DIFFERENTIAL COMPLEXES AND HODGE THEORY ON LOG-SYMPLECTIC MANIFOLDS*

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Abstract. We study certain complexes of differential forms, including 'reverse de Rham' complexes, on (real or complex) Poisson manifolds, especially holomorphic log-symplectic ones. We relate these to the degeneracy divisor and rank loci of the Poisson bivector. In some good holomorphic cases we compute the local cohomology of these complexes. In the Kählerian case, we deduce a relation between the multiplicity loci of the degeneracy divisor and the Hodge numbers of the manifold. We also show that vanishing of one of these Hodge numbers is related to unobstructed deformations of the normalized degeneracy divisor with its induced Poisson structure.

Introduction. One of the interesting features of geometry on (real or complex) Poisson manifolds (X, Π) is the richness of the calculus, which in a sense is twice as rich as on a plain manifold: the usual plus a dual. Interesting differential operators can be constructed using the Poisson bivector Π. One of these is the Koszul-Brylinski operator on differential forms:

$$\partial = d\iota_\Pi - \iota_\Pi d$$

where d is exterior derivative and ι_Π denotes interior multiplication by Π. This is an operator of degree (-1) on differential forms, and Brylinski [1] has shown that it has square zero, hence gives rise to a 'reverse de Rham' complex:

$$... \xrightarrow{\partial} \Omega_X^i \xrightarrow{\partial} \Omega_X^{i-1} \xrightarrow{\partial} ...$$

He has also shown, using the Hodge $*$ operator in the real C^∞ category, that when Π is a symplectic Poisson structure, i.e. everywhere nondegenerate, the reverse de Rham complex is equivalent to the usual de Rham complex, hence computes the cohomology $H^\bullet(X, \mathbb{R})$.

Here we start with the observation that a different set of operators of degree (-1), namely

$$\delta_i = i d\iota_\Pi - (i-1)\iota_\Pi d : \Omega_X^{n+i} \to \Omega_X^{n+i-1}$$

($n \in \mathbb{N}$ fixed, usually as half the dimension of X), can be used to construct a reverse De Rham complex $\Theta^\bullet = \Omega_X^{\dim(X)-\bullet}$ of differential forms called the 'Mahr de Poisson' or MdP complex, in either the C^∞ or holomorphic category (or for that matter, in any setting where d and ι_Π make sense). More generally, for any $\lambda \in \mathbb{C}$, there is a complex $\Theta_\lambda^\bullet$ with differential

$$\delta_{\lambda,j} = (j+\lambda)d\iota_\Pi - (j+\lambda-1)\iota_\Pi d : \Omega^j \to \Omega^{j-1}.$$

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Note that unlike the De Rham complex, the MdP complex need not be acyclic locally where Π degenerates, indeed its local cohomology seems difficult to compute in general. Some special cases will be computed below.

A special feature of the MdP complex $\Theta_X^\bullet$, on a $2n$-dimensional (C^∞ or holomorphic) Poisson manifold (X, Π), not shared by Brylinski's complex and which makes $\Theta_X^\bullet$ amenable to study, is the existence of 'bonding' maps relating it to the de Rham complex:

$$\pi : \Theta_{X, \leq n}^\bullet \to \Omega_{X, \leq n}^\bullet,$$

$$\pi' : \Omega_{X, \geq n}^\bullet \to \Theta_{X, \geq n}^\bullet.$$

It is also possible to construct a pair of hybrid complexes on the top or bottom half of the de Rham groups:

$$\mathcal{ED}^\bullet : \Omega_X^{2n} \xrightarrow{\delta_n} \Omega_X^{2n-1} ... \Omega_X^{n+i} \xrightarrow{\delta_i} \Omega_X^{n+i-1} ... \to \Omega_X^{n+1} \xrightarrow{\delta_1} \Omega_X^n \xrightarrow{d} \Omega_X^{n+1} ... \xrightarrow{d} \Omega_X^{2n},$$

$$\mathcal{DE}^\bullet : \mathcal{O}_X \xrightarrow{d} \Omega_X^1 ... \xrightarrow{d} \Omega_X^n \xrightarrow{\delta_0} \Omega_X^{n-1} ... \Omega_X^{n-i} \xrightarrow{\delta_{-i}} \Omega_X^{n-i-1} ... \to \Omega_X^1 \xrightarrow{\delta_{-n+1}} \mathcal{O}_X,$$

together with a map of complexes

$$\pi : \mathcal{ED}^\bullet \to \mathcal{DE}^\bullet.$$

The mapping cone of π may be thought of as a 'double helix' with strands $\mathcal{ED}^\bullet$ and $\mathcal{DE}^\bullet$ or $\Theta^\bullet$ and $\Omega^\bullet$ [1].

In the case where Π is pseudo-symplectic, i.e. nondegenerate almost everywhere, hence degenerates along a Pfaffian divisor P, these complexes are closely related to a (singular) codimension-1 'kernel foliation' on P (also called 'symplectic foliation" in the literature). In general, this kernel foliation is not 'tame' in the sense that the leaves are Zariski-locally closed (see Example 5.2 below). In fact, leaves can be dense in P.

When X is a compact Kähler manifold, the cohomology of the MdP complex admits a Hodge decomposition like its De Rham analogue. Indeed the 'Hodge diamond' for $\Theta^\bullet$ is just a 90° turn of the usual.

We will concentrate mainly on the case where Π is log-symplectic, i.e. the degeneracy divisor $D = D(\Pi)$ has normal crossings. In that case Π corresponds to a log-symplectic form Φ, i.e. a closed log 2-form whose polar locus coincides with D. For certain purposes it is easier to work directly with Φ rather than Π. In the case where the log-symplectic structure Π satisfies a certain 'residual generality' condition (see §3.5), we will study the image of π via the corresponding log-symplectic form Φ and consequently we will be able to determine the image of π via a simplicial resolution, and hence determine the local cohomology of $\Theta^{n]}$, i.e. the 'upper half' of $\Theta^\bullet$. Curiously, the method does not seem to adapt easily to the case of the lower half $\Theta^{n]}$.

For other work on De Rham-like complexes and degeneracy of log-symplectic Poisson structures, see [11], [8], [7], and [10]. In particular, Polishchuk [11] constructs and analyzes a different differential complex on a Poisson manifold with normal crossings Pfaffian divisor.

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[1] The bonds between the two strands of the DNA molecule are called π bonds

Notations and conventions. We work over $\mathbb{C}$. For a natural number k, $\mathbf{k}$ denotes $\{1, ..., k\}$. For a multi-index $I = (i_1 < ... < i_r)$, $|I|$ denotes the degree, i.e. r. Note the difference between "i", used for indices, and "ι", used for inclusion maps.

1. Preliminaries: twisted log complexes. In this section we study some differential complexes attached to a general log pair (X, D), i.e. a complex manifold endowed with a reduced, locally normal-crossing divisor. No Poisson, symplectic or log-symplectic structure is assumed.

1.1. Minor log complex and compactly supported cohomology. Here we study certain twists of the log complex on a log pair (X, D). Let X be a complex manifold of dimension d endowed with a divisor D with local normal crossings. We remark that in our subsequent application, X will be X_1, normalization of the degeneracy divisor $D(\Pi)$ of of a log symplectic manifold (X, Π), and D will be the double point locus of the map $X_1 \to D(\Pi) \subset X$. ⚠ This will result in a shift of indices!!

Via the inclusion $\Omega^\bullet_X \langle \log D \rangle \subset \Omega^\bullet_X(D)$, we get a graded subgroup

$$\Omega^\bullet_X \langle \log{}^- D \rangle := \Omega^\bullet_X \langle \log D \rangle (-D) \subset \Omega^\bullet_X. \tag{1}$$

Locally, letting $F = x_1...x_k$ be an equation for D, $\Omega^\bullet_X \langle \log{}^- D \rangle$ is generated by differentials of the following form, in which $\mathbf{k}$ denotes $\{1, ..., k\}$ and $J = (j_1 < ... < j_r)$ is a multi-index:

$$\omega_{J,\mathbf{k}} = \prod_{j \in \mathbf{k} \setminus J} x_j \prod_{j \in J} dx_j, \forall J \subset \mathbf{k}, k \le d.$$

It is clear from this, or otherwise, that $(\Omega^\bullet_X \langle \log{}^- D \rangle, d, \wedge)$ is a dg algebra over $\Omega^\bullet_X \langle \log D \rangle$, called the *log-minus* or *minor log* complex associated to D. Given the equation F as above, $\Omega^\bullet_X \langle \log{}^- D \rangle$ can be identified with $\Omega^\bullet_X \langle \log D \rangle^\circ$ which is $\Omega^\bullet_X \langle \log D \rangle$ with twisted differential

$$d^\circ = d + \mathrm{dlog}(F).$$

LEMMA 1.1. *The log-minus complex $\Omega^\bullet_X \langle \log{}^- D \rangle$ is a resolution of the compact-support direct image $\mathbb{C}_{U!} := i_{U!}(\mathbb{C}_U)$ where $i_U : U \to X$ is the inclusion of $U = X \setminus D$.*

Proof. There is a natural map $\mathbb{C}_{U!} \to \mathcal{O}(-D)$ which lifts to a map $\mathbb{C}_{U!} \to \Omega^\bullet_X \langle \log{}^- D \rangle$ and the latter is clearly a quasi-isomorphism over U, so it suffices to prove that $\Omega^\bullet_X \langle \log D \rangle^\circ$ is exact locally at every point of D. To simplify notations we assume D is of maximal multiplicity $k = d$ at the given point; the general case is a product of a maximal-multiplicity case and a zero-multiplicity case, and one can use a Künneth decomposition. Then all the sheaves in the log complex decompose into homogeneous components $S^i_{(m.)}$ indexed by exterior degree i and multi-indices $(m.)$ where $m_i \ge 0$. That is, each local section is an infinite convergent sum of homogeneous components. Note that x_i and dx_i both have degree 1. For any multi-index $(m_1, ..., m_k)$, we set

$$\chi_{(m.)} = \sum m_i dx_i / x_i.$$

Now note that d° maps $S^i_{(m.)}$ to $S^{i+1}_{(m.)}$ and there, in fact, is given by multiplication by $\chi_{(m.)} + \chi_{(1.)}$ where $(1.) = (1, ..., 1)$. Because the latter form is part of a basis of

$S^1_{(0.)} \subset \Omega^\bullet_X \langle \log D \rangle$, multiplication by it clearly defines an exact complex and in fact admits a 'homotopy' operator ι_v given by interior multiplication by the log vector field

$$v = \sum x_i \, \partial_{x_i} \, .$$

This has the property that that the commutator $[\iota_v, d^\circ]$ is a nonzero multiple of the identity on each $S^i_{(m.)}$ term. Therefore $\Omega^\bullet_X \langle \log D \rangle^\circ$ is null-homotopic and exact. $\square$

REMARK 1.2. When D is the exceptional divisor of a resolution of singularities, the minor log complex seems related to the Du Bois complex of the singularity, see [15].

1.2. Augmented minor log complex. We shall need an enlargement of the log-minus complex along the double locus of D, called the *augmented minor* or *log-minus-plus* complex. Let

$$\nu_i : X_i \to D \subset X$$

be the normalization of the i-fold locus of D, and let $D_i \subset X_i$ be the natural normal-crossing divisor on X_i, which maps to the $(i+1)$-fold locus of D. Also set $U_i = X_i \backslash D_i$. This maps to the set of points of multiplicity exactly i on D. Note the natural surjective pullback map for all $i \geq 0$, where $X_0 = X$ etc.

$$\nu_i^* : \Omega^\bullet_{X_i} \to \nu_{i+1 *} \Omega^\bullet_{X_{i+1}}$$

whose kernel is just $\Omega^\bullet_{X_i} \langle \log {}^- D_i \rangle$. We denote by $Z^{[m}$ the truncation of a complex $Z^\bullet$ below degree m (thus $Z^i = 0, i < m$). Set

$$K_0 = \Omega^\bullet_X \langle \log {}^- D \rangle,$$
$$K_1 = (\nu_1^*)^{-1}(\Omega^{[1}_{X_1} \langle \log D_1 \rangle (-D_1)).$$

Thus, K_1 is a subcomplex of $\Omega^\bullet_X$ which coincides with K_0 off X_1 and which, locally at a point of X_1 with branch equation x_k, where D has equation $F = x_1 ... x_k$, is generated by $\Omega^\bullet_X \langle \log {}^- D \rangle$ and by differentials of degree 1 or more, of the form

$$\omega_{I,\ell,\mathbf{k}} = \prod_{i \in \mathbf{k} \backslash I, i \neq \ell} x_i \prod_{i \in I} dx_i = \mathrm{dlog}\,(x)_I F / x_\ell, I \subset \mathbf{k} \backslash \{\ell\}.$$

In the general case, assuming K_i is constructed, we construct K_{i+1} by modifying K_i along X_{i+1} for forms of degree $i+1$ or more, i.e.

$$K_{i+1} = (\nu_{i+1}^*)^{-1}(\Omega^{[i+1}_{X_{i+1}} \langle \log D_{i+1} \rangle (-D_{i+1})).$$

Finally set

$$\Omega^\bullet_X \langle \log {}^\mp D \rangle = K_{d-2}. \tag{2}$$

(note that $K_{d-2} = K_{d-1}$ because $\Omega^\bullet_{X_{d-1}} \langle \log D_{d-1} \rangle (-D_{d_1}) = \Omega^\bullet_{X_{d-1}}$). By construction, $\Omega^\bullet_X \langle \log {}^\mp D \rangle$ is a $\Omega^\bullet_X \langle \log D \rangle$- module endowed with an increasing filtration $F_\bullet$ with graded pieces

$$Gr^{F_\bullet}_i(\Omega^\bullet_X \langle \log {}^\mp D \rangle) = \Omega^{[i}_{X_i} \langle \log {}^- D_i \rangle.$$

Locally at a point of X_r with branch equations $x_j, j \in J, |J| = r$, where D has equation $x_{\mathbf{k}}$, $\Omega_X^\bullet \langle \log^\mp D \rangle$ is generated over $\Omega_X^\bullet \langle \log D \rangle$ by differentials of the form

$$\omega_{I,J,\mathbf{k}} = \mathrm{dlog}(x)_I F / x_J, I \subset \mathbf{k} \setminus J \tag{3}$$

as well as differentials on X_{r-1} whose pullback on X_r is of such form.

As in Lemma 1.1, we can compute the cohomology of the augmented minor log complex:

LEMMA 1.3. *We have*

$$\mathcal{H}^i(\Omega_X^\bullet \langle \log^\mp D \rangle) = \hat{\Omega}^i_{U_i!} := i_{U_i!}(\hat{\Omega}^i_{X_i}), i \geq 0. \tag{4}$$

where $\hat{\Omega}$ denoted closed forms.

Proof. Use the spectral sequence of the filtered complex with $E_1^{p,q} = H^{p+q}(Gr_{-p})$, together with Lemma 1.1. The fact that each ith graded piece has cohomology only in degree i ensures that the spectral sequence degenerates at E_1. $\square$

As a slight generalization of the log-minusplus complex, we have for any $s \geq 0$ a complex $\Omega_X^\bullet \langle \log^{\mp s} D \rangle$ defined as above but with

$$K_{i+1} = (\nu_{i+1}^*)^{-1}(\Omega^{[i+1-s]}_{X_{i+1}} \langle \log D_{i+1} \rangle (-D_{i+1})), i+1 \geq s.$$

We will need this only for $s = 1$ which yields a complex with zeroth term $\mathcal{O}_X(-\nu_1(D_1))$ (recall that D_1 is a divisor on X_1 which maps to a codimension-2 locus on X). Note $\Omega_X^\bullet \langle \log^{\mp s} D \rangle$ admits an increasing filtration with graded pieces $\Omega_{X_i}^\bullet \langle \log^\mp D_i \rangle, i = 0, ..., s$.

1.3. Foliated De Rham complex, log version. With (X, D) a log pair as above, let ψ be a closed log 1-form, nowhere vanishing as such. Then ψ generates an $\Omega_X^\bullet \langle \log D \rangle$-submodule

$$\Omega_\psi^\bullet = \psi \Omega_X^\bullet \langle \log^- D \rangle \subset \Omega_X^\bullet \langle \log^- D \rangle [1]. \tag{5}$$

This is locally the $\Omega_X^\bullet \langle \log D \rangle$-submodule of $\Omega_X^\bullet$ generated by ψF. Thus, sections of $\Omega_\psi^\bullet$ are of the form $\psi F \gamma$ where γ is a log form. Then

$$\Omega_{X/\psi}^\bullet := \Omega_X^\bullet / \Omega_\psi^\bullet [-1] \tag{6}$$

is called the *foliated De Rham complex* associated to ψ. The differential on $\Omega_\psi^\bullet$ is given by

$$d(\psi F \alpha) = \psi F(d\alpha + \mathrm{dlog}(F)\alpha).$$

Consequently, $\Omega_\psi^\bullet$ is a quotient of $\Omega_X^\bullet \langle \log^- D \rangle^{d-1]}$ where $d = \dim(X)$ and $\bullet^{d-1]}$ means truncation in degrees $> d-1$. Locally, choosing an equation F for D, we may identify $\Omega_X^\bullet \langle \log^- D \rangle$ as above with the complex denoted $\Omega_X^\bullet \langle \log D \rangle^\circ$ which is $\Omega_X^\bullet \langle \log D \rangle$ with differential

$$d^\circ = d + \mathrm{dlog}(F).$$

This is defined locally, depending on the choice of local equation F. The kernel of the natural surjection $\Omega_X^\bullet \langle \log^- D \rangle^{d-1]} \to \Omega_\psi^\bullet$ consists of the forms divisible by ψ, hence can be identified with $\Omega_\psi^\bullet [-1]^{d-1]}$.

Continuing in this manner, $\Omega_\psi^\bullet$ admits a left resolution of the form

$$\Omega_X^\bullet \langle \log{}^- D \rangle [-d+1]^{d-1]} \to \ldots\ldots \to \Omega_X^\bullet \langle \log{}^- D \rangle [-1]^{d-1]} \to \Omega_X^\bullet \langle \log{}^- D \rangle^{d-1]} \to \Omega_\psi^\bullet. \quad (7)$$

Note that $*[-i]^{d]} = (*^{d-i})[-i]$. Set

$$K^i = \ker(d^\circ, \Omega_X^i \langle \log D \rangle), i \geq 0.$$

By Lemma 1.1,

$$K^i \simeq \Omega_X^{i-1} \langle \log D \rangle / K^{i-1}, i \geq 1.$$

Locally at a point of D, the latter is true for $i = 0$ as well, in the sense that $K^0 = 0$ while locally at a point of U, $K^0 = \mathbb{C}$. Moreover K^i is the unique nonvanishing cohomology sheaf of $\Omega_X^\bullet \langle \log D \rangle [-i]^{d]}$, and occurs as $\mathcal{H}^d$. We now introduce the following generality hypothesis on our form ψ:

(*) *For each nonnegative integral multi-index* $(m.)$, *the log differentials* ψ *and* $\chi_{(m.)} + \chi_{(1.)}$ *are linearly independent, i.e. generate a free and cofree subsheaf of* $\Omega_X^1 \langle \log D \rangle$, *locally at every point of multiplicity 2 or more on* D.

When ψ is one of the forms ψ_i deduced from a log-symplectic structure, hypothesis (*) is equivalent to the '1-very general' hypothesis introduced in [14], Erratum, hence weaker than the Residual Generality condition in §3.5.

It is essentially clear that a general log 1-form cannot be holomorphically integrated and in 2 or more variables, is not even proportional to an integrable form. Our aim next is to generalize this observation.

Let let $\mathcal{O}_X^\psi$ denote the sheaf of ψ-constant holomorphic functions, i.e. holomorphic functions g such that $dg \wedge \psi = 0$. Locally at $p \in U$ we can write $\psi = dx, \mathcal{O}_p^\psi = \mathbb{C}\{x\}$ for a coordinate x. Similarly, locally over U_1, the smooth part of D, we can write $\psi = dx/x, \mathcal{O}_p^\psi = \mathbb{C}\{x\}$. Also let $U_j \subset X$ denote the set of points where D has multiplicity $\leq j$. More generally, we let $U_{i,j} \subset X_i$ denote the set of points where D_i has multiplicity $\leq j - i$, i.e. the inverse image of the set of point in X where the multiplicity of D is in $[i,j]$. Thus, $U_j = U_{0,j}$.

LEMMA 1.4. *Under hypothesis (*) above, we have*

$$\mathcal{H}^i(\Omega_\psi^\bullet) = \begin{cases} i_{U_1!}(\mathcal{O}^\psi(-D)\psi), i = 0; \\ 0, i > 0. \end{cases} \quad (8)$$

Proof. To begin with, the RHS of (8) clearly maps naturally to the LHS, so it suffices to prove that this map is an isomorphism locally at each point.

Consider first the elementary case of a point $p \in U$. There the quotient complex $\Omega_{X/\psi}^\bullet$ is the usual relative De Rham for the foliation determined by ψ, which is a resolution of $\mathcal{O}_X^\psi$. Then the cohomology sequence of (11) reduces to

$$0 \to \mathbb{C}_X \to \mathcal{O}^\psi \to \mathcal{O}_X.\psi \to 0,$$

the second map being exterior derivative, so we get the result. The case $p \in U_1$ is similar, because there we may assume $\psi = dx/x, F = x$ so $F\psi = dx$ like before.

Now we may assume $p \in D_1$, double locus of D, and show $\Omega_\psi^\bullet$ is exact. We will use hypothesis (*), which says that ψ and $\chi := \chi_{(m.)} + \chi_{(1.)}$ are linearly independent

at $p, \forall(m.)$. For simplicity we assume p is a point of maximum multiplicity, i.e. d, on D. The closed log 1-form ψ can be written in the form

$$\psi = \sum a_i \, d\log(x_i) + dg$$

with a_i constant and g holomorphic. Then replacing x_1 by $\exp(g/a_1)x_1$, we may assume $g = 0$ In particular, ψ is homogeneous of degree 0.

Consider the E_1 'termwise-to-total' spectral sequence associated to the resolution (7). Each resolving term only has $\mathcal{H}^{d-1}$ and that is given by the appropriate K^i. Therefore the entire E_1 page reduces to the following complex (occurring at height $d-1$ in the second quadrant)

$$K^1 \to K^2 \to ... \to K^d \to 0 \tag{9}$$

where the maps are multiplication by ψ. We claim that the larger complex

$$0 \to K^0 \to K^1 \to ... \to K^d \to 0$$

is exact. Now working on a given homogeneous component $S^\bullet_{(m.)}$, d° itself is multiplication by $\chi_{(m.)} + \chi_{(0.)}$ By Assumption (*), the latter section together with ψ forms part of a basis of $S^1_{(0.)}$. Therefore clearly multiplication by ψ, which preserves multidegree, is exact on the kernel (= image) of multiplication by $\chi_{(m.)} + \chi_{(0.)}$, i.e. $K^\cdot$. Therefore the larger complex extending (9) is exact.

Thus, the $E_2 = E_\infty$ page for the complex (9) just reduces to the K^0, sitting in bidegree $(-d, d)$, which yields our claim. $\square$

As an immediate consequence of Lemma 1.4, we conclude

COROLLARY 1.5. *Under hypothesis (*), we have*

$$\mathcal{H}^i(\Omega^\bullet_{X/\psi}) = \mathcal{H}^i(\Omega^\bullet_\psi), i > 0,$$

and there is an exact sequence

$$0 \to \mathbb{C}_X \to \mathcal{H}^0(\Omega^\bullet_{X/\psi}) \to i_{U_1!}(\mathcal{O}^\psi(-D)\psi) \to 0. \tag{10}$$

Proof. Using the long cohomology sequence of

$$0 \to \Omega^\bullet_\psi[-1] \to \Omega^\bullet_X \to \Omega^\bullet_{X/\psi} \to 0, \tag{11}$$

the assertion follows from Lemma 1.4. $\square$

We will require a generalization of Lemma 1.4 to a k-tuple of forms. Thus, with notations as above, let $\psi_1, ..., \psi_k$ be sufficiently general closed log 1-forms on X and set

$$\psi_{\underline{k}} = \psi_1...\psi_k, \Omega^\bullet_{\psi_{\underline{k}}} = \psi_{\underline{k}}\Omega^\bullet_X\langle\log{}^\mp D\rangle.$$

Let $\mathcal{O}^{\psi_{\underline{k}}}$ be the sheaf of holomorphic functions f such that $df \equiv 0 \mod \psi_{\underline{k}}$ (i.e. such that df is in the $\mathcal{O}_X$-module generated by $\psi_1, ..., \psi_k$).

LEMMA 1.6. *Notations as above, we have*

$$\mathcal{H}^i(\Omega^\bullet_{\psi_{\underline{k}}}) = \begin{cases} i_{U_k!}(\mathcal{O}^{\psi_{\underline{k}}}(-D)\psi_{\underline{k}}), i = 0 \\ 0, i > 0. \end{cases} \tag{12}$$

Proof. The proof is by induction on k. The induction step is analogous to the proof of Lemma 1.4, with $\Omega^\bullet_{\psi_{k-1}}$ replacing $\Omega^\bullet_X\langle\log{}^-D\rangle$. $\square$

We shall also need an analogue of Lemma 1.4 for the minusplus complex. Thus set (cf. §1.2)

$$\Omega^\bullet_{\psi+} = \psi\Omega^\bullet_X\langle\log{}^\mp D\rangle.$$

This is a complex that starts with $\psi\mathcal{O}_X(-D)$ in degree 0. Set

$$\Omega^{i,\psi} = \{\alpha \in \Omega^i_{X_i}\langle\log D_i\rangle : d\alpha \equiv 0 \mod \psi\}/\psi\Omega^{i-1}_X\langle\log D\rangle.$$

Thus by definition, $\psi\Omega^{i,\psi}$ consists of the closed forms in $\psi\Omega^i_{X_i}\langle\log D_i\rangle$. Similarly with ψ replaced by $\psi_{\underline{k}}$. These complexes admit a natural increasing filtration with quotients $\Omega^{[k}_{X_k,\psi_{\underline{k}}}$.

LEMMA 1.7. *Hypotheses as above, we have*

$$\mathcal{H}^i(\Omega^\bullet_{\psi+}) = i_{U_{i,i+1}!}(\psi\Omega^{i,\psi}_{X_i}(-D_i)) \tag{13}$$

and more generally

$$\mathcal{H}^i(\Omega^\bullet_{\psi_{\underline{k}}+}) = i_{U_{i,i+k}!}(\psi\Omega^{i,\psi_{\underline{k}}}_{X_i}(-D_i)) \tag{14}$$

Proof. Follows from Lemma 1.4 and Lemma 1.6, using the spectral sequence for a filtered complex, which degenerates at E_2 for support reasons. $\square$

1.4. Simplicial De Rham complex on normal crossing varieties. Let D be a variety with local normal crossings. Thus, D is locally embeddable as a divisor in a manifold X with defining equation $F = x_1...x_m$, where $x_1, ..., x_m$ are part of a local coordinate system (we call these 'adapted' coordinates). Let X_k be the normalization of the k-fold locus of D, with (unramified) natural map

$$p_k : X_k \to D.$$

Thus, a point in X_k is specified by a k-tuple $I \subset \mathbf{m}$ plus a point where $x_i = 0, \forall i \in I$, and X_k is a transverse union of smooth branches X_I corresponding to choices of I. So X_1 is just the normalization of D and X_k generally is the normalization of the k-fold locus of D. Note that there is a natural map

$$\rho_k : p_{k*}\Omega^\bullet_{X_k} \to p_{k+1*}\Omega^\bullet_{X_{k+1}}$$

defined by, for any $(k+1)$-tuple $I = (i_1 < ... < i_{k+1})$,

$$\rho(\omega)_I = \sum_{j=1}^{k+1}(-1)^j\omega_{I\setminus i_j}$$

where ω_J is the restriction of ω on the branch-intersection $X_J := (x_j : j \in J)$. It is easy to check that this is a morphism of complexes and that $\rho_{k+1} \circ \rho_k = 0$ so we get a double complex $(\Omega^\bullet_{X_\bullet}, d, \rho)$, which we will call the *simplicial De Rham complex* associated to D or *simplicial De Rham resolution of* $\Omega^\bullet_D$ (see below).

On the other hand, recall that we have a complex- actually dg algebra, namely $\Omega^\bullet_D$, which is the quotient of $\Omega^\bullet_X$ by the exterior ideal generated by $\mathcal{O}(-D)$ and its

image by d in Ω^1_X, i.e. locally by $F = x_1...x_m$ and $dF = F \sum dx_i/x_i$. The following result is probably well known.

LEMMA 1.8. (i) $\Omega^\bullet_{X_\bullet}$ *is a resolution of* $\Omega^\bullet_D$.
(ii) *There is an exact sequence*

$$0 \to \Omega^\bullet_X \langle \log {}^- D \rangle \to \Omega^\bullet_X \to \Omega^\bullet_D \to 0. \tag{15}$$

Proof. (i) To begin with, there is clearly a map $\Omega^\bullet_D \to \Omega^\bullet_{X_\bullet}$ and it is easy to check locally that this map induces an isomorphism $\Omega^\bullet_D \to \ker(\rho_0)$. It remains to prove that $\ker(\rho_{k+1}) = \mathrm{im}(\rho_k)$ which can also be done locally, so we can choose a local basis. We may assume each constituent ω_I is extended over D_i for all $i \in I$ via some compatible collection of deformation retractions $X_i \to X_I$. Then the required exactness follows by using the following homotopy operator

$$h_k : p_{k*}\Omega^\bullet_{X_k} \to p_{k-1*}\Omega^\bullet_{X_{k-1}}$$

$$(h_k(\omega))_J := \sum_{i \notin J} \mathrm{sgn}(J|i)\omega_{J\cup i},$$

where

$$\mathrm{sgn}(J|i) = (-1)^{|\{r:j_r>i\}|}.$$

(ii) It is easy to check locally that the image of the pullback map

$$\Omega^i_X \to p_{1*}\Omega^i_{X_1}$$

coincides with the kernel of ρ_1, and then that the kernel of the same pullback map coincides with $\Omega^i_X \langle \log {}^- D \rangle$. □

2. A double helix. In what follows we will fix a manifold X of even dimension $2n$ endowed with a Poisson structure Π. Our main interest is in the case where (X, Π) is holomorphic, i.e. X is a complex manifold (of complex dimension $2n$), and Π is holomorphic, and especially where Π is pseudo-symplectic in the sense that on some dense open subset of X, Π is nondegenerate, i.e. dual to a symplectic structure. However, some of the basic constructions apply without the pseudo- symplectic condition and in the real C^∞ case as well.

Brylinski [1] constructed on the sheaves of differential forms on X the structure of a 'reverse de Rham' complex

$$...\Omega^i_X \xrightarrow{\partial} \Omega^{i-1}_X \to ..., \quad \partial = d\iota_\Pi - \iota_\Pi d,$$

where ι_Π denotes interior multiplication by Π (which lowers degree by 2) and d is the usual exterior derivative. In fact, his construction is quite formal and is valid generally in the context of a Poisson structure on a ringed space X/B where B is a ringed space over $\mathbb{Q}$, interpreted as a linear map $\Omega^2_{X/B} \to \mathcal{O}_X$, and where ι_Π is the natural extension of the latter to a degree- (-2) map on $\Omega^\bullet_X$.

Our observation here is first that there exists a *different* reverse de Rham complex, which we call the 'Mahr de Poisson' (MdP) complex $\Theta^\bullet_X$ with differentials not

proportional to Brylinski's, valid in similar generality. An essential feature of the MdP complex not shared by the Brylinski complex is the existence of a 'bonding map'

$$\pi : \Theta_X^{n]} \to \Omega_X^{n]}$$

as well as a dual bonding map

$$\pi' : \Omega_X^{[n} \to \Theta_X^{[n}.$$

Based in this, we will define a pair of hybrid complexes $\mathcal{ED}^{\bullet}, \mathcal{DE}^{\bullet}$, each of type 'half de Rham, half twisted reverse de Rham'. We will then construct a map between these complexes and study its mapping cone, identifying it with an analogous complex built on the *foliated* de Rham/ twisted reverse de Rham complex associated to a 'degeneracy foliation' defined on the degeneracy or Pfaffian divisor of Π. 'Morally speaking', it is the existence of this foliation and its associated foliated de Rham complex, which is a quotient of the de Rham complex of X, that force our twisted reverse de Rham complex to exist, essentially as the kernel of the quotient map. See [2] or [4] for basic information on Poisson structures.

To begin with, define an operator

$$\delta : \Omega_X^{n+i} \to \Omega_X^{n+i-1}, i \in [-n, n],$$
$$\delta = id\iota_\Pi - (i-1)\iota_\Pi d. \tag{16}$$

[To simplify the notation we will sometimes suppress the interior multiplication symbol and simply write this operator as $id\Pi - (i-1)\Pi d$. We will also denote $\iota_\Pi(\omega)$ by $\langle \Pi, \omega \rangle$]. Here n is of course half the dimension of X if X is a Poisson manifold of dimension $2n$, or just an arbitrary natural number if X/B is an arbitrary Poisson ringed space (in which case the construction will of course depend on n). Note that this differential is not proportional to Brylinski's. Then define sheaves $\mathcal{ED}^i, \mathcal{DE}^i, \Theta^i$ by

$$\Theta^i = \Omega_X^{2n-i}, i \in [0, 2n],$$
$$\mathcal{ED}^i = \begin{cases} \Omega_X^{n-i}, i \in [-n, 0] \\ \Omega_X^{n+i}, i \in [1, n], \end{cases} \tag{17}$$
$$\mathcal{DE}^i = \begin{cases} \Omega_X^{n+i}, i \in [-n, 0] \\ \Omega^{n-i}, i \in [1, n]. \end{cases}$$

Note that the maps defined by interior multiplication

$$\iota_{\Pi^k} : \Omega^{n+k} \to \Omega^{n-k} \tag{18}$$

yield for each $i \in [0, n]$ a map

$$\pi : \Theta^i \to \Omega^i \tag{19}$$

and for each $i \in [n, 2n]$ a map

$$\pi' : \Omega^i \to \Theta^i \tag{20}$$

and also for each $i \in [-n, n]$ a map

$$\pi = \iota_{\Pi^{|i|}} : \mathcal{ED}^i \to \mathcal{DE}^i. \tag{21}$$

THEOREM 2.1. *Let Π be a Poisson structure on a ringed space X/B.*
(i) *Endowed with differential δ, $\Theta^\bullet$ is a complex.*
(ii) *Endowed with differential δ in negative degrees and d in nonnegative degrees, $\mathcal{DE}^\bullet$ is a complex.*
(iii) *Endowed with differential d in negative degrees and δ in nonnegative degrees, $\mathcal{ED}^\bullet$ is a complex.*
(iv) *The map π defined above yields morphisms of complexes*

$$\pi : \Theta^\bullet_{n]} \to \Omega^\bullet_{n]},$$
$$\pi' : \Omega^\bullet_{[n} \to \Theta^\bullet_{[n}, \tag{22}$$
$$\pi : \mathcal{ED}^\bullet \to \mathcal{DE}^\bullet$$

where $n], [n$ denote truncation above (resp. below) degree n, which are isomorphisms locally wherever Π is nondegenerate.

The top and bottom squares of π are, respectively:

$$\delta_n = nd\iota_\Pi - (n-1)\iota_\Pi d = nd\iota_\Pi \quad
\begin{array}{ccc}
\Omega^{2n}_X & \overset{\iota_\Pi^n}{\to} & \mathcal{O}_X \\
\downarrow & & \downarrow d \\
\Omega^{2n-1}_X & \overset{\iota_\Pi^{n-1}}{\to} & \Omega^1_X
\end{array}$$

$$\begin{array}{ccc}
\Omega^{2n-1}_X & \overset{\iota_\Pi^{n-1}}{\to} & \Omega^1_X \\
d \downarrow & & \downarrow \\
\Omega^{2n}_X & \overset{\iota_\Pi^n}{\to} & \mathcal{O}_X
\end{array} \quad \delta_{-n+1} = -(n-1)d\iota_\Pi + n\iota_\Pi d = n\iota_\Pi d \tag{23}$$

where $\delta_n = nd\iota_\Pi$ on Ω^{2n}_X and $\delta_{-n+1} = n\iota_\Pi d$ on Ω^1_X. The middle two squares are:

$$d\iota_\Pi \quad
\begin{array}{ccc}
\Omega^{n+1}_X & \overset{\iota_\Pi}{\to} & \Omega^{n-1}_X \\
\downarrow & & \downarrow d \\
\Omega^n_X & = & \Omega^n_X \\
\downarrow d & & \downarrow \\
\Omega^{n+1}_X & \overset{\iota_\Pi}{\to} & \Omega^{n-1}_X.
\end{array} \quad -\iota_\Pi d \tag{24}$$

We will call the mapping cone of π the *dihelical* (double helix) complex associated to Π.

COROLLARY 2.2. *Locally where Π is nondegenerate, $\mathcal{ED}^\bullet$ and $\mathcal{DE}^\bullet$ are exact in degrees $\neq -n$.*

Proof. Follows from exactness in positive degrees of the de Rham complex. □

COROLLARY 2.3. *Assume (X, Π) is holomorphic and Π is generically nondegenerate with Pfaffian divisor $D = [\Pi^n]$. Then $\Theta^\bullet_{n]}$ is isomorphic to the 'augmented twisted truncated Poisson complex'*

$$\mathcal{O}(-D) \to T_X(-D) \to T^2_X(-D)...$$

where the first map sends a function f to the corresponding Hamiltonian vector field $\langle df, \Pi \rangle$ and other maps are $[.,\Pi]$.

Proof. There is a map

$$\langle ., \Pi^{n-i} \rangle : \Omega^{n-i} \to T^{n-i}$$

coming from a morphism of complexes, such that the composition $\Omega^{n+i} \to T^{n-i}$ is interior multiplication by Π^n, which is F times a volume form where F is an equation of D. This composite yields the desired isomorphism of $\Theta_{n]}$ with the augmented twisted truncated Poisson complex (we use 'augmented' because the usual Poisson complex start with T_X and 'twisted' because the terms are (even though the differential is not). □

REMARK. The referee points out that the contraction map above is well known and appears, e.g. in Twisted Poincaré duality between Poisson homology and cohomology, Luo-Wang-Wu, J. Algebra 442 (2015), 484-505.

Thus, the complex $\Theta^{\bullet}$ is not really 'new' but its realization in terms of differential forms makes possible a useful connection with Hodge theory (see §7 below).

The proof of the Theorem uses the following Calculus lemma (for which which the integrability condition $[\Pi, \Pi] = 0$ is essential):

LEMMA 2.4. *We have*

$$d\iota_{\Pi^m} = \iota_{\Pi^{m-1}}(m d\iota_\Pi - (m-1)\iota_\Pi d) \tag{25}$$

$$\iota_{\Pi^m} d = (m\iota_\Pi - (m-1)d\iota_\Pi)\iota_{\Pi^{m-1}}. \tag{26}$$

Proof of Lemma. To prove (25) for $m = 2$ we can use a direct local computation. In a slightly more canonical vein, we may compute, for any differential form ϕ, by definition of Lie derivative

$$L_\Pi \langle \Pi, \phi \rangle = d\langle \Pi^2, \phi \rangle - \langle \Pi, d\langle \Pi, \phi \rangle \rangle.$$

On the other hand by the derivation property of Lie derivative and the fact that $L_\Pi \Pi = [\Pi, \Pi] = 0$, we have

$$L_\Pi \langle \Pi, \phi \rangle = \langle \Pi, L_\Pi \phi \rangle = \langle \Pi, \langle d\langle \Pi, \phi \rangle \rangle \rangle - \langle \Pi^2, d\phi \rangle.$$

Comparing the last two displayed equations yields (25) for $m = 2$, and the general case follows inductively. (26) is proved similarly. □

REMARK. Alternatively in the pseudo-symplectic case, the only case we need here, it suffices to prove the identities (25), (26) on the dense open set where Π is symplectic, which can be done by a simple local calculation.

Proof of Theorem. To prove that $\mathcal{E}\mathcal{D}^{\bullet}$ and $\mathcal{D}\mathcal{E}^{\bullet}$ are complexes, we start with the well-known relation (equivalent to vanishing of the square of the differential in Brylinski's complex):

$$(d\iota_\Pi - \iota_\Pi d)^2 = 0.$$

Expanding, and using $d^2 = 0$ and (25) for $m = 2$ yields

$$d\iota_\Pi d\iota_\Pi = \iota_\Pi d\iota_\Pi d.$$

Then a direct computation yields

$$((i-1)d\iota_\Pi - (i-2)\iota_\Pi d)(id\iota_\Pi - (i-1)\iota_\Pi d) =$$
$$i(i-1)(d\iota_\Pi d\iota_\Pi - \iota_\Pi d\iota_\Pi d) = 0$$

Thus, $\delta^2 = 0$. Together with $d^2 = 0$ and some trivial verifications around the midpoint $i = 0$, this suffices to show that $\Theta^\bullet, \mathcal{ED}^\bullet$ and $\mathcal{DE}^\bullet$ are complexes.

Finally, the proof the π is a morphism of complexes amounts to commutativity of suitable squares and translates exactly to (25) and (26). □

3. Log-symplectic manifolds.

3.1. log-symplectic form. A Poisson manifold (X, Π) of even dimension $2n$ such that the degeneracy divisor $D = D(\Pi) = [\Pi^n]$ has local normal crossings is said to be *log-symplectic*. The Poisson structure Π can equivalently be described via a 'log-symplectic form' Φ. This is the meromorphic (in fact, logarithmic) form defined by

$$\langle \Pi^n, \Phi \rangle = \Pi^{n-1}.$$

Note that

$$\langle \Pi^n, \Phi^i \rangle = \Pi^{n-i}.$$

Also, the maps on meromorphic forms

$$\Omega_{X,mero}^{2n-i} \overset{\langle ., \Pi^{n-i} \rangle}{\to} \Omega_{X,mero}^i, \Omega_{X,mero}^i \overset{.\wedge \Phi^{n-i}}{\to} \Omega_{X,mero}^{2n-i}$$

are inverse to each other. We can write

$$\Pi^n = FV, \Phi^n = F^{-1}V^*$$

where V, V^* are dual generators of T_X^{2n}, Ω_X^{2n} and F is an equation for D. Thus $\langle V^*, \Pi^{n-1} \rangle = F\Phi$ and for any $v \in \wedge^i T_X$ we have

$$\langle \langle V^*, v \rangle, \Pi^{n-i} \rangle = \langle \langle V^*, \Pi^{n-i} \rangle, v \rangle = F \langle \Phi^i, v \rangle, \tag{27}$$

thus the two maps

$$\langle ., \Pi^{n-i} \rangle : \Omega_X^{2n-i} \to \Omega_X^i, \quad F \langle ., \Phi^i \rangle : \wedge^i T_X \to \Omega_X^i \tag{28}$$

are essentially the same under the exterior duality identification

$$\Omega_X^{2n-i}(D) = \Omega_X^{2n-i} \otimes \wedge^{2n} T_X \simeq \wedge^i T_X$$

and in particular they have the same image.

3.2. Log duality. When Π is log symplectic with degeneracy divisor D, we have a 'log-duality' map

$$\langle \Pi, . \rangle : \Omega_X^1 \langle \log D \rangle \to T_X \langle -\log D \rangle.$$

This map is easily seen to be an isomorphism, with inverse $\langle \Phi, . \rangle$, Φ being the corresponding log-symplectic form (compare the proof of Proposition 3.1 below). Another useful map, also called log-duality, is defined as follows.

Consider the map

$$\pi = \langle ., \Pi^i \rangle : \Omega_X^{n+i} \to \Omega_X^{n-i}.$$

This clearly extends to a map, called the *log duality* map,

$$\pi \langle \log D \rangle = \langle ., \Pi^i \rangle : \Omega_X^{n+i} \langle \log D \rangle \to \Omega_X^{n-i} \langle \log D \rangle.$$

The following result was known before, see e.g. [14].

PROPOSITION 3.1. *If Π is log-symplectic, then $\pi \langle \log D \rangle$ is an isomorphism. Moreover, Π defines a nondegenerate alternating form on $\Omega_X^1 \langle \log D \rangle$.*

Proof. $\pi \langle \log D \rangle$ is clearly an isomorphism locally off D and at *smooth* points of D. Thus, $\pi \langle \log D \rangle$ is a morphism of locally free sheaves of the same rank on X, which is an isomorphism off a codimension-2 subset, viz.the singular locus of D. Therefore $\pi \langle \log D \rangle$ is an isomorphism. The proof of the last assertion is similar, based on the fact that the (Pfaffian) degeneracy locus on Π as alternating form on $\Omega_X^1 \langle \log D \rangle$ is of pure codimension 1, hence empty. $\square$

The remainder of this subsection extends duality to the case of the MdP complex $(\Theta^\bullet, \delta)$ defined in the last section and uses the notations of that section.

COROLLARY 3.2. *Define a 'coduality' map of complexes of degree $2n$ as follows*

$$\pi/\pi : (\Theta \langle \log D \rangle^\bullet, \delta) \to (\Omega^\bullet \langle \log P \rangle, d),$$
$$(\pi/\pi)^i = \iota_{\Pi^i} : \Omega^{n+i} \to \Omega^{n-i}, i = n, ..., -n \tag{29}$$

where δ is the MdP differential and for $i < 0$, ι_{Π^i} means the inverse of the isomorphism $\iota_{\Pi^{|i|}}$, i.e. $\iota_{\Phi^{|i|}}$. Then π/π is an isomorphism of complexes.

REMARK 3.3. When Π is *P*-normal, the Proposition can also be proved by a straightforward local computation, using the normal form. Namely, setting

$$d\log(x_I) = \bigwedge_{i \in I} d\log(x_i), dy_J = \bigwedge_{j \in J} dy_j,$$

and $(d\log(x_I) \wedge dy_J)\check{\ }$ denoting the corresponding complementary multi-vectors, we have

$$\iota_{\Pi^i}((d\log(x_I) \wedge dy_J)\check{\ }) = \pm dy_I \wedge d\log(x_J), |I| + |J| = i.$$

Thus,

$$\iota_{\Pi^i} : \Omega_X^{2n-i} \langle \log D \rangle \to \Omega_X^i \langle \log D \rangle$$

sends a basis to a basis, hence is an isomorphism. $\square$

As a consequence, we can write down local generators for the cohomology sheaves of $\Theta^\bullet \langle \log D \rangle$ for Π P-normal in terms of normal coordinates, cf. (45). Fix a point p of multiplicity k on P and a normal coordinate system (x_i, y_i) so that precisely $x_1, ..., x_k$ vanish at p. Set

$$d_i = \mathrm{dlog}(x_i)dy_i, i = 1, ..., k,$$
$$d_i = dx_i dy_i \text{ where } x_i \neq 0,$$
$$d_I = \bigwedge_{i \in I} d_i, \quad \mathrm{dlog}(x)_I = \bigwedge_{i \in I} \mathrm{dlog}(x_i), I \subset [1, k]. \tag{30}$$

We recall (cf. e.g. [3] or [6]) that the local cohomology of the usual log complex is well known by Deligne, Griffiths and others. It is generated over $\mathbb{C}$ by the $\mathrm{dlog}(x)_I$ for various multi-indices $I \subset [1, k]$. Note that the $x_i, i \in I$ are defining equations for a branch of $D^{|I|}$, denoted X_I, and we have

$$\mathcal{H}^i(\Omega_X\langle \log D\rangle) = \begin{cases} \nu_{i*}\mathbb{C}_{X_i}, i = 0, ..., n, \\ 0, i > n. \end{cases} \tag{31}$$

Applying the π/π isomorphism above, we conclude:

COROLLARY 3.4. *Notations as above,* Π *P-normal, we have*

$$\mathcal{H}^i(\Theta_X^\bullet\langle \log D\rangle) = \begin{cases} \nu_{i*}\mathbb{C}_{X_i}, i = 0, ..., n, \\ 0, i > n. \end{cases} \tag{32}$$

If Π *is P-normal and* $(x.)$ *are normal coordinates, then the cohomology admits local generators of the form*

$$d_I \, \mathrm{dlog}(x)_J, \ \ \forall I \coprod J = \{1, ..., k\}, \tag{33}$$

where the latter generator is supported on the local branch X_J *with equations* $x_j, j \in J$.

3.3. Standard form. We return to the case Π arbitrary log-symplectic. The $\pi\langle \log D\rangle$ isomorphism is useful in yielding a standard form for Π and the corresponding log-symplectic form Φ, as follows. Let $F = x_1...x_m$ be a local equation for D where $x_1, ..., x_{2n}$ are local coordinates. Set

$$v_i = \begin{cases} x_i \, \partial_{x_i}, 1 \le i \le m; \\ \partial_{x_i}, m + 1 \le i \le 2n. \end{cases}.$$

These form a local basis for the sheaf of log vector fields $T_X\langle -\log D\rangle$. Let v_i^* be the dual basis for $\Omega_X^1\langle \log D\rangle$ $(= dx_i/x_i$ or $dx_i)$. Then

$$\Pi = \sum a_{ij} v_i v_j, \tag{34}$$

$$\Phi = \sum b_{ij} v_i^* v_j^* \tag{35}$$

where $A = (a_{ij}), B = (b_{ij}) = A^{-1}$ are skew-symmetric and holomorphic. In fact, $B = \frac{1}{F} \wedge^{n-1} A$. These are the matrices of the isomorphism $\pi\langle \log D\rangle$ and its inverse.

3.4. log (co)normal bundle. Here Π is arbitrary log-symplectic.

3.4.1. First order. Notations as above, the natural map induced by inclusion

$$\nu_1^*(T_X\langle -\log D\rangle) \to \nu_1^*(T_X)$$

has an $\mathcal{O}_{X_1}$-invertible kernel, denoted $N_{\log(D)}$, called the *log normal bundle* associated to the normal-crossing divisor D. $N_{\log(D)}$ is dual to the cokernel of the inclusion

$$\nu_1^*(\Omega_X^1) \to \nu_1^*(\Omega_X^1\langle \log D\rangle),$$

hence via residue $N_{\log(D)}$ is globally free with local generator $x_1 \, \partial_{x_1}$ where x_1 is a branch equation for D. We have exact sequences

$$0 \to N_{\log(D)} \to \nu_1^*(T_X \langle -\log D \rangle) \to T_{X_1} \langle -\log D_1 \rangle \to 0,$$
$$0 \to T_{X_1} \langle -\log D_1 \rangle \to \nu_1^*(T_X) \to N_{X_1/X} \to 0. \qquad (36)$$

When D is the polar divisor of a log-symplectic form Φ we denote by $\check{N}_{\log(D)}$ the image of $N_{\log(D)}$ by the log duality map $\pi \langle \log D \rangle$. This is a priori a a line subbundle of $\nu_1^*(\Omega_X^1 \langle \log (\rangle D))$, but in the exact residue sequence

$$0 \to \Omega_{X_1}^1 \langle \log D_1 \rangle \to \nu_1^*(\Omega_X^1 \langle \log D \rangle) \to \mathcal{O}_{X_1} \to 0,$$

clearly the residue map, which is given by interior multiplication by $v_1 = x_1 \, \partial_{x_1}$, is zero on $\check{N}_{\log D}$, so it is actually a line subbundle of $\Omega_{X_1}^1 \langle \log D_1 \rangle$.

Note that unlike the usual conormal, the log conormal is a <u>sub</u>bundle of the log differentials on X_1, and it is naturally isomorphic rather than dual to the log normal. We get a canonical generator of $\check{N}_{\langle \log D \rangle}$, denoted ψ_1. In terms of a standard form $\Phi = \sum b_{ij} \, \mathrm{dlog}(x_i) \, \mathrm{dlog}(x_j)$ as in §3.3, ψ_1 has the form, locally on X_1 where x_1 is a branch equation

$$\psi_1 = \sum_{i=2}^{2n} b_{1i} \, \mathrm{dlog}(x_i) = \langle \Phi, v_1 \rangle. \qquad (37)$$

Note that ψ_1 is a *closed* log form on X_1. It suffices to check this at a general point of X_1, where we may assume (with a different coordinate system) that $\Phi = dx_1 dx_2 / x_1 + \sum_{i=2}^{2n} dx_{2i-1} dx_{2i}$, $v_1 = x_1 \, \partial_{x_1}$ so $\psi_1 = dx_2$ is closed.

3.4.2. Higher order. Essentially the same construction applies to the higher-order loci X_k. Thus, a point in X_k comes equipped with k transverse normal hyper-planes corresponding to k branches of D, which are well-defined up to order. Hence the kernel of

$$\nu_k^*(T_X \langle -\log D \rangle) \to \nu_k^*(T_X)$$

is a flat, integrable rank-k bundle, denoted $N_{\log(D)}^k$, called the log normal bundle of order k. It is locally generated by the log vector fields $x_1 \, \partial_1, ..., x_k \, \partial_k$. Since these are canonical up to order, the log normal bundle becomes trivial after a suitable S_k-cover, and is already trivial if D has *simple* normal crossings. We have exact sequences of locally free $\mathcal{O}_{X_k}$- modules

$$0 \to N_{\log(D)}^k \to \nu_k^*(T_X \langle -\log D \rangle) \to T_{X_k} \langle -\log D_k \rangle \to 0,$$
$$0 \to T_{X_k} \langle -\log D_k \rangle \to \nu_k^*(T_X) \to N_{X_k/X} \to 0. \qquad (38)$$

In the log-symplectic case, $N_{\log(D)}^k$ is isomorphic via log duality to a trivial rank-k subbundle of $\nu_k^*(\Omega_X^1 \langle \log D \rangle)$, denoted $\check{N}_{\log(D)}^k$, with local generators $\psi_i = \langle \Phi, v_i \rangle, i = 1, ..., k$. Locally at a point, X_k admits k divisorial embeddings into transverse branches of X_{k-1}, with associated log conormals $\check{N}_i \subset \Omega_{X_k}^1 \langle \log D_k \rangle$, respectively generated by the ψ_i, and we have $\check{N}_{\log(D)}^k = \bigoplus \check{N}_i$. As in the first-order case, we have $\check{N}_i \subset \Omega_{X_k}^1 \langle \log D_k \rangle$, hence $\check{N}_{\log(D)}^k \subset \Omega_{X_k}^1 \langle \log D_k \rangle$.

Because the ψ_i are closed forms , $\check{N}^k_{\log(D)}$ is an integrable subbundle, i.e. corresponds to a codimension-k foliation. This foliation is known as the kernel or symplectic foliation, due to the following

LEMMA 3.5. *Outside the divisor $D_k \subset X_k$, the conormal bundle $\check{N}^k_{\log(D)}$ coincides with the kernel of the Poisson structure induced on X_k by Π.*

Proof. On $\nu_1^*(\Omega^1_X \langle \log D \rangle)$, Π induces a nondegenerate form, and it pairs the ψ_i with the conormal forms dx_i/x_i. Therefore Π yields a nondegenerate form on the kernel of the natural map $\nu_1^*(\Omega^1_X \langle \log D \rangle)/\check{N}^k_{\log(D)} \to \bigoplus \mathcal{O} dx_i/x_i$, that is $\Omega^1_{X_k} \langle \log D_k \rangle /\check{N}^k_{\log(D)}$. $\square$

3.4.3. Conormal filtration. The subbundle $\check{N}^k_{\log(D)}$ defines, in the usual way, an increasing, length-k filtration $\mathcal{F}^\perp_\bullet$ on $\Omega^\bullet_{X_k} \langle \log D_k \rangle$, called the *conormal filtration* defined by

$$\mathcal{F}^\perp_j \Omega^\bullet_{X_k} \langle \log D \rangle = \wedge^{k-j+1} \check{N}^k_{\log(D)} \Omega^\bullet_{X_k} \langle \log D \rangle.$$

Thus,

$$\mathcal{F}^\perp_j \Omega^\bullet_{X_k} \langle \log D_k \rangle = \sum_{|I|=k-j+1} \psi_I \Omega^\bullet_{X_k} \langle \log D_k \rangle.$$

3.5. The Residual Generality condition. The log-symplectic Poisson structure Π is said to be *residually general*, or to satisfy the *RG condition*, if at every point p of multiplicity m on the degeneracy divisor, and a standard form $\sum a_{ij} v_i v_j$ as above, the matrix $(a_{ij}(p) : i,j \leq m)$ is a general skew-symmetric $m \times m$ matrix. This condition can be obviously rephrased in terms of the corresponding log-symplectic form Φ to say that its polar part is general. The RG condition is stronger, for any $t \leq m$, than the 't-very-general condition introduced in [14], Erratum, hence also than the original 'general position' condition employed in [14].

One consequence of the RG condition is that for any $i \leq m$, the (closed) 1-form $\psi_i = \langle \Phi, v_i \rangle$ pulls back to a general log 1-form on the branch (x_i) of D and in particular its polar divisor coincides exactly with the divisor on (x_i) induced by D, defined by $\prod_{j \neq i} x_i$. Furthermore, any collection of m or fewer elements among the ψ_i. and the standard forms $\mathrm{dlog}(x_1), ..., \mathrm{dlog}(x_m)$ are linearly independent, i.e. are a basis for a locally free and cofree submodule. Consequently, the pullback of any collection of ψ_i to any multiplicity locus X_k are linearly independent..

Note that the RG condition excludes P-normality (unless D is smooth): indeed if Π is P-normal then ψ_i above has no poles at all.

4. Degeneracy, kernel foliation. From now on we restrict attention to the case of a complex pseudo-symplectic Poisson manifold (X, Π) of dimension $2n$. Then the degeneracy locus of Π is a (Pfaffian) divisor $P = [\Pi^n] \in |-K_X|$ (for this section, not necessarily with normal crossings). It is well known that Π descends to a (degenerate) Poisson structure on the smooth part of P: this follows from the fact that the kernel of Π on Ω_X at a smooth point of P contains the conormal line (cf. §3.4 above or [13], proof of Prop. 10). Here we will expand on this. More precise results will be given in §6, under the hypothesis that Π is log-symplectic and residually general.

Define sheaves C^i via the exact sequence

$$0 \to \Omega^{2n-i}_X \xrightarrow{{}^t\Pi^{n-i}} \Omega^i_X \to C^i \to 0. \tag{39}$$

Thus, $C^0 = \mathcal{O}_P, C^n = 0$. Note that each C^i is an $\mathcal{O}_P$-module. Also, the degeneracy ideal $\mathcal{I}_{2k}$ defined by Lima-Pereira [10] is none other than the $(2n - 2k)$-th Fitting ideal of C^1. Moreover, by Theorem 2.1, there are exact diagrams

$$
\begin{array}{ccccccccc}
0 \to & \Omega_X^{2n-i} & \stackrel{\iota_{\Pi^{n-i}}}{\to} & \Omega_X^i & \to & C^i & \to 0 \\
& \downarrow \delta & & \downarrow d & & \downarrow d & \\
0 \to & \Omega_X^{2n-i-1} & \stackrel{\iota_{\Pi^{n-i-1}}}{\to} & \Omega_X^{i+1} & \to & C^{i+1} & \to 0.
\end{array}
\tag{40}
$$

$$
\begin{array}{ccccccccc}
0 \to & \Omega_X^{2n-i} & \stackrel{\iota_{\Pi^{n-i}}}{\to} & \Omega_X^i & \to & C^i & \to 0 \\
& \downarrow d & & \downarrow \delta & & \downarrow \delta & \\
0 \to & \Omega_X^{2n-i+1} & \stackrel{\iota_{\Pi^{n-i+1}}}{\to} & \Omega_X^{i-1} & \to & C^{i-1} & \to 0.
\end{array}
\tag{41}
$$

Thus, we effectively get two mutually reverse complexes:

$$
(C_{n]}^\bullet, d) : C^0 = \mathcal{O}_P \stackrel{d}{\to} C^1 \stackrel{d}{\to} C^2 \stackrel{d}{\to} ... \stackrel{d}{\to} C^{n-1} \to C^n = 0,
\tag{42}
$$

$$
(C_{[n}^\bullet, \delta) : C^n = 0 \to C^{n-1} \stackrel{\delta}{\to} C^{n-2} \stackrel{\delta}{\to} ... \stackrel{\delta}{\to} C^1 \stackrel{\delta}{\to} \mathcal{O}_P.
\tag{43}
$$

As to the interpretation of these, we have Proposition 4.3 below. First, an auxiliary multilinear algebra result.

LEMMA 4.1. *For $i \leq j \leq k$, there exist bilinear forms*

$$
P_{i,j}(.,.,\Pi^i) : \Omega_X^i \times \Omega_X^k \to \Omega_X^k
$$

(linear in Π^j as well), such that

$$
\alpha \wedge \langle \Pi^j, \beta \rangle = \langle \Pi^{j-i}, P_{i,j}(\alpha, \beta, \Pi^i) \rangle, \alpha \in \Omega_X^i, \beta \in \Omega_X^k.
$$

Proof. It suffices to prove this for α completely decomposable, hence by induction we are reduced to the case $i = 1$. There, Note the following:

$$
\begin{aligned}
\langle \Pi^{j-1}, \langle \Pi, \alpha \wedge \beta \rangle \rangle = \langle \Pi^j, \alpha \wedge \beta \rangle &= \alpha \wedge \langle \Pi^j, \beta \rangle \pm j \langle \langle \Pi^{j-1}, \beta \rangle, \langle \Pi, \alpha \rangle \rangle \\
&= \alpha \wedge \langle \Pi^i, \beta \rangle \pm j \langle \Pi^{j-1}, \langle \beta, \langle \Pi, \alpha \rangle \rangle \rangle.
\end{aligned}
$$

Thus, an explicit formula for $P_{1,j}$ is

$$
P_{1,j}(\alpha, \beta, \Pi) = \langle \Pi, \alpha \wedge \beta \rangle \pm j \langle \langle \Pi, \alpha \rangle, \beta \rangle.
$$

$\square$

COROLLARY 4.2. *The image of the morphism $\pi : \Theta_{n]}^\bullet \to \Omega_{n]}^\bullet$ is an exterior ideal closed under d. Hence $(C_{n]}^\bullet, d, \wedge)$ is a sheaf of differential graded algebras.*

Next, we compare the algebra $C_{n]}^\bullet$ to the exterior algebra on C^1:

PROPOSITION 4.3. (i) *There is a canonical map*

$$
\bigwedge_{\mathcal{O}_P}^i C^1 \to C^i.
$$

(ii) *At a smooth point of P, each C^i is locally free over $\mathcal{O}_P$ and we have an exact sequence*

$$0 \to \bigwedge^{2n-i}_{\mathcal{O}_P} C^1 \to \bigwedge^{i}_{\mathcal{O}_P} C^1 \to C^i \to 0 \tag{44}$$

where the first map is induced by $\iota_{\Pi^{n-i}}$.

Proof. (i) results inductively from the commutative diagram

$$
\begin{array}{ccccccc}
\Omega_X^{2n-1} \otimes \Omega_X^i \oplus \Omega_X^1 \otimes \Omega_X^{2n-i} & \xrightarrow{\iota_{\Pi^{n-1}} \otimes \mathrm{id} \oplus \mathrm{id} \otimes \iota_{\Pi^{n-i}}} & \Omega_X^1 \otimes \Omega_X^i & \to & C^1 \otimes C^i & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
\Omega_X^{2n-i-1} & \xrightarrow{\iota_{\Pi^{n-i-1}}} & \Omega_X^{i+1} & \to & C^{i+1} & \to 0
\end{array}
.
$$

Here the left vertical map is ${}^t P_{i,n-1}(.,.,\Pi^{n-i-1}) \oplus P_{1,n-i}(.,.,\Pi^{n-i-1})$ where ${}^t P(\alpha,\beta,.) = P(\beta,\alpha,.)$, and the other vertical maps are just wedge product.

(ii) At a smooth point of P, Π admits a normal form

$$\Pi = x_1 \, \partial_{x_1} \, \partial_{y_1} + \sum_{i=2}^n \partial_{x_i} \, \partial_{y_i} .$$

From this, the assertion follows by an easy computation. □

COROLLARY 4.4. *C^1 is integrable and induces on the smooth part of P a codimension-1 foliation by Poisson submanifolds, called the kernel foliation.*

Proof. Perhaps the easiest way to check the integrability condition is to use the normal form above, which shows that at a smooth point of P, where P has local equation x_1, C^1 is the quotient of Ω_P by the subsheaf generated by dy_1, and thus corresponds to the foliation by level sets of y_1.

The fact that Π descends to the leaves of the foliation follows from the fact that $\iota_\Pi : \Omega_X^1 \to T_X$ vanishes over $P = [\Pi^n]$ on the image of $\iota_{\Pi^{n-1}} : \Omega_X^{2n-1} \to \Omega_X^1$. Alternatively, this can also be proved easily using the normal form above. □

COROLLARY 4.5. *Over the smooth part of P, C^1 coincides with the quotient of Ω_P^1 by the log-conormal bundle (cf. §3.4).*

Proof. Follows from Lemma 3.5. □

The existence of the kernel foliation is not a new result: this foliation coincides with the so-called symplectic foliation associated to the degenerate Poisson structure induced by Π on P. See for instance [10].

We will henceforth denote $C^\bullet_{n]}$ simply by $C^\bullet$.

REMARK 4.6. There is a Π-trace map $\bigwedge^2 C^1 \to \mathcal{O}_P$. The composition $\bigwedge^{2n-2} C^1 \to \bigwedge^2 C^1 \to \mathcal{O}_P$ is nowhere vanishing on the smooth part of P. Therefore on the smooth part of P we can also identify C^2 with the subsheaf of traceless elements of $\bigwedge^2 C^1$.

5. P-normal case, examples. We recall [13], Proposition 7, that P-normal Poisson structures Π can be characterized by the existence of a local coordinate system (called *normal coordinates*) in which Π has the form

$$\Pi = \sum_{i=1}^{k} x_i \, \partial_{x_i} \, \partial_{y_i} + \sum_{i=k+1}^{n} \partial_{x_i} \, \partial_{y_i} \,. \tag{45}$$

In particular, Π is log-symplectic.

EXAMPLE 5.1 (Modified Hilbert schemes). Let S be a smooth surface endowed with a Poisson structure corresponding to a smooth anticanonical curve D. Then Π induces a Poisson structure $\Pi^{[n]}$ on the Hilbert scheme $S^{[n]}$. The Pfaffian divisor P corresponds to the subschemes having a nonempty intersection with D and the kernel foliation has leaves corresponding to subschemes having a fixed intersection point with D so D is the parameter curve and indeed, D is elliptic. Although Π is not P-normal and P does not have normal crossings, $\Pi^{[n]}$ induces a P-normal Poisson structure Π_X on the stratified blow-up X of the incidence stratification on $S^{[n]}$ (see [12]). The components of the Pfaffian divisor of Π_X are birational to $D^{(i)} \times S^{(n-i)} \times \mathbb{P}^{i-1}, i = 1, ..., n$ and the kernel foliation on the latter corresponds to the map to D defined by projection to $D^{(i)}$ followed by the sum map $D^{(i)} \to D$ coming from an addition law on the elliptic curve D (the addition law and the sum map depend on the choice of origin; the fibres do not). This is the map whose derivative is given by

$$(..., \partial_{y_1}, ..., \partial_{y_i}) \mapsto \partial_{y_1} + ... + \partial_{y_i},$$

y_i being induced by a coordinate y on D. Indeed a straightforward derivative calculation shows that at a general point of the latter component, which corresponds to a reduced point-scheme with exactly i points on D, there are local coordinates such that Π_X takes the form

$$u_1 \, \partial_{u_1} \, \partial_{v_1} + \sum_{i=2}^{n} \partial_{u_i} \, \partial_{v_i},$$

where v_1 is the coordinate on $D^{(i)}$ corresponding to $y_1 + ... + y_i$.

EXAMPLE 5.2 (Toric Poisson structures). Let $X = X(\Delta)$ be a smooth projective toric variety, with torus $T \subset X$ acting on X (cf. [5]). Thus Δ is a fan in $N \otimes \mathbb{R}$ where $N = \mathrm{Hom}(\mathbb{C}^*, T)$ is the lattice of 1-parameter subgroups. Since $N_{\mathbb{C}} = N \otimes \mathbb{C}$ is the (abelian) Lie algebra of T and embeds into $H^0(T_X)$, any element of $\wedge^2 N_{\mathbb{C}}$ yields a Poisson structure on X. These structures generically are log-symplectic, with Pfaffian divisor $X \setminus T$, but they are not P-normal. For $X = \mathbb{P}^n$ these structures are studied in [10], where they are called *diagonal*. A general such structure in even dimension satisfies the Residual Generality condition (see §3.5).

Now suppose that $\dim(X) = 2n$ is even and that the fan Δ satisfies the following condition

(*) There is a basis $u_1, v_1, ..., u_n, v_n$ of N such that for any cone $\sigma \in \Delta$ and any $i = 1, ..., n$, either $u_i \notin \sigma$ or $v_i \notin \sigma$.

For any $u \in N$ and $\sigma \in \Delta$, the limit at 0 of the 1-parameter subgroup $\mathbb{C}^* \to T$ corresponding to u lies in the affine patch $X_\sigma \simeq \mathbb{C}^m \times (\mathbb{C}^*)^{n-m}$ iff $u \in \sigma$. Consequently, the assumption $u \notin \sigma$ implies that as vector field, u is nowhere vanishing on X_σ, while

$u \in \sigma$ implies that u on X_σ is a log vector field, of the form $x\, \partial_x$. Thus, condition (*) implies that the Poisson structure

$$\Pi = u_1 v_1 + \ldots + u_n v_n \qquad (46)$$

is P-normal.

Regarding condition (*), note that, as pointed out by Jose Gonzalez, it can always be achieved by subdividing a given fan, which corresponds to replacing a given toric variety by a toric blowup of itself. In particular, there exist many toric blowups of projective space with this property.

EXAMPLE 5.3 (Toric-by-torus structures). Let Z be an n-dimensional smooth projective toric variety with lattice N, and let $u_1, \ldots, u_n$ be a basis for N, viewed as vector fields. Let A be an n-dimensional complex torus and $t_1, \ldots, t_n$ a basis for the constant vector fields on A. Then

$$\Pi = u_1 \wedge t_1 + \ldots + u_n \wedge t_n \in H^0(Z \times A, \wedge^2 T_{Z \times A}) \qquad (47)$$

is clearly a P-normal Poisson structure on $X := Z \times A$. The kernel foliation on X_i is generated by $t_1, \ldots, t_i$, so it is generally not algebraically (or mermorphically) integrable.

It is worth noting that the *smallest* degeneracy locus, i.e. the zero-locus P_n, of a P-normal Poisson structure, has itself a special structure:

PROPOSITION 5.4. *Let Π be a P-normal Poisson structure on a projective $2n$-manifold X and $Y = P_n(\Pi)$ its zero locus. If $Y \neq \emptyset$, then Y admits a surjective map to a nontrivial abelian variety.*

Proof. To begin with, it is well known that Y is endowed with a tangent vector field called Weinstein's modular field [16]. To construct this field directly in our case, and see that it is never zero, note that Π yields a canonical section of $\check{N}_Y \otimes \overset{2}{\bigwedge} T_X$. By the normal form (45), Π lifts to $\check{N}_Y \otimes T_X \otimes T_Y$, because the defining equations of Y are $x_1, \ldots, x_n$, while $y_1, \ldots, y_n$ are coordinates on Y. There is a canonical map

$$\check{N}_Y \otimes T_X \otimes T_Y \to \check{N}_Y \otimes N_Y \otimes T_Y \to T_Y,$$

and again by the normal form (45), the image of Π by the latter map is never zero (with the notation of loc. cit. it has the form $\partial_{y_1} + \ldots + \partial_{y_n}$). Now use the following, probably well-known, result. □

LEMMA 5.5. *Let Y be a smooth projective variety endowed with a nowhere-vanishing vector field v. Then there is an Abelian variety A and a surjective map $Y \to A$, such that v descends to a nonzero constant vector field on A.*

Proof. Consider the Albanese map alb : $Y \to B = \mathrm{Alb}(Y)$. By a result of Matsushima-Lichnerowicz-Lieberman (cf. [9], Thm. 1.5), v induces a nonzero constant vector field on B, which of course preserves the image alb(Y). Consequently, alb(Y) is invariant under a nontrivial abelian subvariety $A_1 \subset B$. Let $A_2 \subset B$ be a complementary abelian subvariety. Thus, $A_1 \to B/A_2 =: A$ is an isogeny. Because alb(Y) contains A_1-orbits, the map $Y \to A$ is clearly surjective, and v descends to a nonzero constant vector field on A. □

REMARK 5.6. In the above situation, it is not necessarily the case that Y admits an action by an abelian variety. Let E be an elliptic curve, L a nontorsion line bundle of degree 0, and $Y = \mathbb{P}_E(L \oplus \mathcal{O})$. For each $a \in E$, the translate of L by a is isomorphic to L, e.g. because L can be defined by constant transition functions. Therefore the automorphism group G of Y fits in an exact sequence

$$1 \to \mathbb{G}_{\mathrm{m}} \to G \to E \to 1$$

which induces an analogous exact sequence on tangent spaces. Then, a nonzero tangent vector to E lifts to a tangent vector to G, which corresponds to a nowhere-vanishing vector field on Y; however, E does not act on Y due to the nontriviality of L.

EXAMPLE 5.1 CONT'D. In the Hilbert scheme example above, $Y = D^{(n)}$, which maps to the elliptic curve D by the sum map.

EXAMPLE 5.3 CONT'D. In the toric-by-torus example above, Y is a disjoint union of copies of the torus A.

6. Local cohomology of upper MdP complex. We now assume till further notice that our log-symplectic Poisson structure satisfies the Residual Generality condition, see §3.5. Our aim is to study the MdP complex $\Theta^{n]}$ and its cohomology, first locally, then in the next section, globally. We study $\Theta^{n]}$ locally via its image by the bonding map π, and we study the latter image in turn via the simplicial resolution as in §1.4. Thus, we denote by $I_k^{\bullet}$ or I_k the pullback of the image of π to X_k. Otherwise, notations are as in §3.1.

6.1. Image of bonding map via simplicial resolution. To begin with, note that by the discussion in §3.1, the image of π on Ω_X^{2n-r} coincides with $F\langle \Phi^r, \wedge^r T_X \rangle$. In particular it follows that I_1^1 is generated locally by the form $F_1 \psi_1 = F\langle \Phi, \partial_1 \rangle$ where $\partial_i = \partial_{x_i}$. Next we will generalize this to higher-degree differentials and the higher strata X_k. Let I_k denote the image of $\mathrm{im}(\pi)$ under the pullback map on differentials attached to the map $X_k \to X$. Working locally at a point of X_k, we decompose the log-symplectic form Φ into its normal and tangential components:

$$\Phi = \Phi_{\perp,k} + \Phi_{=,k} = \sum_{j=1}^{k} \psi_j \, dx_j / x_j + \Phi_{=,k}$$

where $x_1, ..., x_k$ are equations of the branches of D at the point in question and $\Phi_{=,k}$ is a log-symplectic form on X_k itself. Now the contraction of a log form of degree a with a log polyvector field of degree $b \leq a$ is a log form of degree $a - b$ (and thus if $a = b$, a holomorphic function). Hence note that for any (resp. any log) polyvector field u, $\langle \Phi_{=,k}, u \rangle$ is of the form α/x_e (resp. α), where α is a log form on X_k and x_e is a factor of F_k. Note that an expression $F\langle \Phi^r, u_1...u_r \rangle$ can be nonzero on X_k only if the normal fields $\partial_i = \partial_{x_i}, i = 1, ..., k$ all occur among the u_i, so we may assume $u_i = x_i \partial_i, i = 1, ..., k$. In that case, the only term in the binomial expansion of Φ^r that can contribute is $\Phi_{k,=}^{r-k} \Phi_{\perp,k}^k$, which yields

$$\langle \Phi^r, u_1...u_r \rangle = \binom{r}{k} \langle \Phi_{k,=}^{r-k} \psi_1...\psi_k, u_{k+1}...u_r \rangle.$$

The latter is a sum of terms where some number, say a of the u-s are contracted with ψ-s and the remaining $r - k - a$ are contracted with $\Phi_{k,=}^{r-k}$. Note that such a term is

divisible by $\Phi_{k,=}^a$. Thus we can write

$$F\langle \Phi^r, u_1...u_r \rangle = \sum F_{k,I}\alpha_{I,s}\langle \psi_1...\psi_k, w_J \rangle \Phi_{=,k}^s \tag{48}$$

where the w_J are suitable polyvector fields on X_k the α_I are suitable log forms, products of some $\langle \Phi_{=,k}, u_\ell \rangle x_e$, and $F_{k,I} = F_k / \prod_{e\in I} x_e$ is the appropriate factor of F_k; in all the terms appearing, we have $s \geq |J|$. This can be rewritten as

$$F\langle \Phi^r, u_1...u_r \rangle = \sum F_{k,I}\beta_{I,s}\psi_I \Phi_{=,k}^s \tag{49}$$

where $\psi_I = \prod_{i\in I} \psi_i$ and $s \geq k - |I|$. Due to the residual generality hypothesis on Φ, the coefficients β_I are general log forms of their degree when the polyvector field $u_1...u_r$ is chosen generally. Recalling the log-conormal filtration from §3.4, we conclude:

PROPOSITION 6.1. *We have, where $\mathcal{F}_\bullet^\perp$ denotes conormal filtration,*

$$\begin{aligned}
\mathcal{I}_k^\bullet &= \sum_{I \subset \underline{k}, s} \psi_I \Phi_{=,k}^s \Omega_{X_k}^\bullet \langle \log {}^\mp D_k \rangle [-|I| - 2s] \\
&= \sum_{s \geq j-1} \Phi_{k,=}^s \mathcal{F}_j^\perp \Omega_{X_k}^\bullet \langle \log {}^\mp D_k \rangle [-2s].
\end{aligned} \tag{50}$$

6.2. Cohomology. Our goal is to compute the cohomology sheaves of $I_k^\bullet$ for fixed k, and then that of $I^\bullet$, via the simplicial resolution $I_\bullet^\bullet$. To this end, we note first that an expression as in (49) can be normalized. In fact, we may assume that each $\beta_{I,s}$ with $I \neq \emptyset$, when written out in terms on a basis for 1-forms, does not contain any term divisible by any $\psi_i, i \in I$ nor $\Phi_{=,k}$. In the first case the term is zero, while in the second case it can be added to a term attached to $\Phi_{=,k}^{s+1}$. With this proviso, the expression (49) is *unique*.

Next, as in the proof of Lemma 1.4, we may assume that the ψ_i and Φ have constant coefficients, hence I_k can be decomposed into homogeneous components $S_{(m.)}^i$. Now consider a differential $\gamma \in S_{(m.)}^i$ decomposed as in the proof of Lemma 1.4 and normalized as above. Suppose first that the multiplicity μ_k of D_k on X_k at the point in question is greater than $|I|$. Consider a nonzero term $F_{k,I}\beta_{I,s}\psi_I \Phi_{=,k}^s$ with smallest s. Then

$$d(F_{k,I}\beta_{I,s}\psi_I \Phi_{=,k}^s) = F_{k,I}\beta_{I,s}\psi_I \Phi_{=,k}^s \wedge \chi_{(m.)+1_I}$$

where 1_I is the characteristic function of I. Due to the residual general position of the ψ_i, this cannot vanish unless $\beta_{I,s} \wedge \chi_{(m.)+1_I} = 0$, i.e. $\beta_{I,s}$ is divisible by $\chi_{(m.)+1_I}$. Proceeding inductively over s, the same holds for all the β coefficients, hence for γ. This proves exactness of the complex I_k locally over $X_k \setminus U_{k,\mu_k}$.

Now suppose $\mu_k \leq |I|$. Then it is easy to see that

$$F_{k,I}\psi_I = dx_I = \prod_{i\in I} dx_i$$

which is a closed form. Consider again a term $F_{k,I}\beta_{I,s}\psi_I \Phi_{=,k}^s$ with smallest s. Then

$$d(F_{k,I}\beta_{I,s}\psi_I \Phi_{=,k}^s) = d(\beta_{I,s} \prod_{i\in I} dx_i \Phi_{=,k}^s) = \beta_{I,s} \prod_{i\in I} dx_i \Phi_{=,k}^s \chi_{(m.+1_I)}.$$

If $d\gamma = 0$, this vanishes. But clearly this expression can vanish only if $\beta_{I,s}$ is a closed form modulo the coordinates in I, i.e. $d\beta_{I,s}$ is in the complex generated by the $dx_i, i \in I$. Thus, $\beta_{I,s}$ is a section of $\hat{\Omega}_{X_k}^{r,\psi_I}$ for some r. Similarly, or inductively, for terms with higher s. We have proven

PROPOSITION 6.2. *The local cohomology sheaf of the pullback $I_k^\bullet$ of the complex $im(\pi)$ on X_k is as follows, where $\widehat{\bullet}$ indicates closed forms:*

$$\mathcal{H}^i(I_k^\bullet) = \bigoplus_{\substack{|I|+r+2s=i \\ I \subset \underline{k}}} i_{U_{k,k+|I|}!}(\hat{\Omega}_{X_k}^{r,\psi_I}\psi_I\Phi_{=,k}^s)$$

$$= \bigoplus_{\substack{t+2s=i,s\geq j-1 \\ I \subset \underline{k}}} i_{U_{k,k+|I|}!}(\widehat{\mathcal{F}_j^\perp\Omega_{X_k}^t}\Phi_{=,k}^s) \tag{51}$$

Now via the inclusion $I_k^\bullet = im(\pi)_{X_k} \subset \Omega_{X_k}^\bullet$, the complexes I_k for varying k form a double complex resolving $im(\pi) \simeq \Theta^\bullet$ (see the proof of Lemma 1.8), so we study next the maps $I_k \to I_{k+1}$ and their induced maps on cohomology. Thus consider the middle cohomology of the short complex

$$H^i(I_{k-1}) \to H^i(I_k) \to H^i(I_{k+1})$$

with terms given by (51), hence compactly supported respectively over

$$U_{k-1,k-1+|I|}, U_{k,k+|I|}, U_{k+1,k+1+|I|}.$$

Over the common intersection $U_{k+1,k+|I|}$, the complex is exact by the argument of §1.4, the simplicial De Rham resolution. Over $U_{k,k}$, the left map is clearly surjective (and the right term is zero). Over $U_{k+|I|,k|I|}$, the left term vanishes and the middle and right terms consist of differentials on k-fold, resp. $k+1$-fold branch intersections at a point of multiplicity exactly $k+|I|$. Thus the kernel of the right map consist of the differentials that descend from X_k to D_k. This also applies mutatis mutandis to the case $k = 1$. It follows that the spectral sequence for the local cohomology of the double complex $I_\bullet^\bullet$ we have

$$E_2^{i,k} = \bigoplus_{\substack{|I|+r+2s=i \\ I \subset \underline{k}}} i_{U_{k+|I|,k+|I|}!}(\hat{\Omega}_{X_k}^{r,\psi_I}\psi_I\Phi_{=,k}^s).$$

We claim that this spectral sequence degenerates at E_2 Indeed consider an element of $E_2^{i,k}$ represented by the form $a = \beta\prod dx_i\Phi_{=,k}^s$ on $U_{k+|I|,k+|I|}$. The image of a in I_{k+1} can be written as db for some $i-1$-form b with the same I and s. Then the image c of b in I_{k+2} is exact for support reasons. But the class of c is just the image of a under the second-page differential $d_2^{i,k}$ so that differential is zero. Likewise for further pages. Therefore the spectral sequence degenerates at E_2. Consequently we conclude

PROPOSITION 6.3. *The local cohomology $\mathcal{H}^j(I_\bullet^\bullet)$ admits a filtration with graded pieces*

$$\bigoplus_{\substack{|I|+r+2s=i \\ I \subset \underline{k}}} i_{U_{k+|I|,k+|I|}!}(\hat{\Omega}_{X_k}^{r,\psi_I}\psi_I\Phi_{=,k}^s) = \bigoplus_{\substack{t+2s=i,s\geq j-1 \\ I \subset \underline{k}}} i_{U_{k+|I|,k+|I|}!}(\widehat{\mathcal{F}_j^\perp\Omega_{X_k}^t}\Phi_{=,k}^s)$$

for all i, k with $i + k = j$.

This essentially computes the cohomology of the upper MdP complex:

THEOREM 6.4. *Let (X, Π) be a log-symplectic manifold of dimension $2n$ satisfying the Residual Generality condition. The local cohomology of $\Theta_X^{n]}$ is as follows :*
$\mathcal{H}^0(\Theta^\bullet) = i_{U_0!}(\mathbb{C}_{U_0})$;
for $0 < j < n$, $\mathcal{H}^j(\Theta^\bullet)$ has a filtration with graded pieces

$$\bigoplus_{\substack{|I|+r+2s=i \\ I \subset \underline{k}}} i_{U_{k+|I|,k+|I|}!}(\hat{\Omega}_{X_k}^{r,\psi_I} \psi_I \Phi_{=,k}^s) = \bigoplus_{\substack{t+2s=i, s \geq j-1 \\ I \subset \underline{k}}} i_{U_{k+|I|,k+|I|}!}(\widehat{\mathcal{F}_j^\perp \Omega_{X_k}^t} \Phi_{=,k}^s)$$

for $i + k = j - 1$.

Proof. Let $K^\bullet$ be the kernel of the natural surjection $\Omega_X^\bullet \to \Omega_D^\bullet$. Because $\Omega_X^\bullet$ and $\Omega_D^\bullet$ are resolutions of the respective constant sheaves, we have a quasi-isomorphism

$$K^\bullet \sim i_{U_0!}(\mathbb{C}_{U_0}).$$

Consequently we have

$$\mathcal{H}^0(\Theta^{n]}) = i_{U_0!}(\mathbb{C}_{U_0}), \mathcal{H}^i(\Theta^\bullet) \simeq \mathcal{H}^i(I^\bullet), 0 < i < n.$$

Because $C^\bullet$ is a quotient of $\Omega_D^\bullet$ as we have seen, the map $K^\bullet \to \Omega_X^\bullet$ factors through $\Theta^\bullet$ and we have an exact diagram

$$
\begin{array}{ccccccccc}
& & & & 0 & & 0 & & \\
& & & & \downarrow & & \downarrow & & \\
0 & \to & K^\bullet & \to & \Theta^{n]} & \to & I^\bullet & \to & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \to & K^\bullet & \to & \Omega_X^\bullet & \to & \Omega_D^\bullet & \to & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & C^\bullet & = & C^\bullet & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
\tag{52}
$$

Note that $K^0 = \mathcal{O}_X(-D)$ maps isomorphically to $\Theta^0 = \Omega_X^{2n}$, so that $I^0 = 0$. Also, as we have seen in §1.4, $\Omega_D^\bullet$ is quasi-isomorphic to its simplicial resolution $\Omega_{X_\bullet}^\bullet$, which induces a simplicial resolution $I_\bullet^\bullet$, also quasi-isomorphic to $I^\bullet$. Thus

$$\mathcal{H}^i(\Theta^\bullet = \mathcal{H}^i(I_\bullet^\bullet) = \mathcal{H}^{i-1}(\tilde{C}^\bullet), 1 < i < n,$$

and there is an exact sequence

$$0 \to \mathbb{C}_D \to \mathcal{H}^0(\tilde{C}^\bullet) \to \mathcal{H}^1(\Theta^\bullet) \to 0$$

Now the Theorem follows from the preceding Proposition. □

Now recall that for a smooth affine d-dimensional variety Y and a locally free coherent sheaf $\mathcal{F}$, the compact-support cohomology vanishes:

$$H_c^i(Y, \mathcal{F}) = 0, \forall i < d$$

(this is because the compact-support cohomology is the limit of local cohomology supported at points, and the latter vanishes by depth considerations). Now the sheaves

occurring as summands in the Theorem are not themselves coherent but via the De Rham complex they admit a resolution by locally free $\mathcal{O}_{X_k}$ modules. Therefore each such summand on X_k has vanishing H^t for $t < \dim(X_k) = 2n - 2k$ provided $U_{k+|I|}$ is affine, hence in particular if $X_{k+|I|}$ is Fano. Thus we conclude:

COROLLARY 6.5. *Suppose* (X, Π) *satisfies the RG condition and moreover that* X_k *is Fano for* $k \leq a$. *The* $H^i(\Theta_X^\bullet) = 0$ *for* $i \leq a$.

As we shall see, when X is Kählerian the cohomology of $\Theta_X^\bullet$ can be computed in terms of the usual Hodge cohomology of X.

7. Kählerian case, Hodge cohomology. Here we assume that our holomorphic pseudo-symplectic Poisson manifold (X, Π) is compact and Kählerian (or more generally satisfies the $\partial\bar{\partial}$ lemma), Π otherwise arbitrary. This has the usual implications vis-a-vis degeneration of spectral sequences involving sheaves of holomorphic differentials (see for instance [6]). Then similar results can be derived for the MdP and dihelical complexes:

THEOREM 7.1. *The global hypercohomology spectral sequences*

$$E_1^{p,q} = H^q(X, \Theta^p) \Rightarrow H^i(\Theta^\bullet), \tag{53}$$

$$E_1^{p,q} = H^q(X, \mathcal{ED}^p) \Rightarrow H^i(\mathcal{ED}^\bullet), \tag{54}$$

$$E_1^{p,q} = H^q(X, \mathcal{DE}^p) \Rightarrow H^i(\mathcal{DE}^\bullet), \tag{55}$$

all degenerate at E_1.

Proof. It suffices to prove this for $\Theta^\bullet$. Consider a class

$$[\alpha] \in H^i(\Omega_X^j) = H^i(\Theta_X^{2n-j})$$

represented by a harmonic (j, i) form α. Then since $\partial(\alpha) = 0$, $d_1(\alpha) \in H^i(\Omega_X^{j-1})$ is represented by a multiple of $\partial\langle\Pi, \alpha\rangle$, which is ∂-exact and $\bar{\partial}$-closed, hence, by the $\partial\bar{\partial}$ lemma, also $\bar{\partial}$-exact, i.e. null-cohomologous. Hence $d_1([\alpha]) = 0$.

Next, write

$$\delta(\alpha) = (j - n)\,\partial\langle\Pi, \alpha\rangle = \bar{\partial}(\beta)$$

for a $(j-1, i-1)$ form β. Then $d_2([\alpha])$ is represented by

$$\delta(\beta) = (j - 1 - n)\,\partial\langle\Pi, \beta\rangle - (j - 2 - n)\langle\Pi, \partial(\beta)\rangle.$$

Now

$$\bar{\partial}\,\partial(\beta) = \partial\,\bar{\partial}(\beta) = (j - n)\,\partial^2\langle\Pi, \alpha\rangle = 0.$$

Hence $\partial(\beta)$ is $\bar{\partial}$-closed and ∂-exact, hence $\bar{\partial}$- exact. Since Π is holomorphic, $\langle\Pi, \partial(\beta)\rangle$ is also $\bar{\partial}$-exact, hence null-cohomologous.

Next, note

$$\bar{\partial}\,\partial\langle\Pi, \beta\rangle = \partial\,\bar{\partial}\langle\Pi, \beta\rangle = \partial\langle\Pi, \bar{\partial}\beta\rangle = (n - j)\,\partial\langle\partial\langle\Pi, \alpha\rangle\rangle$$
$$= (n - j)\,\partial\langle\Pi, \delta(\alpha)\rangle = (n - j)\delta^2(\alpha) = 0$$

(the next to last equality due to $\partial^2 = 0$). Therefore $\partial\langle\Pi, \beta\rangle$ is again $\bar\partial$-closed and ∂-exact, hence $\bar\partial$ exact, hence null-cohomologous. Thus, $d_2([\alpha]) = 0$. The vanishing of the higher d_r is proved similarly. □

COROLLARY 7.2. (i) $H^i(\Theta^\bullet)$ admits a filtration with quotients

$$H^q(X, \Omega_X^{2n-i+q}), q = 0, ..., i, i = 0, ..., 2n.$$

(ii) $H^i(\mathcal{E}\mathcal{D}^\bullet)$ admits a filtration with quotients

$$H^{n+i-a}(X, \Omega_X^{n+|n-a|}), a = 0, ..., 2n.$$

(iii) $H^i(\mathcal{D}\mathcal{E}^\bullet)$ admits a filtration with quotients

$$H^{n+i-a}(X, \Omega_X^{n-|n-a|}), a = 0, ..., 2n.$$

Thus, the cohomology of $\Theta_X^\bullet$ gives rise to a 'Poisson Hodge diamond' with rows $H_X^{i,2n}, ... H_X^{0,2n-i}, i = 0, ..., 2n$. This diamond is just the usual Hodge diamond of X rotated clockwise by $90°$.

Using Corollary 6.5, we can now conclude

COROLLARY 7.3. *Assume (X, Π) is a compact holomorphic Kählerian log-symplectic manifold such that Π satisfies the RG condition, and that the normalized strata X_k are Fano for $k \le a$. Then the Hodge numbers of X satisfy*

$$h_X^{2n-i,i} = 0, i = 0, ..., a. \tag{56}$$

This might be compared with the following result which not strictly speaking a consequence of the foregoing but related to in in that $\Theta_X^\bullet = T_X^\bullet(-D)$ (see Corollary 2.3). It gives a source of unobstructed odd-dimensional Poisson manifolds.

PROPOSITION 7.4. *Let (X, Π) be a compact holomorphic Kählerian log-symplectic $2n$- dimensional manifold such that the Hodge number $h_X^{1,2n-2} = 0$. Then the normalized degeneracy locus X_1 together with the induced Poisson structure Π_1 have unobstructed deformations and those deformations lift to deformations of (X, Π) inducing locally trivial deformations of the degeneracy divisor.*

Proof. We have an exact sequence

$$0 \to T_X^\bullet(-D) \to T_X^\bullet\langle-\log D\rangle \to j_* T_{X_1}^\bullet \to 0$$

where D is the degeneracy divisor of Π and $j : X_1 \to D \subset X$ is the normalization map.

CLAIM. the induced map

$$\mathbb{H}^1(T_X^\bullet\langle-\log D\rangle) \to \mathbb{H}^1(j_* T_{X_1}^\bullet)$$

is surjective.

Assuming this, a first-order deformation of (X_1, Π_1) given by $v \in \mathbb{H}^1(j_* T_{X_1}^\bullet)$ lifts to a deformation $\tilde v \in \mathbb{H}^1(T_X^\bullet\langle-\log D\rangle)$, i.e. a deformation of (X, Π) inducing a locally trivial deformation of D; as is well known (e.g. [13], p. 1170 and [14], Lemma 1), the latter deformations are unobstructed thanks to Poisson duality and Hodge theory,

hence the given first-order deformation $\tilde{v}$ extends to a formal or analytic arc, hence the same is true of v. Consequently (X_1, Π_1) has unobstructed deformations.

Proof of claim. This follows from the vanishing $\mathbb{H}^2(T_X^\bullet(-D)) = 0$. To see the latter note the exact sequence

$$... \, H^1(T_X^2(-D)) \to \mathbb{H}^2(T_X^\bullet(-D)) \to H^2(T_X(-D)) \, ...$$

Because D is anticanonical, the last group is Serre dual to $H_X^{1,2n-2}$ which vanishes by hypothesis. Similarly the first group is dual to $H_X^{2,2n-2}$ which also vanishes thanks to Hodge symmetry $h^{1,2n-2} = h^{2n-2,1} = h^{2,2n-1}$. $\square$

The condition $h_X^{1,2n-2} = 0$ seems weak and certainly holds for flag manifolds and toric manifolds. Thus we conclude by Example 5.2 that the normalized boundary of an even-dimensional toric variety X carries unobstructed Poisson structures induced from X.

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