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NOWHERE VANISHING PRIMITIVE OF A SYMPLECTIC FORM*

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Abstract. Let M be a manifold with an exact symplectic form ω. Then there is a nowhere vanishing primitive β for ω, i.e. $\omega = \mathrm{d}\beta$.

Key words. global symplectic structure, primitive of symplectic form, open symplectic manifold.

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Blohmann and Weinstein investigate the tangent Lie algebroid TM over a symplectic manifold (M, ω) and they are able to prove that it admits a Hamiltonian structure if and only if the symplectic form ω is exact and there is a nowhere vanishing primitive. That is why they ask later in their paper in full generality whether each manifold with an exact symplectic form admits a nowhere vanishing primitive, i.e. if there exists a nowhere vanishing 1-form β with $\omega = \mathrm{d}\beta$ ([BW18v1, Cor. 6.14]). This paper gives an affirmative answer without any further condition.

THEOREM. *Let M be a manifold with an exact symplectic form ω. Then there is a nowhere vanishing primitive β for ω.*

As an immediate consequence we obtain

COROLLARY. *A manifold with an exact symplectic form admits a nowhere vanishing Liouville vector field.*

REMARK. In this paper all manifolds are tacitly meant to be without boundary.

Proof of the Theorem. Without restriction we may assume M to be connected. Denote by $2n$ the dimension of M and have in mind that the symplectic form fixes an orientation. Exactness of ω implies that M is non-compact, as M is assumed to be without boundary.

Choose a Morse function $\gamma : M \to \mathbb{R}^{>0}$ which is exhaustive, i.e. the sets $\{x \in M \mid \gamma(x) \leq c\}$ are compact for all $c \in \mathbb{R}$. Denote C_0 the discrete set of critical points of γ.

The first step is to construct a primitive θ_1 of ω which is of a particular form around each point $z \in C_0$. For this choose for each $z \in C_0$ a contractible open neighborhood U_z in a way that U_z and $U_{\tilde{z}}$ do not intersect for all $z \neq \tilde{z}$ and such that there are $2n$ global functions x_i with $x_i(z) = 0$ for all $z \in C_0$ providing local coordinates on U_z such that

$$\omega = 2 \cdot \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}x_{i+n} \qquad \text{on } U_z \ . \tag{1}$$

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By assumption there is a primitive θ_0 for ω. Locally on each neighborhood U_z the 1-form

$$\tau_z = \sum_{i=1}^{n} x_i \mathrm{d}x_{i+n} - \sum_{i=1}^{n} x_{i+n} \mathrm{d}x_i$$

is a primitive as well. As U_z is contractible there is a function $f_z : U_z \to \mathbb{R}$ such that

$$\theta_0 - \tau_z = \mathrm{d}f_z \qquad \text{on } U_z \; . \tag{2}$$

Choosing smooth functions $\chi_z : M \to [0,1]$ with compact support in U_z and a relatively compact open neighborhood $V_z \subset U_z$ of z on which χ_z is identically equal to 1 we define the 1-form

$$\theta_1 = \theta_0 - \sum_{z \in C_0} \mathrm{d}(\chi_z \cdot f_z) \; , \tag{3}$$

again a primitive of ω. By construction θ_1 and τ_z are equal on V_z.

The next step will be to modify γ to ν such that the function ν has the same critical points but is of a particular form in the neighborhood of each critical point with respect to the coordinate functions x_i already introduced. This will be done by defining a diffeomorphism $\Psi : M \to M$ fixing C_0 and setting $\nu = \gamma \circ \Psi$.

As γ is a Morse function, the Morse Lemma states that, after possible shrinking of V_z, there are $2n$ global functions y_i with $y_i(z) = 0$ defining local coordinates on V_z for all $z \in C_0$ such that there are integers $0 \leq s_z \leq 2n$ so that

$$\gamma(y) = \sum_{i=1}^{s_z} y_i^2 - \sum_{i=s_z+1}^{2n} y_i^2 + \gamma(z) \qquad \text{on } V_z \tag{4}$$

This choice can be done in a way that the two bases $\mathrm{d}x_1, \ldots, \mathrm{d}x_{2n}$ and $\mathrm{d}y_1, \ldots, \mathrm{d}y_{2n}$ in $T_z^* M$ induce the same chosen orientation, e.g. just by replacing y_1 by $-y_1$ locally in V_z if necessary. After possible further shrinking of V_z there is an orientation preserving diffeomorphism $\psi_z : V_z \to \psi_z(V_z)$ with $\psi_z^* y_i = x_i$ for all $1 \leq i \leq 2n$ and $\psi_z(z) = z$. It is a classical result that there is a relatively compact open neighborhood $V_z' \subset V_z$ of z and an orientation preserving diffeomorphism $\Psi_z : M \to M$ such that $\Psi_z|_{V_z'} = \psi_z|_{V_z'}$ and $\Psi_z|_{M \setminus V_z} = \mathrm{id}\,|_{M \setminus V_z}$. The result may be deduced from [Pal59, Theorem 5.5] when applied to V_z'. So there is a global diffeomorphism $\Psi : M \to M$ defined by

$$\Psi(x) = \begin{cases} \Psi_z(x) & \text{for} \quad x \in V_z \\ x & \text{elsewhere} \end{cases}$$

by which we define the exhaustive Morse function $\nu = \gamma \circ \Psi$ which has the same set C_0 as set of critical points.

In summary, we constructed an exhaustive Morse function ν with discrete set C_0 of critical points and not intersecting relatively compact open neighborhoods V_z' such that for each $z \in C_0$

$$\mathrm{d}\nu = 2 \cdot \sum_{i=1}^{s_z} x_i \mathrm{d}x_i - 2 \cdot \sum_{i=s_z+1}^{2n} x_i \mathrm{d}x_i \qquad \text{on } V_z' \tag{5}$$

and a primitive θ_1 which satisfies

$$\theta_1 = \sum_{i=1}^{n} x_i \mathrm{d}x_{i+n} - \sum_{i=1}^{n} x_{i+n} \mathrm{d}x_i \qquad \text{on } V_z' \ . \tag{6}$$

There is a $K_0 > 0$ such that for all $K \geq K_0$ the system of linear equations

$$\left(\sum_{i=1}^{n} x_i \mathrm{d}x_{i+n} - \sum_{i=1}^{n} x_{i+n} \mathrm{d}x_i \right) + \ K \cdot \left(2 \cdot \sum_{i=1}^{s} x_i \mathrm{d}x_i - 2 \cdot \sum_{i=s+1}^{2n} x_i \mathrm{d}x_i \right) = 0 \tag{7}$$

has $(x_1, \dots, x_{2n}) = (0, \dots, 0)$ as the only solution for all $1 \leq s \leq 2n$. Applying this to the equations (5) and (6) we obtain for all $z \in C_0$

$$\{ x \in V_z' \mid \theta_1(x) + K \cdot \mathrm{d}\nu(x) = 0 \} = \{z\} \qquad \text{for all } K \geq K_0$$

and as K_0 is independent of $z \in C_0$

$$\bigcup_{z \in C_0} \{ x \in V_z' \mid \theta_1(x) + K \cdot \mathrm{d}\nu(x) = 0 \} = C_0 \qquad \text{for all } K \geq K_0 \ .$$

So we constructed an exhaustive function ν such that for all $K \geq K_0$ the primitive

$$\theta_1 + K \cdot \mathrm{d}\nu$$

vanishes inside $V' = \bigcup_{z \in C} V_z'$ exactly on C_0.

The next step will be to define a primitive which does not vanish outside V' at all and inside V' still exactly on C_0. For this we modify ν by a smooth function $\lambda : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ defining $\rho = \lambda \circ \nu$ such that the primitive

$$\beta_0 = \theta_1 + \mathrm{d}\rho \tag{8}$$

vanishes exactly at C_0.

Recall that on the closed subset $M \backslash V'$ the 1-form $\mathrm{d}\nu$ vanishes nowhere. For each $m = 1, 2, \dots$ the set $W_m = \nu^{-1}([m-1, m]) \backslash V'$ is compact using the fact that ν is exhaustive. So there is a $K_m > K_0$ such that

$$\{ w \in W_m \mid \theta_1(w) + K \cdot \mathrm{d}\nu(w) = 0 \}$$

is empty for all $K > K_m$. There is a smooth function $\lambda : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ with $\lambda'(s) > K_m$ for all $s \in [m-1, m]$. Now with $\rho = \lambda \circ \nu$ we get

$$\mathrm{d}\rho(x) = \lambda'(\nu(x)) \cdot \mathrm{d}\nu(x) \tag{9}$$

with $\lambda'(\nu(x))$ chosen sufficiently large that $\beta_0 = \theta_1 + \mathrm{d}\rho$ vanishes on V' exactly on C_0 while outside of V' it does not vanish at all.

So there is a primitive β_0 with discrete vanishing set C_0. The final step now consists in moving away these zeroes successively by induction. For this we will construct a sequence of compact subsets $Z_m \subset M$ with the following properties. The sequence is exhaustive, i.e. $\bigcup_{m \in \mathbb{N}} Z_m = M$, and for each $m \in \mathbb{N}$ the compact set Z_m is contained in the interior of Z_{m+1} and each point in $Z_{m+1} \backslash Z_m$ is joint by a smooth path in $M \backslash Z_m$ to a point in $M \backslash Z_{m+1}$. Further the sequence is initialized such that $Z_0 \cap C_0 = \emptyset$. In order to define Z_m choose a smooth exhaustion function

$\kappa : M \to \mathbb{R}^{>0}$ and a strictly increasing sequence $v_m \in \mathbb{R}^{>0}$ of regular values of κ such that v_m converges to infinity and $\{x \in M \mid \kappa(x) \leq v_0\}$ is non-empty, disjoint to C_0 and has connected complement. Set $Z'_m = \{x \in M \mid \kappa(x) \leq v_m\}$ and let $D_{m1}, \ldots, D_{mj_m}$ be those connected components of $M \backslash Z'_m$ whose closure in M is compact. Note that compactness of Z'_m implies that there are only finitely many sets D_{mj} for fixed m. Hence the sets $Z_m = Z'_m \cup \bigcup_{j=1}^{j_m} \overline{D_{mj}}$ define the exhausting sequence with the desired properties.

We will construct an adapted sequence of 1-forms β_m with vanishing sets $C_m = \{x \in M \mid \beta_m(x) = 0\}$ such that

(1) $\omega = \mathrm{d}\beta_m$

(2) C_m is discrete

(3) $Z_m \cap C_m = \emptyset$ and

(4) $\beta_m|_{Z_{m-1}} = \beta_{m-1}|_{Z_{m-1}}$ for all $m \geq 1$

holds. We may consider Condition (4) to be empty for $m = 0$ such that β_0 satisfies the four conditions. For the induction step consider a 1-form β_m to be given for some $m \in \mathbb{N}$ satisfying the four conditions. Since Z_{m+1} is compact, the set $D_{m+1} = Z_{m+1} \cap C_m$ is finite. By assumption each point in $Z_{m+1} \backslash Z_m$ is joint by a smooth path in $M \backslash Z_m$ to a point in $M \backslash Z_{m+1}$. So we may choose a relatively compact open subset $U_m \subset M \backslash Z_m$ with $U_m \cap C_m = D_{m+1}$ such that the intersection of each connected component of U_m with $(M \backslash Z_{m+1})$ is not empty. There is a symplectomorphism of U_m which maps D_{m+1} to a set contained in $M \backslash Z_{m+1}$ and this symplectomorphism can be chosen such that it differs from the identity only on a relatively compact subset of U_m ([Boo69]). Thus the symplectomorphism can be extended as the identity on $M \backslash U_m$ to a symplectomorphism $\varphi_m : M \to M$. We may define $\beta_{m+1} = \varphi_m^* \beta_m$. By construction $C_{m+1} = \varphi_m^{-1}(C_m) \subset M \backslash Z_{m+1}$ since $C_{m+1} \backslash U_m = C_m \backslash U_m$ and $\varphi_m|_{M \backslash U_m} = \mathrm{id}|_{M \backslash U_m}$.

As the sequence β_m becomes stationary on some neighborhood of each point it converges to a 1-form β with $\omega = \mathrm{d}\beta$. Since β does not vanish in Z_m for all m, the form β is a nowhere vanishing primitive for ω. □

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