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ON THE IWASAWA INVARIANTS OF NON-COTORSION SELMER GROUPS*

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Abstract. We study the variation of Iwasawa invariants of Selmer groups and fine Selmer groups of abelian varieties over $\mathbb{Z}_p$-extensions of a fixed number field K. It is shown that the λ-invariants can be unbounded if the Λ-coranks of the Selmer groups (respectively fine Selmer groups) vary. In contrast, the classical Iwasawa λ-invariants of $\mathbb{Z}_p$-extensions are expected to be bounded, at least for small base fields like imaginary quadratic fields. For fine Selmer groups, the boundedness of λ-invariants is related to the (possible) failure of the weak Leopoldt conjecture.

Key words. Boundedness of Iwasawa invariants, abelian varieties, Selmer groups, fine Selmer groups, weak Leopoldt conjecture.

Mathematics Subject Classification. 11R23.

1. Introduction. Let p be a prime, let K be a number field, and let A be an abelian variety defined over K. We study the variation of Iwasawa invariants of Selmer groups and fine Selmer groups of A over the $\mathbb{Z}_p$-extensions of K. It is well-known that the classical Iwasawa μ-invariants $\mu(K_\infty/K)$ are bounded as K_∞ runs over the $\mathbb{Z}_p$-extensions of K; it is not known whether the same holds true in general for the Iwasawa λ-invariants. In this article, we show that the Iwasawa λ-invariants *of Selmer groups* can be unbounded as one runs over the $\mathbb{Z}_p$-extensions of K.

In the following, we make this more precise. Greenberg has defined in [Gre73] a topology on the set $\mathcal{E}(K)$ of $\mathbb{Z}_p$-extensions of K. For any $K_\infty \in \mathcal{E}(K)$ and every $m \in \mathbb{N}$, we let $\mathcal{E}(K_\infty, m)$ denote the set of $\mathbb{Z}_p$-extensions $\tilde{K}_\infty$ of K which coincide with K_∞ at least up to the m-th layer:

$$\mathcal{E}(K_\infty, m) = \{\tilde{K}_\infty \in \mathcal{E}(K) \mid [(\tilde{K}_\infty \cap K_\infty) : K] \geq p^m\}.$$

Greenberg proved in [Gre73] that $\mathcal{E}(K)$ is compact with respect to this topology, and he investigated the behaviour of classical Iwasawa invariants of $\mathbb{Z}_p$-extensions of K contained in open neighbourhoods $\mathcal{E}(K_\infty, m)$. The classical Iwasawa invariants describe the asymptotic growth of the p-valuations e_n of the class numbers of the intermediate fields $K_n \subseteq K_\infty$:

$$e_n = \mu(K_\infty/K) \cdot p^n + \lambda(K_\infty/K) \cdot n + \nu(K_\infty/K) \quad (1)$$

for each sufficiently large $n \in \mathbb{N}$. The Iwasawa invariants μ, λ and ν depend on the given $\mathbb{Z}_p$-extension K_∞ of K. Greenberg proved that if no prime of K dividing p splits completely in a $\mathbb{Z}_p$-extension K_∞/K, then the μ-invariant is bounded on some neighbourhood $\mathcal{E}(K_\infty, m)$, i.e. the μ-invariant is locally bounded with respect to Greenberg's topology in this case. Moreover, if the μ-invariant vanishes, then the λ-invariant is locally bounded. More generally, we bounded in [Kle17] the λ-invariants also in situations where the μ-invariant is non-zero, provided that the μ-invariant was locally constant.

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It has been shown by Babaĭcev (see [Bab81]) and Monsky (see [Mon81, Theorem II]) that the classical Iwasawa μ-invariant is in fact *absolutely* bounded on the space $\mathcal{E}(K)$ of $\mathbb{Z}_p$-extensions of K, i.e. that there exists a constant $C \in \mathbb{N}$ such that $\mu(K_\infty/K) \leq C$ for each $K_\infty \in \mathcal{E}(K)$. Concerning the λ-invariants, it is not known whether they are bounded as K_∞ runs over the $\mathbb{Z}_p$-extensions of K. Monsky has shown that the boundedness of the classical λ-invariants follows from the validity of *Greenberg's Generalised Conjecture* for K if the composite of all $\mathbb{Z}_p$-extensions of K is a $\mathbb{Z}_p^2$-extension of K (see [Mon81, Theorem IV]). If there exist more than two independent $\mathbb{Z}_p$-extensions of K, then the boundedness of λ-invariants on $\mathcal{E}(K)$ is much stronger than Greenberg's conjecture (see [Kle21b, Example 6.3 and Theorem 6.4]).

In [Kle21a], we studied analogous boundedness questions in suitable neighbourhoods of $\mathbb{Z}_p$-extensions for the Iwasawa invariants of Selmer groups of abelian varieties, under the assumption that the corresponding Selmer groups are Λ-cotorsion (here $\Lambda = \mathbb{Z}_p[[T]]$ denotes the Iwasawa algebra). The main motivation for the present article was to derive similar results in the case where the corresponding Selmer groups are allowed to have positive Λ-corank. It turns out that the influence of the corank of the Selmer groups on the variation of Iwasawa λ-invariants outweighs the actual value of the μ-invariant. In fact, whereas it seems reasonable to believe that the λ-invariants can be locally bounded also if the μ-invariants vary (cf. the results in Section 4), we will derive explicit sufficient conditions for the λ-invariants to be unbounded, depending on the Λ-coranks (see Theorems 1.1 and 1.3 below).

The following is our first result.

Theorem 1.1. *Let A be an abelian variety defined over the number field K, and suppose that A has potentially good ordinary reduction at the primes above p. Let K_∞^c be the cyclotomic $\mathbb{Z}_p$-extension of K.*

If the Selmer group of A over K_∞^c is Λ-cotorsion and has μ-invariant zero, and if there exists a $\mathbb{Z}_p$-extension K_∞ of K such that the Selmer group of A over K_∞ is not *Λ-cotorsion, then the λ-invariants of Selmer groups of A are unbounded as one runs over the $\mathbb{Z}_p$-extensions of K which are contained in $K_\infty^c K_\infty$.*

A natural candidate for the $\mathbb{Z}_p$-extension K_∞/K in Theorem 1.1 is the *anticyclotomic $\mathbb{Z}_p$-extension* K_∞^a of an imaginary quadratic number field K. For example, Bertolini (see [Ber95]) has described settings where the Selmer group of an abelian variety A over K_∞^a is known to be non-cotorsion, see also Remark 4.15 below. This yields the following

Corollary 1.2. *Let $A = E$ be an elliptic curve of conductor N defined over $\mathbb{Q}$, and let K be an imaginary quadratic field such that $\mathcal{O}_K^\times = \{\pm 1\}$. We assume that*

(1) $p \nmid 6N\mathrm{disc}(K)h_K|A/A^0|$, *where h_K denotes the class number of K,* $\mathrm{disc}(K)$ *means the discriminant and A/A^0 denotes the group of connected components of the Néron model of $A = E$ over* $\mathrm{Spec}(\mathcal{O}_K)$*,*
(2) *A has good ordinary reduction at each prime of K above p,*
(3) *the Galois representation* $\rho_p\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(A[p^\infty])$ *is surjective,*
(4) $A(k_v)[p^\infty] = \{0\}$ *for each prime v of K above p (here k_v denotes the finite residue field of the completion K_v of K at v),*
(5) $|A(\mathbb{F}_p)| \not\equiv -1 \,(\mathrm{mod}\ p)$ *if p splits in $K/\mathbb{Q}$, and that*
(6) *every prime dividing the conductor N of A splits in $K/\mathbb{Q}$ (this is the so-called* Heegner hypothesis*).*

Let K_∞^c denote the cyclotomic $\mathbb{Z}_p$-extension of K.

If the μ-invariant of the Selmer group of A over K_∞^c is zero, then the λ-invariant *of the Selmer group of A over K_∞ is unbounded as K_∞ runs over the $\mathbb{Z}_p$-extensions of K.*

By the work of Skinner and Urban [SU14] on the Iwasawa main conjecture, the μ-invariant of the Selmer group of $A = E$ over K_∞^c is in many cases known to coincide with the μ-invariant of a suitable p-adic L-function $L_p(E, s)$, and therefore the vanishing of this μ-invariant can be checked numerically (see also Example 7.14).

The main ingredient which forces the λ-invariants to be unbounded in the above situations is the existence of two suitable $\mathbb{Z}_p$-extensions of K over which the Selmer groups of A have different Λ-coranks (cf. also Section 7.3). As in [Kle21a], information about the Iwasawa invariants of the Iwasawa Selmer module $X = X_A^{(K_\infty)}$ (cf. Section 2) is encoded into the cardinalities of certain quotients X/w_nX, $n \in \mathbb{N}$ (one should have in mind the example $w_n = (T+1)^{p^n} - 1$, in which case the quotient X/w_nX is closely related to the Selmer group over the n-th layer K_n of K_∞). The main novelty in our approach, compared to [Kle21a], is the introduction of a *second parameter*: we consider quotients of the form $X/(\alpha_k, w_n)X$, $k, n \in \mathbb{N}$, where α_k and w_n are relatively prime for each k and n. This idea arised from the wish to obtain finite quotients even if X is not Λ-torsion. For the price of a more involved analysis, it is this extra parameter α_k which enables us to get rid off the dependancy on the μ-invariants, and prove a direct link between the boundedness of λ-invariants and the Λ-coranks of the Selmer groups (see Corollary 4.2 which, together with the denseness results from Section 7.3, is the key ingredient in all our main theorems).

For *fine Selmer groups*, the property of being Λ-cotorsion is strongly related to the *weak Leopoldt conjecture* (see Remark 5.5 below). Therefore our approach also yields the following result concerning fine Selmer groups.

THEOREM 1.3. *Let A be an abelian variety defined over some number field K, and let p be an odd prime. Suppose that the weak Leopoldt conjecture holds for A over the cyclotomic $\mathbb{Z}_p$-extension K_∞^c, and that there exists a $\mathbb{Z}_p$-extension K_∞ of K such that the weak Leopoldt conjecture for A does not hold over K_∞.*

Then the λ-invariant of the fine Selmer group of A, and therefore also the λ-invariant of the Selmer group, is unbounded as one runs over the $\mathbb{Z}_p$-extensions of K which are contained in $K_\infty^c K_\infty$.

Since the Pontryagin duals of fine Selmer groups are closely related to the classical Iwasawa modules, i.e. projective limits of ideal class groups (see [CS05] and also Theorem 7.3 below), and as the classical λ-invariants are conjectured to be bounded on $\mathcal{E}(K)$ at least for small base fields K, Theorem 1.3 provides some evidence in favor of the weak Leopoldt conjecture for A over arbitrary $\mathbb{Z}_p$-extensions of K.

In [GK19], C. Greither and the author have proved a result which is kind of an analogue of Theorem 1.3 for classical Iwasawa modules: if K is a CM-field such that *Leopoldt's conjecture* fails for K, then there exists a degree p-extension L/K such that the following holds: if $\mu(L_\infty^c/L) = 0$ for the cyclotomic $\mathbb{Z}_p$-extension L_∞^c of L, then the λ-invariants of the $\mathbb{Z}_p$-extensions of L are unbounded.

It is known due to work of Babaĭcev (see [Bab82, Corollary 3]) that for any number field K, the weak Leopoldt conjecture (for ideal class groups) holds for a dense subset of $\mathbb{Z}_p$-extensions of K (with respect to Grenberg's topology), including the cyclotomic $\mathbb{Z}_p$-extension, and that the validity of Leopoldt's conjecture for K and p implies that the weak Leopoldt conjecture holds for every $\mathbb{Z}_p$-extension of K. In Section 7, we show that the weak Leopoldt conjecture *for an abelian variety A* holds over a dense subset of $\mathbb{Z}_p$-extensions of K, provided that it is true for the cyclotomic $\mathbb{Z}_p$-extension (see Corollary 7.13).

Now we briefly describe the plan of the article. After recapitulating the necessary background and notation in Section 2, we derive certain auxiliary lemmas in Section 3. In Section 4, we prove the main results concerning *local* bounds of coranks and Iwasawa invariants of Selmer groups, with respect to Greenberg's topology. Most of these results make use of a uniform control theorem (cf. Theorem 3.4) and therefore are formulated under the hypothesis that A has potentially good ordinary reduction at the primes of K above p and that certain hypotheses on the ramified primes and the primes of bad reduction are satisfied (see Definition 3.3). Interestingly, the λ-invariants can be bounded partially without these assumptions; we obtain a rather strong result concerning the common factors of the characteristic power series of the Selmer group with polynomials of the form $w_n = (T+1)^{p^n} - 1$ (see Theorem 4.8 and the corollaries of this theorem).

Similar results can be derived for fine Selmer groups (see Section 5). In Section 6, we briefly discuss the weak Leopoldt conjecture and its impact on the value of coranks of Selmer groups. Finally, in the last section, we give proofs of our main results. The rough idea is as follows. In Sections 4 and 5 we prove that, in our setting, if the λ-invariants of the (fine) Selmer groups of A are bounded in some Greenberg neighbourhood $\mathcal{E}(K_\infty, m)$, then the Λ-corank of the (fine) Selmer groups is *constant* in some (possibly slightly smaller) neighbourhood of K_∞ (via a trick, our estimate of the corank becomes independent of the orders of the kernels and cokernels of the control theorem homomorphism, see Remark 4.3). In other words, if the (fine) Selmer group of A over K_∞ is *not* Λ-cotorsion for some $\mathbb{Z}_p$-extension K_∞ of K, then either the same holds true for all $\mathbb{Z}_p$-extensions contained in some sufficiently small neighbourhood of K_∞ – or the λ-invariants are unbounded as one runs over the $\mathbb{Z}_p$-extensions of K which are close to K_∞. In Section 7 we prove that the hypotheses on the (fine) Selmer group of A over K_∞^c imply that each $\mathbb{Z}_p^2$-extension of K containing K_∞^c contains only finitely many $\mathbb{Z}_p$-extensions for which the (fine) Selmer group is not Λ-cotorsion. In particular, we find $\mathbb{Z}_p$-extensions with cotorsion (fine) Selmer group which are arbitrarily close to K_∞. By the above, this forces the λ-invariants to be unbounded near K_∞.

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2. Notation and background. We fix a prime p and a number field K. A $\mathbb{Z}_p$-*extension* of K is a Galois extension K_∞/K such that $\Gamma := \mathrm{Gal}(K_\infty/K)$ is isomorphic to the group $\mathbb{Z}_p$ of p-adic integers. Any $\mathbb{Z}_p$-extension K_∞ of K is the union of uniquely determined intermediate number fields $K_n := K_\infty^{\Gamma^{p^n}}$ which are cyclic of degree p^n over K, $n \in \mathbb{N}$. The completed group ring $\mathbb{Z}_p[[\Gamma]]$ identifies (non-canonically) with the ring $\Lambda := \mathbb{Z}_p[[T]]$ of formal power series over $\mathbb{Z}_p$ by mapping a fixed topological generator γ of Γ to $1 + T \in \Lambda$.

The basic approach of Iwasawa theory exploits the well-known structure theory of finitely generated Λ-modules. Let X be a finitely generated Λ-module. By the central structure theorem (see [NSW08, Theorem (5.1.10)], X is *pseudo-isomorphic* to a so-called *elementary* Λ-module $E = E_X$ (i.e. there exists a Λ-module homomorphism

$\varphi : X \longrightarrow E_X$ with finite kernel and cokernel). Here E has the form $E = \Lambda^r \oplus E^\circ$ with $r \in \mathbb{N}$ (the Λ-*rank* of X) and

$$E^\circ = \bigoplus_{j=1}^{s} \Lambda/(p^{m_j}) \oplus \bigoplus_{i=1}^{t} \Lambda/(h_i^{n_i})$$

for some $s, t \in \mathbb{N}$ and suitable natural numbers m_j and n_i. Moreover, each $h_i \in \Lambda$ is a so-called distinguished polynomial. In view of [Was97, Lemma 13.8], we may assume that each h_i is irreducible.

We define $\mu(X) := \sum_{j=1}^{s} m_j$, $F_X := p^{\mu(X)} \cdot \prod_{i=1}^{t} h_i^{n_i}$ and $\lambda(X) := \deg(F_X)$. $F_X \in \mathbb{Z}_p[T] \subseteq \Lambda$ is called the *characteristic power series* of X, and $\mu(X), \lambda(X)$ are the *Iwasawa invariants* of X.

If K_∞/K is a $\mathbb{Z}_p$-extension, then the projective limit $X^{(K_\infty)} = \varprojlim_n X^{(K_n)}$ of the p-Sylow subgroups of the ideal class groups of the layers K_n, $n \in \mathbb{N}$, is well-known to be a finitely generated torsion Λ-module. The Iwasawa invariants $\mu(X^{(K_\infty)})$ and $\lambda(X^{(K_\infty)})$ coincide with the coefficients $\mu(K_\infty)$ and $\lambda(K_\infty/K)$ in Iwasawa's class number formula (1) (see [Iwa59]).

For elements $\lambda_1, \ldots, \lambda_k \in \Lambda$ and a finitely generated Λ-module X, we define

$$\mathrm{rank}_{(f_1,\ldots,f_k)}(X) := v_p(|X/((f_1, \ldots, f_k) \cdot X)|),$$

whenever this is finite. The following facts concerning ranks of elementary Λ-modules will be used throughout the article without further notice. If $f \in \Lambda$ is not a unit, then $\Lambda/(f)$ is infinite; if $f_1, f_2 \in \Lambda$ are relatively prime, then $\Lambda/(f_1, f_2)$ is finite (see e.g. [Was97, Lemmas 13.7 and 13.10]).

For any finitely generated abelian group G, we denote by $G[p^\infty]$ the subgroup of p-power torsion elements.

Let now A denote an abelian variety which is defined over K. Having fixed K, p and A, we write $S_p = S_p(K)$ for the set of primes of K above p, and $S_{\mathrm{br}} = S_{\mathrm{br}}(A)$ for the set of primes of K at which A has bad reduction. Both sets are finite, and we will usually (but not always) assume that $S_p \cap S_{\mathrm{br}} = \emptyset$. For any algebraic extension M of K and any finite set S of primes of K, we let $S(M)$ denote the set of primes of M lying above some $v \in S$.

Let K_∞/K be a $\mathbb{Z}_p$-extension with intermediate fields K_n. For each $n \in \mathbb{N}$ and every prime v of K, we consider the localised *Kummer map*

$$\kappa_{n,v} : A(K_{n,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^1(K_{n,v}, A[p^\infty]),$$

where we denote by $K_{n,v}$ the completion of K_n at some prime dividing v. We define the (p-primary subgroup of the) *Selmer group* of A over K_n as

$$\mathrm{Sel}_A(K_n) := \ker\left(H^1(K_n, A[p^\infty]) \longrightarrow \prod_v H^1(K_{n,v}, A[p^\infty])/\mathrm{im}(\kappa_{n,v})\right),$$

where v runs over all primes of K_n. Note that

$$H^1(K_{n,v}, A[p^\infty])/\mathrm{im}(\kappa_{n,v}) \cong H^1(K_{n,v}, A)[p^\infty]$$

by the definition of the Kummer map. We furthermore remark that one can define the Selmer group in a slightly different way: choose any set S of primes which contains

$S_p(K) \cup S_{\mathrm{br}}(A)$ and the infinite primes of K. Then $\mathrm{Sel}_A(K_n)$ is equal to

$$\ker\left(H^1(\mathrm{Gal}(M_S(K)/K_n), A[p^\infty]) \longrightarrow \prod_{v \in S(K_n)} H^1(K_{n,v}, A[p^\infty])/\mathrm{im}(\kappa_{n,v})\right),$$

where $M_S(K)$ denotes the maximal algebraic pro-p-extension of K which is unramified outside of S (see [Mil06, Chapter 1, Corollary 6.6] for a proof of the equivalence of these definitions; for odd p one can drop the assumption that S contains the infinite primes).

We denote by

$$X_A^{(K_n)} := \mathrm{Sel}_A(K_n)^\vee = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Sel}_A(K_n), \mathbb{Q}_p/\mathbb{Z}_p)$$

the *Pontryagin duals*, and we define

$$X_A^{(K_\infty)} := \varprojlim_n X_A^{(K_n)},$$

i.e., $X_A^{(K_\infty)}$ is the Pontryagin dual of $\mathrm{Sel}_A(K_\infty) := \varinjlim_n \mathrm{Sel}_A(K_n)$, where the injective limit is taken with respect to the restriction maps. Then $X_A^{(K_\infty)}$ is a finitely generated Λ-module, $\Lambda \cong \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$ (see [LKM16, Lemma 5.4]).

Now suppose that p is odd. We define the (p-primary subgroup of the) *fine Selmer group* of A over K_n as

$$\mathrm{Sel}_{A,0}(K_n) := \ker\left(H^1(K_n, A[p^\infty]) \longrightarrow \prod_v H^1(K_{n,v}, A[p^\infty])\right),$$

$n \in \mathbb{N}$, where v runs over all primes of K_n. Again, these groups can also be defined with respect to finite sets S containing $S_p(K) \cup S_{\mathrm{br}}$. We denote the Pontryagin duals by $Y_A^{(K_n)} := \mathrm{Sel}_{A,0}(K_n)^\vee$, and we define $Y_A^{(K_\infty)} = \varprojlim Y_A^{(K_n)}$, where the projective limit is taken with respect to the corestriction maps.

For every $n \in \mathbb{N}$, we have an exact sequence

$$0 \longrightarrow \mathrm{Sel}_{A,0}(K_n) \longrightarrow \mathrm{Sel}_A(K_n) \longrightarrow \bigoplus_{v \in S_p(K_n)} A(K_{n,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \tag{2}$$

by [CS05, equation (58) on p. 828], since $A(K_{n,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p = \{0\}$ whenever $v \nmid p$. In particular, $Y_A^{(K_\infty)}$ is a quotient of the Λ-module $X_A^{(K_\infty)}$.

3. Preliminary results. We fix a number field K and a rational prime p. Let K_∞/K be a $\mathbb{Z}_p$-extension, $K_\infty = \bigcup_n K_n$. We write $\Gamma = \mathrm{Gal}(K_\infty/K)$; then $\Gamma_n = \Gamma^{p^n}$ fixes the intermediate field K_n of degree p^n over K, $n \in \mathbb{N}$. Fix an isomorphism $\mathbb{Z}_p[[\Gamma]] \cong \Lambda$, as in Section 2. For any finitely generated Λ-module X and every $n \in \mathbb{N}$, we denote by X_{Γ_n} the quotient of Γ_n-coinvariants of X, i.e. the maximal quotient of X on which Γ_n acts trivially. Then

$$X_{\Gamma_n} = X/(w_n(T) \cdot X),$$

where $w_n(T) = (T+1)^{p^n} - 1$.

Now let A be an abelian variety defined over K. Recall the definition of the (Pontryagin duals of the) Selmer groups $X_A^{(K_\infty)} = \varprojlim_n X_A^{(K_n)}$ from Section 2. The following result is a crucial ingredient of our approach.

THEOREM 3.1 (Mazur's Control Theorem). *Suppose that A has potentially good ordinary reduction at each prime $v \in S_p$. Then the natural maps*

$$\mathrm{pr}_n^{(K_\infty)} \colon (X_A^{(K_\infty)})_{\Gamma_n} \longrightarrow X_A^{(K_n)}$$

have finite kernels and cokernels, the orders of which are bounded as $n \to \infty$.

Proof. See [Maz72], and see also [Gre01, Chapter 4] for a very detailed exposition of the proof. □

It turns out that the finiteness of $\mathrm{coker}(\mathrm{pr}_n)$ for all n holds in greater generality:

LEMMA 3.2. *Let A denote any abelian variety defined over the number field K, and let K_∞/K be a $\mathbb{Z}_p$-extension. Then the cokernels of the maps*

$$\mathrm{pr}_n^{(K_\infty)} \colon (X_A^{(K_\infty)})_{\Gamma_n} \longrightarrow X_A^{(K_n)}$$

are finite for every $n \in \mathbb{N}$.

Proof. We dualise the statement [Lee20, Lemma 2.0.1]. □

Recall from the Introduction that $\mathcal{E}(K_\infty, m)$ denotes the set of $\mathbb{Z}_p$-extensions $\tilde{K}_\infty/K$ which coincide with K_∞ at least up to the m-th layer K_m. The next result shows that the orders of the kernels and cokernels from Mazur's Control Theorem can sometimes be bounded uniformly on $\mathcal{E}(K_\infty, m)$. Since the corresponding hypotheses will play a prominent role in many of the following results, we give them an explicit name.

DEFINITION 3.3. Let K_∞/K be a $\mathbb{Z}_p$-extension, and let A be an abelian variety defined over K. We say that the pair (K_∞, A) *has Property (F)* if
(1) A has potentially good ordinary reduction at each $v \in S_p$,
(2) each prime $v \in S_p$ ramifies in K_∞, and
(3) no prime $v \in S_{\mathrm{br}}$ splits completely in K_∞/K.

We call this collection of hypotheses the Property (F) because it implies that $X_A^{(K_\infty)}$ is a *Fukuda module* with locally bounded Fukuda parameters in the sense of [Kle21a, Definition 3.1 and Theorem 4.5], by the following result.

THEOREM 3.4. *Let K_∞/K be a $\mathbb{Z}_p$-extension. If the pair (K_∞, A) satisfies Property (F), then there exist constants $m, C_1, C_2 \in \mathbb{N}$ such that*

$$|\ker(\mathrm{pr}_n^{(\tilde{K}_\infty)})| \le C_1 \quad \text{and} \quad |\mathrm{coker}(\mathrm{pr}_n^{(\tilde{K}_\infty)})| \le C_2$$

for every $n \in \mathbb{N}$ and each $\tilde{K}_\infty \in \mathcal{E}(K_\infty, m)$.

Proof. Since each prime $v \in S_p$ ramifies in K_∞, the same holds true for any $\tilde{K}_\infty \in \mathcal{E}(K_\infty, m)$, provided that m has been chosen large enough. Moreover, we may derive from [Gre03, Proposition 3.2,(ii)] that $A(K_\infty)[p^\infty]$ is finite, which was one of the assumptions in [Kle21a, Theorem 4.5]. Finally, if m is large enough to ensure that each $v \in S_{\mathrm{br}}$ is inert in K_m/K_{m-1}, then no prime $v \in S_{\mathrm{br}}$ splits completely in any $\tilde{K}_\infty \in \mathcal{E}(K_\infty, m)$. Therefore the assertion follows from [Kle21a, Theorem 4.5]. □

Remark 3.5. One can formulate a version of Theorem 3.4 which does not require the hypothesis that each $v \in S_p$ ramifies in K_∞/K. If one drops this assumption and in addition adds the hypothesis that $A(K_\infty)[p^\infty]$ is finite, then the statement of Theorem 3.4 remains valid for the $\mathbb{Z}_p$-extensions $\tilde{K}_\infty \in \mathcal{E}(K_\infty, m)$ which have the same ramification set as K_∞/K, i.e. each $v \in S_p$ which ramifies in $\tilde{K}_\infty$ must ramify also in K_∞/K and vice versa (see [Kle21a, Theorem 4.5]).

Similarly, one can allow that the primes $v \in S_{\mathrm{br}}$ split completely in K_∞. In order to nevertheless bound the orders of $\ker(\mathrm{pr}_n^{(\tilde{K}_\infty)})$, one has to restrict oneself to the subset $\mathcal{U}(A, K_\infty, m) \subseteq \mathcal{E}(K_\infty, m)$ of $\mathbb{Z}_p$-extensions $\tilde{K}_\infty/K$ such that every $v \in S_{\mathrm{br}}$ which splits completely in K_∞ does split completely also in $\tilde{K}_\infty$. The subset $\mathcal{U}(A, K_\infty, m)$ in general is much smaller than $\mathcal{E}(K_\infty, m)$, but can be non-trivial (cf. also [Kle21a, Remark 4.4]).

We note that the boundedness of $|\mathrm{coker}(\mathrm{pr}_n^{(\tilde{K}_\infty)})|$ does not require any restriction on the primes v of K, as long as $A(K_\infty)[p^\infty]$ is finite.

For later use, we state a result on quotients of Selmer groups.

Lemma 3.6. *Let K_∞/K and A be as in Theorem 3.1. Let $C_1, C_2 \in \mathbb{N}$ be the orders of the kernels and cokernels of $\mathrm{pr}_n^{(K_\infty)}$ for some $n \in \mathbb{N}$, and let $\lambda \in \Lambda$ be an arbitrary element. We consider the induced map*

$$\overline{\mathrm{pr}}_n^{(K_\infty)} \colon X_A^{(K_\infty)}/((w_n(T), \lambda) \cdot X) \longrightarrow X_A^{(K_n)}/(\lambda \cdot X_A^{(K_n)}).$$

Then $|\ker(\overline{\mathrm{pr}}_n^{(K_\infty)})| \leq C_1 \cdot C_2$ and $|\mathrm{coker}(\overline{\mathrm{pr}}_n^{(K_\infty)})| \leq C_2$.

Proof. Apply the snake lemma to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \lambda \cdot (X_A^{(K_\infty)})_{\Gamma_n} \arrow[r] \arrow[d, "\mathrm{pr}_n^{(K_\infty)}"] & (X_A^{(K_\infty)})_{\Gamma_n} \arrow[r] \arrow[d, "\mathrm{pr}_n^{(K_\infty)}"] & (X_A^{(K_\infty)})_{\Gamma_n}/\lambda \cdot (X_A^{(K_\infty)})_{\Gamma_n} \arrow[r] \arrow[d, "\overline{\mathrm{pr}}_n^{(K_\infty)}"] & 0 \\
0 \arrow[r] & \lambda \cdot X_A^{(K_n)} \arrow[r] & X_A^{(K_n)} \arrow[r] & X_A^{(K_n)}/\lambda \cdot X_A^{(K_n)} \arrow[r] & 0.
\end{tikzcd}$$

∎

Recall the definition of $(f_1, \ldots, f_k)$-ranks of Λ-modules from Section 2. We will now study the relation between the ranks of a finitely generated Λ-module X and its corresponding elementary Λ-module E_X. If X is Λ-torsion and $k = 1$, then it has been shown in [Kle17, Proposition 3.4] that

$$\mathrm{rank}_{(f)}(X) \geq \mathrm{rank}_{(f)}(E_X)$$

for every $f \in \Lambda$, provided that (one and therefore both of) the ranks are finite. If the maximal finite Λ-submodule of X is trivial, then $\mathrm{rank}_{(f)}(X) = \mathrm{rank}_{(f)}(E_X)$ in view of [Kle17, equation (3.5)]. Now we consider the case $k = 2$.

Lemma 3.7. *Let X be a finitely generated Λ-module with elementary Λ-module E_X. Let $\alpha, \omega \in \Lambda$ be two relatively prime elements. We write*

$$E_X = \Lambda^r \oplus E_1 \oplus E_2,$$

where $r = \mathrm{rank}_\Lambda(X)$, $\alpha^s \cdot E_2 = \{0\}$ for some $0 \neq s \in \mathbb{N}$, and where multiplication by α is injective on E_1. Then

$$\mathrm{rank}_{(\alpha,\omega)}(X) \geq \mathrm{rank}_{(\alpha,\omega)}(\Lambda^r) + \mathrm{rank}_{(\alpha,\omega)}(E_2).$$

We stress the symmetry in α and ω: the submodule E_2 of E_X° could have been defined also with respect to ω, leading to an analogous inequality.

Proof. Let $\varphi : X \longrightarrow E_X$ be a pseudo-isomorphism with kernel M_1 and cokernel M_2. We start from the exact sequence

$$0 \longrightarrow X/M_1 \longrightarrow E_X \longrightarrow M_2 \longrightarrow 0.$$

The snake lemma implies that the kernel of the first map f in the induced exact sequence

$$X/(M_1 + \alpha X) \xrightarrow{f} E_X/(\alpha E_X) \longrightarrow M_2/(\alpha M_2) \longrightarrow 0$$

equals the image of the (finite) Λ-module $M_2[\alpha]$ under the connecting homomorphism. Let $M_3 \subseteq X$ be the pre-image of $\ker(f)$ under the canonical surjection $X \twoheadrightarrow X/(M_1 + \alpha X)$.

Now we apply the snake lemma again; note that multiplication by ω is injective on $(\Lambda/(\alpha))^r \oplus E_2/(\alpha E_2)$, since $E_2/(\alpha E_2)$ is a direct sum of modules $\Lambda/(h_j)$ such that h_j divides α; by assumption, the two elements $\alpha, \omega \in \Lambda$ are coprime. We obtain an exact sequence

$$\begin{aligned} (E_1/(\alpha E_1))[\omega] \longrightarrow (M_2/(\alpha M_2))[\omega] \xrightarrow{g} X/(M_1 + M_3 + (\alpha,\omega)X) \\ \longrightarrow E_X/((\alpha,\omega)E_X) \longrightarrow M_2/((\alpha,\omega)M_2) \longrightarrow 0. \end{aligned} \tag{3}$$

Since $M_2/(\alpha M_2)$ is finite, the groups $(M_2/(\alpha M_2))[\omega]$ and $M_2/((\alpha,\omega)M_2)$ have the same cardinality. Moreover, as $E_1/(\alpha E_1)$ is finite by the definition of E_1, we have

$$|\ker(g)| \leq |(E_1/(\alpha E_1))[\omega]| = |E_1/((\alpha,\omega)E_1)|.$$

Therefore it follows from the exact sequence (3) that

$$|(\Lambda/(\alpha,\omega))^r| \cdot |E_2/((\alpha,\omega)E_2)| \leq |X/(M_1 + M_3 + (\alpha,\omega)X)| \leq |X/((\alpha,\omega)X)|.$$

□

We prove one final auxiliary

Lemma 3.8. *Let X be a finitely generated Λ-module with elementary Λ-module E_X. We fix a pseudo-isomorphism $\varphi\colon X \longrightarrow E_X$ with finite kernel M_1 and cokernel M_2. Let $f_1, \ldots, f_s \in \Lambda$.*

Then $\mathrm{rank}_{(f_1,\ldots,f_s)}(X) < \infty$ *if and only if* $\mathrm{rank}_{(f_1,\ldots,f_s)}(E_X) < \infty$*. In fact,*

$$\mathrm{rank}_{(f_1,\ldots,f_s)}(E_X) \leq \mathrm{rank}_{(f_1,\ldots,f_s)}(X) + v_p(|M_2|)$$

and

$$\mathrm{rank}_{(f_1,\ldots,f_s)}(X) \leq \mathrm{rank}_{(f_1,\ldots,f_s)}(E_X) + v_p(C^{(s)}),$$

where $C^{(s)} = |M_1| \cdot |M_2|^s$.

Proof. The homomorphism φ induces a map

$$\tilde{\varphi}\colon X/((f_1,\dots,f_s)\cdot X) \longrightarrow E_X/((f_1,\dots,f_s)\cdot E_X).$$

We have a canonical surjection $\operatorname{coker}(\varphi) \twoheadrightarrow \operatorname{coker}(\tilde{\varphi})$. On the other hand, suppose that $\overline{x} \in \ker(\tilde{\varphi})$, and choose a representative $x \in \overline{x}$. Then

$$\varphi(x) = \sum_{i=1}^{s} f_i \cdot \alpha_i$$

for suitable elements $\alpha_i \in E_X$. If each α_i lies in the image of φ, then

$$x \in (f_1,\dots,f_s)\cdot X + \ker(\varphi).$$

This shows that $|\ker(\tilde{\varphi})| \le C^{(s)}$, where $C^{(s)}$ is defined as in the statement of the lemma. □

4. Bounding Iwasawa invariants of Selmer groups. In this section, we bound arithmetical invariants of Selmer groups (as the corank, the Iwasawa invariants and the multiplicities of certain factors of the characteristic power series of the Pontryagin dual) with respect to Greenberg's topology from the Introduction. In other words, we fix a prime p, a number field K and an abelian variety defined over K, and we study the variation of the above invariants for $X_A^{(K_\infty)}$ as K_∞ runs over the $\mathbb{Z}_p$-extensions of K.

THEOREM 4.1. *Let A be an abelian variety defined over the number field K, and let K_∞ be a $\mathbb{Z}_p$-extension of K. Suppose that A has potentially good ordinary reduction at each prime $v \in S_p$. Then there exists a neighbourhood $U = \mathcal{E}(K_\infty, n)$ such that*

$$\operatorname{rank}_\Lambda(X_A^{(\tilde{K}_\infty)}) \le \operatorname{rank}_\Lambda(X_A^{(K_\infty)})$$

for every $\mathbb{Z}_p$-extension $\tilde{K}_\infty$ of K contained in U.

In fact, it suffices to choose n large enough such that

$$p^n \ge \lambda(X_A^{(K_\infty)}) + 1. \tag{4}$$

Proof. Write $X = X_A^{(K_\infty)}$, $\mu = \mu(X)$ and $\lambda = \lambda(X)$ for brevity. We choose $n \in \mathbb{N}$ large enough such that $p^n \ge \lambda + 1$, as in the statement of the theorem.

Recall that $w_n = w_n(T) = (T+1)^{p^n} - 1$, and let $U := \mathcal{E}(K_\infty, n)$. For $\tilde{K}_\infty \in U$ arbitrary and fixed, we write $\tilde{X}_n := X_A^{(\tilde{K}_n)}$, $\tilde{X} := X_A^{(\tilde{K}_\infty)}$, and $\tilde{r} := \operatorname{rank}_\Lambda(\tilde{X})$.

Let $C_i = C_i^{(K_\infty)}$, respectively, $\tilde{C}_i = C^{(\tilde{K}_\infty)}$ $(i = 1, 2)$ be the cardinalities of the kernels and cokernels of $\mathrm{pr}_n^{(K_\infty)}$ and $\mathrm{pr}_n^{(\tilde{K}_\infty)}$, as in Lemma 3.6 (for the fixed number n), and let $C = C^{(s)}$ be as in Lemma 3.8, with $s = 2$ and $X = X_A^{(K_\infty)}$.

In what follows, rank will always denote $\operatorname{rank}_{(\alpha, w_n)}$, where $\alpha \in \Lambda$ is coprime with w_n (we will make a concrete choice below). Then we obtain a chain of inequalities

$$\begin{aligned}
\operatorname{rank}(E_{\tilde{X}}) &\overset{3.8}{\le} \operatorname{rank}(\tilde{X}) + v_p(\tilde{C}) \overset{3.6}{\le} \operatorname{rank}(\tilde{X}_n) + v_p(\tilde{C}_1\tilde{C}_2\tilde{C}) \\
&= \operatorname{rank}(X_A^{(K_n)}) + v_p(\tilde{C}_1\tilde{C}_2\tilde{C}) \overset{3.6}{\le} \operatorname{rank}(X) + v_p(\tilde{C}_1\tilde{C}_2\tilde{C}C_2) \\
&\overset{3.8}{\le} \operatorname{rank}(E_X) + v_p(D),
\end{aligned} \tag{5}$$

where $\tilde{C} = |M_2|$ in the notation from Lemma 3.8, applied to $\tilde{X}$ and $\mathrm{rank}_{(\alpha,w_n)}$, $C^{(2)}$ is defined as in Lemma 3.8 and $D := \tilde{C}_1\tilde{C}_2\tilde{C}C_2C^{(2)}$ for brevity. Note that w_n annihilates both $X_A^{(K_n)}$ and $\tilde{X}_n$ (the topological generators of $\Gamma = \mathrm{Gal}(K_\infty/K)$ and $\tilde{\Gamma} = \mathrm{Gal}(\tilde{K}_\infty/K)$ can be chosen to coincide on $\tilde{K}_n = K_n$), i.e.

$$\mathrm{rank}(X_A^{(K_n)}) = \mathrm{rank}_{(\alpha)}(X_A^{(K_n)}) \quad \text{and} \quad \mathrm{rank}(\tilde{X}_n) = \mathrm{rank}_{(\alpha)}(\tilde{X}_n).$$

The integer D depends on the chosen $\tilde{K}_\infty \in U$.

Now we choose $\alpha = p^k$, where

$$k > \mu p^n + v_p(D). \tag{6}$$

Since the cardinality of $\Lambda/(p^k, w^n)$ is p^{kp^n}, and as $\mathrm{rank}(M) \le \mathrm{rank}_{(p^k)}(M)$ and $\mathrm{rank}(M) \le \mathrm{rank}_{(w_n)}(M)$ for every Λ-module M, provided that all these ranks are finite, we may conclude that

$$\mathrm{rank}(E_X) \le rp^n k + \mu p^n + k\lambda,$$

where $r = \mathrm{rank}_\Lambda(X)$. By assumption (6), it follows that the right hand side of the above chain of inequalities (5) is smaller than $rp^n k + k\lambda + k$.

Now we consider the left hand side. Since $\mathrm{rank}(\Lambda^{\tilde{r}}) = \tilde{r}p^n k$, it follows immediately from the above and the choice of n in (4) that $\tilde{r} \le r$. □

Corollary 4.2. *Let A be an abelian variety defined over the number field K, and let K_∞/K be a $\mathbb{Z}_p$-extension. Suppose that A has potentially good ordinary reduction at the primes of K above p. Suppose that there exists an integer $m \in \mathbb{N}$ such that $\lambda(X_A^{(\tilde{K}_\infty)})$ is bounded on $\mathcal{E}(K_\infty, m)$. Then there exists some $n \ge m$ such that*

$$\mathrm{rank}_\Lambda(X_A^{(\tilde{K}_\infty)}) = \mathrm{rank}_\Lambda(X_A^{(K_\infty)})$$

for every $\tilde{K}_\infty \in \mathcal{E}(K_\infty, n)$.

Proof. Choose $C \in \mathbb{N}$ such that $\lambda(X_A^{(\tilde{K}_\infty)}) \le C$ for every $\tilde{K}_\infty \in \mathcal{E}(K_\infty, m)$, and let $n \ge m$ be large enough such that $p^n \ge C + 1$. Then

$$\mathrm{rank}_\Lambda(X_A^{(\tilde{K}_\infty)}) \le \mathrm{rank}_\Lambda(X_A^{(K_\infty)})$$

for each $\tilde{K}_\infty \in \mathcal{E}(K_\infty, n)$, by Theorem 4.1. On the other hand, since $\tilde{K}_\infty \in \mathcal{E}(K_\infty, n)$ if and only if $K_\infty \in \mathcal{E}(\tilde{K}_\infty, n)$, we have in fact equality of Λ-ranks. □

Remark 4.3. Note that the usage of (p^k, w_n)-ranks for fixed w_n and variable k allows us to make the contributions of X and $\tilde{X}$ to our inequalities independent of the constants C_i, $\tilde{C}_i$ from the control theorem. In fact, only the contributions of the Λ-ranks and the λ-invariants of X and $\tilde{X}$ depend on k.

In the next step, we bound μ-invariants. To this purpose, we use the full Control Theorem in the version of Theorem 3.4. Therefore we will from now on assume that Property (F) (see Definition 3.3) holds for the pair (K_∞, A). Note: in the following result, the corresponding hypotheses may be weakened as described in Remark 3.5.

Theorem 4.4. *Let A be an abelian variety defined over the number field K, and let K_∞ be a $\mathbb{Z}_p$-extension of K. We assume that the pair (K_∞, A) has property*

(F). Then there exists a neighbourhood $U = \mathcal{E}(K_\infty, n)$ of K_∞ such that for every $\mathbb{Z}_p$-extension $\tilde{K}_\infty$ of K contained in U,

$$\mu(X_A^{(\tilde{K}_\infty)}) \le \mu(X_A^{(K_\infty)}) \quad \textit{if} \quad \mathrm{rank}_\Lambda(X_A^{(\tilde{K}_\infty)}) = \mathrm{rank}_\Lambda(X_A^{(K_\infty)}).$$

Proof. Let $m \in \mathbb{N}$ be large enough such that the conclusion of Theorem 3.4 holds for every $\mathbb{Z}_p$-extension $\tilde{K}_\infty \in \mathcal{E}(K_\infty, m)$, and fix corresponding constants $C_1, C_2 \in \mathbb{N}$ and an arbitrary $\tilde{K}_\infty \in \mathcal{E}(K_\infty, m)$.

We use the same notation as in the proof of Theorem 4.1, and we choose $n \ge m$ large enough such that

(i) $n > \mu + 1$, and

(ii) $p^n > n\lambda + v_p(C_1 C_2^2 C)$, where $C = C^{(2)}$ is as in Lemma 3.8, applied to a pseudo-isomorphism $\varphi : X \longrightarrow E_X$ with $s = 2$.

We choose $\alpha = p^n$, and we write

$$E_{\tilde{X}} = \Lambda^{\tilde{r}} \oplus \tilde{E}_1 \oplus \tilde{E}_2,$$

where the decomposition is such that multiplication by α is injective on $\tilde{E}_1$, whereas $\tilde{E}_2$ is annihilated by a suitable power of α (as in Lemma 3.7).

Using Lemma 3.7 instead of the first inequality from Lemma 3.8, we deduce, analogous to inequality (5) in the proof of Theorem 4.1, a chain of inequalities

$$\mathrm{rank}(\tilde{E}_2) + \mathrm{rank}(\Lambda^{\tilde{r}}) \le \mathrm{rank}(E_X) + v_p(C_1 C_2^2 C). \tag{7}$$

Now suppose that $\tilde{r} = r$. Then the summand $rp^n n = \mathrm{rank}(\Lambda^r)$ can be subtracted on both sides of the inequality, and therefore we derive that

$$\mathrm{rank}(\tilde{E}_2) < (\mu + 1)p^n, \tag{8}$$

using the property (ii) of n.

If $\tilde{E}_2$ contains a summand $\Lambda/(p^i)$ for some $i \ge n$, then the rank of $\tilde{E}_2$ is at least np^n, which contradicts to (8) in view of hypothesis (i). Therefore $\tilde{E}_2$ is a direct sum of modules $\Lambda/(p^i)$ with $i < n$, and $\mathrm{rank}(\tilde{E}_2) = \mathrm{rank}_{(w_n)}(\tilde{E}_2) = \tilde{\mu} p^n$. Inequality (8) now implies that $\tilde{\mu} \le \mu$. □

Remark 4.5. Let A and K_∞/K be as in Theorem 4.4. In view of Corollary 4.2, we obtain a stronger result if the λ-invariants are locally bounded in a neighbourhood $\mathcal{E}(K_\infty, m)$ of K_∞. More precisely, if the λ-invariants are bounded on $\mathcal{E}(K_\infty, m)$, then we can choose a neighbourhood $U = \mathcal{E}(K_\infty, m)$ such that

$$\mathrm{rank}_\Lambda(X_A^{(\tilde{K}_\infty)}) = \mathrm{rank}_\Lambda(X_A^{(K_\infty)}) \quad \text{and} \quad \mu(X_A^{(\tilde{K}_\infty)}) \le \mu(X_A^{(K_\infty)})$$

for each $\tilde{K}_\infty \in U$.

If moreover the integers $C^{(2)}$ from Lemma 3.8 (corresponding to pseudo-isomorphisms $\varphi : \tilde{X} \longrightarrow E_{\tilde{X}}$, $\tilde{K}_\infty \in \mathcal{E}(K_\infty, m)$) are bounded, then one can find a neighbourhood $U = \mathcal{E}(K_\infty, m)$ such that

$$\mu(X_A^{(\tilde{K}_\infty)}) = \mu(X_A^{(K_\infty)})$$

for each $\tilde{K}_\infty \in U$.

In the following result, we (partially) bound μ-invariants without any assumption on the Λ-ranks. Let M be any finitely generated Λ-module with elementary Λ-module

$$E = \Lambda^r \oplus \bigoplus_{j=1}^{s} \Lambda/(p^{m_j}) \oplus \bigoplus_{i=1}^{t} \Lambda/(h_i^{n_i}).$$

Then we define

$$\mu^{(k)}(M) := \sum_{j=1}^{t} \min(k, m_j).$$

LEMMA 4.6. *Let A be an abelian variety defined over the number field K, and let K_∞ be a $\mathbb{Z}_p$-extension of K. We assume that the pair (K_∞, A) has Property (F).*

Let $k \in \mathbb{N}$. Then there exists a neighbourhood $U = \mathcal{E}(K_\infty, n)$ of K_∞ such that for every $\mathbb{Z}_p$-extension $\tilde{K}_\infty$ of K contained in U,

$$\mu^{(k)}(X_A^{(\tilde{K}_\infty)}) \le \mu^{(k)}(X_A^{(K_\infty)}) + k \cdot (\mathrm{rank}_\Lambda(X_A^{(K_\infty)}) - \mathrm{rank}_\Lambda(X_A^{(\tilde{K}_\infty)})).$$

Proof. Let m be as in the proof of Theorem 4.4, and choose $n \ge m$ such that

$$p^n > k\lambda + v_p(C_1 C_2^2 C).$$

Considering (p^k, w_n)-ranks, we obtain that

$$\mu^{(k)}(X_A^{(\tilde{K}_\infty)})p^n + kp^n\tilde{r} \le kp^n r + p^n \mu^{(k)}(X_A^{(K_\infty)}) + k\lambda + v_p(C_1 C_2^2 C)$$

for each $\tilde{K}_\infty \in \mathcal{E}(K_\infty, n)$, as in (7). Therefore

$$\mu^{(k)}(X_A^{(\tilde{K}_\infty)})p^n \le p^n(kr - k\tilde{r} + \mu^{(k)}(X_A^{(K_\infty)})) + k\lambda + v_p(C_1 C_2^2 C).$$

The statement follows from the choice of n. □

In [Kle21a], which deals with the rank zero setting, we bounded the Iwasawa λ-invariants of $\mathbb{Z}_p$-extensions $\tilde{K}_\infty/K$ satisfying $\mu(X_A^{(\tilde{K}_\infty)}) = \mu(X)$. It seems, however, that bounding λ-invariants is far more involved if $\mathrm{rank}_\Lambda(X) > 0$ (see also Theorem 7.7 below). Recall that $\lambda(X)$ equals the degree of the characteristic power series $F_X \in \mathbb{Z}_p[T]$. In the following results, we distinguish between the common factors of F_X and some $w_n = (T+1)^{p^n} - 1$ on the one hand and the divisors f of F_X which are coprime with every w_n on the other hand. We start with the second family, proving first a boundedness result along the lines of Lemma 4.6.

Let f be a distinguished polynomial. For a finitely generated Λ-module M with elementary Λ-module

$$E = \Lambda^r \oplus \bigoplus_{j=1}^{s} \Lambda/(f^{m_j}) \oplus \bigoplus_{i=1}^{t} \Lambda/(h_i^{n_i})$$

(here all h_i shall be coprime with f, and $h_i = p$ is possible), we define

$$f^{(k)}(M) := \sum_{j=1}^{s} \min(k, m_j).$$

Theorem 4.7. *Let A be an abelian variety defined over the number field K, and let K_∞ be a $\mathbb{Z}_p$-extension of K. We assume that the pair (K_∞, A) satisfies Property (F).*

Let $k \in \mathbb{N}$, and let $f \in \Lambda$ be a distinguished polynomial which is coprime with each w_n, $n \in \mathbb{N}$. Then there exists a neighbourhood $U = \mathcal{E}(K_\infty, n)$ of K_∞ such that for every $\mathbb{Z}_p$-extension $\tilde{K}_\infty$ of K contained in U,

$$f^{(k)}(X_A^{(\tilde{K}_\infty)}) \leq f^{(k)}(X_A^{(K_\infty)}) + k \cdot (\mathrm{rank}_\Lambda(X_A^{(K_\infty)}) - \mathrm{rank}_\Lambda(X_A^{(\tilde{K}_\infty)})).$$

Proof. Let $R_n = v_p(|\Lambda/(f, w_n)|)$, $n \in \mathbb{N}$ (this corresponds to $p^n = v_p(|\Lambda/(p, w_n)|)$ in the proof of Lemma 4.6). Note that $R_n \to \infty$, $n \to \infty$, and that

$$R_n = \sum_{\zeta : \zeta^{p^n} = 1} v_p(f(\zeta - 1)),$$

where ζ runs over all p^n-th roots of unity in some fixed algebraic closure of $\mathbb{Q}_p$ (we normalise v_p such that $v_p(p) = 1$). In particular,

$$v_p(|\Lambda/(f^i, w_n)|) = i \cdot R_n$$

for all $i, n \in \mathbb{N}$.

Now we choose m as in Lemma 4.6, and we let $n \geq m$ be large enough such that

$$R_n > \mu(X) \deg(f^k) + v_p(|\Lambda/(f, \frac{F_X}{f^{v_f(F_X)}})|) + v_p(C_1 C_2^2 C),$$

where $v_f(F_X)$ denotes the largest integer v such that f^v divides F_X. Using inequality (7) for $\mathrm{rank}_{(f^k, w_n)}$, we may conclude that

$$R_n \cdot f^{(k)}(X_A^{(\tilde{K}_\infty)}) + k\tilde{r}R_n \leq krR_n + \mathrm{rank}_{(f^k, w_n)}(E_X^\circ) + v_p(C_1 C_2^2 C);$$

the lemma follows by choosing n sufficiently large. □

Now we turn to the investigation of common divisors of F_X with some w_n, $n \in \mathbb{N}$, using the notation $f^{(k)}(X)$ from Theorem 4.7.

This can be done even if no control theorem 3.4 is known for A over K_∞. Therefore the following theorem holds for any abelian variety A with potentially good ordinary reduction at p and any $\mathbb{Z}_p$-extension K_∞ of K, i.e. we do not have to assume that the pair (K_∞, A) satisfies Property (F).

On the other hand, we can handle only the case $k = 1$.

Theorem 4.8. *Let A be an abelian variety defined over K, and let K_∞ be a $\mathbb{Z}_p$-extension of K. Suppose that A has potentially good ordinary reduction at each prime $v \in S_p$. Let $n \in \mathbb{N}$ be arbitrary, and let $U = \mathcal{E}(K_\infty, n)$. For each non-trivial divisor f of w_n, we have*

$$f^{(1)}(X_A^{(\tilde{K}_\infty)}) = f^{(1)}(X_A^{(K_\infty)}) + \mathrm{rank}_\Lambda(X_A^{(K_\infty)}) - \mathrm{rank}_\Lambda(X_A^{(\tilde{K}_\infty)})$$

for every $\mathbb{Z}_p$-extension $\tilde{K}_\infty \in U$ of K. In particular, $f^{(1)}(X_A^{(\tilde{K}_\infty)})$ is bounded on U, and

$$f^{(1)}(X_A^{(\tilde{K}_\infty)}) = f^{(1)}(X_A^{(K_\infty)})$$

if in addition $\operatorname{rank}_\Lambda(X_A^{(\tilde{K}_\infty)}) = \operatorname{rank}_\Lambda(X_A^{(K_\infty)})$.

We stress that we do not assume that $\mu(X_A^{(\tilde{K}_\infty)}) = \mu(X_A^{(K_\infty)})$ in this result.

Proof. We fix f, n and $\tilde{K}_\infty \in \mathcal{E}(K_\infty, n)$, and we write $X = X_A^{(K_\infty)}$ and $\tilde{X} = X_A^{(\tilde{K}_\infty)}$ for brevity. Let $r = \operatorname{rank}_\Lambda(X)$ and $\tilde{r} = \operatorname{rank}_\Lambda(\tilde{X})$. For a $\mathbb{Z}_p$-module M, we write $\operatorname{rank}_{\mathbb{Z}_p}(M) := \dim_{\mathbb{Q}_p}(M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$, whenever this is finite.

Since the kernels and cokernels of $\operatorname{pr}_n^{(K_\infty)}$ and $\operatorname{pr}_n^{(\tilde{K}_\infty)}$ are finite by Theorem 3.1,

$$\begin{aligned} f^{(1)}(X_A^{(\tilde{K}_\infty)}) \deg(f) &= \operatorname{rank}_{\mathbb{Z}_p}(\tilde{X}/f\tilde{X}) - \tilde{r}\deg(f) \\ &\overset{3.6}{=} \operatorname{rank}_{\mathbb{Z}_p}(\tilde{X}_n/f\tilde{X}_n) - \tilde{r}\deg(f) \\ &= \operatorname{rank}_{\mathbb{Z}_p}(X_A^{(K_n)}/fX_A^{(K_n)}) - \tilde{r}\deg(f) \\ &= f^{(1)}(X_A^{(K_\infty)})\deg(f) + (r - \tilde{r})\deg(f). \end{aligned}$$

Here we note that Lemma 3.6 can indeed be applied since f divides w_n and therefore $\operatorname{rank}_{(f,w_n)}(M) = \operatorname{rank}_{(f)}(M)$ for every Λ-module M. □

COROLLARY 4.9. *In the situation of Theorem 4.8, if a divisor f of w_n is relatively prime to $F_{X_A^{(K_\infty)}}$, then it is also coprime with $F_{X_A^{(\tilde{K}_\infty)}}$ for each $\tilde{K}_\infty \in \mathcal{E}(K_\infty, n)$ satisfying* $\operatorname{rank}_\Lambda(X_A^{(\tilde{K}_\infty)}) = \operatorname{rank}_\Lambda(X_A^{(K_\infty)})$.

COROLLARY 4.10. *Let A be an abelian variety defined over K, and let K_∞/K be a $\mathbb{Z}_p$-extension. We assume that A has potentially good ordinary reduction at p. If* $\operatorname{rank}_\Lambda(X_A^{(K_\infty)}) = 0$, *then there exists a neighbourhood $U = \mathcal{E}(K_\infty, m)$ such that $T \mid F_{X_A^{(K_\infty)}}$ if and only if $T \mid F_{X_A^{(\tilde{K}_\infty)}}$ for all $\tilde{K}_\infty \in U$.*

Proof. By Theorem 4.1, we can choose an integer $m \in \mathbb{N}$ such that $\operatorname{rank}_\Lambda(X_A^{(\tilde{K}_\infty)}) = 0$ for every $\tilde{K}_\infty \in \mathcal{E}(K_\infty, m)$. Now apply Corollary 4.9. □

We note that the assertion in this special case also follows from Lemma 3.2 because $(X_A^{(K_\infty)})_\Gamma$ is finite if and only if $\operatorname{rank}_\Lambda(X_A^{(K_\infty)}) = 0$ and $T \nmid F_{X_A^{(K_\infty)}}$.

REMARK 4.11. The analogous statement for classical Iwasawa modules, i.e. projective limits of ideal class groups in $\mathbb{Z}_p$-extensions (as defined in Section 2), is not true in this generality. In fact, the corresponding Iwasawa modules $X_A^{(K_\infty)}$ are always Λ-torsion. However, if $T \nmid F_{X_A^{(K_\infty)}}$ for some $\mathbb{Z}_p$-extension K_∞/K, then it is nevertheless possible that $T \mid F_{X_A^{(\tilde{K}_\infty)}}$ for $\mathbb{Z}_p$-extensions $\tilde{K}_\infty$ of K which are arbitrarily close to K_∞ with respect to Greenberg's topology.

The following example is from [Kle19, Remark 4.8]. Let K be imaginary quadratic. We suppose that p splits in $K/\mathbb{Q}$ and that p does not divide the class number of K. It is well-known that there exist two unique $\mathbb{Z}_p$-extensions of K in which only one of the two primes of K above p ramifies; our hypotheses imply that the classical Iwasawa modules are trivial for these two $\mathbb{Z}_p$-extensions of K. On the other hand, in all but these two $\mathbb{Z}_p$-extensions of K, both primes above p are ramified. Since p is totally split in $K/\mathbb{Q}$, it follows from class field theory that the characteristic power series $F_{X_A^{(K_\infty)}}$ is divisible by T (i.e. $X_A^{(K_\infty)}/TX_A^{(K_\infty)}$ is infinite) if K_∞ is different from the two exceptional $\mathbb{Z}_p$-extensions of K mentioned above.

The reason behind this different behaviour of the classical Iwasawa modules is the lack of a control theorem: in the above example, the kernel of the natural map

$$(X^{(K_\infty)})_{\Gamma_n} \longrightarrow X^{(K_n)}$$

has infinite order (here $\Gamma_n = \mathrm{Gal}(K_\infty/K_n)$, as in Theorem 3.1).

We note that common factors of $F_{X_A^{(K_\infty)}}$ with some w_n arise naturally if the Mordell-Weil groups $A(K_n)$ of the layers K_n of K_∞ have bounded ranks. As an illustration, we would like to mention the following interesting example given by Wüthrich.

EXAMPLE 4.12. Let $p = 3$, and $A = E$ denote the elliptic curve defined by the equation

$$E\colon y^2 = x^3 + x^2 - 18x + 25.$$

Then A has good ordinary reduction at p, and Wüthrich has shown that the characteristic power series of $X = X_A^{(\mathbb{Q}_\infty)}$ is given by $w_3 \cdot w_9$ (see [Wut07, Proposition 11.1]).

We recall that for any number field F, we have an exact sequence

$$0 \longrightarrow A(F) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathrm{Sel}_A(F) \longrightarrow Ш_A(F) \longrightarrow 0, \tag{9}$$

where $Ш_A(F)$ denotes the so-called Tate-Shafarevich group of A over F.

If $K_\infty = \bigcup_n K_n$ denotes the $\mathbb{Z}_p$-extension of a number field K, then we define

$$Ш_A(K_\infty) := \varinjlim Ш_A(K_n),$$

where the direct limit is induced by the restriction maps. Note that the Pontryagin dual $Ш_A(K_\infty)^\vee$ is a finitely generated Λ-submodule of $X_A^{(K_\infty)}$.

REMARK 4.13. If $\mathrm{rank}_\Lambda(X_A^{(K_\infty)}) = 0$, then the characteristic power series $F_{X_A^{(K_\infty)}}$ divides $w_n \cdot F_{Ш_A(K_\infty)^\vee}$ for some $n \in \mathbb{N}$.

Indeed, since $\mathrm{rank}_\Lambda(X_A^{(K_\infty)}) = 0$, the exact sequences (9) imply that

$$\mathrm{rank}_\mathbb{Z}(A(K_n)) \leq \lambda(X_A^{(K_\infty)})$$

for every $n \in \mathbb{N}$ (see the proof of [Gre01, Corollary 4.9]). In particular, we can choose $n \in \mathbb{N}$ such that $\mathrm{rank}_\mathbb{Z}(A(K_\infty)) = \mathrm{rank}_\mathbb{Z}(A(K_n))$; therefore $A(K_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is annihilated by w_n. The sequence (9) remains exact if we take direct limits and then Pontryagin duals, therefore obtaining an exact sequence

$$0 \longrightarrow Ш_A(K_\infty)^\vee \longrightarrow X_A^{(K_\infty)} \longrightarrow (A(K_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \longrightarrow 0$$

of Λ-modules. Since $(A(K_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \cong \mathbb{Z}_p^{\mathrm{rank}_\mathbb{Z}(A(K_n))}$ is annihilated by w_n, it follows that

$$F_{X_A^{(K_\infty)}} = F_{Ш_A(K_\infty)^\vee} \cdot f$$

for some divisor f of w_n.

In other words, the Mordell-Weil groups typically contribute factors of some w_n to $F_{X_A^{(K_\infty)}}$; divisors which are coprime with the w_n should stem from the Tate-Shafarevich groups.

Example 4.14. The power series of $Ш_A(K_\infty)^\vee$ can be non-trivial. For example, let $A = E$ be the elliptic curve defined over $\mathbb{Q}$ by the equation

$$E\colon y^2 + xy = x^3 - 6511x - 203353.$$

Then E has good ordinary reduction at $p = 7$, and $\operatorname{rank}_{\mathbb{Z}}(E(\mathbb{Q}_n)) = 0$ for each layer $\mathbb{Q}_n$ of the cyclotomic $\mathbb{Z}_7$-extension $\mathbb{Q}_\infty^c$ of $\mathbb{Q}$. Therefore $F_{X_A^{(K_\infty)}} = F_{Ш_E(K_\infty)^\vee}$. Now

$$v_p(|Ш_E(\mathbb{Q}_n)|) = p^n + 2n - 1$$

for every $n \in \mathbb{N}$ (see [Wut14, p. 13, Example 2]). In view of [Kle21a, Remark 5.4], this means that $\lambda(Ш_A(K_\infty)^\vee) = 2$.

We conclude the current section by mentioning some known results concerning the coranks of Selmer groups, focusing on the good ordinary setting.

Remark 4.15. Mazur (see [Maz72, p. 104]) conjectured the following: if $K_\infty = K_\infty^c$ denotes the *cyclotomic* $\mathbb{Z}_p$-extension of K and if A has good ordinary reduction at the primes $v \in S_p$, then $X_A^{(K_\infty)}$ is Λ-torsion.

If this holds true, then Theorem 4.1 implies that there exists some $n \in \mathbb{N}$ such that $\operatorname{rank}_\Lambda(X_A^{(\tilde{K}_\infty)}) = 0$ for every $\mathbb{Z}_p$-extension $\tilde{K}_\infty$ of K which coincides with the cyclotomic $\mathbb{Z}_p$-extension at least up to the n-th layer. In particular, the statements of Theorems 4.4, 4.7 and 4.8 simplify. Moreover, one can bound λ-invariants via [Kle21a, Theorem 4.11] in this case.

It follows from deep results of Kato and Rohrlich (see [Kat04] and [Roh84]) that Mazur's Conjecture holds true if (under the hypotheses of the conjecture) K is abelian over $\mathbb{Q}$ and $A = E$ denotes an elliptic curve which is defined over $\mathbb{Q}$.

Moreover, if (in addition to the hypotheses of the conjecture) $A = E$ denotes an elliptic curve defined over $\mathbb{Q}$, with complex multiplication by the ring of integers of an imaginary quadratic number field K, and if the prime $p \neq 2$ splits in $K/\mathbb{Q}$, then $X_A^{(K_\infty^c)}$ is Λ-torsion by results of Rubin (see [Rub88, Theorem 4.4]).

On the other hand, it is known that $X_A^{(K_\infty^a)}$ can be non-torsion for the so-called *anticyclotomic* $\mathbb{Z}_p$-extension K_∞^a of an imaginary quadratic number field K; in fact, if E is an elliptic curve defined over $\mathbb{Q}$ with good ordinary reduction at p, and if the primes dividing the conductor of E are split in $K/\mathbb{Q}$ (this is the so-called *Heegner hypothesis*), then it is known that $\operatorname{rank}_\Lambda(X_A^{(K_\infty^a)}) > 0$ by [Ber95, Theorem A] and [Cor02] (here a set of primes p of density zero has to be excluded from consideraton, see Corollary 1.2 and [Ber95, Remark on p. 166]). Note that in this situation the Hasse-Weil L-series $L(E/K, s)$ has an odd order zero at $s = 1$, and that E does *not* have CM by the ring of integers $\mathcal{O}_K$ of K.

Suppose that A has good ordinary reduction at the primes $v \in S_p$, $p \neq 2$, and let K_∞/K be a $\mathbb{Z}_p$-extension such that each prime $v \in S_p$ ramifies in K_∞ and such that no prime $v \in S_{\mathrm{br}}$ splits completely in K_∞. Under these hypotheses, which are for example satisfied for the cyclotomic $\mathbb{Z}_p$-extension of any number field K, Perrin-Riou (see [PR92]) constructed a certain p-adic height pairing on Selmer groups. If this pairing is non-degenerate, then $X_A^{(K_\infty)}$ is Λ-torsion (see also related work by Schneider in [Sch85]).

In the previous remark, we focused on the good ordinary situation. On the other hand, if A has good and *supersingular* reduction at the primes of S_p, then the Λ-module $X_A^{(K_\infty)}$ usually is non-torsion. We will study this situation in more detail in Section 6.

5. Fine Selmer groups. In this section, we prove results for fine Selmer groups which are analogous to Theorem 3.4 and Lemma 3.2. These results can be used in order to obtain bounds on Iwasawa invariants of (Pontryagin duals of) fine Selmer groups, as in Section 4. Recall the definition of fine Selmer groups from Section 2, and the notation introduced at the beginning of Section 3. For simplicity, we will assume throughout this section that $p \neq 2$. Many of the following results are valid without any restriction on the reduction type of A at p.

THEOREM 5.1. *Let A be an abelian variety defined over a number field K, and let K_∞/K be a $\mathbb{Z}_p$-extension, p an odd prime.*

(a) *Then the kernels and cokernels of the canonical projection maps*

$$\mathrm{pr}_{n,0}^{(K_\infty)} : (Y_A^{(K_\infty)})_{\Gamma_n} \longrightarrow Y_A^{(K_n)}$$

are finite and of bounded order as $n \to \infty$.

(b) *Suppose that no prime $v \in S := S_p \cup S_{\mathrm{br}}$ is totally decomposed in K_∞/K, and that $A(K_{\infty,v})[p^\infty]$ is finite for every $v \in S$, where $K_{\infty,v} = \bigcup_n K_{n,v}$ denotes the completion at some prime above v.*
Then there exist constants $m, C_1, C_2 \in \mathbb{N}$ such that

$$|\ker(\mathrm{pr}_{n,0}^{(\tilde{K}_\infty)})| \le C_1 \quad \text{and} \quad |\mathrm{coker}(\mathrm{pr}_{n,0}^{(\tilde{K}_\infty)})| \le C_2$$

for each $n \in \mathbb{N}$ and for every $\tilde{K}_\infty \in \mathcal{E}(K_\infty, m)$.

(c) *Suppose that the pair (K_∞, A) satisfies Property (F) in the sense of Definition 3.3. Then the conclusion of (b) is valid.*

REMARK 5.2. In (c), one can allow primes $v \in S_{\mathrm{br}}$ which are completely split in K_∞/K by restricting to a subset of $\mathcal{E}(K_\infty, m)$, as in Remark 3.5.

Proof of Theorem 5.1. For (a), see [Lim20, Theorem 3.3]. In order to prove (b), we let

$$B = A(K_\infty)[p^\infty],$$

and we similarly define $B^{(\tilde{K}_\infty)}$ for each $\mathbb{Z}_p$-extension $\tilde{K}_\infty$ of K. It follows from (the proof of) [Lim20, Theorem 3.3] that

$$|\mathrm{coker}(\mathrm{pr}_{n,0}^{(\tilde{K}_\infty)})| \le |H^1(\Gamma_n, B^{(\tilde{K}_\infty)})|$$

for each $\tilde{K}_\infty$ and every $n \in \mathbb{N}$. Note that $B^{(\tilde{K}_\infty)}$ might be infinite. We have exact sequences

$$0 \longrightarrow B_{\mathrm{div}}^{(\tilde{K}_\infty)} \longrightarrow B^{(\tilde{K}_\infty)} \longrightarrow C^{(\tilde{K}_\infty)} \longrightarrow 0,$$

where $C^{(\tilde{K}_\infty)}$ is finite and $B_{\mathrm{div}}^{(\tilde{K}_\infty)}$ is divisible. Taking Pontryagin duals, we obtain exact sequences

$$0 \longrightarrow \tilde{C} \longrightarrow \tilde{U} \longrightarrow \tilde{V} \longrightarrow 0, \tag{10}$$

where $\tilde{C} = (C^{(\tilde{K}_\infty)})^\vee$, etc.

Note that $\tilde{V}$ is $\mathbb{Z}_p$-free. Since the kernel of multiplication by w_n on $B^{(\tilde{K}_\infty)}$ is equal to $A(\tilde{K}_n)[p^\infty]$ and therefore is finite, it follows that the cokernel (and therefore, by a dimension argument, also the kernel) of the induced map on $\tilde{V}$ is finite; as $\tilde{V}$ is $\mathbb{Z}_p$-free, we may conclude that actually $\tilde{V}[w_n] = \{0\}$.

Therefore the snake lemma implies that we have injections

$$0 \longrightarrow \tilde{C}_{\Gamma_n} \longrightarrow \tilde{U}_{\Gamma_n} \tag{11}$$

for each $n \in \mathbb{N}$. Now choose $m \in \mathbb{N}$ such that

$$m \geq v_p(|B|) + 1,$$

and suppose that $\tilde{K}_\infty \in \mathcal{E}(K_\infty, m)$. Since $\tilde{U}_{\Gamma_n}$ is isomorphic to the Pontryagin dual of $(B^{(\tilde{K}_\infty)})^{\Gamma_n}$, we may conclude that

$$|H^1(\Gamma_n, \tilde{C})| \overset{(11)}{\leq} |\tilde{U}_{\Gamma_n}| = |(B^{(\tilde{K}_\infty)})^{\Gamma_n}| = |B^{\Gamma_n}| \leq |B|$$

for every $n \leq m$. By the choice of m, this means that there exists $n < m$ such that

$$|H^1(\Gamma_n, \tilde{C})| = |H^1(\Gamma_{n+1}, \tilde{C})|;$$

Nakayama's Lemma implies that

$$|\tilde{C}| = |H^1(\Gamma_n, \tilde{C})| \leq |B|.$$

We have thus shown that $|C^{(\tilde{K}_\infty)}| \leq |B|$ for every $\tilde{K}_\infty \in \mathcal{E}(K_\infty, m)$ (see also the end of the proof of [Kle21a, Theorem 4.5]). Now

$$|H^1(\Gamma_n, B^{(\tilde{K}_\infty)})| = |(B^{(\tilde{K}_\infty)})_{\Gamma_n}| = |\tilde{U}^{\Gamma_n}| = |\tilde{C}^{\Gamma_n}| \leq |\tilde{C}|$$

by the exact sequence (10), since we have already seen that $\tilde{V}^{\Gamma_n} = \tilde{V}[w_n] = \{0\}$. This proves the uniform boundedness of $|\mathrm{coker}(\mathrm{pr}_{n,0}^{(\tilde{K}_\infty)})|$ on $\mathcal{E}(K_\infty, m)$.

Now suppose that m is moreover large enough to ensure that each $v \in S$ is non-split in K_m/K_{m-1}. If $\tilde{K}_\infty \in \mathcal{E}(K_\infty, m)$ is arbitrary, then this means that no prime $v \in S$ splits completely in $\tilde{K}_\infty/K$; in fact the number of primes of $\tilde{K}_\infty$ above S is constant on $\mathcal{E}(K_\infty, m)$.

Following the proof of [Lim20, Theorem 3.3], the order of $\ker(\mathrm{pr}_{n,0}^{(\tilde{K}_\infty)})$ can be bounded by

$$\sum_{v_n \in S(K_n)} |H^1(\Gamma_{v_n}, A(\tilde{K}_{\infty,v_n})[p^\infty])|,$$

where $\Gamma_{v_n} = \mathrm{Gal}(\tilde{K}_{\infty,v_n}/\tilde{K}_{n,v_n})$ (here $\tilde{K}_{n,v_n}$ denotes the completion at v_n, and $\tilde{K}_{\infty,v_n}$ denotes the union of the completions of the finite subextensions $\tilde{K}_l \subseteq \tilde{K}_\infty$, $l \geq n$, at some prime dividing v_n).

For any $v_n \in S(K_n)$, the same reasoning as in the first part of the proof, applied to the $\mathbb{Z}_p$-extension $\tilde{K}_{\infty,v_n}/\tilde{K}_{n,v_n}$, shows that the local Tamagawa factors $|H^1(\Gamma_{v_n}, A(\tilde{K}_{\infty,v_n})[p^\infty])|$ (see also [CS10, Lemma 3.4]) are bounded for $n \geq m$ and $\tilde{K}_\infty \in \mathcal{E}(K_\infty, m)$ (here we apply the hypothesis that $A(K_{\infty,v})[p^\infty]$ is finite for each $v \in S$).

This concludes the proof of (b). Finally, (c) follows from [Kle21a, Theorem 5.13]. $\square$

REMARK 5.3.

(1) Suppose that K_∞ is the cyclotomic $\mathbb{Z}_p$-extension of K, and that A has good reduction at some $v \in S_p$. Then the finiteness of $A(K_\infty)[p^\infty]$ and $A(K_{\infty,v})[p^\infty]$ follows from a result of Imai (see [Ima75]).
(2) The potential finiteness of $A(L_{\infty,v})[p^\infty]$ for infinite extensions $L_{\infty,v}$ of $\mathbb{Q}_p$ has also been studied in other settings; see for example [Oze09] for a result in the potentially good ordinary setting.
(3) If A has potentially good ordinary reduction at the primes $v \in S_p$, and if each $v \in S_p$ ramifies in a $\mathbb{Z}_p$-extension K_∞ of K, then $A(K_\infty)[p^\infty]$ is finite by [Gre03, Proposition 3.2(ii)].
(4) If A has good supersingular reduction at $v \mid p$ and p does not ramify in $K/\mathbb{Q}$, then $A(K_{\infty,v})[p^\infty]$ is actually trivial by [Maz72, Lemma 5.11] (see also [Kle21a, Remark 5.12]).

Using the same approach as in Section 4, we can derive from Theorem 5.1 the following results concerning the ranks and Iwasawa invariants of fine Selmer groups. In statement (c) below, we make use of the notation $f^{(k)}$ from Theorem 4.7.

THEOREM 5.4. *Let p be odd, let K_∞/K be a $\mathbb{Z}_p$-extension, and let A be an abelian variety defined over K.*

(a) *Then there exists a neighbourhood $U = \mathcal{E}(K_\infty, m)$ such that*

$$\operatorname{rank}_\Lambda(Y_A^{(\tilde{K}_\infty)}) \leq \operatorname{rank}_\Lambda(Y_A^{(K_\infty)})$$

for each $\tilde{K}_\infty \in U$.
In fact, it suffices to choose m large enough to ensure that

$$p^m \geq \lambda(Y_A^{(K_\infty)}) + 1.$$

If $\lambda(Y_A^{(\tilde{K}_\infty)})$ is bounded on U, then there exists some $n \geq m$ such that

$$\operatorname{rank}_\Lambda(Y_A^{(\tilde{K}_\infty)}) = \operatorname{rank}_\Lambda(Y_A^{(K_\infty)})$$

for every $\tilde{K}_\infty \in \mathcal{E}(K_\infty, n)$.

(b) *Suppose that the hypotheses of either Theorem 5.1,(b) or Theorem 5.1,(c) are satisfied. Then there exists a neighbourhood $U = \mathcal{E}(K_\infty, n)$ of K_∞ such that*

$$\mu(Y_A^{(\tilde{K}_\infty)}) \leq \mu(Y_A^{(K_\infty)})$$

for each $\tilde{K}_\infty \in U$ satisfying $\operatorname{rank}_\Lambda(Y_A^{(\tilde{K}_\infty)}) = \operatorname{rank}_\Lambda(Y_A^{(K_\infty)})$.

(c) *Suppose that the hypotheses of either Theorem 5.1,(b) or Theorem 5.1,(c) are satisfied, and let $f \in \Lambda$ be a distinguished polynomial which is coprime with each w_n, $n \in \mathbb{N}$. Then there exists a neighbourhood $U = \mathcal{E}(K_\infty, n)$ such that*

$$f^{(k)}(Y_A^{(\tilde{K}_\infty)}) \leq f^{(k)}(Y_A^{(K_\infty)}) + k \cdot (\operatorname{rank}_\Lambda(Y_A^{(K_\infty)}) - \operatorname{rank}_\Lambda(Y_A^{(\tilde{Y}_\infty)})).$$

for each $\tilde{K}_\infty \in U$.

(d) *Let $n \in \mathbb{N}$ be arbitrary, and let $U = \mathcal{E}(K_\infty, n)$. Let f be a non-trivial divisor of w_n, and let $\tilde{K}_\infty \in U$. Then*

$$f^{(1)}(Y_A^{(\tilde{K}_\infty)}) = f^{(1)}(Y_A^{(K_\infty)}) + \operatorname{rank}_\Lambda(Y_A^{(K_\infty)}) - \operatorname{rank}_\Lambda(Y_A^{(\tilde{Y}_\infty)}).$$

In particular, if $\mathrm{rank}_\Lambda(Y_A^{(\tilde{K}_\infty)}) = \mathrm{rank}_\Lambda(Y_A^{(K_\infty)})$*, then*

$$f^{(1)}(Y_A^{(\tilde{K}_\infty)}) = f^{(1)}(Y_A^{(K_\infty)}).$$

(e) *Under the hypotheses of (b), suppose that* $\mathrm{rank}_\Lambda(Y_A^{(K_\infty)}) = 0$ *and* $\mu(Y_A^{(K_\infty)}) = 0$*. Then there exists a neighbourhood* $U = \mathcal{E}(K_\infty, n)$ *such that*

$$\mathrm{rank}_\Lambda(Y_A^{(\tilde{K}_\infty)}) = 0, \quad \mu(Y_A^{(\tilde{K}_\infty)}) = 0 \quad \textit{and} \quad \lambda(Y_A^{(\tilde{K}_\infty)}) \leq \lambda(Y_A^{(K_\infty)})$$

for every $\tilde{K}_\infty \in U$*.*

Proof. For (a), (b), (c) and (d), mimic the proofs of Theorem 4.1, Corollary 4.2 and Theorems 4.4, 4.7 and 4.8. The first part of (e) is a special case of (a) and (b); for the statement concerning the λ-invariants, one can use the arguments from the proof of [Kle21a, Theorem 4.11]. □

Let us discuss situations where $Y_A^{(K_\infty)}$ is Λ-torsion (cf. also Remark 4.15).

REMARK 5.5. Since $Y_A^{(K_\infty)}$ is a quotient of $X_A^{(K_\infty)}$ by definition (see also the exact sequence (2) at the end of Section 2), $Y_A^{(K_\infty)}$ is conjectured to be Λ-torsion if $K_\infty = K_\infty^c$ denotes the cyclotomic $\mathbb{Z}_p$-extension of K and if A has good ordinary reduction at the primes $v \in S_p$; see Remark 4.15 for a list of situations where this conjecture is known to be true.

If A is an abelian variety, $p \neq 2$ and $S = S_p \cup S_{\mathrm{br}}$ (more generally, if S is a finite set of places of K containing S_p and S_{br}), then $Y_A^{(K_\infty)}$ is Λ-torsion if and only if

$$H^2(\mathrm{Gal}(M_S(K)/K_\infty), A[p^\infty]) = 0 \tag{12}$$

(see [Lim17, Lemma 7.1]). Here $M_S(K)$ denotes the maximal algebraic pro-p-extension of K which is unramified outside of S, as in Section 2. The condition (12) is called the *weak Leopoldt conjecture for A over K_∞*. It is conjectured that (12) is true for the cyclotomic $\mathbb{Z}_p$-extension $K_\infty = K_\infty^c$, $p \neq 2$, and any abelian variety A defined over K (see [Gre89, Conjecture 3]). In particular, whereas $X_A^{(K_\infty^c)}$ is conjectured to be Λ-torsion only if A has good ordinary reduction at the primes of K above p, $Y_A^{(K_\infty^c)}$ should be a Λ-torsion module independently of the reduction type of A at p. More recently, the validity of a weak Leopoldt conjecture (12) and its relation to the Λ-ranks of $X_A^{(K_\infty)}$ and $Y_A^{(K_\infty)}$ has been studied more generally for arbitrary $\mathbb{Z}_p$-extensions, see also Section 6 below.

In addition to the cases from Remark 4.15, $Y_A^{(K_\infty)}$ is known to be Λ-torsion if $A = E$ is an elliptic curve defined over $\mathbb{Q}$ having good supersingular reduction at p and $K/\mathbb{Q}$ is abelian, by the deep work of Kato (see [Kat04]); the good supersingular case of a CM elliptic curve over an imaginary quadratic number field K has also been studied by McConnell (see [McC96]).

Moreover, it has been shown by Bertolini (see [Ber01, Theorem 5.4] and [Cor02]) that the weak Leopoldt conjecture holds for elliptic curves $A = E$ over the anticyclotomic $\mathbb{Z}_p$-extension K_∞^a of K, provided that E is defined over $\mathbb{Q}$, has good ordinary reduction at p and that K is an imaginary quadratic field satisfying the Heegner hypothesis (cf. also Remark 4.15); in fact Bertolini has to exclude, for every $A = E$ and K, a set of primes p of density 0 from consideration (see [Ber01, Assumption B]).

For elliptic curves, Wüthrich defined in [Wut07] a p-adic height pairing on the fine Selmer group $\mathrm{Sel}_{A,0}(K_\infty) = (Y_A^{(K_\infty)})^\vee$ over a $\mathbb{Z}_p$-extension K_∞/K, and he proved that $Y_A^{(K_\infty)}$ is Λ-torsion if this pairing is non-degenerate (see [Wut07, Theorem 6.1]). This result does not assume $A = E$ to have good reduction at the primes above p.

We can deduce from Theorem 5.4 that the validity of the weak Leopoldt conjecture (12) for some $\mathbb{Z}_p$-extension is an 'open condition' with regard to Greenberg's topology (defined in the Introduction). More precisely, we have the following

COROLLARY 5.6. *Let A be an abelian variety defined over the number field K, and let K_∞/K be a $\mathbb{Z}_p$-extension (recall that p is odd throughout this section). If the weak Leopoldt conjecture* (12) *holds for A over K_∞, then there exists a natural number m such that* (12) *holds over every $\mathbb{Z}_p$-extension $\tilde{K}_\infty$ of K contained in $\mathcal{E}(K_\infty, m)$.*

Proof. We have $\mathrm{rank}_\Lambda(Y_A^{(K_\infty)}) = 0$ by [Lim17, Lemma 7.1]. Theorem 5.4(a) implies that $Y_A^{(\tilde{K}_\infty)}$ is Λ-torsion for every $\tilde{K}_\infty$ which is contained in some neighbourhood $\mathcal{E}(K_\infty, m)$ of K_∞. □

We note that a similar property is also known for the classical weak Leopoldt conjecture of ideal class groups: the subset of $\mathcal{E}(K)$ of $\mathbb{Z}_p$-extensions of K for which the weak Leopoldt conjecture is true is an open set (see [Bab77, Theorem 3]). In fact this subset is *dense* with respect to Greenberg's topology by [Bab82, Corollary 3]. In Section 7, we will prove that a similar result is true for the weak Leopoldt conjecture for abelian varieties: the corresponding subset of $\mathcal{E}(K)$ is dense provided that the conjecture is known over the cyclotomic $\mathbb{Z}_p$-extension of K (see Corollary 7.13).

6. Coranks of Selmer groups and the weak Leopoldt conjecture. As we have already noticed in Remark 5.5, the weak Leopoldt conjecture for A over a $\mathbb{Z}_p$-extension K_∞/K holds if and only if $Y_A^{(K_\infty)}$ is Λ-torsion. In this section, we study the relation of this conjecture to the Λ-rank of $X_A^{(K_\infty)}$.

Suppose first that $A = E$ is an elliptic curve defined over $\mathbb{Q}$, with good supersingular reduction at each $v \in S_p$, $p \neq 2$. Then the Pontryagin dual $X_A^{(K_\infty)}$ of the Selmer group, K_∞/K any $\mathbb{Z}_p$-extension, is known to be non-torsion. In fact, it follows from [IP06, Proposition 5.3] that

$$\mathrm{rank}_\Lambda(X_A^{(K_\infty)}) \geq [K : \mathbb{Q}] \tag{13}$$

for every $\mathbb{Z}_p$-extension K_∞/K. Moreover, if the weak Leopoldt conjecture (12) holds for $A = E$ over K_∞, then $\mathrm{rank}_\Lambda(X_A^{(K_\infty)}) = [K : \mathbb{Q}]$ (see [IP06, Proposition 6.1]).

For general abelian varieties, M. F. Lim generalised the above results and proved the following

THEOREM 6.1. *Let $p \neq 2$, and let A be a g-dimensional abelian variety defined over a number field K which has good supersingular reduction at each $v \in S_p$. Let K_∞/K be a $\mathbb{Z}_p$-extension.*

Then the weak Leopoldt conjecture (12) *for A over K_∞ is true if and only if*

$$\mathrm{rank}_\Lambda(X_A^{(K_\infty)}) = g \cdot [K : \mathbb{Q}].$$

Proof. See [Lim22, Theorem 2.2]. □

In particular, if A has good supersingular reduction at each $v \in S_p$, then the Λ-corank of the Selmer groups over any $\mathbb{Z}_p$-extension of K should, as long as the weak Leopoldt conjecture holds, be constant. The analogous fact in the good ordinary setting is not true, as already the coranks over the cyclotomic and anticyclotomic $\mathbb{Z}_p$-extensions of an imaginary quadratic base field K can differ (see also Remark 4.15).

In fact, the inequality (13) is also valid in a more general setting:

LEMMA 6.2. *Let $p \neq 2$, let A be an abelian variety of dimension g which is defined over a number field K, and let K_∞/K be a $\mathbb{Z}_p$-extension. We assume that each $v \in S_p$ is totally ramified in K_∞ and that A has good reduction at every $v \in S_p$. Then*

$$\operatorname{rank}_\Lambda(X_A^{(K_\infty)}) \geq g \cdot \sum_{v \in S_p^s} [K_v : \mathbb{Q}_p] - g \cdot \sum_{v \in S_p \setminus S_p^s} [K_v : \mathbb{Q}_p],$$

where $S_p^s \subseteq S_p$ shall denote the subset of primes v of K above p at which A has good supersingular reduction.

In particular, if $S_p = S_p^s$, then $\operatorname{rank}_\Lambda(X_A^{(K_\infty)}) \geq g \cdot [K : \mathbb{Q}]$.

Proof. We can use the same idea as in the proof of [IP06, Proposition 5.3]. Let $S = S_p \cup S_{\mathrm{br}}$, and let $M_S(K)$ denote the maximal algebraic pro-p-extension of K unramified outside of S, as in Section 2. First, it follows from results of Schneider (see [Sch85, §3, Lemma 2] and [Sch87, Proposition on p. 596]) that

$$\operatorname{rank}_\Lambda(H^1(\operatorname{Gal}(M_S(K)/K_\infty), A[p^\infty])^\vee)$$

is greater than or equal to

$$\sum_{v \in S_p} g \cdot [K_v : \mathbb{Q}_p] - \sum_{v \in S_p} \operatorname{rank}_\Lambda(H^1(K_{\infty,v}, A)[p^\infty]^\vee)$$

and that

$$\operatorname{rank}_\Lambda(H^1(K_{\infty,v}, A)[p^\infty]^\vee) = r_v \cdot [K_v : \mathbb{Q}_p]$$

for each $v \in S_p$, where r_v denotes the p-rank of $\tilde{A}(k_v)[p^\infty]$ (here $\tilde{A}$ shall be the reduction of A at v and k_v denotes the residue field of the completion K_v). Note that the set Σ used by Schneider corresponds to the set of all primes of K which ramify in K_∞, and therefore $\Sigma = S_p$ by assumption.

Since A has good reduction at every $v \in S_p$, it follows that $r_v = 0$ if $v \in S_p^s$, and $r_v = g$ otherwise (see also [Lim22, Lemma 2.3]). Therefore

$$\operatorname{rank}_\Lambda(H^1(\operatorname{Gal}(M_S(K)/K_\infty), A[p^\infty])^\vee) \geq \sum_{v \in S_p \setminus S_p^s} g \cdot [K_v : \mathbb{Q}_p].$$

Moreover, it follows from the (S-variant of the) definition of the Selmer group (see Section 2) by taking Pontryagin duals that we have an exact sequence

$$\prod_{v \in S(K_\infty)} H^1(K_{\infty,v}, A)[p^\infty]^\vee \longrightarrow H^1(\operatorname{Gal}(M_S(K)/K_\infty), A[p^\infty])^\vee \twoheadrightarrow X_A^{(K_\infty)}.$$

Note that [Lim22, Lemma 2.3] implies that $\prod_{v \in S(K_\infty), v \nmid p} H^1(K_{\infty,v}, A)[p^\infty]$ is Λ-cotorsion; this even holds if some $v \in S_{\mathrm{br}}$ is completely split in K_∞. The lemma follows because $\operatorname{rank}_\Lambda(H^1(K_{\infty,v}, A)[p^\infty]^\vee) = 0$ for each $v \in S_p^s$. □

REMARK 6.3. Let A be an abelian variety defined over K, and let K_∞/K be a $\mathbb{Z}_p$-extension. We assume that A has good supersingular reduction at each $v \in S_p$, and that each $v \in S_p$ is totally ramified in K_∞. Suppose that the weak Leopoldt conjecture (12) holds for A over K_∞. In view of Theorem 6.1, this means that

$$\mathrm{rank}_\Lambda(X_A^{(K_\infty)}) = g \cdot [K : \mathbb{Q}],$$

where g denotes the dimension of A. In fact, it follows from Corollary 5.6 and Theorem 6.1 that

$$\mathrm{rank}_\Lambda(X_A^{(\tilde{K}_\infty)}) = g \cdot [K : \mathbb{Q}]$$

for all $\tilde{K}_\infty \in \mathcal{E}(K_\infty, m)$, provided that m is sufficiently large.

This simplifies our results from Section 4: for example, the following corollary follows from Theorem 4.8.

Corollary 6.4. *Let A, K_∞/K and $\mathcal{E}(K_\infty, m)$ be as in Remark 6.3.*

$$f^{(1)}(X_A^{(\tilde{K}_\infty)}) = f^{(1)}(X_A^{(K_\infty)})$$

for each non-trivial divisor f of w_n and every $\tilde{K}_\infty \in \mathcal{E}(K_\infty, m)$.

7. Proofs of our main boundedness results. Fix a number field K and a prime p. In this final section we study the (absolute) boundedness of Iwasawa invariants of Selmer groups on the set $\mathcal{E}(K)$ of $\mathbb{Z}_p$-extensions of K. Sometimes we fix a suitable multiple $\mathbb{Z}_p$-extension $\mathbb{L}_\infty$ of K and consider only $\mathbb{Z}_p$-extensions K_∞ of K contained in $\mathbb{L}_\infty$; we write $\mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ for the set of these $\mathbb{Z}_p$-extensions of K.

As mentioned in the Introduction, it has been shown by Greenberg that the set $\mathcal{E}(K)$ is *compact* with respect to the topology defined in the Introduction (see [Gre73, Section 3]); the same proof also implies that the sets $\mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ are compact. Therefore one possible approach for proving the global boundedness of an Iwasawa invariant could be to prove its local boundedness (in the sense of Section 4) near *every* $K_\infty \in \mathcal{E}(K)$. However, as we will see soon (see Section 7.2), this approach seems not feasible in practice.

7.1. Bounding Λ-ranks. We start with a result bounding the Λ-ranks of $X_A^{(K_\infty)}$ and $Y_A^{(K_\infty)}$ for $K_\infty \in \mathcal{E}(K)$.

Lemma 7.1. *Let A be an abelian variety defined over the number field K, and let $p \neq 2$. Then there exists a constant $C_1 \in \mathbb{N}$ such that*

$$\mathrm{rank}_\Lambda(Y_A^{(K_\infty)}) \le C_1$$

for every $K_\infty \in \mathcal{E}(K)$. If A has potentially good ordinary reduction at each prime $v \in S_p$, then there exists a constant $C_2 \in \mathbb{N}$ such that

$$\mathrm{rank}_\Lambda(X_A^{(K_\infty)}) \le C_2$$

for every $K_\infty \in \mathcal{E}(K)$.

Proof. Let K_∞ be an arbitrary $\mathbb{Z}_p$-extension of K. In view of Lemma 3.2 and Theorem 5.1(a), the canonical maps

$$\mathrm{pr}_0^{(K_\infty)} : (X_A^{(K_\infty)})_\Gamma \longrightarrow X_0^{(K_\infty)} = \mathrm{Sel}_A(K)^\vee,$$

$\Gamma = \mathrm{Gal}(K_\infty/K)$, and

$$\mathrm{pr}_{0,0}^{(K_\infty)} : (Y_A^{(K_\infty)})_\Gamma \longrightarrow Y_0^{(K_\infty)} = \mathrm{Sel}_{A,0}(K)^\vee$$

have finite kernels and cokernels. Since $\mathrm{Sel}_A(K)^\vee$ and $\mathrm{Sel}_{A,0}(K)^\vee$ are finitely generated $\mathbb{Z}_p$-modules, it follows that

$$\mathrm{rank}_\Lambda(X_A^{(K_\infty)}) \le \mathrm{rank}_{\mathbb{Z}_p}(X_A^{(K_\infty)}/(T \cdot X_A^{(K_\infty)})) = \mathrm{rank}_{\mathbb{Z}_p}(\mathrm{Sel}_A(K)^\vee)$$

and similarly

$$\mathrm{rank}_\Lambda(Y_A^{(K_\infty)}) \le \mathrm{rank}_{\mathbb{Z}_p}(\mathrm{Sel}_{A,0}(K)^\vee).$$

Of course we also have $\mathrm{rank}_\Lambda(Y_A^{(K_\infty)}) \le \mathrm{rank}_\Lambda(X_A^{(K_\infty)})$ for all $K_\infty \in \mathcal{E}(K)$. □

7.2. Bounding μ-invariants. Now we turn to the μ-invariants. Let A be an abelian variety defined over K. If the (fine) Selmer groups are Λ-cotorsion for every $\mathbb{Z}_p$-extension of K, then we can bound μ-invariants by using the results from [Kle21a], at least in the good ordinary setting. However, without restrictions on the Λ-coranks the task of bounding μ-invariants is quite delicate.

Let $\mathbb{L}_\infty$ be a $\mathbb{Z}_p^2$-extension of K. If we assume that A has good reduction at each $v \in S_p$, then S_{br} will contain only primes v which do not divide p and therefore must be infinitely split in $\mathbb{L}_\infty/K$. In other words, for every $v \in S_{\mathrm{br}}$ there exists a $\mathbb{Z}_p$-extension $K_\infty \subseteq \mathbb{L}_\infty$ of K in which v is totally split. This means that our results on the local boundedness of μ-invariants (i.e. Theorems 4.4 and 5.4,(b) and Lemma 4.6) can not be applied to K_∞.

We can nevertheless (partially) bound the μ-invariants of the $Y_A^{(K_\infty)}$ on $\mathcal{E}(K)$ because the μ-invariants of the fine Selmer groups turn out to be related to the classical Iwasawa μ-invariants of projective limits of ideal class groups in $\mathbb{Z}_p$-extensions. Such a connection has been obtained first by Coates and Sujatha in the following form (see [CS05, Theorem 3.4]).

Theorem 7.2 (Coates, Sujatha). *Let E be an elliptic curve defined over K, let $p \neq 2$, and let $K_\infty = K_\infty^c$ be the cyclotomic $\mathbb{Z}_p$-extension of K. If $E[p] \subseteq E(K)$, then $\mu(Y_A^{(K_\infty)}) = 0$ if and only if the classical μ-invariant $\mu(K_\infty/K)$ vanishes.*

More generally, we can derive from results of Lim and Murty the following result. Let $\mu^{(1)}(Y_A^{(K_\infty/K)})$ be defined as in Lemma 4.6, and let $\mu^{(1)}(K_\infty/K)$ denote the number of summands $\Lambda/(p^{m_j})$, $m_j \in \mathbb{N}$, in the elementary Λ-module attached to the classical Iwasawa module $X^{(K_\infty)} = \varprojlim_n X^{(K_n)}$ of ideal class groups (see Section 2).

Theorem 7.3. *Let A be a d-dimensional abelian variety defined over a number field K, and let $p \neq 2$. We assume that $A[p] \subseteq A(K)$. Let K_∞/K be a $\mathbb{Z}_p$-extension such that no prime $v \in S_p \cup S_{\mathrm{br}}$ splits completely in K_∞. Then*

$$\mu^{(1)}(Y_A^{(K_\infty)}) + \mathrm{rank}_\Lambda(Y_A^{(K_\infty)}) = 2d \cdot \mu^{(1)}(K_\infty/K).$$

More generally, if K_∞/K is any $\mathbb{Z}_p$-extension, then

$$|\mu^{(1)}(Y_A^{(K_\infty)}) + \mathrm{rank}_\Lambda(Y_A^{(K_\infty)}) - 2d \cdot \mu^{(1)}(K_\infty/K)| \le 6d|S_{split}(K_\infty/K)|,$$

where $S_{split}(K_\infty/K) \subseteq S_p \cup S_{\mathrm{br}}(A)$ denotes the subset of primes of bad reduction or above p which split completely in K_∞/K.

Proof. It follows from (the proof of) [LKM16, Theorem 5.1] that

$$|\mathrm{rank}_p(Y_A^{(K_n)}) - 2d \cdot \mathrm{rank}_p(X^{(K_n)})| \leq 6d \cdot |S(K_n)| + 4d,$$

where $S(K_n)$ denotes the set of primes of K_n dividing some $v \in S_p \cup S_{\mathrm{br}}$. If no such prime splits completely in K_∞/K, then $|S(K_n)|$ stays bounded as $n \to \infty$. On the other hand, if $v \in S_p \cup S_{\mathrm{br}}$ is completely split, then the number of primes of K_n above v equals p^n for every $n \in \mathbb{N}$.

Now

$$||Y_A^{(K_\infty)}/((p, \omega_n)Y_A^{(K_\infty)})| - p^{(\mu^{(1)}(Y_A^{(K_\infty)}) + \mathrm{rank}_\Lambda(Y_A^{(K_\infty)})) \cdot p^n}| = \mathcal{O}(1),$$

and

$$||Y_A^{(K_\infty)}/((p, \omega_n)Y_A^{(K_\infty)}| - |Y_A^{(K_n)}/pY_A^{(K_n)}|| = \mathcal{O}(1)$$

in view of Theorem 5.1(a). In other words,

$$|\mathrm{rank}_p(Y_A^{(K_n)}) - (\mu^{(1)}(Y_A^{(K_\infty)}) + \mathrm{rank}_\Lambda(Y_A^{(K_\infty)})) \cdot p^n| = \mathcal{O}(1).$$

Together with the above result of [LKM16], we may conclude that

$$|2d \cdot \mathrm{rank}_p(X^{(K_n)}) - (\mu^{(1)}(Y_A^{(K_\infty)}) + \mathrm{rank}_\Lambda(Y_A^{(K_\infty)}))p^n| \leq t, \tag{14}$$

where $t = 6d|S_{split}(K_\infty/K)| \cdot p^n + C$ for some fixed constant C.

Similarly, one has

$$|\mathrm{rank}_p(X^{(K_n)}) - \mu^{(1)}(K_\infty/K) \cdot p^n| = \mathcal{O}(1),$$

since $X^{(K_\infty)} = \varprojlim X^{(K_n)}$ is known to be Λ-torsion. Indeed, one can use [Was97, Lemma 13.18] as a substitute for Theorem 5.1(a). By this result, letting $e \in \mathbb{N}$ denote the smallest integer such that each prime of K which ramifies in K_∞ is totally ramified in K_∞/K_e, and writing $Y \subseteq X^{(K_\infty)}$ for the kernel of the canonical map $X^{(K_\infty)} \longrightarrow X^{(K_e)}$, one has

$$|X^{(K_n)}/pX^{(K_n)}| = |X^{(K_\infty)}/(p \cdot X^{(K_\infty)} + \nu_{n,e} \cdot Y)|$$

for every $n \geq e$. Here $\nu_{n,e} = \frac{(T+1)^{p^n}-1}{(T+1)^{p^e}-1}$ is a polynomial of degree $p^n - p^e$, respectively, and $Y \subseteq X^{(K_\infty)}$ is of finite index. Therefore

$$|\mathrm{rank}_p(X^{(K_n)}) - v_p(|X^{(K_\infty)}/((p, \nu_{n,e}) \cdot X^{(K_\infty)})|)| = \mathcal{O}(1).$$

Moreover, since

$$|v_p(|X^{(K_\infty)}/((p, \nu_{n,e}) \cdot X^{(K_\infty)})|)| - v_p(|E_{X^{(K_\infty)}}/((p, \nu_{n,e}) \cdot E_{X^{(K_\infty)}})|)| = \mathcal{O}(1)$$

by Lemma 3.8, it can be derived that

$$|\mathrm{rank}_p(X^{(K_n)}) - \mu^{(1)}(K_\infty/K) \cdot (p^n - p^e)| = \mathcal{O}(1). \tag{15}$$

Combining equations (14) and (15), we may conclude that

$$|p^n \cdot (\mu^{(1)}(Y_A^{(K_\infty)}) + \mathrm{rank}_\Lambda(Y_A^{(K_\infty)}) - 2d \cdot \mu^{(1)}(K_\infty/K))|$$

is smaller than or equal to

$$6d \cdot |S_{split}(K_\infty/K)| \cdot p^n + \tilde{C}$$

for some constant $\tilde{C}$ and every sufficiently large $n \in \mathbb{N}$. This completes the proof of the theorem. □

Corollary 7.4. *Let A be a d-dimensional abelian variety defined over a number field K, let $p \neq 2$ and assume that $A[p] \subseteq A(K)$. If K_∞/K is a $\mathbb{Z}_p$-extension such that no prime $v \in S_p \cup S_{br}$ is totally split in K_∞/K, and if $\mu(K_\infty/K) = 0$, then* $\operatorname{rank}_\Lambda(Y_A^{(K_\infty)}) = 0$ *and* $\mu(Y_A^{(K_\infty)}) = 0$.

Corollary 7.5. *Let A be a d-dimensional abelian variety defined over a number field K, let $p \neq 2$ and assume that $A[p] \subseteq A(K)$. Then there exists a constant $B \in \mathbb{N}$ such that*

$$\mu^{(1)}(Y_A^{(K_\infty)}) \leq B$$

for each $\mathbb{Z}_p$-extension K_∞/K.

Proof. It follows from results of Monsky (see [Mon81, Theorem II]) that the classical μ-invariants $\mu(K_\infty/K)$ are bounded as K_∞ runs over the $\mathbb{Z}_p$-extensions of K. Since $\mu^{(1)}(K_\infty/K) \leq \mu(K_\infty/K)$, the result follows from Theorem 7.3. □

Remark 7.6. Using a p^k-analogue of the result of Lim and Murty, one can derive similar upper bounds for the invariants $\mu^{(k)}(Y_A^{(\tilde{K}_\infty)})$ defined as in Lemma 4.6 (such a comparison result will also play a prominent role in forthcoming joint work with K. Müller).

Note: in Lemma 4.6 we were able to bound the invariants $\mu^{(k)}(X_A^{(\tilde{K}_\infty)})$ locally in some neighbourhood $U = \mathcal{E}(K_\infty, n)$ of K_∞ only if A has good ordinary reduction at each $v \in S_p$, each such prime is ramified in K_∞ and no prime $v \in S_{\mathrm{br}}$ is completely split in K_∞/K. It follows from the p^k-analogue of Theorem 7.3 that at least the local boundedness of $\mu^{(k)}(Y_A^{(\tilde{K}_\infty)})$ (i.e. fine Selmer groups instead of Selmer groups) does not require any of these assumptions – in particular no assumption on the reduction type of A at the primes above p is needed.

7.3. Bounding λ-invariants. We now turn to the (un-) boundedness of λ-invariants. First, we derive from Corollary 4.2 the following

Theorem 7.7. *Let A be an abelian variety defined over a number field K, and let K_∞/K be a $\mathbb{Z}_p$-extension. We assume that A has potentially good ordinary reduction at each prime $v \in S_p$. Suppose that for each $n \in \mathbb{N}$ there exists $\tilde{K}_\infty \in \mathcal{E}(K_\infty, n)$ such that* $\operatorname{rank}_\Lambda(X_A^{(\tilde{K}_\infty)}) < \operatorname{rank}_\Lambda(X_A^{(K_\infty)})$.

Then $\lambda(X_A^{(\tilde{K}_\infty)})$ is unbounded on $\mathcal{E}(K_\infty, m)$ for every $m \in \mathbb{N}$.

An analogous statement holds for Pontryagin duals of fine Selmer groups (if $p \neq 2$; here the assumption on the reduction type of A at p is not necessary).

Proof. This follows from Corollary 4.2 (Selmer groups) and Theorem 5.4(a) (fine Selmer groups) via contraposition. □

In order to describe a situation where this theorem can be applied, we prove several auxiliary results concerning the variation of Λ-ranks of $X_A^{(K_\infty)}$ and $Y_A^{(K_\infty)}$ for the $\mathbb{Z}_p$-extensions K_∞/K contained in some fixed $\mathbb{Z}_p^2$-extension $\mathbb{L}_\infty$ of K. To this purpose, we consider also Selmer groups of abelian varieties A (which are defined over K) over $\mathbb{L}_\infty$. These are defined analogously to Selmer groups over $\mathbb{Z}_p$-extensions:

$$\mathrm{Sel}_A(\mathbb{L}_\infty) = \varinjlim_{K\subseteq L\subsetneq \mathbb{L}_\infty} \mathrm{Sel}_A(L) \quad \text{and} \quad \mathrm{Sel}_{A,0}(\mathbb{L}_\infty) = \varinjlim_{K\subseteq L\subsetneq \mathbb{L}_\infty} \mathrm{Sel}_{A,0}(L),$$

where L runs over all finite subextensions of $\mathbb{L}_\infty/K$. We let

$$X_A^{(\mathbb{L}_\infty)} = \varprojlim_{K\subseteq L\subseteq \mathbb{L}_\infty} X_A^{(L)} \quad \text{and} \quad Y_A^{(\mathbb{L}_\infty)} = \varprojlim_{K\subseteq L\subseteq \mathbb{L}_\infty} Y_A^{(L)},$$

where $X_A^{(L)} = \mathrm{Sel}_A(L)^\vee$ and $Y_A^{(L)} = \mathrm{Sel}_{A,0}(L)^\vee$ for each L. Note that this induces a Λ_2-module structure on $X_A^{(\mathbb{L}_\infty)}$ and $Y_A^{(\mathbb{L}_\infty)}$, where $\Lambda_2 = \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{L}_\infty/K)]]$; this completed group ring can be (non-canonically) identified with the ring $\mathbb{Z}_p[[T_1, T_2]]$ of formal power series in two variables over $\mathbb{Z}_p$.

LEMMA 7.8. *Let $\mathbb{L}_\infty$ be a $\mathbb{Z}_p^2$-extension of a number field K, and let A be an abelian variety defined over K. We assume that $\mathbb{L}_\infty$ contains the cyclotomic $\mathbb{Z}_p$-extension K_∞^c of K. Then the following statements hold.*

(a) *If A has good ordinary reduction at the primes above p, and if*

$$\mathrm{rank}_\Lambda(X_A^{(K_\infty^c)}) = \mu(X_A^{(K_\infty^c)}) = 0,$$

then $\mathrm{rank}_{\Lambda_2}(X_A^{(\mathbb{L}_\infty)}) = 0$.

(b) *For fine Selmer groups, if $p \neq 2$, then* $\mathrm{rank}_{\Lambda_2}(Y_A^{(\mathbb{L}_\infty)}) \leq \mathrm{rank}_\Lambda(Y_A^{(K_\infty^c)})$*, independently of the actual value of the latter rank.*

Proof. For elliptic curves, the statement for Selmer groups follows from [CS12, Theorem 2.1] (in the notation of that theorem, we have $L = F = K$, $F_\infty = \mathbb{L}_\infty$ and $L^{cyc} = K_\infty^c$; it follows that $X_A^{(\mathbb{L}_\infty)}$ is finitely generated over $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{L}_\infty/K_\infty^c)]]$, and thus is Λ_2-torsion). We therefore focus on the fine Selmer groups (but note that the same proof could also be used for Selmer groups; in the last step we would then use the fact that the Mordell-Weil ranks of the layers $K_n \subseteq K_\infty^c$ are bounded by the hypotheses in (a)).

Let $\Gamma_\infty = \mathrm{Gal}(\mathbb{L}_\infty/K_\infty^c)$; this Galois group is isomorphic to $\mathbb{Z}_p$. We start from a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sel}_{A,0}(K_\infty^c) \arrow[r] \arrow[d, "g"] & H^1(K_\infty^c, A[p^\infty]) \arrow[r] \arrow[d, "h"] & \prod_v H^1(K_{\infty,v}^c, A[p^\infty]) \arrow[d] \\
0 \arrow[r] & \mathrm{Sel}_{A,0}(\mathbb{L}_\infty)^{\Gamma_\infty} \arrow[r] & H^1(\mathbb{L}_\infty, A[p^\infty])^{\Gamma_\infty} \arrow[r] & \prod_w H^1(\mathbb{L}_{\infty,w}, A[p^\infty])^{\Gamma_{\infty,w}}
\end{tikzcd}$$

Here $K_{\infty,v}^c$ and $\mathbb{L}_{\infty,w}$ denote the localisations of K_∞^c and $\mathbb{L}_\infty$ at primes v and $w \mid v$ (i.e. the unions of the completions of finite subextensions of K_∞^c and $\mathbb{L}_\infty$), $\Gamma_{\infty,w} = \mathrm{Gal}(\mathbb{L}_{\infty,w}/K_{\infty,v}^c)$, and v, w run over all primes of K_∞^c, respectively, $\mathbb{L}_\infty$.

Dualising, we obtain a canonical map

$$f : (Y_A^{(\mathbb{L}_\infty)})_{\Gamma_\infty} \longrightarrow Y_A^{(K_\infty^c)}.$$

Choosing the variables $T_1, T_2 \in \Lambda_2$ properly, we may assume that

$$(Y_A^{(\mathbb{L}_\infty)})_{\Gamma_\infty} = Y_A^{(\mathbb{L}_\infty)}/(T_2 \cdot Y_A^{(\mathbb{L}_\infty)})$$

and $\Lambda_2/(T_2) = \Lambda$. Using the above commutative diagram and the snake lemma, we can bound the kernel of f. First, since the projective dimension of $\Gamma_\infty \cong \mathbb{Z}_p$ is equal to 1, the middle vertical arrow is surjective. This means that the cokernel of g can be bounded in terms of the kernel of the right vertical arrow. By the inflation-restriction sequence, this kernel is given by

$$\prod_w H^1(\Gamma_{\infty,w}, A(\mathbb{L}_{\infty,w})[p^\infty]);$$

here the product runs over the primes w of $\mathbb{L}_\infty$. Note: since no prime of K splits completely in the cyclotomic $\mathbb{Z}_p$-extension K_∞^c, each $v \nmid p$ of K_∞ must split completely in the $\mathbb{Z}_p$-extension $\mathbb{L}_\infty/K_\infty^c$. Therefore $\Gamma_{\infty,w} = \{0\}$ whenever $w \nmid p$, and the product reduces to a finite direct sum over the primes dividing p.

Now fix such a prime w; we assume that $\Gamma_{\infty,w} \neq \{0\}$. Since $U := (A(\mathbb{L}_{\infty,w})[p^\infty])^\vee$ is finitely generated over $\mathbb{Z}_p$ by the Lefschetz principle, the submodule $U^{\Gamma_{\infty,w}}$ is also finitely generated over $\mathbb{Z}_p$ – but this is precisely the Pontryagin dual of

$$H^1(\Gamma_{\infty,w}, A(\mathbb{L}_{\infty,w})[p^\infty]) \cong (A(\mathbb{L}_{\infty,w})[p^\infty])_{\Gamma_{\infty,w}}$$

(recall that $\Gamma_{\infty,w} \cong \mathbb{Z}_p$ by assumption).

We have therefore shown that the kernel of

$$f : Y_A^{(\mathbb{L}_\infty)}/(T_2 \cdot Y_A^{(\mathbb{L}_\infty)}) \longrightarrow Y_A^{(K_\infty^c)}$$

is a finitely generated $\mathbb{Z}_p$-module. Therefore

$$\mathrm{rank}_{\Lambda_2}(Y_A^{(\mathbb{L}_\infty)}) \leq \mathrm{rank}_\Lambda(Y_A^{(\mathbb{L}_\infty)}/(T_2 \cdot Y_A^{(\mathbb{L}_\infty)})) \leq \mathrm{rank}_\Lambda(Y_A^{(K_\infty^c)}).$$

∎

Now we prove a complementary result.

LEMMA 7.9. *Let $\mathbb{L}_\infty$ be a $\mathbb{Z}_p^2$-extension of a number field K, and let A be an abelian variety defined over K. Then $\mathrm{rank}_\Lambda(X_A^{(K_\infty)}) \leq \mathrm{rank}_{\Lambda_2}(X_A^{(\mathbb{L}_\infty)})$ for all but finitely many $\mathbb{Z}_p$-extensions $K_\infty \subseteq \mathbb{L}_\infty$ of K.*

If $p \neq 2$, then an analogous statement holds for X replaced by Y.

Proof. We first consider fine Selmer groups. Let $K_\infty \subseteq \mathbb{L}_\infty$ be a $\mathbb{Z}_p$-extension of K. Using a commutative diagram as in the proof of Lemma 7.8, we investigate the cokernel of

$$f : (Y_A^{(\mathbb{L}_\infty)})_{\Gamma_\infty} \longrightarrow Y_A^{(K_\infty)},$$

$\Gamma_\infty = \mathrm{Gal}(\mathbb{L}_\infty/K_\infty)$, by studying the kernel of

$$g : \mathrm{Sel}_{A,0}(K_\infty) \longrightarrow \mathrm{Sel}_{A,0}(\mathbb{L}_\infty)^{\Gamma_\infty}.$$

By the snake lemma, the kernel of g is contained in the kernel of the middle vertical map h; in view of the inflation-restriction exact sequence, this kernel is equal to $H^1(\Gamma_\infty, A(\mathbb{L}_\infty)[p^\infty])$. Therefore the dual of $\ker(g)$ is a finitely generated $\mathbb{Z}_p$-module and $\mathrm{rank}_\Lambda(Y_A^{(K_\infty)}) \leq \mathrm{rank}_\Lambda((Y_A^{(\mathbb{L}_\infty)})_{\Gamma_\infty})$, where we let $\Lambda = \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$ (in fact, since also $\ker(f)$ is finitely generated over $\mathbb{Z}_p$ by the proof of Lemma 7.8, we have equality of ranks here).

We will now show the following fact, which concludes the proof of the lemma: for all but finitely many $\mathbb{Z}_p$-extensions $K_\infty \subseteq \mathbb{L}_\infty$ of K, the $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$-rank of the quotient

$$(Y_A^{(\mathbb{L}_\infty)})_{\Gamma_\infty}, \quad \Gamma_\infty = \mathrm{Gal}(\mathbb{L}_\infty/K_\infty),$$

is at most equal to $\mathrm{rank}_{\Lambda_2}(Y_A^{(\mathbb{L}_\infty)})$.

Indeed, by a general structure theorem (see [NSW08, Theorem (5.1.10)]), there exists a pseudo-homomorphism (i.e. a Λ_2-module homomorphism φ such that both $\ker(\varphi)$ and $\mathrm{coker}(\varphi)$ are annihilated by two relatively prime elements of Λ_2)

$$\varphi\colon Y_A^{(\mathbb{L}_\infty)} \longrightarrow E$$

for some elementary Λ_2-module of the form

$$E = \Lambda_2^r \oplus \bigoplus_{i=1}^{s} \Lambda_2/(h_i),$$

where $h_i \in \Lambda_2$ for every i, and $r = \mathrm{rank}_{\Lambda_2}(Y_A^{(\mathbb{L}_\infty)})$. The product

$$F_{Y_A^{(\mathbb{L}_\infty)}} := \prod_{i=1}^{s} h_i \in \Lambda_2$$

is called the *characteristic power series* of $Y_A^{(\mathbb{L}_\infty)}$, in analogy with the one-dimensional setting (cf. Section 2).

Let γ_∞ be a topological generator of Γ_∞, and let $T := \gamma_\infty - 1$, then

$$(Y_A^{(\mathbb{L}_\infty)})_{\Gamma_\infty} = Y_A^{(\mathbb{L}_\infty)}/(T \cdot Y_A^{(\mathbb{L}_\infty)}).$$

The map $\varphi\colon Y_A^{(\mathbb{L}_\infty)} \longrightarrow E$ induces an exact sequence

$$M_1 \longrightarrow (Y_A^{(\mathbb{L}_\infty)})_{\Gamma_\infty} \longrightarrow E_{\Gamma_\infty} \longrightarrow M_2 \longrightarrow 0, \tag{16}$$

of finitely generated $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$-modules, where the first and the last term are $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$-torsion because the Λ_2-modules $\ker(\varphi)$ and $\mathrm{coker}(\varphi)$ are annihilated by an element of Λ_2 which is coprime with T. In particular,

$$\mathrm{rank}_{\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]}((Y_A^{(\mathbb{L}_\infty)})_{\Gamma_\infty}) = \mathrm{rank}_{\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]}(E_{\Gamma_\infty}).$$

The latter equals r whenever T does not divide the characteristic power series $F_{Y_A^{(\mathbb{L}_\infty)}}$ of $Y_A^{(\mathbb{L}_\infty)}$, since in this case the image of $F_{Y_A^{(\mathbb{L}_\infty)}}$ under the canonical map

$$\Lambda_2 \twoheadrightarrow \Lambda_2/(T) \cong \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$$

is non-zero, i.e. each summand $\Lambda_2/(h_i, T)$ is then $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$-torsion. We will show that this is true for all but finitely many $\mathbb{Z}_p$-extensions $K_\infty \subseteq \mathbb{L}_\infty$.

Let $K_\infty^{(1)}$ and $K_\infty^{(2)}$ be two independent $\mathbb{Z}_p$-extensions of K spanning $\mathbb{L}_\infty$, i.e. $K_\infty^{(1)} \cap K_\infty^{(2)} = K$ and $K_\infty^{(1)} \cdot K_\infty^{(2)} = \mathbb{L}_\infty$. Let γ_1, γ_2 be topological generators of

$\mathrm{Gal}(\mathbb{L}/K_\infty^{(1)})$ and $\mathrm{Gal}(\mathbb{L}_\infty/K_\infty^{(2)})$. Then for any $\mathbb{Z}_p$-extension $K_\infty \subseteq \mathbb{L}_\infty$ of K different from $K_\infty^{(1)}$ and $K_\infty^{(2)}$, the Galois group $\mathrm{Gal}(\mathbb{L}_\infty/K_\infty)$ is generated topologically by an element $\gamma_1^{p^m} \cdot \gamma_2^{p^n \cdot u}$ with $n, m \in \mathbb{N}$, $u \in \mathbb{Z}_p^\times$ and $n \cdot m = 0$.

Let $E^\circ = \bigoplus_{i=1}^s \Lambda_2/(h_i)$. Since $\mathrm{rank}_{\Lambda_2}(E^\circ) = 0$, $\mathrm{ht}_{\Lambda_2}(\mathrm{Ann}(E^\circ)) \geq 1$. If this height is greater than or equal to 2, then obviously

$$\mathrm{ht}_{\Lambda_1^{(m,n,u)}}((\mathrm{Ann}(E^\circ) + ((\gamma_1^{p^m}\gamma_2^{p^n \cdot u} - 1))/(\gamma_1^{p^m}\gamma_2^{p^n \cdot u} - 1)) \geq 1,$$

where $\Lambda_1^{(m,n,u)} = \Lambda_2/(\gamma_1^{p^m}\gamma_2^{p^n \cdot u} - 1) \cong \Lambda_1$.

From now on, we assume that $\mathrm{ht}_{\Lambda_2}(\mathrm{Ann}(E^\circ)) = 1$.

Lemma 7.10. *Let M be a Λ_2-module such that $\mathrm{ht}_{\Lambda_2}(\mathrm{Ann}(M)) = 1$. Then*

$$\mathrm{ht}_{\Lambda_2}(\mathrm{Ann}(M) + (\gamma_1^{p^m}\gamma_2^{p^n \cdot u} - 1)) \geq 2$$

for all but finitely many $u \in \mathbb{Z}_p^\times$ and $n, m \in \mathbb{N}$ satisfying $n \cdot m = 0$.

Proof. We use an argument analogous to that from the proof of [Kle21b, Lemma 3.1]; we will consider only the case $m = 0$. For each $n \in \mathbb{N}$ and $u \in \mathbb{Z}_p^\times$, we define

$$I_{n,u} := \mathrm{Ann}(M) + (\gamma_1\gamma_2^{p^n \cdot u} - 1),$$

and we denote by $\mathfrak{p}_{n,u} \supseteq I_{n,u}$ a minimal prime ideal. We want to show that $\mathrm{ht}(I_{n,u}) \geq 2$ for all but finitely many n and u.

Let $I := \bigcap_{(n,u):\mathrm{ht}(I_{n,u})=1} \mathfrak{p}_{n,u}$. Then $\mathrm{ht}(I) \geq 1$ because $\mathrm{Ann}(M) \subseteq I$, and we may assume that in fact $\mathrm{ht}(I) = 1$. Then each $\mathfrak{p}_{n,u}$ occurring in the definition of I is a minimal prime ideal of I. Since Λ_2 is a Noetherian ring, we may conclude that only finitely many different prime ideals occur in the definition of I.

Now suppose that $\mathfrak{p}_{n,u} = \mathfrak{p}_{\tilde{n},\tilde{u}}$ for some $(\tilde{n}, \tilde{u}) \neq (n, u)$. Then $\mathfrak{p}_{n,u}$ contains both $\gamma_1 \cdot \gamma_2^{p^n \cdot u} - 1$ and $\gamma_1 \cdot \gamma_2^{p^{\tilde{n}} \cdot \tilde{u}} - 1$. These two elements form a Λ_2-regular sequence, i.e. $\Lambda_2/\mathfrak{p}_{n,u}$ is a finitely generated $\mathbb{Z}_p$-module and therefore $\mathrm{ht}(\mathfrak{p}_{n,u}) \geq 2$. This proves the lemma. □

Returning to the proof of Lemma 7.9, we may conclude that for all but finitely many $\mathbb{Z}_p$-extensions $K_\infty \subseteq \mathbb{L}_\infty$ of K, the quotient $(E^\circ)_{\Gamma_\infty}$, $\Gamma_\infty = \mathrm{Gal}(\mathbb{L}_\infty/K_\infty)$, is a torsion $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$-module.

Finally, the assertion concerning $X_A^{(\mathbb{L}_\infty)}$ can be proved completely analogously: using a similar commutative diagram as for the fine Selmer groups, the kernel of

$$g : \mathrm{Sel}_A(K_\infty) \longrightarrow \mathrm{Sel}_A(\mathbb{L}_\infty)^{\Gamma_\infty}$$

is again contained in $H^1(\Gamma_\infty, A(\mathbb{L}_\infty)[p^\infty])$, and the proof proceeds as above. □

Putting together Lemmas 7.8 and 7.9, we may derive the following

Corollary 7.11. *Let $\mathbb{L}_\infty$ be a $\mathbb{Z}_p^2$-extension of a number field K, and let A be an abelian variety defined over K. We assume that $\mathbb{L}_\infty$ contains the cyclotomic $\mathbb{Z}_p$-extension K_∞^c of K.*

(a) *Suppose that $p \neq 2$. If there exists a $\mathbb{Z}_p$-extension $K_\infty \subseteq \mathbb{L}_\infty$ of K such that*

$$\mathrm{rank}_\Lambda(Y_A^{(K_\infty)}) > \mathrm{rank}_\Lambda(Y_A^{(K_\infty^c)}),$$

then $\lambda(Y_A^{(\tilde{K}_\infty)})$ is unbounded on $\mathcal{E}(K_\infty, m)$ for every $m \in \mathbb{N}$.

(b) *Suppose that A has good ordinary reduction at the primes of K above p. If* $\mathrm{rank}_\Lambda(X_A^{(K_\infty^c)}) = 0$ *and* $\mu(X_A^{(K_\infty^c)}) = 0$*, and if there exists a $\mathbb{Z}_p$-extension* $K_\infty \subseteq \mathbb{L}_\infty$ *such that* $\mathrm{rank}_\Lambda(X_A^{(K_\infty)}) > 0$*, then* $\lambda(X_A^{(\tilde{K}_\infty)})$ *is unbounded on* $\mathcal{E}(K_\infty, m) \cap \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ *for every $m \in \mathbb{N}$.*

Proof. We prove only part (b); part (a) can be proved analogously. It follows from Lemmas 7.8 and 7.9 that $\mathrm{rank}_\Lambda(X_A^{(\tilde{K}_\infty)}) = 0$ for all but finitely many $\mathbb{Z}_p$-extensions $\tilde{K}_\infty \subseteq \mathbb{L}_\infty$ of K. In particular, if $m \in \mathbb{N}$ is large enough, then

$$0 = \mathrm{rank}_\Lambda(X_A^{(\tilde{K}_\infty)}) < \mathrm{rank}_\Lambda(X_A^{(K_\infty)})$$

for each $\tilde{K}_\infty \in \mathcal{E}(K_\infty, m)$. Therefore the assertion of the corollary follows from Theorem 7.7. □

Remark 7.12. On the contrary, it follows from the results of Sections 4 and 5 and [Kle21a] that (in the good ordinary setting) $\lambda(X_A^{(\tilde{K}_\infty/K)})$ is *bounded* on $\mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ if both $\mathrm{rank}_\Lambda(X_A^{(\tilde{K}_\infty)}) = 0$ *and* $\mu(X_A^{(\tilde{K}_\infty)}) = 0$ for each $\tilde{K}_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$; similarly for $\lambda(Y_A^{(\tilde{K}_\infty/K)})$.

Let us stress the following special case of Corollary 7.11,(a):

Corollary 7.13. *Let A be an abelian variety defined over the number field K, and suppose that $p \neq 2$. If the weak Leopoldt conjecture holds for A over K_∞^c, then the subset of $\mathcal{E}(K)$ of $\mathbb{Z}_p$-extensions of K over which the weak Leopoldt conjecture for A holds is dense in $\mathcal{E}(K)$.*

Proof. We may assume that there exists more than one single $\mathbb{Z}_p$-extension of K. By [Lim17, Lemma 7.1], the weak Leopoldt conjecture for A holds over some $\mathbb{Z}_p$-extension K_∞/K if and only if $Y_A^{(K_\infty)}$ is Λ-torsion.

For given $K_\infty^c \neq K_\infty \in \mathcal{E}(K)$ and $n \in \mathbb{N}$, we consider the $\mathbb{Z}_p^2$-extension $K_\infty^c \cdot K_\infty$ of K. In view of Lemmas 7.8 and 7.9, the weak Leopoldt conjecture for A holds over all but finitely many $\mathbb{Z}_p$-extensions of K contained in $\mathcal{E}^{\subseteq \mathbb{L}_\infty}(K) \cap \mathcal{E}(K_\infty, n)$. □

Now we turn to the proofs of our main results.

Proof of Theorem 1.1. This follows immediately from Corollary 7.11, (b). □

Proof of Corollary 1.2. The hypotheses on $A = E$ imply that $\mathrm{rank}_\Lambda(X_A^{(K_\infty^c)}) = 0$ and $\mathrm{rank}_\Lambda(X_A^{(K_\infty^a)}) > 0$ (see also Remark 4.15). Since $\mu(X_A^{(K_\infty^c)}) = 0$ by assumption, it follows from Corollary 7.11, or alternatively from Theorem 1.1, that $\lambda(X_A^{(\tilde{K}_\infty)})$ is unbounded on $\mathcal{E}(K)$ (recall that the composite of all $\mathbb{Z}_p$-extensions of an imaginary quadratic number field K is a $\mathbb{Z}_p^2$-extension $\mathbb{L}_\infty$ of K). □

Before, finally, turning to the proof of Theorem 1.3, we mention a concrete example.

Example 7.14. Let $K = \mathbb{Q}(\sqrt{-7})$. We consider the elliptic curve E defined by

$$E\colon y^2 + y = x^3 - x^2 - 10x - 20.$$

The only prime number dividing the conductor of E is 11, which splits in $K/\mathbb{Q}$. We consider the two primes 37 and 43, both of which are split in $K/\mathbb{Q}$.

Since E is a semistable elliptic curve, it follows from the Iwasawa main conjecture, proven in this case by Skinner and Urban in [SU14], that $\mu(X_A^{(K_\infty^c)})$ equals the μ-invariant of a p-adic L-function $L_p(E,s)$ interpolating the Hasse-Weil L-series for each prime $p \geq 11$ of good ordinary reduction. A computation with PARI/GP (see [PAR21]) shows that the Hasse-L-function is not divisible by either 37 or 43.

In view of the data from [LMF21], the image of the p-adic representation

$$\overline{\rho}_p\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(E[p])$$

contains $\mathrm{SL}_2(\mathbb{F}_p)$ for each prime $p < 1000$ which is different from 5. The results from [Ser72] imply that hypothesis (3) from Corollary 1.2 is satisfied for these primes.

A computation with PARI/GP reveals that $|E(k_v)| = |E(\mathbb{F}_{37})| = 35$ for each $v \in S_{37}(K)$, and that $|E(k_v)| = |E(\mathbb{F}_{43})| = 50$ for both $v \in S_{43}(K)$. Therefore the hypotheses (2), (4) and (5) from Corollary 1.2 are also satisfied, and $\lambda(X_A^{(K_\infty)})$ is unbounded as K_∞ runs over the $\mathbb{Z}_p$-extensions of K, $p \in \{37, 43\}$.

Proof of Theorem 1.3. If the weak Leopoldt conjecture holds for A over K_∞^c, then $\mathrm{rank}_\Lambda(Y_A^{(K_\infty^c)}) = 0$ by [Lim17, Lemma 7.1]. Since, by the second hypothesis, $\mathrm{rank}_\Lambda(Y_A^{(K_\infty^a)}) > 0$, the theorem follows by an application of Corollary 7.11. □

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