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COMPARING THE CARATHÉODORY PSEUDO-DISTANCE AND THE KÄHLER-EINSTEIN DISTANCE ON COMPLETE REINHARDT DOMAINS*

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Abstract. We show that on a certain class of bounded, complete Reinhardt domains in $\mathbb{C}^n$ that enjoy a lot of symmetries, the Carathéodory pseudo-distance and the geodesic distance of the complete Kähler-Einstein metric with Ricci curvature -1 are different.

Key words. Carathéodory pseudo-distance, the Kähler-Einstein distance of Ricci curvature -1, complete Reinhardt domains, the application of Yau's Schwarz lemma.

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1. Introduction and main results. In this paper, we compare the Carathéodory pseudo–distance and the geodesic distance of the complete Kähler–Einstein metric with Ricci curvature -1 on certain complete Reinhardt domains in $\mathbb{C}^n, n \geq 2$. Throughout the paper, we will call the geodesic distance of the complete Kähler–Einstein metric with Ricci curvature -1 by the Kähler–Einstein distance. The Carathéodory pseudo–distance c_M on a complex manifold M is defined by

$$c_M(x, y) := \sup_{f \in \mathrm{Hol}(M,\mathbb{D})} \rho_{\mathbb{D}}(f(x), f(y)).$$

Here, $\rho_{\mathbb{D}}$ denotes the Poincaré distance on the unit disk $\mathbb{D}$ in $\mathbb{C}^1$ and $\mathrm{Hol}(M, \mathbb{D})$ is the collection of all holomorphic functions from M to $\mathbb{D}$. We call c_M the *pseudo*-distance instead of the distance because there could be distinct points $x, y \in M$ such that $c_M(x, y) = 0$.

The CarathéodoryReiffen (pseudo) metric and the Carathéodory pseudodistance are invariant under biholomorphic mappings and completely determined by bounded, non-constant, holomorphic functions. This is the smallest possible invariant metric and the shortest distance between invariant metrics and invariant distances. Consequently, they are particularly relevant to the long-standing conjecture regarding the existence of a bounded, non-constant, holomorphic function on a simply connected, complete, non-compact Kähler manifold whose sectional curvature is negatively pinched. Wu and Yau recently demonstrated [33] that KählerEinstein metric of negative Ricci curvature, the Bergman metric, and the KobayashiRoyden metrics exist and are uniformly equivalent on this class of manifolds. Therefore, it is natural to compare the Carathéodory-Reiffen metric and the Carathéodory distance with these three invariant metrics and their distances (also see [32, 34]).

The complete KählerEinstein metric of negative Ricci curvature and its distance on complex manifolds are invariant under biholomorphic mappings, especially for bounded pseudoconvex domains in $\mathbb{C}^n$. In recent years, a number of discoveries have been made concerning the relationship between the KählerEinstein metric and other invariant metrics (see, for example, [1–10, 13, 16, 18, 21–27, 31–34, 36] and references therein). The outcomes also included the Carathéodory-Reiffen metric and Carathéodory pseudodistance ([15, Proposition 3.1.7], [14, Chapter 2]).

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As an invariant pseudo-distance, the Carathéodory pseudo-distance is less than or equal to the Carathéodory inner-distance [14, Remark 2.7.5], Bergman distance [11, 12, 19], and Kobayashi–Royden distance [14, 15], and there are known instances in which it is strictly smaller than each of these distances (for example, [24, 25, 30]). Due to the Schwarz-Yau Lemma (Section 2), on the other hand, the Carathéodory pseudo-distance is less than or equal to the Kähler-Einstein distance; therefore, it is natural to ask whether the Kähler-Einstein distance and the Carathéodory pseudo-distance are different in general. For strictly pseudoconvex domains in $\mathbb{C}^n$, the concrete formulas of the Kähler-Einstein distance and the Carathéodory pseudo-distance remain unknown. Thus, a comparison of the two distances is not trivial (see [17, 29] and [4, Proposition 5.5, Theorem 7.5] for the case of smoothly bounded, strictly pseudoconvex domains). We introduce a class of complete Reinhardt domains in $\mathbb{C}^n, n \geq 2$ that distinguish between the Kähler-Einstein distance and the Carathéodory pseudo-distance in this paper.

The work of Vigue [30] provides one possible approach for distinguishing the Carathéodory pseudo-distance from the Carathéodory inner-distance. After a clever choice of one point with origin on the diagonal entries on

$$\left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| + |z_2| < 1, |z_1 z_2| < \frac{1}{16}\right\}, \tag{1.1}$$

Vigue distinguished the Carathéodory inner-distance and the Carathéodory pseudo–distance. By extending this idea, we could distinguish the Carathéodory-distance from the Kähler–Einstein distance. We say a domain Ω in $\mathbb{C}^n$ is a complete Reinhardt domain if for any $(z_1, ..., z_n) \in \Omega$ and $(\lambda_1, ..., \lambda_n) \in \overline{\mathbb{D}}^n$, the point $(\lambda_1 z_1, ..., \lambda_n z_n)$ belongs to Ω [14, Remark 2.2.1]. We say $\Omega \subset \mathbb{C}^n$ is compatible with the symmetry by all permutations if Ω satisfies the following: $(z_1, \cdots, z_a, \cdots, z_b, \cdots, z_n) \in \Omega$ if and only if $(z_1, \cdots, z_{\phi(a)}, \cdots, z_{\phi(b)}, \cdots, z_n) \in \Omega$ for any $a, b \in \{1, \cdots, n\}$ and any permutation $\phi : \{1, \cdots, n\} \to \{1, \cdots, n\}$. We denote the ball of radius $R > 0$, centered at the origin in $\mathbb{C}^n$ by B_R. We denote the complete Kähler–Einstein metric with Ricci curvature -1 on Ω by ω_{KE}, and let d_Ω^{KE} be the distance induced by ω_{KE}.

Theorem 1. *Let Ω be a bounded, complete Reinhardt domain in $\mathbb{C}^n, n \geq 2$. Suppose that Ω is compatible with the symmetry by all permutations and there exists $R > 0$ such that Ω is contained in B_R and $\sum_{k=1}^n p_k = R$ for some $(p_1, \cdots, p_n) \in \Omega$. Then*

$$c_\Omega \lneq d_\Omega^{KE}. \tag{1.2}$$

Here, (1.2) *means for any $a, b \in \Omega$, and there exist $p, q \in \Omega$ such that*

$$c_\Omega(a, b) \leq d_\Omega^{KE}(a, b),$$

$$c_\Omega(p, q) < d_\Omega^{KE}(p, q).$$

We apply the theorem of the existence of the complete Kähler–Einstein metric of a negative Ricci curvature on a bounded pseudoconvex domain in $\mathbb{C}^n$ as given in the main theorem in [20]. The hypothesis of Theorem 1 is satisfied in several examples including (1.1) complex ellipsoids [5] and symmetrize polydisks of arbitrary complex dimensions [8, 24] and also others [6]. The proof of Theorem 1 will be presented in Section 3.

Theorem 1 holds on weakly pseudoconvex domains with very nice symmetries. In particular, if Ω is selected as in Theorem 2, and it would be particularly beneficial to obtain a distance comparison between the origin $(0, ..., 0)$ and the diagonal entry $(x, ..., x)$. Especially, the following theorem is implicitly related to [14, Problem 2.10].

THEOREM 2. *Let $n \geq 2$. For each $0 < \epsilon < \frac{1}{n^n}$, define*

$$\mathbb{D}_\epsilon^n := \{(z_1, ..., z_n) \in \mathbb{D}^n : |\Pi_{i=1}^n z_i| < \epsilon\}.$$

Then for any $(x, \cdots, x) \in \mathbb{D}_\epsilon^n$ satisfying $|x| > (\epsilon n)^{\frac{1}{n-1}}$, we have

$$c_{\mathbb{D}_\epsilon^n}((0, ..., 0), (x, ..., x)) < d_{\mathbb{D}_\epsilon^n}^{KE}((0, ..., 0), (x, ..., x)).$$

Note that $\epsilon > 0$ must satisfy $\epsilon < \frac{1}{n^n}$ in order to take $(x, \cdots, x) \in \mathbb{D}_\epsilon^n$ with $|x| > (\epsilon n)^{\frac{1}{n-1}}$. The proof of Theorem 2 will be presented in Section 4 with some additional remarks.

Lastly, one can easily extend the proof of Theorem 1 with the complete Kähler–Einstein metric of Ricci curvature $-\lambda, \lambda > 0$.

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2. Schwarz–Yau lemma. The following generalized Schwarz lemma due to Yau will be used to compare the Carathéodory-distance and the Kähler–Einstein distance.

THEOREM 3 (the Schwarz-Yau lemma, [28, 35]). *Let (M, g) be a complete Kähler manifold with Ricci curvature bounded from below by a negative constant K_1. Let (N, h) be another Hermitian manifold with holomorphic bisectional curvature bounded from above by a negative constant K_2. If there is a non-constant holomorphic map f from M to N, we have*

$$f^* h \leq \frac{K_1}{K_2} g.$$

We can use the upper bound of holomorphic sectional curvature of (N, h) instead of the upper bound of bisectional curvature if N is a Riemann surface [28].

3. $c_\Omega \lneq d_\Omega^{KE}$ for some complete Reinhardt domains Ω in $\mathbb{C}^n, n \geq 2$. The proof of Theorem 1 can be reduced to Lemma 4 which gives the comparison of the Carathéodory pseudo–distance and the Kähler–Einstein distance. Note that by Montel's theorem, for any two points in a complex manifold M, we can always achieve the extremal map $f \in \mathrm{Hol}(M, \mathbb{D})$ with respect to the Carathéodory pseudo–distance. In Lemma 4, $\gamma_{\mathbb{D}}$ is the Poincaré metric on the unit disk and $\omega_{KE}(a)$ is the hermitian inner product on the holomorphic tangent space at $a \in \Omega$.

LEMMA 4. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, and $a, b \in \Omega$, $a \neq b$. Suppose there exists $f \in \mathrm{Hol}(\Omega, \mathbb{D})$ that is extremal for $c_\Omega(a, b)$ such that*

$$\gamma_{\mathbb{D}}(f(a), f'(a)X) < \sqrt{(\omega_{KE}(a)(X, X))}, X \in \mathbb{C}^n - \{0\}. \tag{3.1}$$

Then

$$c_\Omega(a,b) < d_\Omega^{KE}(a,b).$$

Proof. By assumption, continuity of metrics gives ϵ, δ such that

$$\gamma_{\mathbb{D}}(f(z), f'(z)X) + \epsilon\|X\| \le \sqrt{(\omega_{KE}(z)(X,X))}, z \in \mathbb{B}(a,\delta) \Subset \Omega, X \in \mathbb{C}^n.$$

On the other hand, by Theorem 3, we have

$$\gamma_{\mathbb{D}}(f(z), f'(z)X) \le \sqrt{(\omega_{KE}(z)(X,X))}, z \in \Omega, X \in \mathbb{C}^n.$$

Let $\alpha : [0,1] \to \Omega$ be any piecewise C^1-curve joining a and b. Denote by t_0 the maximal $t \in [0,1]$ such that $\alpha([0,t]) \subset \overline{\mathbb{B}(a,\delta)}$. Denote the arc-length of α with respect to the Kähler–Einstein metric by $L_{KE}(\alpha)$. Then

$$\begin{aligned} L_{KE}(\alpha) &= \int_0^{t_0} \sqrt{\omega_{KE}(\alpha(t))(\alpha'(t),\alpha'(t))}dt + \int_{t_0}^1 \sqrt{\omega_{KE}(\alpha(t))(\alpha'(t),\alpha'(t))dt} \\ &\ge \int_0^1 \gamma_{\mathbb{D}}(f(\alpha(t)), f'(\alpha(t))dt + \epsilon \int_0^{t_0} \|\alpha'(t)\|dt \\ &\ge L_{\gamma_{\mathbb{D}}}(f\circ\alpha) + \epsilon\delta \ge \rho_{\mathbb{D}}(f(a), f(b)) + \epsilon\delta = c_\Omega(a,b) + \epsilon\delta. \end{aligned}$$

Hence $d_\Omega^{KE}(a,b) \ge c_\Omega(a,b) + \epsilon\delta$. □

Proof of Theorem 1. The comparison between c_Ω and d_Ω^{KE} can be achieved once some extremal map $f \in \mathrm{Hol}(\Omega, \mathbb{D})$ with respect to the Carathéodory pseudo–distance satisfies the assumption of Lemma 4.

Since Ω is a bounded domain in $\mathbb{C}^n$, we may assume that Ω is contained in the unit ball $B \subset \mathbb{C}^n$ centered at the origin due to the fact that the scaling transformation is a biholomorphism and the Carathéodory pseudo–distance and the Kähler–Einstein distance are preserved under any biholomorphism.

With the global coordinate $(z_1, ..., z_n) \in \Omega$ in $\mathbb{C}^n$, let $\{\frac{\partial}{\partial z_i} | i = 1, ..., n\}$ be the basis on the holomorphic tangent bundle $T_0^{1,0}\Omega$. Without loss of generality, we may assume that $\{\frac{\partial}{\partial z_i} | i = 1, ..., n\}$ are orthonormal with repect to the Euclidean metric and orthogonal with respect to $\omega_{KE}(0)$. With the usual global coordinates $(z_1, \cdots, z_n) \in \Omega$, denote $\frac{\partial}{\partial z_i} = X_i, i = 1, \cdots, n$. We may assume that $0 < \omega_{KE}(0)(X_1, \overline{X}_1) \le \omega_{KE}(0)(X_i, \overline{X}_i), i = 1 \cdots, n$.

We will show there exists a local hypersurface in Ω which is defined by

$$\sqrt{\omega_{KE}(X_1, \overline{X}_1)(0)} \sum_{k=1}^n z_k = 1.$$

From the assumption of Theorem 1, we can take $0 < R \le 1$ such that Ω is contained in B_R and the boundary of Ω touches the boundary of B_R. Then by using a projection map with respect to the first coordinate, we can define the holomorphic map from (Ω, ω_{KE}) to $(\mathbb{D}_R, h)$, where h is the Poincaré metric on $\mathbb{D}_R$ the ball of radius R in $\mathbb{C}^1$. Then by applying Theorem 3, $h(0) \le R^2\omega_{KE}(0)$. This implies

$$1 \le \frac{1}{R} \le R\sqrt{\omega_{KE}(0)(X_1, \overline{X}_1)}. \tag{3.2}$$

In particular, $R \leq 1 \leq R\sqrt{\omega_{KE}(0)(X_1, \overline{X}_1)}$ and the assumption of the existence of $(p_1, \cdots, p_n) \in \Omega$ satisfying $\sum_{k=1}^n p_k = 1$ implies the points $(z_1, \cdots, z_n) \in \Omega$ satisfying $\sqrt{\omega_{KE}(0)(X_1, \overline{X}_1)} \sum_{k=1}^n z_k = 1$. Here, the rescaling of the domain $\Omega \subset B$ and the fact that Ω is a complete Reinhardt domain are applied.

We will control the extremal map with respect to the Carathéodory pseudo–distance $f : \Omega \to \mathbb{D}$ such that

$$c_\Omega((0, \cdots, 0), (x, \cdots, x)) = c_{\mathbb{D}}(f(0, \cdots, 0), f(x, \cdots, x)). \tag{3.3}$$

After acting on the unit disk by an automorphism, we may assume that $f(0, \cdots, 0) = 0$. Also we can replace f by the symmetrization map $\frac{1}{n!}\sum_\sigma f(z_{\sigma(1)}, \cdots, z_{\sigma(n)})$ with all permutations $\sigma : \{1, \cdots, n\} \to \{1, \cdots, n\}$. Around the origin, one can write f as a power series $\sum_{k=1}^n a_k z_k + \sum_{m=2}^\infty P_m(z_1, \cdots, z_n) = \sum_{i=1}^n a_i z_i + f_2(z_1, \cdots, z_n)$, where each $P_m(z_1, \cdots, z_n)$ is a homogeneous polynomial of degree m. Since we use $\frac{1}{n!}\sum_\sigma f(z_{\sigma(1)}, \cdots, z_{\sigma(n)})$, we can replace $\sum_{i=1}^n a_i z_i$ by $a\sum_{k=1}^n z_k$ for some real number a (after acting on the unit disk by an automorphism if necessary). Then by Theorem 3, $f^*\gamma_{\mathbb{D}} \leq \sqrt{\omega_{KE}}$. In particular, we get

$$a \leq \sqrt{\omega_{KE}(0)(X_1, \overline{X}_1)}.$$

Now let's suppose $a = \sqrt{\omega_{KE}(0)(X_1, \overline{X}_1)}$. Since there exists a hypersurface in Ω given by $a\sum_{k=1}^n z_k = \sqrt{\omega_{KE}(X_1, \overline{X}_1)(0)} \sum_{k=1}^n z_k = 1$, we can choose $(u_1, \cdots, u_n) \in \Omega$ such that $\sum_{k=1}^n u_k = \frac{1}{\sqrt{\omega_{KE}(0)(X_1, \overline{X}_1)}}$. Define $g : \mathbb{D} \to \Omega$ by $g(\lambda) := (\lambda u_1, \cdots, \lambda u_n)$. Since Ω is a complete Reinhardt domain, g is a well-defined holomorphic function. Furtheremore, $f \circ g(0) = 0$ and $(f \circ g)'(0) = 1$. Thus $f \circ g = id_{\mathbb{D}}$ by the classical Schwarz's lemma that $f_2(\lambda u_1, \cdots, \lambda u_n) = 0$. Since $\lambda \in \mathbb{D}$ is arbitrary and with other choices of $(u_1, \cdots, u_n)$, we can take a small open set in Ω such that $f_2 = 0$ on this open set. Thus by the identity theorem, $f_2 \equiv 0$ on Ω. Hence $f(z_1, \cdots, z_n) = a\sum_{k=1}^n z_k$, which is impossible since the image of f can't hit the boundary of $\mathbb{D}$. Hence $a < \sqrt{\omega_{KE}(0)(X_1, \overline{X}_1)}$. In particular, this implies

$$|f'(0)X_i| < \sqrt{\omega_{KE}(0)(X_i, \overline{X_i})}, i = 1, \cdots, n. \tag{3.4}$$

Since we showed the above inequality for X_i's, that are orthogonal to the Euclidean metric and $\omega_{KE}(0)$ at the same time, the same inequality holds for arbitrary tangent vectors $X \in \mathbb{C}^n - \{0\}$. Consequently, (3.4) implies the assumption in Lemma 4, and the proof is over. □

Remark 5. A separation between the Carathéodory pseudo–distance and the inner Caratheéodory pseudo–distance was made from similar argument (see [30], [14, Lemma 2.7.8]).

4. $\mathbf{c}_{\mathbb{D}^n_\epsilon}((\mathbf{0}, ..., \mathbf{0}), (\mathbf{x}, ..., \mathbf{x})) \lneq \mathbf{d}^{\mathbf{KE}}_{\mathbb{D}^n_\epsilon}((\mathbf{0}, ..., \mathbf{0}), (\mathbf{x}, ..., \mathbf{x}))$**.** Although the proof of the Thoerem 2 contains the similar argument in the proof of the Theorem 1, we will provide the proof in detail, because the role of two fixed points $(x, ..., x)$ and $(0, ..., 0)$ in $\mathbb{D}^n_\epsilon$ should be justified.

Proof of Theorem 2. As we did in the proof of the Theorem 1, we establish the basic setting first. Fix $\epsilon > 0$, with the global coordinate $(z_1, \cdots, z_n) \in \mathbb{D}^n_\epsilon$ in $\mathbb{C}^n$, let

$\{\frac{\partial}{\partial z_i}|i=1,\cdots,z_n\}$ be the basis on the holomorphic tangent bundle $T_0^{1,0}\mathbb{D}_\epsilon^n$. Without loss of generality, we may assume that $\{\frac{\partial}{\partial z_i}|i=1,\cdots,n\}$ are orthonormal with repect to the Euclidean metric and orthogonal with respect to $\omega_{KE}(0)$. With the global coordinates $(z_1,\cdots,z_n)\in\mathbb{D}_\epsilon^n$, denote $\frac{\partial}{\partial z_i}=X_i, i=1,\cdots,n$. Also, we may assume that $0<\omega_{KE}(0)(X_1,\overline{X}_1)\le\omega_{KE}(0)(X_i,\overline{X}_i), i=1,\cdots,n$. For $z=(z_1,\cdots,z_n)$, define the holomorphic function $h:\mathbb{D}_\epsilon^n\to\mathbb{D}$ by $h(z):=\frac{1}{\epsilon}\Pi_{k=1}^n z_k$.

By the distance-decreasing property of the Carathéodory pseudo–distance,

$$c_{\mathbb{D}}(h(0,\cdots,0),h(x,\cdots,x))\le c_{\mathbb{D}_\epsilon^n}((0,\cdots,0),(x,\cdots,x)).$$

Thus

$$c_{\mathbb{D}}(0,\frac{1}{\epsilon}x^n)\le c_{\mathbb{D}_\epsilon^n}((0,\cdots,0),(x,\cdots,x)). \tag{4.1}$$

From the hypothesis, take $(x,\cdots,x)\in\mathbb{D}_\epsilon^n$ satisfying $(\epsilon n)^{\frac{1}{n-1}}<|x|<1$. We may assume x is a positive real number so that $x\in\mathbb{D}$ satisfies

$$x>(\epsilon n)^{\frac{1}{n-1}}. \tag{4.2}$$

Let $f:\mathbb{D}_\epsilon^n\to\mathbb{D}$ be the extremal map with respect to the Carathéodory pseudo–distance so that

$$c_{\mathbb{D}_\epsilon^n}((0,\cdots,0),(x,\cdots,x))=c_{\mathbb{D}}(f(0,\cdots,0),f(x,\cdots,x)). \tag{4.3}$$

We may assume that $f(0,\cdots,0)=0$. Since $\mathbb{D}_\epsilon^n$ is compatible with the symmetry by all permuations, we may assume that f itself is a symmetrization map so that $f(z)=a\sum_{k=1}^n z_k+\sum_{m=2}^\infty P_m(z)$, each P_m is a homogeneous polynomial of degree m and as in the same argument of the proof of Theorem 1, we get

$$a\le\sqrt{\omega_{KE}(0)(X_1,\overline{X}_1)}.$$

From the description of $\mathbb{D}_\epsilon^n$, we can find $(p_1,\cdots,p_n)\in\mathbb{D}_\epsilon^n$ satisfying $\sum_{k=1}^n z_k=1$ by taking one component with the magnitude almost one and the magnitudes of the other components almost zero. Then as in following the same argument of the proof of Theorem 1, there exists a local hypersurface of $\mathbb{D}_\epsilon^n$ given by

$$\sqrt{\omega_{KE}(X_1,\overline{X}_1)(0)}\sum_{k=1}^n z_k=1.$$

Now we will show $a<\sqrt{\omega_{KE}(0)(X_1,\overline{X}_1)}$. Assume $a=\sqrt{\omega_{KE}(0)(X_1,\overline{X}_1)}$. By (4.1) and (4.3),

$$c_{\mathbb{D}}(0,n\sqrt{\omega_{KE}(0)(X_1,\overline{X}_1)}x)=c_{\mathbb{D}_\epsilon^n}((0,0),(x,x))\ge c_{\mathbb{D}}(0,\frac{1}{\epsilon}x^n).$$

Thus we obtain $x\le\left(n\epsilon\sqrt{(\omega_{KE}(0)(X_1,\overline{X_1})}\right)^{1/n}$. On the other hand, since the projection of $f:\mathbb{D}_\epsilon^n\to\mathbb{D}$ from $\mathbb{D}_\epsilon^n$ to the first coordinate induces the holomoprhic function from $\mathbb{D}$ to $\mathbb{D}$, the classical Schwarz lemma implies $\sqrt{(\omega_{KE}(0)(X_1,\overline{X_1})}=a\le 1$. In particular, we have

$$n\epsilon\sqrt{\omega_{KE}(0)(X_1,\overline{X}_1)}=n\epsilon a\le n\epsilon<1.$$

However, by (4.2), we also have $x > (\epsilon n)^{\frac{1}{n-1}} \geq \left(n\epsilon\sqrt{(\omega_{KE}(0)(X_1, \overline{X_1})}\right)^{1/n}$, which is impossible. Hence $a < \sqrt{\omega_{KE}(0)(X_1, \overline{X}_1)}$.

Then the rest of the proof follows as in the proof of Theorem 1. □

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