha_jdu_j+\alpha_kdu_k$$

is a well-defined locally concave function of $(u_i,u_j,u_k)\in\Omega^E_{ijk}(\widetilde{l})$ with $F_{ijk}(u_i+t,u_j+t,u_k+t)=F_{ijk}(u_i,u_j,u_k)+t\pi$. By Lemma 5.5 and Theorem 4.2, $F_{ijk}(u_i,u_j,u_k)$ defined on $\Omega^E_{ijk}(\widetilde{l})$ could be extended to the following function

$$\widetilde{F}_{ijk}(u_i,u_j,u_k)=\int_{(\overline{u}_i,\overline{u}_j,\overline{u}_k)}^{(u_i,u_j,u_k)}\widetilde{\alpha}_idu_i+\widetilde{\alpha}_jdu_j+\widetilde{\alpha}_kdu_k,$$

which is a C^1-smooth concave function defined on $\mathbb{R}^3$ with $\nabla_u \widetilde{F}_{ijk} = (\widetilde{\alpha}_i, \widetilde{\alpha_j}, \widetilde{\alpha}_k)^T$. Then the following of the proof for the global rigidity for Euclidean vertex scaling is almost the same as the hyperbolic case. We omit the details here.

REMARK 6. In the Euclidean case, similar idea to use Luo's extension theorem 4.2 to extend $F_{ijk}(u_i, u_j, u_k)$ appears in [7], where the extension depends on the simply connectivity of the admissible space $\Omega^E_{ijk}(\widetilde{l})$ and negative semi-definiteness of Λ^E_{ijk} obtained by Luo [17]. Here we provide a unified approach to prove the simply connectivity of the admissible space of conformal factors and the negative definiteness of the Jacobian matrix $[\frac{\partial \alpha_r}{\partial u_s}]_{3\times 3}$ for a triangle in the Euclidean and hyperbolic cases.

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