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003

ON THE STABILITY OF HOMOGENEOUS EINSTEIN MANIFOLDS*

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Abstract. Let g be a G-invariant Einstein metric on a compact homogeneous space $M = G/K$. We use a formula for the Lichnerowicz Laplacian of g at G-invariant TT-tensors to study the stability type of g as a critical point of the scalar curvature function. The case when g is naturally reductive is studied in special detail.

Key words. Einstein, homogeneous, stability.

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1. Introduction. Given a compact connected differentiable manifold M and a transitive action of a compact Lie group G on M, the aim of this paper is to study the stability of G-invariant Einstein metrics on M within the G-invariant setting. It is well known that if $\mathcal{M}_1^G$ denotes the finite-dimensional manifold of all unit volume G-invariant metrics on M, then $g \in \mathcal{M}_1^G$ is Einstein (i.e. $\operatorname{Rc}(g) = \rho g$ for some $\rho \in \mathbb{R}$, which is necessarily positive if G is non-abelian) if and only if g is a critical point of the scalar curvature functional

$$\operatorname{Sc} : \mathcal{M}_1^G \longrightarrow \mathbb{R}.$$

The G-action we have fixed provides a presentation $M = G/K$ of M as a homogeneous space, where $K \subset G$ is the isotropy subgroup at some origin point $o \in M$.

We start by showing in §3 that

$$T_g\mathcal{M}_1^G = T_g \operatorname{Aut}(G/K) \cdot g \oplus \mathcal{TT}_g^G,$$

where $\operatorname{Aut}(G/K) \subset \operatorname{Diff}(M)$ is the Lie group of automorphisms of G taking K onto K, giving rise to trivial variations of g, and $\mathcal{TT}_g^G := (\operatorname{Ker}\delta_g \cap \operatorname{Ker}\operatorname{tr}_g)^G$ is the space of so-called TT-tensors (see §2) which are G-invariant. It is therefore natural to say that an Einstein metric $g \in \mathcal{M}_1^G$ is *G-stable* when the second derivative or Hessian of Sc satisfies that

$$\operatorname{Sc}''_g|_{\mathcal{TT}_g^G} < 0,$$

which in particular implies that g is a local maximum of $\operatorname{Sc} : \mathcal{M}_1^G \longrightarrow \mathbb{R}$. Recall that without assuming G-invariance, g is called *stable* if Sc''_g is negative definite on $\mathcal{TT}_g$, the infinite dimensional space of all unit volume constant scalar curvature (non-trivial) variations of g (see §2).

Some potential applications of establishing the G-stability type of G-invariant Einstein metrics include:

- If g is *G-non-degenerate* (i.e., $\operatorname{Sc}''_g|_{\mathcal{TT}_g^G}$ is non-degenerate), then g is *G-rigid*, in the sense that g is an isolated point in the moduli space $\mathcal{E}_1^G/\operatorname{Aut}(G/K)$ of G-invariant unit volume Einstein metrics on M. The main long standing open question in the subject is whether such moduli space is always finite, which has been conjectured

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to hold in the multiplicity-free isotropy representation case by Böhm, Wang and Ziller in [BWZ] (note that $T_g\mathcal{M}_1^G = \mathcal{TT}_g^G$ in that case and so $\mathcal{E}_1^G$ must itself be finite).
It is worth noticing that since $\mathcal{E}_1^G$ is known to be compact (see [BWZ, Theorem 1.6]), the finiteness of $\mathcal{E}_1^G/\operatorname{Aut}(G/K)$ is equivalent to the G-rigidity of any G-invariant Einstein metric on M. G-non-degeneracy seems to be a generic property, though this is hard to put in a rigorous statement.

- In the case when g is *G-unstable* (i.e., $\operatorname{Sc}''_g(T,T) > 0$ for some $T \in \mathcal{TT}_g^G$), one obtains that g is also unstable relative to the ν-entropy functional introduced by Perelman (see [CH]) and so it is *dynamically unstable*, in the sense that there exists a nontrivial normalized Ricci flow defined on $(-\infty, 0]$ which converges modulo diffeomorphisms to g as $t \to -\infty$ (see [Kr2, Theorem 1.3]). Additionally, it is known that a G-unstable Einstein metric g does not realize the Yamabe invariant of M (see [BWZ, Theorem 5.1]).
G-instability is also an expected behavior, as suggested by the graph theorem [BWZ, Theorem 3.3] and its generalization, the simplicial complex theorem [B1, Theorem 1.5]. However, a rigorous result on this is still lacking.
- Beyond irreducible symmetric metrics and the special case when K is a maximal subgroup of G (see [WZ2, B1]), G-stability is extremely rare if $\dim \mathcal{M}_1^G > 1$, it is considered a mere coincidence or accident by the experts. It is for instance unknown whether there can be two non-homothetic G-stable Einstein metrics for a given G.
- Since the normalized Ricci flow on $\mathcal{M}_1^G$ is precisely the gradient flow of Sc, its dynamical behavior is mostly governed by the G-stability types of their fixed points, the G-invariant Einstein metrics (see [AC] and references therein).

As known, the second variation Sc''_g of the total scalar curvature at any Einstein metric g on M, say with $\operatorname{Rc}(g) = \rho g$, coincides on $\mathcal{TT}_g$ with $\frac{1}{2}(2\rho\operatorname{id} - \Delta_L)$, where Δ_L is the Lichnerowicz Laplacian of g (see §2). In §4, we consider the self-adjoint operator

$$\mathrm{L}_{\mathfrak{p}} = \mathrm{L}_{\mathfrak{p}}(g) : \operatorname{sym}(\mathfrak{p})^K \longrightarrow \operatorname{sym}(\mathfrak{p})^K,$$

defined by Δ_L under the usual identifications, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is any reductive decomposition and $\operatorname{sym}(\mathfrak{p})^K := \{A : \mathfrak{p} \to \mathfrak{p} : A^t = A,\ [\operatorname{Ad}(K), A] = 0\}$. Note that the G-stability type of g is therefore determined by how is the constant 2ρ suited relative to the spectrum of $\mathrm{L}_{\mathfrak{p}}$. We use moving bracket approach techniques to prove the following formula for $\mathrm{L}_{\mathfrak{p}}$:

$$\langle \mathrm{L}_{\mathfrak{p}} A, A\rangle = \tfrac{1}{2}|\theta(A)\mu_{\mathfrak{p}}|^2 + 2\operatorname{tr}\mathrm{M}_{\mu_{\mathfrak{p}}} A^2, \qquad \forall A \in \operatorname{sym}(\mathfrak{p}), \tag{1}$$

where $\mu_{\mathfrak{p}} := \operatorname{pr}_{\mathfrak{p}} \circ [\cdot,\cdot]|_{\mathfrak{p}\times\mathfrak{p}} : \mathfrak{p}\times\mathfrak{p} \longrightarrow \mathfrak{p}$ and the function $\mathrm{M} : \Lambda^2\mathfrak{p}^*\otimes\mathfrak{p} \to \operatorname{sym}(\mathfrak{p})$ is the *moment map* from geometric invariant theory (see [LfL, BL1]) for the representation θ of $\mathfrak{gl}(\mathfrak{p})$ given by

$$\theta(A)\lambda := A\lambda(\cdot,\cdot) - \lambda(A\cdot,\cdot) - \lambda(\cdot,A\cdot), \qquad \forall A \in \mathfrak{gl}(\mathfrak{p}), \quad \lambda \in \Lambda^2\mathfrak{p}^*\otimes\mathfrak{p},$$

that is,

$$\langle \mathrm{M}_{\mu_{\mathfrak{p}}}, A\rangle := \tfrac{1}{4}\langle \theta(A)\mu_{\mathfrak{p}}, \mu_{\mathfrak{p}}\rangle, \qquad \forall A \in \mathfrak{gl}(\mathfrak{p}).$$

This is actually the main part of Ricci curvature, the Ricci operator of the metric g is given by $\operatorname{Ric}(g) = \mathrm{M}_{\mu_{\mathfrak{p}}} - \frac{1}{2}\mathrm{B}_{\mu}$, where $\langle \mathrm{B}_{\mu}\cdot,\cdot\rangle := \mathrm{B}_{\mathfrak{g}}|_{\mathfrak{p}\times\mathfrak{p}}$ and $\mathrm{B}_{\mathfrak{g}}$ denotes the Killing form of the Lie algebra $\mathfrak{g}$.

As a first application of formula (1), we focus in §5 on the case when g is naturally reductive with respect to G and $\mathfrak{p}$. We have in this case that

$$T_g\mathcal{M}_1^G = \mathcal{T}\mathcal{T}_g^G = \operatorname{sym}_0(\mathfrak{p})^K := \{A \in \operatorname{sym}(\mathfrak{p})^K : \operatorname{tr} A = 0\},$$

and furthermore, the operator $\mathrm{L}_\mathfrak{p}$ is non-negative and takes the following simpler form:

$$\mathrm{L}_\mathfrak{p} A := -\tfrac{1}{2}\sum[\operatorname{ad}_\mathfrak{p} X_i, [\operatorname{ad}_\mathfrak{p} X_i, A]], \qquad \forall A \in \operatorname{sym}(\mathfrak{p})^K, \tag{2}$$

where $\{X_i\}$ is any g-orthonormal basis of $\mathfrak{p}$ and $\operatorname{ad}_\mathfrak{p} X_i := \mu_\mathfrak{p}(X_i, \cdot)$ (recall that naturally reductive means that $\operatorname{ad}_\mathfrak{p} X_i$ is skew-symmetric for all i). In particular, if g_B is the Killing left-invariant metric on any compact simple Lie group G, which satisfies $\operatorname{Rc}(g_\mathrm{B}) = \frac{1}{4}g_\mathrm{B}$, then

$$\mathrm{L}_\mathfrak{p}(g_\mathrm{B}) = \tfrac{1}{2}\,\mathrm{C}_{\tau,-\mathrm{B}_\mathfrak{g}},$$

where $\mathrm{C}_{\tau,-\mathrm{B}_\mathfrak{g}}$ is the Casimir operator acting on the representation $\operatorname{sym}(\mathfrak{g})$ of $\mathfrak{g}$ given by $\tau(X)A := [\operatorname{ad} X, A]$. Thus the G-stability type of g_B can be obtained by using representation theory to compute the spectrum of $\mathrm{C}_{\tau,-\mathrm{B}_\mathfrak{g}}$ (see Table 1). We obtain that they are all G-stable, except for $\mathrm{SU}(n)$, $n \geq 3$ and $\mathrm{Sp}(n)$, $n \geq 2$, where g_B is G-neutrally stable of nullity n^2-1 and G-unstable of coindex $\geq \frac{2n(2n-1)}{2}-1$, respectively. The picture in the G-invariant setting is therefore analogous to the general case, which follows from Koiso's results on the stability of irreducible symmetric spaces (see §2).

On the other hand, we use formula (2) to compute the matrix of $\mathrm{L}_\mathfrak{p}$ in the multiplicity-free case in terms of the structural constants of the metric. Given any g-orthogonal decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r$ in $\operatorname{Ad}(K)$-invariant and irreducible subspaces, the numbers

$$[ijk] := \sum_{\alpha,\beta,\gamma} g([X_\alpha^i, X_\beta^j], X_\gamma^k)^2,$$

where $\{X_\alpha^i\}$ is a g-orthonormal basis of $\mathfrak{p}_i$, are invariant under any permutation of ijk by the natural reductivity of g and one has that $\operatorname{Rc}(g) = \rho g$ if and only if

$$\tfrac{b_k}{2} - \tfrac{1}{4d_k}\sum_{i,j}[ijk] = \rho, \qquad \forall k = 1, \dots, r,$$

where $-\mathrm{B}_\mathfrak{g}|_{\mathfrak{p}_k} = b_k g|_{\mathfrak{p}_k}$ and $d_k := \dim \mathfrak{p}_k$. We obtain in §5.2 that the entries of the matrix of $\mathrm{L}_\mathfrak{p}$ with respect to the orthonormal basis $\left\{\frac{1}{\sqrt{d_1}}I_{\mathfrak{p}_1}, \dots, \frac{1}{\sqrt{d_r}}I_{\mathfrak{p}_r}\right\}$ of $\operatorname{sym}(\mathfrak{p})^K$ are given by

$$[\mathrm{L}_\mathfrak{p}]_{kk} = \tfrac{1}{d_k}\sum_{\substack{j\neq k\\ i}}[ijk], \quad \forall k, \qquad [\mathrm{L}_\mathfrak{p}]_{jk} = -\tfrac{1}{\sqrt{d_j}\sqrt{d_k}}\sum_i[ijk], \quad \forall j \neq k. \tag{3}$$

This formula is applied in §6 to prove that the standard metric is G-unstable (and consequently Ricci flow dynamically unstable) on each of the following homogeneous spaces,

- $\mathrm{SU}(nk)/\mathrm{S}(\mathrm{U}(k) \times \cdots \times \mathrm{U}(k))$, $\quad k \geq 1$,
- $\mathrm{Sp}(nk)/\mathrm{Sp}(k) \times \cdots \times \mathrm{Sp}(k)$, $\quad k \geq 1$,
- $\mathrm{SO}(nk)/\mathrm{S}(\mathrm{O}(k) \times \cdots \times \mathrm{O}(k))$, $\quad k \geq 3$,

where the quotients are all n-times products with $n \geq 3$. Note that $\dim \mathcal{M}^G = \frac{n(n-1)}{2}$. We also compute the coindex (see Table 2) and found that the standard metric is a local minimum of $\mathrm{Sc} : \mathcal{M}_1^G \to \mathbb{R}$ in many cases (including $\mathrm{SU}(3)/T^2$) and it is G-degenerate in some others (e.g., $\mathrm{SU}(4)/T^3$).

As a second application of formula (3), we study in §7 the G-stability of the left-invariant Einstein metrics found by Jensen in [J2]. Given any simple Lie group H, one considers the left-invariant metric on H given by

$$g_t = -\mathrm{B}_{\mathfrak{h}}|_{\mathfrak{a}} + t(-\mathrm{B}_{\mathfrak{h}})|_{\mathfrak{k}}, \qquad t > 0,$$

where $K \subset H$ is a semisimple subgroup and $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{k}$ is the $\mathrm{B}_{\mathfrak{h}}$-orthogonal decomposition. g_1 is therefore the Killing metric on H and for each $t \neq 1$, the metric g_t is naturally reductive with respect to $G = H \times K$ (see [Z] or [DZ, Theorem 1]). If we assume that $\mathfrak{a}$ is $\mathrm{Ad}(K)$-irreducible (i.e., H/K is isotropy irreducible), then the isotropy representation of $G/\Delta K$ is mutliplicity-free and consists of $r+1$ $\mathrm{Ad}(K)$-irreducible summands, where $\mathfrak{k} = \mathfrak{k}_1 \oplus \dots \oplus \mathfrak{k}_r$ is a decomposition in simple ideals of $\mathfrak{k}$. Note that therefore $\dim \mathcal{M}_1^G = r$. We also assume that $\mathrm{B}_{\mathfrak{k}_i} = c\,\mathrm{B}_{\mathfrak{h}}|_{\mathfrak{k}_i}$ for any $i = 1, \dots, r$ and some constant c. It is proved in [DZ, Corollary 2, p.44] that $\mathrm{Ric}(g_t) = \rho I$ $(t \neq 1)$ if and only if,

$$t = t_E := \tfrac{dc}{(d+2k)(1-c)}, \qquad 2\rho = \tfrac{c}{2t_E} + \tfrac{(1-c)t_E}{2},$$

where $d = \dim \mathfrak{a}$ and $k := \dim \mathfrak{k}$. The explicit computation of $\mathrm{Spec}(\mathrm{L}_{\mathfrak{p}})$ using (3) shows that every g_{t_E} is G-unstable with coindex r, and in particular, g_{t_E} is always a local minimum. This provides at least one H-unstable (and so Ricci flow dynamically unstable) left-invariant Einstein metric on most simple Lie groups, including one of coindex ≥ 3 on E_6 and one of coindex ≥ 2 on $\mathrm{SO}(2n)$, $\mathrm{Sp}(2n)$, $\mathrm{SU}(n^2)$ and E_7.

Finally, we would like to mention that this is the first of a series of forthcoming papers on G-stability of homogeneous Einstein metrics on compact manifolds. In [LW2], we give a formula for the operator $\mathrm{L}_{\mathfrak{p}}(g)$ for any G-invariant Einstein metric g in terms of its usual structural constants $[ijk]$ with respect to a bi-invariant metric on $\mathfrak{g}$. The formula is used to establish the G-stability types of several Einstein metrics on well-known families of homogeneous spaces, including generalized Wallach spaces and some generalized flag manifolds. On the other hand, we compute in [LL] the G-stability types of all the standard Einstein metrics with G simple obtained in the famous classification by Wang and Ziller in [WZ1].

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2. Stability of compact Einstein manifolds. Einstein metrics on a compact differentiable manifold M, i.e., the Ricci tensor satisfies $\mathrm{Rc}(g) = \rho g$ for some $\rho \in \mathbb{R}$, were first studied by Hilbert, who proved that they are precisely the critical points of the total scalar curvature functional

$$\widetilde{\mathrm{Sc}}(g) := \int_M \mathrm{Sc}(g)\, d\,\mathrm{vol}_g, \tag{4}$$

restricted to the space $\mathcal{M}_1$ of unit volume Riemannian metrics on M (see [B, 4.21]). A fundamental problem is to determine whether a given Einstein metric g is *rigid*, in the sense that any Einstein metric sufficiently close to g (compact open C^∞ topology) is isometric to g up to scaling. Hilbert's variational characterization, beyond being a tool for the existence problem, allows the use of stability theory and calculus of variations in the study of the rigidity of Einstein metrics.

The case of (M, g) being isometric to a round sphere will be excluded in what follows. The tangent space $T_g\mathcal{M} = \mathcal{S}^2(M)$ (symmetric 2-tensors) of the space $\mathcal{M}$ of all Riemannian metrics on M at a metric $g \in \mathcal{M}$ admits the following decomposition (see [B, 4.57]):

$$T_g\mathcal{M} = \big(\mathcal{L}_{\mathfrak{X}(M)}g \oplus C^\infty(M)g\big) \oplus^{\perp_g} \mathcal{TT}_g, \tag{5}$$

where $\perp_g$ denotes orthogonality with respect to the usual L^2 inner product $\langle \cdot, \cdot \rangle_g$ on $\mathcal{S}^2(M)$ defined by g. The three summands are given by:

- $\mathcal{L}_{\mathfrak{X}(M)}g = \operatorname{Im}\delta_g^* = T_g \operatorname{Diff}(M) \cdot g$ is the space of *trivial variations*, where $\mathcal{L}$ denotes Lie derivative. Here $\delta_g : \mathcal{S}^2(M) \to \Omega^1(M)$ is the divergence operator $\delta_g(T) := -\sum_i \nabla_{X_i} T(X_i, \cdot)$, where $\{X_i\}$ is any local orthonormal frame, and δ_g^* is sometimes called the Killing operator as its kernel consists of Killing vector fields. An alternative decomposition is given by $T_g\mathcal{M} = \operatorname{Im}\delta_g^* \oplus^{\perp_g} \operatorname{Ker}\delta_g$.
- $C^\infty(M)g$ is the space of *conformal variations*, i.e., the tangent space at g of the space of metrics which are conformally equivalent to g. Note that $\mathbb{R}g \subset C^\infty(M)g$.
- $\mathcal{TT}_g = \operatorname{Ker}\delta_g \cap \operatorname{Ker}\operatorname{tr}_g$ is the subspace of divergence-free (or transversal) and traceless symmetric 2-tensors, so-called *TT-tensors*.

Let us now assume that g is an Einstein metric on M. If

$$\mathcal{C} := \{g \in \mathcal{M} : \operatorname{Sc}(g) \text{ is a constant function on } M\},$$

then at any $g \in \mathcal{C}$,

$$T_g\mathcal{C} = \big(\mathcal{L}_{\mathfrak{X}(M)}g \oplus \mathbb{R}g\big) \oplus^{\perp_g} \mathcal{TT}_g. \tag{6}$$

Thus $\mathcal{TT}_g$ can also be described as the space of all unit volume constant scalar curvature non-trivial variations of g (see [B, 4.44-4.46]).

We consider the second variation (or Hessian) of $\widetilde{\operatorname{Sc}}$ at g, i.e.,

$$\widetilde{\operatorname{Sc}}''_g(T,T) := \tfrac{d^2}{dt^2}\Big|_0 \widetilde{\operatorname{Sc}}(g+tT), \qquad \forall T \in \mathcal{S}^2(M).$$

Recall that g is a critical point of $\widetilde{\operatorname{Sc}}|_{\mathcal{M}_1}$, so for traceless tensors, this can be computed by using, instead of the line $g + tT$, any smooth curve $g(t) \in \mathcal{M}$ such that $g(0) = g$ and $g'(0) = T$. The following properties of the second variation are well known (see [B, 4.60]):

- Decomposition (5) is orthogonal with respect to $\widetilde{\operatorname{Sc}}''_g$, so its restriction on each of the three summands can be studied separately.
- $\widetilde{\operatorname{Sc}}''_g$ vanishes on $\mathcal{L}_{\mathfrak{X}(M)}g$ and $\widetilde{\operatorname{Sc}}''_g(g,g) = 2\operatorname{Sc}(g)$.
- $\widetilde{\operatorname{Sc}}''_g$ is positive definite on $C^\infty(M)g$.
- $\widetilde{\operatorname{Sc}}''_g|_{\mathcal{TT}_g}$ is negative definite on the orthogonal complement of a (possibly trivial) finite-dimensional vector subspace of $\mathcal{TT}_g$ (i.e., nullity and coindex are both finite).

These facts motivate the definition of the following concepts.

Definition 2.1. Let $g \in \mathcal{M}$ be an Einstein metric. We call g

- Sc-*stable* (or Sc-linearly stable): $\widetilde{\mathrm{Sc}}''_g|_{\mathcal{TT}_g} < 0$ (see [K, Definition 2.7] and [CH, Definition 2.2]). In particular, g is a local maximum of $\widetilde{\mathrm{Sc}}|_{\mathcal{C}_1}$ if $g \in \mathcal{M}_1$, where $\mathcal{C}_1$ is the space of all unit volume constant scalar metrics on M (indeed, by (6), $T_g\mathcal{C}_1 = \mathcal{L}_{\mathfrak{X}(M)}g \oplus^{\perp_g} \mathcal{TT}_g$ and one uses that $\mathcal{TT}_g$ exponentiates into a slice for the Diff(M)-action; see [B, 12.22] or [Kr1, Lemma 2.6.3]). This is actually the definition of g Sc-stable in many papers (e.g., [B2, WW]).
- Sc-*unstable* (or Sc-linearly unstable): $\widetilde{\mathrm{Sc}}''_g(T,T) > 0$ for some $T \in \mathcal{TT}_g$ (see [K, Definition 2.7] and [CH, Definition 2.2]).
- *infinitesimally non-deformable*: $\operatorname{Ker} \mathrm{E}'_g \cap \mathcal{TT}_g = 0$, and otherwise *infinitesimally deformable* (see [B, 12.29]). Here, E'_g is the first variation of the operator

$$\mathrm{E} : \mathcal{M} \longrightarrow \mathcal{S}^2(M), \qquad \mathrm{E}(g) := \mathrm{Rc}(g) - \tfrac{\widetilde{\mathrm{Sc}}(g)}{n} g, \tag{7}$$

so-called the *Einstein operator* (see [B, 12.26]). Note that $g \in \mathcal{M}_1$ is Einstein if and only if $\mathrm{E}(g) = 0$. Each element of $\operatorname{Ker} \mathrm{E}'_g \cap \mathcal{TT}_g$ is called an *infinitesimally Einstein deformation*, which may or may not be the velocity of a genuine Einstein deformation, i.e., a differentiable curve $g(t)$ of Einstein metrics through g.

If $\mathrm{Rc}(g) = \rho g$, then for any $T \in \mathcal{TT}_g$,

$$\widetilde{\mathrm{Sc}}''_g(T,T) = -\tfrac{1}{2}\langle(\Delta_L - 2\rho\,\mathrm{id})T, T\rangle_g \quad \text{and} \quad \mathrm{E}'_g(T) = \tfrac{1}{2}\Delta_L(T) - \rho T,$$

where Δ_L is the *Lichnerowicz Laplacian* of g, given by,

$$\Delta_L T = -\nabla^*\nabla T - 2\,\mathrm{Rm}_g(T,\cdot) + \mathrm{Rc}_g \circ T + T \circ \mathrm{Rc}_g,$$

and $\nabla\nabla^*$ denotes the usual rough Laplacian of g (see [B, 4.64] and [B, 12.28'], respectively). This implies that if $\lambda_L(g)$ denotes the smallest eigenvalue of $\Delta_L|_{\mathcal{TT}_g}$, then the following characterizations hold (cf. [CH, §4] and [WW, §1]):
- g is Sc-stable if and only if $2\rho < \lambda_L(g)$.
- g is Sc-unstable if and only if $\lambda_L(g) < 2\rho$.
- g is infinitesimally non-deformable if and only if $2\rho \notin \mathrm{Spec}\left(\Delta_L|_{\mathcal{TT}_g}\right)$, if and only if $\widetilde{\mathrm{Sc}}''_g|_{\mathcal{TT}_g}$ is non-degenerate.

In particular, stability implies infinitesimal non-deformability (cf. [K, Remark (2) below Definition 2.7]). On the other hand, the fact that any infinitesimally non-deformable Einstein metric is rigid is a strong result by Koiso (see [K, Proposition 3.3] and [B, 12.66]).

After forty years, the stability picture for symmetric spaces has recently been completed.

Theorem 2.2 ([K, GG, SW, S]). *All compact irreducible symmetric spaces are* Sc-*stable, except for*

$$\mathrm{Sp}(n)\ (n \geq 2), \quad \mathrm{Sp}(n)/\mathrm{U}(n)\ (n \geq 3), \quad \mathrm{SO}(5)/(\mathrm{SO}(3) \times \mathrm{SO}(2)),$$
$$\mathrm{Sp}(p+q)/(\mathrm{Sp}(p) \times \mathrm{Sp}(q))\ (p, q \geq 2),$$

which are Sc-*unstable and infinitesimally non-deformable, and*

$$\mathrm{SU}(n)/\mathrm{SO}(n), \quad \mathrm{SU}(2n)/\mathrm{Sp}(n)\ (n \geq 3),$$
$$\mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))\ (p \geq q \geq 2), \quad \mathrm{Sp}(3)/(\mathrm{Sp}(2) \times \mathrm{Sp}(1)),$$
$$F_4/\mathrm{Spin}(9), \quad \mathrm{SU}(n)\ (n \geq 3), \quad E_6/F_4,$$

which are infinitesimally deformable and not Sc-*unstable (i.e.,* $\lambda_L(g) = 2\rho$*), often called* Sc-*neutrally stable.*

The following questions remain open:

- Are the infinitesimally deformable irreducible symmetric metrics local maxima of $\widetilde{\mathrm{Sc}}|_{\mathcal{C}_1}$? The only results we know on this question are that SU(3) and SU($2n$)/Sp(n) are not local maxima (see [J1] and [BWZ, Example 6.7], respectively). We refer to [LW3] for a more detailed treatment of this question.
- Does there exist a Sc-stable Einstein manifold with $\mathrm{Sc} > 0$ which is not symmetric?
- Are the irreducible symmetric spaces

$$\mathrm{SU}(n)/\mathrm{SO}(n), \quad \mathrm{SU}(2n)/\mathrm{Sp}(n), \quad \mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q)), \quad \mathrm{SU}(n), \quad E_6/F_4,$$

rigid? Recently, the space SU($2n+1$) has been shown to be rigid in [BHMW].

Another important kind of stability is ν-entropy stability, relative to the ν-entropy functional $\nu : \mathcal{M} \longrightarrow \mathbb{R}$ introduced by Perelman (see [CH] for the definition). It was proved in [P] that ν is strictly increasing along any Ricci flow solution unless the solution consists of a shrinking gradient Ricci soliton (e.g., an Einstein metric with positive scalar curvature).

Decomposition (5) is also ν''_g-orthogonal and ν''_g also vanishes on $\mathcal{L}_{\mathfrak{X}(M)}g$ (see [CHI, CH]).

Definition 2.3 ([CH, Definition 3.3]). An Einstein metric $g \in \mathcal{M}$ is said to be,

- *ν-stable*: $\nu''_g \leq 0$ (called ν-linearly stable in [WW, Definition 1.2]). Equivalently, $\nu''_g|_{C^\infty(M)g} \leq 0$ and $\nu''_g|_{\mathcal{TT}_g} \leq 0$.
- *strictly ν-stable*: $\nu''_g|_{C^\infty(M)g} < 0$ and $\nu''_g|_{\mathcal{TT}_g} < 0$.
- *neutrally ν-stable*: g is ν-stable and there is a non-zero symmetric 2-tensor T either in $C^\infty(M)g$ or in $\mathcal{TT}_g$ such that $\nu''_g(T,T) = 0$.
- *ν-unstable*: $\nu''_g(T,T) > 0$ for some T either in $C^\infty(M)g$ or $\mathcal{TT}_g$.

Remark 2.4. In particular, if $g \in \mathcal{M}_1$ is strictly ν-stable, then g is a local maximum of ν among conformal variations of g, as well as a local maximum of $\nu|_{\mathcal{C}_1}$ by (6) (this is called ν-stable in [WW, Definition 1.2]).

Let $\lambda(g)$ denote the first eigenvalue of the Laplacian on functions Δ of the metric g (i.e., the Laplace-Beltrami operator).

Theorem 2.5 ([CHI]). *Let (M, g) be a compact Einstein manifold other than the standard sphere, with* $\mathrm{Rc}(g) = \rho g$, $\rho > 0$. *Then,*

(i) $\nu''_g(T,T) > 0$ *for some* $T \in C^\infty(M)g$ *if and only if* $\lambda(g) < 2\rho$ *(see also [CH, Lemma 3.5]).*
(ii) $\nu''_g(T,T) > 0$ *for some* $T \in \mathcal{TT}_g$ *if and only if* $\lambda_L(g) < 2\rho$ *(i.e., g is* Sc-*unstable).*

In particular,

- g is ν-stable if and only if $2\rho \leq \lambda(g)$ and $2\rho \leq \lambda_L(g)$;
- it is neutrally ν-stable if and only if in addition $\lambda(g) = 2\rho$ or $\lambda_L(g) = 2\rho$;
- and g is ν-unstable if and only if either $\lambda(g) < 2\rho$ or $\lambda_L(g) < 2\rho$.

The following notion of stability is more intuitive.

Definition 2.6 ([Kr2, Definition 1.1]). A compact Ricci soliton (M, g) is called *dynamically stable* if for any metric g_0 near g, the normalized Ricci flow starting at g_0 exists for all $t \geq 0$ and converges modulo diffeomorphisms to an Einstein metric near g, as $t \to \infty$. On the other hand, (M, g) is said to be *dynamically unstable* if there exists a nontrivial normalized Ricci flow defined on $(-\infty, 0]$ which converges modulo diffeomorphisms to g as $t \to -\infty$.

Kröncke proved that if a compact shrinking Ricci soliton (M, g) is not a local maximizer of ν (in particular, if g is ν-unstable), then (M, g) is dynamically unstable (see [Kr1, Corollary 6.2.5] or [Kr2, Theorem 1.3]). The following implications for a positive scalar curvature Einstein metric follow:

$$\text{Sc-instability} \Rightarrow \nu\text{-instability} \Rightarrow \text{dynamical instability.}$$

3. Rigidity and stability of homogeneous Einstein manifolds. In this section, we consider a connected differentiable manifold M (not necessarily compact) and assume that M is homogeneous. We also fix the transitive action of a Lie group G on M, which is assumed to be *almost-effective* (i.e., only a discrete subgroup of G acts trivially). This provides a presentation $M = G/K$ of M as a homogeneous space, where $K \subset G$ is the isotropy subgroup at some origin point $o \in M$. Neither G nor K are assumed to be connected.

We denote by $\mathcal{S}^2(M)^G$ the finite-dimensional vector space of all G-invariant symmetric 2-tensors on M, and by $\mathcal{M}^G \subset \mathcal{S}^2(M)^G$, the open cone of G-invariant Riemannian metrics. Note that $\mathcal{M}^G$ is a differentiable manifold with $1 \leq \dim \mathcal{M}^G \leq \frac{n(n+1)}{2}$ and tangent space $T_g\mathcal{M}^G = \mathcal{S}^2(M)^G$ at any $g \in \mathcal{M}^G$, where $n := \dim M$.

3.1. ***G*-rigidity.** The Lie group $\operatorname{Aut}(G/K) \subset \operatorname{Diff}(M)$ of all Lie automorphisms of G taking K onto K acts by pullback on $\mathcal{M}^G$, so each of its orbits consist of pairwise isometric metrics and the orbit $\operatorname{Aut}(G/K) \cdot g$ can be viewed as the trivial G-invariant deformations of a metric $g \in \mathcal{M}^G$. In this way, $\operatorname{Aut}(G/K)$ acts as the natural 'gauge group' in the G-invariant setting.

Remark 3.1. Two G-invariant metrics belonging to different $\operatorname{Aut}(G/K)$-orbits may however be isometric via some $\psi \in \operatorname{Diff}(M)$ which is not an automorphism. This cannot occur for left-invariant metrics on completely solvable Lie groups (see [A]). For G compact, one anyhow has that $T_g \operatorname{Aut}(G/K) \cdot g = T_g(\mathcal{M}^G \cap \operatorname{Diff}(M) \cdot g)$ for any $g \in \mathcal{M}^G$ (see Corollary 3.12 below).

Rigidity of Einstein metrics among $\mathcal{M}^G$ can therefore be naturally defined as follows.

Definition 3.2. An G-invariant Einstein metric g is called *G-rigid* if there exists an open neighborhood U of g in $\mathcal{M}^G$ such that any Einstein $g' \in U$ belongs to $\operatorname{Aut}(G/K) \cdot g$ up to scaling.

In other words, a G-invariant Einstein metric g is G-rigid when g is an isolated point in the moduli space $\overline{\mathcal{E}}^G := \mathcal{E}^G/\mathbb{R}_+ \operatorname{Aut}(G/K)$, where

$$\mathcal{E}^G := \{g \in \mathcal{M}^G : g \text{ is Einstein}\},$$

and $\mathbb{R}_+ := \{a \in \mathbb{R} : a > 0\}$ acts on $\mathcal{M}^G$ by scaling. We note that $\overline{\mathcal{E}}^G = \mathcal{E}_1^G / \operatorname{Aut}(G/K)$, where $\mathcal{E}_1^G := \mathcal{E}^G \cap \mathcal{M}_1^G$ and

$$\mathcal{M}_1^G := \{g' \in \mathcal{M}^G : \det_{\overline{g}} g' = 1\}.$$

Here $\overline{g}$ denotes a fixed background metric in $\mathcal{M}^G$. For G compact, $\mathcal{M}_1^G$ is the space of all G-invariant metrics of a given fixed volume.

The space $\mathcal{E}^G$ is a real semialgebraic subset (i.e., the set of solutions of finitely many polynomial equalities and inequalities) of $\mathcal{S}^2(M)^G$ (see [BWZ, Proposition 1.5]). The following properties therefore follow from classical theorems of Whitney (see e.g. [BCR]):

- $\mathcal{E}^G$ has finitely many connected components.
- There is a (local) stratification of $\mathcal{E}^G$ into real algebraic smooth submanifolds.
- Path components and connected components coincide, as $\mathcal{E}^G$ is locally path-connected.

In the compact case, we have in addition the following major result.

THEOREM 3.3 ([BWZ, Theorem 1.6]). *Let G be a compact Lie group and $M = G/K$ be a connected homogeneous space with finite fundamental group. Then each connected component of $\mathcal{E}_1^G$ is compact, and the set of possible Einstein constants of metrics among $\mathcal{E}_1^G$ is finite.*

In particular, in the compact case, the moduli space $\overline{\mathcal{E}}^G = \mathcal{E}_1^G/\operatorname{Aut}(G/K)$ is also compact and hence $\overline{\mathcal{E}}^G$ is finite if and only if every $g \in \mathcal{M}^G$ is G-rigid. It is an open question whether $\overline{\mathcal{E}}^G$ is always finite. This has been conjectured for the multiplicity-free isotropy representation case in [BWZ], where only finitely many trivial deformations are possible, so conjecturally, $\mathcal{E}_1^G$ is itself a finite set. Classes of compact homogeneous spaces for which $\overline{\mathcal{E}}^G$ is known to be finite include D'Atri-Ziller metrics (see [DZ]), generalized Wallach spaces (see [LNF]) and spaces with only two isotropy summands (see [WZ2]), but it is still open in general for generalized flag manifolds, even for the full flag $\mathrm{SU}(n)/T$ for n large.

On the other hand, a left-invariant Einstein metric on a solvable Lie group G is known to be G-rigid; moreover, $\overline{\mathcal{E}}^G$ is either empty or a singleton (see [H] and [BL1, Corollary 4.3]).

PROPOSITION 3.4. *If an Einstein metric $g \in \mathcal{M}^G$ is not G-rigid, then there exists a smooth path $g : (-\epsilon, \epsilon) \to \mathcal{M}^G$ such that $g(0) = g$, $g(s)$ is Einstein for all s and*

$$g'(0) \perp_g T_g\mathbb{R}_+ \operatorname{Aut}(G/K) \cdot g.$$

REMARK 3.5. It follows from the existence of a slice for the $\mathbb{R}_+ \operatorname{Aut}(G/K)$-action that the path $g(s)$ is transversal to $\mathbb{R}_+ \operatorname{Aut}(G/K)$-orbits for sufficiently small ϵ, in the sense that $g(s) \notin \mathbb{R}_+ \operatorname{Aut}(G/K) \cdot g(s')$ for all $s, s' \in (-\epsilon, \epsilon)$, $s \neq s'$. In other words, $g(s)$ descends to a genuine curve through the class of g in the moduli space $\overline{\mathcal{E}}^G$.

Proof. As an element of $\mathcal{E}^G$, the metric g belongs to a finite number of connected smooth submanifolds contained in $\mathcal{E}^G$, each of which is invariant under the connected component $\operatorname{Aut}(G/K)^0$ of the Lie group $\operatorname{Aut}(G/K)$. Since g is not G-rigid, the dimension of the orbit $\mathbb{R}_+ \operatorname{Aut}(G/K)^0 \cdot g$ is strictly less than the dimension of at least one of these submanifolds, so the existence of the smooth path $g(s)$ follows. □

3.2. Variational principle. The manifold $\mathcal{M}^G$ is itself naturally endowed with a Riemannian metric defined at each $g \in \mathcal{M}^G$ by

$$\langle T, T\rangle_g := \operatorname{tr} A^2, \quad \text{where} \quad T_o = g_o(A\cdot, \cdot), \quad \forall T \in \mathcal{S}^2(M)^G. \tag{8}$$

Note that the linear map $A : T_oM \to T_oM$ is g_o-self-adjoint and $\operatorname{tr}_g T = \operatorname{tr} A$, $\det_g T = \det A$. Equivalently, $\langle T, T\rangle_g := \sum T_o(X_i, X_i)^2$, for any g_o-orthonormal basis $\{X_i\}$ of

T_oM. In particular, $\langle \cdot,\cdot\rangle_g$ is precisely the L^2 metric considered in §2 if M is compact and $g \in \mathcal{M}_1^G$.

In the case when G is unimodular, it is well known (see e.g. [N, H] and [W, (1.11)]) that relative to such metric on $\mathcal{M}^G$, the gradient of the scalar curvature function

$$\operatorname{Sc} : \mathcal{M}^G \to \mathbb{R}, \qquad \operatorname{Sc}(g) := \operatorname{tr}_g \operatorname{Rc}(g),$$

is given by

$$\operatorname{grad}(\operatorname{Sc})_g = -\operatorname{Rc}(g), \qquad \forall g \in \mathcal{M}^G, \tag{9}$$

where $\operatorname{Rc}(g) \in \mathcal{S}^2(M)^G$ is the Ricci tensor of g. Since the tangent space of the submanifold $\mathcal{M}_1^G$ at a metric $g \in \mathcal{M}_1^G$ is precisely

$$(\mathbb{R}g)^{\perp_g} = \left\{T \in \mathcal{S}^2(M)^G : \operatorname{tr}_g T = 0\right\} = \operatorname{Ker}\operatorname{tr}_g \cap \mathcal{S}^2(M)^G,$$

one obtains the following result, which it was first proved by Palais (see [B, 4.23]) for G compact.

Lemma 3.6. *If $M = G/K$ and G is unimodular, then $g \in \mathcal{M}_1^G$ is a critical point of* $\operatorname{Sc}|_{\mathcal{M}_1^G}$ *if and only if g is Einstein.*

This variational characterization has been successfully applied for decades, since the pioneer articles [J1, WZ2], to study the existence of invariant Einstein metrics on homogeneous spaces (see [BWZ, B1, W] and references therein). In this paper, we aim to use the second variation of $\operatorname{Sc} : \mathcal{M}^G \to \mathbb{R}$ to study G-rigidity.

3.3. Trivial variations. According to §3.1, the space of trivial G-invariant variations of a metric $g \in \mathcal{M}^G$ is given by the tangent space $T_g \operatorname{Aut}(G/K)\cdot g \subset \mathcal{S}^2(M)^G$. A distinguished subgroup of $\operatorname{Aut}(G/K)$ is the normalizer $N_G(K)$, which acts on M by $n\cdot(a\cdot o) = I_n(a\cdot o) := nan^{-1}\cdot o$ and on $T_oM \equiv \mathfrak{g}/\mathfrak{k}$ by $n\cdot X := \operatorname{Ad}(n)X$. Alternatively, the Lie group $N := N_G(K)/K$ acts on M by *G-equivariant diffeomorphisms* (i.e., $\psi(a\cdot p) = a\cdot\psi(p)$ for all $a \in G$, $p \in M$) in the following way: $n\cdot(a\cdot o) = R_n(a\cdot o) := an\cdot o$. Thus $N\cdot g$ is contained in the so-called *G-equivariant isometry class* of the metric g, and since $R_n^*g = I_{n^{-1}}^*g$ for any $n \in N$, one obtains that

$$N\cdot g = N_G(K)\cdot g, \qquad \forall g \in \mathcal{M}^G. \tag{10}$$

We consider any *reductive decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the homogeneous space $M = G/K$ (i.e., $\operatorname{Ad}(K)\mathfrak{p} \subset \mathfrak{p}$), where $\mathfrak{g}$ and $\mathfrak{k}$ are respectively the Lie algebras of G and K, which provides the usual identification $T_oM \equiv \mathfrak{p}$. Thus $\mathcal{S}^2(M)^G$ will be often identified, without any further mention, with the vector space of $\operatorname{Ad}(K)$-invariant symmetric 2-forms on $\mathfrak{p}$, and $\mathcal{M}^G$ with the open cone of positive definite ones. For each $X \in \mathfrak{p}$, consider the linear map

$$\operatorname{ad}_{\mathfrak{p}} X := \operatorname{pr}_{\mathfrak{p}} \circ \operatorname{ad} X|_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}, \tag{11}$$

where $\operatorname{pr}_{\mathfrak{p}} : \mathfrak{g} \to \mathfrak{p}$ is the projection on $\mathfrak{p}$ relative to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

As shown in [LW1, Lemma 6.10], at any $g \in \mathcal{M}^G$, the trivial variations space satisfies that

$$T_g \operatorname{Aut}(G/K)\cdot g \subset \left\{g_o(S(D)\cdot,\cdot) : \underline{D} \in \operatorname{Der}(\mathfrak{g}/\mathfrak{k})\right\}, \tag{12}$$

where $S(A) := \frac{1}{2}(A + A^t)$ denotes the symmetric part of a linear map A with respect to g_o and

$$\operatorname{Der}(\mathfrak{g}/\mathfrak{k}) := \{\underline{D} \in \operatorname{Der}(\mathfrak{g}) : \underline{D}(\mathfrak{k}) \subset \mathfrak{k}\}, \qquad \underline{D} = \begin{bmatrix} * & * \\ 0 & D \end{bmatrix}.$$

We note that if

$$\mathfrak{p}_0 := \{X \in \mathfrak{p} : [\mathfrak{k}, X] = 0\},$$

then $\operatorname{ad}\mathfrak{p}_0 \subset \operatorname{Der}(\mathfrak{g}/\mathfrak{k})$ and the Lie algebra of $N_G(K)$ is given by $N_{\mathfrak{g}}(\mathfrak{k}) = \mathfrak{k} \oplus \mathfrak{p}_0$. On the other hand, $g_o(S(\operatorname{ad}_{\mathfrak{p}} \mathfrak{p})\cdot, \cdot) \cap \mathcal{S}^2(M)^G \subset g_o(S(\operatorname{ad}_{\mathfrak{p}} \mathfrak{p})^{\mathfrak{k}}\cdot, \cdot)$, where

$$S(\operatorname{ad}_{\mathfrak{p}} \mathfrak{p})^{\mathfrak{k}} := \{S(\operatorname{ad}_{\mathfrak{p}} X) : X \in \mathfrak{p},\ [\operatorname{ad}\mathfrak{k}|_{\mathfrak{p}}, S(\operatorname{ad}_{\mathfrak{p}} X)] = 0\},$$

and equality holds if K is connected.

Lemma 3.7. *For any $g \in \mathcal{M}^G$, $S(\operatorname{ad}_{\mathfrak{p}} \mathfrak{p})^{\mathfrak{k}} = S(\operatorname{ad}_{\mathfrak{p}} \mathfrak{p}_0)$ and*

$$T_g N \cdot g = g_o(S(\operatorname{ad}_{\mathfrak{p}} \mathfrak{p}_0)\cdot, \cdot).$$

Remark 3.8. In the Lie group case, i.e., $M = G$ and K trivial, we have that $S(\operatorname{ad}_{\mathfrak{p}} \mathfrak{p}_0) = S(\operatorname{ad}\mathfrak{g})$, so it is zero if and only if g is bi-invariant.

Proof. Since $[\operatorname{ad} Z|_{\mathfrak{p}}, S(\operatorname{ad}_{\mathfrak{p}} X)] = S(\operatorname{ad}_{\mathfrak{p}} [Z, X]|_{\mathfrak{p}})$ for any $Z \in \mathfrak{k}$, we obtain that $S(\operatorname{ad}_{\mathfrak{p}} \mathfrak{p}_0) \subset S(\operatorname{ad}_{\mathfrak{p}} \mathfrak{p})^{\mathfrak{k}}$. Conversely, given $S(\operatorname{ad}_{\mathfrak{p}} X) \in S(\operatorname{ad}_{\mathfrak{p}} \mathfrak{p})^{\mathfrak{k}}$, we consider the decomposition $X = X_0 + X_1$, where $X_0 \in \overline{\mathfrak{p}} := \{Y \in \mathfrak{p} : (\operatorname{ad}_{\mathfrak{p}} Y)^t = -\operatorname{ad}_{\mathfrak{p}} Y\}$ and $X_1 \perp \overline{\mathfrak{p}}$. Note that both $\overline{\mathfrak{p}}$ and its orthogonal complement are $\operatorname{ad}\mathfrak{k}|_{\mathfrak{p}}$-invariant subspaces. Thus $[Z, X]$ and $[Z, X_0]$ both belong to $\mathfrak{p}_0$ and so $[Z, X_1] = 0$ for any $Z \in \mathfrak{k}$, from which follows that $S(\operatorname{ad}_{\mathfrak{p}} X) = S(\operatorname{ad}_{\mathfrak{p}} X_1) \in S(\operatorname{ad}_{\mathfrak{p}} \mathfrak{p}_0)$.

The second equality can be proved using (10) as follows. For any $X \in \mathfrak{g}$ such that $[X, \mathfrak{k}] \subset \mathfrak{k}$,

$$\begin{aligned} \tfrac{d}{dt}\big|_0 (I_{\exp tX})^* g &= \tfrac{d}{dt}\big|_0 g_o(\operatorname{Ad}(\exp tX)|_{\mathfrak{p}}\cdot, \operatorname{Ad}(\exp tX)|_{\mathfrak{p}}\cdot) \\ &= g_o(\operatorname{ad} X|_{\mathfrak{p}}\cdot, \cdot) + g_o(\cdot, \operatorname{ad} X|_{\mathfrak{p}}\cdot). = 2g_o(S(\operatorname{ad} X|_{\mathfrak{p}})\cdot, \cdot), \end{aligned}$$

Now if $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$, then $S(\operatorname{ad} X|_{\mathfrak{p}}) = S(\operatorname{ad}_{\mathfrak{p}} X_{\mathfrak{p}})$ (since $\operatorname{ad} X_{\mathfrak{k}}|_{\mathfrak{p}}$ is skew-symmetric) and $[X_{\mathfrak{p}}, \mathfrak{k}] \subset \mathfrak{k} \cap \mathfrak{p} = 0$, i.e., $X_{\mathfrak{p}} \in \mathfrak{p}_0$. □

Assume from now on in this subsection that G is compact, thus M and K are also compact. In this case, it is known that N is the group of all G-equivariant diffeomorphisms of $M = G/K$ (see [Br, Chapter I, Corollary 4.3]) and so N-orbits (or $N_G(K)$-orbits, see (10)) are precisely the equivariant isometry classes. Since $N_G(K)$ and $\operatorname{Aut}(G/K)$ have the same connected components of the identity, an Einstein metric g is G-rigid if and only if any other G-invariant Einstein metric on M near g is equivariantly isometric up to scaling to g. Furthermore, one obtains from Lemma 3.7 the following useful description of the space of trivial G-invariant variations.

Corollary 3.9. *If G is compact, then at any $g \in \mathcal{M}^G$,*

$$T_g \operatorname{Aut}(G/K) \cdot g = T_g N \cdot g = g_o(S(\operatorname{ad}_{\mathfrak{p}} \mathfrak{p}_0)\cdot, \cdot).$$

Contrary to what happens in the Lie group case (see Remark 3.8), the space of trivial variations vanishes in many cases if K is non-trivial:

- If $g \in \mathcal{M}^G$ is *naturally reductive* with respect to G, i.e., there exists a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that $\operatorname{ad}_{\mathfrak{p}} X$ is skew-symmetric for any $X \in \mathfrak{p}$, then $T_g \operatorname{Aut}(G/K) \cdot g = 0$ by Corollary 3.9.
- Another direct consequence of Corollary 3.9 is that $T_g \operatorname{Aut}(G/K) \cdot g = 0$ for any $g \in \mathcal{M}^G$ if the trivial representation does not appear in the $\mathfrak{k}$-isotropy representation of $M = G/K$ (i.e., $\mathfrak{p}_0 = 0$).
- If G is compact and the isotropy representation of G/K is *mutiplicity-free* (i.e., any two different $\operatorname{Ad}(K)$-invariant irreducible subspaces are inequivalent as $\operatorname{Ad}(K)$-representations), e.g., when $\operatorname{rk}(G) = \operatorname{rk}(K)$, then $N_G(K) \cdot g$ is finite and so $T_g \operatorname{Aut}(G/K) \cdot g = 0$ for any $g \in \mathcal{M}^G$. Indeed, the multiplicity-free condition is equivalent to the existence of only finitely many $\operatorname{Ad}(K)$-invariant subspaces of $\mathfrak{p}$, which implies that the connected component $N_G(K)^0$ necessarily leaves invariant any $\operatorname{Ad}(K)$-invariant and irreducible subspace of $\mathfrak{p}$ and consequently $N_G(K)^0$ acts trivially on $\mathcal{M}^G$.

3.4. G-invariant TT-tensors. Recall from §2 the divergence operator δ_g attached to a Riemannian metric g, and the space of TT-tensors $\mathcal{TT}_g = \operatorname{Ker}\delta_g \cap \operatorname{Ker}\operatorname{tr}_g$. The proof of the following lemma is strongly based on the proof of [WW, Lemma 2.2].

Proposition 3.10. *If G is unimodular and $g \in \mathcal{M}^G$, then*

$$\mathcal{S}^2(M)^G = g_o(S(\operatorname{ad}_{\mathfrak{p}} \mathfrak{p}_0)\cdot, \cdot) \oplus^{\perp_g} \operatorname{Ker}\delta_g \cap \mathcal{S}^2(M)^G.$$

Remark 3.11. In particular, $\mathcal{S}^2(M)^G \subset \operatorname{Ker}\delta_g$ and so $\mathcal{TT}_g^G = \mathcal{S}^2(M)^G \cap \operatorname{Ker}\operatorname{tr}_g$ under any of the above three assumptions, where

$$\mathcal{TT}_g^G := \mathcal{S}^2(M)^G \cap \mathcal{TT}_g$$

is the space of all G-invariant TT-tensors.

Proof. Let $\{X_i\}$ be a g_o-orthonormal basis of $\mathfrak{p}$ and extend it to a local frame of Killing vector fields. Consider $T \in \mathcal{S}^2(M)^G$. Then, at the point o we have that

$$\begin{aligned}
\delta_g(T)(X) &= -\sum (\nabla_{X_i} T)(X_i, X) = \sum -X_i(T(X_i,X)) + T(\nabla_{X_i} X_i, X) + T(X_i, \nabla_{X_i} X) \\
&= \sum T(X_i, [X, X_i]) + T(X_i, \nabla_{X_i} X) + T(\nabla_{X_i} X_i, X) \\
&= \sum T(X_i, \nabla_X X_i) + T(\nabla_{X_i} X_i, X) \\
&= \sum g(\nabla_X X_i, X_k) T(X_i, X_k) + \sum g(\nabla_{X_i} X_i, X_k) T(X_k, X).
\end{aligned}$$

It follows from the Koszul formula (recall that $[X_i, X_j]_o = -[X_i, X_j]_{\mathfrak{p}}$, where $[\cdot,\cdot]_{\mathfrak{p}}$ denotes the Lie bracket of $\mathfrak{g}$ restricted and then projected on $\mathfrak{p}$) that the right summand equals

$$\begin{aligned}
\sum g_o([X_k, X_i]_{\mathfrak{p}}, X_i) T(X_k, X) &= \sum_k T(X_k, X) \sum_i g_o([X_k, X_i]_{\mathfrak{p}}, X_i) \\
&= \sum T(X_k, X) \operatorname{tr}\operatorname{ad}_{\mathfrak{p}} X_k = 0,
\end{aligned}$$

since $\operatorname{tr}\operatorname{ad}_{\mathfrak{p}} Y = \operatorname{tr}\operatorname{ad} Y = 0$ for any $Y \in \mathfrak{p}$ as G is unimodular, and the left one gives

$$\begin{aligned}
&-\tfrac{1}{2}\sum g_o([X, X_i]_{\mathfrak{p}}, X_k) T(X_i, X_k) - \tfrac{1}{2}\sum g_o([X_i, X_k]_{\mathfrak{p}}, X) T(X_i, X_k) \\
&+\tfrac{1}{2}\sum g_o([X_k, X]_{\mathfrak{p}}, X_i) T(X_i, X_k) = -\tfrac{1}{2}\sum T([X, X_i]_{\mathfrak{p}}, X_i) - \tfrac{1}{2}\sum T([X, X_k]_{\mathfrak{p}}, X_k) \\
&= -\sum T([X, X_i]_{\mathfrak{p}}, X_i) = -\langle T, g_o(S(\operatorname{ad}_{\mathfrak{p}} X)\cdot, \cdot)\rangle_g.
\end{aligned}$$

Note that the middle term vanishes since $[\cdot,\cdot]_{\mathfrak{p}}$ and T are respectively skew-symmetric and symmetric bilinear forms. Thus a tensor $T \in \mathcal{S}^2(M)^G$ is divergence-free if and only if $T \perp g_o(S(\operatorname{ad}_{\mathfrak{p}} X)\cdot,\cdot)$ for any $X \in \mathfrak{p}$, which is equivalent to $T \perp g_o(S(\operatorname{ad}_{\mathfrak{p}} \mathfrak{p}_0)\cdot,\cdot)$ by Lemma 3.7 and the fact that T is $\operatorname{Ad}(K)$-invariant. □

It follows from Corollary 3.9 and Proposition 3.10 that the space of all G-invariant variations $T_g\mathcal{M}^G = \mathcal{S}^2(M)^G$ admits the following decomposition in the compact case.

Corollary 3.12. *If G is compact, then at any $g \in \mathcal{M}^G$,*

$$T_g\mathcal{M}^G = \mathbb{R}g \oplus^{\perp_g} T_g \operatorname{Aut}(G/K)\cdot g \oplus^{\perp_g} \mathcal{TT}_g^G.$$

Recall from Remark 3.11 that $T_g\mathcal{M}^G = \mathbb{R}g \oplus^{\perp_g} \mathcal{TT}_g^G$ therefore holds in many natural cases. Curiously enough, as far as we know, $S^2 \times S^3 = \mathrm{SO}(4)/\mathrm{SO}(2)$ is the only homogeneous space G/K with $\dim K > 0$ known such that $T_g \operatorname{Aut}(G/K)\cdot g$ is nonzero for a G-invariant Einstein metric g (see [LW2, Example 3.7]).

3.5. ***G*-stability.** Since the function Sc is constant on $\operatorname{Aut}(G/K)\cdot g$, its second variation Sc''_g vanishes on $T_g \operatorname{Aut}(G/K)\cdot g$. Note that $\mathrm{Sc}''(g,g) = 2\,\mathrm{Sc}(g)$. On the other hand, if $g \in \mathcal{M}^G$ is Einstein, then the orbit $\operatorname{Aut}(G/K)\cdot g$ consists of Einstein metrics and so $\mathrm{E}|_{\operatorname{Aut}(G/K)\cdot g} \equiv 0$ and $\mathrm{E}(\mathbb{R}_+ g) = 0$, where

$$\mathrm{E} : \mathcal{M}^G \longrightarrow \mathcal{S}^2(M)^G, \qquad \mathrm{E}(g') := \mathrm{Rc}(g') - \tfrac{\mathrm{Sc}(g')}{n} g',$$

is the Einstein operator or traceless Ricci tensor (cf. (7)).

At each $g \in \mathcal{M}^G$, we consider the following decomposition,

$$T_g\mathcal{M}^G = (\mathbb{R}g \oplus T_g \operatorname{Aut}(G/K)\cdot g) \oplus^{\perp_g} W_g, \tag{13}$$

where W_g is defined as the $\langle\cdot,\cdot\rangle_g$-orthogonal complement of the space $\mathbb{R}g \oplus T_g \operatorname{Aut}(G/K)\cdot g$ of trivial variations. According to Proposition 3.10 and (13), if G is unimodular, then $W_g \subset \mathcal{TT}_g^G$, and if in addition G is compact, then by Corollary 3.12,

$$W_g = \mathcal{TT}_g^G, \tag{14}$$

the vector space of G-invariant TT-tensors.

Remark 3.13. The existence of G-invariant Einstein metrics on $M = G/K$ for a non-compact unimodular G is open. It is proved in [DLM] that G must be semisimple, hence such existence would provide a counterexample to the *Alekseevsky conjecture*: any non-compact and non-flat homogeneous Einstein manifold is isometric to a simply connected solvmanifold (in particular, diffeomorphic to the Euclidean space). After the conclusion of the first version of this paper, a proof of the Alekseevsky conjecture was uploaded to arXiv by C. Böhm and R. Lafuente (see [BL2]).

We are now ready to define the notions of stability and deformability in the G-invariant setting (cf. Definition 2.1).

Definition 3.14. An Einstein metric $g \in \mathcal{M}_1^G$ is said to be,

- *G-stable*: $\mathrm{Sc}''_g|_{W_g \times W_g} < 0$ (in particular, g is a local maximum of $\mathrm{Sc}|_{\mathcal{M}_1^G}$, by using a slice for the $\operatorname{Aut}(G/K)$-action on $\mathcal{M}^G$).

- *G-unstable*: $\mathrm{Sc}''_g(T,T) > 0$ for some $T \in W_g$ (g is a saddle point, unless $\mathrm{Sc}''_g|_{W_g\times W_g} > 0$, see below). The *coindex* is the dimension of the maximal subspace of W_g on which Sc''_g is positive definite.
- *G-non-degenerate*: $\mathrm{Sc}''_g|_{W_g\times W_g}$ non-degenerate (thus g is an isolated critical point up to the $\mathrm{Aut}(G/K)$-action, i.e., g is rigid), and otherwise, *G-degenerate*. The *nullity* is the dimension of the kernel of $\mathrm{Sc}''_g|_{W_g\times W_g}$. Recall from §2 that G-non-degeneracy is equivalent to *G-infinitesimal non-deformability*: $\operatorname{Ker} d\,\mathrm{E}|_g \cap W_g = 0$, where $d\,\mathrm{E}|_g : \mathcal{S}^2(M)^G \to \mathcal{S}^2(M)^G$ is the derivative of E.
- *G-neutrally stable*: $\mathrm{Sc}''_g|_{W_g\times W_g} \leq 0$ and degenerate (i.e., g is G-degenerate and it is not G-unstable). Note that this must hold for any local maximum.
- *G-strongly unstable*: $\mathrm{Sc}''_g|_{W_g\times W_g} > 0$ (g is therefore a local minimum of $\mathrm{Sc}|_{\mathcal{M}_1^G}$).

REMARK 3.15. Recall that the prefix G in the name of the different notions is referring not only to the group G but also to its action on M, which has been fixed at the beginning of the section.

If an Einstein metric $g \in \mathcal{M}^G$ is G-stable, then g is clearly G-non-degenerate, which in turn implies that g is G-rigid by Proposition 3.4. On the other hand, it follows from (14) and §2 that if G is compact, then

$$G\text{-instability} \Rightarrow \text{Sc-instability} \Rightarrow \nu\text{-instability} \Rightarrow \text{dynamical instability},$$

and that non-rigidity also follows from the assumption that the corresponding G-invariant concept holds.

In [WW, Theorems 1.3, 1.4, 1.5], the authors obtained that all Einstein metrics on Aloff-Wallach spaces are G-unstable, as well as any G-invariant Einstein metric on a homogeneous space G/K ((G,K) not a symmetric pair) of dimension ≤ 7, except for $\mathrm{SU}(2)\times\mathrm{SU}(2)$ and the isotropy irreducible $\mathrm{Sp}(2)/\mathrm{SU}(2)$ (see also [SWW]).

REMARK 3.16. In [LW1], the Ricci curvature function

$$\mathrm{Rc} : \mathcal{M}^G \to \mathcal{S}^2(M)^G, \qquad g \mapsto \mathrm{Rc}(g),$$

and its derivative $d\,\mathrm{Rc}|_g : \mathcal{S}^2(M)^G \to \mathcal{S}^2(M)^G$, at each $g \in \mathcal{M}^G$, were used in the study of the prescribed Ricci curvature problem. Given an Einstein metric $g \in \mathcal{M}^G$, say $\mathrm{Rc}(g) = \rho g$, it is easy to see that restricted to $(\mathbb{R}g)^{\perp_g}$, $d\,\mathrm{E}|_g = d\,\mathrm{Rc}|_g - \rho\,\mathrm{id}$. On the other hand, we will show below in §4 that $\mathrm{Sc}''_g(T,T) = \langle(\rho\,\mathrm{id} - d\,\mathrm{Rc}|_g)T,T\rangle_g$, for any $T \in \mathcal{S}^2(M)^G$. Thus the stability type of g is determined by $\operatorname{Spec}\big(d\,\mathrm{Rc}|_g|_{W_g}\big)$. The operator $d\,\mathrm{Rc}|_g$, which restricted to W_g is precisely one half of the Lichnerowicz Laplacian Δ_L when G is compact, was computed in [LW1] in terms of the moment map of the variety of algebras via the moving bracket approach. This is developed in §4.

4. Second variation of the scalar curvature. Given $M^n = G/K$ as in §3, we consider any reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ in order to obtain the usual identifications $T_oM \equiv \mathfrak{p}$ and

$$\mathcal{S}^2(M)^G \leftrightarrow \mathrm{sym}^2(\mathfrak{p})^K, \qquad \mathcal{M}^G \leftrightarrow \mathrm{sym}^2_+(\mathfrak{p})^K,$$

where $\mathrm{sym}^2(\mathfrak{p})^K$ is the vector space of all $\mathrm{Ad}(K)$-invariant symmetric 2-forms on the n-dimensional vector space $\mathfrak{p}$ and $\mathrm{sym}^2_+(\mathfrak{p})^K$ the open cone of positive ones.

REMARK 4.1. It is usual in the literature the choice of $\mathfrak{p}$ as the orthogonal complement of $\mathfrak{k}$ relative to some bi-invariant inner product on $\mathfrak{g}$, which always exists for G compact. However, this choice may hide, among other nice properties, the fact that a metric is naturally reductive with respect to G.

We also fix a background metric $g \in \mathcal{M}^G$ and set $\langle\cdot,\cdot\rangle := g_o \in \operatorname{sym}^2_+(\mathfrak{p})^K$. This allows the following alternative identifications in terms of operators:

$$\operatorname{sym}(\mathfrak{p})^K \ni A \leftrightarrow T = \langle A\cdot,\cdot\rangle \in \operatorname{sym}^2(\mathfrak{p})^K, \qquad \operatorname{sym}_+(\mathfrak{p})^K \ni h \leftrightarrow \langle h\cdot,h\cdot\rangle \in \operatorname{sym}^2_+(\mathfrak{p})^K,$$

where $\operatorname{sym}(\mathfrak{p})$ is the vector space of all self-adjoint (or symmetric) linear maps of $\mathfrak{p}$ with respect to $\langle\cdot,\cdot\rangle$ and $\operatorname{sym}_+(\mathfrak{p})$ the open subset of those which are positive definite. Note that $A \in \operatorname{sym}(\mathfrak{p})$ belongs to $\operatorname{sym}(\mathfrak{p})^K$ if and only if $[\operatorname{Ad}(K)|_{\mathfrak{p}}, A] = 0$ (equivalently, $[\operatorname{ad}\mathfrak{k}|_{\mathfrak{p}}, A] = 0$, if K is connected).

4.1. Ricci curvature. Let μ denote the Lie bracket of $\mathfrak{g}$. We extend $\langle\cdot,\cdot\rangle$ in the usual way to inner products on $\mathfrak{gl}(\mathfrak{p})$ and $\Lambda^2\mathfrak{p}^*\otimes\mathfrak{p}$, respectively:

$$\langle A,B\rangle := \operatorname{tr} AB^t, \qquad \langle\lambda,\lambda\rangle := \sum |\operatorname{ad}_\lambda X_i|^2 = \sum |\lambda(X_i,X_j)|^2,$$

where $\{X_i\}$ is any orthonormal basis of $\mathfrak{p}$ relative to $\langle\cdot,\cdot\rangle$. We also consider the algebra product,

$$\mu_{\mathfrak{p}} := \operatorname{pr}_{\mathfrak{p}} \circ \mu|_{\mathfrak{p}\times\mathfrak{p}} : \mathfrak{p}\times\mathfrak{p} \longrightarrow \mathfrak{p}, \tag{15}$$

where $\operatorname{pr}_{\mathfrak{p}} : \mathfrak{g} \to \mathfrak{p}$ is the projection on $\mathfrak{p}$ relative to $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$, and consider the linear maps $\operatorname{ad}_{\mathfrak{p}} X := \mu_{\mathfrak{p}}(X,\cdot)$, $X \in \mathfrak{p}$, as in (11).

If G is unimodular, then the Ricci operator $\operatorname{Ric}(g)$ of the metric g (see e.g. [LW1, (5)]) is given by

$$\operatorname{Ric}(g) = \mathrm{M}_{\mu_{\mathfrak{p}}} - \tfrac{1}{2}\mathrm{B}_\mu, \tag{16}$$

where $\langle \mathrm{B}_\mu\cdot,\cdot\rangle := \mathrm{B}_{\mathfrak{g}}|_{\mathfrak{p}\times\mathfrak{p}}$, $\mathrm{B}_{\mathfrak{g}}$ denotes the Killing form of the Lie algebra $\mathfrak{g}$ and

$$\langle \mathrm{M}_{\mu_{\mathfrak{p}}}, A\rangle := \tfrac{1}{4}\langle\theta(A)\mu_{\mathfrak{p}},\mu_{\mathfrak{p}}\rangle, \qquad \forall A \in \mathfrak{gl}(\mathfrak{p}). \tag{17}$$

Here θ is the representation of $\mathfrak{gl}(\mathfrak{p})$ given by,

$$\theta(A)\lambda := A\lambda(\cdot,\cdot) - \lambda(A\cdot,\cdot) - \lambda(\cdot,A\cdot), \qquad \forall A\in\mathfrak{gl}(\mathfrak{p}), \quad \lambda \in \Lambda^2\mathfrak{p}^*\otimes\mathfrak{p}. \tag{18}$$

The function $\mathrm{M} : \Lambda^2\mathfrak{p}^*\otimes\mathfrak{p} \to \operatorname{sym}(\mathfrak{p})$ is therefore the *moment map* from geometric invariant theory (see e.g. [BL1] and the references therein) for the representation θ of $\mathfrak{gl}(\mathfrak{p})$. Equivalently,

$$\mathrm{M}_{\mu_{\mathfrak{p}}} = -\tfrac{1}{2}\sum(\operatorname{ad}_{\mathfrak{p}} X_i)^t \operatorname{ad}_{\mathfrak{p}} X_i + \tfrac{1}{4}\sum \operatorname{ad}_{\mathfrak{p}} X_i(\operatorname{ad}_{\mathfrak{p}} X_i)^t, \tag{19}$$

or

$$\langle \mathrm{M}_{\mu_{\mathfrak{p}}} X, X\rangle = -\tfrac{1}{2}\sum\langle\mu_{\mathfrak{p}}(X,X_i),X_j\rangle^2 + \tfrac{1}{4}\sum\langle\mu_{\mathfrak{p}}(X_i,X_j),X\rangle^2, \qquad \forall X\in\mathfrak{p}. \tag{20}$$

It is easy to check that both operators $\mathrm{M}_{\mu_{\mathfrak{p}}}$ and B_μ belong to $\operatorname{sym}(\mathfrak{p})^K$. The main part of the Ricci curvature of g is $\mathrm{M}_{\mu_{\mathfrak{p}}}$, observe that B_μ is just measuring in some sense how far is g from being standard. It follows from (17) and (18) that $\operatorname{tr}\mathrm{M}_{\mu_{\mathfrak{p}}} = \langle \mathrm{M}_{\mu_{\mathfrak{p}}}, I\rangle = -\frac{1}{4}|\mu_{\mathfrak{p}}|^2$ and so by (16),

$$\operatorname{Sc}(g) = -\tfrac{1}{4}|\mu_{\mathfrak{p}}|^2 - \tfrac{1}{2}\operatorname{tr}\mathrm{B}_\mu. \tag{21}$$

We refer to [LfL] for more details on this viewpoint on Ricci curvature.

4.2. Moving bracket approach. Recall that μ is the Lie bracket of $\mathfrak{g}$. Given $h \in \operatorname{sym}_+(\mathfrak{p})^K$, we consider the new Lie algebra $(\mathfrak{g}, \underline{h} \cdot \mu)$, where $\underline{h} \in \mathrm{GL}(\mathfrak{g})$ is defined by $\underline{h}|_{\mathfrak{k}} := I$, $\underline{h}|_{\mathfrak{p}} := h$. Here $\underline{h} \cdot \mu := \underline{h}\mu(\underline{h}^{-1}\cdot, \underline{h}^{-1}\cdot)$ is the usual action of $\mathrm{GL}(\mathfrak{g})$ on $\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$, so $\underline{h} : (\mathfrak{g}, \mu) \to (\mathfrak{g}, \underline{h} \cdot \mu)$ is a Lie algebra isomorphism. Now for any Lie group $G_{\underline{h}\cdot\mu}$ with Lie algebra $(\mathfrak{g}, \underline{h} \cdot \mu)$ such that there is an isomorphism $G \to G_{\underline{h}\cdot\mu}$ with derivative $\underline{h}$, one obtains an isometry between the following Riemannian homogeneous spaces,

$$(G/K, \langle h\cdot, h\cdot\rangle) \longrightarrow (G_{\underline{h}\cdot\mu}/K_{\underline{h}\cdot\mu}, \langle\cdot,\cdot\rangle), \tag{22}$$

where $K_{\underline{h}\cdot\mu}$ is the image of K under the isomorphism. Note that $K_{\underline{h}\cdot\mu}$ is a Lie subgroup of $G_{\underline{h}\cdot\mu}$ with Lie algebra $(\mathfrak{k}, \underline{h} \cdot \mu|_{\mathfrak{k}\times\mathfrak{k}})$ and that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a reductive decomposition for every homogeneous space $G_{\underline{h}\cdot\mu}/K_{\underline{h}\cdot\mu}$, $h \in \operatorname{sym}_+(\mathfrak{p})^K$. Therefore, by varying the Lie brackets as in the right of (22), one is covering the whole set $\mathcal{M}^G$ (see [L] and references therein for further information).

We assume from now on in this section that G is unimodular (see [LW1, §2.2] for the general case). According to (16), for any $h \in \operatorname{sym}_+(\mathfrak{p})^K$, the Ricci operator of $(G_{\underline{h}\cdot\mu}/K_{\underline{h}\cdot\mu}, \langle\cdot,\cdot\rangle)$ is given by

$$\operatorname{Ric}_{\underline{h}\cdot\mu} = \mathrm{M}_{h\cdot\mu_{\mathfrak{p}}} - \tfrac{1}{2}h^{-1}\,\mathrm{B}_\mu\, h^{-1}. \tag{23}$$

Note that $h^{-1}\,\mathrm{B}_\mu\, h^{-1}$ is the Killing form operator of the Lie algebra $(\mathfrak{g}, \underline{h} \cdot \mu)$ and by (17),

$$\langle \mathrm{M}_{h\cdot\mu_{\mathfrak{p}}}, A\rangle := \tfrac{1}{4}\langle\theta(A)(h \cdot \mu_{\mathfrak{p}}), h \cdot \mu_{\mathfrak{p}}\rangle, \qquad \forall A \in \mathfrak{gl}(\mathfrak{p}). \tag{24}$$

It follows from (22) that the Ricci tensor and the Ricci operator of each metric $g_h := \langle h\cdot, h\cdot\rangle \in \mathcal{M}^G$ are respectively given by

$$\operatorname{Rc}(g_h) = \langle h \operatorname{Ric}_{\underline{h}\cdot\mu} h\cdot,\cdot\rangle, \qquad \operatorname{Ric}(g_h) = h^{-1}\operatorname{Ric}_{\underline{h}\cdot\mu} h, \qquad \forall h \in \operatorname{sym}_+(\mathfrak{p})^K,$$

and by (21),

$$\operatorname{Sc}(g_h) = -\tfrac{1}{4}|h \cdot \mu_{\mathfrak{p}}|^2 - \tfrac{1}{2}\operatorname{tr}\mathrm{B}_\mu h^{-2}. \tag{25}$$

In order to study the different types of G-stability and G-deformability (see Definition 3.14), using the moving-bracket approach described above, we consider the functions

$$\overline{\operatorname{Rc}}, \overline{\mathrm{E}} : \operatorname{sym}_+(\mathfrak{p})^K \longrightarrow \operatorname{sym}^2(\mathfrak{p})^K, \qquad \overline{\operatorname{Sc}} : \operatorname{sym}_+(\mathfrak{p})^K \longrightarrow \mathbb{R}, \tag{26}$$

defined by $\overline{\operatorname{Rc}}(h) := \operatorname{Rc}(g_h)$, $\overline{\operatorname{Sc}}(h) := \operatorname{Sc}(g_h)$ and $\overline{\mathrm{E}}(h) := \mathrm{E}(g_h) = \overline{\operatorname{Rc}}(h) - \frac{\overline{\operatorname{Sc}}(h)}{n} g_h$, for any $h \in \operatorname{sym}_+(\mathfrak{p})^K$.

4.3. First variation of Sc. Let $S : \mathfrak{gl}(\mathfrak{p}) \to \operatorname{sym}(\mathfrak{p})$ denote the symmetric part operator $S(A) := \frac{1}{2}(A + A^t)$ relative to $\langle\cdot,\cdot\rangle$.

Lemma 4.2. *At any $h \in \operatorname{sym}_+(\mathfrak{p})^K$, if $h(t) \in \operatorname{sym}_+(\mathfrak{p})^K$, $h(0) = h$, $h'(0) = A$ (e.g., $h(t) = h + tA$ or $h(t) = he^{th^{-1}A}$), then*

$$\overline{\operatorname{Sc}}'_h(A) := \tfrac{d}{dt}\big|_0 \overline{\operatorname{Sc}}(h(t)) = -2\langle \operatorname{Ric}_{\underline{h}\cdot\mu}, S(Ah^{-1})\rangle, \qquad \forall A \in \operatorname{sym}(\mathfrak{p})^K.$$

Remark 4.3. At the background metric g, i.e., $h = I$, in accordance with (9), the following simpler formula holds:

$$\operatorname{Sc}'_g(T) := \tfrac{d}{dt}\big|_0 \operatorname{Sc}(g + tT) = \tfrac{1}{2}\overline{\operatorname{Sc}}'_I(A) = -\langle \operatorname{Ric}_\mu, A\rangle = -\langle \operatorname{Rc}(g), T\rangle_g,$$

for any $A \in \operatorname{sym}(\mathfrak{p})^K$, where $T \in \mathcal{S}^2(M)^G$, $T_o = \langle A\cdot, \cdot\rangle \in \operatorname{sym}^2(\mathfrak{p})^K$.

Proof. We first give the following useful formula, which is easy to prove using (18):

$$\tfrac{d}{dt}\left(h(t)\cdot \mu_{\mathfrak{p}}\right) = \theta\left(h'(t)h(t)^{-1}\right)\left(h(t)\cdot\mu_{\mathfrak{p}}\right). \tag{27}$$

It now follows from (25) and (27) that

$$\begin{aligned}
\tfrac{d}{dt}\big|_0 \overline{\operatorname{Sc}}(h(t)) &= -\tfrac14 \tfrac{d}{dt}\big|_0 |h(t)\cdot\mu_{\mathfrak{p}}|^2 - \tfrac12 \tfrac{d}{dt}\big|_0 \operatorname{tr} \mathrm{B}_\mu\, h(t)^{-2} \\
&= -\tfrac12 \langle \tfrac{d}{dt}\big|_0 h(t)\cdot\mu_{\mathfrak{p}}, h\cdot\mu_{\mathfrak{p}}\rangle - \tfrac12 \operatorname{tr}\mathrm{B}_\mu \tfrac{d}{dt}\big|_0 h(t)^{-2} \\
&= -\tfrac12 \langle \theta(Ah^{-1})h\cdot\mu_{\mathfrak{p}}, h\cdot\mu_{\mathfrak{p}}\rangle + \tfrac12 \operatorname{tr}\mathrm{B}_\mu\, h^{-1}Ah^{-2} + \tfrac12 \operatorname{tr}\mathrm{B}_\mu\, h^{-2}Ah^{-1} \\
&= -2\langle \mathrm{M}_{h\cdot\mu_{\mathfrak{p}}}, Ah^{-1}\rangle + \operatorname{tr} h^{-1}\,\mathrm{B}_\mu\, h^{-1} S(Ah^{-1}) \\
&= -2\langle \operatorname{Ric}_{\underline{h}\cdot\mu}, S(Ah^{-1})\rangle,
\end{aligned}$$

where the last equality follows from (23). □

Since $d\det|_h A = (\det h)\operatorname{tr} Ah^{-1}$, if

$$\operatorname{sym}_+(\mathfrak{p})_1 := \{h \in \operatorname{sym}_+(\mathfrak{p}) : \det h = 1\},$$

then

$$T_h \operatorname{sym}_+(\mathfrak{p})_1^K = \left\{A \in \operatorname{sym}(\mathfrak{p})^K : \operatorname{tr} Ah^{-1} = 0\right\},$$

so the following corollary analogous to Lemma 3.6 follows.

COROLLARY 4.4. *$h \in \operatorname{sym}_+(\mathfrak{p})_1^K$ is a critical point of $\overline{\operatorname{Sc}} : \operatorname{sym}_+(\mathfrak{p})_1^K \longrightarrow \mathbb{R}$ if and only if the metric $g_h \in \mathcal{M}^G$ is Einstein.*

4.4. First variation of Rc. The derivative of the Ricci curvature function at the background metric $g \in \mathcal{M}^G$ ($\langle\cdot,\cdot\rangle = g_o$) was computed in [LW1]. We consider the maps

$$\delta_{\mu_{\mathfrak{p}}} : \mathfrak{gl}(\mathfrak{p}) \longrightarrow \Lambda^2\mathfrak{p}^* \otimes \mathfrak{p}, \qquad \delta_{\mu_{\mathfrak{p}}}^t : \Lambda^2\mathfrak{p}^*\otimes\mathfrak{p} \longrightarrow \mathfrak{gl}(\mathfrak{p}),$$

where $\delta_{\mu_{\mathfrak{p}}}(A) := -\theta(A)\mu_{\mathfrak{p}}$ (see (18)) and $\delta_{\mu_{\mathfrak{p}}}^t$ is the transpose of $\delta_{\mu_{\mathfrak{p}}}$, and define the following operator,

$$\mathrm{L}_{\mathfrak{p}} = \mathrm{L}_{\mathfrak{p}}(g) : \operatorname{sym}(\mathfrak{p}) \longrightarrow \operatorname{sym}(\mathfrak{p}), \qquad \mathrm{L}_{\mathfrak{p}} A := \tfrac12 S\circ\delta_{\mu_{\mathfrak{p}}}^t\delta_{\mu_{\mathfrak{p}}}(A) + A\,\mathrm{M}_{\mu_{\mathfrak{p}}} + \mathrm{M}_{\mu_{\mathfrak{p}}} A. \tag{28}$$

By using (17), it is easy to check that $\mathrm{L}_{\mathfrak{p}}$ satisfies the following properties (see [LW1]):

- $\mathrm{L}_{\mathfrak{p}}$ is a self-adjoint operator.
- $\mathrm{L}_{\mathfrak{p}} I = 0$ since $\delta_{\mu_{\mathfrak{p}}}(I) = \mu_{\mathfrak{p}}$ and $\delta_{\mu_{\mathfrak{p}}}^t\delta_{\mu_{\mathfrak{p}}}(I) = \delta_{\mu_{\mathfrak{p}}}^t(\mu_{\mathfrak{p}}) = -4\,\mathrm{M}_{\mu_{\mathfrak{p}}}$. Thus $\mathrm{L}_{\mathfrak{p}}\operatorname{sym}(\mathfrak{p}) \subset \operatorname{sym}_0(\mathfrak{p}) := \{A \in \operatorname{sym}(\mathfrak{p}) : \operatorname{tr} A = 0\}$ by self-adjointness.
- $\langle \mathrm{L}_{\mathfrak{p}} A, A\rangle = \tfrac12 |\theta(A)\mu_{\mathfrak{p}}|^2 + 2\operatorname{tr}\mathrm{M}_{\mu_{\mathfrak{p}}} A^2 = \tfrac12 \langle \left(\theta(A)^2 + \theta(A^2)\right)\mu_{\mathfrak{p}}, \mu_{\mathfrak{p}}\rangle$, for any $A \in \operatorname{sym}(\mathfrak{p})$.
- $\mathrm{L}_{\mathfrak{p}}\operatorname{sym}(\mathfrak{p})^K \subset \operatorname{sym}(\mathfrak{p})^K$. This follows by a straightforward computation using that $\operatorname{Ad}(z) \in \operatorname{Aut}(\mathfrak{g},\mu)$ and $\operatorname{Ad}(z)|_{\mathfrak{p}}$ is $\langle\cdot,\cdot\rangle$-orthogonal for any $z \in K$.
- Moreover, $\mathrm{L}_{\mathfrak{p}}\operatorname{sym}(\mathfrak{p})^H \subset \operatorname{sym}(\mathfrak{p})^H$ for any $g \in \mathcal{M}^H$, where H is any intermediate subgroup $K \subset H \subset N_G(K)$.

LEMMA 4.5 ([LW1, Lemma 6.1]). *For any* $T \in \mathcal{S}^2(M)^G$, $T_o = \langle A\cdot,\cdot\rangle$, $A \in \operatorname{sym}(\mathfrak{p})^K$,

$$d\operatorname{Rc}|_g T = \tfrac{1}{2} d\overline{\operatorname{Rc}}|_I A = \tfrac{1}{2}\langle \operatorname{L}_{\mathfrak{p}} A\cdot,\cdot\rangle.$$

Since $\Delta_L T = 2d\operatorname{Rc}|_g T$ on $\mathcal{TT}_g$ (see [B, 12.28']), the following formula follows.

COROLLARY 4.6 ([LW1, Corollary 6.7]). *Let $M = G/K$ be a homogeneous space with G compact, endowed with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then the Lichnerowicz Laplacian Δ_L of any G-invariant Riemannian metric g on M is given by*

$$\Delta_L T = \langle \operatorname{L}_{\mathfrak{p}} A\cdot,\cdot\rangle, \qquad \forall T \in \mathcal{TT}_g^G,$$

where $T_o = \langle A\cdot,\cdot\rangle \in \operatorname{sym}^2(\mathfrak{p})^K \equiv \mathcal{S}^2(M)^G$, $A \in \operatorname{sym}(\mathfrak{p})^K$ *and* $\langle\cdot,\cdot\rangle = g_o$.

Recall from §3.4 the computation of the space $\operatorname{Ker}\delta_g \cap \mathcal{S}^2(M)^G$ of G-invariant divergence-free symmetric 2-tensors.

4.5. Second variation of Sc**.** As expected, at an Einstein metric, the second derivative of the scalar curvature is strongly related to the first derivative of the Ricci curvature.

LEMMA 4.7. *Suppose that the background metric g is Einstein, say* $\operatorname{Rc}(g) = \rho g$. *Then, for any* $T \in \mathcal{S}^2(M)^G$, $T_o = \langle A\cdot,\cdot\rangle$, $A \in \operatorname{sym}(\mathfrak{p})^K$,

$$\begin{aligned}\operatorname{Sc}''_g(T,T) =& \tfrac{1}{4}\overline{\operatorname{Sc}}''_I(A,A) := \tfrac{1}{4}\left.\tfrac{d^2}{dt^2}\right|_0 \overline{\operatorname{Sc}}(h(t)) \\ =& -\tfrac{1}{2}\langle \operatorname{L}_{\mathfrak{p}} A, A\rangle + \rho \operatorname{tr} A^2 = \tfrac{1}{2}\langle (2\rho\operatorname{id} - \operatorname{L}_{\mathfrak{p}})A, A\rangle,\end{aligned}$$

where $h(t) \in \operatorname{sym}_+(\mathfrak{p})^K$, $h(0) = I$, $h'(0) = A$.

REMARK 4.8. Alternatively, $\operatorname{Sc}''_g(T,T) = -\frac{1}{4}|\theta(A)\mu_{\mathfrak{p}}|^2 - \frac{1}{2}\operatorname{tr}\operatorname{B}_\mu A^2$, which follows from the fact that $M_{\mu_{\mathfrak{p}}} - \frac{1}{2}\operatorname{B}_\mu = \rho I$.

REMARK 4.9. Since I is a critical point of $\overline{\operatorname{Sc}}|_{\operatorname{sym}_1(\mathfrak{p})^K}$, the value of $\overline{\operatorname{Sc}}''_I(A,A)$ is well defined if $\operatorname{tr} A = 0$, in the sense that it can be computed using any curve $h(t) \in \operatorname{sym}_1(\mathfrak{p})^K$ through I with velocity A. On the other hand, $\overline{\operatorname{Sc}}''_I(I,I) = 4\operatorname{Sc}(I)$, so the formula also holds for $A = I$ and thus $\operatorname{Sc}''_I(A,A)$ is well defined for any A.

Proof. If $h(t) := e^{tA}$, then $\frac{d}{dt}h(t)\cdot\mu_{\mathfrak{p}} = \theta(A)(h(t)\cdot\mu_{\mathfrak{p}})$ by (27), and so

$$\begin{aligned}\left.\tfrac{d^2}{dt^2}\right|_0 \overline{\operatorname{Sc}}(h(t)) =& -\tfrac{1}{2}\left.\tfrac{d}{dt}\right|_0 \langle\theta(A)h(t)\cdot\mu_{\mathfrak{p}}, h(t)\cdot\mu_{\mathfrak{p}}\rangle + \left.\tfrac{d}{dt}\right|_0 \operatorname{tr} Ae^{-2tA}\operatorname{B}_\mu \\ =& -\tfrac{1}{2}\langle\theta(A)^2\mu_{\mathfrak{p}},\mu_{\mathfrak{p}}\rangle - \tfrac{1}{2}\langle\theta(A)\mu_{\mathfrak{p}},\theta(A)\mu_{\mathfrak{p}}\rangle - 2\operatorname{tr} A^2\operatorname{B}_\mu \\ =& -\langle\delta^t_{\mu_{\mathfrak{p}}}\delta_{\mu_{\mathfrak{p}}}(A), A\rangle - 2\operatorname{tr}\operatorname{B}_\mu A^2 \\ =& -2\langle\operatorname{L}_{\mathfrak{p}} A, A\rangle + 4\operatorname{tr}\operatorname{M}_{\mu_{\mathfrak{p}}} A^2 - 2\operatorname{tr}\operatorname{B}_\mu A^2 \\ =& -2\langle\operatorname{L}_{\mathfrak{p}} A, A\rangle + 4\operatorname{tr}\operatorname{Ric}_\mu A^2.\end{aligned}$$

We are using formula (28) in the second last equality. The fact that $\operatorname{Ric}_\mu = \rho I$ concludes the proof. □

4.6. First variation of E. The following formula for the derivative of the Einstein operator follows from Lemma 4.5.

LEMMA 4.10. *If g is Einstein, say* $\mathrm{Rc}(g) = \rho g$, *then*

$$d\,\mathrm{E}\,|_g T = \tfrac{1}{2} d\overline{\mathrm{E}}|_I A = \big\langle \big(\tfrac{1}{2}\,\mathrm{L}_{\mathfrak{p}}\, A - \rho A\big)\,\cdot, \cdot\big\rangle + \tfrac{1}{n}\rho(\operatorname{tr} A)\langle\cdot,\cdot\rangle,$$

for any $T \in \mathcal{S}^2(M)^G$, $T_o = \langle A\cdot, \cdot\rangle$, $A \in \mathrm{sym}(\mathfrak{p})^K$.

In particular, $d\,\mathrm{E}\,|_g = d\,\mathrm{Rc}\,|_g - \rho\,\mathrm{id}$ restricted to $(\mathbb{R}g)^{\perp_g}$.

4.7. Stability in terms of $\mathrm{L}_{\mathfrak{p}}$**.** We assume in this subsection that the background metric $g \in \mathcal{M}^G$ is Einstein. Under the identifications in terms of operators, the decomposition of the space of variations analogous to (13) is the following decomposition of the tangent space $T_I\,\mathrm{sym}_+(\mathfrak{p})^K = \mathrm{sym}(\mathfrak{p})^K$ at the identity map I:

$$T_I\,\mathrm{sym}_+(\mathfrak{p})^K = (\mathbb{R}I \oplus T_I\,\mathrm{Aut}(G/K)\cdot I) \oplus^{\perp} W, \tag{29}$$

where W is the $\langle\cdot,\cdot\rangle$-orthogonal complement of $\mathbb{R}I \oplus \mathrm{Aut}(G/K)\cdot I$ and $\mathrm{Aut}(G/K)$ acts on $\mathrm{sym}_+(\mathfrak{p})^K$ according to the identification $\mathrm{sym}_+(\mathfrak{p})^K \equiv \mathcal{M}^G$. Recall that if G is compact, then $\mathcal{TT}_g^G = \langle W\cdot,\cdot\rangle$ by Corollary 3.12. Note that

$$W \subset \mathrm{sym}_0(\mathfrak{p})^K,$$

and if in addition any of the conditions listed at the end of §3.3 holds, then $W = \mathrm{sym}_0(\mathfrak{p})^K$.

It follows from [LW1, Lemma 6.10] that $d\overline{\mathrm{Rc}}|_I S(D) = 2\rho S(D)$ for any $\underline{D} \in \mathrm{Der}(\mathfrak{g}/\mathfrak{k})$. We therefore obtain from (12) and Lemma 4.5 that

$$\mathrm{L}_{\mathfrak{p}}\,|_{T_I\,\mathrm{Aut}(G/K)\cdot I} = 2\rho\,\mathrm{id}, \tag{30}$$

where $\mathrm{L}_{\mathfrak{p}}$ is the operator attached to the metric g as in (28).

According to Definition 3.14, it follows from Lemmas 4.7 and 4.10 that the G-stability and G-deformability types of the Einstein metric g are both determined by the spectrum of the operator $\mathrm{L}_{\mathfrak{p}}$ restricted to W, which coincides with the Lichnerowicz Laplacian in the compact case (see Corollary 4.6). All this is summarized in the following proposition.

Let $\lambda_{\mathfrak{p}} = \lambda_{\mathfrak{p}}(g)$ and $\lambda_{\mathfrak{p}}^{max} = \lambda_{\mathfrak{p}}^{max}(g)$ denote, respectively, the minimum and maximum eigenvalue of $\mathrm{L}_{\mathfrak{p}} = \mathrm{L}_{\mathfrak{p}}(g)$ restricted to the subspace W defined in (29).

PROPOSITION 4.11. *Let g be a G-invariant metric on a homogeneous space $M = G/K$, where G is unimodular, endowed with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. If g is Einstein, say* $\mathrm{Rc}(g) = \rho g$, *then the following holds:*

(i) *g is G-stable if and only if $2\rho < \lambda_{\mathfrak{p}}$.*
(ii) *g is G-unstable if and only if $\lambda_{\mathfrak{p}} < 2\rho$.*
(iii) *g is G-non-degenerate if and only if G-infinitesimally non-deformable, if and only if $2\rho \notin \mathrm{Spec}\,(\mathrm{L}_{\mathfrak{p}}\,|_W)$.*
(iv) *g is G-neutrally stable if and only if $\lambda_{\mathfrak{p}} = 2\rho$.*
(v) *g is G-strongly unstable if and only if $\lambda_{\mathfrak{p}}^{max} < 2\rho$.*

REMARK 4.12. It follows from Corollary 4.6 that $\lambda_L(g) \leq \lambda_{\mathfrak{p}}(g)$ (see §2).

In the case of a product homogeneous space, i.e., $G = G_1 \times G_2$, $K = K_1 \times K_2$, $K_i \subset G_i$, $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$ and $g = g_1 + g_2$, where g_i is a G_i-invariant metric on $M_i = G_i/K_i$, we obtain that $W = W_1 \oplus W_2 \oplus \mathbb{R}A_0$, where

$$A_0 := (n_2 I_{\mathfrak{p}_1}, -n_1 I_{\mathfrak{p}_2}), \qquad n_i := \dim M_i,$$

and $\mathrm{L}_\mathfrak{p}(g) = \mathrm{L}_{\mathfrak{p}_1}(g_1) + \mathrm{L}_{\mathfrak{p}_2}(g_2)$. Since $A_0 \in W \cap \operatorname{Ker} \mathrm{L}_\mathfrak{p}$, one deduces that $\lambda_\mathfrak{p} \leq 0$ and so any positive scalar curvature homogeneous product Einstein metric g is G-unstable.

5. Naturally reductive case. We consider in this section the case when $g \in \mathcal{M}^G$ is a *naturally reductive* metric on M with respect to G and some reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, i.e., the map $\operatorname{ad}_\mathfrak{p} X : \mathfrak{p} \to \mathfrak{p}$ is skew-symmetric for any $X \in \mathfrak{p}$ (see (11) or (15)). Note that G is necessarily unimodular. We refer to [LW1, §7] and references therein for further information on naturally reductive metrics.

The moment map takes the simpler form

$$\mathrm{M}_{\mu_\mathfrak{p}} = \tfrac{1}{4} \sum (\operatorname{ad}_\mathfrak{p} X_i)^2, \tag{31}$$

and the operator $\mathrm{L}_\mathfrak{p}$ also considerably simplifies in the naturally reductive setting (see Lemma 4.5 and [LW1, Lemma 7.18]):

$$\mathrm{L}_\mathfrak{p} A := -\tfrac{1}{2} \sum [\operatorname{ad}_\mathfrak{p} X_i, [\operatorname{ad}_\mathfrak{p} X_i, A]], \qquad \forall A \in \operatorname{sym}(\mathfrak{p})^K, \tag{32}$$

where $\{X_i\}$ is any orthonormal basis of $(\mathfrak{p}, \langle\cdot,\cdot\rangle)$ and $\langle\cdot,\cdot\rangle = g_o$. Note that $\mathrm{L}_\mathfrak{p} \geq 0$; in particular, $\lambda_\mathfrak{p} \geq 0$. We also recall that $T_g\mathcal{M}^G = \mathbb{R}g \oplus \mathcal{TT}_g^G$ in the compact case, i.e., $W = \operatorname{sym}_0(\mathfrak{p})^K$.

Since $\mathrm{L}_\mathfrak{p} A = 0$ if and only if $[A, \operatorname{ad}_\mathfrak{p} \mathfrak{p}] = 0$, the following conditions are equivalent by results due to Kostant [Ko] (see [LW1, §7.1]):

- $\operatorname{Ker} \mathrm{L}_\mathfrak{p} = \mathbb{R}I$.
- g is, up to scaling, the unique naturally reductive metric on M with respect to G and $\mathfrak{p}$.
- g is holonomy irreducible.
- $(\widetilde{M}, g)$ is de Rham irreducible, where $\widetilde{M}$ denotes the simply connected cover of M.
- $\mathfrak{k}$ is $\mathfrak{g}$-indecomposable, in the sense that there exist no nonzero ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{k} = \mathfrak{k} \cap \mathfrak{g}_1 \oplus \mathfrak{k} \cap \mathfrak{g}_2$ (e.g., if $\mathfrak{g}$ is indecomposable).

5.1. Killing metrics on Lie groups. For $M = G$ a compact semisimple Lie group, we consider the left-invariant metric $g_\mathrm{B} \in \mathcal{M}^G$ defined by $-\mathrm{B}_\mathfrak{g}$, where $\mathrm{B}_\mathfrak{g}$ denotes the Killing form of $\mathfrak{g}$. According to (31), $M_{\mu_\mathfrak{p}} = -\frac{1}{4}\mathrm{C}_{\operatorname{ad}, -\mathrm{B}_\mathfrak{g}} = -\frac{1}{4}I$, the Casimir operator acting on the adjoint representation of $\mathfrak{g}$, and so $\operatorname{Rc}(g_\mathrm{B}) = \frac{1}{4} g_\mathrm{B}$ by (16). On the other hand, $\mathfrak{p} = \mathfrak{g}$ and it follows from (32) that

$$\mathrm{L}_\mathfrak{p} = \tfrac{1}{2}\mathrm{C}_\tau,$$

where $\mathrm{C}_\tau = \mathrm{C}_{\tau, -\mathrm{B}_\mathfrak{g}}$ is the Casimir operator acting on the representation $\operatorname{sym}(\mathfrak{g})$ of $\mathfrak{g}$ given by

$$\tau(X)A := [\operatorname{ad} X, A],$$

i.e., $\mathrm{C}_\tau = -\sum \tau(X_i)^2$, where $\{X_i\}$ is a $-\mathrm{B}_\mathfrak{g}$-orthonormal basis of $\mathfrak{g}$. The first positive eigenvalue λ_τ of C_τ can therefore be computed by using representation theory. We have collected in Table 1 the values of λ_τ for each simple Lie algebra $\mathfrak{g}$, which together with Proposition 4.11, give the following. Note that $\lambda_\mathfrak{p} = \frac{1}{2}\lambda_\tau$ and $2\rho = \frac{1}{2}$.

PROPOSITION 5.1. *Let G be a connected compact simple Lie group and let g_B denote the Killing metric, which is Einstein with* $\operatorname{Rc}(g_\mathrm{B}) = \frac{1}{4} g_\mathrm{B}$.

Type	$\mathfrak{g}$	n	λ_τ	Stab. type
A_1	$\mathfrak{su}(2)$		3	G-stable
A_n	$\mathfrak{su}(n+1)$	$n \geq 2$	1	G-neut. stab.
B_3	$\mathfrak{so}(7)$		$\frac{6}{5}$	G-stable
B_n	$\mathfrak{so}(2n+1)$	$n \geq 4$	$\frac{2n+1}{2n-1}$	G-stable
C_n	$\mathfrak{sp}(n)$	$n \geq 2$	$\frac{n}{n+1}$	G-unstable
D_n	$\mathfrak{so}(2n)$	$n \geq 4$	$\frac{n}{n-1}$	G-stable
E_6	$\mathfrak{e}_6$		$\frac{3}{2}$	G-stable
E_7	$\mathfrak{e}_7$		$\frac{14}{9}$	G-stable
E_8	$\mathfrak{e}_8$		$\frac{8}{5}$	G-stable
F_4	$\mathfrak{f}_4$		$\frac{13}{9}$	G-stable
G_2	$\mathfrak{g}_2$		$\frac{7}{6}$	G-stable

TABLE 1

First eigenvalue λ_τ of the Casimir operator C_τ acting on $\operatorname{sym}(\mathfrak{g})$ with respect to $-\mathrm{B}_\mathfrak{g}$ for a compact simple $\mathfrak{g}$

- *For $G = \mathrm{SU}(n)$, $n \geq 3$, the metric g_B is G-neutrally stable with nullity $n^2 - 1$.*
- *g_B is G-unstable on any $G = \mathrm{Sp}(n)$, $n \geq 2$, with coindex $\geq \frac{2n(2n-1)}{2} - 1$.*
- *In all the remaining cases, g_B is G-stable.*

In particular, g_B is a local maximum of $\mathrm{Sc}\,|_{\mathcal{M}_1^G}$ in most of the cases. The question of whether g_B on $\mathrm{SU}(n)$ is a local maximum of $\mathrm{Sc}\,|_{\mathcal{M}_1^G}$ or not is still open for $n \geq 4$. It was proved in [J1] that it is not for $n = 3$, while it is well known that it is a global maximum for $n = 2$. Concerning $\mathrm{Sp}(n)$, since $\lambda_\mathfrak{p}^{max} = \frac{2n+4}{2(n+1)} > \frac{1}{2} = 2\rho$, g_B is a saddle point of $\mathrm{Sc}\,|_{\mathcal{M}_1^G}$.

This shows that the picture in the G-invariant setting is completely analogous to the general case studied by Koiso in [K], as described at the end of §2. In particular, any bi-invariant metric on any compact simple Lie group G is G-rigid, except possibly for $\mathrm{SU}(n)$, $n \geq 3$. Nevertheless, it was proved in [DG, Theorem 22.3] that on $\mathrm{SU}(n)$, g_B is indeed G-rigid.

5.2. A formula for $\mathrm{L}_\mathfrak{p}$ in terms of structural constants. Let $M = G/K$ be a homogeneous space with G compact and reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Given a non-degenerate $\operatorname{ad}\mathfrak{g}$-invariant symmetric bilinear form Q on $\mathfrak{g}$ such that $Q(\mathfrak{k}, \mathfrak{p}) = 0$ and $Q|_\mathfrak{p} > 0$, we consider the metric $g_Q \in \mathcal{M}^G$ whose value at o is $Q|_\mathfrak{p}$. Thus g_Q is naturally reductive with respect to G and $\mathfrak{p}$.

Remark 5.2. According to [Ko, Theorem 4] (see also [DZ, p.4]), if $\mathfrak{g} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$, then any G-invariant metric on M which is naturally reductive with respect to G and $\mathfrak{p}$ is given in this way for a unique Q.

Recall that g is called *normal* when $Q > 0$, and if in addition G is semi-simple and $Q = -\mathrm{B}_{\mathfrak{g}}$, then g is called *standard*. In particular, if G is simple, then g_Q is necessarily standard (up to scaling).

Given any Q-orthogonal decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r$ in $\mathrm{Ad}(K)$-invariant and irreducible subspaces $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ ($d_i := \dim \mathfrak{p}_i$), we consider the corresponding structural constants given by,

$$[ijk] := \sum_{\alpha,\beta,\gamma} Q([X_\alpha^i, X_\beta^j], X_\gamma^k)^2,$$

where $\{X_\alpha^i\}$ is a Q-orthonormal basis of $\mathfrak{p}_i$. Since g_Q is naturally reductive relative to G and $\mathfrak{p}$, the number $[ijk]$ is invariant under any permutation of ijk.

Recall from (16) that the Ricci operator of g is given by $\mathrm{Ric}(g_Q) = \mathrm{M}_{\mu_{\mathfrak{p}}} - \frac{1}{2}\mathrm{B}_\mu$. Since $\mathrm{M}_{\mu_{\mathfrak{p}}}$ is $\mathrm{Ad}(K)$-invariant, for each k we have that the linear map $\mathrm{M}_{\mu_{\mathfrak{p}}}$ restricted to $\mathfrak{p}_k$ and composed with the orthogonal projection on $\mathfrak{p}_k$ is given by $m_k I_{\mathfrak{p}_k}$ for some $m_k \in \mathbb{R}$. It follows from (20) that

$$m_k = -\tfrac{1}{4d_k} \sum_{i,j} [ijk], \qquad \forall k = 1, \dots, r; \tag{33}$$

indeed,

$$\begin{aligned} m_k d_k &= \operatorname{tr} \mathrm{M}_{\mu_{\mathfrak{p}}}|_{\mathfrak{p}_k} = \sum Q(\mathrm{M}_{\mu_{\mathfrak{p}}} X_\gamma^k, X_\gamma^k) \\ &= -\tfrac{1}{2} \sum Q(\mu_{\mathfrak{p}}(X_\gamma^k, X_\alpha^i), X_\beta^j)^2 + \tfrac{1}{4} \sum Q(\mu_{\mathfrak{p}}(X_\alpha^i, X_\beta^j), X_\gamma^k)^2 \\ &= -\tfrac{1}{2} \sum [kij] + \tfrac{1}{4} \sum [ijk] = -\tfrac{1}{4} \sum [ijk]. \end{aligned}$$

Note that this can alternatively be computed using (31). The irreducibility of $\mathfrak{p}_k$ also gives that $-\mathrm{B}_{\mathfrak{g}}$ restricted to $\mathfrak{p}_k$ equals $b_k Q|_{\mathfrak{p}_k}$ for some $b_k \in \mathbb{R}$ for any k, and consequently, the restriction and projection of B_μ is given by $-b_k I_{\mathfrak{p}_k}$. Note that $b_k \geq 0$, where equality holds if and only if $\mathfrak{p}_k \subset \mathfrak{z}(\mathfrak{g})$, and that if g_Q is standard, then $b_k = 1$ for all k. We therefore obtain that

$$\mathrm{Ric}(g_Q)|_{\mathfrak{p}_k} = \rho_k I_{\mathfrak{p}_k}, \qquad \rho_k = \tfrac{b_k}{2} - \tfrac{1}{4d_k} \sum_{i,j} [ijk], \tag{34}$$

and the Einstein equations become: $\mathrm{Ric}(g_Q) = \rho I$ if and only if

$$\rho_k = \rho, \quad \forall k = 1, \dots, r \quad \text{and} \quad \langle \mathrm{Ric}(g_Q)\mathfrak{p}_i, \mathfrak{p}_j \rangle = 0, \quad \forall i \neq j.$$

We now assume that the isotropy representation of the homogeneous space $M = G/K$ is multiplicity-free. Thus the right-hand side Einstein conditions above automatically hold and

$$\left\{ \tfrac{1}{\sqrt{d_1}} I_{\mathfrak{p}_1}, \dots, \tfrac{1}{\sqrt{d_r}} I_{\mathfrak{p}_r} \right\}$$

is an orthonormal basis of $\mathrm{sym}(\mathfrak{p})^K$. Let $[\mathrm{L}_{\mathfrak{p}}]$ denote the matrix of $\mathrm{L}_{\mathfrak{p}}(g_Q)$ with respect to this basis.

THEOREM 5.3. *Let $g_Q \in \mathcal{M}^G$ be the naturally reductive metric on $M = G/K$ (G compact) attached to a non-degenerate* $\operatorname{ad}\mathfrak{g}$*-invariant symmetric bilinear form Q on $\mathfrak{g}$, and assume that G/K is multiplicity-free. Then, the entries of the matrix $[\mathrm{L}_\mathfrak{p}]$ are given by,*

$$[\mathrm{L}_\mathfrak{p}]_{kk} = \tfrac{1}{d_k} \sum_{\substack{j\neq k\\ i}} [ijk], \quad \forall k, \qquad [\mathrm{L}_\mathfrak{p}]_{jk} = -\tfrac{1}{\sqrt{d_j}\sqrt{d_k}} \sum_i [ijk], \quad \forall j \neq k.$$

REMARK 5.4. It is easy to check that the coordinates vector $[\sqrt{d_1}, \dots, \sqrt{d_r}]^t$ of the identity map is indeed in the kernel of $[\mathrm{L}_\mathfrak{p}]$. Note that the structural constants of the form $[kkk]$ are not involved in the above formulas.

Proof. We fix any Q-orthonormal basis $\{X^i_\alpha\}$ of each $\mathfrak{p}_i$ and denote

$$\operatorname{ad}_\mathfrak{p} X^i_\alpha = \begin{bmatrix} \operatorname{ad}_{\mathfrak{p}_1} X^i_\alpha & (\operatorname{ad}_\mathfrak{p} X^i_\alpha)_{12} & \cdots & (\operatorname{ad}_\mathfrak{p} X^i_\alpha)_{1r} \\ -(\operatorname{ad}_\mathfrak{p} X^i_\alpha)^t_{12} & \operatorname{ad}_{\mathfrak{p}_2} X^i_\alpha & \cdots & (\operatorname{ad}_\mathfrak{p} X^i_\alpha)_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ -(\operatorname{ad}_\mathfrak{p} X^i_\alpha)^t_{1r} & -(\operatorname{ad}_\mathfrak{p} X^i_\alpha)^t_{2r} & \cdots & \operatorname{ad}_{\mathfrak{p}_r} X^i_\alpha \end{bmatrix},$$

where $(\operatorname{ad}_\mathfrak{p} X^i_\alpha)_{jk} : \mathfrak{p}_k \to \mathfrak{p}_j$. We also consider $E_{jk} : \mathfrak{p}_j \to \mathfrak{p}_j$ and $F_{jk} : \mathfrak{p}_k \to \mathfrak{p}_k$ defined by

$$E_{jk} := \sum_{i,\alpha} (\operatorname{ad}_\mathfrak{p} X^i_\alpha)_{jk} (\operatorname{ad}_\mathfrak{p} X^i_\alpha)^t_{jk}, \qquad F_{jk} := \sum_{i,\alpha} (\operatorname{ad}_\mathfrak{p} X^i_\alpha)^t_{jk} (\operatorname{ad}_\mathfrak{p} X^i_\alpha)_{jk}, \qquad \forall j < k.$$

For any diagonal block map

$$A := [a_1 I_{\mathfrak{p}_1}, a_2 I_{\mathfrak{p}_2}, \dots, a_r I_{\mathfrak{p}_r}] \in \operatorname{sym}(\mathfrak{p})^K,$$

a straightforward computation using (32) gives that the k-th block of $\mathrm{L}_\mathfrak{p} A$ is given by

$$\sum_{j<k} (a_k - a_j) F_{jk} + \sum_{k<j} (a_k - a_j) E_{kj}.$$

In particular, for each l,

$$\mathrm{L}_\mathfrak{p} I_{\mathfrak{p}_l} = \left[-E_{1l}, \dots, -E_{l-1,l}, \sum_{j=1}^{l-1} F_{jl} + \sum_{j=l+1}^{r} E_{lj}, -F_{l,l+1}, \dots, -F_{lr} \right]^t.$$

Since $\mathrm{L}_\mathfrak{p} \operatorname{sym}(\mathfrak{p})^K \subset \operatorname{sym}(\mathfrak{p})^K$, this implies that $E_{jk} = e_{jk} I_{\mathfrak{p}_j}$ and $F_{jk} = f_{jk} I_{\mathfrak{p}_k}$ for all $j < k$, for some non-negative $e_{kj}, f_{jk} \in \mathbb{R}$. But $\operatorname{tr} E_{jk} = \operatorname{tr} F_{jk}$, so

$$d_k f_{jk} = \operatorname{tr} F_{jk} = \sum_i [ijk], \qquad e_{jk} = \tfrac{d_k}{d_j} f_{jk}, \qquad \forall j < k,$$

concluding the proof. □

6. Three standard infinite families. In this section, we assume that $M = G/K$ is one of the following:

$$\begin{gathered}\mathrm{SU}(nk)/\mathrm{S}(\mathrm{U}(k)\times\cdots\times\mathrm{U}(k)),\quad k\geq 1,\qquad \mathrm{Sp}(nk)/\mathrm{Sp}(k)\times\cdots\times\mathrm{Sp}(k),\quad k\geq 1;\\ \mathrm{SO}(nk)/\mathrm{S}(\mathrm{O}(k)\times\cdots\times\mathrm{O}(k)),\quad k\geq 3,\end{gathered} \tag{35}$$

where the quotients are all n-times products with $n \geq 3$. The standard block matrix reductive decomposition is given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_{12} \oplus \mathfrak{p}_{13} \oplus \cdots \oplus \mathfrak{p}_{(n-1)n},$$

where every $\mathfrak{p}_{ij} = \mathfrak{p}_{ji}$ (note that always $i \neq j$) has dimension $d = 2k^2, 4k^2, k^2$, respectively, and they are all $\mathrm{Ad}(K)$-irreducible and pairwise inequivalent. Thus G/K is multiplicity-free and $\dim \mathcal{M}^G = \frac{n(n-1)}{2}$.

It is easy to check that $[\mathfrak{p}_{ij}, \mathfrak{p}_{kl}]_{\mathfrak{p}} = 0$ if $\{i,j\}$ and $\{k,l\}$ are either equal or disjoint, and $[\mathfrak{p}_{ij}, \mathfrak{p}_{ik}]_{\mathfrak{p}}$ is nonzero and it is contained in $\mathfrak{p}_{jk}$ for all $j \neq k$. Moreover, a straightforward computation gives that any nonzero structural constant $[ijk]$ as in §5.2 is equal to the same $c = c(G,k,n)$, where $\frac{c}{d}$ is respectively given by

$$\frac{c}{d} = \frac{1}{2n}, \quad \frac{k}{2(nk+1)}, \quad \frac{k}{2(nk-2)}. \tag{36}$$

We consider the standard or Killing metric g_{B} on G/K, i.e., $Q = -\mathrm{B}_{\mathfrak{g}}$ (see §5.2). It follows from (34) that g_{B} is Einstein with

$$2\rho = 1 - \frac{c}{d}(n-2). \tag{37}$$

On the other hand, according to Theorem 5.3,

$$[\mathrm{L}_{\mathfrak{p}}]_{(ij)(ij)} = \frac{c}{d}2(n-2), \qquad [\mathrm{L}_{\mathfrak{p}}]_{(ij)(ik)} = -\frac{c}{d}, \quad \forall j \neq k,$$

and $[\mathrm{L}_{\mathfrak{p}}]_{(ij)(kl)} = 0$ otherwise. This implies that

$$[\mathrm{L}_{\mathfrak{p}}] = \frac{c}{d}\Big(2(n-2)I - \mathrm{Adj}(X)\Big),$$

where $X = J(n,2,1)$ is the Johnson graph with parameters $(n,2,1)$ (see [GR, §1.6]) and $\mathrm{Adj}(X)$ denotes its adjacency matrix. Since the graph is strongly regular with parameters $(\frac{n(n-1)}{2}, 2(n-2), n-2, 4)$ for any $n \geq 4$ (see [GR, §10.1]), it follows from [GR, §10.2] that the spectrum of $\mathrm{Adj}(X)$ is given by

$$2(n-2), \quad n-4, \quad -2, \qquad \text{with multiplicities} \qquad 1, \quad n-1, \quad \frac{n(n-3)}{2},$$

respectively. Thus $\mathrm{Spec}(\mathrm{L}_{\mathfrak{p}}) = \{0, \lambda_{\mathfrak{p}}, \lambda_{\mathfrak{p}}^{max}\}$, where

$$\lambda_{\mathfrak{p}} = \frac{c}{d}n, \qquad \lambda_{\mathfrak{p}}^{max} = \frac{c}{d}2(n-1), \qquad n \geq 4, \tag{38}$$

and have multiplicities $n-1$ and $\frac{n(n-3)}{2}$, respectively.

For $n = 3$, X is the complete graph on 3 vertices and so the spectrum of $\mathrm{Adj}(X)$ equals $\{2, -1\}$, with multiplicities 1 and 2, respectively. Thus $\lambda_{\mathfrak{p}} = \lambda_{\mathfrak{p}}^{max} = \frac{c}{d}3$ and has multiplicity 2 if $n = 3$.

G/K	n	k	Crit.point	coindex
$\mathrm{SU}(3k)/\mathrm{S}(\mathrm{U}(k)^3)$	3	$k \geq 1$	loc.min.	2
$\mathrm{SU}(4k)/\mathrm{S}(\mathrm{U}(k)^4)$	4	$k \geq 1$	G-deg.	3
$\mathrm{SU}(nk)/\mathrm{S}(\mathrm{U}(k)^n)$	$n \geq 5$	$k \geq 1$	saddle	$n-1$
$\mathrm{Sp}(3k)/\mathrm{Sp}(k)^3$	3	$k \geq 1$	loc.min.	2
$\mathrm{Sp}(4k)/\mathrm{Sp}(k)^4$	4	$k \geq 1$	loc.min.	5
$\mathrm{Sp}(5)/\mathrm{Sp}(1)^5$	5	1	loc.min.	9
$\mathrm{Sp}(10)/\mathrm{Sp}(2)^5$	5	2	G-deg.	4
$\mathrm{Sp}(6)/\mathrm{Sp}(1)^6$	6	1	G-deg.	5
$\mathrm{Sp}(kn)/\mathrm{Sp}(k)^n$	$n \geq 5$	otherwise	saddle	$n-1$
$\mathrm{SO}(3k)/\mathrm{S}(\mathrm{O}(k)^3)$	3	$k \geq 3$	loc.min.	2
$\mathrm{SO}(nk)/\mathrm{S}(\mathrm{O}(k)^n)$	$n \geq 4$	$k \geq 3$	saddle	$n-1$

TABLE 2

Coindex and critical point type of the G-unstable Einstein metric g_{B} on each of the spaces given in (35).

The following proposition follows from a straightforward comparison between (36), (37) and (38).

PROPOSITION 6.1. *The standard metric g_{B} on each of the homogeneous spaces given in* (35) *is always G-unstable, and so Ricci flow dynamically unstable. The coindex and type of critical point are given in Table 2. They are all G-non-degenerate, and in particular G-rigid, except*

$$\mathrm{SU}(4k)/\mathrm{S}(\mathrm{U}(k)\times\mathrm{U}(k)\times\mathrm{U}(k)\times\mathrm{U}(k)), \quad k \geq 1, \qquad \mathrm{Sp}(10)/\mathrm{Sp}(2)^5, \qquad \mathrm{Sp}(6)/\mathrm{Sp}(1)^6.$$

We do not know whether g_{B} is still a local minimum in the G-degenerate cases or not.

7. Jensen's metrics. Given a simple Lie group H and a semisimple subgroup $K \subset H$, we consider the $\mathrm{B}_{\mathfrak{h}}$-orthogonal decomposition $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{k}$ and the left-invariant metrics on H defined by

$$g_t = -\,\mathrm{B}_{\mathfrak{h}}\,|_{\mathfrak{a}} + t(-\,\mathrm{B}_{\mathfrak{h}})|_{\mathfrak{k}}, \qquad t > 0.$$

Thus g_1 is the Killing metric on H. On the other hand, it was proved in [Z] (see also [DZ, Theorem 1]) that for each $t \neq 1$, the metric g_t is naturally reductive with respect to $G = H \times K$ (acting on H by $(h,k)\cdot p := hpk^{-1}$) and the reductive decomposition

$$\mathfrak{g} = \Delta\mathfrak{k} \oplus \mathfrak{p}_t, \qquad \mathfrak{p}_t := \mathfrak{p}_{\mathfrak{a}} \oplus \mathfrak{p}_{\mathfrak{k}}, \qquad \mathfrak{p}_{\mathfrak{a}} := (\mathfrak{a}, 0), \qquad \mathfrak{p}_{\mathfrak{k}} := \{(\tfrac{t}{1-t}Z, -Z) : Z \in \mathfrak{k}\}.$$

Indeed, g_t is identified with g_{Q_t}, where Q_t is the non-degenerate $\operatorname{ad}\mathfrak{g}$-invariant bilinear symmetric form on $\mathfrak{g} = (\mathfrak{h}, 0) \oplus (0, \mathfrak{k})$ given by

$$Q_t := -\operatorname{B}_{\mathfrak{h}} + \tfrac{t}{1-t}(-\operatorname{B}_{\mathfrak{h}})|_{\mathfrak{k}},$$

since for any $Z \in \mathfrak{k}$, the Q_t-orthogonal projection of $(0, Z)$ on $\mathfrak{p}_t$ is $(t-1)(\frac{t}{1-t}Z, -Z)$. Note that g_t is normal (i.e., $Q_t > 0$) if and only if $t < 1$. If $\mathfrak{k} = \mathfrak{k}_1 \oplus \dots \oplus \mathfrak{k}_r$ is a $\operatorname{B}_{\mathfrak{h}}$-orthogonal decomposition in simple ideals of $\mathfrak{k}$, then

$$\mathfrak{p}_t := \mathfrak{p}_{\mathfrak{a}} \oplus \mathfrak{p}_1 \oplus \dots \oplus \mathfrak{p}_r, \qquad \mathfrak{p}_i := \{(\tfrac{t}{1-t}Z, -Z) : Z \in \mathfrak{k}_i\}, \quad i = 1, \dots, r, \tag{39}$$

is an $\operatorname{Ad}(\Delta K)$-invariant Q_t-orthogonal decomposition of $\mathfrak{p}_t$.

We assume from now on that $\mathfrak{a}$ is $\operatorname{Ad}(K)$-irreducible (i.e., H/K is isotropy irreducible) and that for some constant c, $\operatorname{B}_{\mathfrak{k}_i} = c\operatorname{B}_{\mathfrak{h}}|_{\mathfrak{k}_i}$ for any $i = 1, \dots, r$. In particular, the summands in (39) are all $\operatorname{Ad}(\Delta K)$-irreducible and most of the times pairwise inequivalent (see [LL, Remark 4.2] for a counterexample). We assume that they are pairwise inequivalent, so $\dim \mathcal{M}_1^G = r$. It is easy to check that the only nonzero structural constants are $[jjj]$, $[j\mathfrak{a}\mathfrak{a}]$ and $[\mathfrak{a}\mathfrak{a}\mathfrak{a}]$ (see §5.2), which are next computed.

Lemma 7.1. *For each $j = 1, \dots, r$,*

$$[jjj] = \tfrac{(2t-1)^2}{t} c d_j, \qquad [j\mathfrak{a}\mathfrak{a}] = t(1-c)d_j, \qquad [\mathfrak{a}\mathfrak{a}\mathfrak{a}] = d - 2(1-c)k,$$

where $d_j := \dim \mathfrak{p}_j = \dim \mathfrak{k}_j$, $d := \dim \mathfrak{p}_{\mathfrak{a}} = \dim \mathfrak{a}$ and $k := \dim \mathfrak{k}$.

Proof. These are straightforward computations which use for $[jjj]$ that

$$I_{\mathfrak{k}_j} = \operatorname{C}_{\operatorname{ad}, -\operatorname{B}_{\mathfrak{k}_j}} = -\sum \left(\operatorname{ad}_{\mathfrak{k}_j} \tfrac{1}{\sqrt{c}} Z_i^j\right)^2,$$

where $\{Z_i^j\}$ is any $-\operatorname{B}_{\mathfrak{h}}$-orthonormal basis of $\mathfrak{k}_j$ (recall that $[jjj] = -\sum_{\alpha} \operatorname{tr}(\operatorname{ad} X_\alpha^j|_{\mathfrak{p}_j})^2$ for any orthonormal basis $\{X_\alpha^j\}_{\alpha=1}^{d_j}$ of $\mathfrak{p}_j$), and for $[j\mathfrak{a}\mathfrak{a}]$ and $[\mathfrak{a}\mathfrak{a}\mathfrak{a}]$ that

$$\sum a_j(X_i)^t a_j(X_i) = (1-c)I_{\mathfrak{k}_j}, \qquad \forall j = 1, \dots, r,$$

where

$$\operatorname{ad}_{\mathfrak{h}} X_i = \begin{bmatrix} \operatorname{ad}_{\mathfrak{a}} X_i & a_1(X_i) & \cdots & a_r(X_i) \\ -a_1(X_i)^t & & & \\ \vdots & & 0 & \\ -a_r(X_i)^t & & & \end{bmatrix}$$

and $\{X_i\}$ is a $-\operatorname{B}_{\mathfrak{h}}$-orthonormal basis of $\mathfrak{a}$. □

According to [DZ, Corollary 2, p.44], if $t \neq 1$, then $\operatorname{Ric}(g_t) = \rho I$ if and only if

$$t = t_E := \frac{dc}{(d+2k)(1-c)}, \qquad 2\rho = \frac{c}{2t_E} + \frac{(1-c)t_E}{2}.$$

We know from [DZ, Theorem 11, (ii), p.35] that

$$c < \frac{d+2k}{2d+2k}, \qquad \text{that is,} \qquad t_E < 1,$$

as the exception $\mathfrak{sp}(n-1) \subset \mathfrak{sp}(n)$ does not appear in this case (see the last paragraph of the proof of [DZ, Corollary 2, p.44]). In particular, g_{t_E} is normal with respect to G and $\mathfrak{p}_{t_E}$.

It follows from Theorem 5.3 and Lemma 7.1 that the matrix of the Lichnerowicz Laplacian $\mathrm{L}_{\mathfrak{p}}(g_{t_E})$ relative to the orthonormal basis

$$\left\{ \tfrac{1}{\sqrt{d}} I_{\mathfrak{p}_{\mathfrak{a}}}, \tfrac{1}{\sqrt{d_1}} I_{\mathfrak{p}_1}, \dots, \tfrac{1}{\sqrt{d_r}} I_{\mathfrak{p}_r} \right\},$$

of $\mathrm{sym}(\mathfrak{p}_{t_E})^{\Delta K}$ is given by

$$[\mathrm{L}_{\mathfrak{p}}] = t_E(1-c) \begin{bmatrix} \frac{k}{d} & -\frac{\sqrt{d_1}}{\sqrt{d}} & \cdots & -\frac{\sqrt{d_r}}{\sqrt{d}} \\ -\frac{\sqrt{d_1}}{\sqrt{d}} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\sqrt{d_r}}{\sqrt{d}} & 0 & \cdots & 1 \end{bmatrix}.$$

Since the characteristic polynomial of $\frac{1}{t_E(1-c)}\,\mathrm{L}_{\mathfrak{p}}$ is $f(x) = x(x-1)^{r-1}(x-(1+\frac{k}{d}))$, we obtain that

$$\mathrm{Spec}(\mathrm{L}_{\mathfrak{p}}) = \left\{0,\ t_E(1-c),\ t_E(1-c)(1+\tfrac{k}{d})\right\},$$

with multiplicities $1, r-1, 1$, respectively, and so

$$\begin{cases} \lambda_{\mathfrak{p}} = t_E(1-c), \quad \lambda_{\mathfrak{p}}^{max} = t_E(1-c)(1+\frac{k}{d}) & r \geq 2, \\ \lambda_{\mathfrak{p}} = \lambda_{\mathfrak{p}}^{max} = t_E(1-c)(1+\frac{k}{d}), & r = 1. \end{cases}$$

Proposition 7.2. *Every g_{t_E} is G-unstable with coindex r (in particular, g_{t_E} is always a local minimum).*

Proof. We have that

$$\lambda_{\mathfrak{p}}^{max} = t_E(1-c)(1+\tfrac{k}{d}) < 2\rho = \frac{c}{2t_E} + \frac{(1-c)t_E}{2},$$

if and only if

$$0 < \frac{c}{2t_E} - t_E(1-c)(\tfrac{1}{2}+\tfrac{k}{d}) = \frac{c}{2t_E} - \frac{c}{2},$$

if and only if $t_E < 1$, as was to be shown. □

If $\mathcal{M}^H$ denotes the huge space of all left-invariant metrics on H, then $\mathcal{M}^G$ is identified with the subset of $\mathcal{M}^H$ of those metrics which are in addition K-invariant. In particular, the Einstein metric g_{t_E} is also H-unstable, that is, unstable as a left-invariant metric on H, and so Ricci flow dynamically unstable. Recall that the H-stability type of the Killing metric g_1 on the Lie group H has been established in Proposition 5.1.

It follows from the lists of isotropy irreducible homogeneous spaces given in [B, Tables 7.102, 7.106, 7.107] that Proposition 7.2 provides at least one H-unstable Einstein left-invariant metric on any simple Lie group, except $\mathrm{Sp}(2n+1)$, $n \geq 4$ and $\mathrm{SO}(n)$ for some odd n's.

The only cases $K \subset H$ with coindex ≥ 2 (i.e., K non-simple) are (see [DZ, p.46]):

$$\begin{aligned}
&\mathrm{SO}(n) \times \mathrm{SO}(n) \subset \mathrm{SO}(2n), \qquad \mathrm{Sp}(n) \times \mathrm{Sp}(n) \subset \mathrm{Sp}(2n),\\
&\mathrm{SU}(n) \times \mathrm{SU}(n) \subset \mathrm{SU}(n^2) \quad \text{(tensor product)},\\
&\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{SU}(3) \subset E_6 \qquad \mathrm{Sp}(3) \times G_2 \subset E_7.
\end{aligned}$$

REFERENCES

[A] D. Alekseevskii, *Conjugacy of polar factorizations of Lie groups*, Mat. Sb., 84 (1971), pp. 14–26; English translation: Math. USSR-Sb., 13 (1971), pp. 12–24.

[AC] S. Anastassiou and I. Chrysikos, *Ancient solutions of the homogeneous Ricci flow on flag manifolds*, Extracta Math., 36:1 (2021), pp. 99–145.

[BHMW] W. Batat, S. Hall, T. Murphy, and J. Waldron, *Rigidity of* SU_n*-type symmetric spaces*, preprint 2021 (arXiv).

[B] A. Besse, *Einstein manifolds*, Ergeb. Math., 10 (1987), Springer-Verlag, Berlin-Heidelberg.

[BCR] J. Bochnak, M. Coste, and M.-F. Roy, *Real Algebraic Geometry*, Ergeb. Math., 36 (1998), Springer-Verlag.

[B1] C. Böhm, *Homogeneous Einstein metrics and simplicial complexes*, J. Diff. Geom., 67 (2004), pp. 79–165.

[B2] C. Böhm, *Unstable Einstein metrics*, Math. Zeit., 250 (2005), pp. 279–286.

[BL1] C. Böhm and R. Lafuente, *Real geometric invariant theory*, in: Differential geometry in the large. Part 1: Geometric evolution equations and curvature flow, Cambridge University Press, Cambridge (2020), pp. 11–49.

[BL2] C. Böhm and R. Lafuente, *Non-compact Einstein manifolds with symmetry*, preprint 2021 (arXiv).

[BWZ] C. Böhm, M. Y. Wang, and W. Ziller, *A variational approach for compact homogeneous Einstein manifolds*, Geom. Funct. Anal., 14 (2004), pp. 681–733.

[Br] G. Bredon, *Introduction to compact transformation groups*, Pure and Applied Mathematics, 46 (1972), Academic Press.

[CHI] H.-D. Cao, R. S. Hamilton, and T. Ilmanen, *Gaussian densities and stability for some Ricci solitons*, preprint 2004 (arXiv).

[CH] H.-D. Cao and C. He, *Linear stability of Perelman's* ν*-entropy on symmetric spaces of compact type*, J. reine angew. Math., 709 (2015), pp. 229–246.

[DZ] J. D'Atri and W. Ziller, *Naturally reductive metrics and Einstein metrics on compact lie groups*, Mem. Amer. Math. Soc., 215 (1979).

[DG] A. Derdzinski and S. Gal, *Indefinite Einstein metrics on simple Lie groups*, Indiana Univ. Math. J., 63 (2014), pp. 165–212.

[DLM] I. Dotti, M. L. Leite, and R. Miatello, *Negative Ricci curvature on complex semisimple Lie grous*, Geom. Dedicata, 17 (1984), pp. 207–218.

[GG] J. Gasqui and H. Goldschmidt, *Radon transforms and spectral rigidity on the complex quadrics and the real Grassmannians of rank two*, J. Reine Angew. Math., 480 (1996), pp. 1–69.

[GR] C. Godsil and G. Royle, *Algebraic graph theory*, GTM, 207 (2001), Springer.

[H] J. Heber, *Noncompact homogeneous Einstein spaces*, Invent. Math., 133 (1998), pp. 279–352.

[J1] G. Jensen, *The Scalar Curvature of Left-Invariant Riemannian Metrics*, Indiana Math. J., 20 (1971), pp. 1125–1144.

[J2] G. Jensen, *Einstein metrics on principal fibre bundles*, J. Diff. Geom., 8 (1973), pp. 599–614.

[K] N. Koiso, *Rigidity and stability of Einstein metrics: the case of compact symmetric spaces*, Osaka J. Math., 17 (1980), pp. 51–73.

[Ko] B. Kostant, *On differential geometry and homogeneous spaces I and II*, Proc. Nat. Acad. Sci. U.S.A., 42 (1956), pp. 258–261 and pp. 354–357.

[Kr1] K. Kröncke, *Stability of Einstein Manifolds*, Ph.D. thesis (2013), Universität Potsdam.

[Kr2] K. Kröncke, *Stability and instability of Ricci solitons*, Calc. Var. PDE., 53 (2015), pp. 265–287.

[LfL] R. Lafuente and J. Lauret, *Structure of homogeneous Ricci solitons and the Alekseevskii conjecture*, J. Diff. Geom., 98 (2014), pp. 315–347.

[LL] E. A. Lauret and J. Lauret, *The stability of standard homogeneous Einstein manifolds*, Math. Z., 303, 16 (2023).

[L] J. Lauret, *The search for solitons on homogeneous spaces*, Abel Symposia, 16 (2019), pp. 147–170, Springer.

[LW1] J. Lauret and C. E. Will, *Prescribing Ricci curvature on homogeneous manifolds*, J. reine angew. Math., 783 (2022), pp. 95–133.

[LW2] J. Lauret and C. E. Will, *On the stability of homogeneous Einstein manifolds II*, J. London Math. Soc., 106 (2022), pp. 3638–3669.

[LW3] J. Lauret and C. E. Will, *Homogeneous Einstein metrics and local maxima of the Hilbert action*, J. Geom. Phys., 178 (2022), 104544.

[LNF] A. Lomshakov, Y. Nikonorov, and E. Firsov, *Invariant Einstein metrics on three-locally-symmetric spaces*, Siberian Adv. Math., 14 (2004), pp. 43–62.

[N] Y.G. Nikonorov, *The scalar curvature functional and homogeneous Einsteinian metrics on Lie groups*, Siberian Math. J., 39 (1998), pp. 504–509.

[P] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, preprint 2002 (arXiv).

[SWW] U. Semmelmann, C. Wang, and M. Y. Wang, *Linear stability of Sasaki Einstein and nearly parallel G_2-manifolds*, preprint 2020 (arXiv).

[SW] U. Semmelmann and G. Weingart, *Stability of compact symmetric spaces*, preprint 2020 (arXiv).

[S] P. Schwahn, *Stability of Einstein metrics on symmetric spaces of compact type*, Ann. Glob. Anal. Geom., in press (arXiv).

[W] M. Y. Wang, *Einstein Metrics from Symmetry and Bundle Constructions: A Sequel*, Adv. Lect. Math., 22 (2012), pp. 253–309, International Press.

[WW] C. Wang and M. Y. Wang, *Instability of some Riemannian manifolds with real Killing spinors*, Comm. Anal. Geom., in press (arXiv).

[WZ1] M. Y. Wang and W. Ziller, *On normal homogeneous Einstein manifolds*, Ann. Sci. École Norm. Sup., 18 (1985), pp. 563–633.

[WZ2] M. Y. Wang and W. Ziller, *Existence and nonexistence of homogeneous Einstein metrics*, Invent. Math., 84 (1986), pp. 177–194.

[Z] W. Ziller, *The Jacobi equation on naturally reductive compact Riemannian homogeneous spaces*, Comment. Math. Helvetici, 52 (1977), pp. 573–590.