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# SIMPLY AND TANGENTIALLY HOMOTOPY EQUIVALENT BUT NON-HOMEOMORPHIC HOMOGENEOUS MANIFOLDS*

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**Abstract.** For each odd integer $r$ greater than one and not divisible by three we give explicit examples of infinite families of simply and tangentially homotopy equivalent but pairwise non-homeomorphic 5-dimensional closed homogeneous spaces with fundamental group isomorphic to $\mathbb{Z}/r$. As an application we construct the first examples of manifolds which possess infinitely many metrics of nonnegative sectional curvature with pairwise non-homeomorphic homogeneous souls of codimension three with trivial normal bundle, such that their curvatures and the diameters of the souls are uniformly bounded. As a by-product, if $L$ is a smooth and closed manifold homotopy equivalent to the standard 3-dimensional lens space then the moduli space of complete smooth metrics of nonnegative sectional curvature on $L \times S^2 \times \mathbb{R}$ has infinitely many components. These manifolds are the first examples of manifolds fulfilling such geometric conditions and they serve as solutions to a problem posed by I. Belegradek, S. Kwasik and R. Schultz.

**Key words.** Homogeneous manifolds, homotopy classification, surgery, non-negative curvature, pinching, souls.

**Mathematics Subject Classification.** 57R22, 53C30, 57R20, 53C21, 57R55.

**1. Introduction.** In 1966 J. Milnor proved in [Mi66] that for each lens space $L$ with $\dim(L) \geq 5$ and fundamental group of order five or greater than six that there exist infinitely many pairwise distinct $h$-cobordisms $(W; L, L')$ over $L$ which are distinguished by the Whitehead torsion of $(W, L)$. In this case the Whitehead torsion is related to the Reidemeister torsion of the boundary components and Milnor's result implied the existence of infinitely many pairwise non-homeomorphic smooth closed manifolds in the $h$-cobordism class of $L$. These manifolds belong to the class of *fake lens spaces* which are quotients of free actions of cyclic groups on odd dimensional spheres.

C.T.C. Wall obtained classification results for the class of fake lens spaces [W99] with fundamental group of odd order which led to infinite sequences of simply homotopy equivalent but pairwise non-homeomorphic smooth closed manifolds.

Let $M$ be a $(4k+3)$-dimensional oriented closed smooth manifold, where $k \geq 1$. S. Chang and S. Weinberger [CW03] implicitly proved that if $\pi_1(M)$ is not torsion-free then there exist infinitely many closed smooth manifolds which are simply and tangentially homotopy equivalent to $M$ but pairwise non-homeomorphic.

The last two examples of infinitely many distinct manifolds sharing the same simple or tangential homotopy type were found by analyzing surgery exact sequences. However, due to the fact that these infinite sequences of simply or tangentially homotopy equivalent but pairwise non-homeomorphic manifolds are constructed in the smooth category the metric properties remain unclear.

This paper touches a fundamental problem in Riemannian Geometry, namely whether metrical properties propagate along surgeries, and more specifically in our case whether non-negative sectional curvature is preserved under the action of the $L$-group.

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A motivation of this paper was to find infinite families of pairwise non-homeomorphic manifolds sharing the same simple and tangential homotopy type. Besides presenting elementary examples of such manifolds, this paper solves Problem 4.8 (i) in [BKS11]) and was referenced in different geometric topological papers: [BH15], [BFK17], and [KS15].

Let $L^{p,q} \to S^2 \times S^2$ be a principal $S^1$-fibre bundle with first Chern class $px + qy$, where $x$ and $y$ are the standard generators of $H(S^2 \times S^2; \mathbb{Z})$ of the first and the second factor of the base space respectively, and let

$$\mathcal{L} := \{L^{p,q} | p, q \in \mathbb{Z}^2 \setminus \{(0,0)\}\}$$

These manifolds form a subclass of the class of lens space bundle over $S^2$.

Let $L^3_p$ be the 3-dimensional lens space with fundamental group isomorphic to $\mathbb{Z}/p$, where the associated action of the fundamental group on the universal cover $S^3$ preserves the $S^1$-fibre. Furthermore, $L^3_p$ is equipped with the induced metric from $S^3$. The manifold $L^{p,q}$ is diffeomorphic to the total space of the $L^3_p$-bundle over $S^2$ with clutching function

$$S^1 \to \mathrm{Isom}(L^3_p), \quad z \mapsto (x \mapsto z^q x),$$

where $\mathrm{Isom}(L^3_p)$ is the group of isometries of $L^3_p$.

THEOREM 1.1. *Let $r, t$ be integers, where $r$ is odd, greater than one and not divisible by three. The set $\mathcal{L}_{r,t} := \{L^{r,(t+kr)r} | k \in \mathbb{Z}\}$ consists of simply and tangentially homotopy equivalent but pairwise non-homeomorphic manifolds on which $SU(2) \times SU(2) \times U(1)$ acts smoothly and transitively.*

In 1972 J. Cheeger and D. Gromoll [CG72] proved that any complete open Riemannian manifold with nonnegative sectional curvature is diffeomorphic to the total space of the normal bundle of a totally geodesic and totally convex submanifold, called a soul.

This fundamental structural result on nonnegatively curved complete Riemannian manifolds led to further questions and results concerning the existence of infinitely large families of codimension $(n - l)$-souls or the moduli space of complete metrics of nonnegative sectional curvature. For example I. Belegradek [B03] constructed the first examples of manifolds admitting infinitely many nonnegatively curved metrics with mutually non-homeomorphic souls of codimension at least five. V. Kapovitch, A. Petrunin and W. Tuschmann found examples of manifolds admitting a sequence of nonnegatively curved metrics with pairwise non-homeomorphic souls of codimension at least eleven but with more geometric control, i.e. they gave uniform upper bounds for the sectional curvature of the open manifolds and the diameter of the souls [KPT05].

J. Milnor proved in [Mi61] that homotopy equivalent 3-dimensional lens spaces share the property that taking the cartesian product with $\mathbb{R}$ yields diffeomorphic manifolds. But one cannot realize an infinite sequence of pairwise homotopy equivalent but non-homeomorphic lens spaces because there are just finitely many lens spaces in each dimension for each finite cyclic group realized as the fundamental group.

A motivation of [BKS11] was to construct infinitely many pairwise homotopy equivalent but non-homeomorphic souls of the smallest possible codimension. They were able to construct infinitely many distinct codimension four souls but not of codimension less than four (Problem 4.8 (i) in [BKS11]).

As an application of Theorem 1.1 we obtain

THEOREM 1.2. *Let $r, q$ be integers such that $r$ is odd, greater than one and not divisible by three. Then there exists a positive constant $D$ independent of $r$ and $q$, such that the manifold $L^{r,qr} \times \mathbb{R} =: M_{r,q}$ possesses an infinite sequence of metrics $\{g^i_{r,q}\}_i$ of nonnegative sectional curvature with pairwise non-homeomorphic souls $\{S^i_{r,qr}\}_i$, such that*

$$0 \leq \sec(M_{r,q}, g^i_{r,q}) \leq 1 \textit{ and } \operatorname{diam}(S^i_{r,q}) \leq D,$$

*where $SU(2) \times SU(2) \times U(1)$ acts smoothly, transitively and isometrically on these souls.*

By applying differential topological arguments we deduce that there is no manifold such that an infinite subset of manifolds in $\{S^i_{r,qr}\}_i$ can be realized as souls of codimension one or two with trivial normal bundle, see Lemma 5.2 (a) and Remark 5.3 (ii).

As a consequence of the previous theorems there is the following theorem.

THEOREM 1.3. *There are infinitely many homotopy equivalent but pairwise non-diffeomorphic Riemannian non-simply connected 5-manifolds with $0 \leq sec \leq 1$ and diameter $\leq D$, where $D$ is a positive constant.*

If we choose $q$ to be zero then $M_{r,q} = L^3_r \times S^2 \times \mathbb{R}$. Let $L$ be a 3-dimensional lens space which is homotopy equivalent to $L^3_r$ then we know from [Mi61] that $L^3_r \times \mathbb{R}$ and $L \times \mathbb{R}$ are diffeomorphic. Hence $L^3_r \times S^2 \times \mathbb{R}$ and $L \times S^2 \times \mathbb{R}$ are diffeomorphic and we obtain as a special case of Theorem 1.2.

COROLLARY 1.4. *Let $r$ be as in the previous theorems and $L$ a lens space homotopy equivalent to $L^3_r$ then $L \times S^2 \times \mathbb{R}$ possesses an infinite sequence of metrics of nonnegative sectional curvature with pairwise non-homeomorphic homogeneous souls such that the geometric implications in Theorem 1.2 hold.*

Let $N$ be a smooth manifold and $\mathfrak{R}^u_{sec\geq 0}(N)$ be the space of complete smooth metrics with topology induced by uniform smooth convergence. If we mod out the action of $\operatorname{Diff}(N)$ on $\mathfrak{R}^u_{sec\geq 0}(N)$ via pullback we obtain the moduli space of complete smooth metrics on $N$ of nonnegative sectional curvature $\mathfrak{M}^u_{sec\geq 0}(N)$. A soul is in general not unique but by V.A. Sharafutdinov [Sh79] any two souls of a metric can be mapped onto each other by a diffeomorphism of the total space. I. Belegradek, S. Kwasik and R. Schultz sharpened this result [BKS11, Theorem 1.4(iii)] by proving that the function that assigns to a nonnegatively curved complete metric on a manifold $N$ the diffeomorphism type of the pair $(N, S_g)$ is constant on connected components of $\mathfrak{M}^u_{sec\geq 0}(N)$. This leads us to the following

COROLLARY 1.5. *If $r$ is as in Theorem 1.1 then $\mathfrak{M}^u_{sec\geq 0}(L^3_r \times S^2 \times \mathbb{R}^3)$ has infinitely many components.*

Let $M$ be a smooth manifold and $G$ a group acting smoothly on $M$. We call two $G$-actions on $M$ equivalent if they differ by a self-diffeomorphism of $M$ through conjugation. The proof of Theorem 1.1 and Theorem 1.2 (see Lemma 5.1) implies

COROLLARY 1.6. *Let $r$ be as in the previous theorems. There exist infinitely many pairwise inequivalent smooth transitive actions of $SU(2) \times SU(2) \times U(1) \times \mathbb{R}^3$ on $L^3_r \times S^2 \times \mathbb{R}$, whereas there exists only one equivalence class of smooth transitive actions of $SU(2) \times SU(2) \times U(1)$ on $L^3_r \times S^2$.*

This paper is structured as follows:

*Section 2 - A homotopy classification:* Let $n, r$ be non-zero integers, where $r$ is chosen as in Theorem 1.1. We show that the manifolds in $\{L^{r,(t+kr)r}\}_k$ lie in the same simple and tangential homotopy type. Therefore we first give a classification of the manifolds in $\mathcal{L}$ up to homotopy. We prove that the tangent bundle of these manifolds is trivial and that their Reidemeister torsion has a simple structure. By applying these facts we show that homotopy equivalences which may occur between these manifolds are tangential and simple. For clarification, let $M$ and $N$ be smooth manifolds and $\tau_M$ and $\tau_N$ the associated tangent bundles. A homotopy equivalence $h : M \to N$ is tangential if $h^*(\tau_N)$ is stably equivalent to $\tau_M$ [M80].

*Section 3 - Distinguishing homeomorphism types:* We use the $\rho$-invariant, calculate it for the manifolds in question and conclude that the manifolds in $\{L^{r,(t+kr)r}\}_k$ are pairwise non-homeomorphic.

*Section 4 - Souls of codimension three:* We show that each manifold in $\mathcal{L}$ is diffeomorphic to the quotient of $SU(2) \times SU(2) \times U(1)$ by an isometric 2-torus action. O'Neill's formula on Riemannian submersions ensures the existence of metrics of non-negative sectional curvature on the corresponding manifolds.

*Section 5 - Remarks and a question:* Here we explain why there is no manifold $M$ such that an infinite subset of manifolds in $\mathcal{L}$ can be realized as souls of $M$ with trivial normal bundle and of codimension less than three. Moreover we raise a question which connects the simple homotopy type of a manifold with the existence of metrics of nonnegative sectional curvature.

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**2. A homotopy classification.** From now on let the manifolds in $\mathcal{L}$ be oriented by orienting the base and the fibre in the standard way and throughout this paper if we speak about spin manifolds we mean oriented manifolds given a certain compatible spin structure. Furthermore, homotopy equivalences, homeomorphisms and diffeomorphisms are orientation preserving.

This section is structured as follows:

1) We start with calculating basic algebraic and differential topological properties of manifolds in $\mathcal{L}$ these become a necessary input for proofing the homotopy classification.
2) The main goal of this section is to proof Theorem 2.2, which gives suficient and neccessary numerical conditions for manifolds in $\mathcal{L}$ for being homotopy equivalent, where the fundamental groups are as assumed in Theorem 1.1. Let $M$ and $N$ be manifolds in $\mathcal{L}$ with $\pi_1(M) = \mathbb{Z}/r = \pi_1(M)$ and let $B_r$ be $K(\mathbb{Z}/r, 1) \times K(\mathbb{Z}, 2)$, where $K(*, *)$ denotes an Eilenberg-Maclane space. Let $\Omega_5^{Spin}(B_r)$ be the bordism group of spin 5-manifolds, where an element in $\Omega_5^{Spin}(B_r)$ is represented by a closed spin 5-manifold $X$ together with a map $s : X \to B_r$, which we denote by $(X, s)$. $(M, f)$ and $(N, g)$ represent the same element in $\Omega_5^{Spin}(B_r)$ if there exists a 6-dimensional smooth spin manifold with boundary equal to the disjoint union of $M$ and $N$ and a map $F : W \to B_r$, which restricted to the boundary components is the map $f$,

$g$ respectively. The strategy for proving Theorem 2.2 is to characterise an element $(M, g)$ in $\Omega_5^{Spin}(B_r)$ in terms of homotopical properties of $g$ and to relate 3-equivalences $(M, g)$ and $(N, f)$ that are bordant in $\Omega_5^{Spin}(B_r)$ to homotopy classification by applying Kreck's modified surgery [K99].

3) By applying differential topological properties mentioned in 1) and Theorem 2.2 we prove that the manifolds in Theorem 1.1 are tangentially homotopy equivalent.

Let $L^{p,q} \in \mathcal{L}$. The homotopy long exact sequence of the $S^1$-fibre bundle over $S^2 \times S^2$ given by the first Chern class $px + qy$ implies that $\pi_1(L^{p,q})$ is isomorphic to $\mathbb{Z}/\mathrm{gcd}(p, q)$. Furthermore since $(p, q) \neq (0, 0)$ we conclude that the universal covering space of $L^{p,q}$ is

$$L^{\frac{p}{gcd(p,q)} \frac{q}{gcd(p,q)}}$$

and again the above mentioned homotopy long exact sequence together with the Hurewicz Theorem implies that $\pi_2(L^{p,q})$ is isomorphic to $\mathbb{Z}$.

Let $\Pi_{p,q} : L^{p,q} \to S^2 \times S^2$ denote the projection map of the fibre bundle. The tangent bundle of $L^{p,q}$, $\tau_{L^{p,q}}$ is the sum of the vertical tangent bundle and the pull-back of the tangent bundle of the base space. If we assume that $\pi_1(L^{p,q})$ has odd order then $H^1(L^{p,q}; \mathbb{Z}/2)$ is trivial and thus $\tau_{L^{p,q}}$ is isomorphic to $\Pi^*_{p,q}(\tau_{S^2 \times S^2}) \oplus \epsilon$, where $\epsilon$ denotes the trivial real line bundle over $L^{p,q}$.

LEMMA 2.1.

(i) *The fundamental group of $L^{p,q}$ acts homotopically trivially on the universal covering space.*

(ii) *If $\pi_1(L^{p,q})$ has odd order then the tangent bundle of $L^{p,q}$ is trivial and $L^{p,q}$ admits a unique spin structure.*

(iii) *The universal covering space $\tilde{L}^{p,q} = L^{\frac{p}{gcd(p,q)} \frac{q}{gcd(p,q)}}$ is diffeomorphic to $S^2 \times S^3$. Thus $\pi_2(L^{p,q}) \cong \mathbb{Z}$.*

*Proof.* (i) The deck-transformations sit inside a circle action. (ii) This follows from the description of $\tau_{L^{p,q}}$ and the fact that $\tau_{S^2} \oplus \epsilon$ is trivial. (iii) Since the universal covering space of $L^{p,q}$ is spin it follows from a result of S. Smale [S62] that the diffeomorphism type only depends on $H_2(\tilde{L}^{p,q}; \mathbb{Z})$. The Gysin sequence of the $S^1$-fibre bundle structure of $L^{\frac{p}{gcd(p,q)} \frac{q}{gcd(p,q)}}$ implies that $H_2(\tilde{L}^{p,q}; \mathbb{Z}) \cong \mathbb{Z}$. Hence $\tilde{L}^{p,q}$ is diffeomorphic to $S^2 \times S$. □

THEOREM 2.2. *Let $r$ be as in Theorem 1.1 and $L^{p,q}$, $L^{p',q'} \in \mathcal{L}$ with $\pi_1(L^{p,q}) \cong \pi_1(L^{p',q'}) \cong \mathbb{Z}/r$. Furthermore let $(m, n)$, $(m', n')$ be pairs of integers such that $m\frac{q}{r} + n\frac{p}{r} = 1 = m'\frac{q'}{r} + n'\frac{p'}{r}$. Then $L^{p,q}$ and $L^{p',q'}$ are homotopy equivalent if and only if there exist $s, s' \in (\mathbb{Z}/r)^*$, $\epsilon, \epsilon' \in \{\pm 1\}$ and $k, k' \in \mathbb{Z}/r$ such that*

$$s^3 \frac{pq}{r^2} \equiv s'^3 \frac{p'q'}{r^2} mod\ r,$$

$$s(\epsilon m + k\frac{p}{r})(\epsilon n - k\frac{q}{r}) \equiv s'(\epsilon' m' + k'\frac{p'}{r})(\epsilon' n' - k'\frac{q'}{r}) mod\ r,$$

$$s^2(\frac{q}{r}(\epsilon m + k\frac{p}{r}) - \frac{p}{r}(\epsilon n - k\frac{q}{r})) \equiv s'^2(\frac{q'}{r}(\epsilon' m' + k'\frac{p'}{r}) - \frac{p'}{r}(\epsilon n' - k'\frac{q'}{r})) mod\ r.$$

Before we give a proof of Theorem 2.2 we gather some further differential and algebraic topological properties of the manifold $L^{p,q}$.

Lemma 2.3. *The R-torsion in the sense of [Mi66, pp.404] is defined for $L^{p,q}$ is $(t-1)^4$, where $t$ represents the fibre.*

*Proof.* This torsion invariant is defined for manifolds with finite cyclic fundamental group that acts trivially on the rational cohomology ring of the universal covering space. By Lemma 2.1 (i) and the fact that $\pi_1(L^{p,q}) \cong \mathbb{Z}/\mathrm{gcd}(p,q)$ these assumptions are fulfilled for $L^{p,q}$. Since $L^{p,q}$ is the total space of an $S^1$-fibre bundle over a 1-connected space it follows from [HKR07, Thm. B] that its $R$-torsion equals $(t-1)^4$. Alternatively, an exlplicit computation can be found in the proof of Theorem 3.4.18 in [Ot09]. □

Lemma 2.4. *Let $r = gcd(p,q)$. The second level of the Postnikov tower of $L^{p,q}$ is*

$$L_r^\infty \times \mathbb{C}P^\infty =: B_r$$

*up to fibrewise homotopy equivalence, where $L_r^\infty$ is the infinite dimensional lens space which is a $K(\mathbb{Z}/r, 1)$ and $\mathbb{C}P^\infty$ is the infinite dimensional complex projective space which is a $K(\mathbb{Z}, 2)$.*

*Proof.* Since $\pi_1(L^{p,q}) \cong \mathbb{Z}/\mathrm{gcd}(p,q)$ the first level of the Postnikov decomposition of $L^{p,q}$ equals $L^\infty_{gcd(p,q)}$. Since $\pi_1(L^{p,q})$ acts trivially on the higher homotopy groups Postnikov theory implies that the second level of the Postnikov tower of $L^{p,q}$ is a $\mathbb{C}P^\infty$-fibration over $L^\infty_{gcd(p,q)}$, which is a pullback of the principal $K(\pi_2(L^{p,q}) \cong \mathbb{Z}, 2)$-fibration over $K(\pi_1(L^{p,q}) \cong \mathbb{Z}/gcd(p,q), 1)$. By obstruction theory the homotopy class of the classifying map of this fibration may be identified with an element of $H(L^\infty_{gcd(p,q)}; \pi_2(L^{p,q}))$ which is trivial. □

Proposition 2.5. *Let $r$ be as in Theorem 1.1 and $N, M$ be closed smooth oriented spin 5-manifolds with vanishing first Pontrjagin classes and $f, g$ maps from $N, M$ to $B_r$ respectively. Then $(N, f)$ and $(M, g)$ represent the same element in $\Omega_5^{Spin}(B_r)$ if and only if*

$$f_*[N] = g_*[M] \in H_5(B_r; \mathbb{Z}).$$

*Proof.* The entry $E^2_{a,b}$ of the $E_2$-term of the *Atiyah Hirzebruch Spectral Sequence* (AHSS) for computing

$$\Omega_\star^{Spin}(L_r^\infty \times \mathbb{C}P^\infty = B_r)$$

is $H_a(L_r^\infty \times \mathbb{C}P^\infty; \Omega_b^{Spin}(*))$. Since $r$ is odd the $E^2$-term for $a + b \leq 6$ looks as follows:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 | 0 | | | | | | |
| 5 | 0 | 0 | | | | | |
| 4 | $\mathbb{Z}$ | $\mathbb{Z}/r$ | $\mathbb{Z}$ | | | | |
| 3 | 0 | 0 | 0 | 0 | | | |
| 2 | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/2$ | | |
| 1 | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/2$ | 0 | |
| 0 | $\mathbb{Z}$ | $\mathbb{Z}/r$ | $\mathbb{Z}$ | $\mathbb{Z}/r$ | $\mathbb{Z}$ | $(\mathbb{Z}/r)^3$ | $\mathbb{Z}$ |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 . |

From [T93, p.7] we know that the differentials in $E^2_{a,b}(L_r^\infty)$ respectively $E^2_{a,b}(\mathbb{C}P^\infty)$ from the first to the second row are the dual of the Steenrod square $Sq^2 : H^*(\cdot; \mathbb{Z}/2) \to$

$H^{*+2}(\cdot;\mathbb{Z}/2)$ precomposed with the reduction map in homology and the differentials from the second to the third row are the dual of $Sq^2$. The differentials $d_5 : E^5_{6,0}(L^\infty_r) \to E^5_{1,4}(L^\infty_r)$ and $d_5 : E^5_{6,0}(\mathbb{C}P^\infty) \to E^5_{1,4}(\mathbb{C}P^\infty)$ are trivial. This observation, the exterior product structure of $E^r_{s,t}(L^\infty_r \times \mathbb{C}P^\infty)$ induced by $E^r_{a,b}(L^\infty_r)$ and $E^r_{c,d}(\mathbb{C}P^\infty)$ and the fact that the differentials obey the Leibniz rule imply that $d_5 : E^5_{6,0}(L^\infty_r \times \mathbb{C}P^\infty) \to E^5_{1,4}(L^\infty_r \times \mathbb{C}P^\infty)$ is trivial. Further details can be found in the proof of Proposition 3.2.2 in [Ot09]. In conclusion $E^\infty_{a,b}(L^\infty_r \times \mathbb{C}P^\infty)$ equals $E^5_{a,b}(L^\infty_r \times \mathbb{C}P^\infty)$ for $a+b = 1, 3, 5$:

$$\begin{array}{ccccccc}
5 & 0 & & & & & \\
4 & & \mathbb{Z}/r & & & & \\
3 & 0 & & 0 & & & \\
2 & & 0 & & 0 & & \\
1 & 0 & & 0 & & 0 & \\
0 & & \mathbb{Z}/r & & (\mathbb{Z}/r)^2 & & (\mathbb{Z}/r)^3 \\
 & & & & & & \\
 & 0 & 1 & 2 & 3 & 4 & 5 \quad .
\end{array}$$

Hence

$$h_1 : \Omega^{Spin}_1(L^\infty_r \times \mathbb{C}P^\infty) \to H_1(L^\infty_r \times \mathbb{C}P^\infty;\mathbb{Z}), \quad [(S,l)] \mapsto l_*[S],$$

is an isomorphism.

Let $K$ be a Kummer surface equipped with its standard orientation. We know from [Mi63] that $K$ generates $\Omega^{Spin}_4(*)$. The construction of the AHSS and its $\infty$-term imply the following extension problem:

$$\Omega^{Spin}_1(L^\infty_r \times \mathbb{C}P^\infty) \overset{\mu_K}{\hookrightarrow} \Omega^{Spin}_5(L^\infty_r \times \mathbb{C}P^\infty) \overset{h_5}{\twoheadrightarrow} H_5(L^\infty_r \times \mathbb{C}P^\infty;\mathbb{Z}) \to 0, \qquad (1)$$

where $\mu_K([(S,l)]) = [(K \times S, l \circ \mathrm{pr}_S)]$ and $h_5([(N,g)]) = g_*[N]$, the Thom map. Let $S^1$ be equipped with the standard orientation and $i : S^1 \to L^\infty_r$ be the inclusion of $S^1$ as the 1-skeleton of $L^\infty_r$. The fact that $h_1$ is an isomorphism implies that $(S^1, i)$ represents a generator of $\Omega^{Spin}_1(L^\infty_r \times \mathbb{C}P^\infty)$.

Let $v_1$ be a generator of $H^1(L^\infty_r \times \mathbb{C}P^\infty;\mathbb{Z}/r)$. We claim that composing the homomorphism

$$n_1 : \Omega^{Spin}_5(L^\infty_r \times \mathbb{C}P^\infty) \to \mathbb{Z}/r, \quad [(N,g)] \mapsto \left\langle \rho_r(p_1(N))g^*(v_1), [N]_{\mathbb{Z}/r} \right\rangle,$$

with $\mu_K$ is an isomorphism:

The Kuenneth theorem implies that $(n_1 \circ \mu_K)([S^1, i])$ equals

$$\left\langle \rho_r(p_1(K)), [K]_{\mathbb{Z}/r} \right\rangle \left\langle i^*(v_1), [S^1]_{\mathbb{Z}/r} \right\rangle,$$

where $\rho_r$ is the mod-$r$-reduction homomorphism in cohomology. It is clear that $\left\langle i^*(v_1), [S^1]_{\mathbb{Z}/r} \right\rangle$ is a generator of $\mathbb{Z}/r$. By the Hirzebruch signature theorem it is known that $\left\langle \frac{p_1}{3}(K), [K] \right\rangle$ equals $\mathrm{sign}(K)$ which is $-16$. Thus

$$\left\langle \rho_r(p_1(K)), [K]_{\mathbb{Z}/r} \right\rangle \equiv -48 \bmod r$$

is a generator of $\mathbb{Z}/r$ if and only if $\gcd(r, 48) = 1$. But $\gcd(r, 48) = 1$ if and only if $r$ is odd and not divisible by 3 which is the case by assumption.

Thus $(n_1 \circ \mu_K)([S^1, i]) \in (\mathbb{Z}/r)^*$ and we can compose $n_1$ with an appropriate isomorphism $\alpha$ from $\mathbb{Z}/r$ to $\Omega_5^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ such that $\alpha \circ n_1$ is a splitting of (1). We conclude that $\Omega_5^{Spin}(L_r^\infty \times \mathbb{C}P^\infty) \cong (\mathbb{Z}/r)^4$ and $(N, f)$, $(M, g)$ represent the same element in $\Omega_5^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ if and only if $n_1(N, f) = n_1(M, g)$ and $f_*[N] = g_*[M]$. Since the first Pontrjagin classes of the underlying manifolds are trivial the proposition follows. □

Proposition 2.6. *Let $r$ be as in Theorem 1.1 and $M$ and $N$ be closed smooth oriented spin 5-manifolds having $B_r$ as the second level of their Postnikov tower. Then $M$ and $N$ are homotopy equivalent if and only if there exist 3-equivalences $g : M \to B_r$ and $f : N \to B_r$ s.t. $g_*[M] = f_*[N]$.*

*Proof.* "⇒": Let $(M, g)$ be a 3-equivalence. If $h : M \to N$ is a homotopy equivalence then $g \circ h^{-1} =: f$ is a 3-equivalence with $g_*[M] = f_*[N]$ .

"⇐": By Proposition 2.5, $[M, g] = [N, f]$ in $\Omega_5^{Spin}(B_r)$. Now we relate bordisms to homotopy classification.

By the given assumption the Postnikov decomposition of $N$ yields maps $f : N \to B_r$ which are 3-equivalences, i.e. they induce isomorphisms on the first and second homotopy groups. Let $f : N \to B_r$, $g : M \to B_r$ be 3-equivalences then $(N, f)$, $(M, g)$ define elements in $\Omega_5^{Spin}(B_r)$. Assume that $(N, f)$, $(M, g)$ represent the same element in $\Omega_5^{Spin}(B_r)$ then there exists a bordism $(W, F)$ between them as described above.

Let $L_6^{s,\tau}(\mathbb{Z}/r, S)$ be the set which consists of stable equivalence classes of weakly based non-singular $(-1)$-quadratic forms over $\Lambda := \mathbb{Z}[\mathbb{Z}/r]$, where stabilization goes by taking orthogonal sum with the $(-1)$-hyperbolic forms and weakly based means that there is a choice of an equivalence class of bases of the underlying free $\Lambda$-module, where two bases are equivalent if the change of basis matrix has trivial Whitehead torsion. The $S$ between the brackets stands for the choice of a so called form parameter [Ba81]. One can easily show that $L_6^{s,\tau}(\mathbb{Z}/r, S)$ is a group with group structure given by taking orthogonal sum. The simple Wall group $L_6^s(\mathbb{Z}/r, S)$ and $L_6^{s,\tau}(\mathbb{Z}/r, S)$ are related to each other by the following exact sequence:

$$0 \to L_6^s(\mathbb{Z}/r, S) \xrightarrow{i} L_6^{s,\tau}(\mathbb{Z}/r, S) \xrightarrow{\tau} \mathrm{Wh}\ (\mathbb{Z}/r),$$

where $i$ is just the canonical inclusion and the map $\tau$ sends stable equivalence classes of weakly based non-singular $(-1)$-quadratic forms over $\Lambda$ to the Whitehead torsion of the matrix representation of the adjoint of this quadratic form with respect to to the chosen weak equivalence class of basis. One can associate to $(W, F)$ an element in $L_6^{s,\tau}(\mathbb{Z}/r, S(N \times I))$, where $S(N \times I)$ is explained below:

Let us recall Wall's definition of a quadratic form on an even dimensional compact manifold. We equip $W$ with a base point and orient it at this point. Wall defines a skew-hermitian form $\lambda$ on the group of regular homotopy classes of immersions of 3-dimensional spheres into $W$ which roughly speaking is given by transversal double point intersections which along two branches are joined with the base point such that this form takes values in $\Lambda$. This form is called the *equivariant intersection form* associated to $W$. Similarly Wall assigns to each immersion $u$ an element $\mu(u) \in \frac{\Lambda}{\langle a+\bar{a}\rangle}$ which is given by self-intersection. If we compose $\mu$ with the quotient map onto $\frac{\Lambda}{\langle a+\bar{a},1\rangle}$ we call the result $\tilde{\mu}$.

By [K99, Prop. 4] we may assume that $F$ is a 3-equivalence and we identify $\pi_1(W)$ with $\mathbb{Z}/r$. For the sake of simplicity we denote $N$ by $N_0$ and $M$ by $N_1$. Since $(W, N_i)$ is 2-connected $\pi_3(W, N_i)$ and $H_3(W, N_i; \Lambda)$ are isomorphic under the relative Hurewicz homomorphism. Poincar duality implies that $H_3(W, N_i; \Lambda)$ is the only possibly non-vanishing homology group of the pair $(W, N_i)$ and by [W99, Lemma 2.3.] it follows that $H_3(W, N_i; \Lambda)$ is a stably free $\Lambda$-module with a preferred equivalence class of $s$-basis (for a definition of $s$-basis see [Mi66, p.369]). If we take connected sum of $W$ with $\#_k(S^3 \times S^3)$ for some $k \in \mathbb{N}$ big enough (see [K99, p.723]) we may assume that $H_3(W, N_i; \Lambda)$ is a free $\Lambda$-module. The intersection form $\lambda : H_3(W, N_0; \Lambda) \times H_3(W, N_1; \Lambda) \to \Lambda$ is unimodular which follows from Poincar duality and [W99, Theorem 2.1.] tells us that this form is even simple if $H_3(W, N_0; \Lambda)$ and $H_3(W, N_1; \Lambda)$ are equipped with preferred bases. Let $K\pi_3(W)$ be $\ker(F_* : \pi_3(W) \to \pi_3(B_r))$, $K\pi_3(N_i)$ be $\ker(f_{i_*} : \pi_3(N_i) \to \pi_3(B_r))$ and $\mathrm{Im}K\pi_3(N_i)$ be the image of $K\pi_3(N_i)$ under the homomorphism which is induced by the inclusion $N_i \hookrightarrow W$. We claim that $\mathrm{Im}K\pi_3(N_0) = \mathrm{Im}K\pi_3(N_1)$ :

Assume there exists an element $x \in \mathrm{Im}K\pi_3(N_0)$ that doesn't lie in $\mathrm{Im}K\pi_3(N_1)$. Then by the homotopy exact sequence associated to $(W, N_1)$ $x$ represents a non-trivial element in $\pi_3(W, N_1)$. As $\lambda$ is non-degenerate and $\pi_3(W) \to \pi_3(W, N_0)$ is surjective there exists a $y \in \pi_3(W)$ such that $\lambda(x, y) \neq 0$. But since $x$ is trivial in $\pi_3(W, N_0)$ we have $\lambda(x, y) = 0$ which is a contradiction. By interchanging the roles of $N_0$ and $N_1$ the claim follows.

The map $\pi_3(W) \to \pi_3(W, N_i)$ induces an isomorphism $\frac{K\pi_3(W)}{\mathrm{Im}K\pi_3(N_i)} \overset{\approx}{\to} \pi_3(W, N_i)$ which is seen with the help of the following diagram,

$$\begin{tikzcd}
 & \pi_4(B, W) \arrow[d] \arrow[dr] & & \\
\pi_3(N_i) \arrow[r] \arrow[dr] & \pi_3(W) \arrow[r] \arrow[d] & \pi_3(W, N_i) \arrow[r] & 0 \\
 & \pi_3(B) \arrow[d] & & \\
 & 0, & &
\end{tikzcd}$$

where the diagonal maps are surjective.

As promised above we explain what $S$ stands for: By $S(W)$ we denote the subgroup of $\Lambda$ which projects onto the image of $\mu$ restricted to $\mathrm{Im}K\pi_3(N_0)$. Since $\mathrm{Im}K\pi_m(N_0) = \mathrm{Im}K\pi_m(N_1)$ $S(W)$ equals $S(N_0 \times I) = S(N_1 \times I)$ and we denote $S(W)$ by $S$.

If we equip $\frac{K\pi_3(W)}{\mathrm{Im}K\pi_3(N_0)}$ with the basis which is induced by the preferred basis on $H_3(W, N_0; \Lambda)$ then by the map $\pi_3(W) \to \pi_3(W, N_i)$ which comes from the inclusion $W \hookrightarrow (W, N_i)$, $\lambda$ induces the form

$$\bar{\lambda} : \frac{K\pi_3(W)}{\mathrm{Im}K\pi_3(N_0)} \times \frac{K\pi_3(W)}{\mathrm{Im}K\pi_3(N_0)} \to \Lambda.$$

This is a unimodular skew-hermitian form and $(\bar{\lambda}, \tilde{\mu})$ represents an element $[(\bar{\lambda}, \tilde{\mu})] =: \Theta(W, F)$ in $L_6^{s,\tau}(\mathbb{Z}/r, S(L^{p,q} \times I))$ which indeed doesn't depend on the choice of a normal bordism $(W, F)$. For every $\Theta \in L_6^{s,\tau}(\mathbb{Z}/r, S)$ there exists a 3-equivalence $(M, f')$ and a bordism $(W', F')$ between $(N, f)$ and $(M, f')$ such that $\Theta(W', F') = \Theta$. Moreover $(M, f')$ is up to $s$-cobordism completely determined by $\Theta$. This is proved in [W99, Theorem 5.8] for the case of simple normal $n$-smoothings into a finite Poincar complex. The proof obviously generalises to our situation. Furthermore $N$ and $\Theta \cdot N$ are homotopy equivalent. □

Remark 2.7. *In section 2.7. of [Ot09] an analysis of* $\Theta(W, F)$ *led to a classification of the manifolds in* $\mathcal{L}$ *with* $|\pi_1|$ *coprime to 6, as assumed in Theorem 1.1., up to diffeomorphism.*

In order to prove Theorem 2.2 we need the following

LEMMA 2.8. *Let $r$ be as in Theorem 1 and $N$, $N'$ be smooth closed spin 5-manifolds with vanishing first Pontrjagin classes and $\pi_1(N) \cong \pi_1(N') \cong \mathbb{Z}/r$, where $\pi_1$ acts trivially on $\pi_2$. Then there exist 3-equivalences $(N,f)$ and $(N',f')$ representing the same element in $\Omega_5^{Spin}(B_r)$ if and only if there exist generators $v \in H^1(N;\mathbb{Z}/r)$, $v' \in H^1(N';\mathbb{Z}/r)$ and $z \in H^2(N;\mathbb{Z})$, $z' \in H^2(N';\mathbb{Z})$ which project to generators of $\frac{H^2(N;\mathbb{Z})}{\text{torsion}}$ and $\frac{H^2(N';\mathbb{Z})}{\text{torsion}}$ respectively such that*

$$\begin{aligned}
&1)\quad \left\langle v(\beta_r(v))^2, [N]_{\mathbb{Z}/r}\right\rangle \equiv \left\langle v'(\beta_r(v'))^2, [N']_{\mathbb{Z}/r}\right\rangle;\\
&2)\quad \left\langle v\beta_r(v)z_r, [N]_{\mathbb{Z}/r}\right\rangle \equiv \left\langle v'\beta_r(v')z'_r, [N']_{\mathbb{Z}/r}\right\rangle;\\
&3)\quad \left\langle vz_r^2, [N]_{\mathbb{Z}/r}\right\rangle \equiv \left\langle v'z_r'^2, [N']_{\mathbb{Z}/r}\right\rangle;
\end{aligned}$$

*where $z_r$ and $z'_r$ are the mod $r$ reductions of $z$ and $z'$ and $[M]_{\mathbb{Z}/r}$, $[N]_{\mathbb{Z}/r}$ denote the mod $r$-reduction of the fundamental class $[M]$ and $[N]$ respectively.*

*Proof.* Proposition 2.6 tells us that $(N,f)$ and $(N',f')$ represent the same element in $\Omega_5^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ if and only if $f_*[N] = f'_*[N']$. We observe that $H_5(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}) \cong H_5(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}/r) \cong H^5(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}/r)$. A basis of $H^5(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}/r)$ is given by $v_1y_r^2$, $v_1(\beta_r(v_1))y_r$, $v_1(\beta_r(v_1))^2$, where $v_1$ is a generator of $H^1(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}/r)$, $y_r$ is a generator of $H^2(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}/r)$ which comes from the mod $r$ reduction of the standard generator $y$ of $H^2(\mathbb{C}P^\infty;\mathbb{Z})$ and $\beta_r$ is the mod $r$ Bockstein homomorphism. We see that $f_*[N] = f'_*[N']$ if and only if $f_*[N]_{\mathbb{Z}/r} = f'_*[N']_{\mathbb{Z}/r}$. But this is the case if and only if

$$\begin{aligned}
\left\langle f^*(v_1(\beta_r(v_1))^2), [N]_{\mathbb{Z}/r}\right\rangle &\equiv \left\langle f'^*(v_1(\beta_r(v_1))^2), [N']_{\mathbb{Z}/r}\right\rangle,\\
\left\langle f^*(v_1(\beta_r(v_1))y_r), [N]_{\mathbb{Z}/r}\right\rangle &\equiv \left\langle f'^*(v_1(\beta_r(v_1))y_r), [N']_{\mathbb{Z}/r}\right\rangle,\\
\left\langle f^*(v_1y_r^2), [N]_{\mathbb{Z}/r}\right\rangle &\equiv \left\langle f'^*(v_1y_r^2), [N']_{\mathbb{Z}/r}\right\rangle.
\end{aligned}$$

We finish the proof by replacing $f^*(v_1)$ by $v$, $f'^*(v_1)$ by $v'$, $f^*(y_r)$ by $z_r$ and $f'^*(y_r)$ by $z'_r$. □

Let $i: S^1 \hookrightarrow L^{p,q}$ be the inclusion of the fibre which preserves the chosen orientation of the fibre. Furthermore let $m,n \in \mathbb{Z}$ such that $m\frac{q}{r} + n\frac{p}{r} = 1$. The Gysin sequence associated to $\Pi_{p,q}$ implies that $H^2(L^{p,q};\mathbb{Z}) \cong \mathbb{Z}/r \oplus \mathbb{Z}$, where $\frac{p}{r}\Pi_{p,q}^*(x) + \frac{q}{r}\Pi_{p,q}^*(y)$ is a generator of the torsion part and $m\Pi_{p,q}^*(x) - n\Pi_{p,q}^*(y)$ is a generator of a $\mathbb{Z}$-summand. By $\alpha$ we denote the *preferred generator* of $H^1(L^{p,q};\mathbb{Z}/r)$ which is characterized by the following property:

$$\left\langle i^*(\alpha), [S^1]_{\mathbb{Z}/r}\right\rangle = 1.$$

Let $v_1$, $z$ be the standard generators of $H^1(L_r^\infty;\mathbb{Z}/r)$, $H^2(\mathbb{C}P^\infty;\mathbb{Z})$ respectively.

This gives rise to a map $g$, which is given as follows:

$$\begin{aligned}
&g^*(v_1) = s\alpha,\\
&\quad g^*(z) = \epsilon(g)(m\Pi_{p,q}^*(x) - n\Pi_{p,q}^*(y)) + k(g,m,n)(\frac{p}{r}\Pi_{p,q}^*(x) + \frac{q}{r}\Pi_{p,q}^*(y)),
\end{aligned}$$

where $s$ is a unit in $\mathbb{Z}/r$, $\epsilon(g) \in \{\pm 1\}$ and $k(g,m,n) \in \mathbb{Z}/r$. We denote the set of triples $\{(s,\epsilon,k) | s \in (\mathbb{Z}/r)^*, \epsilon \in \{\pm 1\}, k \in \mathbb{Z}/r\}$ by $T$.

Lemma 2.9. *Fixing a choice of $m, n \in \mathbb{Z}$ such that $m\frac{q}{r} + n\frac{p}{r} = 1$ then there is a 1-1 correspondence between the set $H$ of homotopy classes of 2-smoothings of $L^{p,q}$ and $T$, where the bijection is given as follows:*

$$\begin{aligned} \mathcal{C} : & H \to T, \\ & [g] \mapsto (s(g), \epsilon(g), k(g, m, n)). \end{aligned}$$

*Proof.* It is clear that $\mathcal{C}$ is injective. We claim that $\mathcal{C}$ is also surjective, i.e. for fixed $m, n \in \mathbb{Z}$ as above any triple $(\epsilon, s, k) \in T$ has a preimage under $\mathcal{C}$. We can write $f$ as $f_1 \times f_2$. The homotopy class $[f_1]$ of $f_1$ can be seen as an element in $H^1(L^{p,q}; \mathbb{Z}/r) \cong \mathbb{Z}/r$.

Any self-automorphism of $\mathbb{Z}/r$ is given by a unit $s$ of $\mathbb{Z}/r$, $(1 \mapsto s)$. Further there is a 1-1 correspondence between self-automorphisms of $\pi_1(L_r^\infty)(\cong \mathbb{Z}/r)$ and homotopy classes of self-maps of $L_r^\infty$. Thus the homotopy classes of self-maps of $L_r^\infty$ correspond to self-automorphisms of $H_1(L_r^\infty; \mathbb{Z})$. Now let $g$ be a self-map of $L_r^\infty$, then from the naturality of the Universal Coefficient Theorem we have:

$$\begin{aligned} g^* : H^1(L_r^\infty; \mathbb{Z}) \cong \text{Hom } (H_1(L_r^\infty; \mathbb{Z}), \mathbb{Z}/r) &\to \text{Hom } (H_1(L_r^\infty; \mathbb{Z}), \mathbb{Z}/r), \\ h &\mapsto g^*(h) = h \circ g_*. \end{aligned}$$

This means that $g^* : H^1(L_r^\infty; \mathbb{Z}) \to H^1(L_r^\infty; \mathbb{Z})$ is an automorphism if and only if $g_* : H_1(L_r^\infty; \mathbb{Z}) \to H_1(L_r^\infty; \mathbb{Z})$ is an automorphism. Thus the set of self-maps of $L_r^\infty$ that induce self-automorphisms on $\pi_1(L_r^\infty)$ is in 1-1 correspondence to $(\mathbb{Z}/r)^*$ which corresponds bijectively to self-automorphisms of $H^1(L_r^\infty; \mathbb{Z})$ and via Whitehead's theorem bijectively to the homotopy classes of self-homotopy equivalences of $L_r^\infty$. Hence by precomposing the $f_1$ in $f = f_1 \times f_2 : L^{p,q} \to L_r^\infty \times \mathbb{C}P^\infty$ by a suitable self-homotopy equivalence one can realize any $s$ as in the above sense.

Without affecting $f_1$, we show that the homotopy class of $f_2 : L^{p,q} \to \mathbb{C}P^\infty$, that realizes $(\epsilon, k)$, induces an isomorphism on $\pi_2$. Therefor we gather some facts:

a) From the Leray-Serre spectral sequence for the fibration $\widetilde{L}^{p,q} \xrightarrow{Pr} L^{p,q} \to L_r^\infty$ we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_{20}^\infty & \longrightarrow & H^2(L^{p,q}; \mathbb{Z}) & \xrightarrow{u} & E_{02}^\infty & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & H^2(L_r^\infty; \mathbb{Z}) & \longrightarrow & H^2(L^{p,q}; \mathbb{Z}) & \xrightarrow{u} & H^2(\widetilde{L}^{p,q}; \mathbb{Z}) & \longrightarrow & 0 \end{array}$$

with $u = Pr^*$. Hence $Pr^* : H^2(L^{p,q}; \mathbb{Z}) \to H^2(\widetilde{L}^{p,q}; \mathbb{Z})$ is surjective with kernel isomorphic to $\mathbb{Z}/r$.

b) Further there exists the following commutative diagram:

$$\begin{tikzcd} \widetilde{L}^{p,q} \arrow[d, "pr"'] \arrow[dr, "\tilde{f}_2"] & \\ L^{p,q} \arrow[r, "f_2"'] & \mathbb{C}P^\infty \end{tikzcd}$$

Applying the $\mathbb{Z}-$cohomology functor $H^2(\cdot; \mathbb{Z})$ we get the following commutative diagram:

$$\begin{tikzcd} H^2(\widetilde{L}^{p,q};\mathbb{Z}) & \\ H^2(L^{p,q};\mathbb{Z}) \arrow[u, "pr^*"] & H^2(\mathbb{C}P^\infty;\mathbb{Z}). \arrow[l, "f_2^*"'] \arrow[ul, "\tilde{f}_2^{\,*}"'] \end{tikzcd}$$

Since $H^2(\tilde{L}^{p,q};\mathbb{Z}) \cong \mathbb{Z}$, $H^2(L^{p,q};\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/r$ and $H^2(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}$ we see that $\tilde{f}_2^*$ must be an isomorphism.

c) Again naturality of the Universal Coefficient Theorem implies that the set of homotopy classes of maps from $\widetilde{L}^{p,q}$ to $\mathbb{C}P^\infty$ that induce isomorphisms on $H^2(\cdot;\mathbb{Z})$ equals the set of homotopy classes of maps that induce isomorphism on $H_2(\cdot;\mathbb{Z})$.

d) Applying the Hurewicz theorem one sees that a map between simply connected CW-complexes that induces isomorphism on $H_2(\cdot;\mathbb{Z})$ also induces isomorphism on $\pi_2(\cdot)$.

So by b) $\tilde{f}_2^*$ is an isomorphism on $H^2(\cdot;\mathbb{Z})$. Then c) implies that $\tilde{f}_2$ induces an isomorphism on $H_2(\cdot;\mathbb{Z})$ which via d) implies that $\tilde{f}_2$ induces isomorphism on $\pi_2(\cdot)$ and thus by b) $f_2$ induces isomorphism on $\pi_2(\cdot)$. Hence $\mathcal{T}$ is surjective. □

*Proof of Theorem 2.2.* By Lemma 2.8 $(L^{p,q}, f)$ and $(L^{p',q'}, f')$ represent the same element in $\Omega_5^{Spin}(B_r)$ if and only if

$$(1') \quad \left\langle f^*(v_1)(\beta_r(f^*(v_1)))^2, [N]_{\mathbb{Z}/r}\right\rangle \equiv \left\langle f'^*(v_1)(\beta_r(f'^*(v_1)))^2, [N']_{\mathbb{Z}/r}\right\rangle;$$

$$(2') \quad \left\langle f^*(v_1 z_r)\beta_r(f^*(v_1)), [N]_{\mathbb{Z}/r}\right\rangle \equiv \left\langle f'^*(v_1 z_r)\beta_r(f'^*(v_1)), [N']_{\mathbb{Z}/r}\right\rangle;$$

$$(3') \quad \left\langle f^*(v_1 z_r^2), [N]_{\mathbb{Z}/r}\right\rangle \equiv \left\langle f'^*(v_1 z_r'^2), [N']_{\mathbb{Z}/r}\right\rangle.$$

Notation. By $x_c, y_c \in H^2(S^2 \times S^2;\mathbb{Z}/c)$ we denote the elements of $H^2(S^2 \times S^2;\mathbb{Z}/c)$ which are the mod $c$ reductions of the elements that come from the standard generator of $H^2(S^2 \times S^2;\mathbb{Z})$ of the first factor and the second factor of $S^2 \times S^2$ respectively

We know that $f^*(\beta_r(v_1)) = \beta_r(f^*(v_1)) = s\beta_r(\alpha)$. Hence in order to compute the Kronecker products above we have to understand what $\beta_r(\alpha)$ is in terms of $\Pi_{p,q}^*(x)$ and $\Pi_{p,q}^*(y)$, i.e. $\beta_r(\alpha) = b_1\Pi_{p,q}^*(x_r) + b_2\Pi_{p,q}^*(y_r)$ for some $b_1, b_2 \in \mathbb{Z}/r$. The cohomological structure of $L_r^\infty \times \mathbb{C}P^\infty$ implies that $\beta_r(\alpha)$ lies in the image of $\rho_r$ restricted to the torsion part of $H^2(L^{p,q};\mathbb{Z})$, i.e.

$$\beta_r(\alpha) = u\left(\frac{p}{r}\Pi_{p,q}^*(x_r) + \frac{q}{r}\Pi_{p,q}^*(y_r)\right) \tag{2}$$

for some $u \in (\mathbb{Z}/r)^*$. We claim that modulo $r$ $b_1$ equals $u_r\frac{p}{r}$ and $b_2$ equals $u_r\frac{q}{r}$. Thus $u = u_r$ for some (universal) $u_r \in (\mathbb{Z}/r)^*$.

*Proof of the last claim.* An idea to obtain information about the $\Pi_{p,q}^*(x_r)$-component of $\beta_r(\alpha)$ is to analyze the 'restricted bundles'

$$S^1 \xrightarrow{i} L^{p,q}|_{S_1^2} \xrightarrow{\Pi_{p,q}^1} S^2,$$

$$S^1 \xrightarrow{j} L^{p,q}|_{S_2^2} \xrightarrow{\Pi_{p,q}^2} S^2,$$

where the first respectively the second fibre bundle is the restriction of the fibre bundle associated to $L^{p,q}$ to the first respectively the second $S^2$-factor. We realize that $L^{p,q}|_{S^1_1}$ respectively $L^{p,q}|_{S^2_2}$ is the familiar standard lens space $L^3_p$ respectively $L^3_q$.

Let $\alpha_p \in H^1(L^3_p;\mathbb{Z}/p)$ such that $\langle i^*(\alpha_p), [S^1]_{\mathbb{Z}/p}\rangle = 1$ and $\alpha_q \in H^1(L^3_q;\mathbb{Z}/q)$ such that $\langle j^*(\alpha_q), [U(1)]_{\mathbb{Z}/q}\rangle = 1$. Let $i_p$ and $i_q$ be the obvious inclusion of $L^3_p$ respectively $L^3_q$ in $L^{p,q}$ then it is clear that $i_p^*(\alpha) =: \alpha_{p,r}$ and $i_q^*(\alpha) =: \alpha_{q,r}$, where $\alpha_{p,r} \in H^1(L^3_p;\mathbb{Z}/r)$ and $\alpha_{q,r} \in H^1(L^3_q;\mathbb{Z}/r)$ are the images of $\alpha_p$ respectively $\alpha_q$ under the corresponding coefficient homomorphism.

Furthermore, the following holds:

$$\begin{aligned}
&i_p^*(\Pi^*_{p,q}(x)) = \Pi^{1^*}_{p,q}(x), i_q^*(\Pi^*_{p,q}(x)) = \Pi^{2^*}_{p,q}(x),\\
&i_p^*(\beta_r(\alpha)) = \beta_r(i_p^*(\alpha)) = \beta_r(\alpha_{p,r}), i_q^*(\beta_r(\alpha)) = \beta_r(i_q^*(\alpha)) = \beta_r(\alpha_{q,r}).
\end{aligned}$$

We conclude from the construction of the maps together with the long exact sequence in $\mathbb{Z}/r$-cohomology for the pairs $(L^{p,q}, L^3_p)$ and $(L^{p,q}, L^3_q)$ that $i_p^*(\Pi^*_{p,q}(y_r))$ and $i_q^*(\Pi^*_{p,q}(x_r))$ vanish. Summarizing the last considerations leads to the following:

$$\begin{aligned}
i_p^*(\beta_r(\alpha)) &= i_p^*(u(\frac{p}{r}\Pi^*_{p,q}(x_r) + \frac{q}{r}\Pi^*_{p,q}(y_r))) = u\frac{p}{r}\Pi^{1^*}_{p,q}(x_r) = \beta_r(\alpha_{p,r}),\\
i_q^*(\beta_r(\alpha)) &= i_q^*(u(\frac{p}{r}\Pi^*_{p,q}(x_r) + \frac{q}{r}\Pi^*_{p,q}(y_r))) = u\frac{p}{r}\Pi^{2^*}_{p,q}(y_r) = \beta_r(\alpha_{q,r}).
\end{aligned}$$

Thus if we knew $\beta_r(\alpha_{p,r})$ and $\beta_r(\alpha_{q,r})$ in terms of $\Pi^{1^*}_{p,q}(x_r)$ respectively $\Pi^{2^*}_{p,q}(y_r)$ then we would know what $\beta_r(\alpha)$ is.

Assume $\beta_p(\alpha_p) = u_p\Pi^{1^*}_{p,q}(x_p)$ for some $u_p \in (\mathbb{Z}/p)^*$. We compare the short exact sequences associated to $\beta_r$ and $\beta_p$:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/r \arrow[r] & \mathbb{Z}/r^2 \arrow[r, "\tilde{\pi}"] & \mathbb{Z}/r \arrow[r] & 0\\
0 \arrow[r] & \mathbb{Z}/p \arrow[r] \arrow[u, "\mathrm{red}_{p,r}"] & \mathbb{Z}/p^2 \arrow[r, "P"] \arrow[u, "\mathrm{red}_{p^2,r}"] & \mathbb{Z}/p \arrow[r] \arrow[u, "\mathrm{red}_{p,r}"] & 0,
\end{tikzcd}$$

where $\mathrm{red}_{\cdot,\cdot}$ denotes the reduction homomorphism. The maps in the squares above commute, hence we get the following commutative diagram:

$$\begin{tikzcd}
H^1(L^3_p;\mathbb{Z}/r) \arrow[r, "\beta_r"] & H^2(L^3_p;\mathbb{Z}/r)\\
H^1(L^3_p;\mathbb{Z}/p) \arrow[r, "\beta_p"] \arrow[u, "\rho_{p,r}"] & H^2(L^3_p;\mathbb{Z}/p), \arrow[u, "\frac{p}{r}\rho_{p,r}"']
\end{tikzcd}$$

where $\rho_{\cdot,\cdot}$ is the "change of coefficient-homomorphism", i.e $\beta_r \circ \rho_{p,r} = \frac{p}{r}\rho_{p,r} \circ \beta_p$. Thus $\beta_r(\alpha_{p,r}) = u_{p,r}\frac{p}{r}\Pi^{1^*}_{p,q}(x_r)$, where $u_{p,r} \in \mathbb{Z}/r$ is the mod-$r$-reduction of $u_p$. It is clear that $u_{p,r}$ is a unit in $\mathbb{Z}/r$. If $\beta_q(\alpha_{q,r}) = u_q\Pi^{2^*}_{p,q}(y_q)$ for some $u_q \in (\mathbb{Z}/q)^*$, then in the same way we obtain: $\beta_r(\alpha_{q,r}) = u_{q,r}\frac{q}{r}\Pi^{2^*}_{p,q}(y_r)$, where $u_{q,r}$ is the mod-$r$-reduction of $u_q$. But (2) implies that $u_{p,r} = u_{q,r} =: u$ and thus $\beta_r(\alpha) = u(\frac{p}{r}\Pi^*_{p,q}(x_r) + \frac{q}{r}\Pi^*_{p,q}(y_r))$.

By definition we have

$$\begin{aligned} f^*(v_1) &= s\alpha, \\ f^*(\beta_r(v_1)) &= su_r(\frac{p}{r}\Pi^*_{p,q}(x) + \frac{q}{r}\Pi^*_{p,q}(y)), \\ f^*(z) &= \epsilon(m\Pi^*_{p,q}(x) - n\Pi^*_{p,q}(y)) + k(\frac{p}{r}\Pi^*_{p,q}(x) + \frac{q}{r}\Pi^*_{p,q}(y)). \end{aligned}$$

Thus

$$\begin{aligned} f^*(v_1 z^2) &= -2s(\epsilon m + k\frac{p}{r})(\epsilon n - k\frac{q}{r})\alpha\Pi^*_{p,q}(xy), \\ f^*(v_1(\beta_r v_1)z) &= us((\epsilon m + k\frac{p}{r})\frac{q}{r} - (\epsilon n - k\frac{q}{r})\frac{p}{r})\alpha\Pi^*_{p,q}(xy), \\ f^*(v_1(\beta_r v_1)^2) &= \frac{u^2 s^3 pq}{r^2}\alpha\Pi^*_{p,q}(xy). \end{aligned}$$

Since $\langle \alpha\Pi^*_{p,q}(xy), [L^{p,q}]_{\mathbb{Z}/r}\rangle$ equals $\langle \alpha, \Pi^*_{p,q}(xy) \cap [L^{p,q}]_{\mathbb{Z}/r}\rangle$ and by the choice of the orientation of $L^{p,q}$ it follows from [G77, Prop. 2] and its proof that $\langle \alpha\Pi^*_{p,q}(xy), [L^{p,q}]_{\mathbb{Z}/r}\rangle \equiv 1 \mathrm{mod}\ r$. Hence since 2 and $u_r$ are units of $\mathbb{Z}/r$ the equations (1')-(3') translate into the congruences stated in Theorem 2.2. □

The manifolds in $\{L^{r,(t+lr)r}\}_l$ lie in the same simple and tangential homotopy type:

Here we have that $p = r$ and $q = (t + lr)r$ thus $\frac{p}{r} = 1$ and $\frac{q}{r} = t + lr$. We choose $m$ to be 0 and $n$ to be 1. From Theorem 2.2 we obtain the following numbers modulo $r$:

$$\begin{aligned} sk(\epsilon - k(t + lr)), \\ s((n + lr)k - (\epsilon - k(t + lr))), \\ s(t + lr). \end{aligned}$$

If we choose $s$ to be 1 and $k$ to be 0 then the three numbers above are modulo $r$ congruent to

$$0, -\epsilon, t$$

respectively which are independent of $l$. Thus the manifolds in $\{L^{r,(t+lr)r}\}_l$ lie in one homotopy type. By [Mi66, Lemma 12.5], Lemma 2.3 and the proof of Theorem 2.2 it follows that there are homotopy equivalences between these manifolds, whose induced map maps the $R$-torsions to each other. And from Lemma 2.1 (ii) it follows that these homotopy equivalences are also tangential. Hence the manifolds in $\{L^{r,(t+lr)r}\}_l$ even lie in the same simple and tangential homotopy type.

**3. Distinguishing homeomorhpism types.** We make use of the so called $\rho$-invariant, which was introduced by M. Atiyah and I. Singer [AS68] as a diffeomorphism invariant for smooth closed non-simply connected manifolds with finite cyclic fundamental group. Let $M$ be a smooth closed non-simply connected and oriented 5-manifold with $\pi_1(M) \cong \mathbb{Z}/r$ being finite and $\tilde{M}$ its universal cover. Assume that there is a 6-dimensional smooth oriented manifold $W$ with boundary $\tilde{M}$ which is equipped with an orientation preserving smooth $\mathbb{Z}/r$-action such that the action coincides with the $\mathbb{Z}/r$-operation on the boundary $\tilde{M}$ given by deck transformation. Let $W_f$ be the fixed point set of the $\mathbb{Z}/r$-action. Assume the equivariant signature [AS68] of $W$ is trivial then the $\rho$-invariant of $M$ associated to a non-trivial element $g$ of $\pi_1(M)$ is defined to be the evaluation of certain characteristic polynomials depending on the Chern-, Pontrjagin classes of the normal bundle of $W_f$ and the Pontrjagin classes of $W_f$, on the (twisted) fundamental class of $W_f$ if $W_f$ is orientable (not orientable) that takes values in $i\mathbb{R}$.

Let $L^{p,q} \in \mathcal{L}$, $r := |\pi_1(L^{p,q})|$, $\bar{p} := p/r$ and $\bar{q} := q/r$. Then we know that the universal covering space of $L^{p,q}$ is $L^{\bar{p},\bar{q}}$, and that the deck transformation on $L^{\bar{p},\bar{q}}$ by $\pi_1(L^{p,q})$ is given by fibrewise rotation by angles corresponding to the $r$'th roots of unity. This perspective yields a canonical identification of $\pi_1(L^{p,q})$ with $\mathbb{Z}/r$. The disc bundle $D^{\bar{p},\bar{q}}$ associated to the $S^1$-fibre bundle structure with the $\mathbb{Z}/r$-action canonically extended serves as a convinient choice of a null-bordism. Furthermore this $\mathbb{Z}/r$-bordism has trivial equivariant signature since on the one hand the $\mathbb{Z}/r$-action is homotopically trivial, as it sits in an $S^1$-action and on the other hand the dimension of the bordism is not divisible by 4 which means that the ordinary signature is trivial. The fixed point set is just $S^2 \times S^2$ and the normal bundle of the fixed point set is isomorphic to the 2-dimensional real vector bundle given by the Euler class $\frac{p}{r}x + \frac{q}{r}y$. Let $g$ be a non-trivial element of $\mathbb{Z}/r$ and $\theta_g$ the rotation angle between 0 and $\pi$ of the action by $g$ then

$$\rho(g, L^{p,q}) = -i\frac{\cos(\frac{\theta_g}{2})}{2r^2\sin^3(\frac{\theta_g}{2})}pq, \tag{3}$$

see e.g. [Ot09, p.88]. From this formula we see that $pq/r^2$ is a diffeomorphism invariant, which shows that the manifolds in $\{L^{r,(n+lr)r}\}_l$ are all pairwise non-diffeomorphic. As the $\rho$-invariant is als a topological invariant [W99, Ch. 14B], we conclude that these manifolds are also pairwise non-homeomorphic but simply and tangentially homotopy equivalent, as was shown in the previous section - this proves the first part of Theorem 1.1.

**4. Souls of codimension three.** Let $L^{p,q} \in \mathcal{L}$ and $(p, q, 1)$ be the matrix representation of the epimorphism, with respect to the standard basis, from $\mathbb{Z}$ to $\mathbb{Z}$, which ist defined by the following assignment rule: $(x_1, x_2, x_3)$ to $px_1 + qx_2 + x_3$. Let $\mathbf{a} := (a_1, a_2, a_3)$, $\mathbf{b} := (b_1, b_2, b_3) \in \mathbb{Z}^3$ be a basis of the kernel of this epimorphism and let $i_{a,b}$ be the following Lie group homomorphism from $U(1) \times U(1)$ into $SU(2) \times SU(2) \times U(1)$:

$$(z_1, z_2) \mapsto \left( \begin{pmatrix} z_1^{a_1} z_2^{b_1} & 0 \\ 0 & z_1^{-a_1} z_2^{-b_1} \end{pmatrix}, \begin{pmatrix} z_1^{a_2} z_2^{b_2} & 0 \\ 0 & z_1^{-a_2} z_2^{-b_2} \end{pmatrix}, z_3^{a_3} z_2^{b_3} \right).$$

It is not hard to show that the image of $i_{a,b}$ does not depend on the choice of a basis of the kernel of $(p, q, 1)$. The following lemma proves the parts in the statements of Theorem 1.1 and 1.2 that involve the group $SU(2) \times SU(2) \times U(1)$ which we denote in this section by $G$.

LEMMA 4.1. *The homogeneous space $G/\mathrm{Im}(i_{a,b})$ is diffeomorphic to $L^{p,q}$.*

*Proof.* The proof is subdivided into the following steps:

(1) First we define the projection map $\Pi : G/\mathrm{Im}(i_{a,b}) \to S^2 \times S^2 \times \mathbb{C}P^0$, where $\mathbb{C}P^0$ is a point and show that the fibre is $S^1$. Then we prove that there is a smooth and free $S^1$-action on $G/\mathrm{Im}(i_{a,b})$ which preserves the fibre.

(2) We prove that there is a bundle isomorphism between $\Pi$ and the sphere bundle $S(E_{p,q})$ of the following complex line bundle:

$$E_{p,q} : (\mathrm{pr}_1^* \gamma_2^p) \otimes (\mathrm{pr}_2^* \gamma_1^q) \otimes \mathrm{pr}_0^* \gamma_0 \to S^2 \times S^2 \times \mathbb{C}P^0,$$

where $\gamma_2$ respectively $\gamma_1$ denotes the tautological bundle over the first and the second factor of the base ($S^2 = \mathbb{C}P^1$) respectively, $\gamma_0$ is the trivial complex line bundle $\mathbb{C}P^0$ and $\mathrm{pr}_i : S^2 \to S^2 \times S^2 \times \mathbb{C}P^0$ is the projection map onto the $i$-th factor ($i \in \{0, 1, 2\}$).

1) Let us denote by $S^3 \times S^3 \times S^1/G_{a,b}$ the quotient of the following smooth $(S^1 \times S^1)$-action on $S^3 \times S^3 \times S^1$:

$$\begin{aligned} G_{a,b} : (S^1 \times S^1) \times (S^3 \times S^3 \times S^1) &\to S^3 \times S^3 \times S^1, \\ (z_1, z_2, ((x_1, x_2), (x_3, x_4), x_5)) &\mapsto (z_1^{a_1} z_2^{b_1}(x_1, x_2), z_1^{a_2} z_2^{b_2}(x_3, x_4), z_1^{a_3} z_2^{b_3} x_5) \end{aligned}$$

induced by the diffeomorphism $SU(2) \xrightarrow{\sim} S^3 \subset \mathbb{C}^2$, $A \mapsto A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. It is clear that $S^3 \times S^3 \times S^1/G_{a,b}$ and $SU(2) \times SU(2) \times U(1)/\mathrm{Im}(i_{a,b})$ are diffeomorphic and we denote this diffeomorphism by $\Phi$. Let $\tilde{\Pi}$ be the map from $S^3 \times S^3 \times S^1/G_{a,b}$ to $S^2 \times S^2 \times \mathbb{C}P^0$ which maps $[(x_1, x_2), (x_3, x_4), x_5]$ to $([x_1 : x_2], [x_3 : x_4], [x_5])$. We define $\Pi$ to be $\tilde{\Pi} \circ \Phi$.

Let $d, e, f \in \mathbb{Z}$ such that $dp + eq + f = 1$. We have the following split short exact sequence

$$1 \to U(1) \times U(1) \xrightarrow{\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}_\star} U(1) \times U(1) \times U(1) \underset{(d,e,f)_\star}{\overset{(p,q,1)_\star}{\rightleftarrows}} U(1) \to 1. \tag{4}$$

From (3) we conclude that $\tilde{\Pi}$ is a $S^1$-fibre bundle, where the $S^1$-action is given by

$$[(x_1, x_2), (x_3, x_4), x_5] * z = [(x_1, x_2) \cdot z^d, (x_3, x_4) \cdot z^e, x_5 \cdot z^f].$$

Hence $\Pi$ is a principal $U(1)$-fibre bundle.

2) The following diagram commutes:

$$\begin{tikzcd} S^3 \times S^3 \times S^1/G_{a,b} \arrow[rr, "\psi"] \arrow[dr, "\tilde{\Pi}"'] & & L^{p,q} \arrow[dl, "\Pi_{p,q}"] \\ & S^2 \times S^2 \times \mathbb{C}P^0 & \end{tikzcd},$$

where

$$\psi([(x_1, x_2), (x_3, x_4), x_5])$$

equals

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{p \text{ copies}} \otimes \underbrace{\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}}_{q \text{ copies}} \otimes x_5$$

and

$$\Pi_{p,q}\left(\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{p \text{ copies}} \otimes \underbrace{\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}}_{q \text{ copies}} \otimes x_5\right)$$

equals

$$([x_1 : x_2], [x_3 : x_4], [x_5]).$$

The assertion follows from the multiplicative property of the (total) Chern classes. □

*Proof of Theorem 1.2.* On the one hand the proof of [BKS11, Prop. 6.7.] implies that the cartesian product of manifolds with $\mathbb{R}$ are diffeomorphic if there exist homotopy equivalences between these manifolds which are tangential and have trivial normal invariant. On the other hand it shows that to any manifold $L^{r,qr}$ in $\{L^{r,(t+kr)r}\}_k$ there exists a subsequence of $\{L^{r,(t+kr)r}\}_k$ of manifolds which are homotopy equivalent, where the homotopy equivalences are tangential and have trivial normal invariants. Thus by Theorem 1.1 there are infinitely many pairwise non-homeomorphic manifolds in $\mathcal{L}$ such that after taking the product with $\mathbb{R}^3$ they become diffeomorphic. The metrics we choose on each such product is the product metric of the submersion metric on the homogeneous quotient, where we choose the standard product metric on $SU(2) \times SU(2) \times U(1)$ and the euclidean metric on $\mathbb{R}$.

As an upper diameter bound we take the diameter $D$ of $SU(2) \times SU(2) \times U(1)$ with respect to the product metric of the standard metrics. The existence of an uniform upper curvature bound follows from an idea by J.-H. Eschenburg [E72], and independently by B. Totaro [To03] on $T^m$-actions which are subactions of isometric and free $T^n$-actions: Namely, O'Neill's formula shows that the sectional curvature of a quotient manifold can be computed locally on $SU(2) \times SU(2) \times U(1)$. The same formula for the curvature formally makes sense for the non-closed subgroup of $T \subset SU(2) \times SU(2) \times U(1)$ associated to any real linear subspace $\mathbb{R}$ in the Lie algebra $\mathbb{R}^3$ of $T$, where we use that $T$ acts freely on $SU(2) \times SU(2) \times U(1)$. The *curvature* so defined is continuous on the compact manifold of all subspaces $\mathbb{R}^2$ of $\mathbb{R}$ and all 2-planes in the tangent bundle of $SU(2) \times SU(2) \times U(1)$ which are orthogonal to the associated foliation of $SU(2) \times SU(2) \times U(1)$ by the $T$-action. Hence there is a uniform upper bound for this curvature function and hence for the curvature of all quotients associated to subtori $T^2 \subset T$. □

REMARK 4.2. *An analysis of the surgery obstruction group, which lies beyond the scope of this paper, shows that even between any two manifolds in* $\{L^{r,(t+kr)r}\}_k$ *there exists simple and tangential homotopy equivalences with trivial normal invariant [Ot09, Corollary 19]. Thus all manifolds in* $\{L^{r,(t+kr)r}\}_k$ *become diffeomorphic after taking product with* $\mathbb{R}$.

COROLLARY 4.3. *Let* $r, n$ *be as in Theorem 1.1 then the* $\{L^{r,(t+kr)r}\}_k$ *consists of pairwise (simply and tangentially) homotopy equivalent but non-diffeomorphic Riemannian manifolds (the metrics are the homogeneous submersion metrics) with* $0 \leq sec \leq 1$ *and diameter* $\leq D$*, where* $D$ *is a positive constant.*

## 5. Remarks and a Question.

Lemma 5.1. *There exists only one smooth transitive action of $SU(2) \times SU(2) \times U(1)$ on $L^3_r \times S^2$ up to conjugation of self-diffeomorphisms of $L^3_r \times S^2$.*

*Proof.* Let $\Phi : SU(2) \times SU(2) \times U(1) \times L^3_r \times S^2 \to L^3_r \times S^2$ be a smooth transitive Lie group action on $L^3_r \times S^2$. Then the isotropy group has to be 2-dimensional, connected and compact. But there is only $S^1 \times S^1$ fulfilling these topological conditions. Thus up to Lie group automorphisms of $SU(2) \times SU(2) \times U(1)$ we are in the situation of the previous section, where we gave a canonical diffeomorphism between the homogeneous quotient and $L^3_r \times S^2$, the total space of the principal $S^1$-fibre bundle over $S^2 \times S^2$ given by the first Chern class $rx$ (see Introduction). But by the formula (3) for the $\rho$-invariant for spaces of such type we see that it is 0 which means that there are no other $S^1$-fibre bundles over $S^2 \times S^2$ which have total spaces diffeomorphic to $L^3_r \times S^2$. □

Lemma 5.2.

(a) *Within the manifolds in $\mathcal{L}$ with odd order fundamental groups there doesn't exist pairs of non-homeomorphic manifolds which can be realized as codimension 1 souls of a fixed manifold.*

(b) *Each sequence of pairwise distinct manifolds in $\mathcal{L}$ contains infinitely many $h$-cobordism classes.*

*Proof.* (a) Since the order of the fundamental groups of the manifolds under considerations is odd any real line bundle has to be trivial. Assume that there are two non-homeomorphic manifolds $L, L'$ which can be realized as codimension 1 souls of a manifold. Then they are $h$-cobordant. Since the manifolds under consideration have finite cyclic fundamental groups acting trivially on the cohomology of their universal covering space it follows from [Mi66] that the $h$-cobordism class is determined by the $R$-torisons of the boundary components. But from Lemma 2.3 it follows that their $R$-torsions are equivalent. The $s$-cobordism theorem implies that $L, L'$ have to be diffeomorphic which is a contradiction.

(b) This follows immediately from the calculation of the $\rho$-invariant in the previous section and the fact that this invariant is an $h$-cobordism invariant. □

Remark 5.3.

(i) *From Lemma 5.1 and the proof of Theorem 1.2 it follows that there exist infinitely many pairwise non-equivalent smooth transitive actions of $SU(2) \times SU(2) \times U(1) \times \mathbb{R}$ on $L^3_r \times S^2 \times \mathbb{R}$ with isotropy group $U(1) \times U(1)$, whereas there is only one class of smooth transitive $SU(2) \times SU(2) \times U(1)$-operations on $L^3_r \times S^2$.*

(ii) *Lemma 5.2 implies that there is no infinite sequence of pairwise non-homeomorphic manifolds in $\mathcal{L}$ which can be realized as codimension 1 or 2 souls.*
*Lemma 5.2 (a) implies that there doesn't exist a sequence of pairwise non-homeomorphic manifolds in $\mathcal{L}$ which can be realized as codimension 1 souls. More complicated spaces as for example lens space bundles over $S^2$ could give pairs of non-homeomorphic manifolds which can be realized as codimension 1 submanifolds of an open manifold. But we doubt that there is an infinite sequence of such spaces.*
*By Lemma 5.2 (b) there is not a sequence in $\mathcal{L}$ which consists of pairwise distinct manifolds such that only finitely many $h$-cobordism classes occur. As a consequence of this observation we deduce from [BKS11, Prop. 5.10] that there doesn't exist an infinite sequence of distinct manifolds in $\mathcal{L}$ which can be*

*realized as codimension 2 souls with trivial normal bundle of a fixed manifold. Another problem is to prove the existence of metrics of nonnegative sectional curvature.*

(iii) *We have seen that for example $L^3_r \times S^2 \times \mathbb{R}$ (for $r$ as in Theorem 1.1) admits metrics of nonnegative sectional curvature with pairwise non-homeomorphic souls. We have seen that the homotopy equivalences between all these souls are tangential and simple. These observations motivate the following question: Let $M$ and $N$ be non-simply connected manifolds and $f : M \to N$ a homotopy equivalence which is simple and tangential. Is it true that if $M$ admits a metric of nonnegative sectional curvature then also $N$ admits a metric of nonnegative sectional curvature?*

(iv) *If we weaken the assumptions in Theorem 1.1 by requiring that $r$ is just odd, then the manifolds in $\{L^{r,(t+kr)r} | k \in \mathbb{Z}\}$ are still tangentially but not necessarily simply homotopy equivalent.*

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