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MODULI SPACE OF IRREGULAR SINGULAR PARABOLIC CONNECTIONS OF GENERIC RAMIFIED TYPE ON A SMOOTH PROJECTIVE CURVE*

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Abstract. We give an algebraic construction of the moduli space of irregular singular connections of generic ramified type on a smooth projective curve. We prove that the moduli space is smooth and give its dimension. Under the assumption that the exponent of ramified type is generic, we give an algebraic symplectic form on the moduli space.

Key words. moduli space, ramified connection, symplectic form.

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Introduction. Let C be a smooth projective curve and E be an algebraic vector bundle on C. Consider an algebraic connection $\nabla\colon E \longrightarrow E\otimes\Omega^1_C(D)$ admitting poles along an effective divisor D on C. ∇ is said to be regular singular at $t \in D$ if ∇ has a simple pole at t. ∇ is said to be irregular singular at $t \in D$ if the pole order of ∇ at t is greater than one. We say that an irregular singular connection ∇ is generic unramified at $t \in D$ if the leading term $\nabla|_t$ of the restriction $\nabla|_{mt}\colon E|_{mt} \longrightarrow E|_{mt} \otimes \Omega_C(D)|_{mt}$ has the distinct eigenvalues, where m is the pole order of ∇ at t. The generic unramified connections are most generic irregular singular connections. The second generic irregular singular connections are generic ramified connections, which is of the following type. We say that an irregular singular connection (E,∇) of rank r is generic ν-ramified at $t \in D$ if its completion $(\hat{E},\hat{\nabla})$ at t is isomorphic to the connection

$$\nabla_\nu\colon \mathbb{C}[[w]] \ \ni \ f \ \mapsto \ df + \nu(w)f \ \in \ \mathbb{C}[[w]] \otimes \frac{dz}{z^m}$$

for $w = z^{\frac{1}{r}}$ and $\nu(w) \in \sum_{l=0}^{mr-r} \mathbb{C}w^l dw/w^{mr-r+1}$. The aim of this paper is to construct the moduli space of generic ramified connections.

It is a classical result by R. Fuchs that Painlevé VI equations can be obtained as the isomonodromic deformation of rank two regular singular connections on $\mathbb{P}^1$ with four poles and other types of Painlevé equations are known to be obtained as generalized isomonodromic deformations of rank two connections on $\mathbb{P}^1$ with irregular singularities ([14], [15], [16]). The space of initial conditions of Painlevé equations are constructed by K. Okamoto in [18] for all types. Their compactifications are classified by H. Sakai in [21], whose geometry characterizes the Painlevé equations. If one wants to formulate the geometry of isomonodromic deformation in a general framework, an appropriate construction of the moduli space of connections is required. Moduli space of regular singular connections are constructed by N. Nitsure in [17], though the moduli space may have singularities. Moduli space of regular singular connections with parabolic structure becomes smooth with a symplectic structure, which is constructed in the joint work [10] with K. Iwasaki and M.-H. Saito and in [9]. In the case of rank two connections on $\mathbb{P}^1$ with 4 regular singular points, the moduli space of regular singular parabolic connections is isomorphic to the space of initial

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conditions of Painlevé VI equations constructed by K. Okamoto. Furthermore, we can construct in [11] a compactified moduli space which is isomorphic to the Sakai's rational surface in [21] or in [20]. We can obtain the geometric Painlevé property (see [12] and [9, Definition 2.4]) of the isomonodromic deformation on the family of moduli spaces and then we can say that the constructed moduli space is the space of initial conditions of the isomonodromic deformation. Note that we cannot assume the underlying vector bundle trivial, even when the base curve is $\mathbb{P}^1$ and the degree of the underlying bundle is zero, for the purpose of obtaining the geometric Painlevé property.

Moduli space of generic unramified connections is analytically constructed by O. Biquard and P. Boalch in [2]. In [3], P. Boalch gives an algebraic construction of the moduli space of unramified irregular singular connections over the trivial bundle on $\mathbb{P}^1$. In a general case involving a higher genus curve, the joint work [13] with M.-H. Saito provides an algebraic construction of the moduli space of unramified irregular singular connections. Again we do not assume the triviality of the underlying bundle even if the base curve is $\mathbb{P}^1$ and the degree of the bundle is zero.

Compared with the unramified case, the algebraic construction of the moduli space of ramified irregular singular connections is difficult. In [7], C. L. Bremer and D. S. Sage construct the algebraic moduli space of ramified connections over a trivial bundle on $\mathbb{P}^1$. They give the characterization of ramified connections via an exhaustive consideration from the representation theory. The method of the construction of the moduli space is similar to that in [3]. P. Boalch constructed in [3] the moduli space of generic unramified irregular singular connections which is locally framed at singular points and called it the extended moduli space. The moduli space of irregular singular connections is obtained as a symplectic reduction of the extended moduli space. K. Hiroe and D. Yamakawa gave in [8] a clear summary of the construction of the moduli space as a symplectic reduction of the extended moduli space. Here we remark that in all these results, the underlying vector bundles on $\mathbb{P}^1$ are assumed to be trivial.

For the character variety side, the moduli spaces were constructed in [4], [5], [6] and [19], which are expected to be related with appropriate moduli spaces of irregular singular connections via the generalized Riemann-Hilbert correspondence.

In this paper, we construct a moduli space of generic ramified connections on a smooth projective curve. For the construction of the moduli space, it is important to give an appropriate formulation of ramified connections. In fact, it is necessary to rephrase the formal data by the data on the restriction $(E,\nabla)|_{mt}$ to each pole divisor, without depending on a framing. First we consider a filtration $E|_{mt} = V_0 \supset V_1 \supset \cdots \supset V_{r-1} \supset zV_0$ whose idea is similar to that in [7]. The filtration is given by $\mathbb{C}[w]/(w^{mr}) \supset (w)/(w^{mr}) \supset \cdots \supset (w^{r-1})/(w^{mr}) \supset (w^r)/(w^{mr})$ when $(\hat{E},\hat{\nabla}) = (\mathbb{C}[[w]],\nabla_\nu)$. Next we introduce quotient free $\mathbb{C}[w]/(w^{mr-r+1})$-module $V_k \otimes \mathbb{C}[w]/(w^{mr-r+1}) \xrightarrow{\pi_k} L_k$ of rank one for each k, whose meaning is a quotient $\nu(w)$-eigenspace of $\nabla|_{mt} \otimes \mathrm{id}$. In order to recover a formal ramified connection, we need not only the genericity condition on $\nu(w)$ but compatibility conditions between the data $\{V_k, L_k, \pi_k\}$. The precise definition is given in Definition 1.2 and Definition 2.1. We should be careful that in our general setting, the true formal data may not be the one given by the exponent if the exponent is not generic. We adopt this general setting, because of the author's hope to get a moduli space as a canonical degeneration of the moduli space of regular singular or unramified irregular singular connections.

Once the formulation of ramified connection is established, the construction of the moduli space becomes a standard task. For the construction of the moduli space in Theorem 2.1, we use a locally closed embedding of the moduli space to the moduli space of parabolic Λ^1_D-triples constructed in [10]. The basic idea of the moduli space construction was inspired by Simpson's works in [23] and [24], which is similar to the GIT construction of the moduli space of vector bundles using the Quot-scheme. The smoothness and the dimension counting of the moduli space in Theorem 3.1 follow from a standard deformation theory, using the idea of the methods in [9] and [13]. In Theorem 4.1, we construct a symplectic form on the moduli space of ramified connections. For the proof of the d-closedness of the symplectic form, we construct a family of moduli spaces of connections, whose special fiber is the moduli space of ramified connections and whose generic fiber is the moduli space of regular singular connections. The d-closedness of the symplectic form on the moduli space of ramified connections follows from that on the moduli space of regular singular connections. A further consideration on the generic unramified irregular singular locus provides a non-degeneracy of the symplectic form.

In the earlier version of the preprint, there was a serious error in the formulation of ramified connection. The author missed the condition (v) of Definition 1.2 or the condition (d) of Definition 2.3. An example concerning this point is given in the appendix.

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1. Formal data of a ramified connection and its paraphrase. In this section we first recall elementary properties of formal connections of ramified type.

Consider the formal power series ring $\mathbb{C}[[z]]$ and denote by $\mathbb{C}((z))$ its quotient field. We say that (W,∇) is a formal connection over $\mathbb{C}[[z]]$ if W is a free $\mathbb{C}[[z]]$-module of finite rank and $\nabla : W \longrightarrow W \otimes \mathbb{C}[[z]] \cdot \frac{dz}{z^m}$ is a $\mathbb{C}$-linear map satisfying $\nabla(fa) = a\otimes df + f\nabla(a)$ for $f \in \mathbb{C}[[z]]$ and $a \in W$. Let w be a variable with $w^l = z$ for a positive integer l. Then the formal power series ring $\mathbb{C}[[w]]$ becomes a free $\mathbb{C}[[z]]$-module of rank l. Throughout this paper, we drop the subscript in the tensor product $W \otimes_{\mathbb{C}[[z]]} \mathbb{C}[[w]]$ and simply write $W \otimes \mathbb{C}[[w]]$.

The following is a fundamental classification theorem of formal irregular singular connections.

Theorem 1.1 (Hukuhara-Turrittin Theorem ([1], Proposition 1.4.1 or [22], Theorem 6.8.1)). *For a formal irregular singular connection (W,∇) over $\mathbb{C}[[z]]$, there is a positive integer l and for a variable w with $w^l = z$, there are $\nu_1(w),\dots,\nu_s(w) \in \sum_{k=0}^{mr-r} \mathbb{C}w^k dw/w^{ml-l+1}$, positive integers $r_1,\dots,r_s$ such that*

$$(W\otimes\mathbb{C}((w)),\nabla\otimes\mathbb{C}((w))) \cong (\mathbb{C}((w))^{r_1}, d+J(\nu_1(w),r_1)) \oplus\cdots\oplus (\mathbb{C}((w))^{r_s}, d+J(\nu_s(w),r_s)),$$

where $d + J(\nu_i(w), r_i) : \mathbb{C}((w))^{r_i} \longrightarrow \mathbb{C}((w))^{r_i}\frac{dz}{z^m}$ is the connection given by

$$\begin{pmatrix} f_1 \\ \vdots \\ f_{r_i}\end{pmatrix} \mapsto \begin{pmatrix} df_1 \\ \vdots \\ df_{r_i}\end{pmatrix} + \left(\begin{pmatrix} \nu_i(w) & 0 & \cdots & 0 \\ 0 & \nu_i(w) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \nu_i(w) \end{pmatrix} + \frac{dw}{w} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \right) \begin{pmatrix} f_1 \\ \vdots \\ f_{r_i}\end{pmatrix}.$$

Here $\nu_i(w) - \nu_j(w) \notin \mathbb{Z}\frac{dw}{w}$ unless $\nu_i(w) = \nu_j(w)$.

We note that $\nu_1(w), \dots, \nu_s(w) \mod \mathbb{Z}\dfrac{dw}{w}$ are invariants of (W, ∇). We say that $\nu_1(w), \dots, \nu_s(w)$ are generalized eigenvalues of ∇ or exponents of ∇. In general, a ramified cover $w \mapsto w^l = z$ is necessary for obtaining the invariants $\nu_1(w), \dots, \nu_s(w)$. If we can take all $\nu_1(w), \dots, \nu_s(w)$ as differential forms in z, we say that ∇ is unramified.

Definition 1.1. We say that a formal connection (V, ∇) over $\mathbb{C}[[z]]$ is formally irreducible if there is no subbundle $0 \neq W \subsetneq V$ satisfying $\nabla(W) \subset W \otimes \dfrac{dz}{z^m}$.

We can see the following proposition immediately by induction on $\operatorname{rank} W$.

Proposition 1.1. *For a formal connection* (W, ∇) *over* $\mathbb{C}[[z]]$ *with* $\operatorname{rank}_{\mathbb{C}[[z]]} W < \infty$*, there is a filtration* $0 = W_s \subset W_{s-1} \subset \cdots \subset W_0 = W$ *such that* $\nabla(W_i) \subset W_i \otimes \dfrac{dz}{z^m}$*, each* W_i/W_{i+1} *is a free* $\mathbb{C}[[z]]$*-module and* $(W_i/W_{i+1}, \nabla_{W_i/W_{i+1}})$ *is formally irreducible, where* $\nabla_{W_i/W_{i+1}}$ *is the connection on* W_i/W_{i+1} *induced by* $\nabla|_{W_i}$.

We will see what data determines the formal connection $(W_i/W_{i+1}, \nabla_{W_i/W_{i+1}})$. So we assume that (W, ∇) is formally irreducible for simplicity. Thanks to the Hukuhara-Turrittin Theorem (Theorem 1.1), there is a ramified cover $w \mapsto w^l = z$ and a generalized eigenvalue $\nu(w) \in \mathbb{C}[[w]]\dfrac{dz}{z^m}$ of $(W \otimes \mathbb{C}((w)), \nabla \otimes \mathbb{C}((w)))$. In other words, there is an eigen vector $v \in W \otimes \mathbb{C}((w))$, that is, $\nabla(v) = \nu(w)v$. If we take a generator σ of $\operatorname{Gal}(\mathbb{C}((w))/\mathbb{C}((z)))$, then the $\mathbb{C}((w))$-subspace generated by $\{\sigma^k(v) | 0 \leq k \leq l-1\}$ descends to a subspace W' of $W \otimes \mathbb{C}((z))$. If we put $\tilde{W}' := W \cap W'$, then $\tilde{W}'$ is a subbundle of W preserved by ∇ and $\tilde{W}' \otimes \mathbb{C}((z)) = W'$. Since (W, ∇) is irreducible, we have $\tilde{W}' = W$. If j is the minimum among $0 < j \leq l$ satisfying $\sigma^j(\nu(w)) = \nu(w)$, then $j | l$ and $\nu(w) \in \mathbb{C}[[w^{\frac{l}{j}}]]\dfrac{dz}{z^m}$. So we may replace w by $w^{\frac{l}{j}} (= z^{\frac{1}{j}})$ and then $\{\sigma^k(\nu(w)) | 0 \leq k \leq l-1\}$ are distinct. We have $l = \operatorname{rank} W$ in this case. For an eigenvector $v \in W \otimes \mathbb{C}((w))$, we have

$$\nabla(\sigma^k(v)) = \sigma^k(\nabla(v)) = \sigma^k(\nu(w)v) = \sigma^k(\nu(w))\sigma^k(v).$$

So $v, \sigma(v), \dots, \sigma^{l-1}(v)$ are linearly independent over $\mathbb{C}((w))$, because they are eigenvectors of ∇ with the distinct eigenvalues $\nu(w), \sigma(\nu(w)), \dots, \sigma^{l-1}(\nu(w))$. So we can see that $W \otimes \mathbb{C}((w))$ has $\sigma^k(\nu(w))$-eigensubspaces, but we can also regard that $W \otimes \mathbb{C}((w))$ has quotient $\sigma^k(\nu(w))$-eigenspaces. Replacing $\nu(w)$ by adding an element of $\mathbb{Z}\dfrac{dw}{w}$, we may assume that there is a surjection $\pi \colon W \otimes \mathbb{C}[[w]] \longrightarrow \mathbb{C}[[w]]$ such that the diagram

$$\begin{array}{ccc} W \otimes \mathbb{C}[[w]] & \xrightarrow{\pi} & \mathbb{C}[[w]] \\ {\scriptstyle \nabla\otimes \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \nabla_\nu} \\ W \otimes \mathbb{C}[[w]] \cdot \dfrac{dz}{z^m} & \xrightarrow{\pi\otimes \mathrm{id}} & \mathbb{C}[[w]]\dfrac{dz}{z^m} \end{array}$$

is commutative, where ∇_ν is given by $\nabla_\nu(f(w)) = df(w) + f(w)\nu(w)$ for $f(w) \in \mathbb{C}[[w]]$. Note that $W \xrightarrow{\pi|_W} \mathbb{C}[[w]]$ is injective, because (W, ∇) is formally irreducible. So $\operatorname{coker}(\pi|_W)$ is of finite length, since $\operatorname{rank}_{\mathbb{C}[[z]]} W = \operatorname{rank}_{\mathbb{C}[[z]]} \mathbb{C}[[w]] = l$.

Conversely, assume that $\nu(w) \in \sum_{s=0}^{mr-r} \mathbb{C}w^s dw/w^{mr-r+1}$ is given for $w = z^{\frac{1}{r}}$ such that $\tau\nu(w) \neq \nu(w)$ for any $\tau \in \mathrm{Gal}(\mathbb{C}((w))/\mathbb{C}((z)))$ other than id. If $\mathbb{C}[[w]] \xrightarrow{q} Q$ is a quotient $\mathbb{C}[[z]]$-module of finite length satisfying the commutative diagram

$$\begin{array}{ccc} \mathbb{C}[[w]] & \xrightarrow{q} & Q \\ \nabla_\nu \downarrow & & \downarrow \nabla_Q \\ \mathbb{C}[[w]] \otimes \frac{dz}{z^m} & \xrightarrow{q\otimes\mathrm{id}} & Q \otimes \frac{dz}{z^m}, \end{array}$$

for some morphism ∇_Q, then a formal connection $\nabla|_{\ker q} \colon \ker q \longrightarrow \ker q \otimes dz/z^m$ is induced. If $0 \neq U \subset \ker q$ is a subbudle preserved by $\nabla|_{\ker q}$, then we have

$$(U, \nabla|_U) \otimes_{\mathbb{C}[[z]]} \mathbb{C}((w)) \subset (\ker q, \nabla|_{\ker q}) \otimes_{\mathbb{C}[[z]]} \mathbb{C}((w)) \cong \bigoplus_{k=0}^{r-1} \big(\mathbb{C}((w)), \nabla_{\sigma^k(\nu(w))}\big),$$

where σ is a generator of $\mathrm{Gal}(\mathbb{C}((w))/\mathbb{C}((z)))$ and $\big(\mathbb{C}((w)), \nabla_{\sigma^k(\nu(w))}\big)$ means the $\sigma^k(\nu(w))$-eigenspace. In particular, we can choose a vector $0 \neq u \in U \otimes \mathbb{C}((w))$ satisfying $\nabla(u) = \sigma^k(\nu(w))u$ for some k. By the assumption, $\nu(w), \sigma(\nu(w)), \ldots, \sigma^{r-1}(\nu(w))$ are mutually distinct. So $u, \sigma(u), \ldots, \sigma^{r-1}(u)$ are linearly independent over $\mathbb{C}((w))$, because they are the eigenvectors with distinct eigenvalues $\sigma^k(\nu(w)), \sigma^{k+1}(\nu(w)), \ldots, \sigma^{r-1+k}(\nu(w))$. Thus $\mathrm{rank}_{\mathbb{C}[[z]]} U = \dim_{\mathbb{C}((w))}(U \otimes \mathbb{C}((w))) = r$ which implies $U = \ker q$. Hence $(\ker q, \nabla|_{\ker q})$ is a formally irreducible connection.

Summarizing the above arguments, we obtain the following proposition:

Proposition 1.2. *A formal connection* (W, ∇) *of rank* r *over* $\mathbb{C}[[z]]$ *is irreducible if and only if it is isomorphic to* $(\ker q, \nabla_\nu|_{\ker q})$*, where* $\mathbb{C}[[w]] \xrightarrow{q} Q$ *is a quotient* $\mathbb{C}[[z]]$*-module of finite length for* $w = z^{\frac{1}{r}}$ *and* $\nu(w) \in \sum_{k=0}^{mr-r} \mathbb{C}w^k dw/w^{mr-r+1}$ *is not fixed by the elements of* $\mathrm{Gal}(\mathbb{C}((w))/\mathbb{C}((z)))$ *other than* id*, such that the diagram*

$$\begin{array}{ccc} \mathbb{C}[[w]] & \xrightarrow{q} & Q \\ \nabla_\nu \downarrow & & \downarrow \nabla_Q \\ \mathbb{C}[[w]] \otimes \frac{dz}{z^m} & \xrightarrow{q\otimes\mathrm{id}} & Q \otimes \frac{dz}{z^m} \end{array}$$

is commutative for a morphism $\nabla_Q \colon Q \longrightarrow Q \otimes dz/z^m$ *satisfying* $Q(f(z)a) = a \otimes df(z) + f(z)a$ $(a \in Q,\ f(z) \in \mathbb{C}[[z]])$*.*

Remark 1.1. Let (W, ∇) be a formal irreducible connection over $\mathbb{C}[[z]]$ of rank r and take a variable w with $w^r = z$. Then we can see from the proof of Proposition 1.2 that there is a surjection $\pi \colon W \otimes \mathbb{C}[[w]] \longrightarrow \mathbb{C}[[w]]$ whose restriction $\pi|_W \colon W \longrightarrow \mathbb{C}[[w]]$ is generically injective with finite length cokernel, such that the diagram

$$\begin{array}{ccc} W \otimes \mathbb{C}[[w]] & \xrightarrow{\pi} & \mathbb{C}[[w]] \\ \nabla\otimes\mathrm{id} \downarrow & & \downarrow \nabla_\nu \\ W \otimes \mathbb{C}[[w]] \otimes \frac{dz}{z^m} & \xrightarrow{\pi\otimes\mathrm{id}} & \mathbb{C}[[w]] \otimes \frac{dz}{z^m} \end{array}$$

is commutative. If σ is a generator of the Galois group $\mathrm{Gal}(\mathbb{C}((w))/\mathbb{C}((z)))$,

$$\sigma^k \circ \pi \circ (1 \otimes \sigma)^{-k} \colon W \otimes \mathbb{C}[[w]] \longrightarrow \mathbb{C}[[w]]$$

is a surjective homomorphism of $\mathbb{C}[[w]]$-modules, which makes the diagram

$$\begin{array}{ccc} W \otimes \mathbb{C}[[w]] & \xrightarrow{\sigma^k\circ\pi\circ(1\otimes\sigma)^{-k}} & \mathbb{C}[[w]] \\ {\scriptstyle \nabla\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \nabla_{\sigma^k(\nu)}} \\ W \otimes \mathbb{C}[[w]] \otimes \dfrac{dz}{z^m} & \xrightarrow{(\sigma^k\circ\pi\circ(1\otimes\sigma)^{-k})\otimes\mathrm{id}} & \mathbb{C}[[w]] \otimes \dfrac{dz}{z^m} \end{array}$$

commutative for $0 \le k \le r-1$.

LEMMA 1.1. *Let (W, ∇) be a formal irreducible connection of rank r over $\mathbb{C}[[z]]$ with a quotient $\pi \colon W \otimes \mathbb{C}[[w]] \longrightarrow \mathbb{C}[[w]]$ for $w^r = z$ satisfying the commutative diagram*

$$\begin{array}{ccc} W \otimes \mathbb{C}[[w]] & \xrightarrow{\pi} & \mathbb{C}[[w]] \\ {\scriptstyle \nabla\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \nabla_{\nu}} \\ W \otimes \mathbb{C}[[w]] \otimes \dfrac{dz}{z^m} & \xrightarrow{\pi\otimes\mathrm{id}} & \mathbb{C}[[w]] \otimes \dfrac{dz}{z^m}, \end{array}$$

where $\nu(w) \in \sum_{l=0}^{mr-r} \mathbb{C}w^k dw/w^{mr-r+1}$. Assume that the $\dfrac{wdw}{w^{mr-r+1}}$-coefficient of $\nu(w)$ is non-zero. Then the restriction

$$\pi|_W \colon W \xrightarrow{\sim} \mathbb{C}[[w]]$$

is in an isomorphism.

Proof. We can write

$$\nu(w) = \nu_0(z) + \nu_1(z)w + \cdots + \nu_{r-1}(z)w^{r-1}$$

with $\nu_k(z) \in \mathbb{C}[z]dz/z^m$. By the assumption, $\nu_1(z)$ is a generator of $\mathbb{C}[[z]]dz/z^m$. So we can write $\nu_1(z) = (c_0 + c_1 z + \cdots + c_{m-2}z^{m-2})dz/z^m$ with $c_0 \neq 0$. Consider the connection

$$\nabla - \nu_0(z)\mathrm{id} \colon\ W \ \ni\ v \ \mapsto\ \nabla(v) - \nu_0(z)v \ \in\ W \otimes \frac{dz}{z^m}.$$

Then we have the commutative diagram

$$\begin{array}{ccc} W \otimes \mathbb{C}[[w]] & \xrightarrow{\pi} & \mathbb{C}[[w]] \\ {\scriptstyle (\nabla-\nu_0(z)\mathrm{id})\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \nabla_{\nu(w)-\nu_0(z)}} \\ W \otimes \mathbb{C}[[w]] \otimes \dfrac{dz}{z^m} & \xrightarrow{\pi\otimes\mathrm{id}} & \mathbb{C}[[w]] \otimes \dfrac{dz}{z^m}. \end{array}$$

In particular, $\pi(W) \subset \mathbb{C}[[w]]$ is preserved by $\nabla_{\nu(w)-\nu_0(z)}$. Since π is surjective, there exists $v \in W$ satisfying $\pi(v) = 1 + wa_0(w)$ for some $a_0(w) \in \mathbb{C}[[w]]$. We can write

$\nu(w) - \nu_0(z) = c_0 w(1 + wb_1(w))dw/w^{mr-r+1}$ for some $b_1(w) \in \mathbb{C}[[w]]$. So we can write

$$\begin{aligned}&\pi((\nabla - \nu_0(z)\mathrm{id})(v)) \\ &= (\nabla_{\nu(w)-\nu_0(z)})(\pi(v)) \\ &= \big(c_0 w + (c_0 b_1(w) + c_0 a_0(w))w^2 + (\text{higher order terms})\big)\, dw/w^{mr-r+1}.\end{aligned}$$

In particular, $\pi(V)$ contains an element of the form $w(1+wa_1(w))$ for some $a_1(w) \in \mathbb{C}[[w]]$. Repeating this procedure, we can construct elements

$$1 + wa_0(w), w(1 + wa_1(w)), \ldots, w^{r-1}(1 + wa_{r-1}(w))$$

of $\pi(W)$. Since all the elements of $\mathbb{C}[[w]]$ can be written as a $\mathbb{C}[[z]]$-linear combination of the above elements, we get the surjectivity of $\pi|_W : W \longrightarrow \mathbb{C}[[w]]$. □

Remark 1.2. In this paper we do not treat formal connections with $\mathrm{coker}(\pi|_W) = Q$ non-trivial. We mainly treat connections satisfying $\mathrm{coker}(\pi|_W) = 0$ and we will say such type of connection a generic ramified type. A precise definition of connections of generic ramified type is given in Definition 2.1, while the essential part of the definition is given in Definition 1.2.

Consider the formal connection $(W, \nabla) \cong (\mathbb{C}[[w]], \nabla_\nu)$. Then we get a filtration

$$W = \hat{V}_0 \supset \hat{V}_1 \supset \cdots \supset \hat{V}_{r-1} \supset \hat{V}_r = z\hat{V}_0$$

by pulling back the filtration $\mathbb{C}[[w]] \supset (w) \supset (w^2) \supset \cdots \supset (w^{r-1}) \supset (w^r)$. There is a canonical surjection $\pi \colon W \otimes_{\mathbb{C}[[z]]} \mathbb{C}[[w]] \longrightarrow \mathbb{C}[[w]]$ which induces commutative diagrams

$$\begin{array}{ccc} \hat{V}_k \otimes_{\mathbb{C}[[z]]} \mathbb{C}[[w]] & \xrightarrow{\pi|_{\hat{V}_k} \otimes \mathrm{id}_{\mathbb{C}[[w]]}} & (w^k) \\ {\scriptstyle \nabla|_{\hat{V}_k} \otimes \mathrm{id}} \big\downarrow & & \big\downarrow {\scriptstyle \nabla_{\nu(w)}|_{(w^k)}} \\ \hat{V}_k \otimes_{\mathbb{C}[[z]]} \mathbb{C}[[w]] \otimes \dfrac{dz}{z^m} & \xrightarrow{\pi|_{\hat{V}_k} \otimes \mathrm{id}_{\mathbb{C}[[w]]}} & (w^k) \otimes \dfrac{dz}{z^m} \end{array}$$

for $0 \le k \le r-1$. Note that the restriction $\nabla_\nu|_{(w^k)}$ is given by

$$\nabla_{\nu(w)}\left(w^k f(w)\right) = d(w^k)f(w) + w^k df(w) + w^k f(w)\nu(w) = w^k \left(\nabla_{\nu(w)} + \frac{k}{r}\frac{dz}{z}\mathrm{id}\right)(f(w)).$$

Let us consider the restriction of the above data to the finite subscheme defined by $z^m = 0$. So the filtration $\hat{V}_k$ induces a filtration

$$W \otimes_{\mathbb{C}[[z]]} \mathbb{C}[z]/(z^m) = V_0 \supset V_1 \supset \cdots \supset V_{r-1} \supset V_r = zV_0.$$

Define

$$\begin{aligned}\bar{\pi}_k &:= (\pi|_{\hat{V}_k \otimes \mathbb{C}[[w]]}) \otimes \mathrm{id}_{\mathbb{C}[w]/(w^{mr-r+1})} \colon V_k \otimes_{\mathbb{C}[z]/(z^m)} \mathbb{C}[w]/(w^{mr-r+1}) \\ &\qquad \longrightarrow (w^k)/(w^{k+mr-r+1}) \\ \bar{\nabla}_k &:= \nabla|_{z^m=0} \colon V_k \longrightarrow V_k \otimes \frac{dz}{z^m}.\end{aligned}$$

Then there are commutative diagrams

$$\begin{array}{ccc} V_k \otimes_{\mathbb{C}[z]/(z^m)} \mathbb{C}[w]/(w^{mr-r+1}) & \xrightarrow{\bar{\pi}_k} & (w^k)/(w^{k+mr-r+1}) \\ {\scriptstyle \bar{\nabla}_k \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \nabla_{\nu(w)+\frac{k}{r}\frac{dz}{z}}} \\ V_k \otimes \dfrac{dz}{z^m} \otimes_{\mathbb{C}[z]/(z^m)} \mathbb{C}[w]/(w^{mr-r+1}) & \xrightarrow{\bar{\pi}_k \otimes \mathrm{id}} & (w^k)/(w^{k+mr-r+1}) \otimes \dfrac{dz}{z^m} \end{array}$$

for $0 \leq k \leq r-1$.

There are compatibility conditions among V_k, π_k for $k = 0, 1 \ldots, r-1$. Later in Proposition 1.3, we will prove that the compatibility conditions can recover a formal ramified connection under a genericity condition on the exponent $\nu(w)$. We summarize the conditions in the following definition. The advantage of the definition is to be fit for the formulation of global moduli of ramified connections.

Definition 1.2. Let w be a variable with $w^r = z$ and take $\nu(w) \in \sum_{l=0}^{mr-r} \mathbb{C} w^l dw/w^{mr-r+1}$. Assume that $(\overline{W}, \overline{\nabla})$ is a pair of a free $\mathbb{C}[z]/(z^m)$-module $\overline{W}$ of rank r and a homomorphism $\overline{\nabla} \colon \overline{W} \longrightarrow \overline{W} \otimes dz/z^m$ of $\mathbb{C}[z]/(z^m)$-modules. We say that $\big((V_k, L_k, \pi_k)_{0 \leq k \leq r-1}, (\phi_k)_{1 \leq k \leq r}\big)$ is a $\nu(w)$-ramified structure on $(\overline{W}, \overline{\nabla})$ if

(i) $\overline{W} = V_0 \supset V_1 \supset \cdots \supset V_{r-1} \supset zV_0 =: V_r$ is a filtration satisfying $\overline{\nabla}(V_k) \subset V_k \otimes dz/z^m$ and $\mathrm{length}(V_k/V_{k+1}) = 1$ for $0 \leq k \leq r-1$,

(ii) $\pi_k \colon V_k \otimes \mathbb{C}[w]/(w^{mr-r+1}) \longrightarrow L_k$ is a quotient free $\mathbb{C}[w]/(w^{mr-r+1})$-module of rank one such that the restriction $\pi_k|_{V_k} \colon V_k \longrightarrow L_k$ is surjective for $0 \leq k \leq r-1$,

(iii) the diagrams

$$\begin{array}{ccc} V_k \otimes \mathbb{C}[w]/(w^{mr-r+1}) & \xrightarrow{\pi_k} & L_k \\ {\scriptstyle \overline{\nabla}|_{V_k} \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \nu(w)+\frac{k}{r}\frac{dz}{z}} \\ V_k \otimes \dfrac{dz}{z^m} \otimes \mathbb{C}[w]/(w^{mr-r+1}) & \xrightarrow{\pi_k \otimes \mathrm{id}} & L_k \otimes \dfrac{dz}{z^m} \end{array}$$

are commutative for $0 \leq k \leq r-1$,

(iv) $\phi_k \colon L_k \longrightarrow L_{k-1}$ are homomorphisms of $\mathbb{C}[w]$-modules with factorizations

$$\phi_k \colon L_k \xrightarrow[\sim]{\psi_k} (w)/(w^{mr-r+2}) \otimes L_{k-1} \longrightarrow wL_{k-1} \hookrightarrow L_{k-1}$$

for isomorphisms $\psi_k \colon L_k \xrightarrow{\sim} (w)/(w^{mr-r+2}) \otimes L_{k-1}$ of $\mathbb{C}[w]$-modules for $1 \leq k \leq r-1$ and $\phi_r \colon (w^r)/(w^{mr+1}) \otimes L_0 \longrightarrow L_{r-1}$ is a $\mathbb{C}[w]$-homomorphism whose image is wL_{r-1} such that the diagram

$$\begin{array}{ccc} (z)/(z^{m+1}) \otimes V_0 \otimes \mathbb{C}[w]/(w^{mr-r+1}) & \xrightarrow{\mathrm{id} \otimes \pi_0} & (w^r)/(w^{mr+1}) \otimes L_0 \\ \downarrow & & \downarrow {\scriptstyle \phi_r} \\ V_{r-1} \otimes \mathbb{C}[w]/(w^{mr-r+1}) & \xrightarrow{\pi_{r-1}} & L_{r-1} \end{array}$$

is commutative and the diagrams

$$\begin{array}{ccc} V_k \otimes \mathbb{C}[w]/(w^{mr-r+1}) & \xrightarrow{\pi_k|_{V_k}} & L_k \\ \downarrow & & \downarrow {\scriptstyle \phi_k} \\ V_{k-1} \otimes \mathbb{C}[w]/(w^{mr-r+1}) & \xrightarrow{\pi_{k-1}|_{V_{k-1}}} & L_{k-1} \end{array}$$

are commutative for $1 \le k \le r-1$,

(v) the composition

$$(w^r)\otimes L_0 \xrightarrow{\phi_r} L_{r-1} \xrightarrow[\sim]{\psi_{r-1}} (w)\otimes L_{r-2} \xrightarrow[\sim]{\mathrm{id}_{(w)}\otimes\psi_{r-2}} (w^2)\otimes L_{r-3} \xrightarrow[\sim]{\mathrm{id}_{(w^2)}\otimes\psi_{r-3}} \cdots \xrightarrow[\sim]{\mathrm{id}_{(w^{r-2})}\otimes\psi_1} (w^{r-1})\otimes L_0 \tag{1}$$

coincides with the homomorphism canonically induced by $(w^r) \longrightarrow (w^{r-1})$.

Remark 1.3. The condition of Definition 1.2 (v) is independent of the choice of the lifts ψ_k of ϕ_k chosen in (iv). Indeed, the composition (1) in Definition 1.2 (v) is determined only by $(\phi_k)_{1\le k\le r}$.

Proposition 1.3. Assume that (W,∇) is a formal connection of rank r over $\mathbb{C}[[z]]$ such that $(W,\nabla)\otimes\mathbb{C}[z]/(z^m)$ has a $\nu(w)$-ramified structure defined in Definition 1.2, where w is a variable with $w^r = z$. Furthermore, assume that the $w\,dw/w^{mr-r+1}$-coefficient of $\nu(w)$ does not vanish. Then there is a formal isomorphism $(W,\nabla) \cong (\mathbb{C}[[w]],\nabla_{\nu(w)})$.

Proof. We write $\nu(w) = (a_0(z)+a_1(z)w+\cdots+a_{r-1}(z)w^{r-1})dz/z^m$ for polynomials $a_0(z),\dots,a_{r-1}(z)$ in z. By the assumption, $a_0(z)$ is of degree at most $m-1$, the degrees of $a_1(z),\dots,a_{r-1}(z)$ are at most $m-2$ and the constant term $a_1(0)$ of $a_1(z)$ does not vanish. Let

$$N'\colon W\otimes\mathbb{C}[z]/(z^{m-1}) \longrightarrow W\otimes\mathbb{C}[z]/(z^{m-1})$$

be the endomorphism of $\mathbb{C}[z]/(z^{m-1})$-module determined by

$$N'\frac{dz}{z^m} = \nabla\otimes\mathrm{id}_{\mathbb{C}[z]/(z^{m-1})} - \nu_0(z)\mathrm{id}_{W\otimes\mathbb{C}[z]/(z^{m-1})}\colon W\otimes\mathbb{C}[z]/(z^{m-1}) \longrightarrow W\otimes\mathbb{C}[z]/(z^{m-1})\otimes\frac{dz}{z^m}.$$

If we put $b(w) := a_1(z)w + a_2(z)w^2+\cdots+a_{r-1}(z)w^{r-1}$, then the diagram

$$\begin{array}{ccc} W\otimes\mathbb{C}[z]/(z^{m-1}) & \xrightarrow[\cong]{\pi_0|_{\bar W}\otimes\mathbb{C}[z]/(z^{m-1})} & L_0/w^{mr-r}L_0 \cong \mathbb{C}[w]/(w^{mr-r}) \\ {\scriptstyle N'}\downarrow & & \downarrow{\scriptstyle b(w)} \\ W\otimes\mathbb{C}[z]/(z^{m-1}) & \xrightarrow[\cong]{\pi_0|_{\bar W}\otimes\mathbb{C}[z]/(z^{m-1})} & L_0/w^{mr-r}L_0 \cong \mathbb{C}[w]/(w^{mr-r}) \end{array}$$

is commutative. Note that

$$p_0 := \pi|_{\overline{W}}\otimes\mathbb{C}[z]/(z^{m-1})\colon W\otimes\mathbb{C}[z]/(z^{m-1}) \longrightarrow L_0/w^{mr-r}L_0$$

is isomorphic, because $\pi_0|_{\overline{W}}\colon \overline{W} = V_0 \longrightarrow L_0 \cong \mathbb{C}[w]/(w^{mr-r+1})$ is surjective by the assumption (ii) of Definition 1.2 and $\mathrm{length}(W\otimes\mathbb{C}[z]/(z^{m-1})) = r(m-1) = \mathrm{length}(L_0/w^{mr-r}L_0)$. Using the condition $a_1(0)\neq 0$, we can see that

$$\begin{aligned} \mathrm{Im}(N')^{r-1} &\not\subset z\big(W\otimes\mathbb{C}[z]/(z^{m-1})\big) \\ \mathrm{Im}(N')^{r} &\subset z\big(W\otimes\mathbb{C}[z]/(z^{m-1})\big). \end{aligned} \tag{2}$$

Assume that there is a subbundle $0 \neq U \subset W$ preserved by ∇. Then $U/z^{m-1}U \subset W/z^{m-1}W$ is a subbundle preserved by N'. In particular, $p_0(U/z^{m-1}U) \subset L_0/w^{mr-r}L_0$ is a $\mathbb{C}[z]/(z^{m-1})$-subbundle preserved by $b(w)$. So $p_0(U/z^{m-1}U)$ contains an element of the form $w^k(1+w\beta(w))$ with $0 \leq k \leq r-1$. Thus we can see that $p_0(U/z^{m-1}U)$ contains $z\bar{W}$, since it is preserved by $b(w)$ satisfying $a_1(0) \neq 0$. Since U is a subbundle of W, we have $U = W$ and we have proved that (W, ∇) is a formal irreducible connection.

By Proposition 1.2 and Remark 1.1, there is $\nu'(w) \in \sum_{l=0}^{mr-r} \mathbb{C}w^l dw/w^{mr-r+1}$ and an injection $W \hookrightarrow \mathbb{C}[[w]]$ such that the induced homomorphism $\pi' \colon W \otimes \mathbb{C}[[w]] \longrightarrow \mathbb{C}[[w]]$ is surjective and that the diagram

$$\begin{array}{ccc} W \otimes \mathbb{C}[[w]] & \xrightarrow{\pi'} & \mathbb{C}[[w]] \\ {\scriptstyle \nabla\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \nabla_{\nu'(w)}} \\ W \otimes \dfrac{dz}{z^m} \otimes \mathbb{C}[[w]] & \xrightarrow{\pi'\otimes\mathrm{id}} & \mathbb{C}[[w]] \otimes \dfrac{dz}{z^m} \end{array}$$

is commutative. Write $\nu'(w) = (a'_0(z) + a'_1(z)w + \cdots + a'_{r-1}(z)w^{r-1})dz/z^m$. If we put

$$b'(w) := a'_0(z) - a_0(z) + a'_1(z)w + \cdots + a'_{r-1}(z)w^{r-1},$$

then π' induces a commutative diagram

$$\begin{array}{ccc} W \otimes \mathbb{C}[z]/(z^{m-1}) & \xrightarrow{\pi'|_W\otimes\mathrm{id}_{\mathbb{C}[z]/(z^{m-1})}} & \mathbb{C}[w]/(w^{mr-r}) \\ {\scriptstyle N'\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle b'(w)} \\ W \otimes \mathbb{C}[z]/(z^{m-1}) & \xrightarrow{\pi'|_W\otimes\mathrm{id}_{\mathbb{C}[z]/(z^{m-1})}} & \mathbb{C}[w]/(w^{mr-r}). \end{array}$$

Since $\pi' \colon W \otimes \mathbb{C}[[w]] \longrightarrow \mathbb{C}[[w]]$ is surjective, $\mathrm{Im}\left(\pi'|_W \otimes \mathrm{id}_{\mathbb{C}[z]/(z^{m-1})}\right)$ contains a generator of $\mathbb{C}[w]/(w^{mr-r})$ as a $\mathbb{C}[w]$-module. Then, we can see from the property (2) of N' that $b'(w)^{r-1} \notin z\mathbb{C}[w]/(w^{mr-r})$ and $b'(w)^r \in z\mathbb{C}[w]/(w^{mr-r})$. So we have $a'_0(z) \equiv a_0(z) \pmod z$ and $a'_1(0) \neq 0$. Thus

$$\pi'|_W \otimes \mathrm{id}_{\mathbb{C}[z]/(z^{m-1})} \colon W \otimes \mathbb{C}[z]/(z^{mr-r}) \longrightarrow \mathbb{C}[w]/(w^{mr-r})$$

is surjective and then it is isomorphic because $\mathrm{length}(W \otimes \mathbb{C}[z]/(z^{m-1})) = \mathrm{length}(\mathbb{C}[w]/(w^{mr-r}))$. Since the composition

$$\mathbb{C}[w]/(w^{mr-r}) \xrightarrow{(\pi'|_W\otimes\mathrm{id}_{\mathbb{C}[z]/(z^{m-1})})\circ p_0^{-1}} \mathbb{C}[w]/(w^{mr-r})$$

is an isomorphism of $\mathbb{C}[w]/(w^{mr-r})$-modules, we have $b(w) \equiv b'(w) \pmod{w^{mr-r}}$, which means $\nu(w) \equiv \nu'(w) \pmod{w^{mr-r}dw/w^{mr-r+1}}$. In particular, $\pi'|_W \colon W \longrightarrow \mathbb{C}[[w]]$ is an isomorphism by Lemma 1.1.

We get a basis of $W \otimes \mathbb{C}[z]/(z^m)$ by pulling back the basis $1, w, \ldots, w^{r-1}$ of $\mathbb{C}[w]/(w^{mr})$ via the isomorphism $\pi'|_W \otimes \mathbb{C}[z]/(z^m)$ and the representation matrix of $\nabla \otimes \mathbb{C}[z]/(z^m)$ with respect to this basis is

$$\begin{pmatrix} \nu'_0(z) & z\nu'_{r-1}(z) & \cdots & z\nu'_1(z) \\ \nu'_1(z) & \nu'_0(z) + \frac{dz}{rz} & \cdots & \nu'_2(z) \\ \vdots & \vdots & \ddots & \vdots \\ \nu'_{r-1}(z) & \nu'_{r-2}(z) & \cdots & \nu'_0(z) + \frac{(r-1)dz}{rz} \end{pmatrix}.$$

In particular, we have $\mathrm{Tr}(\nabla\otimes\mathbb{C}[z]/(z^m))=r\nu_0(z)+(r-1)dz/2z$.

Take a generator $\bar{e}_0$ of L_0 as a $\mathbb{C}[w]/(w^{mr-r+1})$-module. Let $\bar{e}_k$ be the element of L_k corresponding to $w^k\otimes e_0$ via the isomorphism

$$L_k \xrightarrow[\sim]{\psi_1\circ\cdots\circ\psi_k} (w^k)/(w^{k+mr-r+1})\otimes L_0.$$

Since $\pi_k|_{V_k}$ is surjective, we can take an element $e_k\in V_k$ satisfying $\pi_k(e_k)=\bar{e}_k$. Then we have

$$\begin{aligned} w^l\pi_k(e_k) &= (\phi_{k+1}\circ\cdots\circ\phi_{k+l})(\pi_{k+l}(e_{k+l}))=\pi_k(e_{k+l}) \qquad (k+l<r)\\ w^{r-k+l}\pi_k(e_k) &= (\phi_{k-1}\circ\cdots\circ\phi_{r-1}\circ\phi_r\circ(\mathrm{id}\otimes\phi_1)\circ\cdots(\mathrm{id}\otimes\phi_l))(z\otimes\pi_l(e_l))\\ &= \pi_k(ze_l) \qquad (l<k). \end{aligned}$$

So the representation matrix of $\nabla\otimes\mathbb{C}[z]/(z^m)$ with respect to the basis $e_0,\dots,e_{r-1}$ is

$$\begin{pmatrix} \nu_0(z) & z\nu_{r-1}(z) & \cdots & z\nu_1(z)\\ \tilde{\nu}_1(z) & \nu_0(z)+\frac{dz}{rz} & \cdots & \nu_2(z)\\ \vdots & \vdots & \ddots & \vdots\\ \tilde{\nu}_{r-1}(z) & \tilde{\nu}_{r-2}(z) & \cdots & \nu_0(z)+\frac{(r-1)dz}{rz} \end{pmatrix}.$$

whose lower triangular entries $\tilde{\nu}_1(z),\dots,\tilde{\nu}_{r-1}(z)$ satisfy $\tilde{\nu}_k(z)\equiv\nu_k(z)$ $(\mathrm{mod}\ z^{m-1}dz/z^m)$ and have ambiguities in residue parts for $1\le k\le r-1$. In any case, we have $\mathrm{Tr}(\nabla\otimes\mathbb{C}[z]/(z^m))=r\nu_0(z)+(r-1)dz/2z$. So we get $\nu_0(z)\equiv\nu_0'(z)$ $(\mathrm{mod}\ z^m dz/z^m)$. Thus we have $\nu(w)=\nu'(w)$ because

$$\nu(w)\equiv\nu'(w) \quad (\mathrm{mod}\ w^{mr-r+1}dw/w^{mr-r+1})$$

and both of $\nu(w)$ and $\nu'(w)$ belong to $\sum_{l=0}^{mr-r}\mathbb{C}w^l dw/w^{mr-r+1}$. Thus we get a formal isomorphism $\pi'\colon(W,\nabla)\xrightarrow{\sim}(\mathbb{C}[[w]],\nabla_{\nu(w)})$. □

2. Definition and construction of the moduli space of parabolic connections of generic ramified type. Let C be a smooth projective irreducible curve over $\mathrm{Spec}\,\mathbb{C}$ of genus g and $D=\sum_{i=1}^n m_it_i$ be an effective divisor on C with $m_i\ge 1$ for any i. Take integers $r,a,(r^{(i)}_j)^{1\le i\le n}_{0\le j\le s_i-1}$ satisfying $r>0$, $r^{(i)}_j>0$ and $r=\sum_{j=0}^{s_i-1}r^{(i)}_j$ for any i. If $m_i=1$, we further assume that $r^{(i)}_j=1$ for all j. We take a generator z_i of the maximal ideal of $\mathcal{O}_{C,t_i}$ and take a variable $w^{(i)}_j$ satisfying $(w^{(i)}_j)^{r^{(i)}_j}=z_i$. Then the completion $\hat{\mathcal{O}}_{C,t_i}$ of $\mathcal{O}_{C,t_i}$ is isomorphic to the formal power series ring $\mathbb{C}[[z_i]]$ in z_i. The formal power series ring $\mathbb{C}[[w^{(i)}_j]]$ in $w^{(i)}_j$ becomes a free $\mathbb{C}[[z_i]]$-module of rank $r^{(i)}_j$. Note that we have

$$dz_i=r^{(i)}_j(w^{(i)}_j)^{r^{(i)}_j-1}dw^{(i)}_j.$$

We denote by $\mathcal{O}_{m_i t_i}$ the structure sheaf of the effective divisor $m_i t_i$ which is considered as a closed subscheme of C. So we have $\mathcal{O}_{m_i t_i} \cong \mathbb{C}[z_i]/(z_i^{m_i})$. We set

$$
\begin{aligned}
&N((r^{(i)}_j), a) \\
&:= \left\{ \boldsymbol{\nu} = \left(\nu^{(i)}_j(w^{(i)}_j)\right)^{1\leq i\leq n}_{0\leq j\leq s_i-1} \;\middle|\;
\begin{array}{l}
\nu^{(i)}_j(w^{(i)}_j) = \displaystyle\sum_{k=0}^{r^{(i)}_j-1} \nu^{(i)}_{j,k}(z_i)(w^{(i)}_j)^k, \\
\nu^{(i)}_{j,k}(z_i) \in \sum_{l=0}^{m_i-2} \mathbb{C} z_i^l dz_i / z_i^{m_i} \text{ for } 1 \leq k \leq r^{(i)}_j - 1, \\
\nu^{(i)}_{j,0}(z_i) \in \sum_{l=0}^{m_i-1} \mathbb{C} z_i^l dz_i / z_i^{m_i} \text{ and} \\
a + \displaystyle\sum_{i=1}^{n} \sum_{j=0}^{s_i-1} \left(r^{(i)}_j \operatorname{res}_{z^{(i)}_j}(\nu^{(i)}_{j,0}(z_i)) + \frac{r^{(i)}_j - 1}{2} \right) = 0
\end{array}
\right\}
\end{aligned}
$$

and call an element of $N((r^{(i)}_j), a)$ a ramified exponent. Note that a ramified exponent $\boldsymbol{\nu} \in N((r^{(i)}_j), a)$ is determined by the coefficients of $\nu^{(i)}_{j,k}(z_i)$.

Definition 2.1. Take a ramified exponent $\boldsymbol{\nu} = \left(\nu^{(i)}_j(w^{(i)}_j)\right)^{1\leq i\leq n}_{0\leq j\leq s_i-1} \in N((r^{(i)}_j), a)$. We say that a tuple $(E, \nabla, \{l^{(i)}_j, V^{(i)}_{j,k}, L^{(i)}_{j,k}, \pi^{(i)}_{j,k}, \phi^{(i)}_{j,k}\})$ is a parabolic connection of generic ramified type with the exponent $\boldsymbol{\nu}$ if

(i) E is an algebraic vector bundle on C of rank r and degree a,
(ii) $\nabla\colon E \longrightarrow E \otimes \Omega^1_C(D)$ is an algebraic connection admitting poles along D,
(iii) $E|_{m_i t_i} = l^{(i)}_0 \supset l^{(i)}_1 \supset \cdots \supset l^{(i)}_{s_i-1} \supset l^{(i)}_{s_i} = 0$ is a filtration by $\mathcal{O}_{m_i t_i}$-submodules such that $l^{(i)}_j / l^{(i)}_{j+1} \cong \mathcal{O}_{m_i t_i}^{\oplus r^{(i)}_j}$ and $\nabla|_{m_i t_i}(l^{(i)}_j) \subset l^{(i)}_j \otimes \Omega^1_C(D)$ for $1 \leq i \leq n$ and $0 \leq j \leq s_i - 1$,
(iv) $\left((V^{(i)}_{j,k}, L^{(i)}_{j,k}, \pi^{(i)}_{j,k})_{0\leq k\leq r^{(i)}_j-1}, (\phi^{(i)}_{j,k})_{1\leq k\leq r^{(i)}_j} \right)$ is a $\nu^{(i)}_j(w^{(i)}_j)$-ramified structure on $(l^{(i)}_j / l^{(i)}_{j+1}, \nabla^{(i)}_j)$ in the sense of Definition 1.2, where $\nabla^{(i)}_j \colon l^{(i)}_j / l^{(i)}_{j+1} \longrightarrow l^{(i)}_j / l^{(i)}_{j+1} \otimes \Omega^1_C(D)$ is the homomorphism induced by $\nabla|_{m_i t_i}$.

Remark 2.1. If the leading terms $\{\nu^{(i)}_{j,0}(0)\}_{0\leq j\leq s_i-1}$ are distinct for any i and if $\nu^{(i)}_{j,1}(0) \neq 0$ for any i, then we can see by Proposition 1.3 that $(E, \nabla) \otimes \widehat{\mathcal{O}}_{C,t_i} \cong \bigoplus_{j=0}^{s_i-1} \left(\mathbb{C}[[w^{(i)}_j]], \nabla_{\nu(w^{(i)}_j)}\right)$ at each point t_i. Without the condition, we can no longer expect the formal structure to be given by the exponent $\boldsymbol{\nu}$. The above formulation is motivated by the author's hope to construct the moduli space as a canonical degeneration of the full dimensional moduli space of regular singular connections, but we should be careful not to confuse the meaning of the moduli.

Definition 2.2. Take a tuple of rational numbers $\boldsymbol{\alpha} = (\alpha^{(i)}_j)^{1\leq i\leq n}_{1\leq j\leq s_i}$ satisfying

$$0 < \alpha^{(i)}_1 < \alpha^{(i)}_2 < \cdots < \alpha^{(i)}_{s_i} < 1$$

for any i and $\alpha^{(i)}_j \neq \alpha^{(i')}_{j'}$ for $(i,j) \neq (i',j')$. We call $\boldsymbol{\alpha}$ a parabolic weight. We say that a parabolic connection $(E, \nabla, \{l^{(i)}_j, V^{(i)}_{j,k}, L^{(i)}_{j,k}, \pi^{(i)}_{j,k}, \phi^{(i)}_{j,k}\})$ of generic ramified type is $\boldsymbol{\alpha}$-stable (resp. $\boldsymbol{\alpha}$-semistable) if the inequality

$$
\begin{aligned}
&\frac{\deg F + \sum_{i=1}^n \sum_{j=1}^{s_i} \alpha^{(i)}_j \operatorname{length}((F|_{m_i t_i} \cap l^{(i)}_{j-1}) / (F|_{m_i t_i} \cap l^{(i)}_j))}{\operatorname{rank} F} \\
&\quad \underset{(\text{resp. } \leq)}{<} \frac{\deg E + \sum_{i=1}^n \sum_{j=1}^{s_i} \alpha^{(i)}_j \operatorname{length}(l^{(i)}_{j-1} / l^{(i)}_j)}{\operatorname{rank} E}
\end{aligned}
$$

holds for any non-trivial subbundle $0 \neq F \subsetneq E$ satisfying $\nabla(F) \subset F \otimes \Omega^1_C(D)$.

Remark 2.2. If $(E, \nabla, \{l^{(i)}_j, V^{(i)}_{j,k}, L^{(i)}_{j,k}, \pi^{(i)}_{j,k}, \phi^{(i)}_{j,k}\})$ is $\boldsymbol{\alpha}$-stable, we can see that there are only scalar endomorphisms $u \colon E \longrightarrow E$ satisfying $\nabla \circ u = u \circ \nabla$ and $\nabla|_{m_i t_i}(l^{(i)}_j) \subset l^{(i)}_j$ for any i, j.

Remark 2.3. If $s_i = 1$ for some i and if the $\dfrac{w^{(i)}_0 dw^{(i)}_0}{(w^{(i)}_0)^{m_i r^{(i)}_0 - r^{(i)}_0 + 1}}$-coefficient of $\nu^{(i)}_0(w^{(i)}_0)$ is not zero, then (E, ∇) is formally irreducible at t_i by Proposition 1.2 and Proposition 1.3. So a parabolic connection $(E, \nabla, \{l^{(i)}_j, V^{(i)}_{j,k}, L^{(i)}_{j,k}, \pi^{(i)}_{j,k}, \phi^{(i)}_{j,k}\})$ of generic ramified type with the exponent $\boldsymbol{\nu}$ is $\boldsymbol{\alpha}$-stable with respect to any weight $\boldsymbol{\alpha}$ in this case.

In the following, we give the definition of the moduli functor of parabolic connections of generic ramified type in order to make the meaning of the moduli clear.

Definition 2.3. We define a contravariant functor $\mathcal{M}^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j), a)$: $(\mathrm{Sch}/N((r^{(i)}_j), a))^o \longrightarrow (\mathrm{Sets})$ from the category $(\mathrm{Sch}/N((r^{(i)}_j), a))$ of noetherian schemes over $N((r^{(i)}_j), a)$ to the category (Sets) of sets by setting

$$\mathcal{M}^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j), a)(S) = \Big\{(E, \nabla, \{l^{(i)}_j, V^{(i)}_{j,k}, L^{(i)}_{j,k}, \pi^{(i)}_{j,k}, \phi^{(i)}_{j,k}\})\Big\} / \sim,$$

for a noetherian scheme S over $N((r^{(i)}_j), a)$, where

(i) E is a vector bundle on $C \times S$ of rank r and $\deg(E|_{C \times s}) = a$ for any point $s \in S$,

(ii) $\nabla \colon E \longrightarrow E \otimes \Omega^1_{C \times S/S}(D \times S)$ is a relative connection,

(iii) $E|_{m_i t_i \times S} = l^{(i)}_0 \supset l^{(i)}_1 \supset \cdots \supset l^{(i)}_{s_i - 1} \supset l^{(i)}_{s_i} = 0$ is a filtration by $\mathcal{O}_{m_i t_i \times S}$-submodules such that $\nabla|_{m_i t_i \times S}(l^{(i)}_j) \subset l^{(i)}_j \otimes \Omega^1_C(D)$ for any i, j and each $l^{(i)}_j / l^{(i)}_{j+1}$ is a locally free $\mathcal{O}_{m_i t_i \times S}$-module of rank $r^{(i)}_j$,

(iv) for the homomorphism $\nabla^{(i)}_j \colon l^{(i)}_j / l^{(i)}_{j+1} \longrightarrow l^{(i)}_j / l^{(i)}_{j+1} \otimes \Omega^1_C(D)$ induced by $\nabla|_{m_i t_i \times S}$, a tuple $\Big((V^{(i)}_{j,k}, L^{(i)}_{j,k}, \pi^{(i)}_{j,k})_{0 \leq k \leq r^{(i)}_j - 1}, (\phi^{(i)}_{j,k})_{1 \leq k \leq r^{(i)}_j} \Big)$ is a S-flat family of $\nu^{(i)}_j(w^{(i)}_j)$-ramified structure on $(l^{(i)}_j / l^{(i)}_{j+1}, \nabla^{(i)}_j)$ in the following sense, where $\boldsymbol{\nu} = (\nu^{(i)}_j(w^{(i)}_j))$ is a family of ramified exponents determined by the structure morphism $S \longrightarrow N((r^{(i)}_j), a)$:

(a) $l^{(i)}_j / l^{(i)}_{j+1} = V^{(i)}_{j,0} \supset \cdots \supset V^{(i)}_{j, r^{(i)}_j - 1} \supset z_i V^{(i)}_{j,0}$ is a filtration by $\mathcal{O}_{m_i t_i \times S}$-submodules such that $V^{(i)}_{j, r^{(i)}_j - 1} / z_i V^{(i)}_{j,0}$ and $V^{(i)}_{j,k} / V^{(i)}_{j,k+1}$ for $0 \leq k \leq r^{(i)}_j - 2$ are locally free $\mathcal{O}_{t_i \times S}$-modules of rank one and that $\nabla^{(i)}_j(V^{(i)}_{j,k}) \subset V^{(i)}_{j,k} \otimes \Omega^1_C(D)$ for any k,

(b) for $0 \leq k \leq r^{(i)}_j - 1$, $\pi^{(i)}_{j,k} \colon V^{(i)}_{j,k} \otimes \mathcal{O}_S[w^{(i)}_j] / ((w^{(i)}_j)^{m_i r^{(i)}_j - r^{(i)}_j + 1}) \longrightarrow L^{(i)}_{j,k}$ is a locally free quotient $\mathcal{O}_S[w^{(i)}_j] / ((w^{(i)}_j)^{m_i r^{(i)}_j - r^{(i)}_j + 1})$-module of rank one whose restriction $\pi^{(i)}_{j,k}|_{V^{(i)}_{j,k}} \colon V^{(i)}_{j,k} \longrightarrow L^{(i)}_{j,k}$ is surjective for any k and the diagram

$$\begin{array}{ccc} V^{(i)}_{j,k} & \xrightarrow{\pi^{(i)}_{j,k}|_{V^{(i)}_{j,k}}} & L^{(i)}_{j,k} \\ \nabla^{(i)}_j \Big\downarrow & & \Big\downarrow \nu^{(i)}_j(w^{(i)}_j)+\frac{k\,dz_i}{r^{(i)}_j z_i} \\ V^{(i)}_{j,k}\otimes\Omega^1_{C\times S/S}(D\times S) & \xrightarrow{(\pi^{(i)}_{j,k}\otimes\mathrm{id})|_{V^{(i)}_{j,k}\otimes\Omega^1_{C\times S/S}(D)}} & L^{(i)}_{j,k}\otimes\Omega^1_{C\times S/S}(D\times S) \end{array}$$

is commutative,

(c) $\phi^{(i)}_{j,r^{(i)}_j}\colon (z_i)/(z_i^{m_i+1})\otimes L^{(i)}_{j,0}\longrightarrow L^{(i)}_{j,r^{(i)}_j-1}$ and $\phi^{(i)}_{j,k}\colon L^{(i)}_{j,k}\longrightarrow L^{(i)}_{j,k-1}$ for $k=1,\dots,r^{(i)}_j-1$ are $\mathcal{O}_S[w^{(i)}_j]$-homomorphisms such that $\mathrm{Im}(\phi^{(i)}_{j,k})=w^{(i)}_j L^{(i)}_{j,k-1}$ for $k=1,\dots,r^{(i)}_j$ and that the diagrams

$$\begin{array}{ccc} V^{(i)}_{j,k} & \xrightarrow{\pi^{(i)}_{j,k}|_{V^{(i)}_{j,k}}} & L^{(i)}_{j,k} \\ \Big\downarrow & & \Big\downarrow \phi^{(i)}_{j,k} \\ V^{(i)}_{j,k-1} & \xrightarrow{\pi^{(i)}_{j,k-1}|_{V^{(i)}_{j,k-1}}} & L^{(i)}_{j,k-1} \end{array}$$

for $k=1,\dots,r^{(i)}_j-1$ and the diagram

$$\begin{array}{ccc} (z_i)/(z_i^{m_i+1})\otimes V^{(i)}_{j,0} & \xrightarrow{\mathrm{id}\otimes\pi^{(i)}_{j,0}|_{V^{(i)}_{j,0}}} & (z_i)/(z_i^{m_i+1})\otimes L^{(i)}_{j,0} \\ \Big\downarrow & & \Big\downarrow \phi^{(i)}_{j,r^{(i)}_j} \\ V^{(i)}_{j,r^{(i)}_j-1} & \xrightarrow{\pi^{(i)}_{j,r^{(i)}_j-1}|_{V^{(i)}_{j,r^{(i)}_j-1}}} & L^{(i)}_{j,r^{(i)}_j-1} \end{array}$$

are commutative,

(d) there are $\mathcal{O}_S[w^{(i)}_j]$-isomorphisms

$$\psi^{(i)}_{j,k}\colon L^{(i)}_{j,k}\xrightarrow{\sim}(w^{(i)}_j)/((w^{(i)}_j)^{m_i r^{(i)}_j-r^{(i)}_j+2})\otimes L^{(i)}_{j,k-1}$$

for $1\le k\le r^{(i)}_j-1$ such that the composition

$$L^{(i)}_{j,k}\xrightarrow[\sim]{\psi^{(i)}_{j,k}}(w^{(i)}_j)/((w^{(i)}_j)^{m_i r^{(i)}_j-r^{(i)}_j+2})\otimes L^{(i)}_{j,k-1}\longrightarrow w^{(i)}_j L^{(i)}_{j,k-1}\hookrightarrow L^{(i)}_{j,k-1}$$

coincides with $\phi^{(i)}_{j,k}\colon L^{(i)}_{j,k}\longrightarrow L^{(i)}_{j,k-1}$ and that the composition

$$\begin{aligned}\psi^{(i)}_{j,1}\circ\cdots\circ\psi^{(i)}_{j,r^{(i)}_j-1}\circ\phi^{(i)}_{j,r^{(i)}_j}\colon (z_i)/(z_i^{m_i})\otimes L^{(i)}_{j,0}& \\ \longrightarrow((w^{(i)}_j)^{r^{(i)}_j-1})/((w^{(i)}_j)^{m_i r^{(i)}_j})\otimes L^{(i)}_{j,0}&\end{aligned}$$

coincides with the homomorphism induced by $(z_i)/(z_i^{m_i})\longrightarrow((w^{(i)}_j)^{r^{(i)}_j-1})/((w^{(i)}_j)^{m_i r^{(i)}_j})$,

(v) for any geometric point $\operatorname{Spec} K \longrightarrow S$, the fiber $(E, \nabla, \{l_j^{(i)}, V_{j,k}^{(i)}, L_{j,k}^{(i)}, \pi_{j,k}^{(i)}, \phi_{j,k}^{(i)}\}) \otimes K$ is $\boldsymbol{\alpha}$-stable.

Here $(E, \nabla, \{l_j^{(i)}, V_{j,k}^{(i)}, L_{j,k}^{(i)}, \pi_{j,k}^{(i)}, \phi_{j,k}^{(i)}\}) \sim (E', \nabla', \{l_j'^{(i)}, V_{j,k}'^{(i)}, L_{j,k}'^{(i)}, \pi_{j,k}'^{(i)}, \phi_{j,k}'^{(i)}\})$ if there are a line bundle $\mathcal{L}$ on S and

(α) isomorphisms $\theta \colon E \xrightarrow{\sim} E' \otimes \mathcal{L}$ of vector bundles on $C \times S$,

(β) isomorphisms $\vartheta_{j,k}^{(i)} \colon L_{j,k}^{(i)} \xrightarrow{\sim} L_{j,k}'^{(i)} \otimes \mathcal{L}$ of $\mathcal{O}_S[w_j^{(i)}]/((w_j^{(i)})^{m_i r_j^{(i)} - r_j^{(i)} + 1})$-modules for any i, j, k

such that the equalities

$$(\theta \otimes \mathrm{id}) \circ \nabla = \nabla' \circ \theta$$
$$\theta|_{m_i t_i \times S}(l_j^{(i)}) = l_j'^{(i)} \otimes \mathcal{L} \qquad \text{(for any } i, j)$$

hold and that for the induced isomorphism $\theta_j^{(i)} \colon l_j^{(i)}/l_{j+1}^{(i)} \xrightarrow{\sim} (l_j'^{(i)}/l_{j+1}'^{(i)}) \otimes \mathcal{L}$, the equalities

$$\theta_j^{(i)}(V_{j,k}^{(i)}) = V_{j,k}'^{(i)} \otimes \mathcal{L} \qquad (0 \leq k \leq r_j^{(i)} - 1)$$
$$(\pi_{j,k}'^{(i)} \otimes 1)|_{V_{j,k}'^{(i)} \otimes \mathcal{L}} \circ \theta_j^{(i)}|_{V_{j,k}^{(i)}} = \vartheta_{j,k}^{(i)} \circ \pi_{j,k}^{(i)}|_{V_{j,k}^{(i)}} \qquad (0 \leq k \leq r_j^{(i)} - 1)$$
$$\vartheta_{j,k-1}^{(i)} \circ \phi_{j,k}^{(i)} = (\phi_{j,k}'^{(i)} \otimes 1) \circ \vartheta_{j,k}^{(i)} \qquad (1 \leq k \leq r_j^{(i)} - 1)$$
$$\vartheta_{j, r_j^{(i)} - 1}^{(i)} \circ (\mathrm{id}_{(z)} \otimes \phi_{j,0}^{(i)}) = \phi_{j, r_j^{(i)}}'^{(i)} \circ (\mathrm{id}_{(z)} \otimes \vartheta_{j,0}^{(i)})$$

hold.

THEOREM 2.1. *There exists a relative coarse moduli scheme $M_{C,D}^{\boldsymbol{\alpha}}((r_j^{(i)}), a)$ of $\boldsymbol{\alpha}$-stable parabolic connections of generic ramified type over $N((r_j^{(i)}), a)$. Moreover, $M_{C,D}^{\boldsymbol{\alpha}}((r_j^{(i)}), a) \longrightarrow N((r_j^{(i)}), a)$ is a quasi-projective morphism.*

Proof. We consider the moduli functor $\mathcal{X} \colon (\mathrm{Sch}/\mathbb{C})^o \longrightarrow (\mathrm{Sets})$ defined by

$$\mathcal{X}(S) = \left\{ (E, \nabla, \{l_j^{(i)}\}) \right\} / \sim$$

for a noetherian scheme S over $\operatorname{Spec} \mathbb{C}$, where

(a) E is a vector bundle on $C \times S$ of rank r and $\deg(E|_{C \times \{s\}}) = a$ for any $s \in S$,

(b) $\nabla \colon E \longrightarrow E \otimes \Omega^1_{C \times S/S}(D \times S)$ is a relative connection,

(c) $E|_{m_i t_i \times S} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{s_i - 1}^{(i)} \supset l_{s_i}^{(i)} = 0$ is a filtration by $\mathcal{O}_{m_i t_i \times S}$-submodules such that $\nabla|_{m_i t_i \times S}(l_j^{(i)}) \subset l_j^{(i)} \otimes \Omega^1_{C \times S/S}(D \times S)$ for any i, j and that each $l_j^{(i)}/l_{j+1}^{(i)}$ is a locally free $\mathcal{O}_{m_i t_i \times S}$-module of rank $r_j^{(i)}$,

(d) for any geometric point s of S and for any non-trivial subbundle $0 \neq F \subsetneq E|_{C \times s}$ satisfying $\nabla(F) \subset F \otimes \Omega^1_{C \times s}(D \times s)$, the inequality

$$\frac{\deg F + \sum_{i=1}^n \sum_{j=1}^{s_i} \alpha_j^{(i)} \operatorname{length}((F|_{m_i t_i \times s} \cap l_{j-1}^{(i)}|_{m_i t_i \times s})/(F|_{m_i t_i \times s} \cap l_j^{(i)}|_{m_i t_i \times s}))}{\operatorname{rank} F}$$
$$< \frac{\deg(E|_{C \times s}) + \sum_{i=1}^n \sum_{j=1}^{s_i} \alpha_j^{(i)} \operatorname{length}(l_{j-1}^{(i)}|_{m_i t_i \times s}/l_j^{(i)}|_{m_i t_i \times s})}{\operatorname{rank} E}$$

holds.

Here $(E,\nabla,\{l_j^{(i)}\}) \sim (E',\nabla',\{l'^{(i)}_j\})$ if there are a line bundle $\mathcal{L}$ on S and an isomorphism $\theta\colon E \xrightarrow{\sim} E'\otimes\mathcal{L}$ satisfying $\nabla'\theta=(\theta\otimes\mathrm{id})\nabla$ and $\theta|_{m_it_i\times S}(l_j^{(i)})=l'^{(i)}_j\otimes\mathcal{L}$ for any i,j. By a similar argument to that of [13, Theorem 2.1], we can realize a coarse moduli scheme X of $\mathcal{X}$ as a locally closed subscheme of the moduli space of stable parabolic Λ^1_D-triples constructed in [10, Theorem 5.1]. So X is quasi-projective and there is a universal family $(\tilde{E},\tilde{\nabla},\{\tilde{l}_j^{(i)}\})$ over some étale covering $X'\longrightarrow X$. We consider the moduli functor $\mathcal{Y}\colon (\mathrm{Sch}/X')^o\longrightarrow(\mathrm{Sets})$ defined by

$$\mathcal{Y}(S)=\left\{(V_{j,k}^{(i)},L_{j,k}^{(i)},\pi_{j,k}^{(i)})^{1\le i\le n}_{0\le j\le s_i-1,0\le k\le r_j^{(i)}-1}\right\}$$

for a noetherian scheme S over X', where

(e) $l_j^{(i)}/l_{j+1}^{(i)}\otimes\mathcal{O}_S = V_{j,0}^{(i)}\supset V_{j,1}^{(i)}\supset\cdots\supset V^{(i)}_{j,r_j^{(i)}-1}\supset z_iV_{j,0}^{(i)}$ is a filtration by $\mathcal{O}_{m_it_i\times S}$-submodules such that each $V_{j,0}^{(i)}/V_{j,k}^{(i)}$ is flat over S and $\dim_{k(s)}((V_{j,k}^{(i)}/V_{j,k+1}^{(i)})\otimes k(s))=\dim_{k(s)}((V^{(i)}_{j,r_j^{(i)}-1}/z_iV_{j,0}^{(i)})\otimes k(s))=1$ for any $s\in S$,

(f) $\pi_{j,k}^{(i)}\colon V_{j,k}^{(i)}\otimes\mathcal{O}_S[w_j^{(i)}]/((w_j^{(i)})^{m_ir_j^{(i)}-r_j^{(i)}+1})\longrightarrow L_{j,k}^{(i)}$ is a quotient $\mathcal{O}_S[w_j^{(i)}]$-module such that $L_{j,k}^{(i)}$ is a locally free $\mathcal{O}_S[w_j^{(i)}]/((w_j^{(i)})^{m_ir_j^{(i)}-r_j^{(i)}+1})$-module of rank one and the restriction $\pi_{j,k}^{(i)}|_{V_{j,k}^{(i)}}\colon V_{j,k}^{(i)}\longrightarrow L_{j,k}^{(i)}$ is surjective for $k=0,\dots,r_j^{(i)}-1$.

We can see that $\mathcal{Y}$ can be represented by a locally closed subscheme Y of a product of flag schemes over X'. Let $(\tilde{V}_{j,k}^{(i)},\tilde{L}_{j,k}^{(i)},\tilde{\pi}_{j,k}^{(i)})$ be a universal family over Y. We also take a pull-back $\tilde{\boldsymbol{\nu}}=(\tilde{\nu}_j^{(i)})$ to $Y\times N((r_j^{(i)}),a)$ of the universal family over $N((r_j^{(i)}),a)$. There is a maximal locally closed subscheme Y' of $Y\times N((r_j^{(i)}),a)$ such that the composition

$$\begin{aligned}\tilde{V}_{j,k}^{(i)}\otimes\mathcal{O}_{Y'}[w_j^{(i)}]/((w_j^{(i)})^{m_ir_j^{(i)}-r_j^{(i)}+1})&\longrightarrow \tilde{V}_{j,k-1}^{(i)}\otimes\mathcal{O}_{Y'}[w_j^{(i)}]/((w_j^{(i)})^{m_ir_j^{(i)}-r_j^{(i)}+1})\\&\xrightarrow{\tilde{\pi}_{j,k-1}^{(i)}}(\tilde{L}_{j,k-1}^{(i)})_{Y'}\end{aligned}$$

factors through an $\mathcal{O}_{Y'}[w_j^{(i)}]/((w_j^{(i)})^{m_ir_j^{(i)}-r_j^{(i)}+1})$-homomorphism

$$\tilde{\phi}_{j,k}^{(i)}\colon(\tilde{L}_{j,k}^{(i)})_{Y'}\longrightarrow(\tilde{L}_{j,k-1}^{(i)})_{Y'}$$

whose image is $w_j^{(i)}(\tilde{L}_{j,k-1}^{(i)})_{Y'}$ for $1\le k\le r_j^{(i)}-1$, the composition

$$(z_i)/(z_i^{m_i+1})\otimes\tilde{V}_{j,0}^{(i)}\longrightarrow\tilde{V}^{(i)}_{j,r_j^{(i)}-1}\otimes\mathcal{O}_{Y'}[w_j^{(i)}]/((w_j^{(i)})^{m_ir_j^{(i)}-r_j^{(i)}+1})\xrightarrow{\tilde{\pi}^{(i)}_{j,r_j^{(i)}-1}}(\tilde{L}^{(i)}_{j,r_j^{(i)}-1})_{Y'}$$

. factors through an $\mathcal{O}_{Y'}[w_j^{(i)}]/((w_j^{(i)})^{m_ir_j^{(i)}-r_j^{(i)}+1})$-homomorphism

$$\tilde{\phi}^{(i)}_{r_j^{(i)}}\colon(z)/(z^{m_i+1})\otimes(\tilde{L}_{j,0}^{(i)})_{Y'}\longrightarrow(\tilde{L}^{(i)}_{j,r_j^{(i)}-1})_{Y'}$$

whose image is $w_j^{(i)}\tilde{L}^{(i)}_{j,r_j^{(i)}-1}$ and the diagrams

$$\begin{array}{ccc}
\tilde{V}^{(i)}_{j,k}\otimes\mathcal{O}_{Y'}[w_j^{(i)}]/((w_j^{(i)})^{m_ir_j^{(i)}-r_j^{(i)}+1}) & \xrightarrow{\tilde{\pi}^{(i)}_{j,k}\otimes\mathrm{id}} & \tilde{L}^{(i)}_{j,k}\otimes\mathcal{O}_{Y'} \\
\nabla_j^{(i)}\otimes\mathrm{id}\big\downarrow & & \big\downarrow \tilde{\nu}_j^{(i)}(w_j^{(i)})+\frac{k\,dz_i}{r_j^{(i)}z_i} \\
\tilde{V}^{(i)}_{j,k}\otimes\Omega^1_C(D)\otimes\mathcal{O}_{Y'}[w_j^{(i)}]/((w_j^{(i)})^{m_ir_j^{(i)}-r_j^{(i)}+1}) & \xrightarrow{\tilde{\pi}^{(i)}_{j,k}\otimes\mathrm{id}} & \tilde{L}^{(i)}_{j,k}\otimes\Omega^1_C(D)\otimes\mathcal{O}_{Y'}
\end{array}$$

are commutative for all i,j,k.

Consider the product

$$Z=\prod_{i,j,k}\mathrm{Spec}\left(\mathrm{Sym}\left(\mathcal{H}om_{Y'}\left(\mathcal{H}om_{\mathcal{O}_{Y'}[w_j^{(i)}]}\left(\tilde{L}^{(i)}_{j,k},\frac{\left(w_j^{(i)}\right)}{\left((w_j^{(i)})^{m_ir_j^{(i)}-r_j^{(i)}+2}\right)}\otimes\tilde{L}^{(i)}_{j,k-1}\right),\mathcal{O}_{Y'}\right)\right)\right)$$

of affine space bundles. There are universal sections

$$\tilde{\psi}^{(i)}_{j,k}\colon(\tilde{L}^{(i)}_{j,k})_Z\longrightarrow\left((w_j^{(i)})/((w_j^{(i)})^{m_ir_j^{(i)}-r_j^{(i)}+2})\otimes\tilde{L}^{(i)}_{j,k-1}\right)_Z$$

for $1\le k\le r_j^{(i)}-1$. Let Z' be the maximal locally closed subscheme of Z such that $(\tilde{\psi}^{(i)}_{j,k})_{Z'}$ are isomorphic and that the composition

$$(\tilde{L}^{(i)}_{j,k})_{Z'}\xrightarrow{\tilde{\psi}^{(i)}_{j,k}}\left((w_j^{(i)})/((w_j^{(i)})^{m_ir_j^{(i)}-r_j^{(i)}+2})\otimes\tilde{L}^{(i)}_{j,k-1}\right)_{Z'}\longrightarrow\left(w_j^{(i)}\tilde{L}^{(i)}_{j,k-1}\right)_{Z'}$$

coincides with $(\tilde{\phi}^{(i)}_{j,k})_{Z'}$ for $1\le k\le r_j^{(i)}-1$ and any i,j. Consider the group scheme G over Y' whose set of S-valued points is

$$G(S)$$
$$:=\left\{(1+a^{(i)}_{j,k})\in\prod_{i,j,k}\mathcal{O}_S[w_j^{(i)}]/((w_j^{(i)})^{m_ir_j^{(i)}-r_j^{(i)}+1})\;\middle|\;\begin{array}{l}a^{(i)}_{j,k}\in((w_j^{(i)})^{m_ir_j^{(i)}-r_j^{(i)}-1})\text{ and}\\ a^{(i)}_{j,k-1}-a^{(i)}_{j,k}\in((w_j^{(i)})^{m_ir_j^{(i)}-r_j^{(i)}})\\ \text{for }1\le k\le r_j^{(i)}-1\end{array}\right\}$$

for any Y'-scheme S. Then we can see that $Z'\longrightarrow Y'$ is a principal G-bundle. Let Σ be the maximal closed subscheme of Z' such that the composition

$$(\tilde{\psi}^{(i)}_{j,1}\circ\cdots\circ\tilde{\psi}^{(i)}_{j,r_j^{(i)}-1}\circ\tilde{\phi}^{(i)}_{j,r_j^{(i)}})_\Sigma\colon(z_i)/(z_i^{m_i})\otimes(\tilde{L}^{(i)}_{j,0})_\Sigma\longrightarrow((w_j^{(i)})^{r_j^{(i)}-1})/((w_j^{(i)})^{m_ir_j^{(i)}})\otimes(\tilde{L}^{(i)}_{j,0})_\Sigma$$

coincides with the homomorphism induced by $(z_i)/(z_i^{m_i})\longrightarrow((w_j^{(i)})^{r_j^{(i)}-1})/((w_j^{(i)})^{m_ir_j^{(i)}})$. Then Σ is preserved by the action of G. So Σ descends to a closed subscheme M' of Y'.

By the construction, we can see that $M'\longrightarrow X'\times N((r_j^{(i)}),a)$ descends to a quasi-projective morphism $M\longrightarrow X\times N((r_j^{(i)}),a)$, that is,

$$M'\cong M\times_{X\times N((r_j^{(i)}),a)}(X'\times N((r_j^{(i)}),a)).$$

Then M is nothing but the desired moduli space $M^{\boldsymbol{\alpha}}_{C,D}((r_j^{(i)}),a)$. □

3. Smoothness and dimension of the moduli space of parabolic connections of generic ramified type. The aim of this section is to prove the following theorem:

THEOREM 3.1. *The structure morphism $M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a) \longrightarrow N((r^{(i)}_j),a)$ is a smooth morphism and its fiber $M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}$ over $\boldsymbol{\nu} \in N((r^{(i)}_j),a)$ is of equidimension*

$$2r^2(g-1)+2+\sum_{i=1}^{n}(r^2-r)m_i$$

if $M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}} \neq \emptyset$.

Before proving the theorem, we prepare a complex which determines a deformation theory of the moduli space of irregular connections of generic ramified type. From the construction of the moduli space $M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)$ in Theorem 2.1, we can see that there exists an étale surjective morphism $M' \longrightarrow M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)$ and a universal family $(E_{M'},\nabla_{M'},\{l^{(i)}_{M,j},\tilde{V}^{(i)}_{M',j,k},\tilde{L}^{(i)}_{M',j,k},\tilde{\pi}^{(i)}_{M',j,k},\phi^{(i)}_{M',j,k}\})$ of irregular connections of generic ramified type. Define a complex $\mathcal{F}^{\bullet}_{M'}$ of sheaves on $C\times M'$ by

$$\mathcal{F}^0_{M'} = \left\{ u \in \mathcal{E}nd(E_{M'}) \;\middle|\; \begin{array}{l} u|_{m_it_i\times M'}(l^{(i)}_{M',j}) \subset l^{(i)}_{M',j} \text{ for any } i,j \\ \text{and for the induced homomorphism} \\ u^{(i)}_j : l^{(i)}_{M',j}/l^{(i)}_{M',j+1} \longrightarrow l^{(i)}_{M',j}/l^{(i)}_{M',j+1}, \\ u^{(i)}_j(V^{(i)}_{M',j,k}) \subset V^{(i)}_{M',j,k} \text{ and} \\ \bar{\pi}^{(i)}_{M',j,k}((u^{(i)}_j\otimes \mathrm{id})(\ker\bar{\pi}^{(i)}_{M',j,k}))=0 \text{ for any } k \end{array} \right\},$$

$$\mathcal{F}^1_{M'} = \left\{ v \in \mathcal{E}nd(E_{M'})\otimes\Omega^1_C(D) \;\middle|\; \begin{array}{l} v|_{m_it_i\times M'}(l^{(i)}_{M',j}) \subset l^{(i)}_{M',j}\otimes\Omega^1_C(D) \text{ for any } i,j \\ \text{and for the induced homomorphism} \\ v^{(i)}_j : l^{(i)}_{M',j}/l^{(i)}_{M',j+1} \longrightarrow l^{(i)}_{M',j}/l^{(i)}_{M',j+1}\otimes\Omega^1_C(D), \\ v^{(i)}_j(V^{(i)}_{M',j,k}) \subset V^{(i)}_{M',j,k}\otimes\Omega^1_C(D) \text{ for any } k \text{ and} \\ (\bar{\pi}^{(i)}_{j,k}\otimes\mathrm{id})(v^{(i)}_j(V^{(i)}_{M',j,k}))=0 \text{ for any } k \end{array} \right\}, \tag{3}$$

$$\nabla_{\mathcal{F}^{\bullet}_{M'}} : \mathcal{F}^0_{M'} \ni u \;\mapsto\; \nabla u - u\nabla \in \mathcal{F}^1_{M'}.$$

For the proof of Theorem 3.1, we use the morphism

$$\begin{aligned} \det \,:\, M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a) &\longrightarrow M_{C,D}(1,a)\times_{N((1),a)} N((r^{(i)}_j),a) \\ (E,\nabla,\{l^{(i)}_j,V^{(i)}_{j,k},L^{(i)}_{j,k},\pi^{(i)}_{j,k},\phi^{(i)}_{j,k}\}) &\mapsto (\det(E,\nabla),\boldsymbol{\nu}), \end{aligned} \tag{4}$$

where $M_{C,D}(1,a)$ is the moduli space of pairs (L,∇_L) of a line bundle L on C of degree a and a connection $\nabla_L\colon L \longrightarrow L\otimes\Omega^1_C(D)$ and we set

$$N((1),a) = \left\{ (\lambda^{(i)})_{1\le i\le n} \in \mathbb{C}[z_i]/(z_i^{m_i})\frac{dz_i}{z_i^{m_i}} \;\middle|\; a+\sum_{i=1}^{n}\mathsf{res}(\lambda^{(i)})=0 \right\}$$

$$M_{C,D}(1,a) \;\ni\; (L,\nabla_L) \;\mapsto\; (\nabla|_{m_it_i})_{1\le i\le n} \;\in\; N((1),a)$$

$$\begin{aligned} N((r^{(i)}_j),a) \;\ni\; &\left(\nu^{(i)}_j(w^{(i)}_j)\right) \\ &\mapsto \left(\sum_{j=0}^{s_i-1}\left(r^{(i)}_j\nu^{(i)}_{j,0}(z_i) + (r^{(i)}_j-1)dz_i/2z_i \right)\right)_{1\le i\le n} \;\in\; N((1),a). \end{aligned}$$

Recall that $\nu^{(i)}_{j,0}(z_i)$ is determined from $\nu^{(i)}_j(w^{(i)}_j)$ by the equality

$$\nu^{(i)}_j(w^{(i)}_j) = \nu^{(i)}_{j,0}(z_i) + \nu^{(i)}_{j,1}(z_i)w^{(i)}_j + \cdots + \nu^{(i)}_{j,r^{(i)}_j-1}(z_i)(w^{(i)}_j)^{r^{(i)}_j-1}.$$

Note that $M_{C,D}(1,a)$ is an affine space bundle over the Jacobian variety of C and so it is smooth over $N((1),a)$. So it is sufficient to prove the following proposition for the smoothness of $M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)$.

PROPOSITION 3.1. *The morphism* $\det\colon M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a) \longrightarrow M_{C,D}(1,a) \times_{N((1),a)} N((r^{(i)}_j),a)$ *defined in (4) is a smooth morphism.*

Proof. Take an Artinian local ring A over $\mathbb{C}$ with the maximal ideal $\mathfrak{m}$ and an ideal I of A satisfying $\mathfrak{m}I = 0$. Assume that a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A/I & \xrightarrow{\ f\ } & \mathcal{M}^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a) \\ \big\downarrow & & \big\downarrow{\scriptstyle \det} \\ \operatorname{Spec} A & \xrightarrow{\ g\ } & M_{C,D}((1),a) \times_{N((1),a)} N((r^{(i)}_j),a) \end{array}$$

is given. The morphism g corresponds to a tuple $((L,\nabla_L),\tilde{\boldsymbol{\nu}})$, where L is a line bundle on $C\times\operatorname{Spec} A$, $\nabla_L\colon L \longrightarrow L\otimes\Omega^1_{C\times\operatorname{Spec} A/\operatorname{Spec} A}(D\times\operatorname{Spec} A)$ is a relative connection and $\tilde{\boldsymbol{\nu}} = (\tilde{\nu}^{(i)}_{j,k}) \in N((r^{(i)}_j),a)(A)$ satisfies

$$\nabla_L|_{m_it_i\times\operatorname{Spec} A} = \sum_{j=0}^{s_i-1}\Big(r^{(i)}_j\tilde{\nu}^{(i)}_{j,0}(z_i) + (r^{(i)}_j-1)dz_i/2z_i\Big).$$

If we put $\boldsymbol{\nu} = (\nu^{(i)}_j(z_i)) := \tilde{\boldsymbol{\nu}}\otimes A/I$, f corresponds to a flat family $(E,\nabla,\{l^{(i)}_j, V^{(i)}_{j,k}, L^{(i)}_{j,k}, \pi^{(i)}_{j,k}, \phi^{(i)}_{j,k}\})$ of parabolic connections of generic ramified type over A/I with the exponent $\boldsymbol{\nu}$. We can choose an isomorphism $\psi^{(i)}_{j,k}\colon L^{(i)}_{j,k} \xrightarrow{\sim} (w^{(i)}_j)\otimes L^{(i)}_{j,k-1}$ of $(A/I)[w^{(i)}_j]/((w^{(i)}_j)^{m_ir^{(i)}_j-r^{(i)}_j+1})$-modules such that the composition $L^{(i)}_{j,k} \xrightarrow[\sim]{\psi^{(i)}_{j,k}} (w^{(i)}_j)\otimes L^{(i)}_{j,k-1} \longrightarrow w^{(i)}_j L^{(i)}_{j,k-1} \hookrightarrow L^{(i)}_{j,k-1}$ coincides with $\phi^{(i)}_{j,k}$.

Take an affine open covering $\{U_\alpha\}$ of C such that $\sharp\{\alpha | t_i\in U_\alpha\} = 1$ for any i and $\sharp\{i | t_i \in U_\alpha\} \le 1$ for any α. For each i, we take α with $t_i\in U_\alpha$. By shrinking U_α if necessary, we can take a basis $\{e^{(i)}_{j,k}\}$ of $E|_{U_\alpha\times\operatorname{Spec} A/I}$ with the following properties

- $l^{(i)}_j$ is generated by $\left\{ \overline{e^{(i)}_{j',k}|_{m_it_i\times\operatorname{Spec} A/I}} \,\middle|\, j'\ge j, 0\le k\le r^{(i)}_j-1\right\}$
- the induced class $f^{(i)}_{j,k} = \overline{e^{(i)}_{j,k}|_{m_it_i\times\operatorname{Spec} A/I}} \in l^{(i)}_j/l^{(i)}_{j+1}$ belongs to $V^{(i)}_{j,k}$ and
- the image of $\pi^{(i)}_{j,k}(f^{(i)}_{j,k}) \in L^{(i)}_{j,k}$ in $((w^{(i)}_j)^k)\otimes L^{(i)}_{j,0}$ via the isomorphism

  $$L^{(i)}_{j,k} \xrightarrow[\sim]{\psi^{(i)}_{j,k}} (w^{(i)}_j)\otimes L^{(i)}_{j,k-1} \xrightarrow[\sim]{\psi^{(i)}_{j,k-1}} ((w^{(i)}_j)^2)\otimes L^{(i)}_{j,k-2} \xrightarrow[\sim]{\psi^{(i)}_{j,k-2}} \cdots \xrightarrow[\sim]{\psi^{(i)}_{j,1}} ((w^{(i)}_j)^k)\otimes L^{(i)}_{j,0}$$

  is equal to $(w^{(i)}_j)^k\otimes\pi^{(i)}_{j,0}(f^{(i)}_{j,0})$.

From the commutativity of the diagram in Definition 2.3 (b), we can see that the representation matrix of the homomorphism $\nabla^{(i)}_j\colon V^{(i)}_{j,0} \longrightarrow V^{(i)}_{j,0}\otimes\Omega^1_C(D)$ induced by

$\nabla|_{m_i t_i \times \mathrm{Spec}\, A/I}$, with respect to the basis $f^{(i)}_{j,0},\dots,f^{(i)}_{j,r^{(i)}_j-1}$, is given by

$$\begin{pmatrix} \nu^{(i)}_{j,0}(z_i) & z_i\nu^{(i)}_{j,r^{(i)}_j-1}(z_i) & \cdots & z_i\nu^{(i)}_{j,1}(z_i) \\ \nu'^{(i)}_{j,1}(z_i) & \nu^{(i)}_{j,0}(z_i)+\frac{1}{r^{(i)}_j}\frac{dz_i}{z_i} & \cdots & z_i\nu^{(i)}_{j,2}(z_i) \\ \vdots & \vdots & \ddots & \vdots \\ \nu'^{(i)}_{j,r^{(i)}_j-1}(z_i) & \nu'^{(i)}_{j,r^{(i)}_j-2}(z_i) & \cdots & \nu^{(i)}_{j,0}(z_i)+\frac{r^{(i)}_j-1}{r^{(i)}_j}\frac{dz_i}{z_i} \end{pmatrix},$$

where $\nu'^{(i)}_{j,k}(z_i)$ satisfies $\nu'^{(i)}_{j,k}(z_i)\equiv\nu^{(i)}_{j,k}(z_i) \pmod{z_i^{m_i-1}dz_i/z_i^{m_i}}$ for $k=1,\dots,r^{(i)}_j-1$.

We take a free $A[w^{(i)}_j]/((w^{(i)}_j)^{m_i r^{(i)}_j-r^{(i)}_j+1})$-module $\tilde{L}^{(i)}_{j,0}$ of rank one which is a lift of $L^{(i)}_{j,0}$. Then we can determine lifts $\tilde{L}^{(i)}_{j,k}$ of $L^{(i)}_{j,k}$ as free $A[w^{(i)}_j]/((w^{(i)}_j)^{m_i r^{(i)}_j-r^{(i)}_j+1})$-modules of rank one, together with lifts $\tilde{\psi}^{(i)}_{j,k}\colon \tilde{L}^{(i)}_{j,k}\xrightarrow{\sim}(w^{(i)}_j)\otimes\tilde{L}^{(i)}_{j,k-1}$ of $\psi^{(i)}_{j,k}$ for $k=1,\dots,r^{(i)}_j-1$. We define $\tilde{\phi}^{(i)}_{j,k}$ as the composition $\tilde{\phi}^{(i)}_{j,k}\colon \tilde{L}^{(i)}_{j,k}\xrightarrow[\sim]{\tilde{\psi}^{(i)}_{j,k}}(w^{(i)}_j)\otimes\tilde{L}^{(i)}_{j,k-1}\longrightarrow w^{(i)}_j\tilde{L}^{(i)}_{j,k-1}$ and define $\tilde{\phi}^{(i)}_{r^{(i)}_j}\colon (z_i)\otimes\tilde{L}^{(i)}_{j,0}\longrightarrow\tilde{L}^{(i)}_{r^{(i)}_j-1}$ as the composition

$$(z_i)\otimes\tilde{L}^{(i)}_{j,0}\longrightarrow((w^{(i)}_j)^{r^{(i)}_j-1})\otimes\tilde{L}^{(i)}_{j,0}\xrightarrow[\sim]{(\tilde{\psi}^{(i)}_{j,1})^{-1}}((w^{(i)}_j)^{r^{(i)}_j-2})\otimes\tilde{L}^{(i)}_{j,1}$$
$$\xrightarrow[\sim]{(\tilde{\psi}^{(i)}_{j,r^{(i)}_j-1})^{-1}\circ\cdots\circ(\tilde{\psi}^{(i)}_{j,2})^{-1}}\tilde{L}^{(i)}_{r^{(i)}_j-1}.$$

Choose a lift $\epsilon^{(i)}_{j,0}\in\tilde{L}^{(i)}_{j,0}$ of the generator $\pi^{(i)}_{j,0}(f^{(i)}_{j,0})$ of $L^{(i)}_{j,0}$. Let $\epsilon^{(i)}_{j,k}$ be the element of $\tilde{L}^{(i)}_{j,k}$ corresponding to $(w^{(i)}_j)^k\otimes\epsilon^{(i)}_{j,0}$ via the isomorphism

$$\tilde{L}^{(i)}_{j,k}\xrightarrow[\sim]{\tilde{\psi}^{(i)}_{j,k}}(w^{(i)}_j)\otimes\tilde{L}^{(i)}_{j,k-1}\xrightarrow[\sim]{\tilde{\psi}^{(i)}_{j,k-1}}((w^{(i)}_j)^2)\otimes\tilde{L}^{(i)}_{j,k-2}\xrightarrow[\sim]{\tilde{\psi}^{(i)}_{j,k-2}}\cdots\xrightarrow[\sim]{\tilde{\psi}^{(i)}_{j,1}}((w^{(i)}_j)^k)\otimes\tilde{L}^{(i)}_{j,0}.$$

We take a free $\mathcal{O}_{U_\alpha\times\mathrm{Spec}\,A}$-module E_α of rank r with a basis $\{\tilde{e}^{(i)}_{j,k}\}_{0\le j\le s_i-1,0\le k\le r^{(i)}_j-1}$ and an isomorphism $E_\alpha\otimes A/I\xrightarrow[\sim]{\tau_\alpha}E|_{U_\alpha\times\mathrm{Spec}\,A/I}$ sending $\tilde{e}^{(i)}_{j,k}\otimes A/I$ to $e^{(i)}_{j,k}$. We define $\tilde{l}^{(i)}_j\subset E_\alpha|_{m_i t_i\times\mathrm{Spec}\,A}$ as the submodule generated by $\left\{\tilde{e}^{(i)}_{j',k}\,\middle|\, j'\ge j\right\}$. Let $\tilde{f}^{(i)}_{j,k}$ be the image of $\tilde{e}^{(i)}_{j,k}$ in $\tilde{l}^{(i)}_j/\tilde{l}^{(i)}_{j+1}$. We define $\tilde{V}_{j,k}$ as the submodule of $\tilde{l}^{(i)}_j/\tilde{l}^{(i)}_{j+1}$ generated by $\tilde{f}^{(i)}_{j,k},\tilde{f}^{(i)}_{j,k+1},\dots,\tilde{f}^{(i)}_{j,r^{(i)}_j-1},z_i\tilde{f}^{(i)}_{j,0},\dots,z_i\tilde{f}^{(i)}_{j,k-1}$.

Define a homomorphism

$$\tilde{\pi}^{(i)}_{j,k}\colon\tilde{V}^{(i)}_{j,k}\otimes A[w^{(i)}_j]/((w^{(i)}_j)^{m_i r^{(i)}_j-r^{(i)}_j+1})\longrightarrow\tilde{L}^{(i)}_{j,k}$$

by setting

$$\tilde{\pi}^{(i)}_{j,k}(\tilde{f}^{(i)}_{j,l}):=\begin{cases}\epsilon^{(i)}_{j,k} & (\text{when } l=k)\\ (\tilde{\phi}^{(i)}_{j,k+1}\circ\tilde{\phi}^{(i)}_{j,l})(\epsilon^{(i)}_{j,l})=(w^{(i)}_j)^{l-k}\epsilon^{(i)}_{j,k} & (\text{when } l>k)\end{cases}$$
$$\tilde{\pi}^{(i)}_{j,k}(z_i\tilde{f}^{(i)}_{j,l}):=\Big(\tilde{\phi}_{j,k+1}\circ\cdots\circ\tilde{\phi}^{(i)}_{j,r^{(i)}_j}\circ(\mathrm{id}_{(z_i)}\otimes\tilde{\phi}^{(i)}_{j,1})\circ\cdots\circ(\mathrm{id}_{(z_i)}\otimes\tilde{\phi}^{(i)}_{j,l})\Big)(z_i\otimes\epsilon^{(i)}_{j,l})$$
$$=(w^{(i)}_j)^{r^{(i)}_j+l-k}\epsilon^{(i)}_{j,k}\qquad(\text{when } l<k).$$

By the construction, the diagrams

$$\begin{array}{ccc}
\tilde{V}^{(i)}_{j,k}\otimes A[w^{(i)}_j]/((w^{(i)}_j)^{m_i r^{(i)}_j - r^{(i)}_j+1}) & \xrightarrow{\tilde{\pi}^{(i)}_{j,k}} & \tilde{L}^{(i)}_{j,k} \\
\big\downarrow & & \big\downarrow \tilde{\phi}^{(i)}_{j,k} \\
\tilde{V}^{(i)}_{j,k-1}\otimes A[w^{(i)}_j]/((w^{(i)}_j)^{m_i r^{(i)}_j - r^{(i)}_j+1}) & \xrightarrow{\tilde{\pi}^{(i)}_{j,k-1}} & \tilde{L}^{(i)}_{j,k-1}
\end{array}$$

are commutative for $k=1,\dots,r^{(i)}_j-1$ and the diagram

$$\begin{array}{ccc}
(z_i)/(z_i^{m_i+1})\otimes V^{(i)}_{j,0} & \xrightarrow{1\otimes\pi^{(i)}_{j,0}} & (z_i)/(z_i^{m_i+1})\otimes L^{(i)}_{j,0} \\
\big\downarrow & & \big\downarrow \tilde{\phi}^{(i)}_{j,r^{(i)}_j} \\
\tilde{V}^{(i)}_{j,r^{(i)}_j-1}\otimes A[w^{(i)}_j]/((w^{(i)}_j)^{m_i r^{(i)}_j - r^{(i)}_j+1}) & \xrightarrow{\tilde{\pi}^{(i)}_{j,r^{(i)}_j-1}} & \tilde{L}^{(i)}_{j,r^{(i)}_j-1}
\end{array}$$

also commutes. Define a homomorphism $\tilde{l}^{(i)}_j/\tilde{l}^{(i)}_{j+1}\longrightarrow \tilde{l}^{(i)}_j/\tilde{l}^{(i)}_{j+1}\otimes\Omega^1_C(D)$ whose representation matrix with respect to the basis $\tilde{f}^{(i)}_{j,0},\dots,\tilde{f}^{(i)}_{j,r^{(i)}_j-1}$ is

$$\begin{pmatrix}
\tilde{\nu}^{(i)}_{j,0}(z_i) & z_i\tilde{\nu}^{(i)}_{j,r^{(i)}-1}(z_i) & \cdots & z_i\tilde{\nu}^{(i)}_{j,1}(z_i) \\
\tilde{\nu}'^{(i)}_{j,1}(z_i) & \tilde{\nu}^{(i)}_{j,0}(z_i)+\frac{1}{r^{(i)}_j}\frac{dz_i}{z_i} & \cdots & z_i\tilde{\nu}^{(i)}_{j,2}(z_i) \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{\nu}'^{(i)}_{j,r^{(i)}_j-1}(z_i) & \tilde{\nu}'^{(i)}_{j,r^{(i)}_j-2}(z_i) & \cdots & \tilde{\nu}^{(i)}_{j,0}(z_i)+\frac{r^{(i)}_j-1}{r^{(i)}_j}\frac{dz_i}{z_i}
\end{pmatrix},$$

where $\tilde{\nu}'^{(i)}_{j,k}(z_i)$ is a lift of $\nu'^{(i)}_{j,k}(z_i)$ satisfying $\tilde{\nu}'^{(i)}_{j,k}(z_i)\equiv\tilde{\nu}^{(i)}_{j,k}(z_i) \pmod{z_i^{m_i-1}dz_i/z_i^{m_i}}$. Then the diagrams

$$\begin{array}{ccc}
\tilde{V}^{(i)}_{j,k}\otimes A[w^{(i)}_j]/((w^{(i)}_j)^{m_i r^{(i)}_j - r^{(i)}_j+1}) & \xrightarrow{\tilde{\pi}^{(i)}_{j,k}} & \tilde{L}^{(i)}_{j,k} \\
{\scriptstyle \tilde{\nabla}^{(i)}_j|_{\tilde{V}^{(i)}_{j,k}}\otimes\mathrm{id}}\big\downarrow & & \big\downarrow {\scriptstyle \tilde{\nu}^{(i)}_j(w^{(i)}_j)+\frac{k\,dz_i}{r^{(i)}_j z_i}} \\
\tilde{V}^{(i)}_{j,k}\otimes\Omega^1_C(D)\otimes A[w^{(i)}_j]/((w^{(i)}_j)^{m_i r^{(i)}_j - r^{(i)}_j+1}) & \xrightarrow{\tilde{\pi}^{(i)}_{j,k}\otimes 1} & \tilde{L}^{(i)}_{j,k}\otimes\Omega^1_C(D)
\end{array}$$

are commutative for $k=0,\dots,r^{(i)}_j-1$.

Choose an isomorphism $\bigwedge^r E_\alpha \xrightarrow[\sim]{\sigma_\alpha} L|_{U_\alpha\otimes A}$ such that $\sigma_\alpha\otimes A/I$ is the given isomorphism $\bigwedge^r E|_{U_\alpha\otimes A/I}\xrightarrow{\sim} L|_{U_\alpha\otimes A/I}$. We can give a connection $\nabla_\alpha\colon E_\alpha\longrightarrow E_\alpha\otimes\Omega^1_C(D)$ which induces $\tilde{\nabla}^{(i)}_j$ on $\tilde{l}^{(i)}_j/\tilde{l}^{(i)}_{j+1}$ for each i,j. Then we have

$$\mathrm{Tr}\,(\nabla_\alpha|_{m_it_i\times\operatorname{Spec}A})=\sum_{j=0}^{s_i-1}\Big(r^{(i)}_j\tilde{\nu}^{(i)}_{j,0}(z_i)+(r^{(i)}_j-1)dz_i/z_i\Big).$$

So, after adjusting the diagonal entries of ∇_α, we may assume that the connection $\det(E_\alpha,\nabla_\alpha)$ induced by ∇_α on the determinant bundle is transformed to the connection $\nabla_L|_{U_\alpha\times\operatorname{Spec}A}$ via the isomorphism σ_α. Thus we obtain a local parabolic connection $(E_\alpha,\nabla_\alpha,\{\tilde{l}^{(i)}_j,\tilde{V}^{(i)}_{j,k},\tilde{L}^{(i)}_{j,k},\tilde{\pi}^{(i)}_{j,k},\tilde{\phi}^{(i)}_{j,k}\})$ of generic ramified type with the exponent $\tilde{\boldsymbol{\nu}}$ on $U_\alpha\times\operatorname{Spec}A$, which is a lift of the given parabolic connection $(E,\nabla,\{l^{(i)}_j,V^{(i)}_{j,k},L^{(i)}_{j,k},\pi^{(i)}_{j,k},\phi^{(i)}_{j,k}\})\big|_{U_\alpha\otimes A/I}$ of generic ramified type on $U_\alpha\otimes A/I$.

If $t_i\notin U_\alpha$ for any i, then we can easily give a local parabolic connection on $U_\alpha\otimes A$ which is a lift of $(E,\nabla,\{l^{(i)}_j,V^{(i)}_{j,k},L^{(i)}_{j,k},\pi^{(i)}_{j,k},\phi^{(i)}_{j,k}\})|_{U_\alpha\otimes A/I}$. Note that the data $\{\tilde{l}^{(i)}_j,\tilde{V}^{(i)}_{j,k},\tilde{L}^{(i)}_{j,k},\tilde{\pi}^{(i)}_{j,k},\tilde{\phi}^{(i)}_{j,k}\}$ is nothing in this case.

Recall the complex $\mathcal{F}^\bullet_{M'}$ defined in (3) and consider its restriction $\mathcal{F}^\bullet_x:=\mathcal{F}^\bullet_{M'}|_{C\times\{x\}}$, to a point x of M' lying over the given point $\operatorname{Spec}A/\mathfrak{m}\longrightarrow\mathcal{M}^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)$. Define a complex $\mathcal{F}^\bullet_{\mathfrak{sl},x}$ by

$$\begin{aligned}\mathcal{F}^0_{\mathfrak{sl},x}&:=\left\{u\in\mathcal{F}^0_x\;\middle|\;\operatorname{Tr}(u)=0\right\}\\ \mathcal{F}^1_{\mathfrak{sl},x}&:=\left\{v\in\mathcal{F}^1_x\;\middle|\;\operatorname{Tr}(v)=0\right\}\\ \nabla_{\mathcal{F}^\bullet_{\mathfrak{sl},x}}&:=\nabla_{\mathcal{F}^\bullet_x}|_{\mathcal{F}^0_{\mathfrak{sl},x}}:\mathcal{F}^0_{\mathfrak{sl},x}\longrightarrow\mathcal{F}^1_{\mathfrak{sl},x}.\end{aligned}$$

For $U_{\alpha\beta}:=U_\alpha\cap U_\beta$, we take a lift $\theta_{\beta\alpha}:E_\alpha|_{U_{\alpha\beta}\otimes A}\xrightarrow{\sim}E_\beta|_{U_{\alpha\beta}\otimes A}$ of the canonical isomorphism $E_\alpha\otimes A/I|_{U_{\alpha\beta}\otimes A/I}\xrightarrow[\sim]{\tau_\alpha}E|_{U_{\alpha\beta}\otimes A/I}\xrightarrow[\sim]{\tau_\beta^{-1}}E_\beta\otimes A/I|_{U_{\alpha\beta}\otimes A/I}$ such that $\sigma_\beta\circ\det(\theta_{\beta\alpha})=\sigma_\alpha$. We put

$$u_{\alpha\beta\gamma}:=\tau_\alpha(\theta^{-1}_{\gamma\alpha}\theta_{\gamma\beta}\theta_{\beta\alpha}-\mathrm{id})\tau_\alpha^{-1},\quad v_{\alpha\beta}:=\tau_\alpha(\nabla_\alpha-\theta^{-1}_{\beta\alpha}\nabla_\beta\theta_{\beta\alpha})\tau_\alpha^{-1}.$$

Then we have $\{u_{\alpha\beta\gamma}\}\in C^2(\{U_\alpha\},\mathcal{F}^0_{\mathfrak{sl},x}\otimes I)$ and $\{v_{\alpha\beta}\}\in C^1(\{U_\alpha\},\mathcal{F}^1_{\mathfrak{sl},x}\otimes I)$. We can check that $\{u_{\beta\gamma\delta}-u_{\alpha\gamma\delta}+u_{\alpha\beta\delta}-u_{\alpha\beta\gamma}\}=0$ and $\nabla_{\mathcal{F}^\bullet_{\mathfrak{sl},x}}(\{u_{\alpha\beta\gamma}\})=-\{v_{\beta\gamma}-v_{\alpha\gamma}+v_{\alpha\beta}\}$. So we can define an element

$$\omega(E,\nabla,\{l^{(i)}_j,V^{(i)}_{j,k},L^{(i)}_{j,k},\pi^{(i)}_{j,k},\phi^{(i)}_{j,k}\}):=[(\{u_{\alpha\beta\gamma}\},\{v_{\alpha\beta}\})]\in\mathbf{H}^2(\mathcal{F}^\bullet_{\mathfrak{sl},x}\otimes I).$$

in the hyper cohomology group $\mathbf{H}^2(\mathcal{F}^\bullet_{\mathfrak{sl},x}\otimes I)$. Considering a gluing condition of the local parabolic connections $(E_\alpha,\nabla_\alpha,\{\tilde{l}^{(i)}_j,\tilde{V}^{(i)}_{j,k},\tilde{L}^{(i)}_{j,k},\tilde{\pi}^{(i)}_{j,k},\tilde{\phi}^{(i)}_{j,k}\})$, we can see that the vanishing of the element $\omega(E,\nabla,\{l^{(i)}_j,V^{(i)}_{j,k},L^{(i)}_{j,k},\pi^{(i)}_{j,k},\phi^{(i)}_{j,k}\})$ in $\mathbf{H}^2(\mathcal{F}^\bullet_{\mathfrak{sl},x}\otimes I)$ is equivalent to the existence of an A-valued point $(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j,\tilde{V}^{(i)}_{j,k},\tilde{L}^{(i)}_{j,k},\tilde{\pi}^{(i)}_{j,k},\tilde{\phi}^{(i)}_{j,k}\})$ of $\mathcal{M}^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)$ such that

$$(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j,\tilde{V}^{(i)}_{j,k},\tilde{L}^{(i)}_{j,k},\tilde{\pi}^{(i)}_{j,k},\tilde{\phi}^{(i)}_{j,k}\})\otimes A/I\cong(E,\nabla,\{l^{(i)}_j,V^{(i)}_{j,k},L^{(i)}_{j,k},\pi^{(i)}_{j,k},\phi^{(i)}_{j,k}\}).$$

From the hyper cohomology spectral sequence $H^q(\bar{\mathcal{F}}^p_0)\Rightarrow\mathbf{H}^{p+q}(\bar{\mathcal{F}}^\bullet_0)$, we have

$$\begin{aligned}\mathbf{H}^2(\mathcal{F}^\bullet_{\mathfrak{sl},x})&\cong\operatorname{coker}(H^1(\mathcal{F}^0_{\mathfrak{sl},x})\longrightarrow H^1(\mathcal{F}^1_{\mathfrak{sl},x}))\\ &\cong\operatorname{coker}\left(H^0((\mathcal{F}^0_{\mathfrak{sl},x})^\vee\otimes\Omega^1_C)^\vee\longrightarrow H^0((\mathcal{F}^1_{\mathfrak{sl},x})^\vee\otimes\Omega^1_C)^\vee\right)\\ &\cong\ker\left(H^0((\mathcal{F}^1_{\mathfrak{sl},x})^\vee\otimes\Omega^1_C)\xrightarrow{-H^0(\nabla^\dagger)}H^0((\mathcal{F}^0_{\mathfrak{sl},x})^\vee\otimes\Omega^1_C)\right)^\vee.\end{aligned}$$

If we put $(\bar{E},\bar{\nabla},\{\bar{l}^{(i)}_j,\bar{V}^{(i)}_{j,k},\bar{L}^{(i)}_{j,k},\bar{\pi}^{(i)}_{j,k},\bar{\phi}^{(i)}_{j,k}\}):=(E,\nabla,\{l^{(i)}_j,V^{(i)}_{j,k},L^{(i)}_{j,k},\pi^{(i)}_{j,k},\phi^{(i)}_{j,k}\})\otimes A/\mathfrak{m}$, then we can check the equalities

$$(\mathcal{F}^1_{\mathfrak{sl},x})^\vee \otimes \Omega^1_C = \left\{ u \in \mathcal{E}nd(\bar{E}) \;\middle|\; \begin{array}{l} \mathrm{Tr}(u)=0,\ u|_{m_i t_i}(\bar{l}^{(i)}_j) \subset \bar{l}^{(i)}_j \text{ for any } i,j \text{ and} \\ \text{for the induced } u^{(i)}_j : \bar{l}^{(i)}_j/\bar{l}^{(i)}_{j+1} \longrightarrow \bar{l}^{(i)}_j/\bar{l}^{(i)}_{j+1}, \\ u^{(i)}_j(V^{(i)}_{j,k}) \subset V^{(i)}_{j,k} \text{ for any } k \end{array} \right\}$$

$$(\mathcal{F}^0_{\mathfrak{sl},x})^\vee \otimes \Omega^1_C = \left\{ v \in \mathcal{E}nd(\bar{E}) \otimes \Omega^1_C(D) \;\middle|\; \begin{array}{l} \mathrm{Tr}(v)=0 \text{ and } \mathrm{Tr}(v\circ u) \in \Omega^1_C \\ \text{for any } u \in \mathcal{F}^0_{\mathfrak{sl},x} \end{array} \right\},$$

$$\nabla^\dagger \colon (\mathcal{F}^1_{\mathfrak{sl},x})^\vee \otimes \Omega^1_C \ni u \mapsto \nabla u - u\nabla \in (\mathcal{F}^0_{\mathfrak{sl},x})^\vee \otimes \Omega^1_C.$$

We will check that $\nabla^\dagger \colon (\mathcal{F}^1_{\mathfrak{sl},x})^\vee \otimes \Omega^1_C \longrightarrow (\mathcal{F}^0_{\mathfrak{sl},x})^\vee \otimes \Omega^1_C$ is indeed defined. Take $u \in (\mathcal{F}^1_{\mathfrak{sl},x})^\vee \otimes \Omega^1_C$ and $u' \in \mathcal{F}^0_{\mathfrak{sl},x}$. Then there are homomorphisms $\beta^{(i)}_{j,k} \colon L^{(i)}_{j,k} \longrightarrow L^{(i)}_{j,k}$ of $\mathcal{O}_{m_i t_i}$-modules and elements $\gamma^{(i)}_{j,k} \in \mathbb{C}[w^{(i)}_j]/((w^{(i)}_j)^{m_i r^{(i)}_j - r^{(i)}_j + 1})$ satisfying the commutative diagrams

$$\begin{array}{ccc} V^{(i)}_{j,k} & \xrightarrow{u^{(i)}_j} & V^{(i)}_{j,k} \\ {\scriptstyle \pi^{(i)}_{j,k}|_{V^{(i)}_{j,k}}}\downarrow & & \downarrow{\scriptstyle \pi^{(i)}_{j,k}|_{V^{(i)}_{j,k}}} \\ L^{(i)}_{j,k} & \xrightarrow{\beta^{(i)}_{j,k}} & L^{(i)}_{j,k} \end{array} \qquad \begin{array}{ccc} V^{(i)}_{j,k} & \xrightarrow{u'^{(i)}_j} & V^{(i)}_{j,k} \\ {\scriptstyle \pi^{(i)}_{j,k}|_{V^{(i)}_{j,k}}}\downarrow & & \downarrow{\scriptstyle \pi^{(i)}_{j,k}|_{V^{(i)}_{j,k}}} \\ L^{(i)}_{j,k} & \xrightarrow{\gamma^{(i)}_{j,k}} & L^{(i)}_{j,k}, \end{array}$$

where $u^{(i)}_j \colon V^{(i)}_{j,0} \longrightarrow V^{(i)}_{j,0}$ and $u'^{(i)}_j \colon V^{(i)}_{j,0} \longrightarrow V^{(i)}_{j,0}$ are homomorphisms induced by $u|_{m_i t_i}$ and $u'|_{m_i t_i}$, respectively. Then we have

$$\begin{aligned} &\pi^{(i)}_{j,k}((u'^{(i)}_j \nabla^{(i)}_j - \nabla^{(i)}_j u'^{(i)}_j) u^{(i)}_j|_{V^{(i)}_{j,k}}) \\ &= \left(\gamma^{(i)}_{j,k}\big(\nu^{(i)}_j(w^{(i)}_j) + k dz_i/r^{(i)}_j z_i\big) - \big(\nu^{(i)}_j(w^{(i)}_j) + k dz_i/r^{(i)}_j z_i\big)\gamma^{(i)}_{j,k}\right) \beta^{(i)}_{j,k}\pi^{(i)}_{j,k} = 0 \end{aligned}$$

for $k = 0, \ldots, r^{(i)}_j - 1$. So we can see that $\mathrm{Tr}((u'^{(i)}_j \nabla^{(i)}_j - \nabla^{(i)}_j u'^{(i)}_j) u^{(i)}_j) = 0$ and

$$\begin{aligned} &\mathrm{Tr}(u'^{(i)}_j(\nabla^{(i)}_j u^{(i)}_j - u^{(i)}_j \nabla^{(i)}_j)) \\ &= \mathrm{Tr}(u'^{(i)}_j(\nabla^{(i)}_j u^{(i)}_j - u^{(i)}_j \nabla^{(i)}_j)) - \mathrm{Tr}((u'^{(i)}_j \nabla^{(i)}_j - \nabla^{(i)}_j u'^{(i)}_j) u^{(i)}_j) \\ &= -\mathrm{Tr}((u'^{(i)}_j u^{(i)}_j)\nabla^{(i)}_j) + \mathrm{Tr}(\nabla^{(i)}_j (u'^{(i)}_j u^{(i)}_j)) = 0 \end{aligned}$$

for any i,j. Thus we have $\mathrm{Tr}(u' \circ (\nabla u - u\nabla)) \in \Omega^1_C$, $\nabla u - u\nabla \in (\mathcal{F}^0_{\mathfrak{sl},x})^\vee \otimes \Omega^1_C$ and the morphism $\nabla^\dagger \colon (\mathcal{F}^1_{\mathfrak{sl},x})^\vee \otimes \Omega^1_C(D) \longrightarrow (\mathcal{F}^0_{\mathfrak{sl},x})^\vee \otimes \Omega^1_C$ can be defined certainly.

If we take $u \in \ker(H^0((\mathcal{F}^1_{\mathfrak{sl},x})^\vee \otimes \Omega^1_C) \xrightarrow{-\nabla^\dagger} H^0((\mathcal{F}^0_{\mathfrak{sl},x})^\vee \otimes \Omega^1_C)$, then $u : E \longrightarrow E$ is a homomorphism satisfying $\nabla \circ u = (u \otimes \mathrm{id}) \circ \nabla$ and $u|_{m_i t_i}(l^{(i)}_j) \subset l^{(i)}_j$. Since $(E, \nabla, \{l^{(i)}_j\})$ is $\boldsymbol{\alpha}$-stable, we have $u = c \cdot \mathrm{id}$ for some $c \in A/\mathfrak{m}$. So $\mathrm{Tr}(u) = 0$ implies $c = 0$ and $u = 0$. Thus

$$\ker(H^0((\mathcal{F}^1_{\mathfrak{sl},x})^\vee \otimes \Omega^1_C) \xrightarrow{-\nabla_{\mathcal{F}^\bullet_{\mathfrak{sl},x}}} H^0((\mathcal{F}^0_{\mathfrak{sl},x})^\vee \otimes \Omega^1_C)) = 0$$

and $\mathbf{H}^2(\mathcal{F}^\bullet_{\mathfrak{sl},x}) = 0$. In particular, we have $\omega(E, \nabla, \{l^{(i)}_j, V^{(i)}_{j,k}, L^{(i)}_{j,k}, \pi^{(i)}_{j,k}, \phi^{(i)}_{j,k}\}) = 0$, which means that a parabolic connection $(E, \nabla, \{l^{(i)}_j, V^{(i)}_{j,k}, L^{(i)}_{j,k}, \pi^{(i)}_{j,k}, \phi^{(i)}_{j,k}\})$ of generic ramified type over A/I can be lifted to an A-valued point of $\mathcal{M}^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)_{\tilde{\boldsymbol{\nu}}}$. Hence det is a smooth morphism. □

For the proof of Theorem 3.1, it is sufficient to prove the following proposition.

PROPOSITION 3.2. *For any point $x \in M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}(\mathbb{C})$, the dimension of the tangent space $T_{M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}}(x)$ of $M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}$ at x is*

$$2r^2(g-1)+2+\sum_{i=1}^{n}(r^2-r)m_i.$$

Proof. Let $(E,\nabla,\{l^{(i)}_j,V^{(i)}_{j,k},L^{(i)}_{j,k},\pi^{(i)}_{j,k},\phi^{(i)}_{j,k}\})$ be the parabolic connection of generic ramified type with the exponent $\boldsymbol{\nu}$ which corresponds to the point x. Recall the complex $\mathcal{F}^{\bullet}_{M'}$ defined in (3) and consider its restriction $\mathcal{F}^{\bullet}_x=\mathcal{F}^{\bullet}_{M'}\big|_{C\times x}$, where we denote a point of M' lying over x by the same symbol. We will prove that the tangent space $T_{M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}}(x)$ of $M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}$ at x is isomorphic to the hyper cohomology $\mathbf{H}^1(\mathcal{F}^{\bullet}_x)$.

Take a tangent vector $v\in T_{M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}}(x)$ which corresponds to a member

$$(E^v,\nabla^v,\{(l^v)^{(i)}_j,(V^v)^{(i)}_{j,k},(L^v)^{(i)}_{j,k},(\pi^v)^{(i)}_{j,k},(\phi^v)^{(i)}_{j,k}\})\in M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)(\mathbb{C}[\epsilon])$$

such that

$$\begin{aligned}&(E^v,\nabla^v,\{(l^v)^{(i)}_j,(V^v)^{(i)}_{j,k},(L^v)^{(i)}_{j,k},(\pi^v)^{(i)}_{j,k},(\phi^v)^{(i)}_{j,k}\})\otimes\mathbb{C}[\epsilon]/(\epsilon)\\ \cong\ &(E,\nabla,\{l^{(i)}_j,V^{(i)}_{j,k},L^{(i)}_{j,k},\pi^{(i)}_{j,k},\phi^{(i)}_{j,k}\}).\end{aligned}$$

We take an affine open covering $\{U_\alpha\}$ of C as in the proof of Proposition 3.1. Take a lift

$$\varphi_\alpha\colon E^v|_{U_\alpha\otimes\mathbb{C}[\epsilon]}\xrightarrow{\sim}E\otimes\mathbb{C}[\epsilon]|_{U_\alpha\otimes\mathbb{C}[\epsilon]}$$

of the given isomorphism $E^v\otimes\mathbb{C}[\epsilon]/(\epsilon)|_{U_\alpha}\xrightarrow{\sim}E|_{U_\alpha}$ such that the restriction $\varphi_\alpha|_{m_it_i\otimes\mathbb{C}[\epsilon]}$ sends the data $\{(l^v)^{(i)}_j,(V^v)^{(i)}_{j,k},(L^v)^{(i)}_{j,k},(\pi^v)^{(i)}_{j,k},(\phi^v)^{(i)}_{j,k}\}$ to $\{l^{(i)}_j,V^{(i)}_{j,k},L^{(i)}_{j,k},\pi^{(i)}_{j,k},\phi^{(i)}_{j,k}\}\otimes\mathbb{C}[\epsilon]$ if $t_i\in U_\alpha$. We put

$$\begin{aligned}u_{\alpha\beta}&:=\varphi_\alpha\circ\varphi_\beta^{-1}-\mathrm{id}_{E|_{U_{\alpha\beta}\otimes\mathbb{C}[\epsilon]}},\\ v_\alpha&:=(\varphi_\alpha\otimes\mathrm{id})\circ\nabla^v\circ\varphi_\alpha^{-1}-\nabla|_{U_\alpha}\otimes\mathrm{id}_{\mathbb{C}[\epsilon]}.\end{aligned}$$

Then we have $\{u_{\alpha\beta}\}\in C^1(\{U_\alpha\},(\epsilon)\otimes\mathcal{F}^0_x)$, $\{v_\alpha\}\in C^0(\{U_\alpha\},(\epsilon)\otimes\mathcal{F}^1_x)$ and

$$d\{u_{\alpha\beta}\}=\{u_{\beta\gamma}-u_{\alpha\gamma}+u_{\alpha\beta}\}=0,\quad \nabla_{\mathcal{F}^{\bullet}_x}\{u_{\alpha\beta}\}=\{v_\beta-v_\alpha\}=d\{v_\alpha\}.$$

So $[(\{u_{\alpha\beta}\},\{v_\alpha\})]$ gives an element $\Phi(v)$ of $\mathbf{H}^1(\mathcal{F}^{\bullet}_x)$. We can check that the correspondence $v\mapsto\Phi(v)$ gives an isomorphism

$$\Phi\colon T_{M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}}(x)\xrightarrow{\sim}\mathbf{H}^1(\mathcal{F}^{\bullet}_x).$$

From the hyper cohomology spectral sequence $H^q(\mathcal{F}^p_x)\Rightarrow\mathbf{H}^{p+q}(\mathcal{F}^{\bullet}_x)$, we obtain an exact sequence

$$0\longrightarrow\mathbb{C}\longrightarrow H^0(\mathcal{F}^0_x)\longrightarrow H^0(\mathcal{F}^1_x)\longrightarrow\mathbf{H}^1(\mathcal{F}^{\bullet}_x)\longrightarrow H^1(\mathcal{F}^0_x)\longrightarrow H^1(\mathcal{F}^1_x)\longrightarrow\mathbb{C}\longrightarrow 0.$$

So we have

$$\begin{aligned}\dim \mathbf{H}^1(\mathcal{F}_x^\bullet) &= \dim H^0(\mathcal{F}_x^1) + \dim H^1(\mathcal{F}_x^0) - \dim H^0(\mathcal{F}_x^0) - \dim H^1(\mathcal{F}_x^1) + 2\dim_{\mathbb{C}} \mathbb{C} \\ &= \chi(\mathcal{F}_x^1) - \chi(\mathcal{F}_x^0) + 2.\end{aligned}$$

We can calculate $\chi(\mathcal{F}_x^1)$ and $\chi(\mathcal{F}_x^0)$ by its definition as follows:

$$\begin{aligned}\chi(\mathcal{F}_x^1) &= (1-g)\operatorname{rank}\mathcal{F}_x^1 + \deg(\mathcal{F}_x^1) \\ &= r^2(1-g) + r^2(2g-2) + \sum_{i=1}^n \sum_{j'<j} m_i r_{j'}^{(i)} r_j^{(i)} + \sum_{i=1}^n \sum_{j=0}^{s_i-1} \sum_{k=0}^{r_j^{(i)}-1} (r_j^{(i)} - 1 - k) \\ \chi(\mathcal{F}_x^0) &= (1-g)\operatorname{rank}\mathcal{F}_x^0 + \deg(\mathcal{F}_x^0) \\ &= r^2(1-g) - \sum_{i=1}^n \sum_{j'\leq j} m_i r_{j'}^{(i)} r_j^{(i)} \\ &\quad + \sum_{i=1}^n \sum_{j=0}^{s_i-1} (m_i r_j^{(i)} - r_j^{(i)} + 1 + \sum_{k=0}^{r_j^{(i)}-1} 1 + \sum_{k=0}^{r_j^{(i)}-1} (r_j^{(i)} - 1 - k)) \\ &= r^2(1-g) - \sum_{i=1}^n \sum_{j'\leq j} m_i r_{j'}^{(i)} r_j^{(i)} + \sum_{i=1}^n \sum_{j=0}^{s_i-1} m_i r_j^{(i)} + \sum_{i=1}^n \sum_{j=0}^{s_i-1} \sum_{k=0}^{r_j^{(i)}-1} (r_j^{(i)} - 1 - k).\end{aligned}$$

So we have

$$\begin{aligned}&\dim \mathbf{H}^1(\mathcal{F}_x^\bullet) \\ &= \chi(\mathcal{F}_x^1) - \chi(\mathcal{F}_x^0) + 2 \\ &= 2r^2(g-1) + 2 + \sum_{i=1}^n \sum_{j'<j} 2 m_i r_{j'}^{(i)} r_j^{(i)} + \sum_{i=1}^n \sum_{j=0}^{s_i-1} m_i (r_j^{(i)})^2 - \sum_{i=1}^n \sum_{j=0}^{s_i-1} m_i r_j^{(i)} \\ &= 2r^2(g-1) + 2 + \sum_{i=1}^n m_i r(r-1)\end{aligned}$$

and the result follows. □

4. Symplectic form on the moduli space of parabolic connections of generic ramified type for generic exponent. We will give in this section a symplectic form on the moduli space of $\boldsymbol{\alpha}$-stable parabolic connections of generic ramified type when the exponent $\boldsymbol{\nu}$ is generic. For this we enlarge the moduli space to the moduli space of simple parabolic connections of generic ramified type, which is a non-separated algebraic space.

Definition 4.1. We define a functor $\mathcal{M}_{C,D}^{\mathrm{spl}}((r_j^{(i)}), a) \colon (\mathrm{Sch}/N((r_j^{(i)}), a))^o \longrightarrow$ (Sets) by

$$\mathcal{M}_{C,D}^{\mathrm{spl}}((r_j^{(i)}), a)(S) := \left\{ (E, \nabla, \{l_j^{(i)}, V_{j,k}^{(i)}, L_{j,k}^{(i)}, \pi_{j,k}^{(i)}, \phi_{j,k}^{(i)}\}) \right\} / \sim$$

for a noetherian scheme S over $\mathbb{C}$, where $(E, \nabla, \{l_j^{(i)}, V_{j,k}^{(i)}, L_{j,k}^{(i)}, \pi_{j,k}^{(i)}, \phi_{j,k}^{(i)}\})$ is the same as in Definition 2.3 except for replacing the condition (v) by (v') for any geometric point s of S, the equality $\mathrm{End}((E,\nabla,\{l^{(i)}_j\})|_{C\times s})=\mathbb{C}\mathrm{id}_E$ holds, where we define $\mathrm{End}((E,\nabla,\{l^{(i)}_j\})|_{C\times s})$ as the set of the endomorphisms $u\colon E\otimes k(s)\longrightarrow E\otimes k(s)$ satisfying $(\nabla\otimes k(s))\circ u=(u\otimes 1)\circ(\nabla\otimes k(s))$ and $u|_{m_it_i\times s}(l^{(i)}_j\otimes k(s))\subset l^{(i)}_j\otimes k(s)$ for any i,j.

Here the relation $\sim$ is the same as that in Definition 2.3.

The next proposition is much easier than the proof of Theorem 2.1.

Proposition 4.1. *The étale sheafification of $\mathcal{M}^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)$ is represented by an algebraic space $M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)$, locally of finite type over $N((r^{(i)}_j),a)$. Moreover, $M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)$ is smooth over $N((r^{(i)}_j),a)$ whose fiber has dimension $2r^2(g-1)+2+\sum_{i=1}^n(r^2-r)m_i$ if it is non-empty.*

Proof. For each integer m, we consider the subfunctor $\mathcal{M}^{\mathrm{spl,m}}_{C,D}((r^{(i)}_j),a)\subset\mathcal{M}^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)$ such that $(E,\nabla,\{l^{(i)}_j,V^{(i)}_{j,k},L^{(i)}_{j,k},\pi^{(i)}_{j,k},\phi^{(i)}_{j,k}\})\in\mathcal{M}^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}(S)$ lies in $\mathcal{M}^{\mathrm{spl,m}}_{C,D}((r^{(i)}_j),a)(S)$ if $E|_{C\times s}$ is m-regular for any $s\in S$. If we put $N:=h^0(E(m))$, then all m-regular vector bundles of rank r and degree a can be parametrized by a Zariski open set Q of the Quot-scheme $\mathrm{Quot}_{\mathcal{O}_C^{\oplus N}(-m)}$. Take a universal quotient bundle $\mathcal{O}^{\oplus N}_{C\times Q}(-m)\longrightarrow\tilde{E}$. We take the $\mathcal{O}_C$-bimodule Λ^1_D given in [10], section 5.1. Then we can construct a quasi-projective scheme R over Q such that there is a functorial isomorphism $R(S)\cong\mathrm{Hom}(\Lambda^1_D\otimes\tilde{E}_S,\tilde{E}_S)$ for noetherian schemes S over R. The universal family over R corresponds to $(\tilde{\phi}\colon\tilde{E}_R\longrightarrow\tilde{E}_R,\tilde{\nabla}\colon\tilde{E}_R\longrightarrow\tilde{E}_R\otimes\Omega^1_C(D))$. If R' is the subscheme of R where $\tilde{\phi}$ becomes identity, then $\tilde{\nabla}_{R'}\colon\tilde{E}_{R'}\longrightarrow\tilde{E}_{R'}\otimes\Omega^1_C(D)$ becomes a relative connection. We can construct a locally closed subscheme $F_{R'}$ of a flag scheme over R' parameterizing the parabolic structures $(l^{(i)}_j)$ on $\tilde{E}_{R'}$ preserved by $\nabla|_{(m_it_i)_{R'}}$.

By the same argument as in the proof of Theorem 2.1 we can construct a quasi-projective scheme U over $F_{R'}$ parameterizing $\nu^{(i)}_j(w^{(i)}_j)$-ramified structures on $(\tilde{l}^{(i)}_j/\tilde{l}^{(i)}_{j+1},\tilde{\nabla}^{(i)}_j)_{i,j}$, where $(\tilde{l}^{(i)}_j)$ is a universal family of parabolic structure on $\tilde{E}_{F_{R'}}$ and $\tilde{\nabla}^{(i)}_j\colon\tilde{l}^{(i)}_j/\tilde{l}^{(i)}_{j+1}\longrightarrow(\tilde{l}^{(i)}_j/\tilde{l}^{(i)}_{j+1}\otimes\Omega^1_C(D)$ is the homomorphism induced by $\tilde{\nabla}$. There is a canonical morphism $U\longrightarrow\mathcal{M}^{\mathrm{spl,m}}_{C,D}((r^{(i)}_j),a)$ which is formally smooth by construction. The action of $PGL_N(\mathbb{C})$ on $\mathrm{Quot}_{\mathcal{O}_C^{\oplus N}(-m)}$ canonically lifts to a free action on U. We can see that the quotient $M^{\mathrm{spl,m}}_{C,D}((r^{(i)}_j),a):=U/PGL_N(\mathbb{C})$ as an algebraic space represents the étale sheafification of $\mathcal{M}^{\mathrm{spl,m}}_{C,D}((r^{(i)}_j),a)$. There are canonical inclusions $M^{\mathrm{spl,m}}_{C,D}((r^{(i)}_j),a)\hookrightarrow M^{\mathrm{spl,m+1}}_{C,D}((r^{(i)}_j),a)$ and

$$M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a):=\bigcup_{m\geq 0}M^{\mathrm{spl},m}_{C,D}((r^{(i)}_j),a)$$

gives the desired moduli space. Smoothness and the dimension counting follow from the same argument as that of the proof of Theorem 3.1. □

We can take a quasi-finite étale covering $M'_{\boldsymbol{\nu}}\longrightarrow M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}$ such that there is a universal family $(E_{M'_{\boldsymbol{\nu}}},\nabla_{M'_{\boldsymbol{\nu}}},\{l^{(i)}_{M'_{\boldsymbol{\nu}},j},V^{(i)}_{M'_{\boldsymbol{\nu}},j,k},L^{(i)}_{M'_{\boldsymbol{\nu}},j,k},\pi^{(i)}_{M'_{\boldsymbol{\nu}},j,k},\phi^{(i)}_{M'_{\boldsymbol{\nu}},j,k}\})$ over $M'_{\boldsymbol{\nu}}$. Define the complex $\mathcal{F}^\bullet_{M'_{\boldsymbol{\nu}}}$ in the same way as the definition of $\mathcal{F}^\bullet_{M'}$ in (3). We can see that the tangent bundle $T_{M'_{\boldsymbol{\nu}}}$ is isomorphic to $\mathbf{R}^1(\pi_{M'_{\boldsymbol{\nu}}})_*(\mathcal{F}^\bullet_{M'_{\boldsymbol{\nu}}})$, where $\pi_{M'_{\boldsymbol{\nu}}}\colon C\times$

$M'_{\boldsymbol{\nu}} \longrightarrow M'_{\boldsymbol{\nu}}$ is the second projection. We consider the pairing

$$\omega_{M'_{\boldsymbol{\nu}}} : \mathbf{R}^1(\pi_{M'_{\boldsymbol{\nu}}})_*(\mathcal{F}^\bullet_{M'_{\boldsymbol{\nu}}}) \times \mathbf{R}^1(\pi_{M'_{\boldsymbol{\nu}}})_*(\mathcal{F}^\bullet_{M'_{\boldsymbol{\nu}}}) \longrightarrow \mathbf{R}^2(\pi_{M'_{\boldsymbol{\nu}}})_*(\Omega^\bullet_{C\times M'_{\boldsymbol{\nu}}/M'_{\boldsymbol{\nu}}}) \cong \mathcal{O}_{M'_{\boldsymbol{\nu}}}$$
$$([\{u_{\alpha\beta}\},\{v_\alpha\}],[\{u'_{\alpha\beta}\},\{v'_\alpha\}]) \mapsto [(\{\mathrm{Tr}(u_{\alpha\beta}\circ u'_{\beta\gamma})\}, -\{\mathrm{Tr}(u_{\alpha\beta}\circ v'_\beta - v_\alpha\circ u'_{\alpha\beta})\}]$$

We can easily see that $\omega_{M'_{\boldsymbol{\nu}}}$ descends to a pairing

$$\omega_{M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}} : T_{M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}} \times T_{M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}} \longrightarrow \mathcal{O}_{M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}}. \tag{5}$$

Take a tangent vector $v \in T_{M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}}(x)$ corresponding to a $\mathbb{C}[\epsilon]$-valued point

$$(E^v, \nabla^v, \{(l^v)^{(i)}_j, (V^v)^{(i)}_{j,k}, (L^v)^{(i)}_{j,k}, (\pi^v)^{(i)}_{j,k}, (\phi^v)^{(i)}_{j,k}\})$$

of $M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}$. We can see by an easy calculation that $\omega_U(v,v) \in \mathbf{H}^2(\Omega^\bullet_C) \cong \mathbf{H}^2(\mathcal{F}^\bullet_U \otimes k(x))$ is nothing but the obstruction class for the lifting of $(E^v, \nabla^v, \{(l^v)^{(i)}_j, (V^v)^{(i)}_{j,k}, (L^v)^{(i)}_{j,k}, (\pi^v)^{(i)}_{j,k}, (\phi^v)^{(i)}_{j,k}\})$ to a $\mathbb{C}[t]/(t^3)$-valued point of $M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}$. Since $M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}$ is smooth, we have $\omega_U(v,v)=0$. So ω_U is skew-symmetric and we can regard that $\omega_{M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}} \in H^0(\Omega^2_{M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}})$.

Theorem 4.1. *Take* $\boldsymbol{\nu} = (\nu^{(i)}_j(w^{(i)}_j)) \in N((r^{(i)}_j),a)$ *and write*

$$\nu^{(i)}_j(w^{(i)}_j) = \sum_{l=0}^{m_i r^{(i)}_j - r^{(i)}_j} c^{(i)}_{j,l}\,(w^{(i)}_j)^l \frac{dz_i}{z_i^{m_i}} \quad (c^{(i)}_{j,l} \in \mathbb{C}).$$

We assume that $c^{(i)}_{j,1} \neq 0$ *for any* i,j. *Then the* 2*-form* $\omega_{M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}}$ *defined in (5) is symplectic, that is,* $d\omega_{M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}} = 0$ *and* $\omega_{M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}}$ *is non-degenerate.*

Proof. We take a regular parameter z_i of $\mathcal{O}_{C,t_i}$ as a function on an affine open neighborhood U_i of t_i. We may assume that $t_j \notin U_i$ for $j \neq i$. We choose

$$(\kappa_{i,0},\ldots,\kappa_{i,m_i-1}) \in \mathbb{C}^{m_i}$$

satisfying $\kappa_{i,0}=0$ and $\kappa_{i,q} \neq \kappa_{i,q'}$ for $q \neq q'$. If we take a canonical parameter h of $\mathrm{Spec}\,\mathbb{C}[h]$, then $z_i - h\kappa_{i,q}$ becomes a function on $U_i \times \mathrm{Spec}\,\mathbb{C}[h]$ and we can consider its zero scheme

$$D_{i,q} \subset C \times \mathrm{Spec}\,\mathbb{C}[h].$$

If we take some affine open neighborhood H of 0 in $\mathrm{Spec}\,\mathbb{C}[h]$, we may assume that the fibers $\{(D_{i,q})_h | 1 \le i \le n, 0 \le q \le m_i - 1\}$ are distinct $\sum_{i=1}^n m_i$ points for $h \neq 0$. We set

$$\tilde{D}_i := \sum_{q=0}^{m_i-1} (D_{i,q})_H$$
$$\tilde{D} := \sum_{i=1}^{n} \tilde{D}_i,$$

where $(D_{i,q})_H$ means the base change of $D_{i,q}$ by $H \hookrightarrow \operatorname{Spec}\mathbb{C}[h]$. If we write $a_{i,q} := h\kappa_{i,q}$ and $z_{i,q} := z_i - a_{i,q}$, then $z_{i,q}$ becomes a local defining equation of $(D_{i,q})_H$. By construction, we have $(D_{i,0})_H = t_i \times H$ for any i. We put $T := \operatorname{Spec}\mathbb{C}[t] \times H$. Set

$$A^{(i)}_j := \mathcal{O}_{\tilde{D}_i \times T}[W^{(i)}_j]/\big((W^{(i)}_j)^{r^{(i)}_j} - t^{r^{(i)}_j} - z_i\big)$$
$$A^{(i)}_{j,k} := A^{(i)}_j \Big/ \Big(z_{i,1}\cdots z_{i,m_i-1}\big(W^{(i)}_j - \zeta^k_{r^{(i)}_j} t\big)\Big),$$

where $\zeta_{r^{(i)}_j}$ is a primitive $r^{(i)}_j$-th root of unity. Note that

$$\prod_{l=0}^{r^{(i)}_j-1} (W^{(i)}_j - t\zeta^l_{r^{(i)}_j}) = (W^{(i)}_j)^{r^{(i)}_j} - t^{r^{(i)}_j} = z_i$$

and $A^{(i)}_j \otimes k(x_0) \cong \mathbb{C}[w^{(i)}_j]/((w^{(i)}_j)^{m_i r^{(i)}_j})$, for the closed point x_0 of T corresponding to $h = 0$ and $t = 0$. We put

$$\tilde{\nu}^{(i)}_j(W^{(i)}_j) = \sum_{l=0}^{m_i r^{(i)}_j - r^{(i)}_j} c^{(i)}_{j,l}\,(W^{(i)}_j)^l \frac{dz_i}{z_{i,0}z_{i,1}\cdots z_{i,m_i-1}} \in A^{(i)}_j \otimes \Omega^1_{C\times T/T}(\tilde{D}_T).$$

If we take some quasi-finite dominant morphism $T' \longrightarrow T$ with T' normal, we may assume that there are $b^{(j)}_{i,0},\dots,b^{(j)}_{i,m_i-1} \in \mathcal{O}_{T'}$ satisfying $(b^{(j)}_{i,q})^{r^{(i)}_j} = t^{r^{(i)}_j} + a_{i,q}$. Replacing T' by its shrinking, we may assume that

$$\sum_{l=1}^{m_i r^{(i)}_j - r^{(i)}_j} c^{(i)}_{j,l} \cdot (\zeta^{lk}_{r^{(i)}_j} - \zeta^{lk'}_{r^{(i)}_j})(b^{(j)}_{i,q}(u))^{l-1} \neq 0 \tag{6}$$

for any $u \in T'$ and $0 \le k < k' \le r^{(i)}_j - 1$.

We define a moduli functor $\mathcal{M}^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'} \colon (\mathrm{Sch}/T')^o \longrightarrow (\mathrm{Sets})$ from the category of noetherian schemes (Sch/T') over T' to the category of sets by

$$\mathcal{M}^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'}(S) = \Big\{(E,\nabla,\{l^{(i)}_j, V^{(i)}_{j,k}, L^{(i)}_{j,k}, \pi^{(i)}_{j,k}, \phi^{(i)}_{j,k}\})\Big\}/\sim,$$

for a noetherian scheme S over T', where

(i) E is a vector bundle on $C \times S$ of rank r and $\deg(E|_{C\times s}) = a$ for any $s \in S$,

(ii) $\nabla \colon E \longrightarrow E \otimes \Omega^1_{C\times S/S}(\tilde{D} \times_T S)$ is a relative connection,

(iii) $E|_{(\tilde{D}_i)_S} = l^{(i)}_0 \supset l^{(i)}_1 \supset \cdots \supset l^{(i)}_{s_i-1} \supset l^{(i)}_{s_i} = 0$ is a filtration by $\mathcal{O}_{(\tilde{D}_i)_S}$-submodules such that $\nabla|_{(\tilde{D}_i)_S}(l^{(i)}_j) \subset l^{(i)}_j \otimes \Omega^1_{C\times S/S}(\tilde{D} \times_T S)$ for any i, j and that each $l^{(i)}_j / l^{(i)}_{j+1}$ is a locally free $\mathcal{O}_{(\tilde{D}_i)_S}$-module of rank $r^{(i)}_j$,

(iv) $l^{(i)}_j/l^{(i)}_{j+1} = V^{(i)}_{j,0} \supset V^{(i)}_{j,1} \supset \cdots \supset V^{(i)}_{j,r^{(i)}_j-1} \supset z_{i,0}V^{(i)}_{j,0}$ is a filtration by $\mathcal{O}_{(\tilde{D}_i)_S}$-submodules such that $V^{(i)}_{j,r^{i)}_j-1}/z_{i,0}V^{(i)}_{j,0}$ and $V^{(i)}_{j,k}/V^{(i)}_{j,k+1}$ for $k = 0,\dots,r^{(i)}_j - 1$ are locally free $\mathcal{O}_{(D_{i,0})_S}$-modules of rank one and that $\nabla^{(i)}_j(V^{(i)}_{j,k}) \subset V^{(i)}_{j,k} \otimes \Omega^1_{C\times S/S}(\tilde{D}\times_T S)$ for any k, where $\nabla^{(i)}_j \colon l^{(i)}_j/l^{(i)}_{j+1} \longrightarrow l^{(i)}_j/l^{(i)}_{j+1} \otimes \Omega^1_{C\times S/S}(\tilde{D}\times_T S)$ is the homomorphism induced by $\nabla|_{(\tilde{D}_i)_S}$,

(v) $\pi^{(i)}_{j,k}\colon V^{(i)}_{j,k}\otimes A^{(i)}_{j,k}\longrightarrow L^{(i)}_{j,k}$ is a locally free quotient $A^{(i)}_{j,k}\otimes\mathcal{O}_S$-module of rank one such that $p^{(i)}_{j,k}:=\pi^{(i)}_{j,k}|_{V^{(i)}_{j,k}}\colon V^{(i)}_{j,k}\longrightarrow L^{(i)}_{j,k}$ is surjective and the diagram

$$\begin{array}{ccc}
V^{(i)}_{j,k}\otimes A^{(i)}_{j,k} & \xrightarrow{\pi^{(i)}_{j,k}} & L^{(i)}_{j,k} \\
{\scriptstyle \nabla^{(i)}_j\otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \tilde{\nu}^{(i)}_j(W^{(i)}_j)+\frac{k\,dz_i}{r^{(i)}_j z_{i,0}}} \\
V^{(i)}_{j,k}\otimes\Omega^1_{C\times S/S}(\tilde{D}\times_T S)\otimes A^{(i)}_{j,k} & \xrightarrow{\pi^{(i)}_{j,k}} & L^{(i)}_{j,k}\otimes\Omega^1_{C\times S/S}(\tilde{D}\times_T S)
\end{array}$$

is commutative for $0\le k\le r^{(i)}_j-1$,

(vi) $\phi^{(i)}_{j,k}\colon L^{(i)}_{j,k}\longrightarrow L^{(i)}_{j,k-1}$ is an $A^{(i)}_j\otimes\mathcal{O}_S$-homomorphism whose image is $(W^{(i)}_j-\zeta^{k-1}_{r^{(i)}_j}t)L^{(i)}_{j,k-1}$ for $1\le k\le r^{(i)}_j-1$ and $\phi^{(i)}_{j,r^{(i)}_j}\colon (z_{i,0})/(z^{m_i+1}_{i,0})\otimes L^{(i)}_{j,0}\longrightarrow L^{(i)}_{j,r^{(i)}_j-1}$ is an $A^{(i)}_j\otimes\mathcal{O}_S$-homomorphism whose image is $(W^{(i)}_j-\zeta^{r^{(i)}_j-1}_{r^{(i)}_j}t)L^{(i)}_{j,r^{(i)}_j-1}$ such that the diagrams

$$\begin{array}{ccc}
V^{(i)}_{j,k}\otimes A^{(i)}_j & \longrightarrow & V^{(i)}_{j,k-1}\otimes A^{(i)}_j \\
{\scriptstyle \pi^{(i)}_{j,k}}\downarrow & & \downarrow{\scriptstyle \pi^{(i)}_{j,k-1}} \\
L^{(i)}_{j,k} & \xrightarrow{\phi^{(i)}_{j,k}} & L^{(i)}_{j,k-1}
\end{array}$$

for $k=1,\dots,r^{(i)}_j-1$ and the diagram

$$\begin{array}{ccc}
(z_{i,0})/(z^2_{i,0}z_{i,1}\cdots z_{i,m_i-1})\otimes V^{(i)}_{j,0}\otimes A^{(i)}_j & \longrightarrow & V^{(i)}_{j,r^{(i)}_j-1}\otimes A^{(i)}_j \\
{\scriptstyle \mathrm{id}\otimes\pi^{(i)}_{j,0}}\downarrow & & \downarrow{\scriptstyle \pi^{(i)}_{j,r^{(i)}_j-1}} \\
\big(z_{i,0}\big)/\big(z_{i,0}z_{i,1}\cdots z_{i,m_i-1}(W^{(i)}_j-t)\big)\otimes L^{(i)}_{j,0} & \xrightarrow{\phi^{(i)}_{j,r^{(i)}_j}} & L^{(i)}_{j,r^{(i)}_j-1}
\end{array}$$

are commutative,

(vii) $\phi^{(i)}_{j,k}\colon L^{(i)}_{j,k}\longrightarrow (W^{(i)}_j-\zeta^{k-1}_{r^{(i)}_j}t)L^{(i)}_{j,k-1}$ factors through an $A^{(i)}_j\otimes\mathcal{O}_S$-isomorphisms

$$\psi^{(i)}_{j,k}\colon L^{(i)}_{j,k}\xrightarrow{\sim}(W^{(i)}_j-\zeta^{k-1}_{r^{(i)}_j}t)/\Big(\big(W^{(i)}_j-\zeta^{k-1}_{r^{(i)}_j}t\big)\big(W^{(i)}_j-\zeta^{k}_{r^{(i)}_j}t\big)z_{i,1}\cdots z_{i,m_i-1}\Big)\otimes L^{(i)}_{j,k-1}$$

for $1\le k\le r^{(i)}_j-1$ such that the composition

$$(z_{i,0})/\big((W^{(i)}_j-t)z_{i,0}z_{i,1}\cdots z_{i,m_i-1}\big)\otimes L^{(i)}_{j,0}\xrightarrow{\phi^{(i)}_{r^{(i)}_j}}L^{(i)}_{j,r^{(i)}_j-1}\xrightarrow[\sim]{\psi^{(i)}_{j,r^{(i)}_j-1}}$$

$$(W^{(i)}_j-\zeta^{r^{(i)}_j-2}_{r^{(i)}_j}t)/\Big(\big(W^{(i)}_j-\zeta^{r^{(i)}_j-2}_{r^{(i)}_j}t\big)\big(W^{(i)}_j-\zeta^{r^{(i)}_j-1}_{r^{(i)}_j}t\big)z_{i_1}\cdots z_{i_{m_i-1}}\Big)\otimes L^{(i)}_{j,r^{(i)}_j-2}$$

$$\xrightarrow[\sim]{\psi^{(i)}_{r^{(i)}_j-2}}\cdots\xrightarrow[\sim]{\psi^{(i)}_{j,1}}\Big(\prod_{k=0}^{r^{(i)}_j-2}(W^{(i)}_j-\zeta^k_{r^{(i)}_j}t)\Big)\Big/\big(z_{i,0}z_{i,1}\cdots z_{i,m_i-1}\big)\otimes L^{(i)}_{j,0}$$

coincides with the homomorphism canonically induced by

$$
\begin{aligned}
&(z_{i,0})/\big((W_j^{(i)}-t)z_{i,0}z_{i,1}\cdots z_{i,m_i-1}\big)\\
&\longrightarrow \Big(\prod_{k\neq r_j^{(i)}-1}(W_j^{(i)}-\zeta_{r_j^{(i)}}^k t)\Big)\Big/\big(z_{i,0}z_{i,1}\cdots z_{i,m_i-1}\big),
\end{aligned}
$$

(viii) for any point $s\in S$, $(E,\nabla,\{l_j^{(i)}\})\otimes k(s)$ has only scalar endomorphisms.

Here $(E,\nabla,\{l_j^{(i)},V_{j,k}^{(i)},L_{j,k}^{(i)},\pi_{j,k}^{(i)},\phi_{j,k}^{(i)}\})\sim(E',\nabla',\{l_j'^{(i)},V_{j,k}'^{(i)},L_{j,k}'^{(i)},\pi_{j,k}'^{(i)},\phi_{j,k}'^{(i)}\})$ if there are a line bundle $\mathcal{L}$ on S and isomorphisms $\theta\colon E\xrightarrow{\sim}E'\otimes\mathcal{L}$, $\vartheta_{j,k}^{(i)}\colon L_{j,k}^{(i)}\xrightarrow{\sim}L_{j,k}'^{(i)}\otimes\mathcal{L}$ satisfying $\nabla'\circ\theta=(\theta\otimes\mathrm{id})\circ\nabla$, $\theta|_{\tilde{D}_i}(l_j^{(i)})\subset l_j'^{(i)}\otimes\mathcal{L}$ for any i,j and for the induced isomorphism $\theta_j^{(i)}\colon l_j^{(i)}/l_{j+1}^{(i)}\xrightarrow{\sim}l_j'^{(i)}/l_{j+1}'^{(i)}\otimes\mathcal{L}$, $\pi_{j,k}'^{(i)}\circ\theta_j^{(i)}|_{V_{j,k}^{(i)}}=\vartheta_{j,k}^{(i)}\circ\pi_{j,k}^{(i)}|_{V_{j,k}^{(i)}}$, $(\phi_{j,k}'^{(i)}\otimes\mathrm{id}_{\mathcal{L}})\circ\vartheta_{j,k}^{(i)}=\vartheta_{j,k-1}^{(i)}\circ\phi_{j,k}^{(i)}$ and $(\phi'^{(i)}_{j,r_j^{(i)}}\otimes\mathrm{id}_{\mathcal{L}})\circ(\mathrm{id}\otimes\vartheta_{j,0}^{(i)})=\vartheta^{(i)}_{j,r_j^{(i)}-1}\circ\phi^{(i)}_{j,r_j^{(i)}}$.

We can see by the similar argument to that of Proposition 4.1 that the étale sheafification of $\mathcal{M}^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'}$ can be represented by an algebraic space $M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'}$ locally of finite type over T'. If we set

$$
\lambda_i:=\sum_{j=0}^{s_i-1}\sum_{k=0}^{r_j^{(i)}-1}\left(\sum_{l'=0}^{m_i-1}c^{(i)}_{j,r_j^{(i)}l'}(z_i+t^{r_j^{(i)}})^{l'}\frac{dz_i}{z_{i,0}z_{i,1}\cdots z_{i,m_i-1}}+k\frac{dz_{i,0}}{r_j^{(i)}z_{i,0}}\right)
$$

and $\boldsymbol{\lambda}:=(\lambda_i)_{1\le i\le n}$, then there is a moduli scheme $M(1,\boldsymbol{\lambda})$ which represents the functor

$$
\begin{aligned}
&(\mathrm{Sch}/T')\longrightarrow(\mathrm{Sets})\\
&\qquad S\mapsto\left\{(L,\nabla)\;\middle|\;\begin{array}{l}L\text{ is a line bundle on }C\times S\\ \nabla_L\colon L\longrightarrow L\otimes\Omega^1_{C\times S/S}(\tilde{D}_S)\text{ is a relative connection}\\ \text{and }\nabla_L|_{(\tilde{D}_i)_S}=(\lambda_i)_S\text{ for any }i\end{array}\right\}.
\end{aligned}
$$

We can see that $M(1,\boldsymbol{\lambda})$ is an affine space bundle over the relative Jacobian of $C\times T'$ over T' and it is smooth over T'. We can define a morphism $\det\colon M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'}\longrightarrow M(1,\boldsymbol{\lambda})$ by $(E,\nabla)\mapsto\det(E,\nabla)$ and we can prove by the similar argument to that of Proposition 3.1 that this is a smooth morphism. So $M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'}$ is smooth over T'. We take a universal family $(\tilde{E},\tilde{\nabla},\{\tilde{l}_j^{(i)},\tilde{V}_{j,k}^{(i)},\tilde{L}_{j,k}^{(i)},\tilde{\pi}_{j,k}^{(i)},\tilde{\phi}_{j,k}^{(i)}\})$ over some quasi-finite étale covering $M'_{T'}$ of $M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'}$.

First we consider the fiber $(M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'})_x$ over a point $x\in T'$ satisfying $h\neq 0$ for the corresponding point of $T=\operatorname{Spec}\mathbb{C}[t]\times H$. Take $(E,\nabla,\{l_j^{(i)},V_{j,k}^{(i)},L_{j,k}^{(i)},\pi_{j,k}^{(i)},\phi_{j,k}^{(i)}\})\in(M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'})_x$. Note that we have

$$
\begin{aligned}
&A_{j,k}^{(i)}\otimes k(x)\\
&\cong\mathcal{O}_{(D_{i,0})_x}[W_j^{(i)}]/(W_j^{(i)}-\zeta^k_{r_j^{(i)}}t)\oplus\bigoplus_{q=1}^{m_i-1}\mathcal{O}_{(D_{i,q})_x}[W_j^{(i)}]/\big((W_j^{(i)})^{r_j^{(i)}}-(t^{r_j^{(i)}}+a_{iq})_x\big).
\end{aligned}
$$

Consider the case $q\neq 0$. If $(t^{r_j^{(i)}}+a_{i,q})(x)\neq 0$, we have a direct sum decomposition

$$
V_{j,0}^{(i)}\otimes\mathcal{O}_{(\tilde{D}_i)_x}/(z_{i,q}(x))\cong\bigoplus_{k=0}^{r_j^{(i)}-1}L_{j,0}^{(i)}\otimes\mathcal{O}_{(D_{i,q})_x}[W_j^{(i)}]/\big(W_j^{(i)}-\zeta^k_{r_j^{(i)}}b_{i,q}^{(j)}(x)\big),
$$

since $(b^{(j)}_{i,q})^{r^{(i)}_j} = (t^{r^{(i)}_j} + a_{i,q})_x$. Each component $L^{(i)}_{j,0} \otimes \mathcal{O}_{(D_{i,q})_x}[W^{(i)}_j]/\big(W^{(i)}_j - \zeta^k_{r^{(i)}_j} b^{(j)}_{i,q}(x)\big)$ is preserved by $\tilde{\nu}^{(i)}_j(W^{(i)}_j)$ and it is an eigen-space of $\nabla^{(i)}_j$. From the definition of T' given by (6), we can see that

$$\mathsf{res}_{z_{i,q}(x)}\left(\tilde{\nu}^{(i)}_j(W^{(i)}_j) \pmod{W^{(i)}_j - \zeta^k_{r^{(i)}_j} b^{(j)}_{i,q}(x))}\right)$$
$$\neq \mathsf{res}_{z_{i,q}(x)}\left(\tilde{\nu}^{(i)}_j(W^{(i)}_j) \pmod{W^{(i)}_j - \zeta^{k'}_{r^{(i)}_j} b^{(j)}_{i,q}(x))}\right)$$

for $0 \leq k < k' \leq r^{(i)}_j - 1$. So the eigenvalues of $\nabla^{(i)}_j$ on $V^{(i)}_{j,0} \otimes \mathcal{O}_{(\tilde{D}_i)_x}/(z_{i,q}(x))$ are mutually distinct for $q \neq 0$ and $b^{(j)}_{i,q}(x) \neq 0$. For $q \neq 0$ with $(t^{r^{(i)}_j} + a_{i,q})(x) = 0$, we have $b^{(j)}_{i,q}(x) = 0$ and

$$V^{(i)}_{j,0} \otimes \mathcal{O}_{(D_{i,q})_x} \cong \mathcal{O}_{(D_{i,q})_x}[W^{(i)}_j]/((W^{(i)}_j)^{r^{(i)}_j})$$

The linear map $\mathsf{res}_{(D_{i,q})_x}\left(\nabla^{(i)}_j - c^{(i)}_{j,0}\dfrac{dz_i}{z_{i,0}\cdots z_{i,m_i-1}}\right)$ on $V^{(i)}_{j,0} \otimes \mathcal{O}_{(D_{i,q})_x}$ is just the multiplication by $\sum_{l=1}^{m_i r^{(i)}_j - r^{(i)}_j} c^{(i)}_{j,l}\dfrac{dz_i}{z_{i,0}\cdots z_{i,m_i-1}}(W^{(i)}_j)^l$. So we can write

$$\mathsf{res}_{(D_{i,q})_x}\left(\nabla^{(i)}_j|_{(D_{i,q})_x} - \frac{1}{r^{(i)}_j}\operatorname{Tr}(\nabla^{(i)}_j)|_{(D_{i,q})_x}\right)$$
$$= c_1 W^{(i)}_j + c_2 (W^{(i)}_j)^2 + \cdots + c_{r^{(i)}_j - 1}(W^{(i)}_j)^{r^{(i)}_j - 1}$$

with $c_1 \neq 0$. Note that $\mathcal{O}_{(D_{i,q})_x}[W^{(i)}_j]/((W^{(i)}_j)^{r^{(i)}_j})$-module structure on $V^{(i)}_{j,0} \otimes \mathcal{O}_{(D_{i,q})_x}$ is equivalent to giving the action of $c_1 W^{(i)}_j + c_2 (W^{(i)}_j)^2 + \cdots + c_{r^{(i)}_j - 1}(W^{(i)}_j)^{r^{(i)}_j - 1}$ on $V^{(i)}_{j,0} \otimes \mathcal{O}_{(D_{i,q})_x}$ which is of generic nilpotent type and is recovered by $\nabla^{(i)}_j$.

For $q = 0$, we have $L^{(i)}_{j,k} \otimes \mathcal{O}_{(D_{i,0})_x}[W^{(i)}_j]/(W^{(i)}_j - \zeta^k_{r^{(i)}_j} t) \cong \mathbb{C}$. The commutative diagram

$$\begin{array}{ccc}
V^{(i)}_{j,k} \otimes \mathcal{O}_{(D_{i,0})_x} & \xrightarrow{\pi^{(i)}_{j,k}} & L^{(i)}_{j,k} \otimes \mathcal{O}_{(D_{i,0})_x}[W^{(i)}_j]/(W^{(i)}_j - \zeta^k_{r^{(i)}_j} t) \cong \mathbb{C} \\
{\scriptstyle \nabla^{(i)}_j|_{(D_{i,0})_x}}\downarrow & & \downarrow{\scriptstyle \tilde{\nu}^{(i)}_j(W^{(i)}_j) + \frac{k\,dz_i}{r^{(i)}_j z_i}} \\
V^{(i)}_{j,k} \otimes \mathcal{O}_{(D_{i,0})_x} \otimes \Omega^1_C(D) & \xrightarrow{\pi^{(i)}_{j,k}\otimes 1} & L^{(i)}_{j,k} \otimes \mathcal{O}_{(D_{i,0})_x}[W^{(i)}_j]/(W^{(i)}_j - \zeta^k_{r^{(i)}_j} t) \otimes \Omega^1_C(D) \cong \mathbb{C}\cdot\dfrac{dz_i}{z_i}
\end{array}$$

means that $(E, \nabla, \{V^{(i)}_{j,k}, L^{(i)}_{j,k}, \pi^{(i)}_{j,k}\}) \otimes \mathcal{O}_{C,t_i}$ gives a local regular singular parabolic connection at $(D_{i,0})_x$ with the eigenvalues $\mathsf{res}_{z_i}\left(\left(\tilde{\nu}^{(i)}_j(W^{(i)}_j) + \frac{k\,dz_i}{r^{(i)}_j z_{i,0}}\right) \otimes \mathcal{O}_{(D_{i,0})_x}[W^{(i)}_j]/(W^{(i)}_j - \zeta^k_{r^{(i)}_j} t)\right)$. Thus we can see that the fiber $M^{\mathrm{spl}}_{\tilde{\nu},x}$ is nothing but the moduli space of simple regular singular parabolic connections with the assumption that the residue of the connection at $(D_{i,q})_x$ with $b_{i,q}(x) \neq 0$ and $q \neq 0$ is semi-simple of distinct eigenvalues, the residue of the connection at $(D_{i,q})_x$ with $b_{i,q}(x) = 0$ and $q \neq 0$ has a single eigenvalue β and its minimal polynomial is $(T-\beta)^{r^{(i)}_j}$ and regular singular parabolic structure is given by $\big\{V^{(i)}_{j,k}\otimes\mathcal{O}_{(D_{i,0})_x}\big\}_{0\leq k\leq r^{(i)}_j-1}$ at $(D_{i,0})_x$.

Next we consider the fiber $(M^{\mathrm{spl}}_{\tilde{\nu},T'})_y$ over a point $y\in T'$ satisfying $h=0$ and $t\neq 0$ for the corresponding point of $T=\operatorname{Spec}\mathbb{C}[t]\times H$. Note that we have

$$\begin{aligned}A^{(i)}_{j,k}\otimes k(y)&=\mathcal{O}_{(\tilde{D}_i)_y}[W^{(i)}_j]/\big((W^{(i)}_j)^{r^{(i)}_j}-t^{r^{(i)}_j}-z\,,\ z^{m_i-1}(W^{(i)}_j-\zeta^k_{r^{(i)}_j}t)\big)\\&\cong\mathbb{C}[W^{(i)}_j]/\big(((W^{(i)}_j)^{r^{(i)}_j}-t^{r^{(i)}_j})^{m_i-1}(W^{(i)}_j-\zeta^k_{r^{(i)}_j}t)\big)\\&\cong\mathbb{C}[W^{(i)}_j]/\big((W^{(i)}_j-\zeta^k_{r^{(i)}_j}t)^{m_i}\big)\oplus\bigoplus_{k'\neq k}\mathbb{C}[W^{(i)}_j]/\big((W^{(i)}_j-\zeta^{k'}_{r^{(i)}_j}t)^{m_i-1}\big).\end{aligned}$$

We can see by comparing the length that the kernel of the surjection

$$\pi^{(i)}_{j,k}|_{V^{(i)}_{j,k}}:V^{(i)}_{j,k}\longrightarrow L^{(i)}_{j,k}\cong A^{(i)}_{j,k}\otimes k(y)$$

coincides with the image of $\ker\big(V^{(i)}_{j,r^{(i)}_j-1}\xrightarrow{\sim}L^{(i)}_{j,r^{(i)}_j-1}\to L^{(i)}_{j,k}\big)$ via the inclusion $V^{(i)}_{j,r^{(i)}_j-1}\hookrightarrow V^{(i)}_{j,k}$. So we have

$$\ker\big(\pi^{(i)}_{j,k}|_{V^{(i)}_{j,k}}\big)\cong\bigoplus_{k'=k+1}^{r^{(i)}_j-1}\mathbb{C}[W^{(i)}_j]/(W^{(i)}_j-\zeta^{k'}_{r^{(i)}_j}t).$$

Then we can see that $\pi^{(i)}_{j,k}|_{V^{(i)}_{j,k}}$ induces the isomorphism

$$V^{(i)}_{j,0}\supset V^{(i)}_{j,k}\supset V^{(i)}_{j,k}\otimes\mathbb{C}[W^{(i)}_{j,k}]/((W^{(i)}_{j,k}-\zeta^k_{r^{(i)}_j}t)^{m_i})\xrightarrow{\sim}L^{(i)}_{j,k}\otimes\mathbb{C}[W^{(i)}_{j,k}]/((W^{(i)}_{j,k}-\zeta^k_{r^{(i)}_j}t)^{m_i})$$

on the $(W^{(i)}_{j,k}-\zeta^k_{r^{(i)}_j}t)$-torsion parts. So we get a direct sum decomposition

$$V^{(i)}_{j,0}\cong\bigoplus_{k=0}^{r^{(i)}_j-1}L^{(i)}_{j,k}\otimes\mathbb{C}[W^{(i)}_j]/((W^{(i)}_j-\zeta^k_{r^{(i)}_j}t)^{m_i}).$$

On the component $L^{(i)}_{j,k}\otimes\mathbb{C}[W^{(i)}_j]/((W^{(i)}_j-\zeta^k_{r^{(i)}_j}t)^{m_i})$, the operation of $W^{(i)}_j$ is the same as the expansion

$$W^{(i)}_j=\zeta^k_{r^{(i)}_j}\left(t+\frac{z_i}{r^{(i)}_j t}+\cdots\right)\tag{7}$$

in z_i, because of the equalities $(W^{(i)}_j)^{r^{(i)}_j}=t^{r^{(i)}_j}+z_i$ and $\big(W^{(i)}_j-\zeta^k_{r^{(i)}_j}t\big)^{m_i}=0$. So the operation of $\tilde{\nabla}^{(i)}_j\otimes k(y)$ on $L^{(i)}_{j,k}\otimes\mathbb{C}[W^{(i)}_j]/((W^{(i)}_j-\zeta^k_{r^{(i)}_j}t)^{m_i})$ is the substitution of (7) in $\tilde{\nu}^{(i)}_j(W^{(i)}_j)+k\,dz_i/r^{(i)}_j z_i$, whose leading coefficient is

$$\sum_{l=0}^{m_ir^{(i)}_j-r^{(i)}_j}c^{(i)}_{j,l}\zeta^{lk}_{r^{(i)}_j}t^l.$$

These are mutually distinct for $k=0,\dots,r^{(i)}_j-1$, because of the definition of T' given by (6). Thus the fiber $M^{\mathrm{spl}}_{\tilde{\nu},y}$ is nothing but the moduli space of simple unramified irregular singular parabolic connections.

We define a complex $\mathcal{F}^{\bullet}_{M'_{T'}}$ by

$$\mathcal{F}^0_{M'_{T'}}=\left\{u\in\mathcal{E}nd(\tilde{E})\;\middle|\;\begin{array}{l}u|_{\tilde{D}_{M'_{T'}}}(\tilde{l}^{(i)}_j)\subset\tilde{l}^{(i)}_j \text{ for any } i,j \text{ and for the induced}\\ \text{homomorphism } u^{(i)}_j:\tilde{l}^{(i)}_j/\tilde{l}^{(i)}_{j+1}\longrightarrow\tilde{l}^{(i)}_j/\tilde{l}^{(i)}_{j+1},\\ u^{(i)}_j(\tilde{V}^{(i)}_{j,k})\subset\tilde{V}^{(i)}_{j,k} \text{ and } \tilde{\pi}^{(i)}_{j,k}\circ(u^{(i)}_j\otimes\mathrm{id})|_{\ker\pi^{(i)}_{j,k}}=0\\ \text{for any } i,j,k\end{array}\right\}$$

$$\mathcal{F}^1_{M'_{T'}}=\left\{v\in\mathcal{E}nd(\tilde{E})\otimes\Omega^1_{C_{T'}/T'}(\tilde{D}_{T'})_{M'_{T'}}\;\middle|\;\begin{array}{l}v|_{\tilde{D}_{M'_{T'}}}(\tilde{l}^{(i)}_j)\subset\tilde{l}^{(i)}_j\otimes\Omega^1_{C_{T'}/T'}(\tilde{D}_{T'})_{M'_{T'}}\\ \text{for any } i,j \text{ and for the induced}\\ \text{homomorphism}\\ v^{(i)}_j:\tilde{l}^{(i)}_j/\tilde{l}^{(i)}_{j+1}\longrightarrow\\ \quad\tilde{l}^{(i)}_j/\tilde{l}^{(i)}_{j+1}\otimes\Omega^1_{C_{T'}/T'}(\tilde{D}_{T'})_{M'_{T'}},\\ v^{(i)}_j(\tilde{V}^{(i)}_{j,k})\subset\tilde{V}^{(i)}_{j,k}\otimes\Omega^1_{C_{T'}/T'}(\tilde{D}_{T'})_{M'_{T'}}\\ \text{and } (\tilde{\pi}^{(i)}_{j,k}\otimes\mathrm{id})\circ v^{(i)}_j|_{\tilde{V}^{(i)}_{j,k}}=0\\ \text{for any } i,j,k\end{array}\right\}$$

$$\nabla_{\mathcal{F}^{\bullet}_{M'_{T'}}}:\mathcal{F}^0_{M'_{T'}}\ni u\mapsto\tilde{\nabla}u-u\tilde{\nabla}\in\mathcal{F}^1_{M'_{T'}}.$$

Then we can see that the relative tangent bundle $T_{M'_{T'}/T'}$ is isomorphic to $\mathbf{R}^1(\pi_{M'_{T'}})_*(\mathcal{F}^{\bullet}_{M'_{T'}})$. If we define a pairing

$$\omega_{M'_{T'}}:\mathbf{R}^1(\pi_{M'_{T'}})_*(\mathcal{F}^{\bullet}_{M'_{T'}})\times\mathbf{R}^1(\pi_{M'_{T'}})_*(\mathcal{F}^{\bullet}_{M'_{T'}})\longrightarrow\mathbf{R}^2(\pi_{M'_{T'}})_*(\Omega^{\bullet}_{C\times M'_{T'}/M'_{T'}})\cong\mathcal{O}_{M'_{T'}}$$

by setting

$$\omega_{M'_{T'}}\big([(\{u_{\alpha\beta}\},\{v_\alpha\})],[(\{u'_{\alpha\beta}\},\{v'_\alpha\})]\big)=\big[(\{\mathrm{Tr}(u_{\alpha\beta}\circ u'_{\beta\gamma})\},\{\mathrm{Tr}(u_{\alpha\beta}\circ v'_\beta)-\mathrm{Tr}(v_\alpha\circ u'_{\alpha\beta})\}\big],$$

then $\omega_{M'_{T'}}$ descends to a relative 2-form $\omega_{M^{\mathrm{spl}}_{\tilde{\nu},T'}}\in H^0(M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'},\Omega^2_{M^{\mathrm{spl}}_{\tilde{\nu},T'}/T'})$.

The restriction of $\omega_{M^{\mathrm{spl}}_{\tilde{\nu},T'}}$ to the fiber over the point of T' corresponding to $t=0$ and $h=0$ is nothing but the 2-form $\omega_{M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}}$ on the moduli space of parabolic connections of generic ramified type with the exponent $\boldsymbol{\nu}$. The restriction of $\omega_{M^{\mathrm{spl}}_{\tilde{\nu},T'}}$ to the fiber $(M^{\mathrm{spl}}_{\tilde{\nu},T'})_x$ over a point x corresponding to $h\neq 0$ coincides with the 2-form on the moduli space of regular singular parabolic connections defined in [9], section 7. By [9, Proposition 7.2 and Proposition 7.3], $\omega_{M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'}}\big|_{(M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'})_x}$ is non-degenerate and $d\omega_{M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'}}\big|_{(M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'})_x}=0$.

Consider the restriction of $\omega_{M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}}}}$ to the fiber over a point y of T' corresponding to $t\neq 0$ and $h=0$ is the 2-form. Then we can see that the restriction $\omega_{M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'}}\big|_{(M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'})_y}$ coincides with the symplectic form on the moduli space of unramified irregular singular connections given in [13], section 4. By [13, Proposition 4.1], $\omega_{M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'}}\big|_{(M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'})_y}$ is non-degenerate. Since the induced homomorphism

$$\det(\omega_{M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}}}}):\ \det(T_{M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}}}/T'})\longrightarrow\det(\Omega^1_{M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}}}/T'})$$

is isomorphic in codimension one, it is isomorphic on whole $M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}}}$, because $M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'}$ is normal. Thus the relative 2-form $\omega_{M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'}}$ is non-degenerate. The non-degeneracy and d-closedness of $\omega_{M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'}}$ imply the same properties for $\omega_{M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}} = \omega_{M^{\mathrm{spl}}_{\tilde{\boldsymbol{\nu}},T'}}\big|_{M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}}$. □

Restriction of the symplectic form $\omega_{M^{\mathrm{spl}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}}$ to the moduli space $M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}$ gives the following corollary.

Corollary 4.1. *Take an exponent* $\boldsymbol{\nu} \in N((r^{(i)}_j),a)$ *such that the* $\dfrac{w^{(i)}_j dw^{(i)}_j}{(w^{(i)}_j)^{m_i r^{(i)}_j - r^{(i)}_j + 1}}$*-coefficient of* $\nu^{(i)}_{j,k}$ *does not vanish for any* i,j. *Then there is an algebraic symplectic form* $\omega_{M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}}$ *on the moduli space* $M^{\boldsymbol{\alpha}}_{C,D}((r^{(i)}_j),a)_{\boldsymbol{\nu}}$ *of* $\boldsymbol{\alpha}$*-stable parabolic connections of generic ramified type with the exponent* $\boldsymbol{\nu}$.

Appendix. In the earlier version of the preprint, the author missed the condition (v) of Definition 1.2 in the formulation of irregular singular connection of generic ramified type. Indeed, we cannot remove the condition because of the following example.

Consider the case $m = r = 2$ and a connection (E,∇) with $(\hat{E},\hat{\nabla}) \cong (\mathbb{C}[[w]],\nabla_{\nu(w)})$, where $w=\sqrt{z}$, $\nu(w) = w\,dw/w^3$ and

$$\nabla_{\nu(w)}\colon \mathbb{C}[[w]] \ni f(w) \;\mapsto\; df(w) + \nu(w)f(w) \in \mathbb{C}[[w]] \otimes \frac{dz}{z^2}.$$

Let e_0, e_1 be the basis of $\hat{E}$ corresponding to $1, w \in \mathbb{C}[[w]]$. Then a true ramified structure $(V_k, L_k, \pi_k, \phi_k)$, in the sense of Definition 1.2, is given by $V_k = (w^k)/(w^4)$, $L_k = (w^k/w^{3+k})$ for $k=0,1$ and

$$\begin{aligned} \pi_0\colon\ & V_0 \otimes_{\mathbb{C}[z]/(z^2)} \mathbb{C}[w]/(w^3) \ \ni\ f_0(w)e_0 + f_1(w)e_1 \ \mapsto\ f_0(w) + f_1(w)w \ \in\ \mathbb{C}[w]/(w^3) \\ \pi_1\colon\ & V_1 \otimes_{\mathbb{C}[z]/(z^2)} \mathbb{C}[w]/(w^3) \ \ni\ f_1(w)e_1 + f_2(w)ze_0 \ \mapsto\ f_1(w)w + f_2(w)z \ \in\ (w)/(w^4). \end{aligned}$$

However, if we lose the condition (v) of Definition 1.2, we can construct the following unexpected example: Again we put $V_k = (w^k)/(w^4)$ and $L_k = (w^k)/(w^{3+k})$. We define

$$\begin{aligned} \pi'_0\colon V_0 \otimes_{\mathbb{C}[z]/(z^2)} \mathbb{C}[w]/(w^3) &\longrightarrow \mathbb{C}[w]/(w^3) \\ \pi'_1\colon V_1 \otimes_{\mathbb{C}[z]/(z^2)} \mathbb{C}[w]/(w^3) &\longrightarrow (w)/(w^4), \end{aligned}$$

by setting

$$\begin{aligned} \pi'_0(f_0(w)e_0 + f_1(w)e_1) &= f_0(w) + f_1(w)(w+w^2) \\ \pi'_1(f_1(w)e_1 + f_2(w)ze_0) &= f_1(w)(w+w^2) + f_2(w)(z+2zw). \end{aligned}$$

Furthermore, we define the $\mathbb{C}[w]$-homomorphisms

$$\begin{aligned} \phi'_1\colon (w)/(w^4) &\longrightarrow \mathbb{C}[w]/(w^3) \\ \phi'_2\colon (z)/(w^5) &\longrightarrow (w)/(w^4) \end{aligned}$$

by setting

$$\phi'_1(w) = w$$
$$\phi'_2(z) = z + 2wz,$$

which would not satisfy the condition (v) of Definition 1.2. Then we can check that the two squares in the diagram

$$\begin{array}{ccc} V_0 \otimes \mathbb{C}[w]/(w^3) & \xrightarrow{\pi'_0} & \mathbb{C}[w]/(w^3) \\ \uparrow & & \uparrow \phi'_1 \\ V_1 \otimes \mathbb{C}[w]/(w^3) & \xrightarrow{\pi'_1} & (w)/(w^4) \\ \uparrow & & \uparrow \phi'_2 \\ (z) \otimes V_0 \otimes \mathbb{C}[w]/(w^3) & \xrightarrow{\mathrm{id}\otimes\pi'_0} & (z)/(w^5) \end{array}$$

are commutative. If we put $\nu'(w) := (w + w^2)dw/w^3$ which is different from the true exponent, the diagram

$$\begin{array}{ccc} V_0 \otimes \mathbb{C}[w]/(w^3) & \xrightarrow{\pi'_0} & \mathbb{C}[w]/(w^3) \\ {\scriptstyle \nabla|_{z^2=0}\otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \nu'(w)} \\ V_0 \otimes \dfrac{dz}{z^2} \otimes \mathbb{C}[w]/(w^3) & \xrightarrow{\pi'_0} & \mathbb{C}[w]/(w^3) \otimes \dfrac{dz}{z^2} \end{array}$$

is commutative, because

$$\pi'_0(\nabla(e_0)) = \pi'_0\Big(e_1 \otimes \frac{dz}{2z^2}\Big) = (w + w^2)\frac{dw}{w^3} = \nu'(w)\pi'_0(e_0)$$
$$\pi'_0(\nabla(e_1)) = \pi'_0\Big(ze_0 \otimes \frac{dz}{2z^2} + e_1 \otimes \frac{dz}{2z}\Big) = w^2\frac{dw}{w^3} + w\frac{dw}{w} = \nu'(w)(w + w^2) = \nu'(w)\pi'_0(e_1).$$

Similarly, the diagram

$$\begin{array}{ccc} V_1 \otimes \mathbb{C}[w]/(w^3) & \xrightarrow{\pi'_1} & (w)/(w^4) \\ {\scriptstyle \nabla|_{z^2=0}\otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \nu'(w)+\frac{dw}{w}} \\ V_1 \otimes \dfrac{dz}{z^2} \otimes \mathbb{C}[w]/(w^3) & \xrightarrow{\pi'_1} & (w)/(w^4) \otimes \dfrac{dz}{z^2} \end{array}$$

is commutative, because

$$\begin{aligned} \pi'_1(\nabla(e_1)) &= \pi'_1\Big(ze_0 \otimes \frac{dz}{2z^2} + e_1 \otimes \frac{dz}{2z}\Big) \\ &= (z + 2zw)\frac{dw}{w^3} + (w + w^2)z\frac{dw}{w^3} \\ &= (z + 3zw)\frac{dw}{w^3} = (w + w^2)\left((w + w^2)\frac{dw}{w^3} + \frac{dw}{w}\right) = \left(\nu'(w) + \frac{dw}{w}\right)\pi'_1(e_1) \\ \pi'_1(\nabla(ze_0)) &= \pi'_1\Big(ze_1 \otimes \frac{dz}{2z^2} + ze_0 \otimes \frac{dz}{z}\Big) \\ &= z(w + w^2)\frac{dw}{w^3} + (z + 2zw)2z\frac{dw}{w^3} \\ &= zw\frac{dw}{w^3} = (z + 2zw)(w + 2w^2)\frac{dw}{w^3} = \Big(\nu'(w) + \frac{dw}{w}\Big)\pi'_1(ze_0). \end{aligned}$$

So $(V_k, L_k, \pi'_k, \phi'_k)$ satisfies all the other conditions of Definition 1.2 except the condition (v).

REFERENCES

[1] D. G. Babbitt and V. S. Varadarajan, *Local moduli for meromorphic differential equations*, Astérisque, 217 (1989), pp. 169–170.

[2] O. Biquard and P. Boalch, *Wild non-abelian Hodge theory on curves*, Comp. Math. 140:1 (2004), pp. 179–204.

[3] P. Boalch, *Symplectic manifolds and isomonodromic deformations*, Adv. Math. 163:2 (2001), pp. 137–205.

[4] P. Boalch, *Geometry and braiding of Stokes data; Fission and wild character varieties*, Ann. of Math. (2), 179:1 (2014), pp. 301–365.

[5] P. Boalch, *Wild character varieties, points on the Riemann sphere and Calabi's examples*, Representation theory, special functions and Painlevé equationsRIMS 2015, pp. 67–94, Adv. Stud. Pure Math., 76, Math. Soc. Japan, Tokyo, 2018.

[6] P. Boalch and D. Yamakawa, *Twisted wild character varieties*, math:arXiv:1512.0809.

[7] C. L. Bremer and D. S. Sage, *Moduli spaces of irregular singular connections*, Int. Math. Res. Not. IMRN, 2013, no. 8, pp. 1800-1872.

[8] K. Hiroe and D. Yamakawa, *Moduli spaces of meromorphic connections and quiver varieties*, Adv. Math. 266 (2014), pp. 120–151.

[9] M-a. Inaba, *Moduli of parabolic connections on curves and the Riemann-Hilbert correspondence*, J. Algebraic Geom. 22:3 (2013), pp. 407–480.

[10] M-a. Inaba, K. Iwasaki and M.-H. Saito, *Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painlevé equation of type VI, I*, Publ. Res. Inst. Math. Sci., 42:4 (2006), pp. 987–1089.

[11] M-a. Inaba, K. Iwasaki and M.-H. Saito, *Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painlevé equation of type VI. II*, Moduli spaces and arithmetic geometry, pp. 387–432, Adv. Stud. Pure Math., 45, Math. Soc. Japan, Tokyo, 2006.

[12] M-a. Inaba, K. Iwasaki and M.-H. Saito, *Dynamics of the sixth Pailevé equations*, Théories asymptotiques et équations de Painlevé, Sémin. Congr., vol. 14, Soc. Math. France, Paris, 2006, pp. 103–167.

[13] M-a. Inaba and M.-H. Saito, *Moduli of unramified irregular singular parabolic connections on a smooth projective curve*, Kyoto J. Math. 53:2 (2013), pp. 433-482.

[14] M. Jimbo, T. Miwa and K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ-function*, Phys. D, 2:2 (1981), pp. 306–352.

[15] M. Jimbo and T. Miwa, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II*, Phys. D, 2:3 (1981), pp. 407–448.

[16] M. Jimbo and T. Miwa, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. III*, Phys. D, 4:1 (1981/82), pp. 26–46.

[17] N. Nitsure, *Moduli of semistable logarithmic connections*, J. Amer. Math. Soc., 6:3 (1993), pp. 597–609.

[18] K. Okamoto, *Sur les feuilletages associés aux équations du second ordre á points critiques fixes de P. Painlevé*, Japan. J. Math. 5:1 (1979), pp. 1–79.

[19] M. van der Put and M.-H. Saito, *Moduli spaces for linear differential equations and the Painlevé equations*, Ann. Inst. Fourier (Grenoble), 59:7 (2009), pp. 2611-2667.

[20] M.-H. Saito, T. Takebe and H. Terajima, *Deformation of Okamoto-Painlevé pairs and Painlevé equations*, J. Algebraic Geom., 11:2 (2002), pp. 311-362.

[21] H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Comm. Math. Phys. 220:1 (2001), pp. 165–229.

[22] Y. Sibuya, *Linear differential equations in the complex domain: problems of analytic continuation*, Transl. of Math. Monographs Vol. 82, A.M.S, (1990).

[23] C. T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. I*, Inst. Hautes Études Sci. Publ. Math., 79 (1994), pp. 47–129.

[24] C. T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. II*, Inst. Hautes Études Sci. Publ. Math., 80 (1994), pp. 5–79.