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EXACT ALGEBRAIC M(EM)BRANE SOLUTIONS*

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Abstract. Three classes of new, algebraic, zero-mean-curvature hypersurfaces in pseudo-Euclidean spaces are given.

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While everyone thinking about relativistic membrane dynamics will easily find a collapsing round sphere of radius $r(t)$ (= some elliptic function), and perhaps (cp. [1])

$$t - z = \left(\frac{x^2 + y^2}{R^2} - 1\right)\frac{1}{2}R\dot{R} + h\left(\frac{t+z}{2}\right), \tag{1}$$

$\dot{h} = \frac{\dot{R}^2}{2}$, $\frac{1}{2}\dot{R}^2 + \frac{1}{4}R^4 = \text{const.}$, or [2]

$$\begin{aligned} \mathcal{P}(x)\mathcal{P}(y)\mathcal{P}(t) &= \mathcal{P}(z) \\ \mathcal{P}'^2 &= 4\mathcal{P}(\mathcal{P}^2 - 1), \end{aligned} \tag{2}$$

not a single rotationally non-invariant exact time-like solution in 4-dimensional Minkowski space, arising from the motion of a smooth closed 2-dimensional surface, has been described during the 60 years since Dirac attempted to explain the muon as an excited state of the electron [3]. In this note I would like to present 3 classes of new, algebraic, zero-mean-curvature hypersurfaces in $(\mathbb{R}^D, \eta = (\eta_{\mu\nu}) = \text{diag}(1, \pm 1, \dots, \pm 1, -1))$. Consider

$$\begin{aligned} \chi(x) &:= (\alpha \cdot x)^n (x \circ x) =: \psi \cdot \phi \\ x \circ x &:= x^\mu g_{\mu\nu} x^\nu,\ (g_{\mu\nu}) = \text{diag}(1, \varepsilon_a, -1), |\varepsilon_a| = 1 \\ \alpha_\mu &= (1, 0, \dots, 0, 1) \\ a &= 1, \dots, D-2 =: M \\ \mu, \nu &= 0, 1, \dots, D-1. \end{aligned} \tag{3}$$

Straightforwardly one finds

$$\begin{aligned} \Box\chi &:= \eta^{\mu\nu}\partial_\mu\partial_\nu\chi = (4n + 2\gamma)\psi \quad \gamma := \eta^{\mu\nu} g_{\mu\nu} \\ (\partial\chi)^2 &:= \partial_\mu\chi\eta^{\mu\nu}\partial_\nu\chi = 4\psi^2 x^2 + 4n\psi^2\phi. \end{aligned} \tag{4}$$

The mean curvature of the hypersurfaces $\sum_{D-1} := \{x | \chi(x) = C\}$ vanishes if

$$\begin{aligned} &\frac{1}{2}\partial^\mu\chi\partial_\mu((\partial\chi)^2) - (\partial\chi)^2\Box\chi \\ &\quad = (4n(4n + 2\gamma) - (12n^2 + 4n + 8))\psi^3\phi + 8\gamma\psi^3 x^2 \end{aligned} \tag{5}$$

is zero. If $g_{\mu\nu} \neq \eta_{\mu\nu}$ γ must vanish, and $4(n^2 - n - 2) = 0$, i.e. $n = +2$ or -1. If $g_{\mu\nu} = \eta_{\mu\nu}$ the 2 terms can combine (and $\gamma = D$), resulting in $4(n^2 + (2\gamma - 1)n + 2(\gamma - 1)) = 0$,

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hence $n = -1$ or $n = -2(D-1)$. Forgetting about $n = -1$ (linear shifts of the coordinates are always allowed anyway) the two classes of solutions thus obtained are

$$x^\mu x_\mu = (t+z)^{2(D-1)} \cdot C \\ (= x^\mu \eta_{\mu\nu} x^\nu = t^2 + \sum_{a=1}^{D-2} \eta_a (x^a)^2 - z^2) \tag{6}$$

and

$$\phi(x) := x^\mu g_{\mu\nu} x^\nu = \frac{C}{(t+z)^2} \\ (= t^2 + \sum_1^{D-2} \varepsilon_a (x^a) - z^2) \tag{7}$$

where the ε_a, compared with the η_a ($a = 1 \ldots M = D-2$) must have 2 more signs differing compared to those agreeing, (i.e. in this case D must be even). Let me discuss the following (simplest, non-trivial) examples, in ordinary Minkowski space $\mathbb{R}^{1,D-1}$ in some detail:

$$u(x^\mu) := (t^2 - r^2 - z^2) - C(t+z)^{2M+2} \equiv 0 \\ r^2 := x_1^2 + \ldots + x_{D-2=M}^2 \tag{8}$$

and

$$v(x^\mu) := \psi \cdot \phi - C' = (t+z)^2 \cdot (t^2 + r^2 - s^2 - z^2) - C' \equiv 0 \tag{9}$$

where $r^2 = x_1^2 + \ldots + x^2_{q=\frac{D-2}{2}+1}$, $s^2 = x_{q+1}^2 + \ldots + x_{D-2}^2$ (and later only the simplest case, $D = 4$, $s^2 = 0$). Apart from the light-like line $r = 0 = t + z$ (containing the only singular point, $x^\mu = 0$) (8) is space-like if $C < 0$, and time-like if $C > 0$ (to which we now restrict; for (9), which for $C' \neq 0$ is regular, we will later take $C' < 0$, corresponding clearly to time-like; the case $C' > 0$ is somewhat more subtle). Writing (8) as

$$r^2 = \kappa(2t - f(\kappa)) \\ z = -t + \kappa \\ f = f_M(\kappa) := \kappa + C\kappa^{2M+1} \tag{10}$$

suitably parametrizes t-dependent M-dimensional 'axially symmetric' surfaces that are compact and convex (due to f being strictly increasing; $\kappa \in [0, \kappa_M(t)]$), $f(\kappa_M) = 2t > 0$, or, for $t < 0$: $\kappa \in [-\kappa_M, 0]$; note the $t \to -t$, $z \to -z$ hence $\kappa \to -\kappa$ invariance of (8), (9) resp. (6), (7)). As in the literature on relativistic extended objects, the dynamics is usually given assuming an orthonormal parametrization/gauge (ONG) it would be interesting to reparametrize (10) by $\kappa = \kappa(t, \varphi)$ such that the hypersurface described by $r(t, \varphi)$ and $z(t, \varphi)$ moves orthogonal to itself, i.e. satisfying $\dot{r}r' + \dot{z}z' = 0$. Differentiating (10) with respect to t and φ one can then derive

$$\dot{\kappa} = \frac{2\kappa(2t - f + \kappa f')}{(2t - f - \kappa f')^2 + 4\kappa(2t - f)} =: g(t, \kappa), \tag{11}$$

which on the one hand is reassuring (as $g(t, \kappa) \geqslant 0$ on $\mathbb{R}_+ \times [0, \kappa_M]$ resp. $\mathbb{R}_- \times [-\kappa_M, 0]$ and $= 0$ only for $\kappa = z + t = 0$), but also showing that even in the string case where everything is always assumed to be of integrable nature, the ODE (11) does not appear to be easily solvable in explicit terms (let alone the other part of ONG, $\dot{r}^2 + \dot{z}^2 + r^M \frac{(r'^2+z'^2)}{\rho^2} = 1$, yielding –using (11)– an ODE of the form $\frac{\kappa'}{\rho} = h(t,\kappa)$).

Interestingly, the assumption $|\varepsilon_a| = 1$ in (3) was/is actually not necessary, i.e. not the only one that works. Supposing

$$\lambda_{\mu\nu} := g_{\mu\mu'}\eta^{\mu'\nu'}g_{\nu'\nu} = \lambda\eta_{\mu\nu} + (1-\lambda)g_{\mu\nu} \tag{12}$$

$\Box\chi$ will still be $= (4n+2\gamma)\psi$, but

$$(\partial_\mu\chi)^2 = 4\big((n+1-\lambda)\psi^2\phi + \lambda\psi^2 x^2\big), \tag{13}$$

hence

$$\begin{aligned}\frac{1}{2}\partial^\mu\chi\partial_\mu\big((\partial\chi)^2\big) &= \big(\phi\partial^\mu\psi + 2\psi\eta^{\mu\nu}g_{\nu\rho}x^\rho\big)\cdot 4\big(\lambda\psi\partial_\mu\psi x^2 \\ &\quad + \lambda\psi^2\eta_{\mu\nu}x^\nu + (n+1-\lambda)\psi(\phi\partial_\mu\psi + g_{\mu\nu}x^\nu)\big) \\ &= 4\big(\psi^3\phi\big(3n^2 + n(5-4\lambda) + 2(\lambda^2-\lambda+1)\big) \\ &\quad + \lambda\psi^3x^2\big(4n + 2(1-\lambda)\big)\big)\end{aligned} \tag{14}$$

so that (5) vanishes if ($\lambda = \frac{1}{M-1} = \varepsilon_a$)

$$\gamma + \lambda = 1 \quad \text{and} \quad n = n_\pm = (\lambda - \frac{1}{2}) \pm (\lambda + \frac{1}{2}) \tag{15}$$

while $n_- = -1$ is uninteresting (corresponding to linear shifts of some of the coordinates)

$$(t^2 - z^2 + \frac{r^2}{M-1})(t+z)^{\frac{2}{M-1}} = C' \tag{16}$$

shows that (9) is not the only possible generalization of the corresponding $M = 2$ solutions given in [4]. Due to the correspondence found in [5] this will give rise to exact solutions

$$\begin{gathered} t - z = p(\tau, r) = \tau^a P(\tau^c r) \\ a + 2c + 1 = 0, \quad P(w) = \alpha + \beta\frac{w^2}{2} \\ a = -\frac{M+1}{M-1}, \quad \beta = \frac{-1}{M-1} \end{gathered} \tag{17}$$

(as well as $a = 2M+1$, $\beta = 1$) of the hydrodynamical equations

$$\ddot{p} + 2(p'\dot{p}' - \dot{p}p'') = \frac{M-1}{r}(2\dot{p}p' + p'^3), \tag{18}$$

resp.

$$\begin{aligned} a(a-1)P - \frac{3}{4}(a^2-1)wP' + \frac{(a+1)^2}{4}w^2P'' + (Ma+M-2)P'^2 \\ = \frac{M-1}{w}P'^3 + 2a\big(PP'' + (M-1)\frac{PP'}{w}\big). \end{aligned}$$

Note added. For the results presented above one may naturally consider simplified derivations and generalizations. (8) e.g. trivially (translating z and t in opposite directions by $\frac{B}{2}$) generalizes to $u(t,\vec{x},z) = \vec{x}^2 + z^2 - t^2 + B(t+z) + C(t+z)^a =: -q + g(\alpha \cdot x) = 0$, $a = 2M+2 = 2(D-1)$. Inserting $u = f(q) + g(v)$ into $\frac{1}{4}(u^\mu u^\nu u_{\mu\nu} - \partial_\mu u \partial^\mu u \Box u)$ gives $v^2 g'' - avg' + aq$, hence on the hypersurface $u = 0$ reducing to the *linear* ODE $v^2 g'' - avg' + ag = 0$, whose general solution is $g(v) = B\cdot v + C\cdot v^a$. $\partial^\mu u \partial_\mu u = f_\mu f^\mu + g_\mu g^\mu + 2 f_\mu g^\mu = 4x^2 - 4vg'$ on the other hand (on $u = 0$) equals $4(g - vg') = 4C(\alpha\cdot x)^a(1-a) \leqslant 0$ if $C > 0$ (and $= 0$ only for $\alpha \cdot x = t + z = 0$, on $u = 0$ implying $r = 0$), while the normal $\partial_\mu u$ vanishes (on $u = 0$) only at (where the M(em)brane shrinks to) one point, $t = \frac{B}{2} = -z$, $r = 0$.

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