

## NEVANLINNA-TYPE THEORY BASED ON HEAT DIFFUSION\*

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**Abstract.** We obtain an analogue of Nevanlinna theory of holomorphic mappings from a complete and stochastically complete Kähler manifold into a complex projective manifold. When certain curvature conditions are imposed, some Nevanlinna-type defect relations based on heat diffusion are established.

**Key words.** Nevanlinna theory, second main theorem, defect relation, heat diffusion, Ricci curvature.

**Mathematics Subject Classification.** 30D35, 32H30.

**1. Introduction.** In 2010, Atsuji introduced the notion of the so-called Nevanlinna-type functions  $\tilde{T}_x(t)$ ,  $\tilde{N}_x(t, a)$  and  $\tilde{m}_x(t, a)$  for meromorphic functions on complete and stochastically complete Kähler manifolds based on heat diffusion. By means of the techniques from Brownian motion theory (see, e.g., [6, 17, 19]), he obtained an analogue of the Second Main Theorem in Nevanlinna theory:

**THEOREM 1.1** (Atsuji, [4]). *Let  $f$  be a nonconstant meromorphic function on a complete and stochastically complete Kähler manifold  $M$ . Let  $a_1, \dots, a_q$  be distinct points in  $\mathbb{P}^1(\mathbb{C})$ . Assume that  $\tilde{T}_x(t) < \infty$  as  $0 \leq t < \infty$ ,  $\tilde{T}_x(t) \rightarrow \infty$  as  $t \rightarrow \infty$  and  $|\tilde{N}_x(t, \text{Ric})| < \infty$  as  $0 \leq t < \infty$ . Then*

$$\sum_{j=1}^q \tilde{m}_x(t, a_j) + \tilde{N}_1(t, x) \leq 2\tilde{T}_x(t) + 2\tilde{N}_x(t, \text{Ric}) + O(\log^+ \tilde{T}_x(t)) + O(1)$$

holds for all  $t > 0$  outside a subset of finite Lebesgue measure.

To see how Brownian motion can be applied to the Nevanlinna theory, we refer the reader to [1, 2, 3, 4, 5] and refer to [8, 10, 11] also.

In this paper, we shall generalize Theorem 1.1. In doing so, first of all, we will extend the notion of Nevanlinna-type functions for meromorphic functions to the case of holomorphic mappings (see Section 2.1). As a generalization of Theorem 1.1, we prove an analogue of the Second Main Theorem for holomorphic mappings into complex projective manifolds (see Theorem 1.3 below). Furthermore, we discuss some defect relations which can be regarded as an analogue of the classical defect relations in Nevanlinna theory.

In the statement of our main results in what follows,  $M$  is assumed to be a complete and stochastically complete Kähler manifold. We first obtain the following logarithmic derivative lemma:

**THEOREM 1.2.** *Let  $\psi$  be a nonconstant meromorphic function on  $M$  such that  $\tilde{T}_\psi(t, \omega_{FS}) < \infty$  as  $0 \leq t < \infty$ . Then for any  $\delta > 0$ , there exists a subset  $E_\delta \subseteq [0, \infty)$  of finite Lebesgue measure such that*

$$\tilde{m} \left( t, \frac{\|\nabla_M \psi\|}{|\psi|} \right) \leq \frac{3 + \delta}{2} \log^+ \tilde{T}_\psi(t, \omega_{FS}) + O(1)$$

\*Received May 21, 2020; accepted for publication December 13, 2022.

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holds for all  $t > 0$  outside  $E_\delta$ .

The following result is the so-called Second Main Theorem:

**THEOREM 1.3.** *Let  $L$  be a positive line bundle over a complex projective manifold  $N$  with  $\dim_{\mathbb{C}} N \leq \dim_{\mathbb{C}} M$ . Let  $D \in |L|$  be of simple normal crossing type. Let  $f : M \rightarrow N$  be a differentiably non-degenerate holomorphic mapping such that  $\tilde{T}_f(t, L) < \infty$  and  $|\tilde{T}(t, \mathcal{R}_M)| < \infty$  for  $0 \leq t < \infty$ . Then*

$$\begin{aligned} & \tilde{T}_f(t, L) + \tilde{T}_f(t, K_N) + \tilde{T}(t, \mathcal{R}_M) \\ & \leq \tilde{N}_f(t, D) - \tilde{N}_{f', D}(t, 0) + O(\log^+ \tilde{T}_f(t, L)) + O(1) \end{aligned}$$

holds for all  $t > 0$  outside a subset of finite Lebesgue measure.

In 1972s, Griffiths and his co-authors ([7, 14, 15]) devised an equi-distribution theory of holomorphic mappings between algebraic varieties intersecting simple normal crossing type divisors. In our investigations, Theorem 1.3 gives an analogue of Griffiths' equi-distribution theory in terms of heat diffusion.

When  $M$  has non-negative Ricci curvature, we prove a defect relation:

**THEOREM 1.4.** *Let  $L \rightarrow N$  be a positive line bundle over a complex projective manifold  $N$  with  $\dim_{\mathbb{C}} N \leq \dim_{\mathbb{C}} M$ . Let  $D \in |L|$  be of simple normal crossing type. Let  $f : M \rightarrow N$  be a differentiably non-degenerate holomorphic mapping satisfying*

$$\int_1^\infty e^{-\epsilon r^2} dr \int_{B_o(r)} e_{f^*c_1(L, h)}(x) dV(x) < \infty \quad (1)$$

for any  $\epsilon > 0$ . Then

$$\tilde{\delta}_f(D) \leq \left[ \frac{c_1(K_N^*)}{c_1(L)} \right].$$

## 2. First Main Theorem.

**2.1. Dynkin Formula.** Let  $M$  be a Riemannian manifold with Laplace-Beltrami operator  $\Delta_M$ . A Brownian motion  $X_t = (X_t)_{t \geq 0}$  in  $M$  is a heat diffusion process generated by  $\Delta_M/2$ , with transition density function  $p(t, x, y)$  being the minimal positive fundamental solution of the following heat equation

$$\frac{\partial}{\partial t} u(t, x) - \frac{1}{2} \Delta_M u(t, x) = 0.$$

The parabolicity of  $M$  means the recurrence of Brownian motions in  $M$ . We say that  $M$  is stochastically complete if

$$\int_M p(t, x, y) dV(x) = 1$$

holds for all  $x \in M$ . By Grigor'yan's criterion (see [12]),  $M$  is stochastically complete if  $R_M(x) \geq -cr^2(x) - c$  for some constant  $c > 0$ , where  $R_M$  is the pointwise lower bound of Ricci curvature defined by

$$R_M(x) = \inf_{\xi \in T_x M, \|\xi\|=1} \text{Ric}_M(\xi, \xi). \quad (2)$$

Let  $\mathbb{P}_x$  and  $\mathbb{E}_x$  be the law and expectation of  $X_t$  started at  $x$  respectively. The Itô formula (see [2, 16, 17]) states that

$$u(X_t) - u(x) = B \left( \int_0^t \|\nabla_M u\|^2(X_s) ds \right) + \frac{1}{2} \int_0^t \Delta_M u(X_s) dt, \quad \mathbb{P}_x - a.s.$$

for  $u \in \mathcal{C}_b^2(M)$ , where  $B_t$  is the standard Brownian motion in  $\mathbb{R}$  and  $\nabla_M$  is the gradient operator on  $M$ . Take expectation on both sides of the equality, it follows the Dynkin formula

$$\mathbb{E}_x[u(X_t)] - u(x) = \frac{1}{2} \mathbb{E}_o \left[ \int_0^t \Delta_M u(X_s) ds \right] \quad (3)$$

provided that each term makes sense.

**2.2. Nevanlinna-type functions.** Let  $(M, g)$  be a Kähler manifold of complex dimension  $m$ , whose Laplace-Beltrami operator is denoted by  $\Delta_M$  and Kähler form is defined by

$$\alpha = \frac{\sqrt{-1}}{\pi} \sum_{i,j=1}^m g_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$

Let  $L$  be a holomorphic line bundle over a complex projective manifold  $N$ . We denote by  $H^0(N, L)$  the space of all holomorphic global sections of  $L$  over  $N$ , and by  $|L|$  the complete linear system of all effective divisors which are zero divisors of the sections in  $H^0(N, L)$ . Moreover, we use the following standard notations

$$d = \partial + \bar{\partial}, \quad d^c = \frac{\sqrt{-1}}{4\pi} (\bar{\partial} - \partial) \quad \text{so that} \quad dd^c = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}.$$

Let  $X_t$  be a Brownian motion in  $M$  started from a reference point  $o \in M$ . Let  $f : M \rightarrow N$  be a holomorphic mapping satisfying  $f(M) \not\subset \text{Supp} D$ , where  $D \in |L|$  is a given divisor. Equip  $L$  with a Hermitian metric  $h$ , there defines the Chern form  $c_1(L, h) := -dd^c \log h$ . It's trivial to check that  $\Delta_M \log(h \circ f)$  is well defined. Since  $M$  is Kählerian, then

$$\Delta_M \log(h \circ f) = -4m \frac{f^* c_1(L, h) \wedge \alpha^{m-1}}{\alpha^m}. \quad (4)$$

Consider a local trivialization covering  $(\{U_\alpha\}, \{e_\alpha\})$  of  $(L, h)$ . Taking  $0 \neq s \in H^0(N, L)$  and writing  $s = \tilde{s} e_\alpha$  on  $U_\alpha$  locally. We also see that  $\Delta_M \log |\tilde{s} \circ f|^2$  is well defined on  $M$ . By a simple computation, we obtain

$$\begin{aligned} \Delta_M \log |\tilde{s} \circ f|^2 &= 4m \frac{dd^c \log |\tilde{s} \circ f|^2 \wedge \alpha^{m-1}}{\alpha^m}, \\ \Delta_M \log \|s \circ f\|^2 &= \Delta_M \log(h \circ f) + \Delta_M \log |\tilde{s} \circ f|^2. \end{aligned}$$

For a (1,1)-form  $\zeta$  on  $M$ , we use the following convenient symbol

$$e_\zeta = 2m \frac{\zeta \wedge \alpha^{m-1}}{\alpha^m}.$$

According to (4),

$$e_{f^* c_1(L, h)} = -\frac{1}{2} \Delta_M \log(h \circ f).$$

Let  $\omega$  be a (1,1)-form on  $N$ . The *characteristic function* of  $f$  with respect to  $\omega$  is defined by

$$\tilde{T}_f(t, \omega) = \frac{1}{2} \mathbb{E}_o \left[ \int_0^t e_{f^* \omega}(X_s) ds \right].$$

Define  $\tilde{T}_f(t, L) := \tilde{T}_f(t, c_1(L, h))$ , which is well defined up to a bounded term since the compactness of  $N$ .

Let  $s_D$  be the canonical section determined by  $D$ . Suppose that  $\|s_D\| < 1$ , since the compactness of  $N$ . The *proximity function* of  $f$  with respect to  $D$  is defined by

$$\tilde{m}_f(t, D) = \mathbb{E}_o \left[ \log \frac{1}{\|s_D \circ f(X_t)\|} \right].$$

To define the counting function via Brownian motions, we need to assume that  $M$  is stochastically complete. The *counting function* of  $f$  with respect to  $D$  is defined by

$$\tilde{N}_f(t, D) = \lim_{\lambda \rightarrow \infty} \lambda \mathbb{P}_o \left( \sup_{0 \leq s \leq t} \log \frac{1}{\|s_D \circ f(X_s)\|} > \lambda \right). \quad (5)$$

In what follows, we assume that  $L > 0$ . In addition, we also assume that  $f$  is differentially non-degenerate, by which we mean that the Jacobian matrix of  $f$  is of full rank at some point. We first give conditions for  $\tilde{T}_f(t, L) < \infty$  as  $0 < t < \infty$  and  $\tilde{T}_f(t, L) \rightarrow \infty$  as  $t \rightarrow \infty$ .

LEMMA 2.1. *Let  $R_M$  be defined by (2). Each of the following conditions ensures that  $\tilde{T}_f(t, L) < \infty$  as  $0 < t < \infty$ :*

(a)  *$f$  has finite energy, i.e.,*

$$\int_M e_{f^* c_1(L, h)}(x) dV(x) < \infty;$$

(b) *the energy density function  $e_{f^* c_1(L, h)}(x)$  is bounded;*

(c)  *$R_M(x) \geq -k(r(x))$  for a nondecreasing function  $k \geq 0$  on  $[0, \infty)$  with  $k(r)/r^2 \rightarrow 0$  as  $r \rightarrow \infty$  and (1) is assumed;*

(d)  *$R_M(x) \geq -k$  for a constant  $k \geq 0$  with*

$$\int_1^\infty e^{-\epsilon r^2} \sup_{x \in B_o(r)} e_{f^* c_1(L, h)}(x) dr < \infty$$

for any  $\epsilon > 0$ , where  $B_o(r)$  denotes the geodesic ball centered at  $o$  with radius  $r$  in  $M$ .

*Proof.* (a) and (b) are immediate. (c) is proved using the estimate of  $p(t, o, x)$  due to Li-Yau [18]. (d) is confirmed by (c), because the boundedness of Ricci curvature implies that  $\text{Vol}(B_o(r))$  has at most the exponential growth. The arguments here refer to [4], Proposition 6.  $\square$

LEMMA 2.2. *Each of the following conditions ensures that  $\tilde{T}_f(t, L) \rightarrow \infty$  as  $t \rightarrow \infty$ :*

(a) *there exists no nonconstant bounded subharmonic functions on  $M$ . In particular,  $M$  is parabolic;*

(b)  $\text{Ric}_M \geq 0$ .

*Proof.* Since  $L > 0$ , we can identify  $N$  with an algebraic subvariety of  $\mathbb{P}^k(\mathbb{C})$  for some integer  $k > 0$ . Let  $\mathcal{H}_N$  be the restriction of hyperplane line bundle  $\mathcal{H}$  over  $\mathbb{P}^k(\mathbb{C})$  to  $N$ . Note that

$$C_1 \tilde{T}_f(t, \mathcal{H}_N) \leq \tilde{T}_f(t, L) \leq C_2 \tilde{T}_f(t, \mathcal{H}_N) \quad (6)$$

for some constants  $C_1, C_2 > 0$ . Let  $[w_0 : \cdots : w_k]$  stand for the homogeneous coordinate system of  $\mathbb{P}^k(\mathbb{C})$ . Assuming  $w_0 \circ f \neq 0$  without loss of generality. Then

$$u := \log(|w_0 \circ f|^2 + \cdots + |w_k \circ f|^2)$$

is a nonconstant subharmonic function on  $M$ . Since  $f$  is differentially non-degenerate, then (a) implies that

$$\tilde{T}_f(t, \mathcal{H}_N) = \frac{1}{4} \mathbb{E}_o \left[ \int_0^t \Delta_M u(X_s) ds \right] \rightarrow \infty$$

as  $t \rightarrow \infty$ . By (6), we have (a) holds. (b) follows from [4], Proposition 7 (ii) (see the details of arguments in [1]).  $\square$

In general,  $\tilde{N}_f(t, D) \neq 0$  even if  $f$  omits  $D$ . In fact (see [10] for arguments), we have

$$\tilde{N}_f(t, D) \geq \frac{\pi^m}{(m-1)!} \int_{f^*D \cap B_o(r)} g_t(o, x) \alpha^{m-1} \quad (7)$$

provided that both sides of the above inequality make sense, where

$$g_t(o, x) = \int_0^t p(s, o, x) ds.$$

So, to guarantee  $\tilde{N}_f(t, D) = 0$  for  $0 < t < \infty$  if  $f$  omits  $D$ , certain geometric conditions on  $M$  needed to be assumed. Let  $u$  be a nonnegative function on  $M$ . Set

$$\tilde{N}(t, u) = \lim_{\lambda \rightarrow \infty} \lambda \mathbb{P}_o \left( \sup_{0 \leq s \leq t} u(X_s) > \lambda \right).$$

LEMMA 2.3 ([4]). *Assume that the Ricci curvature of  $M$  satisfies  $R_M(x) \geq -c^2 r^2(x) - c$  for all  $x \in M$  and a constant  $c > 0$ , where  $R_M$  is defined by (2). If  $u$  is a nonnegative smooth subharmonic function with*

$$\liminf_{r \rightarrow \infty} \frac{1}{r^2} \log \int_{B_o(r)} \Delta_M u(x) dV(x) < \infty,$$

where  $B_o(r)$  is the geodesic ball centered at  $o$  with radius  $r$  in  $M$ , and if

$$\mathbb{E}_o \left[ \int_0^t \Delta_M u(X_s) ds \right] < \infty$$

for  $0 \leq t < \infty$ , then  $\tilde{N}(t, u) = 0$  for  $0 < t < \infty$ .

THEOREM 2.4. *Let  $(L, h)$  be a positive Hermitian line bundle over  $N$ . Assume that the Ricci curvature of  $M$  satisfies  $R_M(x) \geq -c^2 r^2(x) - c$  for all  $x \in M$  and*

a constant  $c > 0$ , where  $R_M$  is defined by (2). Suppose also that  $\tilde{T}_f(t, L) < \infty$  for  $0 \leq t < \infty$  and

$$\liminf_{r \rightarrow \infty} \frac{1}{r^2} \log \int_{B_o(r)} e_{f^*c_1(L, h)} dV(x) < \infty.$$

If  $f$  omits  $D$ , then  $\tilde{N}_f(t, D) = 0$  for  $0 < t < \infty$ .

*Proof.* The curvature condition means the stochastically completeness of  $M$ . Let  $(\{U_\alpha\}, \{e_\alpha\})$  be a local trivialization covering of  $(L, h)$ . Write  $s_D = \tilde{s}_D e_\alpha$  on  $U_\alpha$  locally. Then

$$\Delta_M \log \frac{1}{\|s_D \circ f\|^2} = 2e_{f^*c_1(L, h)} - \Delta_M \log |\tilde{s}_D \circ f|^2.$$

If  $f$  omits  $D$ , one obtains  $\Delta_M \log |\tilde{s}_D \circ f| = 0$ . Notice that  $c_1(L, h) > 0$ , thus  $-\log \|s_D \circ f\|$  is subharmonic if  $f$  omits  $D$ . Using Lemma 2.3, we have the theorem proved.  $\square$

### 2.3. First Main Theorem.

**THEOREM 2.5.** *Let  $M$  be a stochastically complete Kähler manifold and  $L$  be a positive line bundle over a complex projective manifold  $N$ . Let  $f : M \rightarrow N$  be a holomorphic mapping with  $f(M) \not\subset \text{Supp}(D)$ , where  $D \in |L|$  is a given divisor. If  $\tilde{T}_f(t, L) < \infty$  as  $0 \leq t < \infty$ , then*

$$\tilde{T}_f(t, L) = \tilde{m}_f(t, D) + \tilde{N}_f(t, D) + O(1).$$

*Proof.* Equip  $L$  with a Hermitian metric  $h$  such that  $\omega := c_1(L, h) > 0$ . Set

$$T_\lambda = \inf \left\{ t > 0 : \sup_{0 \leq s \leq t} \log \frac{1}{\|s_D \circ f(X_s)\|} > \lambda \right\}.$$

Let  $(\{U_\alpha\}, \{e_\alpha\})$  be a local trivialization covering of  $(L, h)$ . Write  $s_D = \tilde{s}_D e_\alpha$  locally on  $U_\alpha$ . Then

$$\log \|s_D \circ f\|^2 = \log |\tilde{s}_D \circ f|^2 + \log(h \circ f). \quad (8)$$

Apply Dynkin formula to  $\log \|s_D \circ f\|^{-1}$ , we get

$$\begin{aligned} & \mathbb{E}_o \left[ \log \frac{1}{\|s_D \circ f(X_{t \wedge T_\lambda})\|} \right] \\ &= \frac{1}{2} \mathbb{E}_o \left[ \int_0^{t \wedge T_\lambda} \Delta_M \log \frac{1}{\|s_D \circ f(X_s)\|} ds \right] + \log \frac{1}{\|s_D \circ f(o)\|}, \end{aligned} \quad (9)$$

where  $t \wedge T_\lambda = \min\{t, T_\lambda\}$ . Since  $\log \|s_D \circ f(X_s)\|^{-1}$  has no singularities as  $0 \leq s \leq T_\lambda$  due to the definition of  $T_\lambda$ , it concludes by (8) that

$$\Delta_M \log \frac{1}{\|s_D \circ f(X_s)\|} = -\frac{1}{2} \Delta_M \log(h \circ f(X_s))$$

as  $0 \leq s \leq T_\lambda$ , where we use a fact that  $\log |\tilde{s}_D \circ f|$  is harmonic on  $M \setminus f^{-1}(D)$ . Hence, (9) turns to

$$\mathbb{E}_o \left[ \log \frac{1}{\|s_D \circ f(X_{t \wedge T_\lambda})\|} \right] = -\frac{1}{4} \mathbb{E}_o \left[ \int_0^{t \wedge T_\lambda} \Delta_M \log(h \circ f(X_s)) ds \right] + O(1).$$

Since  $f^*\omega = -dd^c \log(h \circ f)$ , then

$$e_{f^*\omega} = -2m \frac{dd^c \log(h \circ f) \wedge \alpha^{m-1}}{\alpha^m} = -\frac{1}{2} \Delta_M \log(h \circ f). \quad (10)$$

By the monotone convergence theorem, we see from (10) that

$$\begin{aligned} -\frac{1}{4} \mathbb{E}_o \left[ \int_0^{t \wedge T_\lambda} \Delta_M \log(h \circ f(X_s)) ds \right] &= \frac{1}{2} \mathbb{E}_o \left[ \int_0^{t \wedge T_\lambda} e_{f^*\omega}(X_s) ds \right] \\ &\rightarrow \tilde{T}_f(t, L) \end{aligned} \quad (11)$$

as  $\lambda \rightarrow \infty$ , where we use a fact that  $T_\lambda \rightarrow \infty$  a.s. as  $\lambda \rightarrow \infty$  since  $f^{-1}(D)$  is polar. Write the left hand side of (9) as two parts:

$$\text{I} + \text{II} := \mathbb{E}_o \left[ \log \frac{1}{\|s_D \circ f(X_t)\|} : t < T_\lambda \right] + \mathbb{E}_o \left[ \log \frac{1}{\|s_D \circ f(X_{T_\lambda})\|} : T_\lambda \leq t \right].$$

Using the monotone convergence theorem, then

$$\text{I} \rightarrow \tilde{m}_f(r, D) \quad (12)$$

as  $\lambda \rightarrow \infty$ . Moreover, by the definition of  $T_\lambda$ , it is trivial to see that

$$\text{II} = \lambda \mathbb{P}_o \left( \sup_{0 \leq s \leq t} \log \frac{1}{\|s_D \circ f(X_s)\|} > \lambda \right) \rightarrow \tilde{N}_f(t, D) \quad (13)$$

as  $\lambda \rightarrow \infty$ . Combining (11)-(13), we have the desired result.  $\square$

### 3. Second Main Theorem and Defect Relation.

**3.1. Logarithmic Derivative Lemma.** Let  $(M, g)$  be a complete and stochastically complete Kähler manifold of complex dimension  $m$ , with the Kähler form  $\alpha$  and the gradient operator  $\nabla_M$  associated to  $g$ . Let  $X_t$  be the Brownian motion in  $M$  with generator  $\frac{1}{2} \Delta_M$ , started at a fixed point  $o \in M$ , with transition density function  $p(t, o, x)$ .

LEMMA 3.1 (Calculus Lemma). *Let  $k$  be a non-negative function on  $M$  so that  $\mathbb{E}_o[k(X_t)] < \infty$  and  $\mathbb{E}_o[\int_0^t k(X_s) ds] < \infty$  for  $0 \leq t < \infty$ . Then for any  $\delta > 0$ , there exists a set  $E_\delta \subseteq [0, \infty)$  of finite Lebesgue measure such that*

$$\mathbb{E}_o[k(X_t)] \leq \left( \mathbb{E}_o \left[ \int_0^t k(X_s) ds \right] \right)^{1+\delta}$$

holds for all  $t > 0$  outside  $E_\delta$ .

*Proof.* Set  $\gamma(t) := \mathbb{E}_o[\int_0^t k(X_s) ds]$  and  $E_\delta := \{t \in (0, \infty) : \gamma'(t) > \gamma^{1+\delta}(t)\}$ , then  $\gamma'(t) = \mathbb{E}_o[k(X_t)]$ . The claim holds for  $k \equiv 0$ . If  $k \not\equiv 0$ , then we suppose that  $\gamma(1) \neq 0$  without loss of generality. Note that

$$\int_{E_\delta} dt \leq 1 + \int_1^\infty \frac{\gamma'(t)}{\gamma^{1+\delta}(t)} dt \leq 1 + \delta^{-1} \gamma^{-\delta}(1) < \infty.$$

This completes the proof.  $\square$

Let  $\psi$  be a meromorphic function on  $M$ . Define

$$\tilde{m} \left( t, \frac{\|\nabla_M \psi\|}{|\psi|} \right) = \mathbb{E}_o \left[ \log^+ \frac{\|\nabla_M \psi\|}{|\psi|} (X_t) \right],$$

where

$$\|\nabla_M \psi\|^2 = 2 \sum_{i,j=1}^m g^{i\bar{j}} \frac{\partial \psi}{\partial z_i} \overline{\frac{\partial \psi}{\partial z_j}},$$

in which  $(g^{i\bar{j}})$  is the inverse of  $(g_{i\bar{j}})$ . Regarding  $\psi$  as a meromorphic mapping into  $\mathbb{P}^1(\mathbb{C})$ . The characteristic function of  $\psi$  with respect to the Fubini-Study form  $\omega_{FS}$  on  $\mathbb{P}^1(\mathbb{C})$  is defined by

$$\tilde{T}_\psi(t, \omega_{FS}) = \frac{1}{4} \mathbb{E}_o \left[ \int_0^t \Delta_M \log(1 + |\psi(X_s)|^2) ds \right].$$

Adopting the spherical distance  $\|\cdot, \cdot\|$  on  $\mathbb{P}^1(\mathbb{C})$ . The proximity function of  $\psi$  with respect to  $a \in \mathbb{P}^1(\mathbb{C})$  is defined by

$$\tilde{m}_\psi(t, a) = \mathbb{E}_o \left[ \log \frac{1}{\|\psi(X_t), a\|} \right].$$

Again, set

$$\tilde{N}_\psi(t, a) = \lim_{\lambda \rightarrow \infty} \lambda \mathbb{P}_o \left( \sup_{0 \leq s \leq t} \log \frac{1}{\|f(X_s), a\|} > \lambda \right).$$

Using the similar arguments as in the proof of Theorem 2.5, we obtain

$$\tilde{T}_\psi(t, \omega_{FS}) = \tilde{m}_\psi(t, a) + \tilde{N}_\psi(t, a) + O(1).$$

Define a singular metric

$$\Phi = \frac{1}{|\zeta|^2(1 + \log^2 |\zeta|)} \frac{\sqrt{-1}}{4\pi^2} d\zeta \wedge d\bar{\zeta}$$

on  $\mathbb{P}^1(\mathbb{C})$ . A direct computation gives that

$$\int_{\mathbb{P}^1(\mathbb{C})} \Phi = 1, \quad 4m\pi \frac{\psi^* \Phi \wedge \alpha^{m-1}}{\alpha^m} = \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}. \quad (14)$$

Set

$$\tilde{T}_\psi(t, \Phi) = \frac{1}{2} \mathbb{E}_o \left[ \int_0^t e_{\psi^* \Phi}(X_s) ds \right], \quad e_{\psi^* \Phi}(x) = 2m \frac{\psi^* \Phi \wedge \alpha^{m-1}}{\alpha^m}.$$

According to (14), we obtain

$$\tilde{T}_\psi(t, \Phi) = \frac{1}{4\pi} \mathbb{E}_o \left[ \int_0^t \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)} (X_s) ds \right]. \quad (15)$$



LEMMA 3.2. *Let  $\psi$  be a nonconstant meromorphic function on  $M$  such that  $\tilde{T}_\psi(t, \omega_{FS}) < \infty$  as  $0 \leq t < \infty$ . Then for any  $\delta > 0$ , there exists a set  $E_\delta \subseteq [0, \infty)$  of finite Lebesgue measure such that*

$$\mathbb{E}_o \left[ \log^+ \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) \right] \leq (1 + \delta) \log^+ \tilde{T}_\psi(t, \omega_{FS}) + O(1)$$

holds for all  $t > 0$  outside  $E_\delta$ .

*Proof.* By Jensen inequality

$$\begin{aligned} \mathbb{E}_o \left[ \log^+ \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) \right] &\leq \mathbb{E}_o \left[ \log \left( 1 + \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) \right) \right] \\ &\leq \log^+ \mathbb{E}_o \left[ \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) \right] + O(1). \end{aligned}$$

Invoking Lemma 3.1 and (7) and (15), we get

$$\begin{aligned} &\log^+ \mathbb{E}_o \left[ \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) \right] \\ &\leq (1 + \delta) \log^+ \mathbb{E}_o \left[ \int_0^t \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_s) ds \right] \\ &\leq (1 + \delta) \log^+ \int_0^t ds \int_M p(s, o, x) \psi^* \Phi \wedge \alpha^{m-1} + O(1) \\ &\leq (1 + \delta) \log^+ \int_{\mathbb{P}^1(\mathbb{C})} \tilde{N}_\psi(t, \zeta) \Phi(\zeta) + O(1) \\ &\leq (1 + \delta) \log^+ \tilde{T}_\psi(t, \omega_{FS}) + O(1). \end{aligned}$$

□

*Proof of Theorem 1.2.* On the one hand,

$$\begin{aligned} &\tilde{m} \left( t, \frac{\|\nabla_M \psi\|}{|\psi|} \right) \\ &= \frac{1}{2} \mathbb{E}_o \left[ \log^+ \left( \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) (1 + \log^2 |\psi(X_t)|) \right) \right] \\ &\leq \frac{1}{2} \mathbb{E}_o \left[ \log^+ \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) \right] + \frac{1}{2} \mathbb{E}_o \left[ \log^+ (1 + \log^2 |\psi(X_t)|) \right] \\ &\leq \frac{1}{2} \mathbb{E}_o \left[ \log^+ \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) \right] \\ &\quad + \mathbb{E}_o \left[ \log \left( 1 + \log^+ |\psi(X_t)| + \log^+ \frac{1}{|\psi(X_t)|} \right) \right]. \end{aligned}$$

In which, Lemma 3.2 gives that

$$\mathbb{E}_o \left[ \log^+ \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) \right] \leq (1 + \delta) \log^+ \tilde{T}_\psi(t, \omega_{FS}) + O(1).$$

Moreover, by Jensen inequality

$$\begin{aligned} & \mathbb{E}_o \left[ \log \left( 1 + \log^+ |\psi(X_t)| + \log^+ \frac{1}{|\psi(X_t)|} \right) \right] \\ & \leq \log^+ (\tilde{m}_\psi(t, \infty) + \tilde{m}_\psi(t, 0)) + O(1) \\ & \leq \log^+ \tilde{T}_\psi(t, \omega_{FS}) + O(1). \end{aligned}$$

Combining the above, we prove the theorem.  $\square$

**3.2. Second Main Theorem.** Let  $(M, g)$  be a complete and stochastically complete Kähler manifold of complex dimension  $m$ , whose Kähler form is written as

$$\alpha = \frac{\sqrt{-1}}{\pi} \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$

Then

$$\alpha^m = m! \det(g_{i\bar{j}}) \bigwedge_{j=1}^m \frac{\sqrt{-1}}{\pi} dz_j \wedge d\bar{z}_j.$$

Define the Ricci form  $\mathcal{R}_M$  of  $M$  by

$$\mathcal{R}_M = -dd^c \log \det(g_{s\bar{t}}) = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^m R_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

where

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \det(g_{s\bar{t}}).$$

A well-known theorem by S. S. Chern asserts that  $\mathcal{R}_M$  is a real closed smooth  $(1,1)$ -form, which represents the first Chern class of  $M$  in de Rham cohomology group  $H_{\text{DR}}^2(M, \mathbb{R})$ . Let  $s_M$  be the scalar curvature of  $M$ , then

$$s_M = \sum_{i,j} g^{i\bar{j}} R_{i\bar{j}},$$

where  $(g^{i\bar{j}})$  is the inverse of  $(g_{i\bar{j}})$ . A direct computation yields that

$$s_M = -\frac{1}{2} \Delta_M \log \det(g_{s\bar{t}}).$$

Let  $(L, h)$  be a positive Hermitian line bundle over a complex projective manifold  $N$  with  $\dim_{\mathbb{C}} N = n \leq m$ . It defines a volume form  $\Omega = \wedge^n c_1(L, h)$  on  $N$ . Write  $D = \sum_{j=1}^q D_j \in |L|$  into a sum of irreducible components, which is of simple normal crossing type, one can equip every  $L_{D_j}$  ( $1 \leq j \leq q$ ) with a Hermitian metric such that the induced Hermitian metric on  $L = \otimes_{j=1}^q L_{D_j}$  is  $h$ . Taking  $s_j \in H^0(N, L_{D_j})$  satisfying  $(s_j) = D_j$  and  $\|s_j\| < 1$ . On  $N$ , one can define a singular volume form

$$\Phi = \frac{\Omega}{\prod_{j=1}^q \|s_j\|^2}. \quad (16)$$

Set

$$f^* \Phi \wedge \alpha^{m-n} = \xi \alpha^m.$$

Recall that

$$T_\lambda = \inf \left\{ t > 0 : \sup_{0 \leq s \leq t} \log \frac{1}{\|s_D \circ f(X_s)\|} > \lambda \right\}.$$

Introduce

$$\tilde{N}_{f',D}(t, 0) = \lim_{\lambda \rightarrow \infty} \mathbb{E}_o \left[ \log^- \frac{f^* \Omega \wedge \alpha^{m-n}}{\alpha^m}(X_{T_\lambda}) : T_\lambda \leq t \right].$$

Let  $J_f$  denote the set of points in  $M$  such that  $f$  is differentially degenerate, i.e., the rank of Jacobian matrix of  $f$  is not full. Notice that  $T_\lambda \rightarrow 0$  a.s. as  $\lambda \rightarrow \infty$  and the image of  $f$  approaches  $D$  infinitely a.s. as  $\lambda \rightarrow \infty$ , thus one sees that  $\tilde{N}_{f',D}(t, 0)$  measures the size of  $J_f \cap f^{-1}(D)$  counting multiplicities.  $\tilde{N}_{f',D}(t, 0)$  may be divergent unless certain curvature conditions are imposed.

LEMMA 3.3. *If  $\tilde{N}_f(t, D) + \mathbb{E}_o[\log^+ \xi(X_t)] < \infty$  for  $0 \leq t < \infty$ , then*

$$\begin{aligned} & \tilde{T}_f(t, L) + \tilde{T}_f(t, K_N) + \tilde{T}(t, \mathcal{R}_M) \\ & \leq \tilde{N}_f(t, D) - \tilde{N}_{f',D}(t, 0) + \frac{1}{2} \mathbb{E}_o[\log \xi(X_t)] + O(1). \end{aligned}$$

*Proof.* Since

$$\Delta_M \log \|s_D \circ f(X_t)\|^2 = \Delta_M \log h \circ f(X_t)$$

as  $0 \leq t \leq T_\lambda$ , then it yields from Dynkin formula that

$$\begin{aligned} & \frac{1}{2} \mathbb{E}_o[\log \xi(X_{t \wedge T_\lambda})] \\ & = \frac{1}{4} \mathbb{E}_o \left[ \int_0^{t \wedge T_\lambda} \Delta_M \log \xi(X_s) ds \right] + O(1) \\ & = \frac{1}{4} \mathbb{E}_o \left[ \int_0^{t \wedge T_\lambda} 4m \frac{dd^c[\log \xi] \wedge \alpha^{m-n}}{\alpha^m}(X_s) ds \right] + O(1) \\ & = \tilde{T}_f(t \wedge T_\lambda, L) + \tilde{T}_f(t \wedge T_\lambda, K_N) + \tilde{T}(t \wedge T_\lambda, \mathcal{R}_M) + O(1). \end{aligned} \tag{17}$$

On the other hand,

$$\begin{aligned} \mathbb{E}_o[\log \xi(X_{t \wedge T_\lambda})] & = \mathbb{E}_o[\log \xi(X_t) : t < T_\lambda] + \mathbb{E}_o[\log \xi(X_{T_\lambda}) : T_\lambda \leq t] \\ & \leq \mathbb{E}_o[\log^+ \xi(X_t) : t < T_\lambda] + \mathbb{E}_o[\log \xi(X_{T_\lambda}) : T_\lambda \leq t], \end{aligned}$$

where

$$\begin{aligned} & \mathbb{E}_o[\log \xi(X_{T_\lambda}) : T_\lambda \leq t] \\ & = \mathbb{E}_o \left[ \log \frac{1}{\|s_D \circ f(X_{T_\lambda})\|^2} : T_\lambda \leq t \right] + \mathbb{E}_o \left[ \log \frac{f^* \Omega \wedge \alpha^{m-n}}{\alpha^m}(X_{T_\lambda}) : T_\lambda \leq t \right] \\ & \leq 2\lambda \mathbb{P}_o \left( \sup_{0 \leq s \leq t} \log \frac{1}{\|s_D \circ f(X_s)\|} > \lambda \right) + \mathbb{E}_o \left[ \log \frac{f^* \Omega \wedge \alpha^{m-n}}{\alpha^m}(X_{T_\lambda}) : T_\lambda \leq t \right]. \end{aligned}$$

Thus,

$$\begin{aligned} & \frac{1}{2} \mathbb{E}_o [\log \xi(X_{t \wedge T_\lambda})] \\ & \leq \lambda \mathbb{P}_o \left( \sup_{0 \leq s \leq t} \log \frac{1}{\|s_D \circ f(X_s)\|} > \lambda \right) + \frac{1}{2} \mathbb{E}_o [\log^+ \xi(X_t) : t < T_\lambda] \\ & \quad + \frac{1}{2} \mathbb{E}_o \left[ \log \frac{f^* \Omega \wedge \alpha^{m-n}}{\alpha^m}(X_{T_\lambda}) : T_\lambda \leq t \right]. \end{aligned}$$

Since  $\|s_D\| < 1$  and  $T_\lambda \rightarrow \infty$  *a.s.* as  $\lambda \rightarrow \infty$ , it follows from the monotone convergence theorem that

$$\begin{aligned} & \lim_{\lambda \rightarrow \infty} \left[ \lambda \mathbb{P}_o \left( \sup_{0 \leq s \leq t} \log \frac{1}{\|s_D \circ f(X_s)\|} > \lambda \right) + \frac{1}{2} \mathbb{E}_o [\log^+ \xi(X_t) : t < T_\lambda] \right] \\ & = \tilde{N}_f(t, D) + \frac{1}{2} \mathbb{E}_o [\log^+ \xi(X_t)] \end{aligned}$$

and

$$\lim_{\lambda \rightarrow \infty} \mathbb{E}_o \left[ \log \frac{f^* \Omega \wedge \alpha^{m-n}}{\alpha^m}(X_{T_\lambda}) : T_\lambda \leq t \right] = -\tilde{N}_{f', D}(t, 0).$$

Therefore,

$$\lim_{\lambda \rightarrow \infty} \frac{1}{2} \mathbb{E}_o [\log \xi(X_{t \wedge T_\lambda})] \leq \tilde{N}_f(t, D) - \tilde{N}_{f', D}(t, 0) + \frac{1}{2} \mathbb{E}_o [\log^+ \xi(X_t)]. \quad (18)$$

Combining (17) and (18) with conditions, we get

$$\begin{aligned} & \lim_{\lambda \rightarrow \infty} \left[ \tilde{T}_f(t \wedge T_\lambda, L) + \tilde{T}_f(t \wedge T_\lambda, K_N) + \tilde{T}(t \wedge T_\lambda, \mathcal{R}_M) \right] \\ & \leq \tilde{N}_f(t, D) - \tilde{N}_{f', D}(t, 0) + \frac{1}{2} \mathbb{E}_o [\log^+ \xi(X_t)] + O(1) < \infty. \end{aligned} \quad (19)$$

Apply Lebesgue's control convergence theorem to (19), we have the desired result.  $\square$

*Proof of Theorem 1.3.* Following Ru-Wong's arguments (see [21], pp. 231-233; see also [20]), there exist a finite open covering  $\{U_\lambda\}$  of  $N$  and rational functions  $w_{\lambda 1}, \dots, w_{\lambda n}$  on  $N$  for every  $\lambda$  such that  $w_{\lambda 1}, \dots, w_{\lambda n}$  are holomorphic on  $U_\lambda$  and

$$dw_{\lambda 1} \wedge \dots \wedge dw_{\lambda n}(y) \neq 0, \quad \forall y \in U_\lambda;$$

$$U_\lambda \cap D = \{w_{\lambda 1} \cdots w_{\lambda h_\lambda} = 0\}, \quad \exists h_\lambda \leq n.$$

In addition, we can require  $L_{D_j}|_{U_\lambda} \cong U_\lambda \times \mathbb{C}$  for  $\lambda, j$ . On  $U_\lambda$ , we have

$$\Phi = \frac{e_\lambda}{|w_{\lambda 1}|^2 \cdots |w_{\lambda h_\lambda}|^2} \bigwedge_{k=1}^n \frac{\sqrt{-1}}{2\pi} dw_{\lambda k} \wedge d\bar{w}_{\lambda k},$$

where  $\Phi$  is given by (16) and  $e_\lambda$  is a smooth positive function. Let  $\{\phi_\lambda\}$  be a partition of unity subordinate to  $\{U_\lambda\}$ , then  $\phi_\lambda e_\lambda$  is bounded on  $N$ . Set

$$\Phi_\lambda = \frac{\phi_\lambda e_\lambda}{|w_{\lambda 1}|^2 \cdots |w_{\lambda h_\lambda}|^2} \bigwedge_{k=1}^n \frac{\sqrt{-1}}{2\pi} dw_{\lambda k} \wedge d\bar{w}_{\lambda k}.$$

Put  $f_{\lambda k} = w_{\lambda k} \circ f$ , then on  $f^{-1}(U_\lambda)$  we obtain

$$f^* \Phi_\lambda = \frac{\phi_\lambda \circ f \cdot e_\lambda \circ f}{|f_{\lambda 1}|^2 \cdots |f_{\lambda h_\lambda}|^2} \bigwedge_{k=1}^n \frac{\sqrt{-1}}{2\pi} df_{\lambda k} \wedge d\bar{f}_{\lambda k}. \quad (20)$$

Set

$$f^* \Phi_\lambda \wedge \alpha^{m-n} = \xi_\lambda \alpha^m,$$

then we have  $\xi = \sum_\lambda \xi_\lambda$ . Again, set

$$f^* c_1(L, h) \wedge \alpha^{m-1} = \varrho \alpha^m. \quad (21)$$

Then

$$\varrho = \frac{1}{2m} e_{f^* \omega}. \quad (22)$$

For each  $\lambda$  and any  $x \in f^{-1}(U_\lambda)$ , take a local holomorphic coordinate system  $z$  around  $x$ . Since  $\phi_\lambda \circ f \cdot e_\lambda \circ f$  is bounded, it is not very hard to see from (20) and (21) that  $\xi_\lambda$  is bounded from above by  $P_\lambda$ , where  $P_\lambda$  is a polynomial in

$$\varrho, \quad g^{i\bar{j}} \frac{\partial f_{\lambda k}}{\partial z_i} \overline{\frac{\partial f_{\lambda k}}{\partial z_j}} / |f_{\lambda k}|^2, \quad 1 \leq i, j \leq m, \quad 1 \leq k \leq n.$$

It yields that

$$\log^+ \xi_\lambda \leq O\left(\log^+ \varrho + \sum_k \log^+ \frac{\|\nabla_M f_{\lambda k}\|}{|f_{\lambda k}|}\right) + O(1).$$

Thus,

$$\log^+ \xi \leq O\left(\log^+ \varrho + \sum_{k,\lambda} \log^+ \frac{\|\nabla_M f_{\lambda k}\|}{|f_{\lambda k}|}\right) + O(1).$$

By this with Theorem 1.2

$$\begin{aligned} & \frac{1}{2} \mathbb{E}_o [\log \xi(X_t)] \\ & \leq O\left(\sum_{k,\lambda} \mathbb{E}_o \left[\log^+ \frac{\|\nabla_M f_{\lambda k}\|}{|f_{\lambda k}|}(X_t)\right]\right) + O\left(\mathbb{E}_o [\log^+ \varrho(X_t)]\right) + O(1) \\ & \leq O\left(\sum_{k,\lambda} \tilde{m}\left(t, \frac{\|\nabla_M f_{\lambda k}\|}{|f_{\lambda k}|}\right)\right) + O\left(\log^+ \mathbb{E}_o [\varrho(X_t)]\right) + O(1) \\ & \leq O\left(\sum_{k,\lambda} \log \tilde{T}_{f_{\lambda k}}(t, \omega_{FS})\right) + O\left(\log^+ \mathbb{E}_o [\varrho(X_t)]\right) + O(1) \\ & \leq O\left(\log^+ \tilde{T}_f(t, L)\right) + O\left(\log^+ \mathbb{E}_o [\varrho(X_t)]\right) + O(1). \end{aligned}$$

Moreover, Lemma 3.1 and (22) imply that

$$\begin{aligned} \log^+ \mathbb{E}_o [\varrho(X_t)] & \leq (1 + \delta) \log^+ \mathbb{E}_o \left[ \int_0^t \varrho(X_s) ds \right] \\ & = \frac{(1 + \delta)}{2m} \log^+ \mathbb{E}_o \left[ \int_0^t e_{f^* c_1(L, h)}(X_s) ds \right] \\ & \leq \frac{(1 + \delta)}{m} \log^+ \tilde{T}_f(t, L) + O(1). \end{aligned}$$

Combining the above with Lemma 3.3, we prove the theorem.  $\square$

**3.3. Defect Relation.** Let  $L_1, L_2$  be holomorphic line bundles over a complex projective manifold  $N$ . Define

$$\left[ \frac{c_1(L_2)}{c_1(L_1)} \right] = \sup \{ a \in \mathbb{R} : L_2 > aL_1 \}, \quad \overline{\left[ \frac{c_1(L_2)}{c_1(L_1)} \right]} = \inf \{ a \in \mathbb{R} : L_2 < aL_1 \}.$$

It is clear that

$$\left[ \frac{c_1(L_2)}{c_1(L_1)} \right] \leq \liminf_{t \rightarrow \infty} \frac{\tilde{T}_f(t, L_2)}{\tilde{T}_f(t, L_1)} \leq \limsup_{r \rightarrow \infty} \frac{\tilde{T}_f(t, L_2)}{\tilde{T}_f(t, L_1)} \leq \overline{\left[ \frac{c_1(L_2)}{c_1(L_1)} \right]}. \quad (23)$$

Let  $M$  be a complete and stochastically complete Kähler manifold with  $\dim_{\mathbb{C}} M \geq \dim_{\mathbb{C}} N$ , and let  $(L, h)$  be a positive Hermitian line bundle over  $N$ . For  $f : M \rightarrow N$ , a differentiably non-degenerate holomorphic mapping such that  $\tilde{T}_f(t, L) \rightarrow \infty$  as  $t \rightarrow \infty$ , we define the *defect* of  $f$  with respect to  $D$  by

$$\tilde{\delta}_f(D) = 1 - \limsup_{t \rightarrow \infty} \frac{\tilde{N}_f(t, D)}{\tilde{T}_f(t, L)}.$$

**THEOREM 3.4 (Defect relation).** *Assume the same conditions as in Theorem 1.3 and  $\tilde{T}_f(t, L) \rightarrow \infty$  as  $t \rightarrow \infty$ . Then*

$$\tilde{\delta}_f(D) \leq \overline{\left[ \frac{c_1(K_N^*)}{c_1(L)} \right]} - \left[ \frac{\mathcal{R}_M}{f^*c_1(L)} \right].$$

*Proof.* It follows from Theorem 1.3 that

$$1 - \frac{\tilde{N}_f(t, D)}{\tilde{T}_f(t, L)} \leq \frac{\tilde{T}_f(t, K_N^*)}{\tilde{T}_f(t, L)} - \frac{\tilde{T}(t, \mathcal{R}_M)}{\tilde{T}_f(t, L)}.$$

Let  $t \rightarrow \infty$ , then we have the theorem proved.  $\square$

**COROLLARY 3.5.** *Assume the same conditions as in Theorem 3.4. If  $\text{Ric}_M \geq 0$ , then*

$$\tilde{\delta}_f(D) \leq \overline{\left[ \frac{c_1(K_N^*)}{c_1(L)} \right]}.$$

*Proof.* Since  $\text{Ric}_M \geq 0$ , then

$$\left[ \frac{\mathcal{R}_M}{f^*c_1(L)} \right] \geq 0.$$

This proves the corollary.  $\square$

**COROLLARY 3.6.** *Let  $D_j \in |L|$  for  $1 \leq j \leq q$  such that  $\sum_{j=1}^q D_j$  is of simple normal crossing type. Assume the same conditions as in Theorem 3.4. If  $s_M \geq 0$ , then*

$$\sum_{j=1}^q \tilde{\delta}_f(D_j) \leq \frac{1}{q} \overline{\left[ \frac{c_1(K_N^*)}{c_1(L)} \right]}.$$

COROLLARY 3.7. *Let  $D_1, \dots, D_q$  be hypersurfaces in  $\mathbb{P}^n(\mathbb{C})$  of degree  $d_1, \dots, d_q$  such that  $\sum_{j=1}^q D_j$  is of simple normal crossing type. Let  $f : M \rightarrow \mathbb{P}^n(\mathbb{C})$  be a differentiably non-degenerate holomorphic mapping such that  $\tilde{T}_f(t, \omega_{FS}) < \infty$  for  $0 < t < \infty$ . If  $s_M \geq 0$  and*

$$\mathbb{E}_o \left[ \int_0^t s_M(X_s) ds \right] < \infty$$

for  $0 < t < \infty$ , then

$$\sum_{j=1}^q d_j \tilde{\delta}_f(D_j) \leq n + 1.$$

*Proof.* Since  $s_M \geq 0$  implies that  $\mathcal{R}_M \geq 0$ , then it follows  $R_M \geq 0$ . Thus, it yields from Lemma 2.2 that  $\tilde{T}_f(t, L) \rightarrow \infty$  as  $t \rightarrow \infty$ . Furthermore,

$$\begin{aligned} 0 \leq \tilde{T}(t, \mathcal{R}_M) &= -\frac{1}{4} \mathbb{E}_o \left[ \int_0^t \Delta_M \log \det(g_{i\bar{j}})(X_s) ds \right] \\ &= \frac{1}{2} \mathbb{E}_o \left[ \int_0^t s_M(X_s) ds \right] < \infty \end{aligned}$$

and

$$c_1(K_{\mathbb{P}^n(\mathbb{C})}^*) = (n+1)[\omega_{FS}], \quad c_1(L_{D_j}) = d_j[\omega_{FS}].$$

Hence, we have the corollary proved.  $\square$

COROLLARY 3.8. *Let  $a_1, \dots, a_q$  be distinct points in a compact Riemann surface  $S$  of genus  $g$ . Let  $f : M \rightarrow S$  be a differentiably non-degenerate holomorphic mapping such that  $\tilde{T}_f(t, L_{a_1}) < \infty$  for  $0 < t < \infty$ . If  $s_M \geq 0$  and*

$$\mathbb{E}_o \left[ \int_0^t s_M(X_s) ds \right] < \infty$$

for  $0 < t < \infty$ , then

$$\sum_{j=1}^q \tilde{\delta}(a_j) \leq 2 - 2g.$$

If  $M$  is parabolic, namely,  $X_t$  is recurrent, then we obtain

THEOREM 3.9. *Let  $L$  be a positive line bundle over a complex projective manifold  $N$ . Let  $D \in |L|$  be of simple normal crossing type. Let  $f : M \rightarrow N$  be a differentiably non-degenerate holomorphic mapping. If*

$$\int_M s_M^-(x) dV(x) < \infty, \tag{24}$$

then

(a) *Let  $R_M(x) \geq -cr^2(x) - c$  for a constant  $c > 0$ , where  $R_M$  is defined by (2). If  $f$  has finite energy, i.e.,*

$$E(f) := \int_M e_{f^*c_1(L,h)}(x) dV(x) < \infty,$$

then

$$\tilde{\delta}_f(D) \leq \left[ \frac{c_1(K_N^*)}{c_1(L)} \right] + \frac{\int_M s_M^-(x) dV(x)}{E(f)}.$$

(b) Let  $R_M(x) \geq -k(r(x))$  for a nondecreasing function  $k \geq 0$  such that  $k(r)/r^2 \rightarrow 0$  as  $r \rightarrow \infty$ . If (1) is satisfied and  $f$  has infinite energy, then

$$\tilde{\delta}_f(D) \leq \left[ \frac{c_1(K_N^*)}{c_1(L)} \right].$$

*Proof.* From Lemma 2.2, we note that  $\tilde{T}_f(t, L) \rightarrow \infty$  as  $t \rightarrow \infty$ . Ricci curvature assumption implies that  $M$  is stochastically complete, and parabolicity assumption implies that ratio ergodic theorem holds (see [19]). Using ratio ergodic theorem, we get

$$\begin{aligned} \frac{\tilde{T}(t, \mathcal{R}_M)}{\tilde{T}_f(t, L)} &= \frac{\mathbb{E}_o \left[ \int_0^t s_M(X_s) ds \right]}{\mathbb{E}_o \left[ \int_0^t e_{f^*c_1(L, h)}(X_s) ds \right]} \rightarrow \frac{\int_M s_M(x) dV(x)}{\int_M e_{f^*c_1(L, h)}(x) dV(x)} \\ &= \frac{\int_M s_M(x) dV(x)}{E(f)} < \infty \end{aligned}$$

as  $t \rightarrow \infty$ . Thus,  $\tilde{T}(t, L) < \infty$  for  $t < \infty$  and

$$-\left[ \frac{\mathcal{R}_M}{f^*c_1(L)} \right] \leq \frac{\int_M s_M^-(x) dV(x)}{E(f)}.$$

By Theorem 3.4, (a) follows. For (b), we first note that  $\tilde{T}_f(t, L)$  makes sense since Lemma 2.1. By ratio ergodic theorem, we see that (b) holds provided with  $E(f) = \infty$ .  $\square$

If  $M$  is non-parabolic, namely,  $X_t$  is transient, then we obtain

**THEOREM 3.10.** *Assume that (24) holds and  $R_M(x) \geq -k(r(x))$  for a nondecreasing function  $k \geq 0$  satisfying  $k(r)/r^2 \rightarrow 0$  as  $r \rightarrow \infty$ . Let  $L$  be a positive line bundle over a complex projective manifold  $N$  and  $D \in |L|$  be of simple normal crossing type. Let  $f : M \rightarrow N$  be a differentiably non-degenerate holomorphic mapping satisfying (1) and  $\tilde{T}_f(t, L) \rightarrow \infty$  as  $t \rightarrow \infty$ . Then*

$$\tilde{\delta}_f(D) \leq \left[ \frac{c_1(K_N^*)}{c_1(L)} \right].$$

*Proof.* If  $\mathcal{R}_M \geq 0$ , the assertion follows from Theorem 3.4. If  $\mathcal{R}_M < 0$ , then

$$|\tilde{T}(t, \mathcal{R}_M)| = \frac{1}{2} \mathbb{E}_o \left[ \int_0^\infty s_M^-(X_t) dt \right].$$

The non-parabolicity of  $M$  implies that (see [5], Theorem 22)

$$\mathbb{E}_o \left[ \int_0^\infty R_M^-(X_t) dt \right] < \infty,$$



By  $s_M \geq mR_M$ , we see that

$$|\tilde{T}(t, \mathcal{R}_M)| = \frac{1}{2} \mathbb{E}_o \left[ \int_0^\infty s_M^-(X_t) dt \right] \leq \frac{m}{2} \mathbb{E}_o \left[ \int_0^\infty R_M^-(X_t) dt \right] < \infty.$$

Hence,  $\tilde{T}(t, \mathcal{R}_M)$  is bounded. The theorem follows from Theorem 3.4.  $\square$

*Proof of Theorem 1.4.*  $\text{Ric}_M \geq 0$  implies that  $\tilde{T}_f(t, L) \rightarrow \infty$  as  $t \rightarrow \infty$  since Lemma 2.2, and the energy assumption means that  $\tilde{T}_f(t, L) < \infty$  for  $t < \infty$  since Lemma 2.1. By  $\mathcal{R}_M \geq 0$ , the theorem follows from Theorem 3.4.  $\square$

**3.4. The case when  $M$  is an algebraic manifold.** In Section 4, we obtain an analogue of Nevanlinna theory on a wide class of Kähler manifolds. Sometimes, we are more concerned with domain  $M$  which is an algebraic manifold. Consider the algebraic manifold  $M := X \setminus S$ , where  $X$  is a complex projective manifold and  $S$  is a hypersurface of simple normal crossing type in  $X$ . Note that  $M$  is stochastically complete. Let  $S = \sum_{j=1}^r S_j$  be a decomposition into irreducible components. Taking  $\sigma \in H^0(M, L_S)$  and  $\sigma_j \in H^0(M, L_{S_j})$  satisfying  $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r$  and  $(\sigma_j) = S_j$ .

Assume that  $(L_D, \tau) > 0$ , i.e., the Chern form  $c_1(L_D, \tau) > 0$ . We consider situation for the following three typical complete Kähler metrics  $\alpha$  on  $M$  (see [4], pp. 1023), where the Second Main Theorem (Theorem 1.3) still holds.

(I) Projective type:  $\alpha = dd^c \log \|\sigma\|^{-2}$ . Under this metric,  $M$  is parabolic, namely, the Brownian motion is recurrent. However,  $M$  is not stochastically complete. Hence, we cannot ensure the desired property:  $\tilde{N}_f(t, D) = 0$  if  $f$  omits  $D$ .

(II) Euclidean type:  $\alpha = dd^c \|\sigma\|^{-2}$ . Under this metric, the Ricci curvature of  $M$  is bounded and therefore  $\tilde{N}_f(t, D) = 0$  if  $f$  omits  $D$ . Moreover,  $M$  is non-parabolic for  $\dim_{\mathbb{C}} \geq 2$ , i.e., the Brownian motion is transient (see [13]).

(III) Poincaré type:  $\alpha = Cdd^c \log \|\sigma\|^{-2} - \sum_{j=1}^r dd^c \log(\log \|\sigma_j\|^2)^2$ . The metric was introduced by Cornalba-Griffiths [9]. In this case, we can consider a defect relation by choosing a suitable metric  $\tau$  and a constant  $C$ .

LEMMA 3.11. *Assume that  $L > 0$ . Then there exist a constant  $C > 0$  and a Hermitian metric  $\tau$  on  $L$  such that  $\alpha$  satisfies the following properties:*

- (a)  $\alpha$  is complete;
- (b)  $M$  has finite volume with respect to  $\alpha$ ;
- (c)  $\text{Ric}_M$  is bounded. More precisely,  $-2\alpha \leq \mathcal{R}_M < 0$ .

In the above lemma, (b) implies the parabolicity of  $M$ ; (a) and (c) ensures that  $\tilde{N}_f(t, D) = 0$  if  $f$  omits  $D$ .

THEOREM 3.12. *Let  $f : M \rightarrow N$  be a differentiably non-degenerate holomorphic mapping into a complex projective manifold  $N$  with  $\dim_{\mathbb{C}} N \leq \dim_{\mathbb{C}} M$ , where  $M = X \setminus S$  is equipped with a complete Kähler metric  $\alpha$  satisfying the properties of Lemma 3.11. Let  $L$  be a positive line bundle over  $N$ . If  $f$  satisfies (1), then*

$$\tilde{\delta}_f(D) \leq \left[ \frac{c_1(K_N^*)}{c_1(L)} \right] + \frac{4m \cdot \text{Vol}(M)}{E(f)},$$

where  $m = \dim_{\mathbb{C}} M$ .

*Proof.* By Lemma 3.11,  $-2\alpha \leq \mathcal{R}_M < 0$ . It is therefore

$$s_M^- = -2m \frac{\mathcal{R} \wedge \alpha^{m-1}}{\alpha^m} \leq 2m \frac{2\alpha \wedge \alpha^{m-1}}{\alpha^m} = 4m.$$

According to Theorem 3.9, we have the theorem proved.  $\square$

COROLLARY 3.13. *Assume the same condition as in Theorem 3.12. If*

$$\int_M e_{f^*c_1(L,h)} \alpha^m = \infty,$$

then

$$\tilde{\delta}_f(D) \leq \left[ \frac{c_1(K_N^*)}{c_1(L)} \right].$$

**Acknowledgement.** The author is grateful to the referee for his valuable comments on this paper.

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