HIGHER RHO INVARIANT AND DELOCALIZED ETA INVARIANT AT INFINITY[*]

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Abstract. In this paper, we introduce several new secondary invariants for Dirac operators on a complete Riemannian manifold with a uniform positive scalar curvature metric outside a compact set and use these secondary invariants to establish a higher index theorem for the Dirac operators. We apply our theory to study the secondary invariants for a manifold with corner with positive scalar curvature metric on each boundary face.

Key words. Dirac operator, higher index, higher rho invariant at infinity, delocalized eta invariant at infinity, uniform positive scalar curvature, polynomial growth conjugacy class.

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1. Introduction. In this article, we introduce a new theory of secondary invariants for Dirac operators on noncompact spin manifolds endowed with metrics with uniform positive scalar curvature at infinity, i.e. higher rho invariants and the delocalized eta invariants. Let X be a complete spin manifold with metric admitting uniform positive scalar curvature outside a compact set Z, D be the Dirac operator on the universal covering space $\widetilde{X}$ of X, and let G be $\pi_1(X)$, the fundamental group of X. We obtain a higher index formula for the Dirac operator D which expresses the delocalized trace of the higher index in terms of delocalized secondary invariants at infinity, as follows:

THEOREM 1.1. *Let D be the Dirac operator on the universal covering space $\widetilde{X}$ of a spin manifold X with uniform positive scalar curvature outside a compact subset Z defined as above. Let G be the fundamental group of X. Let $g \in G$ be a nontrivial element whose conjugacy class has polynomial growth. Let $ind_G(D)$ be the higher index of the Dirac operator D. The delocalized trace at g of the higher index of D, $\mathrm{tr}_g(\mathrm{ind}_G(D))$ equals the half of the negative delocalized eta invariant at infinity, $\eta_{g,\infty}(D)$, i.e.*

$$\mathrm{tr}_g(\mathrm{ind}_G(D)) = -\frac{1}{2}\eta_{g,\infty}(D). \tag{1.1}$$

Furthermore, we have

$$\frac{1}{2}\eta_{g,\infty}(D) = \lim_{t \to 0} \int_t^\infty \mathrm{tr}_g(e^{-sD_c^- D_c^+} D_c^- [D^+, \psi_2]) ds. \tag{1.2}$$

where D_c is the invertible Dirac operator on $\widetilde{X} \backslash \widetilde{Z}$, and ψ_2 is a G-invariant cutoff function from $\widetilde{X}$ to $[0,1]$, which equals 0 on a cocompact neighbourhood of $\widetilde{Z}$, the

G-Galois covering space of Z, and equals 1 *outside a cocompact set larger than* $\widetilde{Z}$. *The integral on the right hand side of* (1.2) *is independent of the choice of the cutoff function* ψ_2.

The higher index $\text{ind}_G(D)$ of D, on a complete manifold with invertibility condition at infinity, was first introduced in Bunke (cf. [6]). The main significance and subtlety of the higher index formula in Theorem 1.1 is that $\text{tr}_g(\text{ind}_G(D))$ vanishes when X is closed (cf. [29]). Furthermore, if X is a cylindrical manifold obtained from a compact manifold with boundary ∂M, and with a metric with positive scalar curvature on the boundary which is collared near ∂M, then $\text{tr}_g(\text{ind}_G(D))$ equals the delocalized eta invariant of the Dirac operator on ∂M.

Equation (1.1) is obtained by considering the higher rho invariant at infinity. These secondary invariants we introduce demonstrate the nonlocality of the higher index of D. We also apply the higher rho invariant at infinity to analyze the geometry of manifolds with corner of codimension 2 with positive scalar curvature metrics on all boundary faces.

The classical eta invariant is a nonlocal spectral invariant of Dirac type operators (cf. [1, 2, 3, 4]). Let M be a closed odd dimensional spin manifold with a positive scalar curvature metric, D_M be the Dirac operator on M. When the following integral converges, the eta invariant of D_M can be expressed as

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \text{tr}(e^{-t^2 D_M^2} D_M) dt$$

where tr is the operator trace. To take into account of the information of the fundamental group, Lott introduced the delocalized eta invariant (cf. [15]). More precisely, let G be the fundamental group of M, $\widetilde{M}$ be the universal covering space of M, and $\widetilde{D}_M$ be the lifting of D_M to $\widetilde{M}$. Let g be a nontrivial element of G, whose conjugacy class $\langle g \rangle$ has polynomial growth. The delocalized eta invariant of $\widetilde{D}_M$ at g, $\eta_g(\widetilde{D}_M)$, introduced by Lott in [15], is defined by the following integration

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \text{tr}_g(e^{-t^2 \widetilde{D}_M^2} \widetilde{D}_M) dt.$$

Here tr_g is the following trace map

$$\text{tr}_g(A) = \sum_{h \in \langle g \rangle} \int_{\mathcal{F}} A(x, hx) dx,$$

on G-equivariant Schwartz kernels $A \in C^\infty(\widetilde{M} \times \widetilde{M})$, where $\mathcal{F}$ is a fundamental domain of $\widetilde{M}$ under the G-action. One can also define tr_g for continuous group, see [12] for example. There is also a higher generalization of the pairing between tr_g and K-theory of geometric C^*-algebras, introduced by [18], [7], [21], and [25], which is to consider cyclic cocycles.

Since the metric m on M admits positive scalar curvature, it follows from the Lichnerowicz formula that the higher index of D_M, $\text{ind}_G(\widetilde{D}_M)$, in the K-theory of the group C^*-algebra $C_r^*(G)$, is trivial with a specific trivialization. In this case, Higson and Roe proposed to study a secondary invariant in K-theory of a certain C^*-algebra, the higher rho invariant of $\widetilde{D}_M$, $\rho(\widetilde{D}_M)$ (cf. [23, 10]). The higher rho invariant of the Dirac operator has been applied to estimate the lower bound of how many positive scalar curvature metrics a manifold can bear (cf. [20, 33, 32, 34, 29]).

The higher rho invariant is closely related to the delocalized eta invariant. Let g be a nontrivial element of G, whose conjugacy class $\langle g \rangle$ has polynomial growth. In [31], Xie and Yu defined a canonical determinant map τ_g associated to g, and showed that $\tau_g(\rho(\widetilde{D}_M)) = \frac{1}{2}\eta_g(\widetilde{D}_M)$.

The eta invariants and the higher rho invariants appear in the study on the geometry of manifolds with boundary naturally. Let M be an even dimensional spin manifold with boundary N, with the metric m having product structure near N, and admitting positive scalar curvature when restricted to N. Then the eta invariant of the Dirac operator D_N on the boundary, $\eta(D_N)$, is the correction term in the formula of the Fredholm index of the Dirac operator D_{M_∞} on $M_\infty := M \cup \partial M \times [0, \infty)$. Let $\mathrm{index}(D_{M_\infty})$ be the Fredholm index of D_{M_∞}. Then

$$\mathrm{index}(D_{M_\infty}) = \int_M \hat{A}(M) - \frac{\eta(D_N)}{2}, \tag{1.3}$$

where $\hat{A}(M)$ is the $\hat{A}$ genus (cf. [1, 2, 3, 4]). The eta invariant explains the nonlocality of the Fredholm index. Equation 1.3 is often referred to as the Atiyah-Patodi-Singer index theorem. On the other hand, the delocalized eta invariant and the higher rho invariant capture the nonlocality of the higher index of D_{M_∞}. In fact, in [31, 7], Xie-Yu and Chen-Wang-Xie-Yu obtained several higher generalizations of the Atiyah-Patodi-Singer index theorem. In particular, for any nontrivial element $g \in G = \pi_1(M)$ whose conjugacy class is of polynomial growth, they proved

$$\mathrm{tr}_g(\mathrm{ind}_G(\widetilde{D}_{M_\infty})) = -\tau_g(\rho(\widetilde{D}_N)) = -\frac{1}{2}\eta_g(\widetilde{D}_N). \tag{1.4}$$

We mention that the formula

$$\mathrm{tr}_g(\mathrm{ind}_G(\widetilde{D}_{M_\infty})) = -\frac{1}{2}\eta_g(\widetilde{D}_N)$$

was first established in [19] by Piazza and Schick, where their proof employs as a crucial ingredient the specialization of 0-degree forms of the higher Atiyah-Patodi-Singer index formula of Leichtnam and Piazza ([14], [20]). And the formula

$$\tau_g(\rho(\widetilde{D}_N)) = \frac{1}{2}\eta_g(\widetilde{D}_N)$$

was also obtained by Piazza, Schick, and Zenobi in [21].

Since the metric on M_∞ has positive scalar curvature outside the compact set M, the Dirac operator D_{M_∞} is invertible at infinity. In this particular case, the delocalized eta invariant and the higher rho invariant of D_N can be viewed as secondary invariants at infinity associated to D_{M_∞}. The study of the secondary invariants has led to several major breakthroughs to the estimate of the lower bound of the rank of the abelian group formed by the concordance classes of positive scalar curvature metrics (cf. [20, 32, 33, 34, 35]). In the meanwhile, a parallel study to the secondary invariants allows one to estimate the lower bound of the topological structure group ([28]).

The first main result, Theorem 1.1, is a generalization of the above Atiyah-Patodi-Singer index theorem and its higher counterpart to the case of a noncompact complete manifold with uniform positive scalar curvature metric at infinity.

Let X be an n-dimensional complete spin manifold and Z be a compact subset of X, let m be a metric on X, which has uniformly positive scalar curvature outside Z,

and G be the fundamental group of X. Let $\widetilde{X}$ be the universal covering space of X and $\widetilde{Z}$ be the induced G-Galois covering space of Z. In this case, following [24] and [32], one can define the higher index of the Dirac operator D on $\widetilde{X}$ in $K_n(C_r^*(\widetilde{Z}))^G$, which is isomorphic to $K_n(C_r^*(G))$. See Subsections 3.1 and 3.2 for details. The higher index of D is denoted as $\mathrm{ind}_G(D)$ or $\mathrm{ind}_{\widetilde{X}, \widetilde{Z}, T}(D)$, where T is any sufficiently large number. We emphasize that the definition of the higher index of D is independent of T. However, the subscript T manifests the particular choice of a representative class of $\mathrm{ind}_G(D)$. In Subsection 3.3, the higher rho invariant at infinity of D, denoted as $\rho_{\widetilde{X}, \widetilde{Z}, T}(D)$, is defined as the image of $\mathrm{ind}_{\widetilde{X}, \widetilde{Z}, T}(D)$ under the connecting map of a K-theoretic six-term exact sequence associated to a canonical short exact sequence of geometric C^*-algebras. Moreover, for nontrivial element $g \in G$, whose conjugacy class is of polynomial growth, the delocalized eta invariant at infinity of D is simply defined to be

$$2\tau_g(\rho_{\widetilde{X}, \widetilde{Z}, T}(D)). \tag{1.5}$$

As in the case of manifold with cylindrical end, the number in line (1.5) is essential for us to obtain Equation (1.1) in Theorem 1.1.

In the meanwhile, Equation (1.2) in Theorem 1.1 is established by applying the method in [12], which was invented when Hochs, Wang and Wang were to develop a new approach to obtain a refinement of the Atiyah-Patodi-Singer index theorem. See Section 3.6 for details. At the end of the paper, this method is used to obtain an L^2 version of Müller's type Atiyah-Patodi-Singer index theorem associated to the Dirac operator on a manifold with corner endowed with positive scalar curvature metrics on all boundary faces.

The paper is organized as follows. In Section 2, we recall basic concepts, including geometric C^*-algebras and their smooth subalgebras, which will be used later in the paper. In Section 3, we define two secondary invariants at infinity for the Dirac operator on a complete manifold with uniform positive scalar curvature metric outside a compact set, the higher rho invariant at infinity and the delocalized eta invariant at infinity. We establish a formula for the delocalized eta invariant at infinity in Subsection 3.6, together with which we generalize the Atiyah-Patodi-Singer index theorem. In Section 4 we apply the theory developed in Section 3 to study invariants associated to the Dirac operator on a manifold with corner endowed with positive scalar curvature metrics on all boundary faces.

2. Preliminary. In this section, we introduce some notions and concepts used in this paper, including geometric C^*-algebras and their smooth subalgebras. All the groups considered in this paper are finitely generated discrete groups. Denote by $|\cdot|$ the word length metric of a group G constructed using some chosen finite set of generators. For an element g in G, we say its conjugacy class, $\langle g \rangle$ has polynomial growth if there exist constants C and d, such that

$$\sharp\{h \in \langle g \rangle, |h| \leq n\} \leq Cn^d.$$

2.1. Geometric C^*-algebras. In this subsection, we recall definitions of several geometric C^*-algebras, including equivariant Roe, localization and obstruction algebras (see [11, 22, 36, 24, 30] for more details).

Let X be a complete Riemannian manifold where G acts properly, cocompactly and freely. An X-module is a separable Hilbert space equipped with a $*$ representation of $C_0(X)$. It is nondegenerate if the $*$ representation is nondegenerate. It is called

standard if no nonzero function in $C_0(X)$ acts as a compact operator. Let H_X be a standard nondegenerate X-module, where H_X admits a unitary representation of the group G, and the representation of $C_0(X)$ is covariant to the group representation.

We first recall some ingredients for constructing the geometric C^*-algebras.

DEFINITION 2.1. let H_X be an X-module and $T \in B(H_X)$ be a bounded linear operator.

- The propagation of T is defined to be

$$\sup \{d(x,y)| \ (x,y) \in \mathrm{Supp}(T)\},$$

  where $\mathrm{Supp}(T)$ is the complement in $X \times X$ of the set of points $(x, y) \in X \times X$ for which there exist $f, g \in C_0(X)$ satisfying $f(x) \neq 0$, $g(y) \neq 0$ such that $gTf = 0$. T is said to have finite propagation if its propagation is finite. The propagation of a finite propagation operator T is denoted as propagation(T).
- T is said to be locally compact if fT and Tf are compact for all $f \in C_0(X)$.
- T is said to be pseudo-local if $[T, f]$ is compact for all $f \in C_0(X)$.
- Let Z be a G-invariant subspace of X. We say T is supported near Z if there is $\lambda > 0$, such that for all $f \in C_0(X)$ whose support is at least distance λ away from Z, the operators Tf and fT are zero.

The following are definitions of the geometric C^*-algebras that will be used in this paper.

DEFINITION 2.2. Let H_X be a standard (also referred to as ample in the literature) nondegenerate X-module and $B(H_X)$ be the operator algebra of all bounded linear operators on H_X.

- G-equivariant Roe algebra $C^*(X)^G$ is the C^*-algebra generated by locally compact G-invariant operators with finite propagation.
- Let Z be a G-invariant subspace of X. The localized G-equivariant Roe algebra at Z, $C^*(X, Z)^G$, is defined to be the C^*-algebra generated by G-invariant, locally compact operators with finite propagation, supported near Z.
- G-equivariant localization algebra $C_L^*(X)^G$ is the C^*-algebra generated by all bounded, uniformly norm-continuous functions $f : [0, \infty) \to C^*(X)^G$ such that

$$\lim_{t \to \infty} \text{propagation of } f(t) = 0.$$

- The kernel of the evaluation map

$$ev : C_L^*(X)^G \to C^*(X)^G$$
$$f \to f(0)$$

  is called the G-equivariant obstruction algebra, and denoted by $C_{L,0}^*(X)^G$.
- Let Z be a G-invariant subspace of X, then $C_L^*(X, Z)^G$ (resp. $C_{L,0}^*(X, Z)^G$) is defined to be the closed subalgebra of $C_L^*(X)^G$ (resp. $C_{L,0}^*(X)^G$) generated by all elements f such that there exists $c_t > 0$ satisfying that $\lim_{t \to \infty} c_t = 0$, and $\mathrm{Supp}(f(t)) \subset \{(x, z) \in X \times X| \ d((x, z), Z \times Z) \leq c_t\}$.

In general, if Z is a cocompact G-invariant subspace of X, then we have the following isomorphism

$$K_*(C^*(Z)^G) \cong K_*(C^*(X, Z)^G) \cong K_*(C_r^*(G)),$$

where the first isomorphism is induced by the obvious embedding map.

At the same time, by Lemma 3.10 of [36], we know

$$K_*(C_L^*(X, Z)^G) \cong K_*(C_L^*(Z)^G). \tag{2.1}$$

Moreover, by the following two K-theoretic six-term exact sequences

$$
\begin{array}{ccccc}
K_0(C_{L,0}^*(Z)^G) & \longrightarrow & K_0(C_L^*(Z)^G) & \longrightarrow & K_0(C^*(Z)^G) \\
\uparrow & & & & \downarrow \\
K_1(C^*(Z)^G) & \longleftarrow & K_1(C_L^*(Z)^G) & \longleftarrow & K_1(C_{L,0}^*(Z)^G) \\
K_0(C_{L,0}^*(X, Z)^G) & \longrightarrow & K_0(C_L^*(X, Z)^G) & \longrightarrow & K_0(C^*(X, Z)^G) \\
\uparrow & & & & \downarrow \\
K_1(C^*(X, Z)^G) & \longleftarrow & K_1(C_L^*(X, Z)^G) & \longleftarrow & K_1(C_{L,0}^*(X, Z)^G)
\end{array}
$$

and a standard five lemma argument, one can see that

$$K_*(C_{L,0}^*(X, Z)^G) \cong K_*(C_{L,0}^*(Z)^G). \tag{2.2}$$

Zeidler gave a constructive proof of the above isomorphisms in (2.1), and (2.2). See Lemma 3.7 of [38] for details.

2.2. Engel-Samurkaş smooth algebra. In this subsection, we introduce a $*$-algebra defined by Kağan Samurkaş in [26], which is KK-equivalent to $C_r^*(G)$. The construction of this algebra is inspired by Engel [9]. It will be used in computing the paring of the delocalized trace and the higher index.

This algebra depends on the choice of a nontrivial element g of G, whose conjugacy class has polynomial growth. For $S \subset G$ and $f \in l^2(G)$, let $\|f\|_S$ be the l^2 norm of f restricted to S. For $r > 0$, set $B_r(S)$ as

$$\{h \in G, \forall s \in S, d_G(e, hs^{-1}) < r\},$$

where d_G is the word length metric.

Choose $C > 0$ and $d \in \mathbb{N}$ such that for all $k \in \mathbb{N}$,

$$\sharp\{h \in \langle g \rangle, d_G(e, h) = k\} \leq Ck^d.$$

For $a \in B(l^2(G))$ and $r > 0$, define

$$\mu_a(r) = \inf\{c > 0, \forall f \in l^2(G), \|af\|_{G \setminus B_r(\mathrm{supp}(f))} \leq c\|f\|_G\},$$

and

$$\|a\|_g = \inf\{A > 0, \forall r > 0, \mu_a(r) \leq Ar^{-d/2+2}\}.$$

Let $\|\cdot\|_{B(l^2(G))}$ be the operator norm on $B(l^2(G))$. For $f \in \mathbb{C}G$, we also denote by $f \in B(l^2(G))$ the convolution operator by f from the left.

Define $C_g^{pol}(G)$ to be the completion of $\mathbb{C}G$ in the norm

$$\|f\| = \|f\|_{B(l^2(G))} + \|f\|_g.$$

In [26], Samurkaş showed that $C_g^{pol}(G)$ is closed under holomorphic functional calculus. We denote $C_g^{pol}(G)$ by $\mathcal{A}_g(G)$.

Let us recall the delocalized trace map tr_g from the Kağan Samurkaş algebra to $\mathbb{C}$. Denote by $\mathrm{tr}_g : \mathcal{A}_g(G) \to \mathbf{C}$ the delocalized trace given by

$$\mathrm{tr}_g \left(\sum_{\gamma \in G} a_\gamma \gamma \right) = \sum_{h \in \langle g \rangle} a_h \tag{2.3}$$

where $\langle g \rangle$ stands for the conjugacy class of g in G. Moreover, the map tr_g induces the following delocalized trace map on K-theory:

$$\mathrm{tr}_g : K_0(\mathcal{A}_g(G)) \cong K_0(C_r^*(G)) \to \mathbb{C}.$$

At last, we recall the notion of g-trace class operators. Let $\mathcal{F}$ be a fundamental domain of X with respect to the G-action. Compare the following definition to Definition 2.1 of [12].

DEFINITION 2.3. Let T be a G-invariant operator having Schwartz kernel in $C^\infty(X \times X)$ and g be an element of G. Then T is g-trace class if

$$\sum_{h \in \langle g \rangle} \int_{\mathcal{F}} |T(x, hx)| dx$$

converges. The g-trace of T is defined as

$$\mathrm{tr}_g(T) \triangleq \sum_{h \in \langle g \rangle} \int_{\mathcal{F}} T(x, hx) dx.$$

Note that when G is trivial or $g = e$, T is g-trace class if T is trace class, which is equivalent to the condition $\int_M |T|(x, x) dx < \infty$. For a general g distinct from identity, tr_g is not a positive trace, hence positivity is not required in the definition of g-trace class operators.

2.3. The Connes-Moscovici smooth subalgebra. In this subsection, let M be a complete Riemannian manifold with a proper, free and cocompact action by a discrete group G. Let us recall Xie and Yu's construction of a particular smooth dense subalgebra of $C_{L,0}^*(M)^G$ ([31]).

The following construction of a smooth dense subalgebra of $C_r^*(G) \otimes \mathcal{K}$ is due to Connes and Moscovici (cf. [8]). Let $\mathcal{R}$ be the algebra of smoothing operators on M/G. Under the isomorphism $L^2(M/G) \cong l^2(\mathbb{N})$, $\mathcal{R}$ is identified with the algebra of infinite matrices $(a_{ij})_{i,j \in \mathbb{N}}$, such that

$$\forall k, l \in \mathbb{N}, \quad \sup_{i,j} i^k j^l |a_{ij}| < \infty.$$

For a finitely generated discrete group G, let Δ_G be an unbounded operator on $l^2(G)$ defined by

$$\Delta_G g = |g| g, \text{ for } g \in G.$$

Let $\Delta : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the unbounded operator defined by

$$\Delta(\delta_j) = j \delta_j.$$

Denote by $\partial_G = [\Delta_G, \cdot]$ the unbounded derivation of $B(l^2(G))$, and $\widetilde{\partial}_G = \partial_G \otimes I$ the unbounded derivation of $B(l^2(G) \otimes l^2(\mathbb{N}))$. Set

$$\mathcal{B}(M)^G = \{A \in C_r^*(G) \otimes \mathcal{K}, \text{ for all } k \in \mathbb{N}, \ \widetilde{\partial}_G^k(A) \circ (I \otimes \Delta)^2 \text{ is bounded }\},$$

which can be proved to be a smooth dense subalgebra of $C_r^*(G) \otimes \mathcal{K}$. Lemma 2.7 of [31] shows that if $\langle g \rangle$ has polynomial growth, then the map tr_g can be extended to a trace map on $\mathcal{B}(M)^G$ such that for any $A = \sum_g A_g g \in \mathcal{B}(M)^G$, $\mathrm{tr}_g(A)$ equals

$$\sum_{h \in \langle g \rangle} \mathrm{trace}(A_h).$$

Note that this tr_g map induces the same map on $K_0(C_r^*(G))$ with the corresponding one defined in the line (2.3).

In the meanwhile, the following algebra

$$\mathcal{B}_{L,0}(M)^G \triangleq \{f \in C_{L,0}^*(M)^G, \forall t \in [0, \infty), f(t) \in \mathcal{B}(M)^G\}$$

is a smooth dense subalgebra of $C_{L,0}^*(M)^G$. Let $g \in G$ be a nontrivial element whose conjugacy class has polynomial growth. Define the map $\tau_g : \mathcal{B}_{L,0}(M)^G \to \mathbb{C}$ to be

$$\tau_g(A) = \frac{1}{2\pi i} \int_0^\infty \mathrm{tr}_g\left(\frac{dA(t)}{dt} A^{-1}(t)\right) dt.$$

As shown in [31], τ_g induces the following determinant map

$$\tau_g : K_1(\mathcal{B}_{L,0}(M)^G) \cong K_1(C_{L,0}^*(M)^G) \to \mathbb{C}, \tag{2.4}$$

which is also denoted as τ_g.

One may think it is sufficient to recall the definition of either the Engel-Samukarş smooth subalgebra or the Connes-Moscovici smooth subalgebra, however, there is a reason for us not to do so. The advantage of the Engel-Samukarş smooth subalgebra is that it allows us to do analysis in a noncocompact setting and to apply the method of Hochs, Wang and Wang (cf. [12]) to develop the index formula for the Dirac operator on noncocompact complete manifold with uniform positive scalar curvature metric at infinity, while the Connes-Moscovici smooth subalgebra is necessary for us to apply the theory of Xie and Yu on delocalized trace (cf. [31]) to study the higher rho invariant at infinity and define the delocalized eta invariant at infinity, where a cocompact setting is sufficient.

3. Secondary invariants at infinity. In this section, we introduce two new secondary invariants for the Dirac operator on a complete spin manifold endowed with a metric with uniform positive scalar curvature outside a cocompact set, i.e. the higher rho invariant at infinity and the delocalized eta invariant at infinity. We begin with two approaches to the definition of the higher index of the Dirac operator. We also establish a formula for the delocalized eta invariant, along with which we generalize the Atiyah-Patodi-Singer index theorem.

3.1. Higher index. In this subsection, we define the higher index of the Dirac operator on a complete manifold having a metric with uniform positive scalar curvature outside a cocompact set. Higher index of the Dirac operator on a complete manifold with positive scalar curvature at infinity was first introduced by Bunke in

[6]. There is a nice description of this higher index in [5]. However, our construction in this subsection follows [22] [24].

Let X be a complete Riemannian manifold (even dimensional) with a spin structure on which a discrete group G acts freely, properly and preserving the metric. Let $M \subset X$ be a G-invariant subset. Assume that there is a metric μ with uniform positive scalar curvature with strictly positive lower bound $h_0 > 0$ on $X \backslash M$.
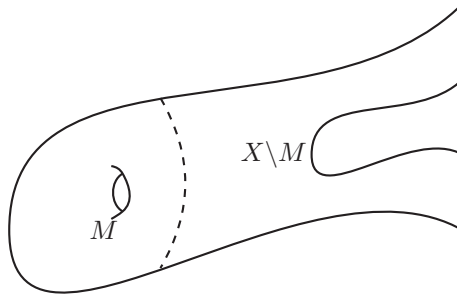

FIG. 1. *Complete manifold X, with G cocompact subset M.*

Let D be the spin Dirac operator associated to the metric μ on X. Denote by D_M, D_c the restriction of D to M and $C := X \backslash M$ respectively. Because X has even dimension, the spinor bundle S is $\mathbf{Z}_2$-graded, i.e. $S = S^+ \oplus S^-$, and the Dirac operator is odd with respect to the grading $L^2(X, S^+) \oplus L^2(X, S^-)$:

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix} \qquad (D^+)^* = D^-.$$

Let $b : (-\infty, +\infty) \to [-1, 1]$ be an odd smooth function such that $b(x) = 1$ when $x > \frac{\sqrt{h_0}}{2}$. Lemma 2.1 and Lemma 2.3 of [24] state that $b(D)$ is pseudo-local, and $b(D)^2 - 1$ lies in $C^*(X, M)^G$. Since b is an odd function, $b(D)$ can also be decomposed as

$$\begin{bmatrix} 0 & b_+ \\ b_- & 0 \end{bmatrix}.$$

Let V be

$$\begin{bmatrix} 1 & b_+ \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -b_- & 1 \end{bmatrix} \begin{bmatrix} 1 & b_+ \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

DEFINITION 3.1. The G-equivariant coarse index of D, $\mathrm{ind}_{G,X,M}(D)$ is defined to be the K-theory class in $K_0(C^*(X, M)^G)$, represented by the formal difference of idempotents

$$V \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} V^{-1} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Furthermore, if M/G is compact, the K-theory class

$$\mathrm{ind}_{G,X,M}(D) \in K_0(C^*(X, M)^G) \cong K_0(C_r^*(G)),$$

is called the higher index of D, and denoted as $\mathrm{ind}_G(D)$.

Note that direct computation shows that

$$V \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} V^{-1} = \begin{bmatrix} (1 - b_+ b_-) b_+ b_- + b_+ b_- & (2 - b_+ b_-) b_+ (1 - b_- b_+) \\ b_- (1 - b_+ b_-) & (1 - b_- b_+)^2 \end{bmatrix}.$$

3.2. Higher index: a different approach. In this subsection, we introduce a new approach to define the higher index of the Dirac operator on a complete manifold with metric admitting uniform positive scalar curvature at infinity. This approach is necessary for us to define the higher rho invariant at infinity. This approach is inspired by [32], [24], and has been used to define the higher index in [39]. See Willett and Yu's recently published book [30] for more details.

Set $X_{[r,s]}$, $r, s \in [0, \infty]$, to be the G-invariant subset of X defined by

$$\{x \in X | r \le \mathrm{dist}(x, M) \le s\}.$$

For simplicity, $X_{[0,s]}$, $s < \infty$, is denoted as $X_{\le s}$, and $X_{[r,\infty]}$, $r > 0$, is denoted as $X_{\ge r}$. Note that $X_{[0,\infty]} = X$. The set $X_{(r,s)}$ are defined similarly.

Let $\chi : (-\infty, \infty) \to [-1, 1]$ be a normalizing function satisfying the following conditions,

1) χ is a smooth odd function, such that $\lim\limits_{s \to \pm\infty} \chi(s) = \pm 1$.

2) $\lim\limits_{s \to 0} \frac{\chi(s)}{s} = 1$.

3) $\hat{\chi}$ is a compactly supported distribution on $\mathbb{R}$, such that its support contains 0, and the diameter of its support is bounded by a real number δ, i.e. $\mathrm{diam}(\mathrm{Supp}(\hat{\chi})) \le \delta$.

4) Let $b : (-\infty, \infty) \to [-1, 1]$ be the normalizing function such that $b(x) = 1$ when $x > \frac{\sqrt{h_0}}{2}$. Then $\|\chi - b\| \le \epsilon$, where ϵ is a positive number less than $\frac{1}{10000^{10000}}$.

Furthermore, let $T \ge 1$ be a real number.

Define F_T to be the self-adjoint bounded operator

$$\chi(TD) = \int_{-\infty}^{\infty} \hat{\chi}(s) e^{2\pi i s T D} ds.$$

Since χ is an odd function, F_T is odd with respect to the grading $L^2(X, S^+) \oplus L^2(X, S^-)$:

$$F_T = \begin{bmatrix} 0 & U_T \\ U_T^* & 0 \end{bmatrix}.$$

Let N be a sufficiently large integer such that for any $x \in [-1000, 1000]$, there are

$$|\sum_{n=1}^{N} \frac{(2\pi i)^n}{n!}| \le \frac{\epsilon}{1000000}, \quad |\sum_{n=N}^{\infty} \frac{(2\pi i x)^n}{n!}| \le \frac{\epsilon}{1000000}.$$

Let $r > 0$ be a positive number such that

$$Nr \le \frac{\delta}{100}. \tag{3.1}$$

For any $n \in \mathbb{N}$, we choose a G-invariant locally finite open cover $\{U_{n,j}\}_j$, and a G-invariant partition of unity $\{\phi_{n,j}\}_j$ subordinate to $\{U_{n,j}\}_j$, such that

1) if $U_{n,j} \subset X_{\leq 100T\delta}$, then the diameter of $U_{n,j}$ is less than $\frac{r}{2^n}$.
2) for any fixed n, there are precisely two open sets $U_{n,j}$ such that $U_{n,j} \bigcap X_{\geq 100T\delta}$ is nonempty. For convenience, we denote them by $W_{n,1} = X_{(100T\delta - \frac{r}{2^n}, 110T\delta)}$ and $W_2 = X_{>100T\delta}$.

Set

$$F_{T,n} = \sum_j \sqrt{\phi_{n,j}} F_T \sqrt{\phi_{n,j}}, \ \forall n \in \mathbb{N},$$

and

$$F_T(t) = (n+1-t)F_{T,n} + (t-n)F_{T,n+1}.$$

For any t, $F_T(t)$ is odd with respect to the grading $L^2(X, S^+) \oplus L^2(X, S^-)$:

$$F_T(t) = \begin{bmatrix} 0 & U_T(t) \\ U_T^*(t) & 0 \end{bmatrix}.$$

As shown by [37, Lemma 2.6], $\|U_T(t)\| \leq 4\|U_T\| = 4$. Consider

$$W_T(t) = \begin{bmatrix} 1 & U_T(t) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -U_T^*(t) & 1 \end{bmatrix} \begin{bmatrix} 1 & U_T(t) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Note that F_T has finite propagation $T\delta$.
Decompose $L^2(X, S)$ into $L^2(X_{\leq 300T\delta}, S) \oplus L^2(X_{\geq 300T\delta}, S)$.

LEMMA 3.2. *Write*

$$U_T(t)U_T^*(t) = \begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{bmatrix} \ and \ U_T^*(t)U_T(t) = \begin{bmatrix} C_{11}(t) & C_{12}(t) \\ C_{21}(t) & C_{22}(t) \end{bmatrix}$$

with respect to the decomposition

$$L^2(X, S) = L^2(X_{\leq 300T\delta}, S) \oplus L^2(X_{\geq 300T\delta}, S).$$

Then one has the following inequalities

$$\|B_{12}(t)\| = \|B_{21}(t)\| \leq \epsilon, \ \|B_{22}(t) - 1\| \leq \epsilon,$$
$$\|C_{12}(t)\| = \|C_{21}(t)\| \leq \epsilon, \ \|C_{22}(t) - 1\| \leq \epsilon$$

for any $t \in [0, \infty)$.

Proof. By assumption, D_c^2 is bounded below by $\frac{h_0}{4}$. Therefore it has a Friedrich's extension E on the Hilbert space $L^2(X \backslash M, S)$, which is a selfadjoint operator bounded below by $\frac{h_0}{4}$. For the normalizing function χ, define an operator $\chi^2(T\sqrt{E})$.
Since the propagation of F_T is controlled by $T\delta$, for all $f \in L^2(X_{\geq 120T\delta}, S)$, we have

$$F_T(t)(f) \in L^2(X_{\geq 119T\delta}, S), \quad F_T(t)^2(f) \in L^2(X_{\geq 118T\delta}, S).$$

By a standard finite propagation argument (cf. the proof of [24, Lemma 2.5]), we have

$$\|F_T(t)^2(f) - f\| = \|\chi^2(T\sqrt{E})(f) - f\| \leq \epsilon\|f\|,$$

where the last inequality follows from condition 4) in the definition of χ, which says that χ^2 is closed to b^2, a function equals 1 on the spectrum of $T\sqrt{E}$. This implies the Lemma. $\square$

Now let us consider the formal difference

$$W_T(t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W_T^{-1}(t) - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

By definition of the boundary map of the K-theory six-term exact sequence, this formal difference represents a K-theory class in $K_0(C^*(X)^G)$, which is not isomorphic to $K_0(C^*_r(G))$. To define the equivariant coarse index of D, we construct an almost idempotent sufficiently close to

$$W_T(t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W_T(t)^{-1}$$

in the operator norm.

Note that

$$W_T^{-1}(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -U_T(t) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ U_T^*(t) & 1 \end{bmatrix} \begin{bmatrix} 1 & -U_T(t) \\ 0 & 1 \end{bmatrix}.$$

Direct computation shows that

$$W_T(t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W_T(t)^{-1}$$

$$= \begin{bmatrix} (1 - U_T(t)U_T^*(t))U_T(t)U_T^*(t) + U_T(t)U_T^*(t) & (2 - U_T(t)U_T^*(t))U_T(t)(1 - U_T^*(t)U_T(t)) \\ U_T(t)^*(1 - U_T(t)U_T^*(t)) & (1 - U_T^*(t)U_T(t))^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - (1 - U_T(t)U_T^*(t))^2 & (2 - U_T(t)U_T^*(t))U_T(t)(1 - U_T^*(t)U_T(t)) \\ U_T^*(t)(1 - U_T(t)U_T^*(t)) & (1 - U_T^*(t)U_T(t))^2 \end{bmatrix}.$$

Set

$$Z_1(t) = \begin{bmatrix} 1 - B_{11}(t) & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } Z_2(t) = \begin{bmatrix} 1 - C_{11}(t) & 0 \\ 0 & 0 \end{bmatrix}.$$

By Lemma 3.2, we have

$$\|Z_1(t) - (1 - U_T(t)U_T^*(t))\| \le 3\epsilon, \text{ and } \|Z_2(t) - (1 - U_T^*(t)U_T(t))\| \le 3\epsilon.$$

Define $P_T(t)$ to be the operator

$$\begin{bmatrix} 1 - Z_1^2(t) & (2 - U_T(t)U_T^*(t))U_T(t)Z_2(t) \\ U_T^*(t)Z_1(t) & Z_2^2(t) \end{bmatrix}. \tag{3.2}$$

Then we have

$$\|P_T(t) - W_T(t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W_T(t)^{-1}\| \le 2000\epsilon,$$

which implies

$$\|P_T(t) - P_T(t)^2\|$$

$$\leq \|P_T(t) - W_T(t)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W_T(t)^{-1}\| + \|(W_T(t)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W_T(t)^{-1})^2 - P_T(t)^2\|$$

$$\leq \|P_T(t) - W_T(t)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W_T(t)^{-1}\| + \|W_T(t)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W_T(t)^{-1} - P_T(t)\|\|P_T(t)\|$$

$$+\|W_T(t)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W_T(t)^{-1}\|\|W_T(t)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W_T(t)^{-1} - P_T(t)\|$$

$$\leq 5 \times 2000\epsilon = 10000\epsilon.$$

The following lemma explains why we construct $P_T(t)$ in Equation (3.2).

PROPOSITION 3.3. *Let $P_T(t)$ be as in Equation (3.2), then $P_T(t)$ preserves the decomposition*

$$L^2(X, S) = L^2(X_{\leq 300T\delta}, S) \oplus L^2(X_{\geq 300T\delta}, S).$$

Moreover, $P_T(t)$ has the form

$$P_T(t) = \begin{bmatrix} P'_T(t) & 0 \\ 0 & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

with respect to the decomposition

$$L^2(X, S) = L^2(X_{\leq 300T\delta}, S) \oplus L^2(X_{\geq 300T\delta}, S).$$

Proof. The lemma is proved by direct computation.

For any $f \in L^2(X_{\geq 300T\delta}, S)$, by definitions of Z_1 and Z_2, we have

$$Z_1^2(t)f = 0,$$
$$(2 - U_T(t)U_T^*(t))U_T(t)Z_2(t)f = 0,$$
$$U_T^*(t)Z_1(t)f = 0,$$
$$Z_2^2(t)f = 0.$$

On the other hand, for any $f \in L^2(X_{\leq 300T\delta}, S)$, we have

$$Z_1(t)f \in L^2(X_{\leq 300T\delta}, S),$$
$$Z_1^2(t)f \in L^2(X_{\leq 300T\delta}, S),$$
$$Z_2(t)f \in L^2(X_{\leq 300T\delta}, S),$$
$$Z_2^2(t)f \in L^2(X_{\leq 300T\delta}, S).$$

For any t, by definition of $U_T(t)$, we know that the propagation of $U_T(t)$ and $U_T^*(t)$ is less than $T\delta$. Thus one can see

$$(2 - U_T(t)U_T^*(t))U_T(t)Z_2(t)f \in L^2(X_{\leq 300T\delta}, S),$$
$$U_T^*(t)Z_1(t)f \in L^2(X_{\leq 300T\delta}, S).$$
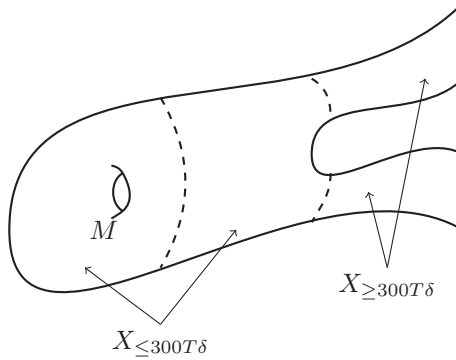
This completes the proof. □

FIG. 2. *For all $t \in [0, \infty)$, the operator $P_{X,M,T}(t)$ is supported on $X_{\leq 300T\delta}$.*

Since $\|P_T'(t)^2 - P_T'(t)\| \leq 10000\epsilon$, the spectrum of $P_T'(t)$ lies in

$$O(0, 20000\epsilon) \cup O(1, 20000\epsilon).$$

Note that 20000ϵ is far less than $\frac{1}{4}$, which implies that

$$O(0, 20000\epsilon) \cap O(1, 20000\epsilon) = \emptyset.$$

Thus

$$P_T'(t) - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

defines a class in $K_0(C^*(X_{\leq 300T\delta})^G)$. In fact, set $\mathcal{H} \equiv 0$ on $O(0, 20000\epsilon)$ and $\mathcal{H} \equiv 1$ on $O(1, 20000\epsilon)$, then

$$\mathcal{H}(P_T')(t) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\mathcal{H}(z)}{z - P_T'(t)} dz,$$

where $\mathcal{C}$ is a contour surrounding the spectrum of $P_T'(t)$ in $O(0, 20000\epsilon) \cup O(1, 20000\epsilon)$, defines a genuine idempotent.

DEFINITION 3.4. By Proposition 3.3, for any $t \in [0, \infty)$,

$$P_T'(t) - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

represents an element in $K_0(C^*(X_{\leq 300T\delta})^G)$, which is denoted by $\mathrm{ind}_{X,M,T}(D)$. (see Figure 2).

When M/G is compact, $\mathrm{ind}_{X,M,T}(D)$ defines an element in $K_0(C_r^*(G))$ and coincides with the higher index defined in Definition 3.1.

In addition, it is straightforward to see that

$$\|P_T'(t) - \mathcal{H}(P_T')(t)\| \leq 20000\epsilon.$$

Then we have the following estimate which is useful in the next subsection:

$$\|e^{2\pi i P'_T(t)} - e^{2\pi i \mathcal{H}(P'_T)(t)}\|$$

$$= \|\frac{1}{2\pi i}\int_{\|z\|=4}\frac{e^{2\pi i z}}{z - P'_T(t)}dz - \frac{1}{2\pi i}\int_{\|z\|=2}\frac{e^{2\pi i z}}{z - \mathcal{H}(P'_T)(t)}dz\|$$

$$\leq 20000\epsilon \times \frac{4}{2\pi}\int_{\|z\|=4} 1 dz$$

$$= 2^5 \times 10^5\epsilon \leq 500000\epsilon.$$

3.3. Higher rho invariant at infinity. In this subsection, we define the higher rho invariant at infinity. This invariant gives a K-theoretic demonstration of the nonlocality of the higher index of the Dirac operator on noncompact spin manifold endowed with a metric admitting uniform positive scalar curvature at infinity.

The following theorem is necessary in order to introduce the definition of the higher rho invariant at infinity.

THEOREM 3.5. *Let* $\mathrm{ind}_{X,M,T}(D)$ *be as in Definition 3.4, then* $\partial(\mathrm{ind}_{X,M,T}(D))$ *represents an element in* $K_1(C^*_{L,0}(X_{[90T\delta,300T\delta]})^G)$, *where* ∂ *is the connecting map in the following exact sequence*

$$\begin{array}{ccc} K_0(C^*_{L,0}(X_{\leq 300T\delta})^G) \longrightarrow K_0(C^*_L(X_{\leq 300T\delta})^G) \longrightarrow K_0(C^*(X_{\leq 300T\delta})^G) \\ \uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ K_1(C^*(X_{\leq 300T\delta})^G) \longleftarrow K_1(C^*_L(X_{\leq 300T\delta})^G) \longleftarrow K_1(C^*_{L,0}(X_{\leq 300T\delta})^G) \end{array}$$

Proof. Fix $t = 0$. To compute $\partial(\mathrm{ind}_{X,M,T}(D))$, one needs to lift $P_{X,M,T}(0)$ to $C^*_L(X_{\leq 300T\delta})^G$ first.

Recall that N is an integer sufficiently large such that for any $x \in [-1000, 1000]$,

$$|\sum_{n=1}^{N}\frac{(2\pi i)^n}{n!}| \leq \frac{\epsilon}{1000000}, \quad |\sum_{n=N}^{\infty}\frac{(2\pi i x)^n}{n!}| \leq \frac{\epsilon}{1000000}.$$

and r is a positive number such that

$$Nr \leq \frac{\delta}{100}. \tag{3.3}$$

Thus for any $x \in [-1000, 1000]$, we have

$$|1 + \sum_{n=1}^{N}\frac{(2\pi i x)^n}{n!} - e^{2\pi i x}| = |\sum_{n=N}^{\infty}\frac{(2\pi i x)^n}{n!}| < \frac{\epsilon}{2}.$$

Furthermore, there is

$$|1 + \sum_{n=1}^{N}\frac{(2\pi i x)^n}{n!} - e^{2\pi i x} - \sum_{n=1}^{N}\frac{(2\pi i)^n}{n!}x| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

However, we have

$$\sum_{n=1}^{N}\frac{(2\pi i x)^n}{n!} - \sum_{n=1}^{N}\frac{(2\pi i)^n}{n!}x = \sum_{n=2}^{N}\left(\sum_{j=0}^{n-2}\frac{(2\pi i)^n}{n!}x^j\right)(x^2 - x).$$

These above equalities imply that

$$\left|1 + \sum_{n=2}^{N}\left(\sum_{j=0}^{n-2}\frac{(2\pi i)^n}{n!}x^j\right)(x^2 - x) - e^{2\pi i x}\right| \le \epsilon.$$

For each $t \in [0, \infty)$, consider a G-invariant partition of unity $\{\zeta_t, \theta_t\}$ on $X_{\le 300T\delta}$ satisfying

1) $\zeta_t + \theta_t = 1$.
2) for $n \le t < n+1$, $\zeta_t \equiv 1$ on $X_{\le 100T\delta - r/2^{n-1}}$ and $\zeta_t \equiv 0$ on $X_{\ge 100T\delta - r/2^n}$.

Let $\xi \in C^\infty(-\infty, \infty)$ be a decreasing function such that $\xi|_{[-\infty,1]} \equiv 1$ and $\xi|_{[2,\infty]} \equiv 0$.

By definition, one can verify that the path $t \mapsto P'_{X,M,T}(t)$ forms a lifting of $P_{X,M,T}(0)$ in $C^*_L(X_{\le 300T\delta})^G$, where $P'_{X,M,T}(t)$ is defined as

$$\begin{cases} (1-t)P_{X,M,T}(0) + t(\sqrt{\zeta_0}P_{X,M,T}(0)\sqrt{\zeta_0} + \sqrt{\theta_0}P_{X,M,T}(0)\sqrt{\theta_0}), & t \in [0,1], \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \sqrt{\zeta_{t-1}}\left(P_{X,M,T}(t-1) - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\sqrt{\zeta_{t-1}} \\ \quad + \xi(t)\sqrt{\theta_{t-1}}\left(P_{X,M,T}(t-1) - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\sqrt{\theta_{t-1}} & , \quad t \in [1,\infty). \end{cases}$$

Then we have

$$\partial(\mathrm{ind}_{X,M,T}(D)) = [e^{2\pi i P'_{X,M,T}(\cdot)}].$$

Define a path of invertible operators $u_T(t)$ as

$$1 + \sum_{n=2}^{N}\left(\sum_{j=0}^{n-2}\frac{(2\pi i)^n}{n!}(P'_{X,M,T}(t))^j\right)\left((P'_{X,M,T}(t))^2 - (P'_{X,M,T}(t))\right).$$

It is straight-forward to verify that

$$\|u_T(t) - e^{2\pi i P'_{X,M,T}(t)}\| \le \epsilon.$$

Thus $u_T(t)$ is invertible for any t and the path

$$u'_T(t) = \begin{cases} (1-t) + tu_T(0) & t \in [0,1] \\ u_T(t-1) & t \in [1,\infty) \end{cases}$$

represents the same element in $K_1(C^*_{L,0}(X)^G)$ as the path $e^{2\pi i P'_{X,M,T}(\cdot)}$. In other words, we have

$$\partial(\mathrm{ind}_{X,M,T}(D)) = [u'_T(t)].$$

Note that the propagation of $u'_T(\cdot)$ restricted to $X_{\le 98T\delta}$ is less than $\frac{\delta}{100}$ by the choice of r in line (3.3). For any $h \in L^2(X_{\le 80T\delta}, S)$ and any $t \in [0, \infty)$, we have both $(u'_T(t))^2 h \in L^2(X_{\le 98T\delta}, S)$. Recall that

$$\mathcal{H}(P'_T)(t) = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\mathcal{H}(z)}{z - P'_T(t)}dz$$

is a path of genuine idempotent obtained from $P'_T(t)$ as above. Thus $\forall h \in L^2(X_{\leq 80T\delta}, S)$, we have

$$\|u'_T(t)h - h\|$$
$$= \|u'_T(t)h - e^{2\pi i \mathcal{H}(P'_{X,M,T})(t)}h\|$$
$$\leq \|u'_T(t)h - e^{2\pi i P'_{X,M,T}(t)}h\| + \|e^{2\pi i P'_{X,M,T}(t)}h - e^{2\pi i \mathcal{H}(P'_{X,M,T})(t)}h\|$$
$$\leq \epsilon + 500000\epsilon \leq 1000000\epsilon.$$

By the same reason, for any $h \in L^2(X_{[80T\delta, 95T\delta]}, S)$, we have

$$\|u'_T(t)h - h\| = \|u'_T(t)h - e^{2\pi i P'_{X,M,T}(t)}h\| \leq 1000000\epsilon.$$

Finally, for all $h \in L^2(X_{[95T\delta, 300T\delta]}, S)$, $u'_T(t)h$ belongs to $L^2(X_{[90T\delta, 300T\delta]}, S)$. The above computation shows that with respect to the decomposition

$$L^2(X_{\leq 300T\delta}, S) = L^2(X_{\leq 90T\delta}, S) \oplus L^2(X_{[90T\delta, 300T\delta]}, S),$$

the path $u'_T(t)$ can be perturbed to the following path of invertible operators

$$\begin{bmatrix} I & 0 \\ 0 & u_{X,M,T}(t) \end{bmatrix},$$

such that

$$\left\| u'_T(t) - \begin{bmatrix} I & 0 \\ 0 & u_{X,M,T}(t) \end{bmatrix} \right\| \leq 4 \times 1000000\epsilon.$$

Hence we have

$$\partial(\mathrm{ind}_{X,M,T}(D)) = [u_{X,M,T}(\cdot)], \tag{3.4}$$

where the path $u_{X,M,T}(t)$ is supported on $X_{[90T\delta, 300T\delta]}$ (see Figure 3). The proof is then complete. $\square$
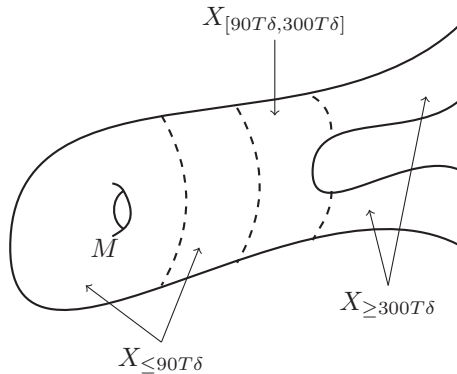

FIG. 3. *for all $t \in [0, \infty)$, operator $u_{X,M,T}(t)$ is supported on $X_{[90T\delta, 300T\delta]}$.*

DEFINITION 3.6.   For any T sufficiently large, the element in $K_1(C^*_{L,0}(X_{[90T\delta,300T\delta]})^G)$ represented by $u_{X,M,T}(\cdot)$ (defined in Equation (3.4)) is the higher rho invariant at infinity, denoted by $\rho_{X,M,T}(D)$.

The reason why we call it the higher rho invariant at infinity, is that we can push its support as far as possible from any base point of X.

Certainly, the obstruction algebra $C^*_{L,0}(X_{[90T\delta,300T\delta]})^G$ and the K-group $K_1(C^*_{L,0}(X_{[90T\delta,300T\delta]})^G)$ depend on T, so does the higher rho invariant $\rho_{X,M,T}(D)$. However, the dependence is actually not a problem for Definition 3.6. In fact, consider $T < T'$, under the following two embeddings

$$\iota : K_1(C^*_{L,0}(X_{[90T\delta,300T\delta]})^G) \to K_1(C^*_{L,0}(X_{\leq 300T'\delta})^G)$$
$$\iota' : K_1(C^*_{L,0}(X_{[90T'\delta,300T'\delta]})^G) \to K_1(C^*_{L,0}(X_{\leq 300T'\delta})^G),$$

it follows directly from the definition that

$$\iota(\rho_{X,M,T}(D)) = \iota'(\rho_{X,M,T'}(D)).$$

Different from the classical higher rho invariant, the higher rho invariant at infinity is an obstruction of a manifold having a metric with uniform positive scalar curvature. More precisely, if X bears a metric with uniform positive scalar curvature, then $\rho_{X,M,T}(D)$ is trivial. Moreover, the higher rho invariant at infinity is designed to differentiate uniform positive scalar curvature metrics at infinity. We will discuss this point in a forthcoming paper.

3.4. Delocalized eta invariant at infinity. In this subsection, we define the delocalized eta invariant at infinity, which measures the nonlocality of higher index of the Dirac operators on noncompact spin manifolds endowed with a metric admitting positive scalar curvature at infinity numerically.

Consider the connecting map

$$\partial : K_0(C^*_r(G)) \to K_1(C^*_{L,0}(X_{[90T\delta,300T\delta]})^G)$$

in the K-theory six-term exact sequence associated to the short exact sequence

$$0 \to C^*_{L,0}(X_{[90T\delta,300T\delta]})^G \to C^*_L(X_{[90T\delta,300T\delta]})^G \to C^*(X_{[90T\delta,300T\delta]})^G \to 0.$$

Here $K_0(C^*_r(G))$ is identified with $K_0(C^*(X_{[90T\delta,300T\delta]})^G)$. The following lemma is similar to Lemma 3.9 of [31], which shows that the delocalized trace on $K_0(C^*_r(G))$, defined in line (2.3), and the determinant map, defined in line (2.4), are compatible with the K-theory boundary map. Note that although $X_{[90T\delta,300T\delta]}$ is not a complete manifold, it can be embedded into a complete manifold with proper, free and cocompact G-action. For example, let $N \subset X$ be a manifold with boundary which contains $X_{\leq 400T\delta}$, then a perturbation around ∂N makes $N \cup N^{op}$ into a complete smooth manifold, in which one can embed $X_{[90T\delta,300T\delta]}$. Thus for any nontrivial element $g \in G$ whose conjucgay class has polynomial growth, the determinant map

$$\tau_g : K_1(C^*_{L,0}(X_{[90T\delta,300T\delta]})^G) \to \mathbb{C}$$

is still well defined.

LEMMA 3.7. *Let $g \in G$ be a nontrivial element whose conjugacy class has polynomial growth, then the following diagram commutes:*

$$
\begin{array}{ccc}
K_0(C_r^*(G)) & \xrightarrow{\ \partial\ } & K_1(C_{L,0}^*(X_{[90T\delta,300T\delta]})^G) \\
{\scriptstyle \mathrm{tr}_g} \downarrow & & \downarrow {\scriptstyle -\tau_g} \\
\mathbb{C} & \xrightarrow{\hspace{4cm}} & \mathbb{C}.
\end{array}
$$

Proof. For any element $[p] \in K_0(C_r^*(G)) \cong K_0(C^*(X_{[90T\delta,300T\delta]})^G)$, $\partial[p] = [u(\cdot)]$ is defined by

$$u(t) = e^{2\pi i a(t)}, t \in [0, \infty),$$

where $a(t)$ is a lift of p in $C_L^*(X_{[90T\delta,300T\delta]})^G$ with $a(0) = p$. By definition, we have

$$\tau_g(u) = \frac{1}{2\pi i} \int_0^\infty \mathrm{tr}_g(u(t)^{-1} u'(t)) dt = \int_0^\infty \mathrm{tr}_g(a'(t)) dt = \lim_{t \to \infty} \mathrm{tr}_g(a(t)) - \mathrm{tr}_g(a(0)).$$

Since g is not the identity element, $\mathrm{tr}_g(a(t)) = 0$ when t is sufficiently large such that the propagation of $a(t)$ is small enough. It follows that

$$\tau_g(\partial[p]) = \tau_g([u]) = -\mathrm{tr}_g([p]).$$

The lemma is then proved. □

By definition, Lemma 3.7 implies that

$$\mathrm{tr}_g(\mathrm{ind}_G(D)) = -\tau_g(\rho_{X,M,T}(D)).$$

Inspired by Theorem 4.3 of [31], we define the delocalized higher eta invariant at infinity as follows.

DEFINITION 3.8. Let $g \in G$ be a nontrivial element. The delocalized eta invariant at $g \in G$ at infinity, denoted by $\eta_{g,\infty}(D)$, is defined to be

$$\eta_{g,\infty}(D) := -2\mathrm{tr}_g(\mathrm{ind}_G(D)) = 2\tau_g(\rho_{X,M,T}(D)).$$

3.5. A formula for the higher index. In this subsection, we establish a formula for the higher index of the Dirac operator we considered above for the purpose of computing the delocalized eta invariant defined in Definition 3.8. In Subsections 3.5 and 3.6, we assume in addition that X has bounded geometry, and consider only finitely generated discrete group. This is due to a special choice of parametrix of the Dirac operator.

Compare Subsection 3.5 and 3.6 with [13].

Consider the Dirac operator on X (even dimensional) which is odd with respect to the grading $L^2(X, S^+) \oplus L^2(X, S^-)$:

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix} \qquad (D^+)^* = D^-.$$

Set $Q(t) := \frac{1 - e^{-t D^- D^+}}{D^- D^+} D^-, t \geq 0$. Operators $Q(t)$ are well defined since X is a complete manifold, $D^- D^+$ admits a unique self-adjoint extension and the functional calculus can be applied. Recall D_c is the restriction of D to $X \backslash M$. Choose

a parametrix for D_c^+ to be its inverse $Q_c := (D_c^- D_c^+)^{-1} D_c^-$. Operators Q_c are well defined since $D_c^- D_c^+$ is bounded below by a positive number. The functional calculus is taken on the self-adjoint extension of $D_c^- D_c^+$ by extending it to a complete manifold containing $X \backslash M$ and having a uniform positive scalar curvature metric.

Without loss of generality, we assume that M is a manifold with boundary ∂M, and assume that

$$(-1, 1) \times \partial M = \{x \in X, \mathrm{d}(x, \partial M) \leq 1\}$$

is a tubular neighbourhood of ∂M in X. Let ψ_1 be a smooth function from X to $[0, 1]$ such that $\psi_1 \equiv 1$ on $X_{[0, \frac{1}{4}]}$ and $\psi_1 \equiv 0$ on $X_{\geq \frac{3}{4}}$. Set $\psi_2 = 1 - \psi_1$. Let φ_1, φ_2 be smooth functions from X to $[0, 1]$ such that $\varphi_1 \equiv 1$ on $X_{[0, \frac{7}{8}]}$ and $\varphi_1 \equiv 0$ on $X_{\geq 1}$, while $\varphi_2 \equiv 0$ on $X_{[0, \frac{1}{16}]}$ and $\varphi_2 \equiv 1$ on $X_{\geq \frac{1}{8}}$. Note that $\varphi_j \psi_j = \psi_j$ and $[D, \varphi_j]\psi_j = 0$, $j = 1, 2$.
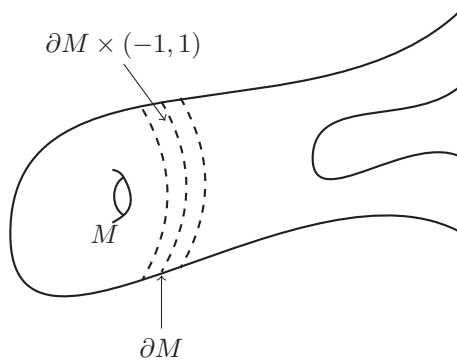

FIG. 4. *Tubular neighbourhood of ∂M.*

We construct parametrices for D^+ as follows,

$$R = \varphi_1 Q \psi_1 + \varphi_2 Q_c \psi_2,$$
$$R' = \psi_1 Q \varphi_1 + \psi_2 Q_c \varphi_2.$$

Denote

$$S_0 = 1 - R D^+,$$
$$S_1 = 1 - D^+ R,$$
$$S_0' = 1 - R' D^+.$$

Denote by $C^*(X, M)^G$ the equivariant Roe algebra supported near M.

The proof for the following Lemma goes verbatim as the one for Lemma 5.2 in [12] and Lemma 3.3 in [13].

LEMMA 3.9. *The operators S_0, S_1, S_0' have smooth kernels and*

$$S_0, S_1, S_0' \in C^*(X, M)^G.$$

Furthermore, by the same arguments in [12] and [13], one can show that S_0^2, S_1^2, $(S_0')^2$ and S_0, S_1, S_0' are all g-trace classes with smooth Schwartz kernel when the

conjugacy class of g has polynomial growth, and when the manifold X is assumed to have bounded geometry and when the lower bound of D_c^2 is large enough. Note that we have assumed that the positive scalar curvature outside compact subset M is uniformly bounded below.

The following proposition has been proven by Hochs, Wang, and Wang in [13, Lemma 5.2]. For the sake of the completion, we still sketch the idea of the proof below.

PROPOSITION 3.10. *The higher index for D is given by*

$$\text{ind}_G(D) = \begin{bmatrix} S_0^2 & S_0(S_0+1)R \\ S_1 D^+ & 1 - S_1^2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Proof. This can be seen by choosing the real function $b : (-\infty, \infty) \to [-1, 1]$ in Subsection 3.1 to be

$$b(x) = \begin{cases} \sqrt{1 - f^2(x)} & x \geq 0 \\ -\sqrt{1 - f^2(x)} & x < 0 \end{cases},$$

where $f : (-\infty, \infty) \to [-1, 1]$ is a real function defined as

$$f(x) = \begin{cases} 0 & x \geq \frac{\sqrt{h_0}}{2} \\ 1 - \frac{2x}{\sqrt{h_0}} & x \in [0, \frac{\sqrt{h_0}}{2}] \\ 1 + \frac{2x}{\sqrt{h_0}} & x \in [-\frac{\sqrt{h_0}}{2}, 0] \\ 0 & x < -\frac{\sqrt{h_0}}{2} \end{cases}.$$

$\square$

3.6. Formula for the delocalized eta-invariant at infinity. The main goal of this subsection is to prove a formula for the delocalized eta invariant at infinity in light of the techniques developed in [12], along with which we establish an Atiyah-Patodi-Singer type index theorem for Dirac operator on manifold with a metric with uniform positive scalar curvature at infinity. We mention that with the technique developed in [27], one can generalize the main result of this subsection to the case of proper actions.

THEOREM 3.11. *Let g be an element of G whose conjugacy class has polynomial growth. Then*

$$\text{tr}_g(\text{ind}_G(D)) = \int_{M^g} I(g) - \lim_{t \to 0} \int_t^\infty \text{tr}_g(e^{-sD_c^- D_c^+} D_c^- [D^+, \psi_2]) ds, \quad (3.5)$$

where M^g is the fixed point submanifold of g, and $I(g)$ is the integrand as in the Atiyah-Segal-Singer fixed point theorem. In particular, if $g \neq e$, we have

$$\frac{1}{2}\eta_{g,\infty}(D) = -\text{tr}_g(\text{ind}_G(D)) = \lim_{t \to 0} \int_t^\infty \text{tr}_g(e^{-sD_c^- D_c^+} D_c^- [D^+, \psi_2]) ds. \quad (3.6)$$

The second integral on the right hand side of (3.5) and the integral on the right hand side of (3.6) are independent from the choice of the cutoff function ψ_2.

REMARK. For technical reasons, when $g \neq e$, the lower bound of D_c^2 is assumed to be large enough so that S_i^2 is g-trace class [13]. This can be achieved by scaling the manifold globally.

We divide the proof of Theorem 3.11 into the following series of lemmas.

LEMMA 3.12. *We have*

$$S_0 = \phi_1(1 - QD^+)\psi_1 + \phi_1 Q[D^+, \psi_1] + \phi_2 Q_c[D^+, \psi_2], \tag{3.7}$$
$$S_0' = \psi_1(1 - QD^+)\phi_1 + \psi_1 Q[D^+, \phi_1] + \psi_2 Q_c[D^+, \phi_2], \tag{3.8}$$
$$S_1 = \phi_1(1 - D^+Q)\psi_1 - [D^+, \phi_1]Q\psi_1 - [D^+, \phi_2]Q_c\psi_2. \tag{3.9}$$

Proof. Direct computation. □

From now on, to emphasize the dependence of operators Q (resp. Q_c, S_0, S_0', S_1) on t, we write $Q(t)$ (resp. $Q_c(t)$, $S_0(t)$, $S_0'(t)$, $S_1(t)$) for Q (resp. Q_c, S_0, S_0', S_1).

LEMMA 3.13. *We have*

$$\mathrm{tr}_g(S_0^2) - \mathrm{tr}_g(S_1^2) = \mathrm{tr}_g(S_0) - \mathrm{tr}_g(S_1).$$

Proof. To begin with, note that

$$\mathrm{tr}_g(S_0) - \mathrm{tr}_g(S_0^2) = \mathrm{tr}_g(S_0 - S_0^2) = \mathrm{tr}_g(S_0 R D^+).$$

By definition, $S_0 R D^+ = R S_1 D^+$. Moreover, applying trace property of tr_g, we obtain

$$\mathrm{tr}_g(R S_1 D^+) = \mathrm{tr}_g(S_1 D^+ R) = \mathrm{tr}_g(S_1(1 - S_1)) = \mathrm{tr}_g(S_1) - \mathrm{tr}_g(S_1^2).$$

Hence the lemma follows. □

LEMMA 3.14.

$$\lim_{t \to 0^+} [\mathrm{tr}_g(S_0') - \mathrm{tr}_g(S_1)] = \int_{M^g} I(g)$$

Proof. By Lemma 3.12, one can see

$$\mathrm{tr}_g(S_0') = \mathrm{tr}_g(\psi_1(1 - QD^+)\phi_1 + \psi_1 Q[D^+, \phi_1] + \psi_2 Q_c[D^+, \phi_2]).$$

However, since $[D^+, \phi_1]\psi_1 = [D^+, \phi_2]\psi_2 = 0$,

$$\mathrm{tr}_g(S_0') = \mathrm{tr}_g(\psi_1(1 - QD^+)\phi_1) = \mathrm{tr}_g((1 - QD^+)\psi_1).$$

Similarly,

$$\mathrm{tr}_g(S_1) = \mathrm{tr}_g((1 - D^+Q)\psi_1).$$

Then a standard heat kernel argument shows that

$$\lim_{t \to 0^+} [\mathrm{tr}_g(S_0') - \mathrm{tr}_g(S_1)] = \int_{M^g} I(g).$$

This proves the Lemma. □

LEMMA 3.15.

$$\lim_{t \to 0^+} [\mathrm{tr}_g(S_0) - \mathrm{tr}_g(S_0')] = - \lim_{t \to 0^+} \int_t^\infty \mathrm{tr}_g(e^{-sD_c^- D_c^+} D_c^- [D^+, \psi_2]) ds$$

Proof. By Lemma 3.12, the following equality

$$\mathrm{tr}_g(S_0) - \mathrm{tr}_g(S_0') = \mathrm{tr}_g(\phi_1 Q[D^+, \psi_1]) + \mathrm{tr}_g(\phi_2 Q_c[D^+, \psi_2])$$

holds. Construct a new parametrix for D_c

$$Q_c' := \frac{1 - e^{-tD_c^- D_c^+}}{D_c^- D_c^+} D_c^-.$$

Choose $\phi \in C^\infty(X)$ such that $\phi = 1$ on the support of $[D^+, \psi_j], j = 1, 2$ and $\phi = 0$ on the support of $1 - \phi_j$. This implies that

$$\mathrm{tr}_g(\phi_1 Q[D^+, \psi_1] - \phi_2 Q_c'[D^+, \psi_1]) \tag{3.10}$$
$$= \mathrm{tr}_g(\phi(Q - Q_c')[D^+, \psi_1]) + \mathrm{tr}_g((1 - \phi)(\phi_1 Q - \phi_2 Q_c')[D^+, \psi_1]) \tag{3.11}$$
$$= \mathrm{tr}_g(\phi(Q - Q_c')[D^+, \psi_1]). \tag{3.12}$$

Hence we have

$$\mathrm{tr}_g(\phi(Q - Q_c')[D^+, \psi_1]) \tag{3.13}$$
$$= \mathrm{tr}_g(\phi(Q(t_0) - Q_c'(t_0))[D^+, \psi_1]) \tag{3.14}$$
$$= \mathrm{tr}_g(\phi \int_0^{t_0} e^{-sD^- D^+} D^- - e^{-sD_c^- D_c^+} D_c^- ds[D^+, \psi_1]) \tag{3.15}$$
$$= \mathrm{tr}_g(\int_0^{t_0} \phi(e^{-sD^- D^+} D^- - e^{-sD_c^- D_c^+} D_c^-)[D^+, \psi_1] ds). \tag{3.16}$$

However, by a finite propagation argument, one can show

$$\phi(e^{-sD^- D^+} D^- - e^{-sD_c^- D_c^+} D_c^-)[D^+, \psi_1] = 0$$

when s is sufficiently small. Thus

$$\mathrm{tr}_g(\phi_1 Q[D^+, \psi_1] - \phi_2 Q_c'[D^+, \psi_1]) = \mathrm{tr}_g(\phi(Q - Q_c')[D^+, \psi_1]) = 0.$$

On the other hand, we have

$$\lim_{t \to 0^+} [\mathrm{tr}_g(\phi_2 Q_c[D^+, \psi_2] - \phi_2 Q_c'[D^+, \psi_2])] \tag{3.17}$$
$$= \lim_{t \to 0^+} [\mathrm{tr}_g(\phi_2(Q_c - \frac{1 - e^{-tD_c^- D_c^+}}{D_c^- D_c^+} D_c^-)[D^+, \psi_2])] \tag{3.18}$$
$$= \lim_{t \to 0^+} [\mathrm{tr}_g(\phi_2(Q_c - \frac{1 - e^{-tD_c^- D_c^+}}{D_c^- D_c^+} D_c^- D_c^+ Q_c)[D^+, \psi_2])] \tag{3.19}$$
$$= \lim_{t \to 0^+} [\mathrm{tr}_g(\phi_2 e^{-tD_c^- D_c^+} Q_c[D^+, \psi_2])] \tag{3.20}$$
$$= -\lim_{t \to 0^+} [\mathrm{tr}_g(\phi_2 \int_t^\infty e^{-sD_c^- D_c^+} D_c^- dt[D^+, \psi_2])] \tag{3.21}$$
$$= -\lim_{t \to 0^+} \int_t^\infty \mathrm{tr}_g(e^{-sD_c^- D_c^+} D_c^-[D^+, \psi_2]) dt. \tag{3.22}$$

Note that $[D^+, \psi_1] + [D^+, \psi_2] = 0$, thus we have

$$\lim_{t \to 0^+} [\operatorname{tr}_g(S_0) - \operatorname{tr}_g(S_0')]$$

$$= \lim_{t \to 0^+} [\operatorname{tr}_g(\phi_1 Q[D^+, \psi_1]) + \operatorname{tr}_g(\phi_2 Q_c[D^+, \psi_2])]$$

$$= \lim_{t \to 0^+} [[\operatorname{tr}_g(\phi_1 Q[D^+, \psi_1]) - \operatorname{tr}_g(\phi_2 Q_c'[D^+, \psi_1])]$$

$$+ \lim_{t \to 0^+} [\operatorname{tr}_g(\phi_2 Q_c[D^+, \psi_2]) - \operatorname{tr}_g(\phi_2 Q_c'[D^+, \psi_2])]$$

$$= -\lim_{t \to 0^+} \int_t^\infty \operatorname{tr}_g(e^{-tD_c^- D_c^+} D_c^-[D^+, \psi_2]) dt.$$

The lemma is then proved. $\square$

The following lemma is a direct result of Proposition 4.1 and Proposition 5.1 of [13].

LEMMA 3.16. *By the isomorphism between $C_r^*(G) \otimes \mathcal{K}$ and $C^*(X, M)^G$, and basic index theory, we have*

$$\operatorname{tr}_g(\operatorname{ind}_G D) = \operatorname{tr}_g(S_0^2) - \operatorname{tr}_g(S_1^2).$$

Proof of Theorem 3.11. Combine Lemma 3.13, 3.14, 3.15 and 3.16, we obtain

$$\operatorname{tr}_g(\operatorname{ind}_G(D)) = \operatorname{tr}_g(S_0^2) - \operatorname{tr}_g(S_1^2) \tag{3.23}$$

$$= \operatorname{tr}_g(S_0) - \operatorname{tr}_g(S_1) \tag{3.24}$$

$$= \operatorname{tr}_g(S_0) - \operatorname{tr}_g(S_0') + \operatorname{tr}_g(S_0') - \operatorname{tr}_g(S_1) \tag{3.25}$$

$$= \int_{M^g} I(g) - \lim_{t \to 0} \int_t^\infty \operatorname{tr}_g(e^{-sD_c^- D_c^+} D_c^-[D^+, \psi_2]) dt. \tag{3.26}$$

The proof is completed. $\square$

4. Müller's Atiyah-Patodi-Singer type theorem. In this section, we apply the technique from Subsection 3.6 to develop a Atiyah-Patodi-Singer type theorem on manifold with corner of codimension 2. This subsection can be viewed as an L^2 generalization of Müller's extraordinary pioneer work in [16] and [17] in the particular case of manifolds with metric with uniform positive scalar curvature at infinity.

Let us first fix some notions. Let M be an even dimensional manifold with corner with $\partial M = \partial_1 M \cup \partial_2 M$, where $\partial_1 M$ and $\partial_2 M$ are manifolds with common boundary, i.e. $\partial\partial_1 M = \partial\partial_2 M = Y$. The metric of M is collared near ∂M and have positive scalar curvature on $\partial_1 M$ and $\partial_2 M$. Furthermore, the metric on $\partial_1 M$ and $\partial_2 M$ is collared near Y. Let G be a discrete group acting on M properly, freely and cocompactly by isometries.

Let $Z_i, i = 1, 2$ be $\partial_i M \cup (Y \times \mathbb{R}_+)$ and $Z_{i, \leq T}, i = 1, 2$ be $\partial_i M \cup (Y \times [0, T])$. Denote $C_i, i = 1, 2$ as $\partial_i M \times \mathbb{R}_+, i = 1, 2$ and $C_{i, \leq T}, i = 1, 2$ as $\partial_i M \times [0, T], i = 1, 2$. Let C_0 be $Y \times \mathbb{R}_+^2$ and $C_{0, \leq T}$ be $Y \times [0, T] \times [0, T]$. Let $M_i, i = 1, 2$ be $M \cup C_i, i = 1, 2$ and $M_{i, \leq T}, i = 1, 2$ be $M \cup C_{i, \leq T}, i = 1, 2$.

Set a complete manifold

$$X = M_1 \cup (Z_2 \times \mathbb{R}_+) = M_2 \cup (Z_1 \times \mathbb{R}_+) \tag{4.1}$$

and

$$X_{\leq T} = M_{1, \leq T} \cup Z_{1, \leq T} \times [0, T] = M_{2, \leq T} \cup Z_{2, \leq T} \times [0, T]. \tag{4.2}$$
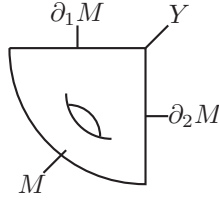
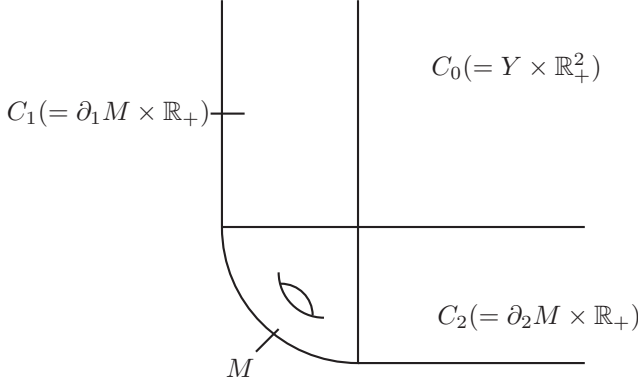FIG. 5. *Manifold with corner of codimension 2.*

FIG. 6. *Complete manifold X.*

We assume in addition G is a finitely generated discrete group acting on M, properly, cocompactly, and freely by isometries.

Let σ be the grading operator on the spinor bundle of X.

Recall that $Z_1 \times \mathbb{R}_+ = C_1 \cup C_0$ and $Z_2 \times \mathbb{R}_+ = C_2 \cup C_0$ (see Figures 6 and 7).
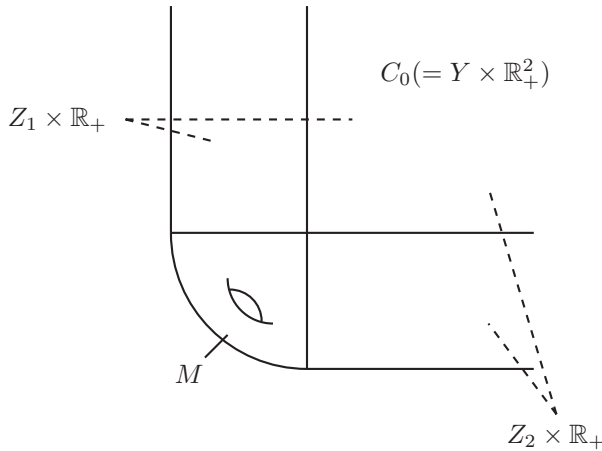

FIG. 7. *$Z_1 \times \mathbb{R}_+$ and $Z_2 \times \mathbb{R}_+$.*

Let $\psi_1 : \mathbb{R}_+ \to [0, 1]$ be a smooth function such that $\psi_1 \equiv 1$ on $(0, \frac{1}{4})$ and $\psi_1 \equiv 0$ on $(\frac{3}{4}, \infty)$. Set $\psi_2 = 1 - \psi_1$. Let $\varphi_1, \ \varphi_2 : \mathbb{R}_+ \to [0, 1]$ be smooth functions such that $\varphi_1 \equiv 1$ on $(0, \frac{7}{8})$ and $\varphi_1 \equiv 0$ on $(1, \infty)$, while $\varphi_2 \equiv 0$ on $(0, \frac{1}{16})$ and $\varphi_2 \equiv 1$ on $(\frac{1}{8}, \infty)$.

Note that $\varphi_j\psi_j = \psi_j$ and $\varphi_j'\psi_j = 0$, $j = 1, 2$. Denote $\psi_{i,j}$ by $\psi_j \otimes \chi_{Z_i}$, and $\varphi_{i,j}$ by $\varphi_j \otimes \chi_{Z_i}$ for $i, j = 1, 2$, where χ_{Z_i} is the characteristic function of Z_i. Note that the range of $\varphi_{i,j}$ is contained in $[0, 1]$.

Let D_X be the Dirac operator on X, $D_{Z_i \times \mathbb{R}_+}$ and D_0 be the Dirac operator on $Z_i \times \mathbb{R}_+$ and C_0 respectively. Let Q_i be defined using $(D^-_{Z_i \times \mathbb{R}_+} D^+_{Z_i \times \mathbb{R}_+})^{-1} D^-_{Z_i \times \mathbb{R}_+}$, the inverse of $D^+_{Z_i \times \mathbb{R}_+}$, and Q_0 be defined using the inverse of D_0^+. The existence of Q_0, Q_1, Q_2 are ensured by the uniform positive scalar curvature on all boundary pieces of M. For any $t \in [0, \infty)$, set $Q(t)$ as

$$\frac{1 - e^{-tD_X^- D_X^+}}{D_X^- D_X^+} D_X^-,$$

$Q_i'(t)$ as

$$\frac{1 - e^{-tD^-_{Z_i \times \mathbb{R}_+} D^+_{Z_i \times \mathbb{R}_+}}}{D^-_{Z_i \times \mathbb{R}_+} D^+_{Z_i \times \mathbb{R}_+}} D^-_{Z_i \times \mathbb{R}_+},$$

and $Q_0'(t)$ as

$$\frac{1 - e^{-tD_0^- D_0^+}}{D_0^- D_0^+} D_0^-.$$

Similarly as in Section 3.5, we construct two paramatices as following

$$R(t) = \varphi_{1,1}\varphi_{2,1}Q(t)\psi_{1,1}\psi_{2,1} + \varphi_{1,2}Q_1(t)\psi_{1,2} + \varphi_{2,2}Q_2(t)\psi_{2,2} - \varphi_{1,2}\varphi_{2,2}Q_0(t)\psi_{1,2}\psi_{2,2},$$
$$R'(t) = \psi_{1,1}\psi_{2,1}Q(t)\varphi_{1,1}\varphi_{2,1} + \psi_{1,2}Q_1(t)\varphi_{1,2} + \psi_{2,2}Q_2(t)\varphi_{2,2} - \psi_{1,2}\psi_{2,2}Q_0(t)\varphi_{1,2}\varphi_{2,2}.$$

Write

$$S_0(t) = 1 - R(t)D_X^+,$$
$$S_0'(t) = 1 - R'(t)D_X^+,$$
$$S_1(t) = 1 - D_X^+ R(t).$$

From

$$\psi_{1,1} + \psi_{1,2} = 1 \text{ and } \psi_{2,1} + \psi_{2,2} = 1,$$

we obtain

$$\psi_{1,1}\psi_{2,1} + \psi_{1,2} + \psi_{2,2} - \psi_{1,2}\psi_{2,2} = 1.$$

Replacing 1 in S_0, S_0', S_1 by $\psi_{1,1}\psi_{2,1} + \psi_{1,2} + \psi_{2,2} - \psi_{1,2}\psi_{2,2}$, and direct computation shows that

$$\begin{aligned}
S_0(t) &= \varphi_{1,1}\varphi_{2,1}e^{-tD_X^- D_X^+}\psi_{1,1}\psi_{2,1} + \varphi_{1,1}\varphi_{2,1}Q(t)[D_X^+, \psi_{1,1}\psi_{2,1}] \\
&\quad + \varphi_{1,2}Q_1(t)[D_X^+, \psi_{1,2}] + \varphi_{2,2}Q_2(t)[D_X^+, \psi_{2,2}] - \varphi_{1,2}\varphi_{2,2}Q_0(t)[D_X^+, \psi_{1,2}\psi_{2,2}], \\
S_0'(t) &= \psi_{1,1}\psi_{2,1}e^{-tD_X^- D_X^+}\varphi_{1,1}\varphi_{2,1} + \psi_{1,1}\psi_{2,1}Q(t)[D_X^+, \varphi_{1,1}\varphi_{2,1}] \\
&\quad + \psi_{1,2}Q_1(t)[D_X^+, \varphi_{1,2}] + \psi_{2,2}Q_2(t)[D_X^+, \varphi_{2,2}] - \psi_{1,2}\psi_{2,2}Q_0(t)[D_X^+, \varphi_{1,2}\varphi_{2,2}], \\
S_1(t) &= \varphi_{1,1}\varphi_{2,1}e^{-tD_X^- D_X^+}\psi_{1,1}\psi_{2,1} + [D_X^+, \varphi_{1,1}\varphi_{2,1}]Q(t)\psi_{1,1}\psi_{2,1} \\
&\quad + [D_X^+, \varphi_{1,2}]Q_1(t)\psi_{1,2} + [D_X^+, \varphi_{2,2}]Q_2(t)\psi_{2,2} - [D_X^+, \varphi_{1,2}\phi_{2,2})]Q_0(t)\psi_{1,2}\psi_{2,2}.
\end{aligned}$$

Note that S_0, S_0', S_1 and their squares are e-trace class operators with smooth Schwartz Kernels. Then taking the trace of the K-theoretic index, we obtain

$$\mathrm{tr}_e(\mathrm{ind}_G(D_M)) = \mathrm{tr}_e(S_0^2(t)) - \mathrm{tr}_e(S_1^2(t)). \tag{4.3}$$

Exactly the same argument of Lemma 3.13 shows that

$$\mathrm{tr}_e(S_0^2(t)) - \mathrm{tr}_e(S_1^2(t)) = \mathrm{tr}_e(S_0(t)) - \mathrm{tr}_e(S_1(t)). \tag{4.4}$$

For the same reason as in the proof of Lemma 3.14, we have

$$\lim_{t\to0^+} (\mathrm{tr}_e(S_0'(t)) - \mathrm{tr}_e(S_1(t))) = \int_M \hat{A}(M).$$

Similarly as in the proof of Lemma 3.15, one can also obtain

$$\lim_{t\to0^+} \mathrm{tr}_e(S_0(t) - S_0'(t))$$
$$= \lim_{t\to0} \mathrm{tr}_e((Q_1(t) - Q_1'(t))\sigma\psi_{1,2}') + \lim_{t\to0} \mathrm{tr}_e((Q_2(t) - Q_2'(t))\sigma\psi_{2,2}')$$
$$- \lim_{t\to0} \mathrm{tr}_e(\varphi_{1,2}\varphi_{2,2}(Q_0(t) - Q_0'(t))[D_X^+, \psi_{1,2}\psi_{2,2}])$$
$$= \lim_{t\to0} \mathrm{tr}_e((Q_1(t) - Q_1'(t))\sigma\psi_{1,2}') + \lim_{t\to0} \mathrm{tr}_e((Q_2(t) - Q_2'(t))\sigma\psi_{2,2}')$$
$$- \lim_{t\to0} \mathrm{tr}_e((Q_0(t) - Q_0'(t))\sigma\psi_{1,2}'\psi_{2,2}) - \lim_{t\to0} \mathrm{tr}_e((Q_0(t) - Q_0'(t))\sigma\psi_{1,2}\psi_{2,2}').$$

Let $\chi_{\psi_{i,2}=1}, i = 1, 2$ be the characteristic function on the set $\{x \in X | \psi_{i,2}(x) = 1\}$, and $\chi_{Z_i \times \mathbb{R}_+}, i = 1, 2$ be the characteristic function on $Z_i \times \mathbb{R}_+$. Then we have

$$\lim_{t\to0^+} \mathrm{tr}_e(S_0(t) - S_0'(t))$$
$$= \lim_{t\to0} \mathrm{tr}_e((Q_1(t) - Q_1'(t))\sigma\psi_{1,2}' - (Q_0(t) - Q_0'(t))\sigma\psi_{1,2}'\chi_{\psi_{2,2}=1})$$
$$+ \lim_{t\to0} \mathrm{tr}_e((Q_2(t) - Q_2'(t))\sigma\psi_{2,2}' - (Q_0(t) - Q_0'(t))\sigma\psi_{2,2}'\chi_{\psi_{1,2}=1})$$
$$- \lim_{t\to0} \mathrm{tr}_e((Q_0(t) - Q_0'(t))\sigma(\psi_{1,2}'\psi_{2,2}(1 - \chi_{\psi_{2,2}=1}) + \psi_{1,2}\psi_{2,2}'(1 - \chi_{\psi_{1,2}=1})))$$
$$= \lim_{t\to0} \mathrm{tr}_e((Q_1(t) - Q_1'(t))\sigma\psi_{1,2}' - (Q_0(t) - Q_0'(t))\sigma\psi_{1,2}'\chi_{Z_2 \times \mathbb{R}_+})$$
$$+ \lim_{t\to0} \mathrm{tr}_e((Q_2(t) - Q_2'(t))\sigma\psi_{2,2}' - (Q_0(t) - Q_0'(t))\sigma\psi_{2,2}'\chi_{Z_1 \times \mathbb{R}_+})$$
$$- \lim_{t\to0} \mathrm{tr}_e(\sigma(\psi_{1,2}'\psi_{2,2}(1 - \chi_{\psi_{2,2}=1}) + \psi_{1,2}\psi_{2,2}'(1 - \chi_{\psi_{1,2}=1}))(Q_0(t) - Q_0'(t)))$$
$$- \lim_{t\to0} \mathrm{tr}_e(\sigma(\psi_{1,2}'(\chi_{Z_2 \times \mathbb{R}_+} - \chi_{\psi_{2,2}=1}) + \psi_{2,2}'(\chi_{Z_1 \times \mathbb{R}_+} - \chi_{\psi_{1,2}=1}))(Q_0(t) - Q_0'(t))).$$

LEMMA 4.1. *We have*
1)

$$[\mathrm{tr}_e((Q_1(t) - Q_1'(t))\sigma\psi_{1,2}' - (Q_0(t) - Q_0'(t))\sigma\psi_{1,2}'\chi_{Z_2 \times \mathbb{R}_+})]$$
$$= -\frac{1}{\sqrt{4\pi}} \int_t^\infty \frac{1}{\sqrt{s}} \mathrm{tr}_e(e^{-sD_{Z_1}^2} D_{Z_1} - e^{-sD_{Y \times \mathbb{R}_+}^2} D_{Y \times \mathbb{R}_+}) ds$$

2)

$$[\mathrm{tr}_e((Q_2(t) - Q_2'(t))\sigma\psi_{2,2}' - (Q_0(t) - Q_0'(t))\sigma\psi_{2,2}'\chi_{Z_1 \times \mathbb{R}_+})]$$
$$= -\frac{1}{\sqrt{4\pi}} \int_t^\infty \frac{1}{\sqrt{s}} \mathrm{tr}_e(e^{-sD_{Z_2}^2} D_{Z_2} - e^{-sD_{Y \times \mathbb{R}_+}^2} D_{Y \times \mathbb{R}_+}) ds$$

Proof. It is sufficient to prove the first item. By the proof of Lemma 3.15, we have

$$[\mathrm{tr}_e((Q_1(t) - Q_1'(t))\sigma\psi_{1,2}' - (Q_0(t) - Q_0'(t))\sigma\psi_{1,2}'\chi_{Z_2\times\mathbb{R}_+})]$$

$$= -\int_t^\infty tr(e^{-sD_{\overline{Z}_1\times\mathbb{R}_+}^- D_{Z_1\times\mathbb{R}_+}^+} D_{\overline{Z}_1\times\mathbb{R}_+}^- - e^{-sD_0^- D_0^+} D_0^- \chi_{Z_2\times\mathbb{R}_+})\sigma\psi_{1,2}' ds.$$

Note that

$$D_{Z_1\times\mathbb{R}_+} = \begin{bmatrix} 0 & D_{Z_1\times\mathbb{R}_+}^- \\ D_{Z_1\times\mathbb{R}_+}^+ & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} + D_{Z_1} \\ -\frac{\partial}{\partial x} + D_{Z_1} & 0 \end{bmatrix},$$

and

$$D_0 = \begin{bmatrix} 0 & D_0^- \\ D_0^+ & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} + D_{Y\times\mathbb{R}_+} \\ -\frac{\partial}{\partial x} + D_{Y\times\mathbb{R}_+} & 0 \end{bmatrix}.$$

Hence

$$\int_t^\infty \mathrm{tr}_e(e^{-sD_{\overline{Z}_1\times\mathbb{R}_+}^- D_{Z_1\times\mathbb{R}_+}^+} D_{\overline{Z}_1\times\mathbb{R}_+}^- - e^{-sD_0^- D_0^+} D_0^- \chi_{Z_2\times\mathbb{R}_+})\sigma\psi_{1,2}' ds$$

$$= \int_t^\infty \mathrm{tr}_e(e^{s\frac{\partial^2}{\partial^2 x}}(e^{-sD_{Z_1}^2} D_{Z_1} - e^{-sD_{Y\times\mathbb{R}_+}^2} D_{Y\times\mathbb{R}_+})\psi_{1,2}' ds$$

$$- \int_t^\infty \mathrm{tr}_e(e^{s\frac{\partial^2}{\partial^2 x}} \frac{\partial}{\partial x}(e^{-sD_{Z_1}^2} - e^{-sD_{Y\times\mathbb{R}_+}^2})\psi_{1,2}' ds$$

$$= \int_t^\infty \mathrm{tr}_e(e^{s\frac{\partial^2}{\partial^2 x}}\psi_2')\mathrm{tr}_e(e^{-sD_{Z_1}^2} D_{Z_1} - e^{-sD_{Y\times\mathbb{R}_+}^2} D_{Y\times\mathbb{R}_+}) ds$$

$$- \int_t^\infty \mathrm{tr}_e(e^{s\frac{\partial^2}{\partial^2 x}} \frac{\partial}{\partial x}\psi_2')\mathrm{tr}_e(e^{-sD_{Z_1}^2} - e^{-sD_{Y\times\mathbb{R}_+}^2}) ds.$$

However, the kernel of $e^{s\frac{\partial^2}{\partial^2 x}}$ on $\mathbb{R}$ equals

$$\kappa_{e^{s\frac{\partial^2}{\partial^2 x}}}(u, u') := \frac{1}{\sqrt{4\pi s}} e^{\frac{-|u-u'|^2}{4s}},$$

while the kernel of $e^{s\frac{\partial^2}{\partial^2 x}} \frac{\partial}{\partial x}$ equals

$$\kappa_{e^{s\frac{\partial^2}{\partial^2 x}} \frac{\partial}{\partial x}}(u, u') := -\frac{1}{\sqrt{4\pi s}} \frac{|u-u'|}{4s} e^{\frac{-|u-u'|^2}{4s}}.$$

Hence

$$\mathrm{tr}_e(e^{s\frac{\partial^2}{\partial^2 x}}\psi_2') = \int_{-\infty}^\infty \kappa_{e^{s\frac{\partial^2}{\partial^2 x}}}(x, x)\psi_2' dx = \frac{1}{\sqrt{4\pi s}}$$

and

$$\mathrm{tr}_e(e^{s\frac{\partial^2}{\partial^2 x}} \frac{\partial}{\partial x}\psi_2') = \int_{-\infty}^\infty \kappa_{e^{s\frac{\partial^2}{\partial^2 x}} \frac{\partial}{\partial x}}(x, x)\psi_2' dx = 0.$$

This completes the proof. □

LEMMA 4.2. *We have*

1) $\mathrm{tr}_e(\sigma(\psi'_{1,2}\psi_{2,2}(1 - \chi_{\psi_{2,2}=1}) + \psi_{1,2}\psi'_{2,2}(1 - \chi_{\psi_{1,2}=1}))(Q_0(t) - Q'_0(t))) = 0$

2) $\mathrm{tr}_e(\sigma(\psi'_{1,2}(\chi_{Z_1 \times \mathbb{R}_+} - \chi_{\psi_{2,2}=1}) + \psi'_{2,2}(\chi_{Z_2 \times \mathbb{R}_+} - \chi_{\psi_{1,2}=1}))(Q_0(t) - Q'_0(t))) = 0$

Proof. Actually we have

$$\mathrm{tr}_e(\sigma\psi'_{1,2}\psi_{2,2}(1 - \chi_{\psi_{2,2}=1})(Q_0(t) - Q'_0(t))) = 0 \tag{4.5}$$

$$\mathrm{tr}_e(\sigma\psi_{1,2}\psi'_{2,2}(1 - \chi_{\psi_{1,2}=1})(Q_0(t) - Q'_0(t))) = 0 \tag{4.6}$$

$$\mathrm{tr}_e(\sigma\psi'_{1,2}(\chi_{Z_1 \times \mathbb{R}_+} - \chi_{\psi_{2,2}=1})(Q_0(t) - Q'_0(t))) = 0 \tag{4.7}$$

$$\mathrm{tr}_e(\sigma\psi'_{2,2}(\chi_{Z_2 \times \mathbb{R}_+} - \chi_{\psi_{1,2}=1})(Q_0(t) - Q'_0(t))) = 0 \tag{4.8}$$

We will prove Equation (4.5) only, the other three are totally parallel.

Now, we have

$$\mathrm{tr}_e((Q_0(t) - Q'_0(t))\sigma\psi'_{1,2}\psi_{2,2}(1 - \chi_{\psi_{2,2}=1}))$$
$$= \int_t^\infty \mathrm{tr}_e(e^{-sD_0^- D_0^+} D_0^- \sigma\psi'_{1,2}\psi_{2,2}(1 - \chi_{\psi_{2,2}=1}))ds.$$

By definition, we have

$$D_0^- = \begin{bmatrix} \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & D_Y^- \\ -D_Y^+ & -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \end{bmatrix}$$

$$D_0^+ = \begin{bmatrix} -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & -D_Y^- \\ D_Y^+ & \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \end{bmatrix}.$$

Direct computation shows

$$\mathrm{tr}_e(e^{-sD_0^- D_0^+} D_0^- \sigma\psi_{1,2}\psi'_{2,2}(1 - \chi_{\psi_{1,2}=1}))$$

$$= \mathrm{tr}_e(e^{-s\frac{\partial^2}{\partial^2 x}} \frac{\partial}{\partial x}\psi'_2)\mathrm{tr}_e(e^{-s\frac{\partial^2}{\partial^2 y}} \sigma\psi_2(1 - \chi_{\psi_2=1}))\mathrm{tr}_e(\begin{bmatrix} e^{-sD_Y^- D_Y^+} & 0 \\ 0 & e^{-sD_Y^+ D_Y^-} \end{bmatrix})$$

$$+ \mathrm{tr}_e(e^{-s\frac{\partial^2}{\partial^2 x}} \psi'_2)\mathrm{tr}_e(e^{-s\frac{\partial^2}{\partial^2 y}} \frac{\partial}{\partial y}\psi_2(1 - \chi_{\psi_2=1}))\mathrm{tr}_e(\begin{bmatrix} e^{-sD_Y^- D_Y^+} & 0 \\ 0 & e^{-sD_Y^+ D_Y^-} \end{bmatrix})$$

$$+ \mathrm{tr}_e(e^{-s\frac{\partial^2}{\partial^2 x}} \psi'_2)\mathrm{tr}_e(e^{-s\frac{\partial^2}{\partial^2 y}} \psi_2(1 - \chi_{\psi_2=1}))\mathrm{tr}_e(\begin{bmatrix} 0 & e^{-sD_Y^- D_Y^+} D_Y^- \\ -e^{-sD_Y^+ D_Y^-} D_Y^+ & 0 \end{bmatrix})$$

In the proof of Lemma 4.1, we have shown that

$$\mathrm{tr}_e(e^{-s\frac{\partial^2}{\partial^2 x}} \frac{\partial}{\partial x}\psi'_2) = \mathrm{tr}_e(e^{-s\frac{\partial^2}{\partial^2 y}} \frac{\partial}{\partial y}\psi_2(1 - \chi_{\psi_2=1})) = 0.$$

At the same time, we have

$$\mathrm{tr}_e(\begin{bmatrix} 0 & e^{-sD_Y^- D_Y^+} D_Y^- \\ -e^{-sD_Y^+ D_Y^-} D_Y^+ & 0 \end{bmatrix}) = 0.$$

The proof is completed. □

In a word, by the above argument and Lemmas 4.1 and 4.2, we have

THEOREM 4.3. *The following equality holds:*

$$\mathrm{tr}_e(\mathrm{ind}_G D_M) = \int_M I(e) - \lim_{t \to 0} \frac{1}{\sqrt{4\pi}} \int_t^\infty \mathrm{tr}_e(e^{-sD_{Z_1}^2} D_{Z_1} - e^{-sD_{Y \times \mathbb{R}_+}^2} D_{Y \times \mathbb{R}_+}) \frac{1}{\sqrt{s}} ds$$

$$- \lim_{t \to 0} \frac{1}{\sqrt{4\pi}} \int_t^\infty \mathrm{tr}_e(e^{-sD_{Z_2}^2} D_{Z_2} - e^{-sD_{Y \times \mathbb{R}_+}^2} D_{Y \times \mathbb{R}_+}) \frac{1}{\sqrt{s}} ds.$$

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