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ON THE GLOBAL MODULI OF CALABI-YAU THREEFOLDS[*]

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Abstract. In this note we initiate a program to obtain global descriptions of Calabi-Yau moduli spaces, to calculate their Picard group, and to identify within that group the Hodge line bundle. We do this here for several Calabi-Yau's obtained in [DW09] as crepant resolutions of the orbifold quotient of the product of three elliptic curves. In particular we verify in these cases a recent claim of [GHKSST16] by noting that a power of the Hodge line bundle is trivial – even though in most of these cases the Picard group is infinite.

Key words. Calabi-Yau, moduli space, Hodge line bundle, Bagger-Witten.

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1. Introduction.

Moduli spaces M_X of (complex structures on) a compact Calabi-Yau manifold X are central to superstring compactifications, mirror symmetry, conformal field theory, and numerous other branches of both geometry and physics. They have the familiar complexity of moduli spaces; in particular, there is a stacky version as well as an underlying coarse moduli space. The stack is smooth. It is often easier to understand, and we will deal primarily with it.

While a lot is known about the local structure of these moduli spaces, there are surprisingly few examples where the global geometry is fully understood, and much of this is for moduli spaces of low complex dimensions, one or two.

Of particular interest is an understanding of the Hodge line bundle λ on M_X, whose fiber above the (isomorphism class of) a particular X is the line $H^0(X, \omega_X)$ of top holomorphic forms on X. Some startling predictions have appeared in recent physics literature [GHKSST16], to the effect that Calabi-Yau moduli spaces M_X always admit a (globally defined) Kähler potential. Our results verify these predictions in all cases that we consider. In fact, we show that the Hodge line bundle is not trivial, but some finite power of it is trivial. This implies the existence of a global Kähler potential.

Over M_X there is a universal family $\pi_X : \mathbb{X} \to M_X$ whose fiber above the isomorphism class $[X]$ of some X is that X itself. The intermediate Jacobians $J(X)$ fit into a family $\mathbb{J}_X \to M_X$. In [DM96] it is shown that the pull back $\widetilde{\mathbb{J}_X} \to \widetilde{\mathbb{M}_X}$ is an analytically completely integrable system. Here $\widetilde{\mathbb{M}_X}$ is the space of pairs (X, α) where X is a complex structure on a Calabi-Yau of a given deformation class, and α is a holomorphic volume form on X. In other words, $\widetilde{\mathbb{M}_X}$ is the complement in L_X of the zero section. This system is integrable analytically but not algebraically: the fibers are polarized complex tori, but the polarization is not positive definite, so the fibers are not abelian varieties. (Instead, the polarization is Lorentzian, with $h^{2,1}$ positive directions and a single negative direction corresponding to $h^{3,0}$.) Physically, this corresponds to the fact that $\widetilde{\mathbb{J}_X}$ approximates, in a large limit, the Ramond moduli space

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of the theory. The new prediction implies that $\widetilde{\mathbb{M}_X}$ is just the product $M_X \times \mathbb{C}^*$ (up to a finite cover or a more general covering with constant transition functions), and the symplectic form on the integrable system $\widetilde{\mathbb{J}_X}$ over it splits globally into horizontal and vertical summands. (When X is not necessarily compact, it is possible for the $\mathbb{J}_X$ to be abelian varieties; this happens if the negative direction, i.e. a holomorphic volume form, coincides with a vanishing cycle. In fact [DDP07], for each Riemann surface C and each ADE group G it is possible to construct a family of non-compact Calabi-Yau threefolds whose $\mathbb{J}_X$ recovers the Hitchin system $H_{C,G}$. These results have been extended recently to include the non-simply laced groups BCFG [Beck16].)

It therefore seems worthwhile to construct some global moduli spaces M_X and to test the physics predictions for them. In the case that X is an elliptic curve, the results are well known to mathematicians, and have been summarized for physicists in [GS16]. The purpose of this work is to describe some global moduli spaces M_X and to determine their Hodge bundles λ for some genuine, three-dimensional compact Calabi-Yaus. The moduli spaces we consider themselves will also be three dimensional.

The examples we consider are crepant resolutions $\overline{X}$ of orbifolds

$$X := Y/G$$

of the product $Y := E_1 \times E_2 \times E_3$ of three elliptic curves by the action of a finite group G. The latter contains the subgroup

$$G_S \subset G$$

of its ‘shifts’ or translations, with a quotient group

$$G_T := G/G_S.$$

Elements of G_T are called twists. We consider the case where G_T is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, acting (up to translations) by nontrivial elliptic involutions (sign changes) on an even number of the E_i. It turns out [DW09] that all such groups G, and orbifolds X, can be described explicitly.

In that work, a particular class of such group actions was designated ‘essential’. It was shown that any orbifold X of this type is isomorphic to one whose group is essential. Essential groups were shown to be abelian, isomorphic to the product $G_S \times G_T$ of their shift and twist parts, and all their non-trivial elements are of order 2. So, essentially, the shift group G_S must be a subgroup of the group of points of order 2 in Y, acting by translations. It is therefore isomorphic to $(\mathbb{Z}/2)^r$ for some r, $0 \leq r \leq 6$. This r is called the rank of G. The full essential group G is then isomorphic to $(\mathbb{Z}/2)^{r+2}$.

All such orbifolds have been classified in [DW09]. They fall into 35 types. Most of these are known to be in distinct families, but as explained in [DW09] and reviewed in our appendix, there are a few gaps where the possible existence of isomorphisms is still not known. In fact, one pair from the list in [DW09] has since been shown to be isomorphic, cf. [FRTV13]. After explaining the required notation, we recall that classification and comment on its current state in the Appendix, in Table 3.

In this work we focus on ten of the families in this table, the ones whose moduli spaces are three dimensional. For each of these there is a canonical identification:

$$H^3(X, \mathbb{Q}) \cong \otimes_{1=1,2,3} H^1(E_i, \mathbb{Q}). \tag{1}$$

The best known example, denoted (0-1) in [DW09], was originally studied by [VW95]. In that case the group action has 48 fixed points, leading to singularities of X that need to be resolved (by a crepant resolution). There are four cases where the group action is free, leading to smooth quotients of Hodge numbers $(3, 3)$. In [DW09] these are denoted (0-4), (1-5), (1-11), and (2-12). (The notation (r-i) means that the group G_S has rank r, and that this is the i-th case listed in [DW09] with that given r.) There are six further cases, including the [VW95] orbifold (0-1), where the group does have some fixed points but the number of complex moduli happens to still be $h^{2,1} = 3$. We tabulate these ten orbifolds with $h^{2,1} = 3$, giving their symbol from [DW09], their Hodge numbers, whatever alternative descriptions are available, and some references where they are either analyzed or used:

(0-1)	(51,3)	The basic Vafa-Witten orbifold [VW95], also a Borcea-Voisin BV(18, 4, 0) [Bor97, Voi93].
(0-4)	(3,3)	Occurs in [Ig54, Ue75, OS01, BCDP, T]
(1-1)	(27,3)	A $(\mathbb{Z}/2)$ free quotient of the basic Vafa-Witten orbifold
(1-5)	(3,3)	A $(\mathbb{Z}/2)$ free quotient of (0-4) [DW09, T]
(1-11)	(3,3)	Another $(\mathbb{Z}/2)$ free quotient of (0-4) [DW09, T]
(2-1)	(15,3)	A $(\mathbb{Z}/2)$ free quotient of (1-1)
(2-9)	(27,3)	An orbifold of the $SO(12)$ torus, related to B_{NAHE+} free fermion model [Fa92, Fa93, DF04, DW09]
(2-12)	(3,3)	A $(\mathbb{Z}/2)$ free quotient of (1-11) [DW09, T]
(3-5)	(15,3)	A $(\mathbb{Z}/2)$ free quotient of (2-9)
(4-1)	(15,3)	Related to 'enhanced' B_{NAHE+} free fermion model [Fa92, Fa93, DF04, DW09]

Notice that any projective crepant resolution of any of these is a genuine $\mathcal{N} = 1$ Calabi-Yau threefold, in the sense that its h^1 vanishes and it has precisely a one-dimensional space of holomorphic three-forms or covariantly constant spinors. For the four cases with Hodge numbers $(3, 3)$, the holonomy is a proper subgroup of $SU(3)$, in fact a finite subgroup. The remaining cases, where $h^{2,1} = 3, h^{1,1} > 3$, involve some blowups, and their holonomy is all of $SU(3)$.

Below we give a global description of the moduli space M_X in each case in which $h^{2,1} = 3$. In doing so, we will also describe a connected component $M_{\overline{X}}$ (isomorphic to M_X) of the moduli space of complex structures on a particular resolution $\overline{X}$. What we will find is that M_X is isomorphic to a global quotient of $\mathbb{H}^3$, where $\mathbb{H}$ is the complex upper half plane. We will prove that the Picard group of M_X is a finitely generated abelian group, and calculate the rank of its free part in each case. Despite the fact that $\mathrm{Pic}(M_X)$ will often be infinite, we will prove that the Hodge bundle has finite order in each of these cases. In fact, we will calculate an explicit trivialization of a tensor power of the bundle, and compute the Kähler potential on M_X. In summary:

THEOREM 1.1. *Let E_1, E_2, E_3 denote elliptic curves, and let $G \subseteq \mathrm{Aut}(E_1 \times E_2 \times E_3)$ denote an essential group of automorphisms (in the sense of [DW09], Definition 1.1.1). Let $X = (E_1 \times E_2 \times E_3)/G$ denote the quotient orbifold, and let M_X denote the connected component of the moduli stack of complex structures on X containing X. Assume that $h^{2,1}(X) = 3$.*

(1) *We have*

$$Pic(M_X) \simeq \mathbb{Z}^a \times A$$

where a is given in Table 2 below (as $dim_K(H^2(H', K))$) for each case and A is some (case dependent) 6-torsion finite abelian group.

(2) *The class of the Hodge line bundle* $\lambda \in Pic(M_X)$ *is non-trivial, and torsion. Furthermore, the moduli stack* M_X *admits a globally defined Kähler potential.*

2. The moduli spaces.

2.1. Notation and Terminology.

$V :=$ a fixed, 2-dimensional vector space over the field of 2 elements

$S := SL(V) \cong SL(2, \mathbb{Z}/2),$ acting linearly on V.

$B_i \subset S :=$ a Borel subgroup, stabilizer of a non-zero vector in $V,\ i = 1, 2, 3.$

$E_i :=$ elliptic curves, $i = 1, 2, 3.$

$E_i[2] :=$ the subgroup of points of order 2 in E_i

$Y := E_1 \times E_2 \times E_3$

$Y[2] :=$ the subgroup of points of order 2 in Y

$l_i :=$ a level 2 structure on E_i, i.e. an isomorphism $l_i := V \overset{\sim}{\to} E_i[2]$

$l := \Pi_{i=1}^{3} l_i : V^{\oplus 3} \overset{\sim}{\to} Y[2],$ the induced level 2 structure on Y

$M(2) :=$ moduli stack of elliptic curves with level 2 structure

$M(4) :=$ moduli stack of elliptic curves with level 4 structure

$M :=$ moduli stack of elliptic curves $= M(2)/S$

$\Gamma := SL(2, \mathbb{Z})$

$A := (\mathbb{R}/\mathbb{Z})^2 \rtimes \Gamma$

$A^{(3)} := A^3 \rtimes S_3$

$G_{max} := \mathrm{Ker}\ ((V \times \mathbb{Z}/2)^3 \to (\mathbb{Z}/2)^3 \to \mathbb{Z}/2)$

$G :=$ a subgroup of G_{max} mapping onto $\mathrm{Ker}\left((\mathbb{Z}/2)^3 \to \mathbb{Z}/2\right)$

$G_S := \mathrm{Ker}\left(G \to (\mathbb{Z}/2)^3 \subset (\mathbb{Z}/2)^3\right) \subset V^{\oplus 3},$ the 'shift' part of G

$G_T :=$ a subgroup of G mapping isomorphically to $G/G_S,$ the 'twist' part of G

$H := N_{A^{(3)}}(G)$

$H' := H/G$

$H_{max} := (\mathbb{Z}/4)^6 \rtimes \Gamma^3 \rtimes S_3 \leq A^{(3)},$ where $(\mathbb{Z}/4)^6 \leq (\mathbb{R}/\mathbb{Z})^6$ is the set of points of order 4.

$\mathbb{H} :=$ Complex upper half plane

$\mathcal{U} :=$ A family of elliptic curves $(\mathbb{H} \times \mathbb{C})/\mathbb{Z}^2$ over $\mathbb{H}$

$X := Y/G$

$M_X^+ :=$ moduli stack of complex structures on X. (Its dimension is $h^{2,1}(X)$.)

$M_X :=$ The connected component of M_X^+ containing the actual quotients $X = Y/G$.

$\overline{X} :=$ a Calabi-Yau threefold, crepant resolution of X.

$M_{\overline{X}} :=$ moduli stack of the Calabi-Yau threefold $\overline{X}$ (Its dimension is again $h^{2,1}(X)$). It is a covering of M_X, the finite fiber parametrizing Kähler resolutions $\overline{X}$ of a given X. This is the space we care about!

$M_{\overline{X},\mathrm{central}} :=$ The connected component of $M_{\overline{X}}$ containing $\overline{X}$.

$\mathbb{X} :=$ The universal Calabi-Yau over M_X.

Here are some remarks regarding our terminology. Given a complex algebraic variety X, there is a moduli stack M_X^{alg} parameterizing deformations of X over pointed schemes (separated and of finite type) over $\mathbb{C}$. More precisely, a deformation of X over (S, s_0) is a flat and proper morphism $f : \mathcal{X} \to S$, together with an isomorphism $\phi : f^{-1}(s_0) \cong X$. Note that we do not require the data of a polarization on our varieties (that is, a fixed ample invertible sheaf $\mathcal{L}$ on $\mathcal{X}$), so standard results about the existence of a quasi-projective coarse moduli scheme [Vie95] would not be available. When X is a smooth compact Calabi-Yau variety, the Bogomolov-Tian-Todorov theorem ([Bog81, Ti87, To89]) implies that X has unobstructed deformations, and hence M_X^{alg} is smooth. In the cases of this paper, X will be Calabi-Yau with singularities in at worst codimension 2, in which case Theorem 4 of [Ra93] implies that M_X^{alg} is still smooth (Ran actually establishes the more general result that any Calabi-Yau weak orbiKleinfold nonsingular in codimension 2 has unobstructed deformations).

The stack M_X^{alg} determines an underlying analytic stack (see [BN05], section 3, for basic definitions regarding complex analytic stacks), which we will refer to as M_X for the rest of this paper. For example, the moduli stack M of elliptic curves over $\mathbb{C}$ has an underlying analytic stack, which is the quotient $[\mathbb{H}/SL(2,\mathbb{Z})]$. Our goal is to provide a similar global description of M_X and ultimately $M_{\overline{X}}$, for $\overline{X}$ a smooth crepant resolution of X.

Finally, in this paper, the Picard group $\mathrm{Pic}(M_X)$ will always be understood in the stack-theoretic sense. Note that this group is in general *not* isomorphic to the Picard group of the underlying coarse moduli space. For example, if M denotes the moduli stack of ellptic curves, then $\mathrm{Pic}(M) \simeq \mathbb{Z}/12$, while the Picard group of its coarse moduli space $\mathbb{A}^1$ is trivial.

2.2. More details: the spaces and the maps between them. The moduli spaces we consider fit into a sequence of maps:

$$\begin{tikzcd}
 & M_{\overline{X},\text{central}} \arrow[r, hook] \arrow[d] & M_{\overline{X}} \arrow[dl] \\
\mathbb{H}^3 \arrow[r, "f"] & M_X &
\end{tikzcd}$$

We will describe each of these spaces and the maps between them.
$\mathbb{H}$ denotes the complex upper half plane. Over $\mathbb{H}$, there is a family $\mathcal{U}$ of elliptic curves

$$\mathcal{U} = (\mathbb{H} \times \mathbb{C})/\mathbb{Z}^2$$

together with a global section $s : \mathbb{H} \to \mathcal{U}$ given by $s(\tau) = (\tau, 0)$. (Recall that an elliptic curve $(E, 0)$, often abbreviated to just E, is a smooth genus 1 curve E with a marked point $0 \in E$.) The fiber of $\mathcal{U}$ over a point $\tau \in \mathbb{H}$ is the complex torus $\mathbb{C}/\Lambda_\tau$ with the origin $s(\tau)$. Here, Λ_τ is the lattice in $\mathbb{C}$ generated by 1 and τ. If M denotes the moduli stack of elliptic curves, the family $\mathcal{U}$ defines a map $\mathbb{H} \to M$. In fact, as is well known, M is the stacky quotient of $\mathbb{H}$ by the group

$$\Gamma = \mathrm{SL}(2, \mathbb{Z}),$$

which acts on $\mathbb{H}$ by fractional linear transformations. Note that over the space $\mathbb{H}^3$ we have the family $\mathcal{U}^3 \to \mathbb{H}^3$ of abelian varieties, which will be cruical for later developments.

Throughout the paper, G refers to an abelian group isomorphic to $(\mathbb{Z}/2)^{r+2}$, $0 \le r \le 6$, acting on the product $Y = E_1 \times E_2 \times E_3$ of three elliptic curves $\{E_i\}_{i=1,2,3}$ according to one of the 36 entries in Table 1 of [DW09]. G is given as an extension

$$1 \longrightarrow G_S \longrightarrow G \longrightarrow G_T \longrightarrow 1,$$

where the subgroup $G_S \approx (\mathbb{Z}/2)^r$ of "shifts" acts on Y by translation, so G_S can be identified with a subgroup of $Y[2]$, while each nontrivial element of the group $G_T \approx (\mathbb{Z}/2)^2$ (of "twists") acts, modulo some translations, as inversion $y_i \mapsto -y_i$ on two of the three E_i.

Now, fix a group G as in Table 1 of [DW09]. G acts naturally on the family $\mathcal{U}^3 \to \mathbb{H}^3$ (acting trivially on the base). Therefore, we have an induced family of complex orbifolds

$$\eta : \mathcal{U}^3/G \to \mathbb{H}^3.$$

The fiber of η over the triple $(\tau_i)_{i=1}^3$ is the toroidal orbifold

$$\left(\prod_{i=1}^{3} \mathbb{C}/\Lambda_{\tau_i}\right)/G.$$

Since X appears as the fiber of η over some triple $(\tau_1, \tau_2, \tau_3) \in \mathbb{H}^3$, η determines an analytic map $f : \mathbb{H}^3 \to M_X$. We will refer to the family $\mathcal{U}^3/G$ as $f^*\mathbb{X}$.

Finally, we let $\overline{X}$ denote any Calabi-Yau resolution of the possibly singular X, and let $M_{\overline{X}}$ be the moduli space of complex structures on $\overline{X}$. There is a forgetful map $M_{\overline{X}} \to M_X$, whose degree is equal to the number of Kähler resolutions of X.

2.3. The plan. We will construct a discrete group H' acting on $\mathbb{H}^3$. By lifting this action to $\mathcal{U}^3/G$, we will factor the map f through a map $\bar{f} : [\mathbb{H}^3/H'] \to M_X$. We will then prove that $\bar{f}$ induces an isomorphism between $[\mathbb{H}^3/H']$ and the moduli stack M_X. Then, we will analyze the Picard groups of M_X and $M_{\overline{X}}$, and study the Hodge bundle λ in particular.

3. Automorphisms of $f^*\mathbb{X}$.

3.1. Group actions. In this section we will identify a group H' acting on both $f^*\mathbb{X}$ and $\mathbb{H}^3$. It is this action that will induce the map $\bar{f}$ mentioned in section 2.3.

Firstly, the group $T^2 = (\mathbb{R}/\mathbb{Z})^2$ acts on $\mathcal{U}$ by translations. To be precise, let $\epsilon = (\epsilon_0, \epsilon_1) \in T^2$ and $(\tau, z) \in \mathcal{U}$. Then

$$\epsilon \cdot (\tau, z) := (\tau, z + \epsilon_0 + \epsilon_1 \tau).$$

As we have mentioned, there is an action of Γ on $\mathbb{H}$. If $\gamma \in \Gamma$ is given by

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$\gamma \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

This action of Γ can be lifted to an action on $\mathcal{U}$ by defining

$$\gamma \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, (c\tau + d)^{-1} z\right).$$

This action of Γ on $\mathcal{U}$ normalizes the previous action of T^2, in the sense that if $\gamma \in \Gamma$ and $\epsilon \in T^2$, then there exists an $\epsilon' \in T^2$ such that for every $(\tau, z) \in \mathcal{U}$ we have

$$\gamma \cdot \epsilon \cdot \gamma^{-1} \cdot (\tau, z) = \epsilon' \cdot (\tau, z).$$

(Explicitly, $\epsilon' = a\epsilon_0 - b\epsilon_1, -c\epsilon_0 + d\epsilon_1$.) Therefore Γ acts on T^2 by sending ϵ to ϵ', and we may form the extension

$$1 \longrightarrow T^2 \longrightarrow A \longrightarrow \Gamma \longrightarrow 1$$

as a semi-direct product $A = T^2 \rtimes \Gamma$. The previous actions of T^2 and Γ on $\mathcal{U}$ are now combined into a single action of A.

Now, A^3 acts diagonally on $\mathcal{U}^3$. There is also an S_3 action on $\mathcal{U}^3$ by permutation of the three factors, as well as an S_3 action on A^3, so we form the extension

$$1 \longrightarrow A^3 \longrightarrow A^{(3)} \longrightarrow S_3 \longrightarrow 1.$$

So $A^{(3)}$ acts on $\mathcal{U}^3$, combining all of the previous actions. Since the group G acts on $\mathcal{U}^3$ by translations of points of order 2 and elliptic inversions, it embeds naturally into $A^{(3)}$. Therefore we may form the normalizer

$$H := N_{A^{(3)}}(G),$$

and subsequently the quotient

$$H' := H/G.$$

Since H normalizes G, its action on $\mathcal{U}^3$ descends to an action on $f^*\mathbb{X} = \mathbb{X} \times_{M_X} \mathbb{H}^3$, the pullback of the universal Calabi-Yau $\mathbb{X} \to M_X$ to $\mathbb{H}^3$. We remark that since G acts trivially on $\mathbb{H}^3$ and $f^*\mathbb{X}$, H' acts on both of these spaces. We denote the respective actions of H' on $\mathcal{U}^3$ and $f^*\mathbb{X}$ by a and $\widetilde{a}$.

Proposition 3.1. *Let* $h = (\epsilon, \gamma, \sigma)$ *denote an element of* $A^{(3)}$*, where* $\epsilon \in T^6$, $\gamma \in \Gamma^3$*, and* $\sigma \in S_3$*. Then if* $h \in H$*, we must have* $\epsilon \in (\mathbb{Z}/4)^6$*, the subgroup of points of* T^6 *of order dividing 4.*

Proof. Let $g \in G$. For hgh^{-1} to belong to G, its T^6 component must be a point of order 2. We can write $g \in A^{(3)}$ as $(\delta, \iota, 1)$ where ι is a pure twist ($-I$ on an even number of factors) and δ has order 2. Firstly, we observe that $(0, 1, \sigma)g(0, 1, \sigma^{-1})$ also has the form $(\delta', \iota', 1)$ for δ' a point of order 2 and ι' a pure twist. Therefore, we assume without loss of generality that $\sigma = 1$. Then we have

$$hgh^{-1} = (\epsilon, \gamma, 1)(\delta, \iota, 1)(-\gamma^{-1} \cdot \epsilon, \gamma, 1).$$

By assumption on G, there is a component of ι acting by $-I$. Referring to this component as the ith, we will have that the ith component of the above element is

$$-(-\epsilon_i) + \delta_i + \epsilon_i = \delta_i + 2\epsilon_i.$$

As we mentioned, this element must have order 2. Since δ_i has order 2, it follows that ϵ_i has order dividing 4. □

Let H_{max} denote the $A^{(3)}$ subgroup given by $((\mathbb{Z}/4)^2 \rtimes \Gamma)^3 \rtimes S_3$. The previous proposition shows that $H \leq H_{max}$ in all cases, hence the name.

Let $A(2)$ denote the subset of A of elements of the form (ϵ, γ), where ϵ has order 2 and γ lies in the principal congruence subgroup $\Gamma(2)$ of Γ.

Proposition 3.2. *$A(2)^3$ is a normal subgroup of H_{max}, isomorphic to the direct product $(\mathbb{Z}/2)^6 \times \Gamma(2)^3$. Furthermore, $A(2)^3$ is always contained in H.*

Proof. Firstly, we must verify that $A(2)$ is a subgroup of A. We note that for any $(\tau, z) \in \mathcal{U}^3$ and $\gamma \in \Gamma(2)$, $\epsilon \in (\mathbb{Z}/2)^2$, an easy calculation shows that

$$\gamma \cdot \epsilon \cdot \gamma^{-1} \cdot (\tau, z) = (\tau, z + \epsilon).$$

Therefore, $A(2)$ is indeed closed under the product. Furthermore, the subgroups $(\mathbb{Z}/2)^2$ and $\Gamma(2)$ embed naturally into $A(2)$, and the above calculation shows that they commute. Since they clearly generate $A(2)$ and intersect trivially, we have shown that $A(2) \cong (\mathbb{Z}/2)^2 \times \Gamma(2)$. For normality, we begin by checking that $A(2)$ is normal in $(\mathbb{Z}/4)^2 \rtimes \Gamma$. let $\gamma' \in \Gamma$, $\epsilon' \in (\mathbb{Z}/4)^6$, and $\gamma \in \Gamma(2)$, $\epsilon \in (\mathbb{Z}/2)^6$. Firstly,

$$(\gamma', 0)(\gamma, \epsilon)(\gamma'^{-1}, 0) = (\gamma'\gamma\gamma'^{-1}, \gamma' \cdot \epsilon).$$

By the normality of $\Gamma(2)$ in Γ, $\gamma'\gamma\gamma'^{-1} \in \Gamma(2)$. Furthermore, γ' must send order 2 points to order 2 points. Therefore, Γ normalizes $A(2)$. Next, we consider

$$(1, \epsilon')(\gamma, \epsilon)(1, -\epsilon') = (\gamma, -\gamma \cdot \epsilon' + \epsilon + \epsilon').$$

Then $2(\gamma \cdot \epsilon' + \epsilon + \epsilon') = (\gamma \cdot (2\epsilon') + 2\epsilon')$. Since $\gamma \in \Gamma(2)$, it fixes points of order 2. Hence this expression is equal to $4\epsilon' = 0$. So we have established that $A(2)$ is normal in $(\mathbb{Z}/4)^2 \rtimes \Gamma$. Since $A(2)^3$ is clearly preserved by permutation, it follows that $A(2)^3 \trianglelefteq H_{max}$, as claimed. Since every element of $\Gamma(2)$ fixes points of order 2 in T^2 by conjugation, we see that G lies in the center of $A(2)^3$. In particular, $A(2)^3 \leq H$. □

Based on the previous proposition, we can define a *finite* group $L = H/A(2)^3$. L is a subgroup of $L_{max} := ((\mathbb{Z}/2)^2 \rtimes S)^3 \rtimes S_3$, where $S = \mathrm{SL}(2, \mathbb{Z}/2)$ is the quotient $\Gamma/\Gamma(2)$.

3.2. Computation of L.

In this section we explain how to compute the finite group L, and record the results in all cases in which $h^{2,1}(X) = 3$. The technique is straightforward. Since L is by definition the image of H under the quotient map $H_{max} \to L_{max}$, we need to determine which elements $l \in L_{max}$ admit a lift $\widetilde{l} \in H_{max}$ which normalizes G. Note that if $\widetilde{l}$ and $\widetilde{l'}$ are two distinct lifts of l, then $\widetilde{l}$ normalizes G if and only if $\widetilde{l'}$ normalizes G. This result follows since $\widetilde{l}^{-1}\widetilde{l'} \in A(2)^3$, which commutes with G.

Therefore, one simply needs to iterate over the elements of $L_{max} = (\mathbb{Z}/2)^6 \rtimes S^3 \rtimes S_3$, choose a lift for each element, and determine whether or not this element normalizes G. Since L_{max} and G are finite, this algorithm can be easily implemented on a computer [1]. To subsequently determine H, one then finds the preimage of L in H_{max}.

We will not need a complete description of L for the main results of the paper. The translations by points of order 2, given by $(\mathbb{Z}/2)^6$, form a normal subgroup of L_{max}. Let $p : L_{max} \to S^3 \rtimes S_3$ be the quotient map, and $L_0 := p(L) \leq S^3 \rtimes S_3$. Since L_0 is the group that we will need to know to compute the Picard group, we will record it in all cases. We will also record the group H when it has a particularly

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[1] For example, see `https://github.com/mmacerato/Global-Moduli-of-Calabi-Yau-Threefolds/blob/main/normalizers.py`

simple form. We illustrate the relationships between the groups introduced so far in the following diagram.

$$\begin{tikzcd}
N_{H_{max}}(G) = H \arrow[r, hook] \arrow[d, two heads] & H_{max} = (\mathbb{Z}/4)^6 \rtimes \Gamma^3 \rtimes S_3 \arrow[d, two heads] \\
L \arrow[r, hook] \arrow[d, two heads] & L_{max} = H_{max}/A(2)^3 \cong (\mathbb{Z}/2)^6 \rtimes S^3 \rtimes S_3 \arrow[d, two heads] \\
L_0 \arrow[r, hook] & S^3 \rtimes S_3
\end{tikzcd}$$

We fix some notation. S acts faithfully on the vector space $V = (\mathbb{F}_2)^2$, so we will use this action to refer to elements of S. Namely, we label the nonzero elements of V as $\{\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$. There is an isomorphism $S \cong S_3$, as S acts to permute these three nonzero elements. We let $B_i \leq S$ be the stabilizer of the ith element. The subgroup $\widetilde{B}_i \leq S^3$ refers to the set of triples with one identity element, and two elements of B_i. $S \leq S^3$ refers to the diagonal subgroup. We let Γ_i and $\widetilde{\Gamma}_i$ refer to the preimages of B_i and $\widetilde{B}_i$ under the quotient $\Gamma \to S$. The results, which we explained how to obtain earlier, are given in all cases with $h^{2,1} = 3$ except (4-1) by Table 1.

Case	$L_0 = p(L)$	H
(0-1)	$S^3 \rtimes S_3$	$(\mathbb{Z}/2)^6 \rtimes \Gamma^3 \rtimes S_3$
(0-4)	$B_1^3 \rtimes S_3$	–
(1-1)	$B_2^3 \rtimes S_3$	$(\mathbb{Z}/2)^6 \rtimes \Gamma_2^3 \rtimes S_3$
(1-5)	$\widetilde{B}_2 \rtimes S_3$	–
(1-11)	$(B_2^2 \times B_1) \rtimes \langle(1\ 2)\rangle$	–
(2-1)	$S \rtimes S_3$	$(\mathbb{Z}/2)^6 \rtimes \Gamma \rtimes S_3$
(2-9)	$B_1^3 \rtimes S_3$	$(\mathbb{Z}/2 \times \mathbb{Z}/4)^3 \rtimes \Gamma_1^3 \rtimes S_3$
(2-12)	$(1 \times B_1^2) \rtimes \langle(2\ 3)\rangle$	–
(3-5)	$\widetilde{B}_1 \rtimes S_3$	$(\mathbb{Z}/2 \times \mathbb{Z}/4)^3 \rtimes \widetilde{\Gamma} \rtimes S_3$

TABLE 1

The group L_0 in cases with $h^{2,1} = 3$

For case (4-1), L_0 is not a semi-direct product. We will note that L_0 fits into an extension

$$1 \longrightarrow N \longrightarrow L_0 \longrightarrow S_3 \longrightarrow 1,$$

where $N \leq S^3$ has the property that each projection $\pi_i : S^3 \to S$ induces an isomorphism $N \cong S$. However, N is not the diagonally embedded copy of S in S^3.

4. Global geometry of M_X. We have described a map $f : \mathbb{H}^3 \to M_X$. In order to obtain a map $\bar{f} : [\mathbb{H}^3/H'] \to M_X$, we must lift the action of H' on $\mathbb{H}^3$ to an action on the family $f^*\mathbb{X}$. Of course, we have also done this by defining the action $\widetilde{a}$. Our objective now is to prove that $\bar{f}$ is an isomorphism. The strategy will be to show that $\bar{f}$ is proper and étale, and then prove that it has degree 1. We begin with a few lemmas.

LEMMA 4.1. *Assume that three elliptic curves E_i are pairwise non-isogenous, Y is their product, $\pi : Y \to X := Y/G$ the quotient by a G-action on Y, and $\pi' : Y' \to X'$*

another quotient in the same family (i.e. with same G). Then any isomorphism $X \to X'$ lifts to an isomorphism $Y \to Y'$.

Proof. Note that if $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$, the action of G on Y induces an action on $E_i \times E_j$, and X maps to the quotient $(E_i \times E_j)/G$. The generic fiber is isomorphic to E_k. Indeed, any element of G that acts trivially on $E_i \times E_j$ must be a shift, in G_S. But the reduction explained in [DW09] (following Definition 1.1.1) allows us to assume that the group G of automorphisms of X is essential, or non-redundant, meaning that it does not contain a translation by a nonzero $x \in E_k$.

So our X has three elliptic fibrations, which are distinct because the fibers E_1, E_2, E_3 are assumed non-isogenous. Further, these are the *only* elliptic fibrations on X: in fact, their lifts to Y are the only elliptic fibrations there. To see this, consider a genus 1 fibration $\pi : Y \to B$, and let E be a generic fiber. Note that the fixed locus of elements of G is at most one-dimensional, so π^{-1} of the generic E does not meet the fixed locus. The inverse image $\pi^{-1}(E)$ of E in $E_1 \times E_2 \times E_3$ is then an unramified cover of E, so each of its connected components E' is itself an elliptic curve in $E_1 \times E_2 \times E_3$. The assumption about non-isogeny of the E_i implies that E' can map onto at most one of the E_i. It follows that E' is isomorphic to one of the E_i and is embedded in the expected way: parallel to one of the three coordinates. So the same holds for E.

The isomorphism $X \to X'$ must therefore take an elliptic fibration of X with fiber E_i to an elliptic fibration of X' with the same fiber E_i. It therefore lifts to an isomorphism $Y \to Y'$ as claimed. □

Remark 4.1. Note that the isomorphism $Y \to Y'$ obtained at the end of the theorem above is only a complex isomorphism, *not* an isomorphism of abelian varieties. That is, it may fail to map the origin of Y to that of Y'. It is precisely for that reason that the group H_{max}, and hence H, contains a translation component $(\mathbb{Z}/4)^6$.

Lemma 4.2. *Assume that three elliptic curves E_i are pairwise non-isogenous, Y is their product, $\pi : Y \to X := Y/G$ the quotient by a G-action on Y, and $\pi' : Y' \to X'$ another quotient in the same family (i.e. with same G). Let $\phi : X \to X'$ be an isomorphism. Furthermore, assume that we have fixed isomorphisms $\psi_1 : \eta^{-1}(\tau) \cong X$ and $\psi_2 : \eta^{-1}(\tau') \cong X'$, for some $\tau, \tau' \in \mathbb{H}^3$. Then there exists a unique $h' \in H'$, such that*

$$\widetilde{a}(h')|_{\eta^{-1}(\tau)} \circ \psi_1 = \psi_2 \circ \phi.$$

That is, the restriction of the action of h' on $f^*\mathbb{X}$ to $\eta^{-1}(\tau)$ induces the isomorphism ϕ.

Proof. By the above lemma, we have an element $\sigma \in S_3$ together with complex isomorphisms $\phi_i : E_i \xrightarrow{\sim} E_{\sigma(i)}$, such that $\widetilde{\phi} = \prod_{i=1}^{3} \phi_i$ induces ϕ. Any isomorphism of two elliptic curves has a lift to an automorphism of the universal cover, $\mathbb{C}$. Since any automorphism of $\mathbb{C}$ is an affine transformation $z \mapsto az + b$, we can find a unique point $\epsilon_i \in T^2$ such that $\epsilon_i^{-1} \circ \phi_i$ is origin preserving. In that case, $\epsilon_i^{-1} \circ \phi_i$ is an isomorphism of elliptic curves, and is therefore induced uniquely by the action of some $\gamma \in \Gamma$ on the universal family $\mathcal{U}$ (since $M \cong [\mathbb{H}/\Gamma]$). Therefore, the action of $h = (\epsilon, \gamma, \sigma) \in A^{(3)}$ on $f^*\mathbb{X}$ induces the isomorphism ϕ. Since the action of h descends to $f^*\mathbb{X} = \mathcal{U}^3/G$, we must have $h \in N_{A^{(3)}}(G) = H$.

Now for uniqueness. An element h will act trivially on a fiber of $f^*\mathbb{X}$ if and only if it belongs to G. To see this latter claim, assume that $h = (\epsilon, \gamma, \sigma)$ acts trivially on the fiber $X \cong \eta^{-1}(\tau) \subset f^*\mathbb{X}$. The action of h on X comes from its action on Y, a fiber of $\mathcal{U}^3$. Since h acts trivially on X, we must have for any $y \in Y$ that there exists a $g_y \in G$ such that $g_y \cdot y = h \cdot y$. Since $h \cdot y$ depends smoothly on y, so must $g_y \cdot y$. Since G is finite, we then have $g_y = g$, a constant. But then hg^{-1} acts trivially on Y. This clearly implies that $hg^{-1} = 1$, and $h \in G$. Taking h' to be the image of h in H' then proves the claim. □

Let $O \subset \mathbb{H}^3$ be the subset of triples $(\tau_i)_{i=1}^3$ with the propery that the three elliptic curves $E_{\tau_i} = \mathbb{C}/\Lambda_{\tau_i}$ are non-isogeneous. O is a dense subset of $\mathbb{H}^3$. Since O is stable under the action of H', its image $\mathcal{O}$ in the quotient $[\mathbb{H}^3/H']$ is then a dense open substack.

Theorem 4.3. *The map $\bar{f} : \mathcal{O} \to M_X$ is a monomorphism of stacks.*

Proof. We must show that $\bar{f}$ is a fully faithful functor. That is, assume that we are given an isomorphism

$$\begin{tikzcd}
\phi^*(f^*\mathbb{X}) \arrow[r, "\sim"] \arrow[d] & \psi^*(f^*\mathbb{X}) \arrow[d] \\
S \arrow[r, "\sim"] & S'
\end{tikzcd}$$

of pullbacks of the family $f^*\mathbb{X}$ to complex spaces S and S' via maps $\phi : S \to \mathbb{H}^3$ and $\psi : S' \to \mathbb{H}^3$. Then we must show that there exists a unique $h' \in H'$ such that the following diagram commutes.

$$\begin{tikzcd}
\phi^*(f^*\mathbb{X}) \arrow[r, "\sim"] \arrow[d] & \psi^*(f^*\mathbb{X}) \arrow[d] \\
f^*\mathbb{X} \arrow[r, "\widetilde{a}(h')"] & f^*\mathbb{X}
\end{tikzcd}$$

Fix $s \in S$ (not necessarily the basepoint). Let X_s be the fiber of $\phi^*(f^*\mathbb{X})$ over s. Let s' be the image of s in S', and X'_s the fiber of $\psi^*(f^*\mathbb{X})$ over s'. Then ϕ (resp. ψ) maps X_s (resp. X'_s) isomorphically onto some fiber X (resp. X') of $f^*\mathbb{X}$. Therefore, we obtain an induced isomorphism $X \to X'$. But by Lemma 4.2, this means that we can find an element $h'_s \in H'$ whose action on $f^*\mathbb{X}$ induces the isomorphism $X \to X'$. We can construct such an h'_s for every $s \in S$. But since H' is discrete and acts continuously, we must have $h'_s = h'$, a constant. Therefore, the claim follows. □

Remark 4.2. For the purpose of proving that $\bar{f}$ is étale and proper, it is permitted to replace $[\mathbb{H}^3/H']$ by a finite étale cover. We have a subgroup $\Gamma(4)^3 \leq H$. This subgroup intersects G trivially, since $-I \notin \Gamma(4)$. Therefore, the quotient map $H \to H'$ induces an isomorphism between $\Gamma(4)^3$ and some subgroup of H'. In this way, we can regard $\Gamma(4)^3$ as a subgroup of H', and hence $M(4)^3$ as a finite étale cover of $[\mathbb{H}^3/H']$. For now, we will work with the induced map $M(4)^3 \to M_X$.

Theorem 4.4. *The induced map $M(4)^3 \to M_X$ has an everywhere injective differential, and in the 10 cases when $h^{2,1} = 3$ it is étale.*

Proof. Consider the sum of the the three lines

$$H^{1,0}(E_i) \otimes H^{1,0}(E_j) \otimes H^{0,1}(E_k)$$

as $\{i,j,k\}$ runs over cyclic permutations of $\{1,2,3\}$. The orbifold map $E_1\times E_2\times E_3 \to X$ embeds the sum of these three lines as the 'bulk sector,' a direct summand of $H^{2,1}(X)$. The assumption $h^{2,1}=3$ implies that in those cases, the sum is actually isomorphic to $H^{2,1}(X)$. By contracting with top forms we switch to tangent spaces, finding that the sum of the three $H^1(E_i,T_{E_i})$ equals (all of or a direct summand of) $H^1(X,T_X)$. This shows that $\bar{f}$ has injective differential in general, and in the 10 cases when $h^{2,1}=3$ it is a local isomorphism. □

Theorem 4.5. *The induced map $M(4)^3 \to M_X$ is proper.*

Proof. Over $M(4)^3$ there are three universal bundles whose fibers, respectively, are the three elliptic E_i. By the valuative criterion for properness, we need to show that if we have an algebraic family X_t of X's parametrized by t in a regular algebraic curve Δ (or, intuitively, in the unit disc) and a lift $Y_t \to X_t$ over the generic point of Δ (respectively, a family of lifts parametrized by t in the punctured disc) then we can complete the curve of lifts (respectively, fill in with a $Y_0 \to X_0$). The lift gives us three maps E_i from the punctured disc to $M(4)$, hence to M. We see by the same valuative criterion that each of these maps extends to the compactification: $E_i : \Delta \to \overline{M}$. We claim that the central fibers $E_{i,0}$ must be non-singular. This follows immediately from the canonical identification (1) of $H^3(X_t,\mathbb{Q})$ with $\otimes_{1=1,2,3}H^1(E_{t,i},\mathbb{Q})$: The monodromy on $H^3(X_t,\mathbb{Q})$, as t goes around the punctured disc, is trivial since the family extends over Δ. So it must be trivial also on each $H^1(E_{t,i},\mathbb{Q})$, so the elliptic curves cannot degenerate. The family of $Y_t,\ t\in\Delta$ is therefore topologically trivial, so the G action extends to Y_0 with quotient X_0, proving the theorem. □

Corollary 4.6. *The map $\bar{f} : [\mathbb{H}^3/H'] \to M_X$ is an isomorphism in the 10 cases in which $h^{2,1}=3$.*

Proof. We have shown that $\bar{f}$ is étale, hence unramified. Since it has degree 1 on the open substack $\mathcal{O}$ (Theorem 4.3), the same must be true everywhere, and hence $\bar{f}$ is an open embedding. But we have shown that it is also proper. By the connectedness of M_X, $\bar{f}$ is an isomorphism. □

5. Crepant resolutions and the global geometry of $M_{\overline{X}}$. In this section we briefly review the crepant resolutions $\overline{X} \to X$, such that $\overline{X}$ is Calabi-Yau. In the four cases with $h^{1,1}=h^{2,1}=3$ (namely (0-4), (1-5), (1-11), (2-12)), G acts freely on Y, so no resolution is needed. However, the other six examples of $h^{2,1}=3$ all have singularities in their quotient spaces.

5.1. Local structure. The fixed-point loci of our orbifolds X are curves, possibly with trident singularities. There are two local pictures. At a singular point of a curve of fixed points, X is locally of the form $\mathbb{C}^3/(\mathbb{Z}/2)^2$, with generators acting as

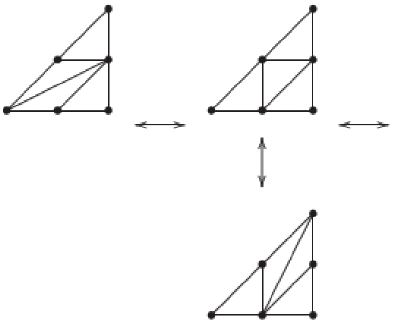
$$\begin{aligned} g_1 &: \ (x,y,z) \mapsto (x,-y,-z), \\ g_2 &: \ (x,y,z) \mapsto (-x,y,-z). \end{aligned}$$

The curve of fixed points looks locally like the trident $xyz=0$, with the three branches $C_1,\ C_2,\ C_3$ consisting of fixed points of $g_1,\ g_2$, and $g_3 := g_1g_2$. (We can always arrange that the local curve C_i is parallel to the global elliptic factor E_i of Y.) At a smooth point of the curve of fixed points, X is locally of the form $\mathbb{C}^3/(\mathbb{Z}/2)$, with generator one of the g_i. So X looks there like the product of C_i (=one of the three axes) with an A_1 surface singularity in the two other directions.

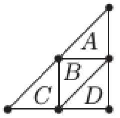
We will focus on the first case, $\mathbb{C}^3/(\mathbb{Z}/2)^2$. Its coordinate ring is

$$\mathbb{C}[x^2, y^2, z^2, xyz] \;=\; \mathbb{C}[a, b, c, d]/(abc = d^2). \tag{2}$$

It has four crepant resolutions. One way to see this is to take advantage of the fact that this singularity is a toric variety, and we can represent the flops graphically as cross-sections of a toric fan as follows:

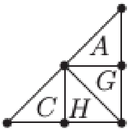


Local coordinates on the central resolution are given by:



$$\begin{aligned} A &= \mathbb{C}[b, c, d_{11}],\\ B &= \mathbb{C}[c_{11}, d_1, t],\\ C &= \mathbb{C}[a, b, w],\\ D &= \mathbb{C}[a, b_1, c], \end{aligned}$$

where $tw = 1$, $c_{11}d_{11} = 1$, $b_1 d_1 = 1$, while local coordinates on one of the outlying resolutions are given by:



$$\begin{aligned} A &= \mathbb{C}[b, c, d_{11}],\\ G &= \mathbb{C}[b, c_{11}, t],\\ H &= \mathbb{C}[a, d_1, w],\\ C &= \mathbb{C}[a, b_1, c]. \end{aligned}$$

The three outlying crepant resolutions as well as the central resolution can be described as successive blowups. Let i, j, k be a permutation of $1, 2, 3$. First blow X up along a smooth surface containing C_i and C_j, two of the components of the curve of fixed points in X. (E.g. for $i, j, k = 1, 3, 2$, the surface is $b = d = 0$.) Second, blow up along a smooth surface containing the remaining curve C_k. (In the above case, we could take this second surface to be $c = d/b = 0$.) This turns out to leave an isolated singularity, a conifold. So, third and finally, take a crepant resolution of the conifold. There are two such resolutions; one will be the k-th outer resolution, and the other, independently of the permutation i, j, k, will be the central resolution.

5.2. Global structure. Globally there are further subtleties to get a Kähler resolution. Consider for example the case (0-1). This has curves of A_1 singularities (along sixteen copies of each of the three elliptic curves E_i). These intersect at 64 fixed points, each of the form of the $(\mathbb{Z}/2)^2$ quotient above. There is thus a total of 4^{64} possible resolutions. However, not all of these are Kähler. We have not checked the total number of Kähler resolutions; it is probably large [2].

In principle, whatever the number n of Kähler crepant resolutions, the forgetful map $M_{\overline{X}} \to M_X$ is a covering of this degree n.

Note that preimage of M_X in $M_{\overline{X}}$ is itself reducible: it has a central component $M_{\overline{X},\text{central}}$, specified by performing the central local resolution at each point of the zero-dimensional stratum. It has one, two or three additional components of small degrees over M_X, specified by performing the type-i local resolution at each point p of the zero-dimensional stratum. Recall that each local curve C_i at each point p is parallel to the global elliptic factor E_i of Y. So on a given $X = Y/G$ it makes sense to perform the same type-i local resolution at all points p. On the other hand, as we vary X in its moduli M_X, monodromy may permute the E_i: depending on the image H_X/H'_X of H_X in S_3, all three of these choices may therefore be on the same component (of degree three over M_X), or on separate components (of degree one), or two can come together with the third staying separate.

In any event, the map $M_{\overline{X},\text{central}} \to M_X$ of connected components is an isomorphism in all ten cases with $h^{2,1} = 3$.

6. Picard groups and Hodge bundles. In this section we will study the Picard groups and Hodge line bundles of M_X when $h^{2,1} = 3$. By Theorem 4.5, we have that $M_X \cong [\mathbb{H}^3/H']$. Our strategy is to prove that the structure sheaf $\mathcal{O}_{M_X}$ is acyclic, and apply the exponential sequence on M_X. We will then prove that $\mathrm{Pic}(M_X)$ is isomorphic to the group cohomology $H^2(H', \mathbb{Z})$, which we will analyze in detail.

6.1. Picard Groups. We begin with a few lemmas on analytic quotient stacks. The first is straightforward and serves mainly to remind the reader of some terminology.

LEMMA 6.1. *Let X be a smooth complex manifold, and T a group acting virtually freely and holomorphically on X (i.e., T has a finite index subgroup T_0 acting freely on X). Let $\mathcal{X}$ denote the quotient stack $[X/T]$. Then the categorical quotient $X_{mod} = X/T$ exists as a complex analytic space, and the natural map $f : \mathcal{X} \to X_{mod}$ is a coarse moduli space for $\mathcal{X}$. Furthermore, $f_*\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_{X_{mod}}$.*

Proof. We have in particular that T acts properly discontinuously on X. Then [Car57], $X_{mod} = X/T$ is a normal analytic space. The map $f : \mathcal{X} \to X_{mod}$ is induced by the natural map $X \to X_{mod}$, since the latter is T equivariant. To see that the map is a coarse moduli space, note that any map ϕ from $\mathcal{X}$ to a space Y factors through a map $X/T \to Y$, since any such map ϕ is by definition a T equivariant map $X \to Y$. Furthermore, it is immediate that there is a bijection between the closed points of $\mathcal{X}$ and X/T.

For the last claim, note that a section of $f_*\mathcal{O}_{\mathcal{X}}$ over an open subset $U \subset X_{mod}$ is by definition a holomorphic function $\pi^{-1}(U) \to \mathbb{C}$ (for π the quotient map $X \to X_{mod}$) which is T invariant. However, by the universal property of the quotient, this is the same thing as a section of $\mathcal{O}_{X_{mod}}$ over U. □

We recall the following definition.

DEFINITION 6.1 ([GR04]). A complex analytic space $(S, \mathcal{O}_S)$ is called *Stein* if it satisfies Cartan's Theorem B, namely every coherent analytic sheaf of $\mathcal{O}_S$-modules

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[2] P. Aspinwall conjectures that it is on the order of 2^{48}.

on S is acyclic.

LEMMA 6.2. *Keeping the notation from the previous lemma, let $\mathcal{O}_{\mathcal{X}}$ denote the structure sheaf of $\mathcal{X}$. If X_{mod} is a Stein space, then $\mathcal{O}_{\mathcal{X}}$ is acyclic (i.e. $H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$ for $i > 0$).*

Proof. The group $H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is by definition the ith equivariant sheaf cohomology of $\mathcal{O}_X$ over X with respect to the action of T. In other words, it is the ith right derived functor of the composition $(-)^T \circ \Gamma$, where Γ is the global sections functor and $(-)^T$ is the T-invariants functor. Therefore, we have a spectral sequence

$$E_2^{p,q} = H^p(T, H^q(X, \mathcal{O}_X)) \implies H^{p+q}(\mathcal{X}, \mathcal{O}_{\mathcal{X}}).$$

Firstly, assume that T is finite. Since $H^q(X, \mathcal{O}_X)$ is a $\mathbb{C}$ vector space, we have that $E_2^{p,q} = 0$ for $p > 0$. In particular, there is an isomorphism

$$H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \cong H^0(T, H^2(X, \mathcal{O}_X)) \cong H^i(X, \mathcal{O}_X)^T.$$

Now, since T is finite, the quotient map $\pi : X \to X/T$ is a finite holomorphic mapping. Therefore by [GR04] (Section I.1, Theorem 5), π_* induces an isomorphism on coherent sheaf cohomology and

$$H^i(X, \mathcal{O}_X) \cong H^i(X_{mod}, \pi_* \mathcal{O}_X).$$

Furthermore, since π is proper, we also have that $\pi_* \mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_{X_{mod}}$ modules ([GR04], Section I.3, Theorem 3). Since X_{mod} is Stein, Cartan's Theorem B ensures that $\pi_* \mathcal{O}_X$ is then acyclic. Putting everything together, we obtain $H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$ for $i > 0$.

Now, relax the assumption that T is finite. By replacing X by X/T_0 and T by T/T_0, we apply the result above to obtain the claim. □

We return now to M_X. As a consequence of the first lemma, the analytic space $\mathbb{H}^3/H'$ is a coarse moduli space for M_X. We denote it by C_X.

LEMMA 6.3. *The analytic space C_X is a Stein space.*

Proof. Firstly, we remark that C_X is isomorphic to $\mathbb{H}^3/H$, since G acts trivially on $\mathbb{H}^3$. Since H is a finite index subgroup of $H_{max} = (\mathbb{Z}/4)^6 \rtimes \Gamma^3 \rtimes S_3$, we have a finite holomorphic mapping $C_X \to C_{max}$, where C_{max} is the quotient $\mathbb{H}^3/H_{max}$. Let us determine C_{max}. We already know that $\mathbb{H}/\Gamma \cong \mathbb{C}$, via the j-invariant. Since $(\mathbb{Z}/4)^6$ acts trivially on $\mathbb{H}^3$, we are reduced to computing $\mathbb{C}^3/S_3$, *i.e.* the third symmetric power $\mathbb{C}^{(3)}$. But it is well known that $\mathbb{C}^{(3)} \cong \mathbb{C}^3$ (for example, by regarding the $\mathbb{C}^3$ as the vector space of monic degree 3 polynomials, the map sending $(z_1, z_2, z_3) \in \mathbb{C}^3$ to $(z - z_1)(z - z_2)(z - z_3) \in \mathbb{C}^3$ is a quotient map). We remark that C_{max} is Stein. But since $C_X \to C_{max}$ is a finite holomorphic mapping, we conclude [GR04] that C_X is Stein. □

PROPOSITION 6.4. *The sheaf $\mathcal{O}_{M_X}$ is acyclic.*

Proof. By Lemma 6.3, C_X is Stein. By Lemma 6.2, the proposition follows. □

Now we can prove the main theorem of this section.

THEOREM 6.5. *In each case with $h^{2,1} = 3$, we have $Pic(M_X) \cong H^2(H', \mathbb{Z})$, where the latter is the second group cohomology of H' with trivial action on $\mathbb{Z}$.*

Proof. On M_X, we have the exponential sequence of sheaves

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{M_X} \longrightarrow \mathcal{O}^*_{M_X} \longrightarrow 1.$$

This sequence induces a long exact cohomology sequence, which in particular contains the following section.

$$\cdots \to H^1(M_X, \mathcal{O}_{M_X}) \to H^1(M_X, \mathcal{O}^*_{M_X}) \xrightarrow{c_1} H^2(M_X, \mathbb{Z}) \to H^2(M_X, \mathcal{O}_{M_X}) \to \cdots$$

By Proposition 6.4, $H^i(M_X, \mathcal{O}_{M_X}) = 0$ for $i > 0$. Hence by the exactness of the sequence, the map c_1 is an isomorphism. So

$$\mathrm{Pic}(M_X) \cong H^2(M_X, \mathbb{Z}).$$

Now, M_X is the quotient stack $[\mathbb{H}^3/H']$. Therefore, there is an isomorphism

$$H^2(M_X, \mathbb{Z}) \cong H^2(\mathbb{H}^3, H', \mathbb{Z}),$$

where the latter group is the equivariant sheaf cohomology of $\mathbb{H}^3$ with respect to the sheaf $\mathbb{Z}$ and group H'. Since $\mathbb{H}^3$ is contractible, we have $H^i(\mathbb{H}^3, \mathbb{Z}) = 0$ for $i > 0$, and hence there is an isomorphism ([Gro57], Proposition 5.2.4 (5.2.5 in English translation))

$$H^2(\mathbb{H}^3, H', \mathbb{Z}) \cong H^2(H', \mathbb{Z}),$$

□

Now we begin to analyze the group $H^2(H', \mathbb{Z})$.

6.2. Group cohomology.

Let us recall some basic elements of group cohomology. A reference for group cohomology is [Br82]. Throughout this subsection, G is an arbitrary group.

Recall that for any group G, the group cohomology functor $H^i(G, -) : \mathrm{Mod}_G \to \mathrm{Ab}$ (for Mod_G the category of G modules, and Ab the category of abelian groups) is defined as the ith right derived functor of $(-)^G : \mathrm{Mod}_G \to \mathrm{Mod}_G$, which sends a G module M to its G invariant submodule M^G (we may regard $(-)^G$ as a functor into the category of abelian groups, since G acts trivially on M^G). Say that G lies in an extension

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1.$$

Then for any G-module M, M^N acquires the structure of a Q module. Indeed, $(-)^G = (-)^Q \circ (-)^N$. Therefore, the Grothendieck spectral sequence in this case becomes

$$E_2^{p,q} = H^p(Q, H^q(N, M)) \implies H^{p+q}(G, M).$$

This spectral sequence is known as the *Lyndon-Hochschild-Serre* sequence. The associated 5-term exact sequence is

$$1 \to H^1(Q, M^N) \to H^1(G, M) \to H^1(N, M)^Q \to H^2(Q, M^N) \to H^2(G, M),$$

which is known as the *inflation-restriction* sequence. The map $H^1(Q, M^N) \to H^1(G, M)$ is known as the *inflation map* and $H^1(G, M) \to H^1(N, M)^Q$ is known as the *restriction map.*

For any abelian group M, we can form $H^i(G,M)$ by equipping M with the trivial G action. If no action of G is specified, this is what we will mean.

We also recall two useful facts. If G acts trivially on M, then $H^1(G,M) \cong \mathrm{Hom}(G,M)$. Also, if M is an G module which happens to be a K vector space, where K is a field such that $\mathrm{Char}(K)$ does not divide the order of G, then $H^i(G,M) = 0$ for $i > 0$.

6.3. Computing $H^2(H',\mathbb{Z})$.

In this section, we will carry out the main group theoretic calculation.

Lemma 6.6. *$H^2(H',\mathbb{Z})$ is a finitely generated abelian group.*

Proof. We begin with a general statement. Let N be a normal subgroup of a group T, with quotient Q. Assume that Q is finite, and that the integral cohomology groups of N are finitely generated. Then we show that $H^i(T,\mathbb{Z})$ is finitely generated. We have the cohomological Lyndon-Hochschild-Serre spectral sequence:

$$E_2^{p,q} = H^p(Q, H^q(N,\mathbb{Z})) \implies H^{p+q}(T,\mathbb{Z})$$

Since $H^q(N,\mathbb{Z})$ is finitely generated by hypothesis, and Q is finite, the groups $E_2^{p,q}$ are finite for any $p, q \in \mathbb{Z}$. Therefore, $H^{p+q}(T,\mathbb{Z})$ admits a finite step filtration with finitely generated quotients. It follows that $H^{p+q}(T,\mathbb{Z})$ is finitely generated (since an extension of finitely generated abelian groups is finitely generated).

Now we apply the lemma. First, $A(2)$ is an extension of a finite group by $\Gamma(2)$, hence has finitely generated integral cohomology. Furthermore, we have an exact sequence

$$1 \longrightarrow A(2)^3 \longrightarrow H \longrightarrow L \longrightarrow 1.$$

Hence, H has finitely generated integral cohomology. Next, consider the sequence

$$1 \longrightarrow G \longrightarrow H \longrightarrow H' \longrightarrow 1.$$

The inflation map $H^i(H',\mathbb{Z}) \to H^i(H,\mathbb{Z})$ has cokernel isomorphic to a subgroup of $H^i(G,\mathbb{Z})^{H'}$, which is a finite group. The kernel is isomorphic to some subgroup of $H^{i-1}(G,\mathbb{Z})$, which is also finite. Since $H^i(H,\mathbb{Z})$ is finitely generated, we conclude that $H^i(H',\mathbb{Z})$ is finitely generated as well. □

The previous lemma allows us to decompose $H^2(H',\mathbb{Z})$ into a free part, $\mathbb{Z}^k$, and a torsion part. By the universal coefficient theorem, we can calculate the rank k as the dimension of the vector space $H^2(H',\mathbb{Q})$ over $\mathbb{Q}$. For the rest of this section, let K be a field of characteristic not equal to 2 or 3.

Lemma 6.7. *The inflation map $H^i(H',K) \to H^i(H,K)$ is an isomorphism for all i.*

Proof. Since the order of $|G|$ (2^{r+2}) is invertible in K, we have $H^i(G,K) = 0$ for $i > 0$. There is an inflation-restriction exact sequence

$$1 \to H^i(H',K) \to H^i(H,K) \to H^i(G,K)^{H'} \to \cdots$$

By the vanishing of the cohomology of G, the claim follows. □

Note that $\Gamma(2)$ contains a free subgroup F_2 on two generators [Sa47], generated by

$$f_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \ f_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

From now on, F_2 denotes this particular free subgroup. In fact, $\Gamma(2) \cong \mathbb{Z}/2 \times F_2$, where $\mathbb{Z}/2$ is generated by $-I$.

Lemma 6.8. *Recall that H lies in an extension*

$$1 \longrightarrow A(2)^3 \longrightarrow H \longrightarrow L \longrightarrow 1.$$

The Lyndon-Hochschild-Serre spectral sequence associated to this exact sequence induces an isomorphism $H^2(H,K) \cong H^0(L, H^2(A(2)^3, K))$.

Proof. The terms of this spectral sequence are

$$E_2^{p,q} = H^p(L, H^q(A(2)^3, K)) \implies H^{p+q}(H,K).$$

L is a subgroup of $L_{max} = (\mathbb{Z}/2)^6 \rtimes S^3 \rtimes S_3$, which has order $2^{10}3^2$. Therefore, $|L|$ is invertible in K. Furthermore, $H^q(A(2)^3, K)$ is a K vector space, so we have the vanishing of $E_2^{p,q}$ for $p > 0$. Hence, there is an isomorphism

$$H^2(H,K) \cong H^0(L, H^2(A(2)^3, K)) \cong H^2(A(2)^3, K)^L.$$

□

Lemma 6.9. *The restriction map $\rho : H^2(A(2)^3, K) \to H^2(F_2^3, K)$ is an isomorphism. Furthermore, let $p : L_{max} \to S^3 \rtimes S_3$ be the quotient by $(\mathbb{Z}/2)^6$. Then, there is an induced action of $L_0 := p(L)$ on $H^2(F_2^3, K)$, so that*

$$H^2(A(2)^3, K)^L \cong H^2(F_2^3, K)^{L_0},$$

under the previously mentioned isomorphism ρ.

Proof. For the first claim, we know that $A(2)^3 \cong (\mathbb{Z}/2)^6 \times F_2^3$. Since Char $K \neq 2$, the restriction map ρ is an isomorphism. Via this isomorphism, we can define an action of L_{max} on $H^2(F_2^3, K)$. For the second claim, we must show that $(\mathbb{Z}/2)^6 \leq L_{max}$ acts trivially. Firstly, we use the Künneth formula to decompose $H^2(F_2^3, K)$.

$$H^2(F_2^3, K) \cong \bigoplus_{i+j+k=2} H^i(F_2, K) \otimes H^j(F_2, K) \otimes H^k(F_2, K) \tag{3}$$

Note that $H^2(F_2, K) = 0$. Therefore, it suffices to show that $(\mathbb{Z}/2)^2$ acts trivially on $H^1(F_2, K)$. An element of the latter group is a homomorphism $\phi : F_2 \to K$. Now we unwind the definition of the action of $(\mathbb{Z}/2)^2 \leq L_{max}$ on $H^1(F_2, K) \cong \mathrm{Hom}(F_2, K)$. Firstly, we extend ϕ trivially to a homomorphism $\widetilde{\phi} : A(2) \to K$ by defining it to vanish on $(\mathbb{Z}/2)^2 \leq A(2)$. Then, for $\epsilon \in (\mathbb{Z}/2)^2 \leq L_{max}$, we have $\epsilon \cdot \phi = \widetilde{\phi}(\epsilon \cdot (-) \cdot \epsilon^{-1})$. But $\widetilde{\phi}(\epsilon) = 0$ by construction, so the claim follows. Therefore, the action of L on $H^2(F_2^3, K)$ factors through an action of L_0. □

By the formula (3), we have $H^2(F_2^3, K) \cong K^{12}$. Our problem is now reduced to calculating the L_0 invariant submodule of $M = H^2(F_2^3, K)$. To do this, let us explain in detail how $L_0 \leq S^3 \rtimes S_3$ acts on M.

Let us first work out the action of S on $H^1(F_2, K)$. To fix notation, let

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, r = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

So, $\bar{s}, \bar{t}, \bar{r} \in S$ generate the subgroups B_3, B_1, and B_2 respectively. We introduced before the elements f_1 and f_2 generating F_2. We now define a basis $\{\hat{f}_1, \hat{f}_2\}$ for $H^1(F_2, K)$, where $\hat{f}_i(f_j) = \delta_{ij} \in K$. So, we can work out the action of the elements $\bar{s}, \bar{t}, \bar{r}$ on $\hat{f}_i$. Firstly, it is easy to check that the following relations hold.

$$\begin{array}{lll} sf_1s^{-1} = f_2^{-1}, & tf_1t^{-1} = f_1, & rf_1r^{-1} = -f_2f_1^{-1}, \\ sf_2s^{-1} = f_1^{-1}, & tf_2t^{-1} = -f_1, f_2^{-1} & rf_2r^{-1} = f_2. \end{array}$$

From now on, we will abuse notation and refer to $\bar{s}, \bar{t}, \bar{r}$ as s, t, r. Then, using the equations above, we have the following.

$$\begin{array}{lll} s \cdot \hat{f}_1 = -\hat{f}_2 & t \cdot \hat{f}_1 = \hat{f}_1 & r \cdot \hat{f}_1 = \hat{f}_2 - \hat{f}_1 \\ s \cdot \hat{f}_2 = -\hat{f}_1 & t \cdot \hat{f}_2 = \hat{f}_1 - \hat{f}_2 & r \cdot \hat{f}_2 = \hat{f}_2 \end{array}$$

So, we have described the S module structure on $H^1(F_2, K)$. S acts trivially on $H^0(F_2, K) = K$. Then, each term $H^i(F_2, K) \otimes H^j(F_2, K) \otimes H^k(F_2, K)$ (which we denote by $H^{i,j,k}$) in the decomposition (3) becomes an S^3 module via the external tensor product. $H^2(F_2^3, K)$ is then the direct sum of these representations. Lastly, $S_3 \leq S^3 \rtimes S_3$ acts by permutation of the three summands in (3). Now we come to the main theorem of this section.

THEOREM 6.10. *The group $H^2(H', K)$ is given in Table 2 in each of the ten cases with $h^{2,1}(X) = 3$.*

Case	$\dim_K(H^2(H', K))$
(0-1)	0
(0-4)	1
(1-1)	1
(1-5)	1
(1-11)	2
(2-1)	1
(2-9)	1
(2-12)	3
(3-5)	1
(4-1)	1

TABLE 2
Cohomology of H' with coefficients in K

Proof. By the previous lemmas, we are reduced to computing $H^2(F_2^3, K)^{L_0}$. The previous remark described the L_0 module structure on $H^2(F_2^3, K)$. We will break into cases, using the results of section 3.2 on the form of L_0. In all of the ten cases, L_0 is an extension of a group $T \leq S_3$ by a normal subgroup $N \leq S^3$. Therefore, to find the L_0 invariants, we will firstly determine the N invariant submodule of $H^2(F_2^3, K)$, and subsequently the T invariant submodule of the $H^2(F_2^3, K)^N$.

As we've said, N acts on each term of the sum (3). A general element of $H^{1,1,0}$ has the form

$$f = \sum_{i,j=1}^{2} \lambda_{ij} \hat{f}_i \otimes \hat{f}_j \otimes 1.$$

We retain the 1 in the expression above in order to distinguish the summands $H^{1,1,0}$, $H^{1,0,1}$, and $H^{0,1,1}$. For concreteness let us work with this factor $H^{1,1,0}$. We will write down some general results that will be used in each case. Namely, we determine the general form of such an element f which is invariant under $(s,1,1)$, $(1,s,1)$, $(t,1,1)$, $(1,t,1)$, $(r,1,1)$, $(1,r,1)$, $(s,s,1)$, $(r,r,1)$, or $(t,t,1)$ respectively. In order, the elements f are given by

$$\lambda_{11}\hat{f}_1 \otimes \hat{f}_1 \otimes 1 + \lambda_{12}\hat{f}_1 \otimes \hat{f}_2 \otimes 1 - \lambda_{11}\hat{f}_2 \otimes \hat{f}_1 \otimes 1 - \lambda_{12}\hat{f}_2 \otimes \hat{f}_2 \otimes 1,$$

$$\lambda_{11}\hat{f}_1 \otimes \hat{f}_1 \otimes 1 - \lambda_{11}\hat{f}_1 \otimes \hat{f}_2 \otimes 1 + \lambda_{21}\hat{f}_2 \otimes \hat{f}_1 \otimes 1 - \lambda_{21}\hat{f}_2 \otimes \hat{f}_2 \otimes 1,$$

$$\lambda_{11}\hat{f}_1 \otimes \hat{f}_1 \otimes 1 + \lambda_{12}\hat{f}_1 \otimes \hat{f}_2 \otimes 1,$$

$$\lambda_{11}\hat{f}_1 \otimes \hat{f}_1 \otimes 1 + \lambda_{21}\hat{f}_2 \otimes \hat{f}_1 \otimes 1,$$

$$\lambda_{21}\hat{f}_2 \otimes \hat{f}_1 \otimes 1 + \lambda_{22}\hat{f}_2 \otimes \hat{f}_2 \otimes 1,$$

$$\lambda_{12}\hat{f}_1 \otimes \hat{f}_2 \otimes 1 + \lambda_{22}\hat{f}_2 \otimes \hat{f}_2 \otimes 1,$$

$$\lambda_{11}\hat{f}_1 \otimes \hat{f}_1 \otimes 1 + \lambda_{12}\hat{f}_1 \otimes \hat{f}_2 \otimes 1 + \lambda_{12}\hat{f}_2 \otimes \hat{f}_1 \otimes 1 + \lambda_{11}\hat{f}_2 \otimes \hat{f}_2 \otimes 1,$$

$$\lambda_{11}\hat{f}_1 \otimes \hat{f}_1 \otimes 1 + \lambda_{12}\hat{f}_1 \otimes \hat{f}_2 \otimes 1 + \lambda_{12}\hat{f}_2 \otimes \hat{f}_1 \otimes 1 - 2\lambda_{12}\hat{f}_2 \otimes \hat{f}_2 \otimes 1,$$

$$-2\lambda_{12}\hat{f}_1 \otimes \hat{f}_1 \otimes 1 + \lambda_{12}\hat{f}_1 \otimes \hat{f}_2 \otimes 1 + \lambda_{12}\hat{f}_2 \otimes \hat{f}_1 \otimes 1 + \lambda_{22}\hat{f}_2 \otimes \hat{f}_2 \otimes 1.$$

Checking these expressions is easy, if tedious. Now we are ready for a case by case analysis.

Case (0-1). $L_0 = S^3 \rtimes S_3$. We begin by finding the S^3 invariant elements of $H^{1,1,0}$. Imposing invariance under $(s,1,1)$ and $(1,s,1)$ implies that an invariant f has the form

$$f = \lambda(\hat{f}_1 \otimes \hat{f}_1 \otimes 1 - \hat{f}_1 \otimes \hat{f}_2 \otimes 1 - \hat{f}_2 \otimes \hat{f}_1 \otimes 1 + \hat{f}_2 \otimes \hat{f}_2 \otimes 1).$$

But then invariance under $(t,1,1)$ requires $\lambda_{21} = 0$, so $f = 0$. The same analysis applies to $H^{1,0,1}$ and $H^{0,1,1}$. Therefore, $H^2(A(2)^3, K)^L = 0$.

Cases (0-4), (1-1), (2-9). In each case, L_0 has the form $B_i^3 \rtimes S_3$. We may assume without loss of generality that $i = 3$ by relabelling basis elements, allowing us to consider the three cases at once. We begin by computing the B_3^3 invariant submodule of $H^{1,1,0}$. Imposing invariance under $(s,1,1)$ and $(1,s,1)$ yields that a general invariant f has the form

$$f=\lambda(\hat{f}_1\otimes\hat{f}_1\otimes 1-\hat{f}_1\otimes\hat{f}_2\otimes 1-\hat{f}_2\otimes\hat{f}_1\otimes 1+\hat{f}_2\otimes\hat{f}_2\otimes 1),$$

for any $\lambda\in K$. Applying the same analysis to $H^{1,0,1}$ and $H^{0,1,1}$ allows us to conclude that B_3^3 invariant submodule of $H^2(H',K)$ has the form $K\oplus K\oplus K$. Then S_3 acts to permute these factors (sending e.g. $\hat{f}_1\otimes\hat{f}_1\otimes 1$ to $\hat{f}_1\otimes 1\otimes\hat{f}_1$), so we conclude that $H^2(A(2)^3,K)=K$, the diagonal.

Case (2-12). $L_0=(1\times B_1^2)\rtimes S_3$. We begin by computing the $1\times B_1^2$ invariant submodule of $H^{1,1,0}$. We must impose invariance under $(1,t,1)$. A general invariant f then has the form

$$f=\lambda_{11}\hat{f}_1\otimes\hat{f}_1\otimes 1+\lambda_{21}\hat{f}_2\otimes\hat{f}_1\otimes 1.$$

Therefore the $1\times B_1^2$ invariant submodule of $H^{1,1,0}$ has dimension 2. The same analysis applies to $H^{1,0,1}$. However, for $H^{0,1,1}$, we must impose invariance under both $(1,t,1)$ and $(1,1,t)$. Therefore a general invariant of $H^{0,1,1}$ has the form

$$f=\lambda(1\otimes\hat{f}_1\otimes\hat{f}_1-1\otimes\hat{f}_1\otimes\hat{f}_2-1\otimes\hat{f}_2\otimes\hat{f}_1+1\otimes\hat{f}_2\otimes\hat{f}_2).$$

So, the $1\times B_1^2$ invariants of $H^{0,1,1}$ have dimension 1. Now, S_2 acts to permute the factors $H^{1,1,0}$ and $H^{1,0,1}$. Therefore, $H^2(A(2)^3,K)^L=K^2\times K=K^3$.

Case (2-1). $L_0=S\rtimes S_3$. Here $S\leq S^3$ is the diagonal. We start by finding the S invariant submodule of $H^{1,1,0}$. We impose invariance under (s,s,s), (t,t,t), and (r,r,r). A general invariant element f must have the form

$$f=\lambda(-2\hat{f}_1\otimes\hat{f}_1\otimes 1+1\otimes\hat{f}_1\otimes\hat{f}_2\otimes 1+\hat{f}_2\otimes\hat{f}_1\otimes 1-2\hat{f}_2\otimes\hat{f}_2\otimes 1).$$

The same analysis applies to finding the S invariants of $H^{1,0,1}$ and $H^{0,1,1}$. They have the same form. So, the S invariant submodule of $H^2(A(2)^3)$ has the form $K\oplus K\oplus K$, and S_3 permutes these factors. Therefore the S_3 invariants are the diagonal, and $H^2(A(2)^3,K)^L=K$.

Cases (1-5), (3-5). $L_0=\widetilde{B}_i\rtimes S_3$. We may assume without loss of generality that $i=3$. This case is actually identical to that of (0-4), (1-1), (2-9). Indeed, to find the $\widetilde{B}_3$ invariant submodule of $H^{1,1,0}$ we need to impose invariance under $(s,1,s)$ and $(1,s,s)$. Since these elements act in the same way as $(s,1,1)$ and $(1,s,1)$ on $H^{1,1,0}$, the $\widetilde{B}_3$ invariants are the same as the B_3^2 invariants. Proceeding with the same analysis as in those former cases, we find that $H^2(A(2)^3,K)^L=K$.

Case (1-11). $L_0=(B_2^2\times B_1)\rtimes S_2$. We start by finding the $B_2^2\times B_1$ invariants of $H^{1,1,0}$. By imposing invariance under $(r,1,s)$ and $(1,r,s)$ we see that a general invariant element f has the form

$$f=\lambda\hat{f}_2\otimes\hat{f}_2\otimes 1.$$

Now, we find that $B_2^2\times B_1$ invariants of $H^{0,1,1}$. We impose invariance under $(1,t,1)$ and $(1,1,r)$. Invariance under $(1,t,1)$ yields $\lambda_{21}=\lambda_{22}=0$. Invariance under $(1,1,r)$ yields $\lambda_{11}=0$. So, a general invariant element has the form

$$f=\lambda(1\otimes\hat{f}_1\otimes\hat{f}_2).$$

The same analysis applies to $H^{1,0,1}$. So, the $B_2^2 \times B_1$ invariant submodule of $H^2(A(2)^3, K)$ has the form $K \oplus K \oplus K$. S_2 acts to permute the last two factors, so we conclude that $H^2(A(2)^3, K)^L \cong K^2$.

Case (4-1). In this case, L_0 is not a semidirect product, but we explained earlier that it lies in an extension of S_3 by a group N. N is generated by the elements (t, s, r), (s, r, t), and (r, t, s). We begin by finding the N invariants in $H^{1,1,0}$. Imposing invariance under (t, s, r) and (s, r, t) forces an element f to have the form

$$f = \lambda(\hat{f}_1 \otimes \hat{f}_1 \otimes 1 - 2\hat{f}_1 \otimes \hat{f}_2 \otimes 1 + \hat{f}_2 \otimes \hat{f}_1 \otimes 1 + \hat{f}_2 \otimes \hat{f}_2 \otimes 1).$$

It is then easy to show that such an f is already invariant under (s, r, t). The same analysis applies to $H^{1,0,1}$ and $H^{0,1,1}$. Therefore the N invariant submodule of $H^2(A(2)^3, K)$ has the form $K \oplus K \oplus K$, and S_3 acts to permute the three factors. Therefore, $H^2(A(2)^3, K)^L = K$. This concludes the proof of the claimed results. □

COROLLARY 6.11. *$H^2(H', \mathbb{Z})$ has no p torsion for $p > 3$.*

Proof. The group $H^2(H', \mathbb{Z}/p)$ has the form $(\mathbb{Z}/p)^a$, for some $a \in \mathbb{N}$. In contains a factor of $\mathbb{Z}/p$ for each (i) factor of $\mathbb{Z}$ in $H^2(H', \mathbb{Z})$, (ii) factor of $\mathbb{Z}/p^k$ in $H^2(H', \mathbb{Z})$ $(k \geq 1)$ and (iii) each factor of $\mathbb{Z}/p^k$ in $H^3(H', \mathbb{Z})$. The previous theorem establishes that $\dim_{\mathbb{Q}}(H^2(H', \mathbb{Q})) = \dim_{\mathbb{Z}/p}(H^2(H', \mathbb{Z}/p))$ for $p > 3$. Hence, there are as many $\mathbb{Z}/p$ factors in $H^2(H', \mathbb{Z}/p)$ as there are factors of $\mathbb{Z}$ in $H^2(H', \mathbb{Z})$. In particular, there is no p torsion in $H^2(H', \mathbb{Z})$. □

COROLLARY 6.12. *The group Pic(M_X), for X such that $h^{2,1} = 3$, is isomorphic to $\mathbb{Z}^a \times A$, where a is given in Table 2 above as $\dim_K(H^2(H', K))$ for each case and A is some (case dependent) finite abelian group of order $2^n 3^m$, $n, m \in \mathbb{N}$. In particular, the Picard group of M_X is infinite in 9 out of the 10 cases.*

6.4. Hodge Bundles. In this section, we will give two proofs that the Hodge bundle over M_X has finite order. The first proof is as follows.

THEOREM 6.13. *Let λ denote the Hodge bundle over M_X, for $h^{2,1}(X) = 3$. Then λ is nontrivial, and has finite order.*

Proof. To show that a line bundle L over an analytic stack $\mathcal{X}$ has finite order, it suffices to exhibit a finite étale cover $\ell : \mathcal{Y} \to \mathcal{X}$ such that $\ell^* L$ has finite order. In our case, as we've discussed in the remark prior to Theorem 4.4, there is a finite étale cover $\ell : M(4)^3 \to M_X$. It is easy to see that the pullback $\ell^*\lambda$ is the Hodge bundle over $M(4)^3$. Since the Hodge bundle over $M(4)$ has finite order (it is the pullback of the Hodge bundle on $M = M_{1,1}$, which has order 12), the second claim follows.

Now we must show that λ is nontrivial. Let $\pi : \mathbb{H}^3 \to [\mathbb{H}^3/H'] = M_X$ denote the quotient map. A global section of λ is determined by a global section of the H'-equivariant line bundle $\pi^*\lambda$. Consider the subgroup $\Gamma(2)^3 \subset H$. The map $\Gamma(2)^3 \to H \to H'$ makes $\pi^*\lambda$ into a $\Gamma(2)^3$-equivairant line bundle, which in turn gives the descent data for the bundle $\rho_2 \boxtimes \rho_2 \boxtimes \rho_2$ on $[\mathbb{H}^3/\Gamma(2)^3] = M(2)^3$, where ρ_2 is the Hodge bundle on $M(2)$. To see that this line bundle is nontrivial, it is sufficient to observe that ρ_2 itself is nontrivial.

A global section of ρ_2 is a weakly modular form $f : \mathbb{H} \to \mathbb{C}$ of weight 1 and level 2 on $\mathbb{H}$. Any such form not only has a zero, but is in fact identically zero: since $-I \in \Gamma(2)$, and $-I$ acts trivially on $\mathbb{H}$, we must have $f(\tau) = -f(\tau)$. This implies that $f = 0$. □

For the second approach, we will explicitly construct a trivializing section of $\lambda^{\otimes 12}$. To do so, we must give a global nonvanishing section of $\pi^*\lambda$ over $\mathbb{H}^3$ which is H' equivariant. To be precise, a line bundle over $[\mathbb{H}^3/H']$ is an H' equivariant line bundle over $\mathbb{H}^3$, *i.e.* a line bundle L over $\mathbb{H}^3$ together with an H' action. In these terms, the definition of the Hodge bundle λ over M_X is as follows. The associated line bundle over $\mathbb{H}^3$ is denoted $\pi^*\lambda$, and is by definition equal to the pushforward of the relative canonical bundle $\Omega^3_{\mathcal{U}^3/\mathbb{H}^3}$ to $\mathbb{H}^3$. The H' action on this bundle is inherited from the H' action on $\Omega^3_{\mathcal{U}^3}$, which is obtained by taking the differential of the action of an element $h \in H'$ on $\mathcal{U}^3$. Then to give a trivialization of λ is to give a trivialization of $\pi^*\lambda$ which is equivariant with respect to the H' actions on $\mathbb{H}^3$ and $\pi^*\lambda$.

So, we will seek a global nonvanishing section of $\pi^*\lambda$ of the form

$$f(\tau_1,\tau_2,\tau_3)dz_1 \wedge dz_2 \wedge dz_3,$$

for f a nonvanishing analytic function on $\mathbb{H}^3$. Before we write down such a section, recall that the modular discriminant $\Delta : \mathbb{H} \to \mathbb{C}$ is defined to be η^{24}, where η is the Dedekind η function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n),$$

with $q = e^{2\pi i \tau}$. Note that Δ is a modular form of weight 12.

THEOREM 6.14. *The section* $\sigma : \mathbb{H}^3 \to (\pi^*\lambda)^{\otimes 12}$ *defined by*

$$\sigma = \left(\prod_{i=1}^{3} \Delta(\tau_i)\right)(dz_1 \wedge dz_2 \wedge dz_3)^{\otimes 12}$$

descends to a trivialization of $\lambda^{\otimes 12}$ *over* M_X.

Proof. To check that σ is equivariant under H', it suffices to show that it is equivariant under Γ^3 and S_3 (since the translations $(\mathbb{Z}/4)^6$ certainly act trivially on σ). The claim of equivariance under S_3 is immediate, since the only effect of a permutation is to introduce a sign coming from the wedge product $dz_1 \wedge dz_2 \wedge dz_3$. Equivariance under Γ^3 follows from the fact that Δ is modular of weight 12. Since Δ is also nonvanishing, this concludes the proof that σ defines a trivializing section of $\lambda^{\otimes 12}$. □

We proceed to write down a globally defined Kähler potential on M_X. We consider the following function defined on $\mathbb{H}^3$

$$\mathcal{K} = \log \|\eta(\tau_1)\eta(\tau_2)\eta(\tau_3)\|^2.$$

Clearly, $\mathcal{K}$ is invariant under the action of S_3. Furthermore, while η is not invariant under Γ (i.e. it is not modular), it satisfies $\eta(\gamma\tau) = \epsilon\eta(\tau)$, where $\gamma \in \Gamma$ and ϵ is some 12th root of unity. Therefore, $\mathcal{K}$ is invariant under the action of H' on $\mathbb{H}^3$. Hence, it is well defined on the moduli space M_X. We can then define

$$\omega = \frac{i}{2}\partial\bar{\partial}\mathcal{K},$$

which defines a (1,1) form on M_X. Since this form is positive definite, we see that ω defines a Kähler form on M_X. ω is in fact the curvature of the Hodge bundle. Indeed, we may define a C^∞ section of λ given by

$$\widetilde{\sigma} = \|\eta(\tau_1)\eta(\tau_2)\eta(\tau_3)\|^2 \, dz_1 \wedge dz_2 \wedge dz_3.$$

This section is well defined for the same reason that $\mathcal{K}$ is well defined. If h denotes the Weil-Petersson norm of $\widetilde{\sigma}$, then (after potentially rescaling $\widetilde{\sigma}$ by a constant) we have $\mathcal{K} = \log h$. Hence, ω is the curvature of λ with respect to the Weil-Petersson metric.

7. Open problems. In general, it seems that very little is known about global Calabi-Yau moduli spaces. Is their Picard group always finitely generated? If not, is the Hodge line bundle still of finite order? Can the Hodge bundle ever be divisible? Are the coarse moduli spaces always affine?

Clearly, it would be useful to have a global description of more examples. The cases previously understood are 1-dimensional (for example the mirror of the quintic threefold) or 2-dimensional. What about general toric hypersurfaces and complete intersections? An obvious starting point might be the quintic threefold itself. However, the large symmetry groups present (and the large dimension) may make this case particularly dificult.

At the opposite extreme, one might prefer to consider Calabi-Yau hypersurfaces in particularly ugly (or: random) toric varieties, ones whose only symmetries come from the torus action. In that case, one can hope to write down a normal form and get a global description of the moduli space. An example of such a normal form is in [CDLW07]. These authors describe a two-dimensional moduli space of lattice-polarized K3s, which are compactifications of the Inose family. These can in turn be described also as hypersurfaces in weighted projective space $W\mathbb{P}(5, 6, 22, 33)$.

More generally, it may be possible to describe the global geometry and Picard groups of the analogous moduli spaces for Borcea-Voisin CYs, using the fact that moduli spaces of complex structures of K3s and lattice-polarized K3s are locally homogeneous spaces. In particular, one should be able to do this with the family in [CDLW07] to get another three-dimensional Calabi-Yau moduli space.

A question closer to our actual results concerns the components of $M_{\overline{X}}$ and their Picard groups. We have seen that $M_{\overline{X}}$ has finitely generated Picard group; what about the non-central components? Likewise, is there a more direct way to compute $\mathrm{Pic}\,(M_{\overline{X}})$ that could obtain the 2 and 3 torsion subgroups?

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Appendix A. Summary of results from [DW09]. In this section, in Table 3, we summarize the results we need from [DW09], listing group actions and pertinent properties of crepant resolutions of the resulting quotient.

As in [DW09], a group action labelled (r-n) refers to an action of the group $(\mathbb{Z}/2)^{r+2}$, where $G_T \cong (\mathbb{Z}/2)^2$ and $G_S \cong (\mathbb{Z}/2)^r$. The number n merely indexes different actions of the same group. We take the periods of the i-th elliptic curve to be $1, \tau_i$. (In [DW09] the periods were doubled to $2, 2\tau_i$ in order to avoid halves in the half-periods. In our current notation, the half-periods are $1/2, \tau_i/2$.) A symbol such as $0\pm$, $1\pm$, or $\tau\pm$ denotes a reflection plus a translation by a half period on one elliptic factor. Explicitly, $0\pm$ indicates that the generator acts as

$$z \mapsto \pm z,$$

$1\pm$ indicates that the generator acts as

$$z \mapsto \pm z + 1/2,$$

and $\tau\pm$ indicates that the generator acts as

$$z \mapsto \pm z + \tau/2.$$

An element of the twist group G_T is denoted by a triple of such symbols (with an even number of negative signs.) The entries in the shift subgroup G_S are pure translations by (half of) the indicated amount; we drop the unneeded ±. For example, $(\tau, \tau, 0)$ indicates that the generator acts as

$$z_1 \mapsto z_1 + \tau/2, \quad z_2 \mapsto z_2 + \tau/2, \quad z_3 \mapsto z_3.$$

We also list the Hodge numbers of a crepant resolution, as well as the fundamental group π_1 of the same. Possible fundamental groups are denoted as follows, in the same notation as [DW09][Table 1]:

A: the extension of $\mathbb{Z}/2$ by $\mathbb{Z}^2$,
B: any extension of $(\mathbb{Z}/2)^2$ by $\mathbb{Z}^6$,
C: $\mathbb{Z}/2$,
D: $(\mathbb{Z}/2)^2$.

The available information is displayed in two tables in [DW09]. A complete list, including 36 types, is given in Table 1. It is still not known exactly which pairs from that list may coincide. The possible coincidences were summarized in a second table (on page 17 of [DW09]), listing all undistinguished cases that may or may not turn out to coincide: this list consisted of seven pairs and one triple of items from Table 1 of [DW09]. A subsequent work [FRTV13] used a computer search to carefully analyze all those equivalences of distinct entries in Table 1 of [DW09] that happen to be induced by affine linear transformations (twists and shifts) of the product of 3 tori. That work confirmed the completeness of the list in Table 1 of [DW09], and showed that precisely one pair of the previously undistinguished cases (items (3-1) vs (3-2)) coincided under such an equivalence. In the table below, we have therefore deleted entry (3-2). As far as we know, no progress has been made concerning the status of the other possible coincidences listed in the table on page 17 of [DW09]: we have no information regarding the possible existence of exotic isomorphisms that are not induced by twists and shifts of the product of tori. We are grateful to Patrick Vaudrevange for drawing our attention to the work [FRTV13] and explaining it to us.

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Label	G_T	G_S	$(h^{1,1},h^{2,1})$	π_1
(0-1)	$(0+,0-,0-),(0-,0+,0-)$		$(51,3)$	0
(0-2)	$(0+,0-,0-),(0-,0+,1-)$		$(19,19)$	0
(0-3)	$(0+,0-,0-),(0-,1+,1-)$		$(11,11)$	A
(0-4)	$(1+,0-,0-),(0-,1+,1-)$		$(3,3)$	B
(1-1)	$(0+,0-,0-),(0-,0+,0-)$	(τ,τ,τ)	$(27,3)$	C
(1-2)	$(0+,0-,0-),(0-,0+,\tau-)$	(τ,τ,τ)	$(15,15)$	0
(1-3)	$(0+,0-,0-),(0-,0+,1-)$	(τ,τ,τ)	$(11,11)$	C
(1-4)	$(0+,0-,0-),(0-,1+,1-)$	(τ,τ,τ)	$(7,7)$	A
(1-5)	$(1+,0-,0-),(0-,1+,1-)$	(τ,τ,τ)	$(3,3)$	B
(1-6)	$(0+,0-,0-),(0-,0+,0-)$	$(\tau,\tau,0)$	$(31,7)$	0
(1-7)	$(0+,0-,0-),(0-,0+,1-)$	$(\tau,\tau,0)$	$(11,11)$	C
(1-8)	$(0+,0-,0-),(0-,1+,0-)$	$(\tau,\tau,0)$	$(15,15)$	0
(1-9)	$(0+,0-,0-),(0-,1+,1-)$	$(\tau,\tau,0)$	$(7,7)$	A
(1-10)	$(1+,0-,0-),(0-,1+,0-)$	$(\tau,\tau,0)$	$(11,11)$	A
(1-11)	$(1+,0-,0-),(0-,1+,1-)$	$(\tau,\tau,0)$	$(3,3)$	B
(2-1)	$(0+,0-,0-),(0-,0+,0-)$	$(1,1,1),(\tau,\tau,\tau)$	$(15,3)$	D
(2-2)	$(0+,0-,0-),(0-,0+,1-)$	$(1,1,1),(\tau,\tau,\tau)$	$(9,9)$	C
(2-3)	$(0+,0-,0-),(0-,0+,0-)$	$(1,1,1),(\tau,\tau,0)$	$(17,5)$	C
(2-4)	$(0+,0-,0-),(0-,0+,1-)$	$(1,1,1),(\tau,\tau,0)$	$(11,11)$	0
(2-5)	$(0+,0-,0-),(0-,0+,\tau-)$	$(1,1,1),(\tau,\tau,0)$	$(7,7)$	D
(2-6)	$(0+,0-,0-),(0-,0+,0-)$	$(1,1,1),(\tau,1,0)$	$(19,7)$	0
(2-7)	$(0+,0-,0-),(0-,0+,\tau-)$	$(1,1,1),(\tau,1,0)$	$(9,9)$	C
(2-8)	$(0+,0-,0-),(0-,\tau+,\tau-)$	$(1,1,1),(\tau,1,0)$	$(5,5)$	A
(2-9)	$(0+,0-,0-),(0-,0+,0-)$	$(0,1,1),(1,0,1)$	$(27,3)$	0
(2-10)	$(0+,0-,0-),(0-,0+,\tau-)$	$(0,1,1),(1,0,1)$	$(11,11)$	0
(2-11)	$(0+,0-,0-),(0-,\tau+,\tau-)$	$(0,1,1),(1,0,1)$	$(7,7)$	A
(2-12)	$(\tau+,0-,0-),(0-,\tau+,\tau-)$	$(0,1,1),(1,0,1)$	$(3,3)$	B
(2-13)	$(0+,0-,0-),(0-,0+,0-)$	$(1,1,0),(\tau,\tau,0)$	$(21,9)$	0
(2-14)	$(0+,0-,0-),(0-,0+,1-)$	$(1,1,0),(\tau,\tau,0)$	$(7,7)$	D
(3-1)	$(0+,0-,0-),(0-,0+,0-)$	$(0,\tau,1),(\tau,1,0),(1,0,\tau)$	$(12,6)$	0
(3-3)	$(0+,0-,0-),(0-,0+,0-)$	$(1,1,0),(\tau,\tau,0),(1,\tau,1)$	$(17,5)$	0
(3-4)	$(0+,0-,0-),(0-,0+,\tau-)$	$(1,1,0),(\tau,\tau,0),(1,\tau,1)$	$(7,7)$	C
(3-5)	$(0+,0-,0-),(0-,0+,0-)$	$(0,1,1),(1,0,1),(\tau,\tau,\tau)$	$(15,3)$	C
(3-6)	$(0+,0-,0-),(0-,0+,\tau-)$	$(0,1,1),(1,0,1),(\tau,\tau,\tau)$	$(9,9)$	0
(4-1)	$(0+,0-,0-),(0-,0+,0-)$	$(0,\tau,1),(\tau,1,0),(1,0,\tau),(1,1,1)$	$(15,3)$	0

Table 3
Summary of Table 1 of [DW09].

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