

OBSTRUCTIONS TO REPRESENTATIONS UP TO HOMOTOPY AND IDEALS*

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Abstract. This paper considers the Pontryagin characters of graded vector bundles of finite rank, in the cohomology vector spaces of a Lie algebroid over the same base. These Pontryagin characters vanish if the graded vector bundle carries a representation up to homotopy of the Lie algebroid. In other words, strong obstructions to the existence of a representation up to homotopy on a graded vector bundle of finite rank are found. In particular, if a graded vector bundle $E_0[0] \oplus E_1[1] \rightarrow M$ carries a 2-term representation up to homotopy of a Lie algebroid $A \rightarrow M$, then all the (classical) A -Pontryagin classes of E_0 and E_1 must coincide.

This paper generalises as well Bott’s vanishing theorem to the setting of Lie algebroid representations (up to homotopy) on arbitrary vector bundles. As an application, the main theorems induce new obstructions to the existence of infinitesimal ideal systems in a given Lie algebroid.

Key words. Lie algebroids, representations up to homotopy, connections up to homotopy, Pontryagin classes, graded vector bundles, Bott vanishing theorem, infinitesimal ideal systems, fibrations of Lie algebroids.

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1. Introduction. Representations up to homotopy of Lie algebroids were found by Arias Abad and Crainic [2] to be a convenient geometric setting for defining the adjoint representation of a Lie algebroid. They showed in [1] that the *adjoint representation up to homotopy* is the right notion of adjoint representation of a Lie algebroid since it can be used to define its Weil algebra. The precursor notion of *strong homotopy representation* could be found already much earlier in [40], in the context of constrained Poisson algebras – incidentally, in the study of *ideals* in constrained Poisson algebras. Further, 2-term representations up to homotopy are *super-representations* in the sense of Quillen [37].

Gracia-Saz and Mehta found in [18] that these 2-representations are equivalent to splittings of VB-algebroids. This latter insight in particular led in the last ten years to advances in the study of VB-algebroids with an additional geometric structure – see [8], [22, 24, 20], [23, 26, 27], [17], [29], [15], [38] among others. Representations up to homotopy, in particular 2-representations, were further richly studied in e.g. [5], [7], [34], [41], [3], [26], [4].

Obstructions to the existence of n -representations. Recall the definition of an *n -term representation up to homotopy* [2], also called *flat superconnection* in [18] following [37], but named here *n -representation* for short.

DEFINITION ([2, 18]). Let $A \rightarrow M$ be a Lie algebroid. Then an **n -representation** of A is a graded vector bundle $\underline{E} = E_0[0] \oplus \dots \oplus E_{n-1}[n-1] \rightarrow M$ with an operator¹

$$\mathcal{D}: \Omega(A, \underline{E})_{\bullet} \rightarrow \Omega(A, \underline{E})_{\bullet+1}$$

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¹Here, $\Omega(A, \underline{E})_{\bullet}$ is the space of A -forms with values in \underline{E} , see Section 2.1.

that increases the total degree by 1 and satisfies $\mathcal{D}^2 = 0$ as well as

$$\mathcal{D}(\omega \wedge \eta) = \mathbf{d}_A \omega \wedge \eta + (-1)^l \omega \wedge \mathcal{D}\eta \tag{1}$$

for $\omega \in \Omega^l(A)$ and $\eta \in \Omega(A, \underline{E})_\bullet$.

An **n -connection** (or *n -term connection up to homotopy*) of a Lie algebroid A on a graded vector bundle $\underline{E} = E_0[0] \oplus \dots \oplus E_{n-1}[n-1] \rightarrow M$ is defined to be an operator \mathcal{D} as in the definition above, but without the condition $\mathcal{D}^2 = 0$, see e.g. [37, 18, 33].

The A -Pontryagin classes of a vector bundle E measure “the failure of E to have a flat A -connection” – or in other words to carry a representation of A . Therefore, it is natural to ask if there are *characteristic classes of a graded vector bundle* $\underline{E} = E_0[0] \oplus \dots \oplus E_{n-1}[n-1] \rightarrow M$ that measure its failure to carry an n -representation of a Lie algebroid A .

This paper explores the fact that the Chern-Weil construction of Pontryagin characters carries over almost word by word to the setting of n -connections, if the *graded trace* on² $\underline{\text{End}}(\underline{E})$ replaces the trace on endomorphisms of an ordinary vector bundle [37, 33]. In short, given an n -connection, its curvature \mathcal{D}^2 is (graded) $\Omega^\bullet(A)$ -linear and “equals” a form $R_{\mathcal{D}} \in \Omega(A, \underline{\text{End}}(\underline{E}))_\bullet$ of total degree 2. The graded trace $\widehat{\text{str}}(R_{\mathcal{D}}^l)$ of the l -th power of this form is just an element of $\Omega^{2l}(A)$, with $\mathbf{d}_A(\widehat{\text{str}}(R_{\mathcal{D}}^l)) = 0$, hence defining a cohomology class

$$\left[\widehat{\text{str}}(R_{\mathcal{D}}^l) \right] \in H^{2l}(A),$$

called here the **l -th Pontryagin character of the graded vector bundle**. These classes, for $l \in \mathbb{N}$, do not depend on the choice of the n -connection on \underline{E} , and they generate together the **A -Pontryagin algebra of the graded vector bundle \underline{E}** , as an \mathbb{R} -subalgebra of $H^\bullet(A)$. Obviously the A -Pontryagin algebra of \underline{E} vanishes if \underline{E} carries an n -representation of A .

A connection $\nabla: \Gamma(A) \times \Gamma(\underline{E}) \rightarrow \Gamma(\underline{E})$ that preserves the grading is an example of an n -connection of A on \underline{E} . Therefore the generators above of $\text{Pont}_A^\bullet(\underline{E})$ are alternating sums of the classical Pontryagin characters of the terms E_i of E , $i = 0, \dots, n-1$. This immediately yields the following theorem, which seems to have been overlooked so far in the literature.

THEOREM A. *Let $\underline{E} = E_0[0] \oplus \dots \oplus E_{n-1}[n-1]$ be a graded vector bundle over a smooth manifold M , and let $A \rightarrow M$ be a Lie algebroid. If there exists an n -representation \mathcal{D} of A on \underline{E} , then the Pontryagin characters $\sigma_A^l(E_i)$, $l > 1$, of the vector bundles E_i , $i = 0, \dots, n-1$, satisfy the equations*

$$\sum_{i=0}^{n-1} (-1)^i \sigma_A^l(E_i) = 0 \in H^{2l}(A) \tag{2}$$

for all $l > 1$.

In particular, for a graded vector bundle with grading concentrated in degrees 0 and 1, this theorem gives a simple obstruction to the existence of a 2-representation.

²The notation $\underline{\text{End}}$ emphasises the fact that the endomorphism space of a graded vector bundle \underline{E} is considered itself as a graded vector bundle.

THEOREM B. *Let E_0 and E_1 be smooth vector bundles over M , and let $A \rightarrow M$ be a Lie algebroid. If there is a 2-representation of A on $E_0[0] \oplus E_1[1]$, then the A -Pontryagin classes of E_0 and E_1 are equal:*

$$p_A^l(E_0) = p_A^l(E_1) \in H^l(A)$$

for all $l \geq 0$.

Using the adjoint representation up to homotopy of a Lie algebroid $A \rightarrow M$, which is a 2-representation of A on $A[0] \oplus TM[1]$, this yields the following result.

THEOREM C. *Let A be a vector bundle over a smooth manifold M , and let $\rho: A \rightarrow TM$ be a vector bundle morphism over the identity. If $A \rightarrow M$ carries a Lie algebroid structure with anchor ρ , then the Pontryagin classes of A and TM satisfy*

$$\rho^*(p^l(A)) = \rho^*(p^l(TM)) \in H^l(A)$$

for all $l \geq 1$.

This is in fact a special case of the following theorem, which is proved using a similar method.

THEOREM D. *Let A and B be Lie algebroids over M . If there is a Lie algebroid morphism $\phi: B \rightarrow A$ over the identity on M , then*

$$\phi^* p_A^l(A) = p_B^l(A) = p_B^l(B) = \phi^* p_A^l(B) \in H^l(B)$$

for all $l \geq 1$.

Theorems A, B, C and D are proved in Section 4.4.

Bott's vanishing theorems and obstructions to the existence of ideals in Lie algebroids. The starting point of this paper is actually *Bott's vanishing theorem* [6] on Pontryagin classes and foliations:

THEOREM ([6]). *Let M be a smooth manifold and let F be a subbundle of codimension q of TM . If F is involutive, then the Pontryagin spaces*

$$\text{Pont}^l(TM/F) \subseteq H^l(M)$$

of TM/F are all trivial for $l > 2q$.

Since an involutive subbundle $F \subseteq TM$ is always represented on the normal bundle TM/F via the Bott connection [6], this theorem is a special case of the following result (see Section 3.1 and Theorem 4.15 for the counterpart of this theorem in the context of representations up to homotopy).

THEOREM E. *Let E be a smooth vector bundle over a smooth manifold M and let A be a Lie algebroid over M . If there exists a Lie subalgebroid B of A of codimension q with a linear representation $\nabla: \Gamma(B) \times \Gamma(E) \rightarrow \Gamma(E)$, then the A -Pontryagin spaces*

$$\text{Pont}_A^l(E) \subseteq H^l(A)$$

are all trivial for $l > 2q$.

The generalisation of this theorem to the setting of Pontryagin algebras of a graded vector bundle is given by Theorem 4.15. Although it does not yet lead to

additional obstruction results for particular examples, its proof is given in detail for completeness and future applications.

The author's original motivation for proving Theorem E is her search for topological obstructions to the existence of ideals in Lie algebroids. Jointly with Ortiz, the author identified in [29] what they consider the "right notion" of ideals in Lie algebroids. These objects are called *infinitesimal ideal systems* and defined as follows.

DEFINITION 1.1 ([29], [19]). *Let $(q: A \rightarrow M, \rho, [\cdot, \cdot])$ be a Lie algebroid, $F \subseteq TM$ an involutive subbundle, $J \subseteq A$ a subbundle over M such that $\rho(J) \subseteq F$, and ∇ a flat partial F -connection on A/J with the following properties:*

- (1) *If $a \in \Gamma(A)$ is ∇ -parallel³, then $[a, j] \in \Gamma(J)$ for all $j \in \Gamma(J)$.*
- (2) *If $a, b \in \Gamma(A)$ are ∇ -parallel, then $[a, b]$ is also ∇ -parallel.*
- (3) *If $a \in \Gamma(A)$ is ∇ -parallel, then $\rho(a)$ is ∇^F -parallel, where*

$$\nabla^F: \Gamma(F) \times \Gamma(TM/F) \rightarrow \Gamma(TM/F), \quad \nabla_X^F \bar{Y} = \overline{[X, Y]}$$

is the Bott connection associated to F .

*Then the triple (F, J, ∇) is an **infinitesimal ideal system** in A .*

The first axiom implies immediately that $J \subseteq A$ is a subalgebroid of A . Infinitesimal ideal systems are an infinitesimal version of the *ideal systems* in [21, 31] – the latter are exactly the kernels of fibrations of Lie algebroids [21]. Infinitesimal ideal systems already appear in [19] (not under this name) in the context of geometric quantization as the infinitesimal version of polarizations on groupoids. Moreover, the special case where $F = TM$ has been studied independently in [13] in relation with a modern approach to Cartan's work on pseudogroups.

Consider an involutive subbundle $F \subseteq TM$ and the Bott connection $\nabla^F: \Gamma(F) \times \Gamma(TM/F) \rightarrow \Gamma(TM/F)$ associated to it. Then the triple (F, F, ∇^F) is an infinitesimal ideal system in the Lie algebroid TM . Therefore, Bott's vanishing theorem provides an obstruction result for this particular class of infinitesimal ideal systems. The general goal of this paper is to find adequate generalisations of Bott's vanishing theorem, yielding obstructions to the existence of infinitesimal ideal systems in a given Lie algebroid $A \rightarrow M$ in terms of the Pontryagin classes of A and TM .

The following result (see Propositions 5.4 and 5.5) gives the first information that can be extracted from Theorem E and the definition of an infinitesimal ideal system.

THEOREM F. *Let (F, J, ∇) be an infinitesimal ideal system in a Lie algebroid $A \rightarrow M$. Let s be the codimension of J in A and let q be the codimension of F in TM . Then*

- (1) *the Pontryagin spaces $\text{Pont}^l(A/J)$ and $\text{Pont}^l(TM/F)$ in $H^\bullet(M)$ all vanish for $l > 2q$, and*
- (2) *the Pontryagin spaces $\text{Pont}_A^l(A/J)$ and $\text{Pont}_A^l(TM/F)$ in $H^\bullet(A)$ all vanish for $l > 2 \min\{s, q\}$.*

However, this result turns out to be rather unsatisfactory on its own because it uses only very few of the axioms of an infinitesimal ideal system: (1), (2) and (3) in the definition are not used in the proof of this proposition. These three axioms ensure [15] that an infinitesimal ideal system in a Lie algebroid $A \rightarrow M$ defines a subrepresentation $J[0] \oplus F[1]$ of the adjoint representation up to homotopy of A on

³A section $a \in \Gamma(A)$ is said to be **∇ -parallel** if $\nabla_X \bar{a} = 0$ for all $X \in \Gamma(F)$. Here, \bar{a} is the class of a in $\Gamma(A/J) \simeq \Gamma(A)/\Gamma(J)$.

$A[0] \oplus TM[1]$, after the choice of a suitable connection. Theorem B hence translates this fact to the context of A -Pontryagin classes of F and J . More precisely, the results in [15] and Theorem B lead to further obstructions to the existence of an infinitesimal ideal system in a Lie algebroid A (see Theorem G):

THEOREM G. *Let $(A \rightarrow M, \rho, [\cdot, \cdot])$ be a Lie algebroid. If (F, J, ∇) is an infinitesimal ideal system in A , then*

$$p_A^l(J) = p_A^l(F)$$

for all $l \geq 1$.

Outline of the paper. Section 2 recalls in detail the Chern-Weil construction of the Pontryagin classes of a vector bundle, using the powerful modern language exposed in [14]. The author recommends here as well the reference [42], which summarises in a beautiful manner the construction of characteristic classes associated to vector bundles and principal bundles, as well as some of their applications in geometry and topology.

Section 3 proves the first generalisation of Bott’s vanishing theorem in [6], and proves a refinement of it in the case where an appropriate Atiyah class vanishes.

Section 4 studies connections up to homotopy and the Pontryagin algebras of graded vector bundles of finite rank. The obstruction to the existence of representations up to homotopy is also proved there, as well as Bott’s vanishing theorem for graded vector bundles.

Section 5 finally applies the prior results to the study of characteristic classes defined by infinitesimal ideal systems in Lie algebroids.

Throughout the paper, all the considered manifolds and vector bundles are assumed to be smooth.

Outlook. The construction of the A -Pontryagin algebra of a graded vector bundle presented here can be extended to a construction of the $(\mathcal{M}, \mathcal{Q})$ -Pontryagin algebra of a graded vector bundle, for a Lie n -algebroid $(\mathcal{M}, \mathcal{Q})$. This is the subject of a project in progress that is joint with Theocharis Papantonis.

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Terminology. Let $A \rightarrow M$ be a Lie algebroid. The A -Pontryagin characters of a vector bundle should not be confused with its A -Pontryagin classes. They are another set of generators of the A -Pontryagin algebra of the vector bundle. To be precise, they are the images under the Chern-Weil morphism of the polynomials $X \mapsto \text{tr}(X^i)$, $i \in \mathbb{N}$ – see Section 2.3. In the complex case, these polynomials yield the Chern characters which, up to multiplications with some complex coefficients, sum up to the Chern character of the complex vector bundle.

2. Preliminaries. This section recalls the modern definition of Pontryagin classes of a vector bundle. It begins with some background on linear connections on vector bundles and the associated calculus on differential forms. The second subsection recalls the definition of the Pontryagin classes of a vector bundle. In this section, the main reference is [14], but A -Pontryagin classes of a vector bundle were defined in [16].

2.1. Notation and vector-valued forms. Given a Lie algebroid $A \rightarrow M$, the Lie algebroid cohomology defined by the complex $(\Omega^\bullet(A), \mathbf{d}_A)$ is written $H^\bullet(A)$. The cohomology $H^\bullet(TM)$ is just written $H^\bullet(M)$ since the cohomology of the standard Lie algebroid $TM \rightarrow M$ is just the de Rham cohomology of M .

Let A be a Lie algebroid over a manifold M and let $E \rightarrow M$ be a vector bundle. Then $\Omega^\bullet(A, E) := \Gamma(\wedge^\bullet A^* \otimes E)$. If $A = TM$ is the standard tangent Lie algebroid, then $\Omega^\bullet(TM, E)$ is written $\Omega^\bullet(M, E)$ for simplicity. The degree of a (degree-homogeneous) element ω of $\Omega^\bullet(A, E)$ is written $|\omega| \in \mathbb{N}$.

For $K \in \Omega^l(A, \text{Hom}(E, E'))$, the graded $\Omega^\bullet(A)$ -linear operator $\widehat{K}: \Omega^\bullet(A, E) \rightarrow \Omega^{\bullet+l}(A, E')$ is defined by $\widehat{K}(\omega) = K \wedge \omega$, i.e.

$$\widehat{K}(\omega)(a_1, \dots, a_{s+l}) = \sum_{\sigma \in \mathfrak{S}_{(l,s)}} (-1)^\sigma K(a_{\sigma(1)}, \dots, a_{\sigma(l)})(\omega(a_{\sigma(l+1)}, \dots, a_{\sigma(l+s)}))$$

for $\omega \in \Omega^s(A, E)$ and $a_1, \dots, a_{s+l} \in \Gamma(A)$. Here, $\mathfrak{S}_{(l,s)}$ is the set of (l, s) -shuffles, i.e. the permutations $\sigma \in S_{l+s}$ such that $\sigma(1) < \dots < \sigma(l)$ and $\sigma(l+1) < \dots < \sigma(l+s)$.

The space of graded- $\Omega^\bullet(A)$ -linear operators $\Omega(A, E) \rightarrow \Omega(A, E')$ is denoted by $\text{Hom}_{\Omega^\bullet(A)}^\bullet(\Omega(A, E), \Omega(A, E'))$. That is, an element \mathcal{K} of $\text{Hom}_{\Omega^\bullet(A)}^s(\Omega(A, E), \Omega(A, E'))$, for $s \geq 0$, is a map $\mathcal{K}: \Omega^\bullet(A, E) \rightarrow \Omega^{\bullet+s}(A, E')$ satisfying $\mathcal{K}(\omega \wedge \eta) = (-1)^{s|\omega|} \omega \wedge \mathcal{K}(\eta)$ for all $\omega \in \Omega^\bullet(A)$ and $\eta \in \Omega^\bullet(A, E)$.

The map $\Omega^\bullet(A, \text{Hom}(E, E')) \rightarrow \text{Hom}_{\Omega^\bullet(A)}^\bullet(\Omega(A, E), \Omega(A, E'))$ given by $K \mapsto \widehat{K}$ is a bijection [2], with inverse sending $\mathcal{K}: \Omega^\bullet(A, E) \rightarrow \Omega^{\bullet+s}(A, E')$ to $\mathcal{K}_0 \in \Omega^s(A, \text{Hom}(E, E'))$ defined by

$$\mathcal{K}_0(a_1, \dots, a_s)(e) = \mathcal{K}(e)(a_1, \dots, a_s)$$

for $a_1, \dots, a_s \in \Gamma(A)$ and $e \in \Gamma(E) = \Omega^0(A, E)$.

Let now $\underline{E} = \bigoplus_{z \in \mathbb{Z}} E_z[z]$ be a graded vector bundle over M . As always, the $\Omega^\bullet(A)$ -module of \underline{E} -valued forms $\Omega(A, \underline{E})_\bullet$ has a total grading given by $\deg \eta = j + l$ for $\eta \in \Omega^j(A, E_l)$. Here also, there is a bijection between elements $K \in \Omega(A, \underline{\text{Hom}}(\underline{E}, \underline{F}))_s$ and graded- $\Omega^\bullet(A)$ -linear operators $\mathcal{K}: \Omega(A, \underline{E})_\bullet \rightarrow \Omega(A, \underline{F})_{\bullet+s}$ that increase the total degree by s . An element $K \in \Omega(A, \underline{\text{Hom}}(\underline{E}, \underline{F}))_s$ can be written

$$K = \sum_{i=0}^s \sum_{j-l=s-i} K^{i,l,j} \in \bigoplus_{i=0}^s \bigoplus_{j-l=s-i} \Omega^i(A, \text{Hom}(E_l, F_j)),$$

i.e. with $K^{i,l,j}$ the component of K in $\Omega^i(A, \text{Hom}(E_l, F_j))$ for i, l, j . The corresponding $\widehat{K} \in \text{Hom}_{\Omega^\bullet(A)}(\Omega(A, \underline{E}), \Omega(A, \underline{F}))_s$ is given by

$$\widehat{K} = \sum_{i=0}^s \sum_{j-l=s-i} \widehat{K^{i,l,j}},$$

with $\widehat{K^{i,l,j}}: \Omega^\bullet(A, E_l) \rightarrow \Omega^{\bullet+i}(A, F_j)$ defined as before. The inverse to the map $\widehat{}$ is easily defined as above.

Finally, the graded commutator of degree-homogeneous elements $K_1, K_2 \in \Omega(A, \widehat{\text{End}}(E))_\bullet$ can now be defined by

$$[\widehat{K_1}, \widehat{K_2}] = [\widehat{K_1}, \widehat{K_2}] = \widehat{K_1} \circ \widehat{K_2} - (-1)^{|K_1| \cdot |K_2|} \widehat{K_2} \circ \widehat{K_1}.$$

2.2. Linear connections on vector bundles, and vector valued forms.

Let $E \rightarrow M$ be a vector bundle, and let $(A \rightarrow M, \rho, [\cdot, \cdot])$ be a Lie algebroid over the same base. Then a linear A -connection $\nabla: \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$ is equivalent to an operator $\mathbf{d}_\nabla: \Omega^\bullet(A, E) \rightarrow \Omega^{\bullet+1}(A, E)$ satisfying

$$\mathbf{d}_\nabla(\omega \wedge \eta) = (\mathbf{d}_A \omega) \wedge \eta + (-1)^l \omega \wedge \mathbf{d}_\nabla \eta \tag{3}$$

for $\omega \in \Omega^l(A)$ and $\eta \in \Omega^\bullet(A, E)$. Given ∇ , the operator \mathbf{d}_∇ is defined by (3) and by

$$\mathbf{d}_\nabla e = \nabla \cdot e \in \Omega^1(A, E)$$

for $e \in \Gamma(E) = \Omega^0(A, E)$. For instance, if $E = \mathbb{R} \times M$ with the canonical flat A -connection $\nabla_a f = \mathcal{L}_{\rho(a)}(f)$, then $\Omega^\bullet(A, E) \simeq \Omega^\bullet(A)$ and $\mathbf{d}_\nabla =: \mathbf{d}_A$, which satisfies in addition $\mathbf{d}_A^2 = 0$ and defines the Lie algebroid cohomology $H^\bullet(A)$. In general,

$$\mathbf{d}_\nabla^2 = \widehat{R_\nabla}: \Omega^\bullet(A, E) \rightarrow \Omega^{\bullet+2}(A, E),$$

with $R_\nabla \in \Omega^2(A, \text{End}(E))$ the curvature tensor of ∇ .

Let E_1 and E_2 be vector bundles over M , and let ∇^1 and ∇^2 be linear A -connections on E_1 and E_2 , respectively. The reader is invited to check (see also [14]) that for $K \in \Omega^s(A, \text{Hom}(E_1, E_2))$,

$$\mathbf{d}_{\nabla^2} \circ \widehat{K} - (-1)^s \widehat{K} \circ \mathbf{d}_{\nabla^1} = \widehat{\mathbf{d}_{\nabla^{\text{Hom}}} K}, \tag{4}$$

where $\nabla^{\text{Hom}}: \Gamma(A) \times \Gamma(\text{Hom}(E_1, E_2)) \rightarrow \Gamma(\text{Hom}(E_1, E_2))$ is defined by $(\nabla_a^{\text{Hom}} \phi)(e) = \nabla_a^2(\phi(e)) - \phi(\nabla_a^1 e)$ for $a \in \Gamma(A)$ and $e \in \Gamma(E_1)$. If $E_1 = E_2$ and $\nabla^1 = \nabla^2$, then

$$[\mathbf{d}_\nabla, \widehat{K}] = \mathbf{d}_\nabla \circ \widehat{K} - (-1)^k \widehat{K} \circ \mathbf{d}_\nabla = \widehat{\mathbf{d}_{\nabla^{\text{End}}} K}. \tag{5}$$

The trace operator $\text{tr}: \Gamma(\text{End}(E)) \rightarrow C^\infty(M)$ can be understood as an element of $\Omega^0(A, \text{Hom}(\text{End}(E), \mathbb{R}))$, and so defines as above an $\Omega^\bullet(A)$ -linear map $\widehat{\text{tr}}: \Omega^\bullet(A, \text{End}(E)) \rightarrow \Omega^\bullet(A)$ that preserves the degree.

Equip $\mathbb{R} \times M$ as above with the flat A -connection $\mathcal{L}: \Gamma(A) \times C^\infty(M) \rightarrow C^\infty(M)$, and the vector bundle $\text{End}(E)$ with the connection induced by $\nabla: \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$. Then the induced connection

$$\nabla^{\text{Hom}}: \Gamma(A) \times \Gamma(\text{Hom}(\text{End}(E), \mathbb{R})) \rightarrow \Gamma(\text{Hom}(\text{End}(E), \mathbb{R}))$$

applied to the trace operator reads

$$(\nabla_a^{\text{Hom}} \widehat{\text{tr}})(\phi) = \mathcal{L}_{\rho(a)}(\text{tr}(\phi)) - \text{tr}(\nabla_a^{\text{End}} \phi)$$

for $a \in \Gamma(A)$ and $\phi \in \Gamma(\text{End}(E))$.

LEMMA 2.1. *With the choices of connections above, $\nabla_a^{\text{Hom}} \widehat{\text{tr}} = 0$ for all $a \in \Gamma(A)$.*

Proof. Take a local frame (e_1, \dots, e_k) of E over an open set $U \subseteq M$ and consider the dual local frame $(\epsilon_1, \dots, \epsilon_k)$ of E^* . It is easy to see that for each $i, j = 1, \dots, k$ and each $a \in \Gamma_U(A)$:

$$\begin{aligned} (\nabla_a^{\text{Hom}} \text{tr})(e_i \otimes \epsilon_j) &= \mathcal{L}_{\rho(a)}(\delta_{ij}) - \text{tr}(\nabla_a^{\text{End}}(e_i \otimes \epsilon_j)) = - \sum_{s=1}^k \langle \epsilon_s, \nabla_a^{\text{End}}(e_i \otimes \epsilon_j)(e_s) \rangle \\ &= - \sum_{s=1}^k \langle \epsilon_s, \nabla_a(\delta_{js}e_i) - \langle \epsilon_j, \nabla_a e_s \rangle e_i \rangle \\ &= - \langle \epsilon_j, \nabla_a e_i \rangle + \langle \epsilon_j, \nabla_a e_i \rangle = 0. \end{aligned}$$

Since the sections $e_i \otimes \epsilon_j$, $i, j = 1, \dots, k$, of $\text{End}(E)$ build a frame for $\text{End}(E)$ over U , this shows that $\nabla_a^{\text{Hom}} \text{tr} = 0$. \square

Lemma 2.1 and (4) yield the equality

$$\mathbf{d}_A \circ \widehat{\text{tr}} = \widehat{\text{tr}} \circ \mathbf{d}_{\nabla^{\text{End}}} : \Omega^\bullet(A, \text{End}(E)) \rightarrow \Omega^{\bullet+1}(A). \quad (6)$$

2.3. A-Pontryagin characters of a vector bundle. As before, consider a Lie algebroid $A \rightarrow M$, and a vector bundle $E \rightarrow M$ of rank k , with a linear A -connection $\nabla : \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$. Let $R_\nabla \in \Omega^2(A, \text{Hom}(TM, A))$ be the curvature of ∇ .

Define for $i \geq 1$ the form $R_\nabla^i \in \Omega^{2i}(A, \text{End}(E))$ by

$$\widehat{R_\nabla^i} = \widehat{R_\nabla}^i = \mathbf{d}_\nabla^{2i} \in \text{End}_{\Omega^\bullet(A)}(\Omega^\bullet(A, E)).$$

Then (5) shows $\mathbf{d}_{\nabla^{\text{End}}} \widehat{R_\nabla^i} = [\mathbf{d}_\nabla, \widehat{R_\nabla^i}] = [\mathbf{d}_\nabla, \widehat{R_\nabla}^i] = [\mathbf{d}_\nabla, \mathbf{d}_\nabla^{2i}] = 0$, and so with (6):

$$\mathbf{d}_A(\widehat{\text{tr}}(R_\nabla^i)) = \widehat{\text{tr}}(\mathbf{d}_{\nabla^{\text{End}}} R_\nabla^i) = 0. \quad (7)$$

Therefore, $\widehat{\text{tr}}(R_\nabla^i)$ defines a cohomology class in $H^{2i}(A)$.

LEMMA 2.2. *Let $E \rightarrow M$ be a vector bundle and let $A \rightarrow M$ be a Lie algebroid. Then the cohomology class $[\widehat{\text{tr}}(R_\nabla^i)] \in H^{2i}(A)$ does not depend on the choice of A -connection ∇ on E , for $i \geq 1$.*

DEFINITION 2.3. *Let E be a vector bundle over M and let $A \rightarrow M$ be a Lie algebroid.*

(1) *Choose any linear A -connection ∇ on E . The cohomology classes*

$$\sigma_A^i(E) := [\widehat{\text{tr}}(R_\nabla^i)] \in H^{2i}(A),$$

for $i \geq 0$, are called the A -Pontryagin characters of E .

(2) *The A -Pontryagin algebra of E is the \mathbb{R} -subalgebra $\text{Pont}_A^\bullet(E) \subseteq H^\bullet(A)$ generated by the A -Pontryagin characters in (1).*

This proof of Lemma 2.2 is standard; in the context of Lie algebroid Pontryagin characters, it is due to [16] following a classical method. The proof is omitted here, but done later in the more general setting of Pontryagin algebras defined by connections up to homotopy (see Proposition 4.6, and Appendix A); in the same manner as in [37] for superconnections.

Note that the A -Pontryagin characters of E are not its A -Pontryagin classes, but another set of generators of its A -Pontryagin algebra, see the next paragraph. The

terminology ‘characters’ is in line with the definition of the *Chern characters* of a complex vector bundle.

The Pontryagin algebra is also called the *characteristic algebra* in [42]. It is easy to see that $\text{Pont}_A^l(E) = 0$ for l an odd number. It is a standard fact that even $\text{Pont}_A^l = 0$ for l not divisible by 4. For completeness, Bott’s proof of this fact [6] is quickly recalled here. Equip the vector bundle E with a smooth metric (i.e. a positive definite fibrewise pairing), and take the A -connection $\nabla: \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$ to be *metric*: $\langle \nabla_a e, e' \rangle + \langle e, \nabla_a e' \rangle = \mathcal{L}_{\rho(a)} \langle e, e' \rangle$ for $a \in \Gamma(A)$, $e, e' \in \Gamma(E)$. Then it is easy to check that $\langle R_\nabla(a, b)e, e' \rangle = -\langle e, R_\nabla(a, b)e' \rangle$ for all $a, b \in \Gamma(A)$, $e, e' \in \Gamma(E)$, and inductively

$$\langle R_\nabla^i(a_1, b_1, a_2, b_2, \dots, a_i, b_i)e, e' \rangle = (-1)^i \langle e, R_\nabla^i(a_1, b_1, a_2, b_2, \dots, a_i, b_i)e' \rangle$$

for $i \geq 1$. Then immediately $\widehat{\text{tr}}(R_\nabla^i) = 0$ for i odd, and so $\text{Pont}_A^{2i}(E) = 0$ for i odd.

Finally, the **A -Pontryagin classes of the vector bundle E** can be defined; see e.g. [42] for detailed explanations. Consider $\text{Gl}(k, \mathbb{R})$ -invariant polynomial functions $p: \mathfrak{gl}(k, \mathbb{R}) \rightarrow \mathbb{R}$, i.e. such that for all $g \in \text{Gl}(k, \mathbb{R})$ and $X \in \mathfrak{gl}(k, \mathbb{R})$

$$p(gXg^{-1}) = p(X).$$

The $\text{Gl}(k, \mathbb{R})$ -invariant polynomials on $\mathfrak{gl}(k, \mathbb{R})$ form an \mathbb{R} -algebra, which is generated as an \mathbb{R} -algebra by the polynomials $\Sigma_0, \Sigma_1, \dots$ defined by

$$\Sigma_i(X) = \text{trace}(X^i)$$

for all $X \in \mathfrak{gl}(k, \mathbb{R})$ (see for instance [6]). It follows from Definition 2.3 that each $\text{Gl}(k, \mathbb{R})$ -invariant polynomial p on $\mathfrak{gl}(k, \mathbb{R})$ defines a closed form $p(R_\nabla) \in \Omega^\bullet(A)$ and an element $[p(R_\nabla)] \in H^\bullet(A)$. More precisely, if $p = q(\Sigma_{i_1}, \dots, \Sigma_{i_l}) \in \mathbb{R}[\Sigma_1, \Sigma_2, \dots]$, then

$$p(R_\nabla) = q(\Sigma_{i_1}(R_\nabla), \dots, \Sigma_{i_l}(R_\nabla)).$$

For instance, $p = \Sigma_2 - (\Sigma_1)^2$ gives $p(R_\nabla) = \widehat{\text{tr}}(R_\nabla^2) - \widehat{\text{tr}}(R_\nabla) \wedge \widehat{\text{tr}}(R_\nabla)$. This defines the Chern-Weil morphism of \mathbb{R} -algebras

$$\text{cw}_A(E): \text{Sym}^\bullet(\mathfrak{gl}(k, \mathbb{R}))^{\text{Gl}(k, \mathbb{R})} \rightarrow H^{2\bullet}(A), \quad p \mapsto [p(R_\nabla)].$$

The \mathbb{R} -subalgebra $\text{Pont}_A^\bullet(E) \subseteq H^\bullet(A)$ is the image of this morphism, i.e. the subalgebra of all cohomology classes $[p(R_\nabla)]$ defined by $\text{Gl}(k, \mathbb{R})$ -invariant polynomial p on $\mathfrak{gl}(k, \mathbb{R})$.

For i a positive integer, the characteristic polynomial

$$\det(\lambda \cdot I_k + X) = \sum_{i=0}^k f_i(X) \lambda^{k-i} \tag{8}$$

defines homogeneous polynomials f_i of degree i on $\mathfrak{gl}(k, \mathbb{R})$, for $k \geq i \geq 0$. These polynomials are obviously $\text{Gl}(k, \mathbb{R})$ -invariant, and so for each $i \geq 1$, the **i -th A -Pontryagin class of E** can be defined as

$$p_A^i(E) := \left[f_{2i} \left(\frac{i}{2\pi} R_\nabla \right) \right] \in H^{4i}(A),$$

for any choice of connection $\nabla: \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$. The A -Pontryagin classes of E generate together $\text{Pont}_A^\bullet(E)$ (see for instance [6]). The **total A -Pontryagin class** of E is defined by

$$p_A(E) = \left[\det \left(I_k + \frac{i}{2\pi} R_\nabla \right) \right] = 1 + p_A^1(E) + p_A^2(E) + \dots + p^{\lfloor \frac{k}{2} \rfloor} \in \text{Pont}_A^\bullet(E).$$

REMARK 2.4. Given an ordinary linear connection $\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ on a vector bundle E of rank k , a Lie algebroid $A \rightarrow M$ defines a linear A -connection $\nabla^A: \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$ by $\nabla_a^A e = \nabla_{\rho(a)} e$. It is easy to see that

$$[p(R_{\nabla^A})] = \rho^*[p(R_\nabla)] \in H^\bullet(A)$$

for any $\text{Gl}(k, \mathbb{R})$ -invariant polynomial p on $\mathfrak{gl}(k, \mathbb{R})$. Here, ρ^* is the cochain map

$$\rho^*: (\Omega^\bullet(M), \mathbf{d}) \rightarrow (\Omega^\bullet(A), \mathbf{d}_A),$$

$\rho^*(\omega)(a_1, \dots, a_s) = \omega(\rho(a_1), \dots, \rho(a_s))$ for $\omega \in \Omega^s(M)$ and $a_1, \dots, a_s \in \Gamma(A)$.

As observed by Fernandes in [16], this yields $\text{Pont}_A^\bullet(E) = \rho^*(\text{Pont}^\bullet(E))$, or more precisely $\text{cw}_A(E) = \rho^* \circ \text{cw}(E)$.

3. Bott’s vanishing theorem in a more general setting. This section rephrases Bott’s proof of the vanishing Pontryagin classes of the normal bundle to an involutive subbundle of the tangent [6]. Since the decisive object is the Bott connection, i.e. a flat F -connection on a vector bundle TM/F , that can be extended to a linear TM -connection in order to define Pontryagin characters or classes, one can easily prove a similar result for the existence of a flat partial connection on a general smooth vector bundle. Further, the construction is adapted to the more general A -Pontryagin classes of a vector bundle E .

3.1. Bott’s vanishing theorem. Let A be a Lie algebroid over a manifold M , and let B be a subalgebroid of A over M . Let n be the rank of A , and let q be the codimension of B in A . Let E be a vector bundle over M , with a *flat B -connection* ∇ . It is not difficult to see⁴ that ∇ can be *extended* to an A -connection $\tilde{\nabla}: \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$, satisfying

$$\tilde{\nabla}_b e = \nabla_b e \tag{9}$$

for all $b \in \Gamma(B)$ and $e \in \Gamma(E)$.

Define the space $I^\bullet(B) \subseteq \Omega^\bullet(A)$ as the ideal in $\Omega^\bullet(A)$ generated by the 1-forms vanishing on B . That is, it is generated by the sections of the *annihilator* $B^\circ \subseteq A^*$ of B . It is explicitly given by $I^0(B) = \{0\} \subseteq \Omega^0(A) = C^\infty(M)$ and

$$I^r(B) = \{\omega \in \Omega^r(A) \mid \omega(b_1, \dots, b_r) = 0 \text{ for all } b_1, \dots, b_r \in \Gamma(B)\}$$

for $r \geq 1$.

Choose an open set $U \subseteq M$ trivialising A and B . That is, there is a smooth frame (a_1, \dots, a_n) for A over U such that (a_{q+1}, \dots, a_n) is a smooth frame for B . Consider the dual frame $(\alpha_1, \dots, \alpha_n)$ of A^* over U . By construction, $(\alpha_1, \dots, \alpha_q)$ is a smooth

⁴Just find a complement C of B in A and any A -connection ∇^A on E . Then set $\tilde{\nabla}_{b+c} e = \nabla_b e + \nabla_c^A e$ for all $b \in \Gamma(B)$ and $c \in \Gamma(C)$.

frame for B° over U . Since $I^\bullet(B)$ is generated as an ideal by $\Gamma(B^\circ)$, for $r \geq 1$, an element ω of $I_U^{r-1}(B)$ can be written as

$$\omega = \sum_{i=1}^q \omega_i \wedge \alpha_i$$

with $\omega_i \in \Omega_U^{r-1}(A)$. Therefore, since B° has rank q , the wedge product

$$(I^\bullet(B))^{q+1} = \underbrace{I^\bullet(B) \wedge \dots \wedge I^\bullet(B)}_{q+1 \text{ times}}$$

must necessarily vanish.

It is easy to see that (9) implies

$$R_{\widehat{\nabla}}(b, b')e = R_{\nabla}(b, b')e = 0$$

for $b, b' \in \Gamma(B)$ and all $e \in \Gamma(E)$, and so $R_{\widehat{\nabla}} \in I^2(B) \otimes_{C^\infty(M)} \Gamma(\text{End}(E))$. This implies $R_{\widehat{\nabla}}^i \in (I^2(B))^i \otimes_{C^\infty(M)} \Gamma(\text{End}(E))$ and so $\widehat{\text{tr}}(R_{\widehat{\nabla}}^i) \in (I^2(B))^i$. More generally, for p a $\text{Gl}(k, \mathbb{R})$ -invariant polynomial of degree d on $\mathfrak{gl}(k, \mathbb{R})$, the $2d$ -form $p(R_{\nabla}) \in \Omega^{2d}(A)$ is an element of $(I^2(B))^d$ and so $p(R_{\nabla}) = 0$ for $d > q$.

As a summary, this section has proved the following result.

THEOREM E. *Let E be a vector bundle over a manifold M and let A be a Lie algebroid over M . If there exists a Lie subalgebroid B of A of codimension q with a linear representation $\nabla: \Gamma(B) \times \Gamma(E) \rightarrow \Gamma(E)$, then the Pontryagin spaces*

$$\text{Pont}_A^l(E) \subseteq H^l(A)$$

are all trivial for $l > 2q$.

Using Remark 2.4, this yields the following obstruction result in terms of the classical Pontryagin spaces of E .

COROLLARY 3.1. *Let E be a vector bundle over a manifold M and let A be a Lie algebroid over M . If there exists a Lie subalgebroid B of A of codimension q with a linear representation $\nabla: \Gamma(B) \times \Gamma(E) \rightarrow \Gamma(E)$, then the Pontryagin spaces*

$$\text{Pont}^l(E) \subseteq H^l(M)$$

all lie in the kernel of $\rho^*: H^\bullet(M) \rightarrow H^\bullet(A)$ for $l > 2q$.

If a Lie algebroid A has a subalgebroid B of codimension q ; then B is represented on A/B via the flat Bott-connection

$$\nabla^B: \Gamma(B) \times \Gamma(A/B) \rightarrow \Gamma(A/B), \quad \nabla_b^B \bar{a} = \overline{[b, a]}.$$

Hence $\text{Pont}_A^l(A/B) \subseteq H^l(A)$ is trivial for $l > 2q$. This yields obstructions to a subalgebroid structure on $B \subseteq A$ of codimension q .

However, in the case $A = TM$, the algebroid $B =: F$ is in fact more than just a subalgebroid: it carries as well an infinitesimal ideal system (F, F, ∇^F) [29]. The goal of this paper is the generalisation of Bott's vanishing theorem [6] as a statement on ideals.

3.1.1. Massey products. As already emphasised in [6], Theorem E shows more than the vanishing of the Pontryagin classes $p_A^l(E)$ for $l > 2q$. It shows the vanishing of all A -characteristic classes of E defined by invariant, degree-homogeneous polynomials of degree $d > q$. In [39], Bott’s vanishing theorem is refined as we explain now in our general setting.

Let $A \rightarrow M$ be a Lie algebroid and $[\alpha], [\beta], [\gamma] \in H^\bullet(A)$ be classes such that

$$[\alpha] \wedge [\beta] = 0 \quad \text{and} \quad [\beta] \wedge [\gamma] = 0.$$

Then $\alpha \wedge \beta = \mathbf{d}_A \omega$ and $\beta \wedge \gamma = (-1)^{|\alpha|} \mathbf{d}_A \eta$ for some forms ω and $\eta \in \Omega^\bullet(A)$. As a consequence, $\mathbf{d}_A(\omega \wedge \gamma) = \alpha \wedge \beta \wedge \gamma = \mathbf{d}_A(\alpha \wedge \eta)$, which shows that the class

$$\langle [\alpha], [\beta], [\gamma] \rangle := [\omega \wedge \gamma - \alpha \wedge \eta] \in H^\bullet(A)$$

is defined. As mentioned in [6], this is called the **Massey triple product** [32] of $[\alpha], [\beta], [\gamma] \in H^\bullet(A)$; it is well-defined up to an element of the ideal generated by $[\alpha]$ and $[\gamma]$ in $H^\bullet(A)$.

Consider the situation of Theorem E and take any three classes $[\alpha], [\beta]$ and $[\gamma]$ in $\text{Pont}_A^\bullet(E)$ such that $|\alpha| + |\beta| > 2q$ and $|\beta| + |\gamma| > 2q$. Then $\alpha, \beta, \gamma \in \Omega^\bullet(A)$ can be chosen $\alpha = p_\alpha(R_\nabla)$, $\beta = p_\beta(R_\nabla)$ and $\gamma = p_\gamma(R_\nabla)$ for ∇ as in the proof of Theorem E and $p_\alpha, p_\beta, p_\gamma \in \text{Gl}(k, \mathbb{R})$ -invariant polynomials on $\mathfrak{gl}(k, \mathbb{R})$ of degrees $|\alpha|/2, |\beta|/2$ and $|\gamma|/2$, respectively – where k is the rank of E . Then by definition⁵, $\alpha \wedge \beta = (p_\alpha \cdot p_\beta)(R_\nabla)$, which must vanish by the proof of Theorem E and $|\alpha| + |\beta| > 2q$, and in the same manner $\beta \wedge \gamma = 0$. Then by definition, $\langle [\alpha], [\beta], [\gamma] \rangle = 0$. This proves the following theorem, which is attributed to Shulman in [6].

THEOREM 3.2. *Let E be a vector bundle over a manifold M and let A be a Lie algebroid over M . If there exists a Lie subalgebroid B of A of codimension q with a linear representation $\nabla: \Gamma(B) \times \Gamma(E) \rightarrow \Gamma(E)$, then for all $[\alpha], [\beta]$ and $[\gamma]$ in $\text{Pont}_A^\bullet(E)$ such that $|\alpha| + |\beta| > 2q$ and $|\beta| + |\gamma| > 2q$,*

$$\langle [\alpha], [\beta], [\gamma] \rangle = 0.$$

3.2. Reducible vector bundles – a short discussion. Consider a fibration of vector bundles

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ q_E \downarrow & & \downarrow q_{E'} \\ M & \xrightarrow{f} & M' \end{array}$$

i.e. a fibrewise surjective vector bundle morphism ϕ over a smooth surjective submersion f . Assume that E and E' have the same rank, so that ϕ restricted to each fibre is a bijection. If f has connected fibres, then M' can be identified with the leaf space of the involutive subbundle $T^f M := \ker(Tf) \subseteq TM$ and the morphism ϕ defines a flat $T^f M$ -connection [29] $\nabla: \Gamma(T^f M) \times \Gamma(E) \rightarrow \Gamma(E)$ by

$$\nabla_X e = 0 \text{ for all } X \in \Gamma(T^f M) \quad :\Leftrightarrow \quad \exists e' \in \Gamma(E') : \phi \circ e = e' \circ f.$$

⁵The dot \cdot denotes the product of polynomial functions.

That is, the ∇ -flat sections of E are the sections of E that are ϕ -projectable to sections of E' .

Conversely, consider a vector bundle $E \rightarrow M$, an involutive subbundle $F \subseteq TM$ and a flat connection $\nabla: \Gamma(F) \times \Gamma(E) \rightarrow \Gamma(E)$. If F is simple, i.e. if the space M/F of leaves of F is smooth, and if ∇ has no holonomy, then they induce a fibration of vector bundles [29]

$$\begin{array}{ccc} E & \xrightarrow{\pi} & E/\nabla \\ q_E \downarrow & & \downarrow [q_E] \\ M & \xrightarrow{\pi_M} & M/F \end{array}$$

where E/∇ is the quotient of E by parallel transport.

We say that E is n -reducible if there is a fibration of vector bundles

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ q_E \downarrow & & \downarrow q_{E'} \\ M & \xrightarrow{f} & M' \end{array}$$

such that $\dim M' = n$ and $\text{rank } E = \text{rank } E'$. Then the Pontryagin classes of E of degree greater than n must necessarily vanish. This is because the Gauss map g_E of E then factors as $g_E = g_{E'} \circ f$, and so $\text{Pont}^\bullet(E) = f^* \text{Pont}^\bullet(E')$. Therefore, in that case, Bott’s vanishing theorem (Theorem E) is satisfied even with n as lower bound.

Pontryagin classes are invariants of a vector bundle that vanish if it is trivializable. In particular, the Pontryagin classes of E of rank k all vanish if there is a smooth morphism of vector bundles

$$\begin{array}{ccc} E & \xrightarrow{\pi} & \mathbb{R}^k \\ q_E \downarrow & & \downarrow \\ M & \longrightarrow & \{\text{pt}\} \end{array}$$

that restricts to an isomorphism on each fibre. The consideration above shows that much finer geometrical information can be extracted from Pontryagin classes, and that they could be seen as obstructions to (constant rank) fibrations to low dimensional manifolds. For instance, if $\text{Pont}^l(E) \neq \{0\}$ for some $l \geq 4$, then the vector bundle E is not 1-reducible, and if $\text{Pont}^l(E) \neq \{0\}$ for some $l > 4$, then the vector bundle E is not 2-reducible, etc.

3.3. Bott’s vanishing theorem and the Atiyah class. If $E \rightarrow M$ has a flat F -connection, but F is not simple (i.e. M/F is not a smooth manifold) or the holonomy of ∇ is not trivial, then the vector bundle E still is “infinitesimally symmetric along F ”, but only prove Bott’s vanishing theorem with lower bound $2q$ can be proved. However, following ideas by Molino [36] (see also [30]), Theorem E holds with the lower bound q instead of $2q$ if the *Atiyah class* of the connection vanishes. On the other hand, the new, more general version of Bott’s vanishing theorem in Theorem E, might be useful in the search for examples where E has a flat F -connection, with F of codimension q , but its k -th Pontryagin class does not vanish for some $k > \frac{q}{2}$.

Let A be a Lie algebroid over a manifold M , and let B be a subalgebroid of A over M , of codimension q . Let E be a vector bundle over M , as before with a flat B -connection ∇ . Take again an extension $\tilde{\nabla}: \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$ of ∇ as in (9). Then define the form $\omega_{\tilde{\nabla}} \in \Omega^1(B, \text{Hom}(A/B, \text{End}(E)))$ by

$$\omega_{\tilde{\nabla}}(b, \bar{a})(e) = R_{\tilde{\nabla}}(b, a)e.$$

The flat B -connection ∇ on E and the flat Bott-connection $\nabla^B: \Gamma(B) \times \Gamma(A/B) \rightarrow \Gamma(A/B)$ combine to a flat B -connection ∇^{Hom} on $\text{Hom}(A/B, \text{End}(E))$, and then $\mathbf{d}_{\nabla^{\text{Hom}}}\omega_{\tilde{\nabla}} = 0$ [36, 10, 25].

The class $\alpha_{\nabla} = [\omega_{\tilde{\nabla}}] \in H^1(B, \text{Hom}(A/B, \text{End}(E)))$ is called the **Atiyah class of the representation of $B \subseteq A$ on E** . It does not depend on the choice of the extension $\tilde{\nabla}$ of ∇ , and it is zero if and only if there is an extension $\tilde{\nabla}$ such that $R_{\tilde{\nabla}}(b, a) = 0$ for all $b \in \Gamma(B)$ and all $a \in \Gamma(A)$ [36, 10, 25]. That is, $\alpha_{\nabla} = 0$ if and only if there is an extension $\tilde{\nabla}$ such that $R_{\tilde{\nabla}} \in \Gamma(\wedge^2 B^\circ \otimes \text{End}(E))$.

For all $l \geq 0$ the form $\widehat{\text{tr}}(R_{\tilde{\nabla}}^l)$ is a section of $\wedge^{2l} B^\circ$ and so $\widehat{\text{tr}}(R_{\tilde{\nabla}}^l) = 0$ for $2l > q$. This proves the following theorem.

THEOREM 3.3. *Let E be a vector bundle over a manifold M and let A be a Lie algebroid over M . If there exists a Lie subalgebroid B of A of codimension q with a linear representation $\nabla: \Gamma(B) \times \Gamma(E) \rightarrow \Gamma(E)$ with vanishing Atiyah class $\alpha_{\nabla} \in H^1(B, \text{Hom}(A/B, \text{End}(E)))$, then the Pontryagin spaces*

$$\text{Pont}_A^l(E) \subseteq H^l(A)$$

are all trivial for $l > q$.

If $A = TM$, $B =: F$, and ∇ is defined as in the previous section by a fibration to a vector bundle over M/F , then the Atiyah class α_{∇} vanishes (see [25]). With Section 3.2, this yields the following corollary.

COROLLARY 3.4. *Let E be a vector bundle over a manifold M . If there exists an involutive subbundle F of TM of codimension q with a flat connection $\nabla: \Gamma(F) \times \Gamma(E) \rightarrow \Gamma(E)$ such that*

$$\begin{array}{ccc} E & \xrightarrow{\pi} & E/\nabla \\ q_E \downarrow & & \downarrow [q_E] \\ M & \xrightarrow{\pi_M} & M/F \end{array}$$

is a smooth fibration of vector bundles, then the Atiyah class $\alpha_{\nabla} \in H^1(F, \text{Hom}(TM/F, \text{End}(E)))$ vanishes and the Pontryagin spaces $\text{Pont}^l(E) \subseteq H^l(M)$ are all trivial for $l > q$.

4. Pontryagin algebras of graded vector bundles. This section studies connections up to homotopy on graded vector bundles, and explains how Pontryagin or characteristic algebras are defined by those objects, in the same manner as the classical Pontryagin algebras of a vector bundle are defined by linear connections on it [37, 33].

4.1. The graded trace operator. In the following, consider a Lie algebroid $(A \rightarrow M, \rho, [\cdot, \cdot])$, and a graded vector bundle $\underline{E} = \bigoplus_{z \in \mathbb{Z}} E_z[z]$ over the same manifold M , with grading concentrated in finitely many degrees (i.e. all but finitely many of the vector bundles E_z , $z \in \mathbb{Z}$ are trivial). Equivalently, \underline{E} has finite (total) rank.

The **graded trace operator** $\text{str}: \Gamma(\underline{\text{End}}(\underline{E})) \rightarrow C^\infty(M)$, i.e.

$$\text{str} \in \Omega^0(A, \text{Hom}(\underline{\text{End}}(\underline{E}), \mathbb{R}))$$

is defined by

$$\text{str}(\phi) = (-1)^i \text{tr}(\phi)$$

for $\phi \in \Gamma(\text{End}(E_i))$. It yields a (graded) $\Omega^\bullet(A)$ -linear map

$$\widehat{\text{str}}: \Omega(A, \underline{\text{End}}(\underline{E}))_\bullet \rightarrow \Omega^\bullet(A).$$

The operator $\widehat{\text{str}}$ vanishes by definition on $\Omega^\bullet(A, \underline{\text{End}}(\underline{E})_i)$ for all $i \neq 0$, and so only ‘sees’ the part $\Omega^\bullet(A, \underline{\text{End}}(\underline{E})_0)$ of $\Omega(A, \underline{\text{End}}(\underline{E}))_\bullet$.

The signs are chosen such that for $K_1 \in \Omega^0(A, \text{Hom}(E_i, E_j)) = \Gamma(\text{Hom}(E_i, E_j))$ and $K_2 \in \Omega^0(A, \text{Hom}(E_j, E_i)) = \Gamma(\text{Hom}(E_j, E_i))$, i.e. with compositions $K_1 \circ K_2 \in \Gamma(\text{End}(E_j))$ and $K_2 \circ K_1 \in \Gamma(\text{End}(E_i))$:

$$\begin{aligned} \text{str}(K_1 \circ K_2) &= (-1)^j \text{tr}(K_1 \circ K_2) = (-1)^j \text{tr}(K_2 \circ K_1) \\ &= (-1)^{i+j} \text{str}(K_2 \circ K_1) = \text{str}((-1)^{(j-i)(i-j)} K_2 \circ K_1) \\ &= \text{str}((-1)^{|K_1| \cdot |K_2|} K_2 \circ K_1) \end{aligned}$$

since $i+j$ and $(j-i)(i-j) = 2ij - j^2 - i^2$ have the same parity. That is,

$$\text{str}[K_1, K_2] = 0 \tag{10}$$

for $K_1 \in \Gamma(\text{Hom}(E_i, E_j))$ and $K_2 \in \Gamma(\text{Hom}(E_j, E_i))$. More generally, this yields

$$\widehat{\text{str}}([K_1, K_2]) = 0 \tag{11}$$

for $K_1, K_2 \in \Omega(A, \underline{\text{End}}(\underline{E}))_\bullet$, see also [37].

4.2. Connections up to homotopy. The notion of superconnection dates back to Quillen [37]. Connections up to homotopy appeared in [18] in the more recent literature. The notion of connection up to homotopy defined by Crainic in [11, 12] is a different⁶ one.

Let $A \rightarrow M$ be a Lie algebroid and let $\underline{E} \rightarrow M$ be a graded vector bundle of finite rank, i.e. the grading is concentrated in finitely many degrees. Then a **connection up to homotopy** of A on \underline{E} is an operator

$$\mathcal{D}: \Omega(A, \underline{E})_\bullet \rightarrow \Omega(A, \underline{E})_{\bullet+1}$$

⁶There, a connection up to homotopy on a 2-term complex (E, ∂) of vector bundles

$$E^0 \begin{array}{c} \xrightarrow{\partial} \\ \xleftarrow{\partial} \end{array} E^1$$

is an \mathbb{R} -bilinear map $\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ such that $\partial \circ \nabla = \nabla \circ \partial$, that satisfies as usual the Leibniz condition in the second argument, but which is not $C^\infty(M)$ -linear in the $\mathfrak{X}(M)$ -entry. Instead, the failure of the $C^\infty(M)$ -linearity is measured by the commutator of ∂ with a map $H_\nabla: C^\infty(M) \times \Gamma(E) \rightarrow \Gamma(\text{End}(E))$, which is \mathbb{R} -linear and local in its entries.

that increases the total degree by 1 and satisfies

$$\mathcal{D}(\omega \wedge \eta) = \mathbf{d}_A \omega \wedge \eta + (-1)^{|\omega|} \omega \wedge \mathcal{D}\eta \quad (12)$$

for $\omega \in \Omega^\bullet(A)$ and $\eta \in \Omega(A, \underline{E})_\bullet$.

If $\underline{E} = E_0[0] \oplus \dots \oplus E_{n-1}[n-1]$, then a connection up to homotopy of A on \underline{E} is called for simplicity **an n -connection**. Of course, an n -connection \mathcal{D} is an n -representation, i.e. an n -term representation up to homotopy in the sense of [2] if in addition $\mathcal{D}^2 = 0$.

EXAMPLE 4.1 (Degree-preserving connections are connections up to homotopy). Let $\underline{E} \rightarrow M$ be a graded vector bundle of finite rank. Choose for all $z \in \mathbb{Z}$ a linear A -connection $\nabla^z: \Gamma(A) \times \Gamma(E_z) \rightarrow \Gamma(E_z)$. Then the connections define together a connection up to homotopy

$$\mathcal{D}: \Omega(A, \underline{E})_\bullet \rightarrow \Omega(A, \underline{E})_{\bullet+1}$$

by $\mathcal{D}(\omega) = \mathbf{d}_{\nabla^z} \omega \in \Omega^{\bullet+1}(A, E_z)$ for $\omega \in \Omega^\bullet(A, E_z)$.

EXAMPLE 4.2 (2-connections in more detail). Take $\underline{E} = E_0[0] \oplus E_1[1]$ over M and $A \rightarrow M$ a Lie algebroid. Then a 2-connection

$$\mathcal{D}: \Omega(A, \underline{E})_\bullet \rightarrow \Omega(A, \underline{E})_{\bullet+1}$$

is completely defined by its values

$$\mathcal{D}(e_0) \in \Omega^1(A, E_0) \oplus \Omega^0(A, E_1) \quad \text{and} \quad \mathcal{D}(e_1) \in \Omega^1(A, E_1) \oplus \Omega^2(A, E_0)$$

for arbitrary $e_0 \in \Gamma(E_0)$ and $e_1 \in \Gamma(E_1)$. It is easy to check that

$$\mathcal{D}(e_0) = \mathbf{d}_{\nabla^0} e_0 + \partial(e_0) \quad \text{and} \quad \mathcal{D}(e_1) = \mathbf{d}_{\nabla^1} e_1 + \widehat{K}(e_1)$$

for $\nabla^i: \Gamma(A) \times \Gamma(E_i) \rightarrow \Gamma(E_i)$ linear connections, $i = 0, 1$, a vector bundle morphism $\partial: E_0 \rightarrow E_1$ over the identity, i.e. $\partial \in \Omega^0(A, \text{Hom}(E_0, E_1))$, and $K \in \Omega^2(A, \text{Hom}(E_1, E_0))$.

In general, connections up to homotopy can be described as follows.

PROPOSITION 4.3. *Let $A \rightarrow M$ be a Lie algebroid and let $\underline{E} \rightarrow M$ be a graded vector bundle of finite rank. Then a connection up to homotopy*

$$\mathcal{D}: \Omega(A, \underline{E})_\bullet \rightarrow \Omega(A, \underline{E})_{\bullet+1}$$

can always be written

$$\mathcal{D} = \mathbf{d}_\nabla + \widehat{D}$$

with a linear connection $\nabla: \Gamma(A) \times \Gamma(\underline{E}) \rightarrow \Gamma(\underline{E})$ that preserves the grading as in Example 4.1, and $D \in \Omega(A, \underline{\text{End}}(\underline{E}))_1$. The connection ∇ and the form D can even be chosen such that

$$D \in \bigoplus_{s \neq 1} \Omega^s(A, \underline{\text{End}}(\underline{E})_{1-s}).$$

Proof. Take any degree-preserving connection $\nabla: \Gamma(A) \times \Gamma(\underline{E}) \rightarrow \Gamma(\underline{E})$ as in Example 4.1. Then $\mathcal{D} - \mathbf{d}_\nabla: \Omega(A, \underline{E})_\bullet \rightarrow \Omega(A, \underline{E})_{\bullet+1}$ is easily seen to be graded $\Omega^\bullet(A)$ -linear. Hence $\mathcal{D} - \mathbf{d}_\nabla = \widehat{D}$ for a $D \in \Omega(A, \underline{\text{End}}(\underline{E}))_1$.

Now write $D = \sum_{s \in \mathbb{Z}} D_s \in \bigoplus_{s \in \mathbb{Z}} \Omega^s(A, \underline{\text{End}}(\underline{E})_{1-s})$. Then $D_1 \in \Omega^1(A, \underline{\text{End}}(\underline{E})_0)$ and so $\nabla' := \nabla + D_1$ is a new connection on \underline{E} that preserves the grading, such that $\mathcal{D} = \mathbf{d}_{\nabla'} + \widehat{D} = \mathbf{d}_{\nabla'} + \widehat{D - D_1}$. \square

Finally, a connection up to homotopy of A on \underline{E} defines an induced connection up to homotopy

$$\mathcal{D}_{\text{End}}: \Omega(A, \underline{\text{End}}(\underline{E}))_\bullet \rightarrow \Omega(A, \underline{\text{End}}(\underline{E}))_{\bullet+1}$$

of A on $\underline{\text{End}}(\underline{E})$ by

$$\widehat{\mathcal{D}_{\text{End}}(K)} = \mathcal{D} \circ \widehat{K} - (-1)^{|K|} \widehat{K} \circ \mathcal{D}$$

for all $K \in \Omega(A, \underline{\text{End}}(\underline{E}))_\bullet$ and $e \in \Gamma(\underline{E})$. That is, as before,

$$[\mathcal{D}, \widehat{K}] := \mathcal{D} \circ \widehat{K} - (-1)^{|K|} \widehat{K} \circ \mathcal{D} = \widehat{\mathcal{D}_{\text{End}}(K)} \quad (13)$$

for all $K \in \Omega(A, \underline{\text{End}}(\underline{E}))_\bullet$. More generally, if \mathcal{D} is a connection up to homotopy of A on \underline{E} and \mathcal{D}' is a connection up to homotopy of A on \underline{E}' , then define the induced connection up to homotopy

$$\mathcal{D}_{\text{Hom}}: \Omega(A, \underline{\text{Hom}}(\underline{E}, \underline{E}'))_\bullet \rightarrow \Omega(A, \underline{\text{Hom}}(\underline{E}, \underline{E}'))_{\bullet+1}$$

of A on $\underline{\text{Hom}}(\underline{E}, \underline{E}')$ by

$$\widehat{\mathcal{D}_{\text{Hom}}(K)} = \mathcal{D}' \circ \widehat{K} - (-1)^{|K|} \widehat{K} \circ \mathcal{D}'$$

for all $K \in \Omega(A, \underline{\text{Hom}}(\underline{E}, \underline{E}'))_\bullet$.

As in the case of superconnections, this yields the following lemma [37], see also [33].

LEMMA 4.4. *In the situation above,*

$$\widehat{\text{str}} \circ \mathcal{D}_{\text{End}} = \mathbf{d}_A \circ \widehat{\text{str}}. \quad (14)$$

Proof. Write the connection up to homotopy \mathcal{D} as in Proposition 4.3 as

$$\mathcal{D} = \mathbf{d}_\nabla + \widehat{D}$$

with $\nabla: \Gamma(A) \times \Gamma(\underline{E}) \rightarrow \Gamma(\underline{E})$ a linear connection that preserves the grading, and $D \in \Omega(A, \underline{\text{End}}(\underline{E}))_1$. Then for $K \in \Omega(A, \underline{\text{End}}(\underline{E}))_\bullet$:

$$\widehat{\mathcal{D}_{\text{End}}(K)} = \mathbf{d}_\nabla \circ \widehat{K} + \widehat{D} \circ \widehat{K} - (-1)^{|K|} \widehat{K} \circ \mathbf{d}_\nabla - (-1)^{|K|} \widehat{K} \circ \widehat{D} = \widehat{\mathbf{d}_{\nabla \text{End}}(K)} + \widehat{[D, K]}.$$

This yields $\mathcal{D}_{\text{End}}(K) = \mathbf{d}_{\nabla \text{End}}(K) + [D, K]$ and so by (11)

$$\widehat{\text{str}}(\mathcal{D}_{\text{End}}(K)) = \widehat{\text{str}}(\mathbf{d}_{\nabla \text{End}}(K)). \quad (15)$$

The connection $\mathbf{d}_{\nabla \text{End}}$ and the flat connection $\mathbf{d}_A: \Omega^\bullet(A) \rightarrow \Omega^{\bullet+1}(A)$ yield as before the connection

$$\mathbf{d}_{\nabla \text{Hom}}: \Omega(A, \underline{\text{Hom}}(\underline{\text{End}}(\underline{E}), \mathbb{R}))_\bullet \rightarrow \Omega(A, \underline{\text{Hom}}(\underline{\text{End}}(\underline{E}), \mathbb{R}))_{\bullet+1}.$$

Equation (15) and the proof of Lemma 2.1 now give

$$\mathbf{d}_A \circ \widehat{\text{str}} - \widehat{\text{str}} \circ \mathcal{D}_{\text{End}} = \mathbf{d}_A \circ \widehat{\text{str}} - \widehat{\text{str}} \circ \mathbf{d}_{\nabla \text{End}} = \widehat{\mathbf{d}_{\nabla \text{Hom}} \text{str}} = 0.$$

\square

4.3. Curvature of a connection up to homotopy, and Pontryagin characters. Now if \mathcal{D} is a connection up to homotopy of A on \underline{E} , then (12) implies immediately

$$\mathcal{D}^2(\omega \wedge \eta) = \omega \wedge \mathcal{D}^2\eta$$

for $\omega \in \Omega^\bullet(A)$ and $\eta \in \Omega(A, \underline{E})_\bullet$. That is, \mathcal{D}^2 is (graded) $\Omega^\bullet(A)$ -linear and there is a unique $R_{\mathcal{D}} \in \Omega(A, \underline{\text{End}}(\underline{E}))_2$ with $\mathcal{D}^2 = \widehat{R}_{\mathcal{D}}$. The form $R_{\mathcal{D}} \in \Omega(A, \underline{\text{End}}(\underline{E}))_2$ is the **curvature form of \mathcal{D}** .

Of course, an n -connection is an n -representation if and only if its curvature form vanishes. As before, define $R_{\mathcal{D}}^i \in \Omega(A, \underline{\text{End}}(\underline{E}))_{2i}$ by

$$\widehat{R}_{\mathcal{D}}^i = \widehat{R}_{\mathcal{D}}^i = \mathcal{D}^{2i}$$

for $i \geq 1$. The Bianchi identity

$$\mathcal{D}_{\text{End}} R_{\mathcal{D}}^i = 0 \tag{16}$$

then holds for all $i \geq 1$ since

$$\widehat{\mathcal{D}_{\text{End}} R_{\mathcal{D}}^i} = [\mathcal{D}, \widehat{R_{\mathcal{D}}^i}] = [\mathcal{D}, \mathcal{D}^{2i}] = \mathcal{D}^{2i+1} - (-1)^{2i} \mathcal{D}^{2i+1} = 0.$$

As a consequence, the curvature form $R_{\mathcal{D}}$ satisfies

$$\mathbf{d}_A(\widehat{\text{str}}(R_{\mathcal{D}}^i)) \stackrel{(14)}{=} \widehat{\text{str}}(\mathcal{D}_{\text{End}}(R_{\mathcal{D}}^i)) \stackrel{(16)}{=} 0$$

for all $i \geq 1$.

EXAMPLE 4.5. In the situation of Example 4.1, it is easy to see that for each $i \geq 1$

$$R_{\mathcal{D}}^i = \sum_{z \in \mathbb{Z}} R_{\nabla^z}^i \in \bigoplus_{z \in \mathbb{Z}} \Omega^{2i}(A, \text{End}(E_z)) \subseteq \Omega(A, \underline{\text{End}}(\underline{E}))_{2i}.$$

In this case, all the results follow easily from the considerations in §2.3, and

$$\widehat{\text{str}}(R_{\mathcal{D}}^i) = \sum_{z \in \mathbb{Z}} (-1)^z \widehat{\text{tr}}(R_{\nabla^z}^i)$$

which is obviously a \mathbf{d}_A -closed element of $\Omega^{2i}(A)$ by (7). This is already observed in [37] in the context of superconnections.

Now one can construct as before the Pontryagin algebras defined by the powers of the curvature form.

PROPOSITION 4.6. *Choose a graded vector bundle \underline{E} of finite rank over a manifold M , and a Lie algebroid A over M . Then the cohomology classes*

$$\left[\widehat{\text{str}}(R_{\mathcal{D}}^i) \right] \in H^{2i}(A)$$

do not depend on the choice of the connection up to homotopy \mathcal{D} on \underline{E} .

As observed in [33], the proof of Proposition 4.6 follows the standard techniques, exactly as done in [37] in the situation of superconnections. For the convenience of the reader, it is carried out in detail in Appendix A.

DEFINITION 4.7. Choose a graded vector bundle \underline{E} of finite rank over a manifold M , and a Lie algebroid A over M . Then the **A -Pontryagin algebra of the graded vector bundle \underline{E}**

$$\text{Pont}_A^\bullet(\underline{E}) \subseteq H^\bullet(A)$$

is the subalgebra generated by the **A -Pontryagin characters of \underline{E}**

$$\sigma_A^i(\underline{E}) := \left[\widehat{\text{str}}(R_{\mathcal{D}}^i) \right] \in H^{2i}(A), \quad i \geq 1,$$

defined by any choice of connection up to homotopy \mathcal{D} of A on \underline{E} .

Here also, it is easy to show using Example 4.1 and Proposition 4.6 that

$$\text{Pont}_A^\bullet(\underline{E}) = \rho^* \text{Pont}^\bullet(\underline{E}).$$

As usual, $\text{Pont}^\bullet(\underline{E})$ denotes the TM -Pontryagin algebra of \underline{E} .

REMARK 4.8. This paper does not define *Pontryagin classes* of a graded vector bundle as images of special invariant polynomials under a suitable Chern-Weil homomorphism – this is not needed for the obstruction theorems below. However, consider a graded vector bundle $\underline{E} = \bigoplus_{z \in \mathbb{Z}} E_z[z]$ and set $\underline{V} := \bigoplus_{z \in \mathbb{Z}} \mathbb{R}^{\text{rank } E_z}[z]$, a (finite dimensional) graded \mathbb{R} -vector space. Set $\mathcal{A}(\underline{V}) \subseteq \mathcal{P}(\underline{\mathfrak{gl}}(\underline{V}))$ to be the subalgebra of polynomials that is generated by the polynomials

$$\phi \mapsto \text{str}(\phi^l),$$

for $l \geq 1$. Then there is an obvious Chern-Weil homomorphism $\mathcal{A}(\underline{V}) \rightarrow \text{Pont}_A^\bullet(\underline{E})$ of \mathbb{R} -algebras, but $\mathcal{A}(\underline{V})$ cannot be understood as a subalgebra of the $\underline{\text{Gl}}(\underline{V})$ -invariant polynomials on $\underline{\mathfrak{gl}}(\underline{V})$ since for $\phi \in \underline{\mathfrak{gl}}(\underline{V})$, $A \in \underline{\text{Gl}}(\underline{V})$ and $l \geq 1$:

$$\text{str}((A\phi A^{-1})^l) = (-1)^{|A|+l|\phi|\cdot|A|} \text{str}(\phi^l)$$

by (10).

EXAMPLE 4.9. In the situation of Example 4.2,

$$\mathcal{D}: \Omega(A, E_0[0] \oplus E_1[1])_\bullet \rightarrow \Omega(A, E_0[0] \oplus E_1[1])_{\bullet+1}$$

equals

$$\mathcal{D} = \mathbf{d}_\nabla + \widehat{\partial} + \widehat{K}$$

with $\nabla: \Gamma(A) \times \Gamma(\underline{E}) \rightarrow \Gamma(\underline{E})$ a linear connection that preserves the degree, $\partial \in \Gamma(\text{Hom}(E_0, E_1)) = \Omega^0(A, \text{End}(\underline{E})_1)$ and $K \in \Omega^2(A, \text{Hom}(E_1, E_0)) = \Omega^2(A, \text{End}(\underline{E})_{-1})$.

Then

$$\begin{aligned} \mathcal{D}^2 &= \mathbf{d}_\nabla^2 + \mathbf{d}_\nabla \circ \widehat{\partial} + \mathbf{d}_\nabla \circ \widehat{K} + \widehat{\partial} \circ \mathbf{d}_\nabla + \widehat{\partial} \circ \widehat{K} + \widehat{K} \circ \mathbf{d}_\nabla + \widehat{K} \circ \widehat{\partial} \\ &= \mathbf{d}_\nabla^2 + \left[\mathbf{d}_\nabla, \widehat{\partial} \right] + \left[\mathbf{d}_\nabla, \widehat{K} \right] + \left[\widehat{\partial}, \widehat{K} \right] = \widehat{R}_\nabla + \widehat{\mathbf{d}}_{\nabla \text{End}} \widehat{\partial} + \widehat{\mathbf{d}}_{\nabla \text{End}} \widehat{K} + \widehat{[\partial, K]}. \end{aligned} \tag{17}$$

In this equation, $R_\nabla + [\partial, K] \in \Omega^2(A, \text{End}(\underline{E})_0)$, $\mathbf{d}_{\nabla \text{End}} \partial \in \Omega^1(A, \text{End}(\underline{E})_1)$ and $\mathbf{d}_{\nabla \text{End}} K \in \Omega^3(A, \text{End}(\underline{E})_{-1})$. This shows that the 2-connection is a 2-representation if and only if [2, 18]

$$R_{\nabla^0} + K \circ \partial = 0, \quad R_{\nabla^1} + \partial \circ K = 0, \quad \nabla^1 \circ \partial = \partial \circ \nabla^0 \quad \text{and} \quad \mathbf{d}_{\nabla \text{End}} K = 0.$$

The form $\widehat{\text{str}}(\mathcal{D}^2)$ is

$$\widehat{\text{str}}(R_\nabla + [\partial, K]) = \widehat{\text{tr}}(R_{\nabla^0} + K \circ \partial) - \widehat{\text{tr}}(R_{\nabla^1} + \partial \circ K).$$

The form $\widehat{\text{str}}(\mathcal{D}^4)$ is the graded trace of

$$R_\nabla^2 + R_\nabla \wedge [\partial, K] + [\partial, K] \wedge R_\nabla + [\partial, K]^2 + (\mathbf{d}_{\nabla^{\text{End}}}\partial) \wedge (\mathbf{d}_{\nabla^{\text{End}}}K) + (\mathbf{d}_{\nabla^{\text{End}}}K) \wedge (\mathbf{d}_{\nabla^{\text{End}}}\partial),$$

etc.

4.4. Application: Obstructions to the existence of an n -representation.

Example 4.1 shows that a degree-preserving linear A -connection on \underline{E} is an example of an A -connection up to homotopy on \underline{E} . Consider a graded vector bundle \underline{E} of finite rank k over a manifold M , and a Lie algebroid A over M , and set $E := \bigoplus_{z \in \mathbb{Z}} E_z$. If \underline{E} is concentrated in even degrees, then by Proposition 4.6 and Example 4.5, the Pontryagin characters of Definition 2.3 satisfy

$$\sigma_A^i(\underline{E}) = \sigma_A^i(E)$$

for all $i \geq 1$. If \underline{E} has grading in odd degrees only,

$$\sigma_A^i(\underline{E}) = -\sigma_A^i(E) \in H^\bullet(A),$$

for all $i \geq 1$. That is, the Pontryagin algebra of the graded vector bundle \underline{E} is then just the Pontryagin algebra of the vector bundle E obtained by forgetting the grading on \underline{E} .

This shows that Pontryagin algebras of graded vector bundles only lead to new information if the grading is on mixed odd and even degrees. In general, Proposition 4.6, Example 4.1 and Example 4.5 lead to the following formula.

COROLLARY 4.10. *Let $\underline{E} = \bigoplus_{z \in \mathbb{Z}} E_z$ be a graded vector bundle of finite rank over a manifold M , and let $A \rightarrow M$ be a Lie algebroid. Then for $l \geq 1$, the A -Pontryagin characters $\sigma_A^l(\underline{E})$ of \underline{E} equal*

$$\sigma_A^l(\underline{E}) = \sum_{z \in \mathbb{Z}} (-1)^z \sigma_A^l(E_z) \in H^{2l}(A). \tag{18}$$

Proof. Choose linear connections $\nabla: \Gamma(A) \times \Gamma(E_z) \rightarrow \Gamma(E_z)$ for each $z \in \mathbb{Z}$ and let \mathcal{D} be the induced connection up to homotopy of A on \underline{E} as in Example 4.1. Then by Proposition 4.6 and Example 4.5:

$$\sigma_A^l(\underline{E}) = \left[\widehat{\text{str}}(R_{\mathcal{D}}^l) \right] = \sum_{z \in \mathbb{Z}} (-1)^z \left[\widehat{\text{tr}}(R_{\nabla^z}^l) \right] = \sum_{z \in \mathbb{Z}} (-1)^z \sigma_A^l(E_z).$$

□

REMARK 4.11. Using the formula in the last corollary, it is again easy to show that $\text{Pont}_A^l(\underline{E}) \neq 0$ implies $l = 4z$ for some $z \in \mathbb{N}$.

Corollary 4.10 gives a necessary condition for the existence of an n -representation on a given graded vector bundle $\underline{E} = E_0[0] \oplus \dots \oplus E_{n-1}[n-1]$.

THEOREM A. *Let $\underline{E} = E_0[0] \oplus \dots \oplus E_{n-1}[n-1]$ be a graded vector bundle over a manifold M , and let $A \rightarrow M$ be a Lie algebroid. If there exists an n -representation*

\mathcal{D} of A on \underline{E} , then the A -Pontryagin characters $\sigma_A^l(E_i)$, $l > 1$, of the vector bundles E_i , $i = 0, \dots, n - 1$, satisfy the equations

$$\sum_{i=0}^{n-1} (-1)^i \sigma_A^l(E_i) = 0 \in H^{2l}(A) \tag{19}$$

for all $l > 1$.

Proof. Since there is an n -connection \mathcal{D} with $\mathcal{D}^2 = 0$, the left-hand side of (18) vanishes. \square

THEOREM B. *Let E_0 and E_1 be vector bundles over M , and let $A \rightarrow M$ be a Lie algebroid. If there is a 2-representation of A on $E_0[0] \oplus E_1[1]$, then*

$$\text{Pont}_A^\bullet(E_0) = \text{Pont}_A^\bullet(E_1) \subseteq H^\bullet(A).$$

More precisely, the A -Pontryagin classes of E_0 equals the A -Pontryagin classes of E_1 .

Proof. In this case, (19) yields immediately

$$\sigma_A^l(E_0) = \sigma_A^l(E_1)$$

for all $l \geq 0$. Therefore, since the generators of the Pontryagin algebras are equal, the Pontryagin algebras and the Pontryagin classes of E_0 and E_1 must be equal. \square

The reader acquainted with the equivalence of decomposed VB-algebroids with 2-representations [18], and of decomposed double Lie algebroids with matched pairs of 2-representations [17] might find interesting the two following corollaries of Theorem B.

COROLLARY 4.12. *Let B and C be vector bundles over M , and let $(A \rightarrow M, \rho, [\cdot, \cdot])$ be a Lie algebroid. If there is a VB-algebroid $(D \rightarrow B, A \rightarrow M)$ with core C , then the total Pontryagin classes coincide:*

$$p_A(B) = p_A(C) \in H^\bullet(A).$$

*That is, $\rho^*p^l(B) = \rho^*p^l(C)$ for all $l \geq 1$.*

COROLLARY 4.13. *Let C be a vector bundle over M , and let $A \rightarrow M$ and $B \rightarrow M$ be two Lie algebroids. If there is a double Lie algebroid (D, A, B, M) with core C , then*

$$p_A(C) = p_A(B) \in H^\bullet(A), \quad \text{and} \quad p_B(C) = p_B(A) \in H^\bullet(B).$$

4.4.1. Example: the double 2-representation defined by a connection.

Let $A \rightarrow M$ be a Lie algebroid and E a vector bundle over M . Then any linear A -connection $\nabla: \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$ defines as follows a representation up to homotopy of A on $E[0] \oplus E[1]$, see [2, 18]. The operator

$$\mathcal{D}: \Omega(A, E[0] \oplus E[1])_\bullet \rightarrow \Omega(A, E[0] \oplus E[1])_{\bullet+1}$$

is defined by⁷

$$\mathcal{D}(e_0) = \mathbf{d}_\nabla(e_0) + e_0 \in \Omega^1(A, E[0]) \oplus \Omega^0(A, E[1])$$

⁷Here, e_0 is considered first a section of $E[0]$, but its image under the degree 1 operator $\text{id}: E[0] \rightarrow E[1]$ is considered a section of $E[1]$.

for $e_0 \in \Omega^0(A, E[0]) = \Gamma(E[0])$, and

$$\mathcal{D}(e_1) = \mathbf{d}_\nabla(e_1) - \widehat{R_\nabla}(e_1) \in \Omega^1(A, E[1]) \oplus \Omega^2(A, E[0])$$

for $e_1 \in \Omega^0(A, E[1]) = \Gamma(E[1])$. Here, R_∇ is seen as an element of $\Omega^2(A, \text{Hom}(E[1], E[0])) \subseteq \Omega(A, \underline{\text{End}}(\underline{E}))_1$. It is easy to check that $\mathcal{D}^2 = 0$ – use (17) in Example 4.9 with $\omega = -R_\nabla \in \Omega^2(A, \text{Hom}(E[1], E[0]))$ and $\partial = \text{id}_E: E[0] \rightarrow E[1]$. This representation up to homotopy is called the *double representation up to homotopy* of A on E [2, 18].

Let $K \subseteq E$ be a vector subbundle. Take an A -connection ∇^K on K and an A -connection $\bar{\nabla}$ on E/K . Then $K \oplus E/K \simeq E$ and the sum $\nabla^K + \bar{\nabla}$ defines an A -connection on E . The A -Pontryagin characters of E , K and E/K satisfy

$$\sigma_A^i(E) = \sigma_A^i(K) + \sigma_A^i(E/K) \in H^{2i}(A)$$

for $i \geq 0$. This is usually formulated as $p_A(E) = p_A(K) \wedge p_A(E/K)$ (see e.g. [35, 42]). In other words the generators of $\text{Pont}_A^\bullet(E/K)$ are given by

$$\sigma_A^i(E/K) = \sigma_A^i(E) - \sigma_A^i(K) \in H^{2i}(A) \tag{20}$$

for $i \geq 0$. Likewise, the linear A -connections on K and on E define together a 2-connection \mathcal{D} of A on $K[0] \oplus E[1]$. Hence, the A -Pontryagin characters of $K[0] \oplus E[1]$ are

$$\widehat{\text{str}}(R_{\mathcal{D}}^i) = \sigma_A^i(K) - \sigma_A^i(E) \in H^{2i}(A) \tag{21}$$

for $i \geq 0$. Up to a sign, they equal the generators of $\text{Pont}_A^\bullet(E/K)$. This yields the following proposition.

PROPOSITION 4.14. *Let $E \rightarrow M$ be a vector bundle, and let A be a Lie algebroid over M . Let $K \subseteq E$ be a vector subbundle of E . Then*

$$\text{Pont}_A^\bullet(K[0] \oplus E[1]) = \text{Pont}_A^\bullet(E/K). \tag{22}$$

4.4.2. Example: the adjoint 2-representation of a Lie algebroid. Let $A \rightarrow M$ be a Lie algebroid with anchor ρ and Lie bracket $[\cdot, \cdot]$. Then any choice of linear connection $\nabla: \mathfrak{X}(M) \times \Gamma(A) \rightarrow \Gamma(A)$ defines as follows a representation up to homotopy of A on $A[0] \oplus TM[1]$, see [2, 18]. The operator

$$\mathcal{D}_{\text{ad}}: \Omega(A, A[0] \oplus TM[1])_\bullet \rightarrow \Omega(A, A[0] \oplus TM[1])_{\bullet+1}$$

is defined by

$$\mathcal{D}_{\text{ad}}(a) = \mathbf{d}_{\nabla^{\text{bas}}}(a) + \rho(a) \in \Omega^1(A, A[0]) \oplus \Omega^0(A, TM[1])$$

for $a \in \Omega^0(A, A[0]) = \Gamma(A)$, and

$$\mathcal{D}_{\text{ad}}(X) = \mathbf{d}_{\nabla^{\text{bas}}}(X) - \widehat{R_\nabla^{\text{bas}}}(X) \in \Omega^1(A, TM[1]) \oplus \Omega^2(A, A[0])$$

for $X \in \Omega^0(A, TM[1]) = \mathfrak{X}(M)$. Here, $R_\nabla^{\text{bas}} \in \Omega^2(A, \text{Hom}(TM, A))$ is defined by

$$R_\nabla^{\text{bas}}(a, b)X = -\nabla_X[a, b] + [\nabla_X a, b] + [a, \nabla_X b] + \nabla_{\nabla_b^{\text{bas}} X} a - \nabla_{\nabla_a^{\text{bas}} X} b$$

for $a, b \in \Gamma(A)$ and $X \in \mathfrak{X}(M)$, and the two basic connections

$$\nabla^{\text{bas}}: \Gamma(A) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M) \quad \text{and} \quad \nabla^{\text{bas}}: \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$$

are defined by

$$\nabla_a^{\text{bas}} X = [\rho(a), X] + \rho(\nabla_X a), \quad \nabla_a^{\text{bas}} b = [a, b] + \nabla_{\rho(b)} a$$

for $a, b \in \Gamma(A)$ and $X \in \mathfrak{X}(M)$. A computation using (17) shows $R_{\mathcal{D}_{\text{ad}}} = \mathcal{D}_{\text{ad}}^2 = 0$.

This representation up to homotopy is called the *adjoint representation up to homotopy* of A [2, 18]. The following result follows from Theorem B

THEOREM C. *Let A be a vector bundle over a manifold M , and let $\rho: A \rightarrow TM$ be a vector bundle morphism over the identity. If $A \rightarrow M$ carries a Lie algebroid structure with anchor ρ , then*

$$\rho^*(p^l(A)) = \rho^*(p^l(TM)) \in H^{4l}(A)$$

for all $l \geq 1$.

4.4.3. Example: the 2-representations defined by a morphism of Lie algebroids. More generally, let $A \rightarrow M$ and $B \rightarrow M$ be two Lie algebroids, with a Lie algebroid morphism $\phi: B \rightarrow A$ over the identity on M . Then any choice of linear connection $\nabla: \Gamma(A) \times \Gamma(B) \rightarrow \Gamma(B)$ defines as follows a representation up to homotopy of B on $B[0] \oplus A[1]$ – see [28].

The operator

$$\mathcal{D}: \Omega(B, B[0] \oplus A[1])_{\bullet} \rightarrow \Omega(B, B[0] \oplus A[1])_{\bullet+1}$$

is defined by

$$\mathcal{D}(b) = \mathbf{d}_{\nabla\phi}(b) + \phi(b) \in \Omega^1(B, B[0]) \oplus \Omega^0(B, A[1])$$

for $b \in \Omega^0(B, B[0]) = \Gamma(B)$, and

$$\mathcal{D}(a) = \mathbf{d}_{\nabla\phi}(a) - \widehat{R_{\nabla}^{\phi}}(a) \in \Omega^1(B, A[1]) \oplus \Omega^2(B, B[0])$$

for $a \in \Omega^0(B, A[1]) = \Gamma(A)$. Here, $R_{\nabla}^{\phi} \in \Omega^2(A, \text{Hom}(X, A))$ is defined by

$$R_{\nabla}^{\phi}(b_1, b_2)a = -\nabla_a[b_1, b_2] + [\nabla_a b_1, b_2] + [b_1, \nabla_a b_2] + \nabla_{\nabla_{b_2} a} b_1 - \nabla_{\nabla_{b_1} a} b_2$$

for $b_1, b_2 \in \Gamma(B)$ and $a \in \Gamma(A)$, and the two connections

$$\nabla^{\phi}: \Gamma(B) \times \Gamma(B) \rightarrow \Gamma(B) \quad \text{and} \quad \nabla^{\phi}: \Gamma(B) \times \Gamma(A) \rightarrow \Gamma(A)$$

are defined by

$$\nabla_{b_1}^{\phi} b_2 = [b_1, b_2] + \nabla_{\phi(b_2)} b_1, \quad \nabla_b^{\phi} a = [\phi(b), a] + \phi(\nabla_a b)$$

for $b_1, b_2 \in \Gamma(B)$ and $a \in \Gamma(A)$. A computation shows $\mathcal{D}^2 = 0$ and so $R_{\mathcal{D}} = 0$. The following result follows then from Theorem B.

THEOREM D. *Let A and B be Lie algebroids over M . If there is a Lie algebroid morphism $\phi: B \rightarrow A$ over the identity on M , then*

$$p_B^l(A) = p_B^l(B)$$

for all $l \geq 1$.

Vaisman defines characteristic classes of morphisms of Lie algebroids in [43]; by considering the graphs of these morphisms. The result above does not consider these classes; but it would be interesting to compare the two approaches.

4.5. Bott’s vanishing theorem for graded vector bundles. This section proves a more general formulation of Bott’s vanishing theorem [6] and of Theorem E, on Lie subalgebroids with n -representations.

For $B \subseteq A$ a subalgebroid, the space $\Omega(B, \underline{E})_\bullet$ can be (non-canonically) embedded as follows as a $C^\infty(M)$ -submodule of $\Omega(A, \underline{E})_\bullet$. Fix a splitting $p: A \rightarrow B$ of the inclusion $\iota: B \hookrightarrow A$. Then the $C^\infty(M)$ -linear map $p^*: \Omega(B, \underline{E})_\bullet \rightarrow \Omega(A, \underline{E})_\bullet$ is defined by

$$p^*(\omega)(a_1, \dots, a_s) = \omega(p(a_1), \dots, p(a_s))$$

for $\omega \in \Omega^s(B, E_i)$ and $a_j \in A$ for $j = 1, \dots, s$. In the same manner,

$$p^*: \Omega(B, \underline{\text{End}}(\underline{E}))_\bullet \rightarrow \Omega(A, \underline{\text{End}}(\underline{E}))_\bullet$$

is defined.

Similarly, the inclusion $\iota: B \rightarrow A$ induces the $C^\infty(M)$ -linear map

$$\iota^*: \Omega(A, \underline{E})_\bullet \rightarrow \Omega(B, \underline{E})_\bullet,$$

defined by $(\iota^*\omega)(b_1, \dots, b_s) = \omega(b_1, \dots, b_s)$ for $\omega \in \Omega^s(A, E_l)$. By construction, $\iota^* \circ p^* = \text{Id}_{\Omega(B, \underline{\text{End}}(\underline{E}))_\bullet}$.

Let now $A \rightarrow M$ be a Lie algebroid and let $\underline{E} = E_0[0] \oplus \dots \oplus E_{n-1}[n-1]$ be a graded vector bundle over M . Let k be the (total) rank of \underline{E} . Assume that there is a Lie subalgebroid $B \subseteq A$ of codimension q , with an n -representation

$$\mathcal{D}: \Omega(B, \underline{E})_\bullet \rightarrow \Omega(B, \underline{E})_{\bullet+1}.$$

Then, as in Proposition 4.3, the n -representation \mathcal{D} equals $\mathcal{D} = \mathbf{d}_\nabla + \widehat{D}$, with a B -connection ∇ on \underline{E} preserving the grading, and a form $D \in \Omega(B, \underline{\text{End}}(\underline{E}))_1$. Using the second part of Proposition 4.3, assume without loss of generality that D has no component in $\Omega^1(B, \underline{\text{End}}(\underline{E})_0)$. Extend the B -connection ∇ on \underline{E} to an A -connection $\widetilde{\nabla}$ on \underline{E} that preserves the grading, and extend the form D to the form $p^*(D) \in \Omega(A, \underline{\text{End}}(\underline{E}))_1$.

Then

$$\widetilde{\mathcal{D}} = \mathbf{d}_{\widetilde{\nabla}} + \widehat{p^*(D)}: \Omega(A, \underline{E})_\bullet \rightarrow \Omega(A, \underline{E})_{\bullet+1}$$

is an n -connection of A on \underline{E} . Take $\omega \in \Omega^s(A, E_l)$. Then

$$\widetilde{\mathcal{D}}(\omega) = \sum_{i=0}^{\text{rank } A} (\widetilde{\mathcal{D}}\omega)_i \in \bigoplus_{i=0}^{\text{rank } A} \Omega^i(A, E_{s+l+1-i})$$

and easy computations yield the following identities:

- For $i = s + 1$:

$$\begin{aligned} (\widetilde{\mathcal{D}}\omega)_i(b_1, \dots, b_i) &= (\mathbf{d}_{\widetilde{\nabla}}\omega)(b_1, \dots, b_i) = \mathbf{d}_\nabla(\iota^*\omega)(b_1, \dots, b_i) \\ &= (\mathcal{D}(\iota^*\omega))_i(b_1, \dots, b_i) \end{aligned}$$

for $b_1, \dots, b_i \in \Gamma(B)$, and

- For $i \neq s + 1$:

$$\begin{aligned} (\widetilde{\mathcal{D}}\omega)_i(b_1, \dots, b_i) &= (\widehat{p^*(D)}\omega)_i(b_1, \dots, b_i) = (\widehat{D}(\iota^*\omega))_i(b_1, \dots, b_i) \\ &= (\mathcal{D}(\iota^*\omega))_i(b_1, \dots, b_i) \end{aligned}$$

for $b_1, \dots, b_i \in \Gamma(B)$.

This proves $\iota^* \circ \tilde{\mathcal{D}} = \mathcal{D} \circ \iota^*$ and as a consequence $\iota^* \circ \tilde{\mathcal{D}}^2 = \mathcal{D}^2 \circ \iota^*$. Therefore, the equality $\mathcal{D}^2 = 0$ yields $\iota^*(R_{\tilde{\mathcal{D}}}) = \iota^*(\tilde{\mathcal{D}}^2) = 0$. That is,

$$R_{\tilde{\mathcal{D}}} \in \left(I^\bullet(B) \otimes_{C^\infty(M)} \Gamma(\text{End}(\underline{E})) \right)_2 = \bigoplus_{j \geq 1} I^j(B) \otimes_{C^\infty(M)} \Gamma(\text{End}(\underline{E})_{2-j}),$$

and so

$$R_{\tilde{\mathcal{D}}}^l \in (I^\bullet(B))^l \otimes_{C^\infty(M)} \Gamma(\text{End}(\underline{E}))$$

for all $l \geq 1$. This yields $R_{\tilde{\mathcal{D}}}^l = 0$ for $l > q$, and, as in the classical case, the following theorem.

THEOREM 4.15. *Let $A \rightarrow M$ be a Lie algebroid and let $\underline{E} = E_0[0] \oplus \dots \oplus E_{n-1}[n-1]$ be a graded vector bundle over M . Assume that there is a Lie subalgebroid $B \subseteq A$ of codimension q , with an n -representation*

$$\mathcal{D}: \Omega(B, \underline{E})_\bullet \rightarrow \Omega(B, \underline{E})_{\bullet+1}.$$

Then the A -Pontryagin spaces of the graded vector bundle \underline{E}

$$\text{Pont}_A^l(\underline{E}) \subseteq H^l(A)$$

all vanish for $l > 2q$.

EXAMPLE 4.16. Let $E \rightarrow M$ be a vector bundle, and let A be a Lie algebroid over M . Let $K \subseteq E$ be a vector subbundle of E and let $B \subseteq A$ be a subalgebroid. Consider a linear A -connection ∇ on E , that preserves K . Define the linear B -connection $\bar{\nabla}: \Gamma(B) \times \Gamma(E/K) \rightarrow \Gamma(E/K)$ by $\bar{\nabla}_b \bar{e} = \overline{\nabla_b e}$ for all $b \in \Gamma(B)$ and $e \in \Gamma(E)$, where $\bar{e} \in \Gamma(E/K)$ is the class of the section e .

The connection $\bar{\nabla}$ is flat if and only if the 2-representation of A on $E[0] \oplus E[1]$ defined by ∇ as in §4.4.1 restricts to a 2-representation of B on $K[0] \oplus E[1]$; see [15]. Then, by Theorem 4.15,

$$\text{Pont}_A^l(K[0] \oplus E[1]) \subseteq H^l(A)$$

all vanish for $l > 2q$. By (22), this is a reformulation in the graded setting of Theorem E applied to $B \subseteq A$ and the flat connection $\bar{\nabla}$ on E/K .

5. Infinitesimal ideal systems and Pontryagin classes. The main motivation for the results above was the search for obstructions to the existence of infinitesimal ideal systems in a given Lie algebroid, in terms of the A and TM -Pontryagin classes of A and TM . This section first recalls some of the main examples of infinitesimal ideal systems. Then the first and second subsections present the obtained obstructions.

Recall that infinitesimal ideal systems are defined as in Definition 1.1. The three main classes of examples of infinitesimal ideal systems are the following.

EXAMPLE 5.1 (The usual notion of ideals in Lie algebroids). An ideal I in a Lie algebroid $A \rightarrow M$ is a subbundle over M such that $[a, i] \in \Gamma(I)$ for all $i \in \Gamma(I)$ and all $a \in \Gamma(A)$. The inclusion $I \subseteq \ker(\rho)$ follows immediately and shows that this definition of an ideal is very restrictive. These ideals, called here *naive ideals*,

correspond obviously to the ideal systems $(F = 0, J = I, \nabla = 0)$ in A . In particular, an ideal in a Lie algebra is an infinitesimal ideal system.

EXAMPLE 5.2 (The Bott connection). Consider an involutive subbundle $F \subseteq TM$ and the Bott connection

$$\nabla^F : \Gamma(F) \times \Gamma(TM/F) \rightarrow \Gamma(TM/F), \quad \nabla_X^F \bar{Y} = \overline{[X, Y]}$$

associated to it. Then it is straightforward to check that the triple (F, F, ∇^F) is an infinitesimal ideal system in the Lie algebroid TM .

EXAMPLE 5.3 (The ideal system associated to a fibration of Lie algebroids). Let

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A' \\ q_A \downarrow & & \downarrow q_{A'} \\ M & \xrightarrow{f} & M' \end{array}$$

be a fibration of Lie algebroids, i.e. the map f is a surjective submersion (with connected fibers) and $\varphi^! : A \rightarrow f^!A'$ is a surjective vector bundle morphism over the identity on A .

Then $J := \ker(\varphi) \subseteq A$ is a subalgebroid of A and $F = T^J M \subseteq TM$ is an involutive subbundle. The equality $Tf \circ \rho = \rho' \circ \varphi$ yields immediately $\rho(J) \subseteq F$.

Define a connection $\nabla^\varphi : \Gamma(F) \times \Gamma(A/J) \rightarrow \Gamma(A/J)$ by setting $\nabla_X^\varphi \bar{a} = 0$ for all sections $a \in \Gamma(A)$ that are φ -related to some section $a' \in \Gamma(A')$, i.e. such that $\varphi \circ a = a' \circ f$. Then the properties of the Lie algebroid morphism (φ, f) imply that (F, J, ∇^φ) is an infinitesimal ideal system in A .

Conversely, if the leaf space M/F of the foliation defined by F is smooth and ∇ has no holonomy, then a Lie algebroid A over M can be “quotiented out” by an infinitesimal ideal system (F, J, ∇) [29], just as a Lie algebra modulo an ideal gives a new Lie algebra. More precisely let (F, J, ∇) be an infinitesimal ideal system in a Lie algebroid A . Assume that $\bar{M} = M/F$ is a smooth manifold and that ∇ has trivial holonomy. Then the quotient defined by parallel transport along the leaves of F , $(A/J)/\nabla$, inherits a Lie algebroid structure over M/F such that the canonical projections $\pi : A \rightarrow (A/J)/\nabla$ and $\pi_M : M \rightarrow M/F$ define a fibration of Lie algebroids [29].

5.1. Pontryagin classes associated to an infinitesimal ideal system. First of all, since an infinitesimal ideal system consists among other ingredients of an involutive subbundle $F \subseteq TM$ and a flat F -connection on A/J , the following proposition is immediate.

PROPOSITION 5.4. *Let (F, J, ∇^i) be an infinitesimal ideal system in a Lie algebroid $A \rightarrow M$. Let q be the codimension of F in TM . Then the Pontryagin spaces $\text{Pont}^r(A/J)$ and $\text{Pont}^r(TM/F)$ are all trivial for $r > 2q$.*

Next, it is easy to see that J is a subalgebroid of A . The Bott connection associated to $J \subseteq A$ is the flat J -connection ∇^J on A defined by

$$\nabla^J : \Gamma(J) \times \Gamma(A/J) \rightarrow \Gamma(A/J), \quad \nabla_j^J \bar{a} = \overline{[j, a]} \tag{23}$$

for $j \in \Gamma(J)$ and $a \in \Gamma(A)$. In addition, there is a flat J -connection on TM/F , defined by

$$\nabla: \Gamma(J) \times \Gamma(TM/F) \rightarrow \Gamma(TM/F), \quad \nabla_j \overline{X} = \overline{[\rho(j), X]} = \nabla_{\rho(j)}^F \overline{X} \quad (24)$$

for $j \in \Gamma(J)$ and $X \in \mathfrak{X}(M)$. This, Proposition 5.4 and Remark 2.4 yield the following result.

PROPOSITION 5.5. *Let (F, J, ∇^i) be an infinitesimal ideal system in a Lie algebroid $A \rightarrow M$. Let s be the codimension of J in A and let q be the codimension of F in TM . Then the Pontryagin spaces $\text{Pont}_A^r(A/J)$ and $\text{Pont}_A^r(TM/F)$ are all trivial for $r > 2 \min\{s, q\}$.*

Of course, Propositions 5.4 and 5.5 can be refined using Theorem 3.3 and the Atiyah classes defined by extensions of the four flat connections (23), (24), the Bott connection ∇^F and the infinitesimal ideal system connection ∇^i .

5.2. Finer obstructions. The obstructions found above are too “rough” for being really meaningful – the proofs use very little of the structure of infinitesimal ideal systems. This section uses the Pontryagin algebras of graded vector bundles in order to find further (finer!) obstructions to the existence of infinitesimal ideal systems in a given Lie algebroid.

In order to do this, let us recall some results found in [15]. Let $A \rightarrow M$ be a Lie algebroid. Let $F \subseteq TM$ be an involutive subbundle and let $J \subseteq A$ be a smooth subbundle. Let $\nabla: \Gamma(F) \times \Gamma(A/J) \rightarrow \Gamma(A/J)$ be a flat connection, and let $\tilde{\nabla}: \mathfrak{X}(M) \times \Gamma(A) \rightarrow \Gamma(A)$ be an extension of ∇ . That is, $\tilde{\nabla}_X j \in \Gamma(J)$ for all $X \in \Gamma(F)$ and $j \in \Gamma(J)$ and the induced quotient connection equals ∇ . Recall from §4.4.3 that $\tilde{\nabla}$ defines the two basic connections

$$\tilde{\nabla}^{\text{bas}}: \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A), \quad \tilde{\nabla}^{\text{bas}}: \Gamma(A) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$$

and the basic curvature $R_{\tilde{\nabla}}^{\text{bas}} \in \Omega^2(A, \text{Hom}(TM, A))$ – that is, $\tilde{\nabla}$ defines the adjoint representation $\text{ad}_{\tilde{\nabla}}$ as in §4.4.3.

Then (F, J, ∇) is an infinitesimal ideal system in A if and only if [15]:

- (1) $\rho(J) \subseteq F$;
- (2) The basic connection $\tilde{\nabla}^{\text{bas}}: \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$ preserves J ;
- (3) The basic connection $\tilde{\nabla}^{\text{bas}}: \Gamma(A) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ preserves F ;
- (4) The basic curvature $R_{\tilde{\nabla}}^{\text{bas}} \in \Omega^2(A, \text{Hom}(TM, A))$ restricts to an element of $\Omega^2(A, \text{Hom}(F, J))$.

That is, (F, J, ∇) is an infinitesimal ideal system in A if and only if the adjoint 2-representation $\text{ad}_{\tilde{\nabla}}$ of A on $A[0] \oplus TM[1]$ defined by the anchor and the basic connections and curvature restricts to a 2-representation of A on $J[0] \oplus F[1]$. Theorem 4.12 yields immediately the following result.

THEOREM G. *Let $(A \rightarrow M, \rho, [\cdot, \cdot])$ be a Lie algebroid. Let $J \subseteq A$ and $F \subseteq TM$ be vector subbundles. If F is involutive and there is a flat F -connection on A/J such that (F, J, ∇) is an infinitesimal ideal system, then*

$$p_A^l(J) = p_A^l(F) \in H^{4l}(A)$$

for all $l \geq 1$.

EXAMPLE 5.6. Example 5.1 and the last proposition show that if $I \subseteq A$ is an ideal, then $\text{Pont}_A^\bullet(I) = \{0\}$. This is easy to see directly since A is represented on I by the Lie bracket.

In the situation of Example 5.2, the statement of the last proposition is trivial since $J = F$. However, Example 5.3 and the last proposition show that if $\varphi: A \rightarrow A'$ is a fibration of Lie algebroids over a smooth submersion $f: M \rightarrow M'$, then with $T^f M := \ker(Tf) \subseteq TM$:

$$p_A^l(T^f M) = p_A^l(\ker \varphi) \in H^{4l}(A)$$

for all $l \geq 1$.

REMARK 5.7. *Obstructions to the holomorphy of a complex Lie algebroid bracket on a holomorphic vector bundle.* Let $A \rightarrow M$ be a complex Lie algebroid; i.e. a complex vector bundle, equipped with a \mathbb{C} -Lie algebra bracket on $\Gamma(A)$, that is anchored by a \mathbb{C} -linear morphism $\rho: A \rightarrow T_{\mathbb{C}}M$, see [9].

The constructions in Sections 2.3 and 4.3 work the same for an A -connection up to homotopy on a complex graded vector bundle $\underline{E} \rightarrow M$ and yield the A -Chern characters of \underline{E} in $H^\bullet(A)$ —as the classical Chern characters of a complex vector bundle E are defined by \mathbb{C} -linear TM -connections on E .

Now let $A \rightarrow M$ be a Lie algebroid and assume that A and TM both have complex fibers. More precisely, assume the existence of vector bundle morphisms $j: A \rightarrow A$ and $J_M: TM \rightarrow TM$ over the identity on M , with $j^2 = -\text{id}_A$ and $J_M^2 = -\text{id}_{TM}$. In particular, M has an almost complex structure. Consider the complexification $A_{\mathbb{C}} \rightarrow M$ of the Lie algebroid A . That is, $A_{\mathbb{C}}$ is a complex Lie algebroid, with bracket the complexification of the bracket on sections of A : for $a_1, a_2 \in \Gamma(A)$ and $z_1, z_2 \in \mathbb{C}$,

$$[a_1 \otimes z_1, a_2 \otimes z_2]_{\mathbb{C}} = [a_1, a_2] \otimes z_1 z_2.$$

The anchor $\rho_{\mathbb{C}}: A_{\mathbb{C}} \rightarrow T_{\mathbb{C}}M$ is the complexification of the anchor ρ of A . Denote as usual the i and $-i$ eigenspaces of the complexification $J_M^{\mathbb{C}}: T_{\mathbb{C}}M \rightarrow T_{\mathbb{C}}M$ of J_M by $T^{1,0}M$ and $T^{0,1}M$, and the i and $-i$ eigenspaces of the complexification $j_{\mathbb{C}}: A_{\mathbb{C}} \rightarrow A_{\mathbb{C}}$ of j by $A^{1,0}$ and $A^{0,1}$.

Assume next that J_M is integrable and that the vector bundle $A \rightarrow M$ is a holomorphic vector bundle (with complex fibers defined by j), such that the Lie bracket sends holomorphic sections of A to a holomorphic section of A , and the anchor sends holomorphic sections of A to holomorphic vector fields on M . That is, assume that $A \rightarrow M$ is a *holomorphic Lie algebroid*. Then the Dolbeault operator $\bar{\partial}: \Gamma(T^{0,1}M) \times \Gamma(A^{1,0}) \rightarrow \Gamma(A^{1,0})$ defines a complex infinitesimal ideal system $(T^{0,1}M, A^{0,1}, \bar{\partial})$ in $A_{\mathbb{C}}$, see [20]. Here, $A^{1,0}$ is identified with $A_{\mathbb{C}}/A^{0,1}$.

As a consequence, the complex analogue of Theorem G yields the following theorem.

THEOREM 5.8. *Let $A \rightarrow M$ be a Lie algebroid and assume that A and TM both have complex fibers; $j: A \rightarrow A$ and $J_M: TM \rightarrow TM$ with $j^2 = -\text{id}_A$ and $J_M^2 = -\text{id}_{TM}$. Consider the complexification $A_{\mathbb{C}} \rightarrow M$ of the Lie algebroid A .*

If $A \rightarrow M$ is a holomorphic Lie algebroid, then all the $A_{\mathbb{C}}$ -Chern classes of $A^{0,1}$ and of $T^{0,1}M$ must coincide.

Appendix A. Proof of Proposition 4.6. Let \mathcal{D} and \mathcal{D}' be two connections up to homotopy of a Lie algebroid $A \rightarrow M$ on a graded vector bundle $\underline{E} = \bigoplus_{k \in \mathbb{Z}} E_z[z] \rightarrow M$ of finite rank. The difference $\mathcal{D}' - \mathcal{D}$ is graded- $\Omega^\bullet(A)$ -linear and there exists an element $D \in \Omega(A, \underline{\text{End}}(\underline{E}))_1$ such that $\mathcal{D}' - \mathcal{D} = \hat{D}$. For each $t \in [0, 1]$ set

$\mathcal{D}_t = \mathcal{D} + t\widehat{D}$. Then \mathcal{D}_t is a connection up to homotopy of A on \underline{E} for all $t \in [0, 1]$, with $\mathcal{D}_0 = \mathcal{D}$ and $\mathcal{D}_1 = \mathcal{D}'$. Its curvature at time t reads $\widehat{R}_{\mathcal{D}_t} = \mathcal{D}_t^2 = (\mathcal{D} + t\widehat{D})^2 = \mathcal{D}^2 + t[\mathcal{D}, \widehat{D}] + \frac{1}{2}t^2[\widehat{D}, \widehat{D}]$, which leads to

$$\frac{d}{dt}\widehat{R}_{\mathcal{D}_t} = [\mathcal{D}, \widehat{D}] + t[\widehat{D}, \widehat{D}] = [\mathcal{D}_t, \widehat{D}] \stackrel{(13)}{=} \widehat{\mathcal{D}_{t, \text{End}}D},$$

and so to $\frac{d}{dt}R_{\mathcal{D}_t} = \mathcal{D}_{t, \text{End}}D$. Next, this implies

$$\frac{d}{dt}R_{\mathcal{D}_t}^i = \sum_{s=1}^i R_{\mathcal{D}_t}^{(s-1)} \wedge \mathcal{D}_{t, \text{End}}D \wedge R_{\mathcal{D}_t}^{i-s}$$

and so

$$\begin{aligned} \frac{d}{dt}\widehat{\text{str}}(R_{\mathcal{D}_t}^i) &= i \cdot \widehat{\text{str}}(R_{\mathcal{D}_t}^{i-1} \wedge \mathcal{D}_{t, \text{End}}D) \stackrel{(16)}{=} i \cdot \widehat{\text{str}}(\mathcal{D}_{t, \text{End}}(R_{\mathcal{D}_t}^{i-1} \wedge D)) \\ &\stackrel{(14)}{=} i \cdot \mathbf{d}_A(\widehat{\text{str}}(R_{\mathcal{D}_t}^{i-1} \wedge D)). \end{aligned}$$

Using this, conclude that

$$\widehat{\text{str}}(R_{\mathcal{D}}^i) - \widehat{\text{str}}(R_{\mathcal{D}'}^i) = \mathbf{d}_A \int_0^1 i \cdot (\widehat{\text{str}}(R_{\mathcal{D}_t}^{i-1} \wedge D)) dt,$$

and so $\widehat{\text{str}}(R_{\mathcal{D}}^i)$ and $\widehat{\text{str}}(R_{\mathcal{D}'}^i)$ define the same cohomology class in $H^{2i}(A)$.

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