Asian J. Math.
Vol. 25, No. 6, pp. 897–910, December 2021

006

CLOSED G_2-STRUCTURES WITH A TRANSITIVE REDUCTIVE GROUP OF AUTOMORPHISMS*

FABIO PODESTÀ[†] AND ALBERTO RAFFERO[‡]

Dedicated to Dmitri V. Alekseevsky on the occasion of his 80th birthday

Abstract. We provide the complete classification of seven-dimensional manifolds endowed with a closed non-parallel G_2-structure and admitting a transitive reductive group G of automorphisms. In particular, we show that the center of G is one-dimensional and the manifold is the Riemannian product of a flat factor and a non-compact homogeneous six-dimensional manifold endowed with an invariant strictly symplectic half-flat SU(3)-structure.

Key words. closed G_2-structure, automorphism.

Mathematics Subject Classification. 53C10, 57S20.

1. Introduction. A closed G_2-structure on a seven-dimensional manifold M is given by a definite 3-form φ satisfying the condition $d\varphi = 0$. Closed G_2-structures appeared in [11] as one of the sixteen natural classes of G_2-structures, and 7-manifolds endowed with these structures provide a fruitful setting where to construct metrics with holonomy G_2 (see e.g. [3, 6, 15, 16, 18, 19, 23]). However, very little is currently known about general properties of these manifolds, see for instance [3, 4] for curvature properties, [5, 7, 8, 10, 12, 20, 21, 22] for examples consisting of Lie groups with left-invariant closed G_2-structures, and [9] for a compact example obtained resolving the singularities of a 7-orbifold.

In our previous work [26], we investigated the properties of the automorphism group $\mathrm{Aut}(M, \varphi) := \{f \in \mathrm{Diff}(M) \mid f^*\varphi = \varphi\}$ when M is compact and the closed G_2-structure φ is not parallel with respect to the Levi Civita connection of the corresponding Riemannian metric g_φ. In particular, we proved that the compact Lie group $\mathrm{Aut}(M, \varphi)$ has abelian Lie algebra with dimension bounded above by $\min\{6, b_2(M)\}$. As a consequence, we showed that there are no compact homogeneous 7-manifolds endowed with an invariant closed non-parallel G_2-structure, i.e., admitting a transitive Lie subgroup $G \subseteq \mathrm{Aut}(M, \varphi)$.

In the non-compact setting, besides the examples on Lie groups mentioned above, it is possible to obtain homogeneous examples on the product of the circle (or the real line) with a non-compact homogeneous 6-manifold endowed with an invariant strictly symplectic half-flat SU(3)-structure (ω, ψ). In these examples, the closed G_2-structure is given by $\varphi = \omega \wedge ds + \psi$, where s is the coordinate on the one-dimensional factor, and the transitive Lie group is *reductive*, i.e., its Lie algebra is the direct sum of a semisimple and an abelian ideal (note that any such Lie group cannot act simply transitively, see [12]). This naturally leads to the question whether these examples exhaust the class of such homogeneous manifolds when a reductive group

*Received July 2, 2020; accepted for publication August 16, 2021. The authors were supported by GNSAGA of INdAM and by the project PRIN 2017 "Real and Complex Manifolds: Topology, Geometry and Holomorphic Dynamics".

[†]Dipartimento di Matematica e Informatica "U. Dini", Università degli Studi di Firenze, Viale Morgagni 67/a, 50134 Firenze, Italy (fabio.podesta@unifi.it).

[‡]Dipartimento di Matematica "G. Peano", Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy (alberto.raffero@unito.it).

of automorphisms acts transitively. Here, we answer this question positively, proving the following result.

THEOREM 1.1. *Let* M *be a seven-dimensional manifold endowed with a closed non-parallel* G_2*-structure* φ*, and assume that there exists a transitive Lie subgroup* $G \subseteq \mathrm{Aut}(M, \varphi)$*. If* G *is reductive and acts irreducibly on* M*, then* M *is non-compact and*

i) *the group* G *has one-dimensional center and its semisimple part* G_s *is (locally) isomorphic to either* $\mathrm{SU}(2,1)$ *or* $\mathrm{SO}(4,1)$*;*
ii) *the universal cover of* M *is isomorphic to the product* $\mathcal{O}\times\mathbb{R}$*, where* $\mathcal{O}$ *is a coadjoint orbit of* G_s *endowed with a* G_s*-invariant strictly symplectic half-flat structure* (ω, ψ)*, and the product* $\mathcal{O}\times\mathbb{R}$ *is endowed with the induced* G_2*-structure* $\varphi = \omega \wedge ds + \psi$*.*

We recall that a transitive action of a Lie group G is called *irreducible* when no proper normal Lie subgroup of G acts transitively (see e.g. [24, p. 75] for terminology). Note that this assumption is not restrictive, as normal subgroups of reductive Lie groups are still reductive (see e.g. [2]).

We emphasize the following consequence of the above theorem.

COROLLARY 1.2. *Let* M *be a non-compact* 7*-manifold endowed with a closed non-parallel* G_2*-structure* φ*. If there exists a transitive Lie subgroup* $G \subseteq \mathrm{Aut}(M, \varphi)$*, then* G *cannot be semisimple.*

This work is structured as follows. In Section 2, we briefly review closed G_2-structures and some related facts about their automorphisms. In Section 3, we prove our main Theorem 1.1. The proof involves several arguments from the theory of Lie algebras and their representations together with more geometric considerations.

Notation. Lie groups and their Lie algebras will be indicated by capital and gothic letters, respectively. If a Lie group G acts on a manifold M, for every $X \in \mathfrak{g}$ we will denote by $\widehat{X}$ the vector field on M induced by the one-parameter subgroup $\exp(tX)$.

The abbreviation $e^{ijk\cdots}$ for the wedge product of covectors $e^i \wedge e^j \wedge e^k \wedge \cdots$ is used throughout the paper.

2. Preliminaries. A G_2-structure on a seven-dimensional manifold M is characterized by the existence of a 3-form $\varphi \in \Omega^3(M)$ which is *definite*, namely at each point p of M

$$0 \neq \iota_v\varphi \wedge \iota_v\varphi \wedge \varphi \in \Lambda^7(T_pM)^*, \quad \forall\, v \in T_pM \smallsetminus \{0\}.$$

Such a 3-form φ gives rise to an orientation on M and to a unique Riemannian metric g_φ such that

$$g_\varphi(v, w)\mathrm{vol}_{g_\varphi} = \frac{1}{6}\, \iota_v\varphi \wedge \iota_w\varphi \wedge \varphi,$$

for all $v, w \in T_pM$. A G_2-structure φ is said to be *parallel* if $\nabla^{g_\varphi}\varphi = 0$, where ∇^{g_φ} denotes the Levi Civita connection of g_φ. By [11], this is equivalent to φ being both *closed* ($d\varphi = 0$) and *coclosed* ($d *_\varphi \varphi = 0$). It is well-known that the Riemannian metric induced by a parallel G_2-structure is Ricci-flat.

We focus on the case when the G_2-structure φ is closed and non-parallel, namely $d\varphi = 0$ and $d *_\varphi \varphi \neq 0$, and we assume the existence of a connected Lie subgroup

$G \subseteq \mathrm{Aut}(M, \varphi)$ acting transitively on M. Then, M is necessarily non-compact by [26, Cor. 2.2], and we can write $M = G/H$, where the isotropy subgroup $H := G_p$ at some fixed point $p \in M$ is compactly embedded in G. As M is not compact, we may suppose that it is simply connected and, therefore, that H is connected.

From now on, we assume that the group G is non-compact and reductive. In particular, the Lie algebra $\mathfrak{g}$ of G can be written as $\mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{z}$, where $\mathfrak{g}_s$ is the Lie algebra of the maximal semisimple connected subgroup G_s of G and $\mathfrak{z}$ is the center of $\mathfrak{g}$. The given G_2-structure φ is not parallel, thus the metric g_φ is not flat, and the Lie algebra $\mathfrak{g}$ is not abelian, i.e., $\mathfrak{g}_s$ is not trivial.

Since ideals of reductive Lie algebras are also reductive (see e.g. [2]), we may also suppose that the G-action is irreducible. The following lemma, which will be also useful in the sequel, gives further restrictions on the isotropy subgroup.

LEMMA 2.1. *Suppose that* $\mathfrak{g} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ *for some non-trivial ideals* $\mathfrak{s}_1, \mathfrak{s}_2$ *of* $\mathfrak{g}$*, and denote by* $p_i : \mathfrak{g} \to \mathfrak{s}_i$*,* $i = 1, 2$*, the corresponding projections. Then,*

$$p_i(\mathfrak{h}) \neq \mathfrak{s}_i, \quad i = 1, 2.$$

In particular, the isotropy subgroup H *is contained in the semisimple subgroup* G_s*.*

Proof. Assume that $p_1(\mathfrak{h}) = \mathfrak{s}_1$, and let S_2 be the connected Lie subgroup of G with Lie algebra $\mathfrak{s}_2$. Then,

$$\dim(S_2 \cdot p) = \dim \mathfrak{s}_2 - \dim(\mathfrak{h} \cap \mathfrak{s}_2) = \dim \mathfrak{s}_2 - (\dim \mathfrak{h} - \dim \mathfrak{s}_1) = \dim \mathfrak{g} - \dim \mathfrak{h} = \dim(M)$$

implies that S_2 has an open orbit in M, hence it is transitive on M (see e.g. [17, p. 178]), contradicting the irreducibility of the G-action. To prove that $H \subseteq G_s$, let us consider the projection $pr : \mathfrak{g} \to \mathfrak{z}$ along $\mathfrak{g}_s$, and let us suppose that $\mathfrak{a} := pr(\mathfrak{h}) \neq \{0\}$. We then get a contradiction by putting $\mathfrak{s}_1 := \mathfrak{a}$ and $\mathfrak{s}_2 := \mathfrak{g}_s \oplus \mathfrak{b}$, where $\mathfrak{b} \subseteq \mathfrak{z}$ is a subspace with $\mathfrak{z} = \mathfrak{a} \oplus \mathfrak{b}$. □

The isotropy subgroup H is compactly embedded in G but it is not necessarily compact. We can then consider the closure $\overline{H}$ of H in the isometry group $\mathrm{Iso}(M, g_\varphi)$ or, equivalently, in the closed subgroup $\mathrm{Aut}(M, \varphi) \subseteq \mathrm{Iso}(M, g_\varphi)$. Notice that $\overline{H}$ is compact, as it is a closed subgroup of the compact group $\mathrm{Iso}(M, g_\varphi)_p$, and that H is a normal subgroup of $\overline{H}$. Since $\overline{H} \subset \mathrm{Aut}(M, \varphi)$, it embeds into G_2 via the isotropy representation at p. We discuss some useful properties of H and $\overline{H}$ in the next lemma.

LEMMA 2.2.

i) *The Lie algebra* $\mathfrak{h}$ *is a proper subalgebra of* $\mathfrak{g}_2$*;*
ii) *if the Lie subalgebra* $\overline{\mathfrak{h}} \subset \mathfrak{g}_2$ *is proper and non-abelian, then its dimension belongs to* $\{3, 4, 6, 8\}$ *and it is isomorphic to* $\mathfrak{so}(3)$*,* $\mathfrak{u}(2)$*,* $\mathfrak{so}(4)$ *and* $\mathfrak{su}(3)$*, respectively. In all these cases,* H *is compact;*
iii) *if* H *is not compact, then* $\dim H = 1$*.*

Proof.

i) If $H \cong G_2$, then (M, g_φ) is a two-point homogeneous space, as G_2 acts transitively on the unit 6-sphere (see e.g. [14]). By [13, Thm. 3], (M, g_φ) is then isometric to the Euclidean space $\mathbb{R}^7$. This implies that φ is parallel.
ii) It is well-known (see e.g. [25]) that maximal subalgebras of maximal rank in $\mathfrak{g}_2$ are given by $\mathfrak{su}(3)$ or $\mathfrak{so}(4)$, while $\mathfrak{so}(3)$ appears as the only maximal subalgebra of rank one. Moreover, a maximal subalgebra of $\mathfrak{su}(3)$ or $\mathfrak{so}(4)$ is isomorphic to $\mathfrak{u}(2)$ or $\mathfrak{so}(3)$. As $\mathfrak{h}$ is an ideal of $\overline{\mathfrak{h}}$, our claim follows.

iii) If H is not compact, then $\overline{\mathfrak{h}}$ must be either $\mathfrak{g}_2$ or abelian. If $\overline{\mathfrak{h}} = \mathfrak{g}_2$, it is simple and therefore $\mathfrak{h}$ is trivial by point i). Consequently, $\overline{\mathfrak{h}}$ is abelian and $\dim \overline{\mathfrak{h}} \leqslant 2$. If $\dim \mathfrak{h} = 2$, then $\mathfrak{h} = \overline{\mathfrak{h}}$ and H is compact. Thus, the only possibility is $\dim \mathrm{H} = 1$. □

3. Proof of the Main Theorem. In this section, we prove the Theorem 1.1. The proof is divided into three parts, according to $\mathfrak{g}$ being simple, semisimple not simple, and not semisimple.

3.1. Case $\mathfrak{g}$ simple. Using Lemma 2.2, we see that $\dim(\mathfrak{h}) \in \{0, 1, 2, 3, 4, 6, 8\}$. Therefore, we need to single out those real simple algebras whose dimensions belong to the set $\{7, 8, 9, 10, 11, 13, 15\}$. We recall that a real simple Lie algebra is either a real form or the realification of a complex simple Lie algebra. A direct inspection of the list of complex simple Lie algebras shows that $\dim(\mathfrak{g}) \in \{8, 10, 15\}$ and that $\mathfrak{g}$ is the real form of one of the following complex simple Lie algebras

$$\mathfrak{sl}(3,\mathbb{C}),\ \mathfrak{so}(5,\mathbb{C}),\ \mathfrak{sl}(4,\mathbb{C}).$$

When $\dim(\mathfrak{g}) = 15$, i.e., $\mathfrak{g}^c = \mathfrak{sl}(4,\mathbb{C})$, we see that $\dim(\mathfrak{h}) = 8$, hence $\mathfrak{h} \cong \mathfrak{su}(3)$. This forces $\mathfrak{g} = \mathfrak{su}(3,1)$ with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_o \oplus \mathfrak{n}$, where $\mathfrak{n}_o \cong \mathbb{R}$ and $\mathfrak{n} \cong \mathbb{C}^3$ is the standard SU(3)-module. The $\mathfrak{h}$-invariant 3-forms on the tangent space $T_p\mathrm{M} \cong \mathfrak{n}_o \oplus \mathfrak{n}$ lie in the modules $\mathfrak{n}_o^* \otimes (\Lambda^2\mathfrak{n}^*)^{\mathfrak{h}}$ and $(\Lambda^3\mathfrak{n}^*)^{\mathfrak{h}}$. We may select a basis $\{e_1, \ldots, e_6\}$ of $\mathfrak{n}$ and a basis vector e_7 of $\mathfrak{n}_o$ with $e_{2i} = [e_7, e_{2i-1}]$, $i = 1, 2, 3$. Let $\{e^1, \ldots, e^6, e^7\}$ be the corresponding dual basis of $(\mathfrak{n}_o \oplus \mathfrak{n})^*$. Then, the space $(\Lambda^3\mathfrak{n}^*)^{\mathfrak{h}}$ is generated by the forms

$$\gamma_1 := e^{135} - e^{146} - e^{236} - e^{245}, \quad \gamma_2 := e^{136} + e^{145} + e^{235} - e^{246},$$

and the module $\mathfrak{n}_o^* \otimes (\Lambda^2\mathfrak{n}^*)^{\mathfrak{h}}$ is spanned by $e^{127} + e^{347} + e^{567}$. Noting that $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{h} \oplus \mathfrak{n}_o$, we see that for any closed invariant 3-form ϕ we have

$$0 = d\phi(e_7, e_1, e_3, e_5) = -\phi(e_2, e_3, e_5) - \phi(e_1, e_4, e_5) - \phi(e_1, e_3, e_6),$$

so that ϕ has no component along γ_2. A similar argument shows that ϕ has no component along γ_1. This implies that ϕ cannot be definite, and the case $\mathfrak{g}^c = \mathfrak{sl}(4,\mathbb{C})$ can be ruled out.

We are then left with the cases $\mathfrak{g}^c = \mathfrak{sl}(3,\mathbb{C})$, $\mathfrak{so}(5,\mathbb{C})$. The possible pairs $(\mathfrak{g}, \mathfrak{h})$ corresponding to these Lie algebras are given in Table 1, where it is also specified how $\mathfrak{h}$ sits inside the maximal compactly embedded subalgebra of $\mathfrak{g}$ up to conjugation. We discuss each possibility separately. As we will see, in all cases there are no invariant 3-forms that are both closed and definite.

Case n.1. We have a reductive decomposition $\mathfrak{sl}(3,\mathbb{R}) = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h} \cong \mathfrak{so}(2)$. Note that $\mathfrak{h} \subset \mathfrak{so}(3)$, so that H is compact and $\mathfrak{h}$ can be assumed to be generated by any element in $\mathfrak{so}(3)$ up to conjugation. The tangent space $\mathfrak{m}$ splits into the sum of four ad($\mathfrak{h}$)-invariant submodules $\mathfrak{m} \cong \bigoplus_{i=0}^{3} \mathfrak{m}_i$, with $\dim(\mathfrak{m}_0) = 1$ and $\dim(\mathfrak{m}_i) = 2$, for $i = 1, 2, 3$. We can fix the following basis of the modules: $\mathfrak{m}_0 = \mathrm{Span}\{e_1\}$, $\mathfrak{m}_i = \mathrm{Span}\{e_{2i}, e_{2i+1}\}$, $i = 1, 2, 3$, and $\mathfrak{h} = \mathrm{Span}\{e_8\}$, where

$$e_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ e_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ e_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ e_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ e_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

n.	$\mathfrak{h}$	$\mathfrak{g}$	note
1	$\mathbb{R}$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{h} \subset \mathfrak{so}(3)$
2	$\mathbb{R}$	$\mathfrak{su}(2,1)$	$\mathfrak{h} \subset \mathfrak{u}(2) \subset \mathfrak{su}(2,1)$
3	$\mathfrak{so}(3)$	$\mathfrak{so}(3,2)$	$\mathfrak{h} \subset \mathfrak{so}(3) \oplus \mathfrak{so}(2) \subset \mathfrak{so}(3,2)$
4	$\mathfrak{so}(3)$	$\mathfrak{so}(4,1)$	$\mathfrak{h} = \mathfrak{so}(3) \oplus \{0\} \subset \mathfrak{so}(4) \subset \mathfrak{so}(4,1)$
5	$\mathfrak{so}(3)$	$\mathfrak{so}(4,1)$	$\mathfrak{h} = \mathfrak{so}(3)_{\text{diag}} \subset \mathfrak{so}(3) \oplus \mathfrak{so}(3) = \mathfrak{so}(4) \subset \mathfrak{so}(4,1)$

TABLE 1
Possible pairs $(\mathfrak{g},\mathfrak{h})$ *when* $\mathfrak{g}^c = \mathfrak{sl}(3,\mathbb{C})$ *or* $\mathfrak{g}^c = \mathfrak{so}(5,\mathbb{C})$.

The space $(\Lambda^3\mathfrak{m})^{\mathfrak{h}}$ has dimension seven and it is generated by the forms

$$\begin{gathered}\gamma_1 = e^{123}, \quad \gamma_2 = e^{145}, \quad \gamma_3 = e^{167},\\ \gamma_4 = e^{124} + e^{135}, \quad \gamma_5 = e^{125} - e^{134},\\ \gamma_6 = e^{246} - e^{257} - e^{347} - e^{356}, \quad \gamma_7 = e^{247} + e^{256} + e^{346} - e^{357}.\end{gathered}$$

The generic $\mathrm{ad}(\mathfrak{h})$-invariant 3-form is then given by $\phi = \sum_{i=1}^{7} a_i\,\gamma_i$, with $a_i \in \mathbb{R}$. Using the expression of the Chevalley-Eilenberg differential, we see that

$$d\phi(e_3, e_5, e_6, e_7) = -a_3.$$

On the other hand, we have

$$\iota_{e_7}\phi \wedge \iota_{e_7}\phi \wedge \phi = 6\left(a_6^2 + a_7^2\right) a_3\, e^{1234567}.$$

This shows that any closed invariant 3-form $\phi \in (\Lambda^3\mathfrak{m})^{\mathfrak{h}}$ cannot be definite.

Case n.2. The isotropy subalgebra $\mathfrak{h} \cong \mathbb{R} \subset \mathfrak{u}(2) \subset \mathfrak{su}(2,1) = \mathfrak{g}$ can be assumed to be spanned by $e_8 := \mathrm{diag}(i\alpha, i\beta, -i(\alpha+\beta))$, with $\alpha, \beta \in \mathbb{R}$ and $\alpha^2 + \beta^2 \neq 0$. Note that H might be non-compact (cf. point iii) of Lemma 2.2). We consider a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, and we observe that an $\mathrm{ad}(\mathfrak{h})$-irreducible decomposition of the tangent space $T_p\mathrm{M} \cong \mathfrak{m}$ is given by

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3,$$

with $\dim\mathfrak{m}_0 = 1$ and $\dim\mathfrak{m}_i = 2$, $i = 1,2,3$. We choose the following basis for the submodules

$$\mathfrak{m}_0 : \ e_1 = \begin{pmatrix} i(-2\beta-\alpha) & 0 & 0\\ 0 & i\,(2\alpha+\beta) & 0\\ 0 & 0 & i\,(\beta-\alpha)\end{pmatrix}, \quad \mathfrak{m}_1 : \ \left\{ e_2 = \begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ e_3 = \begin{pmatrix} 0 & i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right\},$$
$$\mathfrak{m}_2 : \ \left\{ e_4 = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix},\ e_5 = \begin{pmatrix} 0 & 0 & i\\ 0 & 0 & 0\\ -i & 0 & 0\end{pmatrix}\right\}, \quad \mathfrak{m}_3 : \ \left\{ e_6 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},\ e_7 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & i\\ 0 & -i & 0\end{pmatrix}\right\},$$

and we easily obtain

$$\mathrm{ad}(e_8)e_2 = (\alpha-\beta)\,e_3, \quad \mathrm{ad}(e_8)e_4 = (2\alpha+\beta)\,e_5, \quad \mathrm{ad}(e_8)e_6 = (\alpha+2\beta)\,e_7.$$

The $\mathrm{ad}(\mathfrak{h})$-module $\mathfrak{m}_0$ is trivial, and the following possibilities occur for $\mathfrak{m}_i$, $i = 1,2,3$:
a) the modules $\mathfrak{m}_i$ are non-trivial and precisely two of them are equivalent;

b) two of the modules $\mathfrak{m}_i$ are non-trivial and equivalent, while the remaining one is trivial;
c) the modules $\mathfrak{m}_i$ are non-trivial and mutually inequivalent.

Now, in each case a) - c) we can determine the expression of the generic closed invariant 3-form on $\mathfrak{m}$ as follows. First, we consider the generic invariant 3-form $\phi \in (\Lambda^3\mathfrak{m}^*)^{\mathfrak{h}}$ and we compute its exterior derivative $d\phi$ using the expression of the Chevalley-Eilenberg differential (and a software for symbolic computations, e.g. Maple, if needed). Then, we solve the system of linear equations in the coefficients of ϕ arising from the condition $d\phi = 0$. In detail, we obtain

a) $\dim\left((\Lambda^3\mathfrak{m})^{\mathfrak{h}}\right) = 7$. If $\mathfrak{m}_1 \cong \mathfrak{m}_2$, then the generic closed invariant 3-form is
$$\phi = a_1\left(e^{124} - e^{135}\right) + a_2\left(e^{125} + e^{134}\right) + a_3\left(e^{247} - e^{256} + e^{346} + e^{357}\right),$$
and it is not definite. The other possible cases $\mathfrak{m}_1 \cong \mathfrak{m}_3$ and $\mathfrak{m}_2 \cong \mathfrak{m}_3$ are dealt with similarly.

b) $\dim\left((\Lambda^3\mathfrak{m})^{\mathfrak{h}}\right) = 13$. If $\mathfrak{m}_1$ is trivial, the generic closed invariant 3-form is
$$\begin{aligned}\phi = {} & a_1\left(-3(e^{146} + e^{157}) + e^{245} - e^{267}\right) + a_2\left(-3\left(e^{147} - e^{156}\right) - e^{345} + e^{367}\right)\\ & + a_3\left(e^{247} - e^{256} + e^{346} + e^{357}\right),\end{aligned}$$
and it is not definite. The cases $\mathfrak{m}_2$ or $\mathfrak{m}_3$ trivial are analogous.

c) $\dim\left((\Lambda^3\mathfrak{m})^{\mathfrak{h}}\right) = 5$ and the generic closed invariant 3-form is
$$\phi = a_1\left(-e^{247} + e^{256} - e^{346} - e^{357}\right),$$
and it is not definite.

Case n.3. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k} = \mathfrak{so}(3) + \mathfrak{so}(2)$. Then, $\mathfrak{h} = \mathfrak{so}(3)$ is the semisimple part of $\mathfrak{k}$ and $\mathfrak{v} := \mathfrak{so}(2)$ is the center of $\mathfrak{k}$. The $\mathrm{ad}(\mathfrak{h})$-module $\mathfrak{p}$ splits as the sum $\mathfrak{p} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$, where $\mathfrak{n}_1 \cong \mathfrak{n}_2 \cong \mathbb{R}^3$ are equivalent modules. We may select a basis $\{e_1, e_2, e_3\}$ of $\mathfrak{n}_1$ and a basis vector e_7 of $\mathfrak{v}$ in such a way that $\{e_{i+3} := [e_i, e_7]\}_{i=1,2,3}$ is a basis of $\mathfrak{n}_2$. The space of $\mathrm{ad}(\mathfrak{h})$-invariant 3-forms on $T_p\mathrm{M} \cong \mathfrak{v} \oplus \mathfrak{p}$ decomposes into the sum of five one-dimensional submodules as follows
$$\Lambda^3(\mathfrak{v} \oplus \mathfrak{p})^{\mathfrak{h}} = \Lambda^3\mathfrak{n}_1 \oplus (\Lambda^2\mathfrak{n}_1 \otimes \mathfrak{n}_2)^{\mathfrak{h}} \oplus (\mathfrak{n}_1 \otimes \Lambda^2\mathfrak{n}_2)^{\mathfrak{h}} \oplus \left(\mathfrak{v} \otimes (\mathfrak{n}_1 \otimes \mathfrak{n}_2)^{\mathfrak{h}}\right) \oplus \Lambda^3\mathfrak{n}_2.$$
From this, we immediately see that a basis of invariant 3-forms is given by
$$\gamma_1 := e^{123},\ \gamma_2 := e^{126} - e^{135} + e^{234},\ \gamma_3 := e^{156} - e^{246} + e^{345},\ \gamma_4 := e^{147} + e^{257} + e^{367},\ \gamma_5 := e^{456}.$$
The generic invariant 3-form $\phi = \sum_{i=1}^5 a_i\gamma_i$ satisfies
$$\iota_{e_7}\phi \wedge \iota_{e_7}\phi \wedge \phi = -6\, a_4^3\, e^{1234567},$$
and we have
$$d\phi(e_1, e_2, e_4, e_5) = 2\, a_4.$$
Consequently, any closed invariant 3-form is not definite.

Case n.4. We consider the Lie algebra

$$\mathfrak{so}(4,1) = \left\{ \left(\begin{smallmatrix} 0 & v \\ {}^t v & A \end{smallmatrix} \right) \mid A \in \mathfrak{so}(4),\ v \in \mathbb{R}^4 \right\},$$

and the ideals of $\mathfrak{so}(4) \subset \mathfrak{so}(4,1)$ given by

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c & -b \\ -b & -c & 0 & a \\ -c & b & -a & 0 \end{pmatrix},\ a,b,c \in \mathbb{R} \right\}, \quad \mathfrak{p} = \left\{ \begin{pmatrix} 0 & r & s & t \\ -r & 0 & -t & s \\ -s & t & 0 & -r \\ -t & -s & r & 0 \end{pmatrix},\ r,s,t \in \mathbb{R} \right\},$$

so that $\mathfrak{so}(4) = \mathfrak{h} \oplus \mathfrak{p}$, $\mathfrak{h} \cong \mathfrak{so}(3)$. We have the $\mathrm{ad}(\mathfrak{h})$-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} \oplus \mathfrak{n}$, where $\mathfrak{n} \cong \mathbb{R}^4$ via the map $\mathfrak{g} \ni \left(\begin{smallmatrix} 0 & v \\ {}^t v & 0 \end{smallmatrix} \right) \mapsto v \in \mathbb{R}^4$. Moreover, we have

$$[\mathfrak{h}, \mathfrak{p}] = 0, \quad [\mathfrak{n}, \mathfrak{n}] = \mathfrak{h} + \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{n}] \subseteq \mathfrak{n}.$$

We select the following basis of $\mathfrak{p}$

$$e_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad e_3 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

with the standard relations $[e_i, e_j] = 2\epsilon_{ijk} e_k$, for $i,j,k \in \{1,2,3\}$. Moreover, we consider the canonical basis of $\mathbb{R}^4 \cong \mathfrak{n}$ and we denote it by $\{e_4, e_5, e_6, e_7\}$.

The space of $\mathrm{ad}(\mathfrak{h})$-invariant 3-forms $\Lambda^3(\mathfrak{p} \oplus \mathfrak{n})^{\mathfrak{h}}$ can be decomposed as

$$\Lambda^3(\mathfrak{p} \oplus \mathfrak{n})^{\mathfrak{h}} = \Lambda^3 \mathfrak{p} \oplus \left(\mathfrak{p} \otimes (\Lambda^2 \mathfrak{n})^{\mathfrak{h}} \right).$$

A basis of $(\Lambda^2 \mathfrak{n})^{\mathfrak{h}}$ is given by

$$\omega_1 := e^{45} - e^{67}, \quad \omega_2 := e^{46} + e^{57}, \quad \omega_3 := e^{47} - e^{56},$$

so that a basis of invariant 3-forms is $\{e^{123},\ e^i \wedge \omega_j\}_{i,j=1,2,3}$. We consider the generic invariant 3-form

$$\begin{aligned} \phi = {} & a_1 e^{123} + a_2 e^1 \wedge \omega_1 + a_3 e^1 \wedge \omega_2 + a_4 e^1 \wedge \omega_3 + a_5 e^2 \wedge \omega_1 + a_6 e^2 \wedge \omega_2 \\ & + a_7 e^2 \wedge \omega_3 + a_8 e^3 \wedge \omega_1 + a_9 e^3 \wedge \omega_2 + a_{10} e^3 \wedge \omega_3, \end{aligned}$$

and we notice that

$$\iota_{e_1} \phi \wedge \iota_{e_1} \phi \wedge \phi = -6\, a_1 \left(a_2^2 + a_3^2 + a_4^2 \right) e^{1234567}.$$

Now, we have

$$\begin{aligned} & d\phi(e_2, e_3, e_6, e_7) = \frac{1}{2} a_1 + 2a_2 - 2a_6 - 2a_{10}, d\phi(e_1, e_3, e_5, e_7) = \frac{1}{2} a_1 - 2a_2 + 2a_6 - 2a_{10}, \\ & d\phi(e_1, e_2, e_5, e_6) = \frac{1}{2} a_1 - 2a_2 - 2a_6 + 2a_{10}, d\phi(e_4, e_5, e_6, e_7) = a_2 + a_6 + a_{10}. \end{aligned}$$

Thus, if ϕ is closed, then $a_1 = a_2 = a_6 = a_{10} = 0$ and ϕ is not definite.

Case n.5. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k} = \mathfrak{so}(4)$ and $\mathfrak{h} \cong \mathfrak{so}(3) \subset \mathfrak{k}$. We can consider the $\mathrm{ad}(\mathfrak{h})$-invariant decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{n}_1$, with $[\mathfrak{n}_1, \mathfrak{n}_1] \subseteq \mathfrak{h}$. The $\mathrm{ad}(\mathfrak{h})$-module $\mathfrak{p}$ splits as $\mathfrak{p} = \mathfrak{n}_2 \oplus \mathfrak{v}$, where $\mathfrak{v} = \mathbb{R}V$ for some $V \in \mathfrak{p}$ satisfying the following properties

$$[V, \mathfrak{n}_1] = \mathfrak{n}_2, \ [V, \mathfrak{n}_2] = \mathfrak{n}_1, \ [\mathfrak{n}_2, \mathfrak{n}_2] \subseteq \mathfrak{h}.$$

We choose the following basis for $\mathfrak{h}$

$$e_8 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&-1&0&0\\ 0&0&0&0&0 \end{pmatrix}, \quad e_9 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&-1&0&0 \end{pmatrix}, \quad e_{10} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&-1&0 \end{pmatrix},$$

and the following basis for the irreducible summands of the tangent space $\mathfrak{m} \cong \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{v}$

$$\mathfrak{n}_1 : e_1 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&-1&0&0&0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&-1&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&1&0&0\\ 0&-1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix},$$

$$\mathfrak{n}_2 : e_4 = \begin{pmatrix} 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix},$$

$$\mathfrak{v} : e_7 = \begin{pmatrix} 0&1&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}.$$

We then see that $(\Lambda^3\mathfrak{m})^{\mathfrak{h}}$ splits into the sum of the following one-dimensional submodules

$$(\Lambda^3\mathfrak{m})^{\mathfrak{h}} \cong \Lambda^3\mathfrak{n}_1 \oplus \Lambda^3\mathfrak{n}_2 \oplus (\Lambda^2\mathfrak{n}_1 \otimes \mathfrak{n}_2)^{\mathfrak{h}} \oplus (\mathfrak{n}_1 \otimes \Lambda^2\mathfrak{n}_2)^{\mathfrak{h}} \oplus (\mathfrak{n}_1 \otimes \mathfrak{n}_2 \otimes \mathfrak{v})^{\mathfrak{h}},$$

and a basis of invariant 3-forms is given by

$$\gamma_1 = e^{123}, \ \gamma_2 = e^{456}, \ \gamma_3 = e^{126} - e^{135} + e^{234}, \ \gamma_4 = e^{156} - e^{246} + e^{345}, \ \gamma_5 = e^{147} + e^{257} + e^{367}.$$

The generic invariant 3-form $\phi = \sum_{i=1}^{5} a_i\gamma_i$ satisfies

$$\iota_{e_7}\phi \wedge \iota_{e_7}\phi \wedge \phi = -6\,(a_5)^3\, e^{1234567}.$$

If ϕ is closed, then

$$0 = d\phi(e_1, e_2, e_4, e_5) = d\phi(e_1, e_3, e_4, e_6) = d\phi(e_2, e_3, e_5, e_6) = 2\,a_5.$$

Thus, no closed invariant 3-form on $\mathfrak{m}$ can be definite.

3.2. Case $\mathfrak{g}$ semisimple not simple.

We begin with the following.

Lemma 3.1. *Let $\mathfrak{g}$ be semisimple, not simple. Then, any simple factor of $\mathfrak{g}$ appears in the following list:*
- real forms of $\mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{sl}(3,\mathbb{C})$, $\mathfrak{so}(5,\mathbb{C})$;
- the realification of $\mathfrak{sl}(2,\mathbb{C})$.
Moreover, the dimension of the isotropy subalgebra $\mathfrak{h}$ belongs to $\{2, 4, 6\}$.

Proof. By Lemma 2.2 we have that $\dim(\mathfrak{h}) \leqslant 8$, whence $\dim(\mathfrak{g}) \leqslant 15$. The possible simple factors of $\mathfrak{g}$ can be deduced from a direct inspection of the list of complex simple Lie algebras with dimension at most 15 (see e.g. [14, Ch. X, Table

IV]), together with the fact that $\mathfrak{g}$ is not simple. From this, we see that $\dim(\mathfrak{h}) \geqslant 2$, as there are no semisimple real algebras of dimension 7 and a semisimple real Lie algebra of dimension 8 is simple. Noting that $\dim(\mathfrak{h}) = 3$ cannot occur as $\dim(\mathfrak{g}) = 10$ implies that $\mathfrak{g}$ is simple, we have that either $\dim(\mathfrak{h}) = 2$ or $\dim(\mathfrak{h}) \geqslant 4$. The dimensions $\dim(\mathfrak{h}) = 5, 7$ are ruled out using Lemma 2.2, as $\mathfrak{h}$ is an ideal of $\overline{\mathfrak{h}}$. If $\dim(\mathfrak{h}) = 8$, again by Lemma 2.2 we have that $\mathfrak{h} \cong \mathfrak{su}(3)$ and $\dim(\mathfrak{g}) = 15$. We write $\mathfrak{g} = \bigoplus_{i=1}^{k} \mathfrak{s}_i$ as a sum of its simple factors, with $\dim(\mathfrak{s}_i) \in \{3, 6, 8, 10\}$, $i = 1, \ldots, k$, $k \geqslant 2$. As $\mathfrak{h}$ is simple, at least one simple factor of $\mathfrak{g}$ has dimension 8 or 10 so that $\sum_{i=1}^{k} \dim \mathfrak{s}_i \neq 15$. Therefore, $\dim(\mathfrak{h}) = 8$ is excluded. □

The previous lemma allows us to describe all possibilities for the pair $(\mathfrak{g}, \mathfrak{h})$. They are listed in Table 2.

n.	$\mathfrak{h}$	$\mathfrak{g}$	note
1	$2\mathbb{R}$	$3\mathfrak{s}$	$\mathfrak{s}^c = \mathfrak{sl}(2, \mathbb{C})$
2	$2\mathbb{R}$	$\mathfrak{s}_1 \oplus \mathfrak{s}_2$	$\mathfrak{s}_1^c = \mathfrak{sl}(2, \mathbb{C}), \mathfrak{s}_2 = \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$
3	$\mathfrak{u}(2)$	$\mathfrak{s}_1 \oplus \mathfrak{s}_2$	$\mathfrak{s}_1^c = \mathfrak{sl}(2, \mathbb{C}), \mathfrak{s}_2^c = \mathfrak{sl}(3, \mathbb{C})$
4	$\mathfrak{so}(4)$	$\mathfrak{s}_1 \oplus \mathfrak{s}_2$	$\mathfrak{s}_1^c = \mathfrak{sl}(2, \mathbb{C}), \mathfrak{s}_2^c = \mathfrak{so}(5, \mathbb{C})$

TABLE 2
Possible pairs $(\mathfrak{g}, \mathfrak{h})$ when $\mathfrak{g}$ is semisimple and not simple.

The following proposition rules out the cases n.2 and n.4 of Table 2. We will deal with the remaining pairs separately.

PROPOSITION 3.2. *The pairs $(\mathfrak{g}, \mathfrak{h})$ appearing as n.2 and n.4 in Table 2 correspond to homogeneous spaces with no invariant G_2-structures.*

Proof. In case n.2, the isotropy subgroup H embeds as a maximal torus in G_2. Hence, the tangent space T_pM contains three inequivalent real 2-dimensional H-modules. Now, as any abelian subspace of $\mathfrak{s}_1$ is one-dimensional, we see that $\mathfrak{h}$ projects onto a non trivial compactly-embedded subalgebra $\mathfrak{l}$ of $\mathfrak{s}_2$. Up to an inner automorphism, we may suppose that $\mathfrak{l} \subset \mathfrak{u} \cong \mathfrak{su}(2)$, where $\mathfrak{s}_2 = \mathfrak{u} \oplus i\mathfrak{u}$ is a Cartan decomposition. Hence, $\mathfrak{h} \subseteq \mathfrak{s}_1 \oplus \mathfrak{l} = \mathfrak{h} \oplus \mathfrak{q}$, for some $\mathrm{ad}(\mathfrak{h})$-invariant submodule $\mathfrak{q}$. Considering an $\mathrm{ad}(\mathfrak{h})$-invariant decomposition $\mathfrak{u} = \mathfrak{l} \oplus \mathfrak{n}$, we see that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \oplus \mathfrak{n} \oplus i\mathfrak{u}$, showing that the isotropy representation of $\mathfrak{h}$ contains $\mathfrak{n}$ with multiplicity two, a contradiction.

In case n.4, the projection of $\mathfrak{h}$ into $\mathfrak{s}_1$ is not surjective by Lemma 2.1 and, therefore, it is trivial. Thus, the linear isotropy representation has a fixed point set of dimension at least 3. On the other hand, the existence of an invariant G_2-structure implies that $\mathfrak{h}$ embeds into $\mathfrak{g}_2$, and the fact that $SO(4) \subset G_2$ has trivial fixed point set in $\mathbb{R}^7$ gives a contradiction. □

In the following propositions, we consider the remaining cases n.1 and n.3.

PROPOSITION 3.3. *In case n.1, there exists no invariant closed G_2-structure.*

Proof. Let $\mathfrak{g} = \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \mathfrak{s}_3$, where $\mathfrak{s}_j^c \cong \mathfrak{sl}(2, \mathbb{C})$, and suppose there exists an invariant G_2-structure. It then follows that the isotropy $\mathfrak{h}$ can be realized as a maximal abelian subalgebra of $\mathfrak{g}_2$. Hence, as an $\mathfrak{h}$-module we have $T_p\mathrm{M} \cong V_o \oplus \bigoplus_{j=1}^{3} V_j$, where

V_o is a one-dimensional trivial module, while $V_j \cong \mathbb{R}^2$, $j = 1, 2, 3$, are mutually inequivalent irreducible submodules. This implies that each projection of $\mathfrak{h}$ into the simple factors $\mathfrak{s}_j$ of $\mathfrak{g}$ is not trivial, otherwise the isotropy representation would have a trivial module of dimension at least three.

If we select $A := \mathrm{diag}(i, -i) \in \mathfrak{sl}(2, \mathbb{C})$, we can suppose that

$$\mathfrak{h} = \{(\alpha_1(v)A, \alpha_2(v)A, \alpha_3(v)A) \in \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \mathfrak{s}_3 \mid v \in \mathfrak{h}\},$$

for suitable nonzero $\alpha_j \in \mathfrak{h}^*$, $j = 1, 2, 3$, with $\alpha_j \neq \pm\alpha_k$ if $j \neq k$, and $\sum_j \alpha_j = 0$, as $\mathfrak{h}$ embeds into $\mathfrak{su}(3) \subset \mathfrak{g}_2$. We also fix $\mathfrak{p}_j \cong \mathbb{R}^2$ in $\mathfrak{s}_j$, with $\mathfrak{s}_j = \mathbb{R}A \oplus \mathfrak{p}_j$ being a Cartan decomposition.

Now, if we set $V := (A, A, A) \in \mathfrak{g}$ and $\mathfrak{v} := \mathbb{R}V$, then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v} \oplus \bigoplus_{j=1}^3 \mathfrak{p}_j$ and the tangent space $T_p\mathrm{M}$ identifies with $\mathfrak{m} := \mathfrak{v} \oplus \bigoplus_{j=1}^3 \mathfrak{p}_j$. Consequently, we have

$$(\Lambda^3\mathfrak{m})^{\mathfrak{h}} \cong \left(\bigoplus_{j=1}^3 \mathfrak{v} \otimes \Lambda^2\mathfrak{p}_j\right) \oplus (\mathfrak{p}_1 \otimes \mathfrak{p}_2 \otimes \mathfrak{p}_3)^{\mathfrak{h}}.$$

We now fix a basis $\{e_1, \ldots, e_7\}$ of $\mathfrak{m}$ with $e_7 := V$, $e_1, e_2 \in \mathfrak{p}_1$, $e_3, e_4 \in \mathfrak{p}_2$ and $e_5, e_6 \in \mathfrak{p}_3$ so that $\mathrm{ad}(V)|_{\mathfrak{p}_j} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$, for $j = 1, 2, 3$. Then, with respect to the dual basis $\{e^1, \ldots, e^7\}$, the forms

$$\gamma_1 := e^{127}, \quad \gamma_2 := e^{347}, \quad \gamma_3 := e^{567},$$
$$\gamma_4 := e^{135} - e^{146} - e^{236} - e^{245}, \quad \gamma_5 := e^{145} + e^{136} + e^{235} - e^{246},$$

span the space of $\mathrm{ad}(\mathfrak{h})$-invariant 3-forms on $\mathfrak{m}$. Any such ϕ can be written as $\phi = \sum_{j=1}^5 a_j\gamma_j$, for $a_j \in \mathbb{R}$. If ϕ is closed, then

$$0 = d\phi(e_7, e_1, e_3, e_5) = -2(\phi(e_2, e_3, e_5) + \phi(e_1, e_4, e_5) + \phi(e_1, e_3, e_6)) = -6a_5.$$

Similarly, we get $a_4 = 0$. Therefore, $\phi \in \mathrm{Span}(\gamma_1, \gamma_2, \gamma_3)$ and it cannot be definite. □

PROPOSITION 3.4. *In case n.3, there exists no invariant closed* G_2*-structure.*

Proof. Suppose there exists an invariant closed G_2-structure. We let $\mathfrak{h} := \mathfrak{h}_s \oplus \mathbb{R}Z$, where $\mathfrak{h}_s \cong \mathfrak{su}(2)$ and Z generates the center of $\mathfrak{h}$, and we denote by $\mathrm{pr}_j : \mathfrak{g} \to \mathfrak{s}_j$, $j = 1, 2$, the projections onto the simple factors of $\mathfrak{g}$. By Lemma 2.1, we may suppose that $\mathrm{pr}_1(\mathfrak{h}) \neq \mathfrak{s}_1$. We claim that $\mathrm{pr}_1(\mathfrak{h}) \neq \{0\}$. Indeed, if $\mathfrak{h} \subset \mathfrak{s}_2$, then the fixed point set of the isotropy representation would be at least 3-dimensional, while H embeds as a maximal rank subgroup of G_2, whence its fixed point set is at most one-dimensional. Therefore, $\mathrm{pr}_1(\mathfrak{h}_s) = \{0\}$ and $A := \mathrm{pr}_1(Z) \neq 0$. We can also suppose that $A = \mathrm{diag}(i, -i) \in \mathfrak{sl}(2, \mathbb{C})$, with $\mathfrak{s}_1 = \mathbb{R}A \oplus \mathfrak{p}$ a Cartan decomposition.

We now claim that $\mathrm{pr}_2(Z) \neq 0$. Indeed, otherwise $Z = A \in \mathfrak{s}_1$ would act trivially on a five-dimensional subspace of the tangent space $T_p\mathrm{M} \cong \mathfrak{g}/\mathfrak{h}$. The torus T^1 generated by Z embeds into G_2, hence into $\mathrm{SU}(3) \subset \mathrm{G}_2$ up to conjugation. As any non-trivial element of $\mathrm{SU}(3)$ has a fixed point set in $\mathbb{C}^3$ of complex dimension at most one, we see that the fixed point set of T^1 in $\mathbb{R}^7$ has real dimension at most three. This gives a contradiction.

The ideal $\mathfrak{s}_2$ is isomorphic to one of $\mathfrak{su}(3), \mathfrak{su}(1, 2), \mathfrak{sl}(3, \mathbb{R})$, and we claim that the last possibility cannot occur. Indeed $\mathfrak{h}_s \cong \mathfrak{so}(3) \subset \mathfrak{s}_2 \cong \mathfrak{sl}(3, \mathbb{R})$ would have a trivial centralizer in $\mathfrak{s}_2$, while $\mathrm{pr}_2(Z) \neq 0$. Therefore, we can suppose that $\mathfrak{h}_s =$

$\{\mathrm{diag}(0,A) \in \mathfrak{sl}(3,\mathbb{C}) \mid A \in \mathfrak{su}(2)\}$ and $B := \mathrm{pr}_2(Z) = \mathrm{diag}(2ia, -ia - ia) \in \mathfrak{sl}(3,\mathbb{C})$, for some nonzero $a \in \mathbb{R}$. Then, we can fix an $\mathrm{ad}(\mathfrak{h})$-invariant decomposition $\mathfrak{s}_2 = (\mathfrak{h}_s \oplus \mathbb{R}B) \oplus \mathfrak{n}$, and we may consider some nonzero $V \in \mathrm{Span}\{A, B\}$ so that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v} \oplus \mathfrak{p} \oplus \mathfrak{n}, \quad \mathfrak{v} := \mathbb{R}V,$$

is an $\mathrm{ad}(\mathfrak{h})$-invariant decomposition of $\mathfrak{g}$ and $T_p\mathrm{M}$ can be identified with $\mathfrak{m} := \mathfrak{v}\oplus\mathfrak{p}\oplus\mathfrak{n}$. We choose

$$V = \mathrm{diag}(i, -i) \oplus \mathrm{diag}(2bi, -bi, -bi), \quad b \neq 0, a.$$

We let $e_7 := V$, and we select a basis $\{e_1, \dots, e_4\}$ of $\mathfrak{n}$ and a basis $\{e_5, e_6\}$ of $\mathfrak{p}$ so that

$$\begin{aligned} [e_1, e_2]_{\mathfrak{m}} &= \frac{\eta}{b-a}\, e_7, \quad [e_3, e_4]_{\mathfrak{m}} = \frac{\eta}{b-a}\, e_7, \\ [e_5, e_6]_{\mathfrak{m}} &= \frac{2a\varepsilon}{a-b} e_7, \quad [e_i, e_j]_{\mathfrak{m}} = 0,\ i = 1, 2, j = 3, 4, \end{aligned}$$

$$\mathrm{ad}(e_7)|_{\mathfrak{p}} = \mathrm{ad}(Z)|_{\mathfrak{p}} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad \mathrm{ad}(e_7)|_{\mathfrak{n}} = \begin{pmatrix} 0 & -3b & 0 & 0 \\ 3b & 0 & 0 & 0 \\ 0 & 0 & 0 & -3b \\ 0 & 0 & 3b & 0 \end{pmatrix} = \frac{b}{a}\, \mathrm{ad}(Z)|_{\mathfrak{n}},$$

where $\varepsilon, \eta = \pm 1$ according to the Lie algebras $\mathfrak{s}_1, \mathfrak{s}_2$ being of compact or non-compact type. We have the following $\mathrm{ad}(\mathfrak{h})$-invariant decomposition

$$(\Lambda^3\mathfrak{m})^{\mathfrak{h}} = (\mathfrak{v} \otimes \Lambda^2\mathfrak{p}) \oplus (\mathfrak{v} \otimes (\Lambda^2\mathfrak{n})^{\mathfrak{h}}) \oplus (\mathfrak{p} \otimes \Lambda^2\mathfrak{n})^{\mathfrak{h}}.$$

A straightforward computation shows that

$$\dim(\mathfrak{p} \otimes \Lambda^2\mathfrak{n})^{\mathfrak{h}} = \begin{cases} 2, & \text{if } a = \frac{1}{3}, \\ 0, & \text{otherwise.} \end{cases}$$

Let us denote by $\{e^1, \dots, e^7\}$ the dual 1-forms. When $a \neq \frac{1}{3}$, a basis of $(\Lambda^3\mathfrak{m})^{\mathfrak{h}}$ is given by

$$\gamma_1 := e^{567}, \quad \gamma_2 := e^{127} + e^{347}.$$

Clearly, these forms do not span any definite 3-form. When $a = \frac{1}{3}$, a basis of invariant 3-forms is given by

$$\gamma_1, \quad \gamma_2, \quad \gamma_3 := e^{135} + e^{146} + e^{236} - e^{245}, \quad \gamma_4 := e^{136} - e^{145} - e^{246} - e^{235}.$$

In this case, the generic $\mathrm{ad}(\mathfrak{h})$-invariant 3-form ϕ can be written as $\phi = \sum_{j=1}^4 c_j\gamma_j$, for some $c_j \in \mathbb{R}$. If ϕ is closed, then

$$0 = d\phi(e_7, e_5, e_1, e_3) = -2\,\phi(e_1, e_3, e_6) - 3b\,\phi(e_2, e_3, e_5) - 3b\,\phi(e_1, e_4, e_5) = (6b - 2)\, c_4,$$

whence $c_4 = 0$, as $b \neq a = \frac{1}{3}$. Similarly, from $d\phi(e_1, e_3, e_6, e_7) = 0$, we obtain $c_3 = 0$. It then follows that ϕ is not definite. □

3.3. Case $\mathfrak{g}$ not semisimple. In this last case, we have $\mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{z}$, with $\mathfrak{z}$ non-trivial. We start noting that for every $Z \in \mathfrak{z}$ the 2-form $\omega_Z := \iota_{\widehat{Z}}\varphi$ is G-invariant and closed, as $d\omega_Z = \mathcal{L}_{\widehat{Z}}\varphi - \iota_{\widehat{Z}}d\varphi = 0$. Consequently, if $X, Y \in \mathfrak{g}_s$ and $V \in \mathfrak{z}$, we have

$$0 = d\omega_Z\left(\widehat{X}, \widehat{Y}, \widehat{V}\right) = \omega_Z\left(\left[\widehat{X}, \widehat{Y}\right], \widehat{V}\right).$$

As $\mathfrak{g}_s$ is semisimple, it satisfies $\mathfrak{g}_s = [\mathfrak{g}_s, \mathfrak{g}_s]$. Therefore

$$\omega_Z\left(\widehat{A}, \widehat{V}\right) = 0, \quad \forall\, A \in \mathfrak{g}_s,\ V \in \mathfrak{z}. \tag{3.1}$$

Let G_s denote the connected Lie subgroup of G with Lie algebra $\mathfrak{g}_s$. The G_s-orbit $\mathcal{O} := \mathrm{G}_s \cdot p \cong \mathrm{G}_s/\mathrm{H}$ is a proper submanifold of M, and we may select a nonzero $Z \in \mathfrak{z}$ so that $\widehat{Z}_p \notin T_p\mathcal{O}$. We claim that the pull-back of ω_Z to $\mathcal{O}$ is an invariant symplectic form. Indeed, if $X \in \mathfrak{g}_s$ satisfies $\omega_Z\left(\widehat{X}_p, \widehat{Y}_p\right) = 0$ for all $Y \in \mathfrak{g}_s$, then from (3.1) we see that $\widehat{X}_p$ must lie in the kernel of $\omega_Z|_p$. Thus, $\widehat{X}_p$ must be a multiple of $\widehat{Z}_p$, hence zero. Therefore, $\dim\mathcal{O} \in \{2, 4, 6\}$.

Lemma 3.5. *The orbit $\mathcal{O}$ has dimension six.*

Proof. We first prove that $\dim\mathcal{O} \geqslant 4$. Let $\hat{\mathfrak{z}}|_p := \left\{\widehat{V}_p \in T_p\mathrm{M} \mid V \in \mathfrak{z}\right\}$. If $\dim\mathcal{O} = 2$, then we have $\dim\hat{\mathfrak{z}}|_p = 5$. Given a nonzero $X \in \mathfrak{g}_s$, by (3.1) we know that $\iota_{\widehat{X}}\varphi(V_1, V_2) = 0$ for all $V_1, V_2 \in \hat{\mathfrak{z}}|_p$. Now $\iota_{\widehat{X}}\varphi$ is non-degenerate on $U := \mathrm{Span}\{\widehat{X}_p\}^\perp$ and it vanishes on the subspace $U \cap \hat{\mathfrak{z}}_p$, which has dimension at least 4, a contradiction.

Suppose now that $\dim\mathcal{O} = 4$. Since $\mathfrak{h} \subset \mathfrak{g}_s$ and $\mathcal{O}$ is symplectic, $\mathfrak{h}$ is not trivial (see e.g. [1]), and a maximal abelian subalgebra of $\mathfrak{h}$ has dimension $\mathrm{rk}(\mathfrak{g}_s^c) \geqslant 2$. It follows that there exists a 2-torus T^2 in H whose fixed point set in $T_p\mathrm{M}$ has dimension 3. On the other hand, T^2 embeds as a maximal torus of G_2, which has a one-dimensional fixed point set in $\mathbb{R}^7$, a contradiction. □

Since the G-action is irreducible and $\dim\mathcal{O} = 6$, we necessarily have $\dim(\mathfrak{z}) = 1$, so that $\mathfrak{z} = \mathrm{Span}\{Z\}$. We claim that $\widehat{Z}_p$ belongs to the orthogonal complement of $T_p\mathcal{O}$ in $T_p\mathrm{M}$ with respect to the invariant Riemannian metric g_φ. Indeed, we may consider an $\mathrm{ad}(\mathfrak{h})$-invariant decomposition $\mathfrak{g}_s = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m} \cong T_p\mathcal{O}$. The invariant symplectic form on $\mathcal{O}$ corresponds to an Ad(H)-invariant symplectic form on $\mathfrak{m}$ which can be written as $B(\mathrm{ad}(W)\cdot, \cdot)$ for a unique $W \in \mathfrak{g}_s$, where B denotes the Cartan-Killing form of $\mathfrak{g}_s$. Moreover, $\mathfrak{h}$ coincides with the centralizer of W in $\mathfrak{g}_s$ (see, for instance, [1]). Consequently, we have $\mathfrak{m}^{\mathrm{H}} = \{0\}$. This implies that the orthogonal projection of $\widehat{Z}_p$ on $T_p\mathcal{O}$, being invariant under $\mathrm{ad}(\mathfrak{h})$, is trivial.

Let ψ denote the closed G_s-invariant 3-form on $\mathcal{O}$ obtained by pulling back the invariant closed G_2-structure φ on M. To conclude the proof of the main theorem, we need to show that the pair (ω_Z, ψ) defines a G_s-invariant SU(3)-structure on the six-dimensional homogeneous space $\mathcal{O} = \mathrm{G}_s/\mathrm{H}$. Since both ω_Z and ψ are closed and φ is not parallel, the SU(3)-structure will be strictly symplectic half-flat, namely $d{*}\psi \neq 0$, where $*$ is the Hodge operator relative to the metric induced by (ω_Z, ψ). In particular, the orbit $\mathcal{O}$ is non-compact (see [27, Prop. 4.2]).

Now, identifying the invariant closed G_2-structure φ on M with the corresponding $\mathrm{ad}(\mathfrak{h})$-invariant definite 3-form φ on $\mathfrak{m} \oplus \mathfrak{z} \cong T_p\mathrm{M}$, we see that $\varphi = \omega_Z \wedge \eta + \psi$, where $\eta \in \mathfrak{z}^*$ is dual to Z. Since φ is definite, the pair of $\mathrm{ad}(\mathfrak{h})$-invariant forms (ω_Z, ψ) on $\mathfrak{m}$ defines an SU(3)-structure.

Summing up, the orbit $\mathcal{O} = \mathrm{G}_s \cdot p$ is a non-compact G_s-homogeneous six-dimensional manifold endowed with an invariant strictly symplectic half-flat SU(3)-structure. By the classification result [27, Thm. 5.1], we have that the pair $(\mathrm{G}_s, \mathrm{H})$ is (locally) isomorphic to either $(\mathrm{SO}(4,1), \mathrm{U}(2))$ or $(\mathrm{SU}(2,1), \mathrm{T}^2)$. We recall that the classification of all invariant strictly symplectic half-flat SU(3)-structures on these homogeneous spaces is also given in [27, Thm. 5.1].

REFERENCES

[1] M. Bordemann, M. Forger and H. Römer, *Kähler Manifolds: paving the way towards new supersymmetric Sigma Models*, Comm. Math. Phys., 102 (1986), pp. 605–647.

[2] N. Bourbaki, *Lie groups and Lie algebras. Chapters 1–3*, Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1989.

[3] R. L. Bryant, *Some remarks on G_2-structures*, in "Proceedings of Gökova Geometry-Topology Conference 2005", pp. 75–109. Gökova Geometry/Topology Conference (GGT), Gökova, 2006.

[4] R. Cleyton and S. Ivanov, *On the geometry of closed G_2-structures*, Comm. Math. Phys., 270:1 (2007), pp. 53–67.

[5] D. Conti and M. Fernández, *Nilmanifolds with a calibrated G_2-structure*, Differential Geom. Appl., 29:4 (2011), pp. 493–506.

[6] A. Corti, M. Haskins, J. Nordström, and T. Pacini, *G_2-manifolds and associative submanifolds via semi-Fano 3-folds*, Duke Math. J., 164:10 (2015), pp. 1971–2092.

[7] M. Fernández, *An example of a compact calibrated manifold associated with the exceptional Lie group G_2*, J. Differential Geom., 26:2 (1987), pp. 367–370.

[8] M. Fernández, *A family of compact solvable G_2-calibrated manifolds*, Tohoku Math. J. (2), 39:2 (1987), pp. 287–289.

[9] M. Fernández, A. Fino, A. Kovalev, and V. Muñoz, *A compact G_2-calibrated manifold with first Betti number $b_1 = 1$*, https://arxiv.org/abs/1808.07144.

[10] M. Fernández, A. Fino, and V. Manero, *G_2-structures on Einstein solvmanifolds*, Asian J. Math., 19:2 (2015), pp. 321–342.

[11] M. Fernández and A. Gray, *Riemannian manifolds with structure group G_2*, Ann. Mat. Pura Appl. (4), 132 (1982), pp. 19–45.

[12] A. Fino and A. Raffero, *Closed G_2-structures on non-solvable Lie groups*, Rev. Mat. Complut., 32:3 (2019), pp. 837–851.

[13] S. Helgason, *Differential operators on homogeneous spaces*, Acta Math., 102 (1959), pp. 239-299.

[14] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, Inc., 1978

[15] D. D. Joyce, *Compact Riemannian 7-manifolds with holonomy G_2. I, II*, J. Differential Geom., 43:2 (1996), pp. 291–328, 329–375.

[16] D. D. Joyce and S. Karigiannis, *A new construction of compact torsion-free G_2-manifolds by gluing families of Eguchi-Hanson spaces*, J. Differential Geom., 117:2 (2001), pp. 255–343.

[17] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol. I*, Interscience Publishers, New York-London, 1963.

[18] A. Kovalev, *Twisted connected sums and special Riemannian holonomy*, J. Reine Angew. Math., 565 (2003), pp. 125–160.

[19] A. Kovalev and N.-H. Lee, *K3 surfaces with non-symplectic involution and compact irreducible G_2-manifolds*, Math. Proc. Cambridge Philos. Soc., 151 (2011), pp. 193–218.

[20] J. Lauret, *Laplacian solitons: questions and homogeneous examples*, Differential Geom. Appl., 54(B) (2017), pp. 345–360.

[21] J. Lauret and M. Nicolini, *Extremally Ricci pinched G_2-structures on Lie groups*, https://arxiv.org/abs/1902.06375. To appear in Comm. Anal. Geom.

[22] J. Lauret and M. Nicolini, *The classification of ERP G_2-structures on Lie groups*, Ann. Mat. Pura Appl., 199 (2020), pp. 2489–2510.

[23] J. D. Lotay, *Geometric Flows of G_2 Structures*, in: "Lectures and Surveys on G_2-Manifolds and Related Topics". Fields Institute Communications, vol 84. Springer, New York, NY, 2020.

[24] A. L. Onishchik, *Topology of transitive transformation groups*, Johann Ambrosius Barth Verlag GmbH, Leipzig, 1994.

[25] A. L. Onishchik and E. B. Vinberg, *Lie groups and Lie algebras III*, Encyclopaedia Math.

Sci., 41, Springer-Verlag Berlin Heidelberg, 1994.

[26] F. Podestà and A. Raffero, *On the automorphism group of a closed* G_2*-structure*, Q. J. Math., 7:1 (2019), pp. 195–200.

[27] F. Podestà and A. Raffero, *Homogeneous symplectic half-flat 6-manifolds*, Ann. Global Anal. Geom., 55:1 (2019), pp. 1–15.