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MOMENT MAP FOR COUPLED EQUATIONS OF KÄHLER FORMS AND CURVATURE*

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Abstract. In this paper we introduce two new systems of equations in Kähler geometry: The coupled p equation and the generalized coupled cscK equation. We motivate the equations from the moment map pictures, prove the uniqueness of solutions and find out the obstructions to the solutions for the second equation. We also point out the connections between the coupled cscK equation, the coupled Kähler Yang-Mills equations and the deformed Hermitian Yang-Mills equation.

Moreover, using this moment map, we can show the Mabuchi functional for the generalized coupled cscK equation, and a special case of the coupled Kähler Yang-Mills equations and the deformed Hermitian Yang-Mills equation are convex along the smooth geodesic, which is different from the one using the moment map picture from the gauge group. In our case, the geodesic is given by the natural metric on the product of smooth Kähler potential $\mathcal{K}(X,\omega_0) \times \cdots \times \mathcal{K}(X,\omega_k)$.

Key words. Kähler geometry, moment map, differential geometry, partial differential equations.

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1. Introduction.

1.1. Motivation. Over the years, many important equations in complex geometry have been given moment map interpretations. A few examples of equations with moment map interpretations are the cscK equation ([Don00] and [Fuj92]), the coupled Kähler Yang-Mills equation ([ACGFGP13]) and the coupled constant scalar curvature ([DP20]). In this paper, we combine the moment maps for the latter two together with some new ideas to define a new type of canonical metrics. We begin by recalling the definition of the coupled Khler-Yang-Mills equation.

DEFINITION 1.1 ([ACGFGP13]). Let P be a principal $U(k)$ bundle on a Kähler manifold (X, ω_X), A be a connection on P, and F_A be the curvature which is an $Lie(G)$-valued 2 form. Then the coupled Kähler equation is given by

$$\begin{array}{rcl} \alpha_0 S_g + \alpha_1 \bigwedge^2 F_A \wedge F_A & = & c \\ \bigwedge F_A & = & z \end{array}$$

where $z \in Lie(G)$ is invariant under the adjoint $U(k)$ action and c is a constant, which depended on the topological constraint on P and $\alpha_0, \alpha_1, [\omega]$. Also, we need the integrability condition

$$F_A^{0,2} = 0.$$

If $P = U(1)^k$, then the Lie algebra is $u(1) \oplus \cdots \oplus u(1)$, and the F_A can be represented as

$$F_A = \omega_1 + \cdots + \omega_k,$$

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where $\omega_1, ..., \omega_k$ are L-valued Kähler forms on X, which can be realized as Kähler forms on X. In this special case, the moment map equation is given by

$$\begin{array}{rcl}\alpha_0 S_g + \alpha_1 \sum_{i=1}^{k} \frac{\omega_i^2 \wedge \omega_0^{n-2}}{\omega_0^n} & = & c_0 \\ \mathrm{Tr}_{\omega_0}(\omega_1) & = & c_1 . \\ \vdots & = & \vdots \\ \mathrm{Tr}_{\omega_0}(\omega_k) & = & c_k . \end{array}$$

In [HWN18], Hultgren and Witt Nyström introduced another type of canonical metrics: the coupled Kähler Einstein equation. This equation was later generalized by Datar and Pingali ([DP20]) to the coupled cscK equation:

Definition 1.2 ([DP20]). Let X be a Kähler manifold and $\omega_0, ..., \omega_k$ be Kähler forms on X, and let $\omega = \sum_{i=0}^{k} \omega_i$. Then the coupled cscK equation is given by

$$\frac{\omega_0^n}{\mathrm{vol}(\omega_0)} = \cdots = \frac{\omega_k^n}{\mathrm{vol}(\omega_k)}$$
$$S_{\omega_0} = \mathrm{Tr}_{\omega_0}\omega + c,$$

here c is the topological constant depending on the class of ω_i and $\mathrm{Ric}(\omega)$. If $c = 0$, then this reduces to the coupled Kähler-Einstein equation.

Both the cKYM equation and the ccscK equation are moment map equations. For both setups, the domains are in a subspace of $\mathcal{Y} \subset \mathcal{J} \times \mathcal{A}$, which for each (J, A), $A \in \Omega_J^{1,1}(ad(P))$ (See [ACGFGP13], [DP20]). Notice that in order to get the topological constraint, the setups are in the complexifed orbit. But we can study deformation of ths solutions if we consider the bigger subspace $\mathcal{Y}$. When P is a principal $U(1)$ bundle, the moment map for the coupled Kähler Yang-Mills equation is

$$\begin{aligned}\mu_{cKYM}(J, A)(\varphi, \xi) = & \int_X \varphi \left(c - S(J) - \alpha_2 \frac{\omega_X^{n-2} \wedge \omega_A^2}{\omega_X^n} + \alpha_1 \frac{\omega_X^{n-1} \wedge \omega_A}{\omega_X^n} \right) \frac{\omega_X^n}{n!} \\ & + \int_X \theta_A \xi \wedge \left(\alpha_1 z - \alpha_2 \frac{\omega_X^{n-1} \wedge \omega_A}{\omega_X^n} \right) \omega_X^n ;\end{aligned}$$

and the moment map for coupled cscK equation is

$$\mu_{ccscK}(J, A)(H_{\xi,0}, H_{\xi,A}) = \int_X H_{\xi,0} \left(c - S(J) + \frac{\omega_X^{n-1} \wedge \omega_A}{\omega_X^n} \right) + \int_X H_{\xi,A} \left(\frac{\omega_A^n}{\omega_X^n} - z \right)$$

We will now explain how both these moment maps can be constructed using the moment map for the cscK metrics together with a new construction involving maps between two symplectic manifolds (X, ω_X) and (Y, ω_Y) which are diffeomorphic to each others.

Definition 1.3 (Defintion 2.2). We denote the map

$$\mu_p : \mathrm{Map}(X, Y; p)^+ \to Lie(\mathrm{Ham}(X, \omega_X) \times \mathrm{Ham}(Y, \omega_Y))^*$$

by

$$\mu_{p,\omega_X\omega_Y}(f) = \frac{n}{n-p}\left(c_1\frac{\omega_X^n}{n!} - \frac{\omega_X^{n-p-1}\wedge f^*\omega_Y^{p+1}}{(n-p-1)!(p+1)!}, c_2\frac{\omega_Y^n}{n!} - \frac{f_*\omega_X^{n-p}\wedge\omega_Y^p}{(n-p)!p!}\right), \quad (1)$$

where $c_1 = \dfrac{\int_X \omega_X^{n-p-1}\wedge f^*\omega_Y^{p+1}}{\int_X \omega_X^n}$, $c_2 = \dfrac{\int_Y f_*\omega_X^{n-p}\wedge\omega_Y^p}{\int_Y \omega_Y^n}$.

We also have the classical moment map: Denote $\mathcal{J}_{int}(X)$ be the space of all integrable almost complex structure, and let

$$\mathcal{J}(X,\omega_0) := \{J \in \mathcal{J}_{int}(X) | \omega_0(\bullet,\bullet) = \omega_0(J\bullet, J\bullet), \omega_0(\bullet, J\bullet) > 0\}$$

be the space of integrable almost complex structure compactible with ω_0. The metric $g_J := \omega(\bullet, J\bullet)$ induces a pairing on $T_J\mathcal{J}(X,\omega_0)$, and

$$\Omega_{\mathcal{J}}(\delta_1 J, \delta_2 J) := \int_X g_J(\delta_1 J, \delta_2 J)\frac{\omega^n}{n!}.$$

Then the map

$$\mu_{\mathcal{J}}(J) = (S_J - \underline{S_J})\frac{\omega_0^n}{n!} = \mathrm{Ric}(X,J)\wedge\frac{\omega_0^{n-1}}{(n-1)!} - S\frac{\omega_n}{n!}$$

is a moment map corresponding to $(\mathcal{J}(X,\omega_0), \Omega_{\mathcal{J}})$ (see [Don97], [Fuj92]), where

$$\underline{S_J} = \frac{1}{Vol(X,\omega^n)}\int_X S_J\frac{\omega_0^n}{n!}$$

is the average of S_J.

As X is diffeomorphic to Y, if we take $\omega_A = f^*\omega_Y$, then under suitable domain,

1) $\mu_{cKYM}(J,A) = 0$ iff $\mu_{\mathcal{J}}(J) + c_1\mu_1^*(f) - c_2\mu_0^*(f) = 0$ for some suitable constant c_1, c_2;
2) $\mu_{ccscK}(J,A) = 0$ iff $\mu_{\mathcal{J}}(J) + c\mu_0(f) = 0$ for some suitable constant c.

Notice that with a suitable choice of domain and symplectic form, the sum of two moment maps can also be a moment map. Therefore, we unify the ccscK equations and coupled Kähler Yang Mills equation into one general moment map setup, namely, the sum of different moment maps μ_p with the standard moment map $\mu_{\mathcal{J}}$. Moreover, using the same idea, we reconstruct the moment map for deformed Hermitian Yang Mills equation (dHYM) (see [CXY17]) and the coupled dHYM ([SS19]) in section 3.4.

1.2. Construction. We will now explain how to choose the symplectic form, the domain and the range to make the sum of two moment maps a moment map in general by considering the construction for the moment map $\mu_{\mathcal{J}} + \mu_0$, i.e, the moment map for ccscK, as an example.

Step 1 Define

$$\begin{aligned}\mu_{\mathcal{J},0}\colon & \mathcal{J}(X,\omega_0)\times \mathrm{Map}(X,X)\\ & \to Lie(\mathrm{Ham}(X,\omega_0))^* \oplus Lie(\mathrm{Ham}(X,\omega_0)\times\mathrm{Ham}(X,\omega_1))^*\end{aligned}$$

by

$$\mu_{\mathcal{J},0}(J,f) := (\mu_{\mathcal{J}}, \mu_0).$$

We need to show that this is a moment map for the Kähler form

$$\Omega_{\mathcal{J},0} := \Omega_{\mathcal{J}} + \Omega_0$$

For this moment map, the range contains more equations than we want, and the domain J and f has no relation. We will fix this issue by the following steps.

Step 2 Consider the subgroup $H \cong \mathrm{Ham}(X, \omega_0) \times \mathrm{Ham}(X, \omega_1)$, and the embedding map $\iota : H \to \mathrm{Ham}(X, \omega_0) \times \mathrm{Ham}(X, \omega_0) \times \mathrm{Ham}(X, \omega_1)$ by

$$\iota(\sigma, \eta) = (\sigma^{-1}, \sigma, \eta)$$

It induces a map $\iota^* : Lie(\mathrm{Ham}(X, \omega_0) \times \mathrm{Ham}(X, \omega_0) \times \mathrm{Ham}(X, \omega_1))^* \to Lie(H)^*$, and the map

$$\iota^* \circ \mu_{\mathcal{J},0} = \left(\frac{\omega_0^{n-1} \wedge (-\mathrm{Ric}(\omega_0, J) + f^*\omega_1 - c_1\omega_0)}{(n-1)!}, \frac{f_*\omega_0^n - c_2\omega_1^n}{n!} \right)$$

is also a moment map.

Step 3 In order to make sure the solution indeed is Kähler, we consider the subspace

$$\mathcal{Y}_0 := \{(J, f) | DfJDf^{-1} \in \mathcal{J}(X, \omega_1)\},$$

and we need to show that this space has the following properties:

a) It is closed under the action of H;
b) It is a smooth manifold. If we want the solutions to be Kähler, we need this space to be a Kähler manifold.

Then $f^*\omega_1$ is J invariant and hence it is a Kähler form. But our theory also need the domain to be the complexified orbit space $H^{\mathbb{C}} \cdot (J_0, f_0)$. Notice that this space is equivalent to

$$\begin{gathered}\{(\varphi_0, \varphi_1) \in \mathrm{Map}(X, X) | \varphi_i^*\omega_i = \omega_i + \sqrt{-1}\partial\bar{\partial}h_i \\ \text{for some } h_i \in PSH(X, \omega_i), i = 0, 1\}.\end{gathered}$$

We will show that the solution of $\iota^* \circ \mu_{\mathcal{J},0}|_{H^{\mathbb{C}}\cdot(J_0,f_0)} = 0$ is equivalent to the solution of ccscK equation in the Kähler class $[\omega_0], [\omega_1]$.

Remark 1.4.

1) we may also choose $\Omega_{\mathcal{J},0;a_1,a_2} := a_1\Omega_{\mathcal{J}} + a_2\Omega_0$ in step 1 for some positive number a_1, a_2 to affect the constant of the outcome moment map, that is,

$$\iota^* \circ \mu_{\mathcal{J},0} = \left(\frac{\omega_0^{n-1} \wedge (-a_1\mathrm{Ric}(\omega_0, J) + a_2f^*\omega_1 + c_1'\omega_0)}{(n-1)!}, \frac{a_2f_*\omega_0^n - c_2'\omega_1^n}{n!} \right)$$

but a_1, a_2 need to be positive so that $\Omega_{\mathcal{J},0}$ is still a symplectic (or Kähler form if it is J invariant).

2) Notice that the embedding is not unique. For example ,it may also be (σ, σ, η), $(\sigma, \sigma^{-1}, \eta)$ or $(\sigma^{-1}, \sigma, \eta^{-1})$. These embedding change part of the sign of the moment map. For example, if we change the embedding to be (σ, σ, η), then the moment map becomes

$$\iota^* \circ \mu_{\mathcal{J},0} = \left(\frac{\omega_0^{n-1} \wedge (-a_1\mathrm{Ric}(\omega_0, J) - a_2f^*\omega_1 - c_1'\omega_0)}{(n-1)!}, \frac{a_2f_*\omega_0^n - c_2'\omega_1^n}{n!} \right).$$

3) Notice that if $(J, f) \in \mathcal{Y}_0$, then $(J^{-1}, f) = (-J, f) \in \mathcal{Y}_0$. This implies the above choices of embedding won't affect the space $\mathcal{Y}_0$. However, the corresponding Mabuchi functional will be the same.

In general, the main technical part for this set up is to find the correct domain space (which is $\mathcal{Y}_0$ here). We need a space that is closed under the action and is a Kähler submanifold. In section 5, we will discuss the difficulties of finding the suitable complex submanifold of $\mathrm{Map}(X, X)$ for the moment map μ_p.

Similarly, for coupled Kähler Yang-Mills equation, we first construct

$$\mu_{01} = (\mu_{\mathcal{J}} + a_1\mu_0^* - a_2\mu_1^*)|_{cKYM_{01}} = \iota^* \circ (\mu_{\mathcal{J}}, \mu_0^*, \mu_1^*)|_{\mathcal{YM}_{01}}$$

using step 1 and step 2 with a suitable embedding restricted in a suitable subspace $\mathcal{YM}_{01}^+$. The subspace we take in step 3 should be

$$\begin{aligned}\mathcal{YM}_{01} \subset \{&(J, f, g) \in \mathcal{J}(X, \omega_0) \times \mathrm{Map}(X_1, X_0; n-2)^+ \times \mathrm{Map}(X_1, X_0; n-1)^+ | g \\ &= f^{-1}, DfJDf^{-1} \in \mathcal{J}(X, \omega_1)\},\end{aligned}$$

such that

$$\Omega_{\mathcal{J},01;\alpha_0,\alpha_1,\alpha_2} = \alpha_0\Omega_{\mathcal{J}} - \alpha_1\Omega_0^* + \alpha_2\Omega_1^* > 0\}.$$

Here μ^* and Ω^* are defined in Definition 2.9, as we need

$$\int_X (\alpha_1\omega_1 \wedge \omega_0^{[n-1]} - \alpha_2\omega_0^{[n]}) = 0,$$

where $\omega^{[k]} = \dfrac{\omega^k}{k!}$. If we take the undual one, $\alpha_0\Omega_{\mathcal{J}} - \alpha_1\Omega_0 + \alpha_2\Omega_1$ must not be positive.

1.3. Main result. To discuss the main result, We first define the coupled p equation.

Definition 1.5 (coupled p equation). Let $(X, \omega_X), (Y, \omega_Y)$ be symplectic manifolds which are diffeomorphic to each other, $0 \leq p \leq n-1$ and let $\mathrm{Map}(X, Y; p)^+$ be the space of diffeomorphism such that $\omega_X^{n-p} \wedge f^*\omega_Y^p$ is a volume form (see definition (2.1)). Then the couple equation p is given by

$$\mu_p = 0,$$

where $\mu_p : \mathrm{Map}(X, Y; p)^+ \to Lie(\mathrm{Ham}(X, \omega_X) \times \mathrm{Ham}(Y, \omega_Y))^*$ is defined by

$$\mu_p(f) := \left(c_1\frac{\omega_X^n}{n!} - \frac{\omega_X^{n-p-1} \wedge f^*\omega_Y^{p+1}}{(n-p-1)!(p+1)!}, \quad c_2\frac{\omega_Y^n}{n!} - \frac{f_*\omega_X^{n-p} \wedge \omega_Y^p}{(n-p)!p!} \right),$$

with $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{aligned}\int_X \frac{\omega_X^{n-p-1} \wedge f^*\omega_Y^{p+1}}{(n-p-1)!(p+1)!} &= c_1\int_X \frac{\omega_X^n}{n!};\\ \int_Y \frac{f_*\omega_X^{n-p} \wedge \omega_Y^p}{(n-p)!p!} &= c_2\int_Y \frac{\omega_Y^n}{n!}.\end{aligned}$$

After that, we will study the procedure of combining the moment maps μ_p and $\mu_{\mathcal{J}}$ by a special case which we call the generalized ccscK equation:

DEFINITION 1.6. Let X be a compact Kähler manifold with Kähler forms $\omega_0, ..., \omega_k$. Then we define the generalised ccscK equation to be the following:

$$\begin{cases} \sum_{i=0}^{k} \left(\frac{\omega_{i,\varphi_i}^{p_i+1}}{(p_i+1)!} \wedge \frac{\omega_{0,\varphi_0}^{n-p_i-1}}{(n-p_i-1)} \right) - \mathrm{Ric}(\omega_{0,\varphi_0}, J_0) \wedge \frac{\omega_0^{n-1}}{(n-1)!} - c_0 \frac{\omega_{0,\varphi_0}^n}{n!} & = & 0 \\ \frac{\omega_{0,\varphi_0}^{n-p_1}}{(n-p_1)!} \wedge \frac{\omega_{1,\varphi_1}^{p_1}}{p_1!} - c_1 \frac{\omega_{1,\varphi_1}^n}{n!} & = & 0 \\ \vdots & & \\ \frac{\omega_{0,\varphi_0}^{n-p_k}}{(n-p_k)!} \wedge \frac{\omega_{k,\varphi_k}^{p_k}}{p_k!} - c_k \frac{\omega_{k,\varphi_k}^n}{n!} & = & 0. \end{cases}$$

We will show that this system of equations has a moment map setup. Moreover, there exists a underlying space which has a Kähler structure and is compatible with the action. Hence, by considering the orbit space

$$\mathcal{O}_{J,\vec{f}} := \left(\prod_{i=0}^{k} \mathrm{Ham}_{J_i}^{\mathbb{C}}(X_i, \omega_i) \right) \cdot (J, f_1, ..., f_k).$$

Then the moment map equation is given by theorem 3.11:

THEOREM 1.7. *Consider the moment map $\mu_{\vec{p}} : \mathcal{O}_{J,\vec{f}} \to Lie(H_0 \times ... \times H_k)^*$ defined by theorem 3.9 restricted on $\mathcal{O}_{J,\vec{f}}$. Then $\mu_{\vec{p}} = 0$ has a solution iff the generalized ccscK equation has a solution $(\varphi_0, \cdots, \varphi_k)$.*

In particular, if $X_0 = \cdots = X_k$, $f_1 = f_2 = \cdots = f_k = id$, $\vec{p} = (0, ..., 0)$, then this is the ccscK equation with the classes fixed. Also, using similar idea, we can get an alternate setup for the coupled Kähler Yang-Mills equation (see [ACGFGP13]) for the case $G = U(1)^k$.

1.4. Application. As a result, similar consequences in [Don00], [Wan04] (see also [PS04], [PS10], [Szé10], [LS15], [ACGFGP13]) can be applied for generalised ccscK:

1) Corollary 4.2: If the solution of $\mu_{\mathcal{J},p} = 0$ exists, then $\bigcap_{i=0}^{k} Aut(X_i, L_i)$ is reductive.
2) Corollary 4.4: Futaki invariant is given by $\langle \mu_{\mathcal{J},p}, \xi \rangle$; and if solution exists, then Futaki variant are 0.
3) Corollary 4.5: Calabi functional is defined by $||\mu_{\mathcal{J},p}||^2$; and if $||\mu||^2$ have a critical point and Futaki invariant vanished implies $\mu = 0$ has a solution.
4) Corollary 4.9: The Mabuchi functional can be defined. (See definition 4.8) This functional is geodesic convex along the geodesic $e^{\sqrt{-1}\xi} \cdot p$, where $\xi \in Lie(G)$, and the minimums (if exists) are the solutions of $\mu = 0$.
5) Corollary 4.10: If the manifold X is a toric variety, then the $(S^1)^n$ invariant solution is unique (if it exists).

REMARK 1.8. Notice that if $G = Ham(X, \omega)$, there is no complexified group $G^{\mathbb{C}}$. We can still define an orbit space, but the uniqueness of the solution still need to investigate. If the orbit is geodesically convex, i.e., any two points can be connected by the geodesic $e^{\sqrt{-1}\xi}\cdot p$, then the solution is unique. However, in general, by [Dar14], $Ham^{\mathbb{C}}(X,\mathbb{R})$ is not geodesically convex. Hence the uniqueness still need to study.

Denote $\mathcal{K}(X,\omega_i) := \{h_i \in C^\infty(X,\mathbb{R}) | \omega_{i,h_i} := \omega_i + \sqrt{-1}\partial\bar{\partial}h_i > 0\}$. The smooth gedosic we defined is given by $(h_{0,t}, h_{1,t}, \cdots, h_{k,t}) \subset \mathcal{K}(X,\omega_0)\times\cdots\mathcal{K}(X,\omega_k)$ such that for all $0 \le i \le k$,

$$\ddot{h}_{i,t} = |\nabla \dot{h}_{i,t}|^2_{\omega_i}.$$

Consider $k=1$ case. In [ACGFGP13], denote the space of metic on the line bundle L to be $\mathcal{H}(L)$, then the geodesic (Proposition 3.17 of [ACGFGP13]) is given by $(h,H) \in \mathcal{K}(X,\omega)\times\mathcal{H}(L)$.

$$\ddot{h}_{0,t} = |\nabla \dot{h}_{0,t}|^2_{\omega_0},$$

$$\ddot{H}_t - 2d\dot{H}_t(JX_{\dot{h}_{0,t}}) + \sqrt{-1}F_{H_t}(X_{\dot{h}_{0,t}}, JX_{\dot{h}_{0,t}}).$$

Notice that the second equation is twisted by the Kähler potential h, while in our note, the geodesic are independent by each other. Therefore, in this note, we show that the functional is convex along different geodesic, which is more natural in the space of $\mathcal{K}(X_{\omega_0})\times\cdots\mathcal{K}(X,\omega_k)$.

1.5. More result of μ_p. After the above applications, we will study if the couple p equation can be viewed as a moment map in a Kähler manifold. Unfortunately, there is no Kähler submanifold in the domain which is closed under the action of $\mathrm{Ham}(X,\omega_X)\times\mathrm{Ham}(Y,\omega)$, and Ω is non -degenerated . The best result is in Proposition 5.7, which implies that μ_p is a pseudo moment map in $\mathcal{X}^+_{f,p}$ (as $\Omega_p|_{\mathcal{X}^+_{f,p}}$ may be degenerated).

After that, we give a special case for the moment map μ_p is still a moment map when X is a submanifold of Y.

Finally, in the appendix, we will give a rough idea about the relation between this setup and the setup in [ACGFGP13] and [DP20].

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2. moment map for coupled equation p. In this section, we will define a class of moment map μ_p on a open subset of $\mathrm{Map}(X,Y;p)^+ \subset \mathrm{Map}(X,Y)$, with a sympectic form Ω_p on $\mathrm{Map}(X,Y;p)^+$. To do so, first, we define the domain $\mathrm{Map}(X,Y;p)^+$ and the symplectic form Ω_p on $\mathrm{Map}(X,Y;p)^+$:

DEFINITION 2.1. Let $(X,\omega_X),(Y,\omega_Y)$ be two compact symplectic manifolds which are diffeomorphic to each other. We define $(\mathrm{Map}(X,Y;p)^+,\Omega_p)$ to be the space

$$\mathrm{Map}(X,Y;p)^+ := \{f \in \mathrm{Diffeo}(X,Y) | \omega_X^{n-p} \wedge f^*\omega_Y^p > 0\}.$$

that is, $\omega_X^{n-p} \wedge f^*\omega_Y^p$ is a volume form, with the symplectic form

$$\Omega_p(\delta_1 f,\delta_2 f) := \frac{1}{(n-p)!p!}\int_X \omega_Y(\delta_1 f,\delta_2 f)\omega_X^{n-p} \wedge f^*\omega_Y^p,$$

where $\delta_1 f,\delta_2 f \in T_f(\mathrm{Map}(X,Y)) = f^*(TY) := \{s_f : X \to TY | |s_f|_x \in T_{f(x)}Y\}$.

Notice that Ω_p is a symplectic form on $\mathrm{Map}(X,Y;p)$ as ω_Y is non degenerate and closed. Also, we have a group $H := \mathrm{Ham}(X,\omega_X) \times \mathrm{Ham}(Y,\omega_Y)$ acts on $\mathrm{Map}(X,Y;p)$ defined by

$$(\sigma,\eta)\cdot f := \eta \circ f \circ \sigma^{-1},$$

where $\mathrm{Ham}(X,\omega_X)$ and $\mathrm{Ham}(Y,\omega_Y)$ are the Hamiltonian groups with respect to ω_X and ω_Y respectively. Also, $\mathrm{Map}(X,Y;p)^+$ is an open set in $\mathrm{Map}(X,Y;p)$, hence it is also a symplectic manifold.

We will first show that the Hamiltonian action on $\mathrm{Map}(X,Y;p)$ is closed in $\mathrm{Map}(X,Y;p)^+$. Then up to constants, we can define a map μ_p by

DEFINITION 2.2. We denote the map

$$\mu_p : \mathrm{Map}(X,Y;p)^+ \to \mathit{Lie}(\mathrm{Ham}(X,\omega_X)\times \mathrm{Ham}(Y,\omega_Y))^*$$

by

$$\mu_{p,\omega_X\omega_Y}(f) = \frac{n}{n-p}\left(c_1\frac{\omega_X^n}{n!} - \frac{\omega_X^{n-p-1}\wedge f^*\omega_Y^{p+1}}{(n-p-1)!(p+1)!}, \frac{f_*\omega_X^{n-p}\wedge\omega_Y^p}{(n-p)!p!} - c_2\frac{\omega_Y^n}{n!}\right), \tag{2}$$

where $c_1 = \dfrac{\int_X \omega_X^{n-p-1}\wedge f^*\omega_Y^{p+1}}{\int_X \omega_X^n}$, $c_2 = \dfrac{\int_Y f_*\omega_X^{n-p}\wedge\omega_Y^p}{\int_Y \omega_Y^n}$.

We will show that this is a moment map corresponding to $(\mathrm{Map}(X,Y;p)^+,\Omega_p)$. In particular, if we take $(X,\omega_X) = (X,\omega_0)$ and $(Y,\omega_Y) = (X,\omega_1)$, then we will get the moment map for coupled equation p.

LEMMA 2.3. *The group action H on $\mathrm{Map}(X,Y;p)$ is closed in $\mathrm{Map}(X,Y;p)^+$. Also, Ω_p is invariant under the action of H for $p = 0,\ldots,n-1$.*

Proof. Let $\varphi : X \to \mathbb{R}$ be a test function, that is $\varphi \geq 0$ is a smooth function, and there exists $x \in X$ such that $\varphi(x) > 0$

Let $f \in \mathrm{Map}(X,Y;p)^+$ and denote $u = \sigma^{-1}(x)$. Then

$$\begin{aligned}\int_X \varphi(x)\omega_X^{n-p} \wedge (\eta \circ f \circ \sigma^{-1})^*\omega_Y^p &= \int_X \varphi(x)\omega_X^{n-p} \wedge (\sigma^{-1})^* f^*\eta^*\omega_Y^p \\ &= \int_X \varphi(u)\sigma^*\omega_X^{n-p} \wedge f^*\eta^*\omega_Y^p|_u \\ &= \int_X \varphi(u)\omega_X^{n-p} \wedge f^*\omega_Y^p|_u \\ &= \int_X \varphi(x)\omega_X^{n-p} \wedge f^*\omega_Y^p > 0.\end{aligned}$$

Hence $\eta \circ f \circ \sigma^{-1} \in \mathrm{Map}(X,Y;p)^+$. Also,if we choose φ such that $\varphi(x) = 1$, by the above calculation, we can see that the volume is unchanged. Finally, notice that $(\sigma,\eta)\cdot(\delta f)|_x = 9D\eta)(\delta f)|_{\sigma^{-1}(x)}$. Hence

$$\begin{aligned}&\int_X \omega_Y(D\eta)(\delta_1 f),(D\eta)(\delta_2 f))|_{\eta\circ f\circ\sigma^{-1}(x)}\omega_X^{n-p} \wedge (\sigma^{-1})^* f^*\eta^*\omega_Y^p \\ =&\int_X \omega_Y((D\eta)(\delta_1 f),(D\eta)(\delta_2 f))|_{\eta\circ f(x)}\sigma^*\omega_X^{n-p} \wedge f^*\sigma_Y^*\omega_Y^p \\ =&\int_X \eta^*\omega_Y((\delta_1 f),(\delta_2 f))|_{f(x)}\sigma^*\omega_X^{n-p} \wedge f^*\eta^*\omega_Y^p \\ =&\int_X \omega_Y((\delta_1 f),(\delta_2 f))|_{f(x)}\omega_X^{n-p} \wedge f^*\omega_Y^p.\end{aligned}$$

□

REMARK 2.4. The proof also applies to the case $p = n$. Indeed, $\mathrm{Sym}(X,\omega_X) \times \mathrm{Sym}(Y,\omega_Y) \subset \mathrm{Sym}(\mathrm{Map}(X,Y),\Omega_p)$ for all $p = 0,...,n$.

Before we prove the first main theorem, we first prove a technical lemma.

LEMMA 2.5. *Let (X,α,β) be a symplectic manifold with symplectic forms α,β. Denote $\gamma_p := \alpha^{n-1-p}\wedge\beta^p$. If $\alpha^{n-p}\wedge\beta^p > 0$, then for any $u,v \in TX$,*

$$n\iota_u\alpha \wedge \iota_v\beta \wedge \gamma_p = -\beta(u,v)\alpha\wedge\gamma_p.$$

Similarly, we have

$$n\iota_u\alpha \wedge \iota_v\alpha \wedge \gamma_p = -\alpha(u,v)\alpha\wedge\gamma_p.$$

Proof. We prove it on local coordinate. As $\alpha\wedge\gamma_p > 0$, it is a volume form, so if we denote $\alpha = A_{ij}$, then for any 2 form $\eta = d_{ij}$,

$$\frac{n\eta\wedge\gamma_p}{\alpha\wedge\gamma_p} = d_{ij}A^{ji},$$

where A^{ji} is the inverse matric of A_{ij}. As a result, if we denote $u = u^i$, $v = v^i$, $\beta = B_{ij}$, then

$$\frac{n\iota_u\alpha\wedge\iota_v\beta\wedge\gamma_p}{\alpha\wedge\gamma_p} = u^iA_{ij}v^kB_{kl}A^{lj} = u^lv^kB_{kl} = -\beta(u,v).$$

the second statement follows from the same proof. □

Notice that We now give the first theorem of this note.

THEOREM 2.6. *Let $0 \leq p \leq n-1$, then the map $\mu : \mathrm{Map}(X, Y; p)^+ \to$ $Lie(\mathrm{Ham}(X, \omega_X) \times \mathrm{Ham}(Y, \omega_Y))^*$ defined by*

$$\mu_p(f) := \frac{n}{n-p}\left(c_1 \frac{\omega_X^n}{n!} - \frac{\omega_X^{n-1-p}}{(n-1-p)!} \wedge f^* \frac{\omega_Y^{p+1}}{(p+1)!}, c_2 \frac{\omega_Y^n}{n!} - \frac{f_* \omega_X^{n-p}}{(n-p)!} \wedge \frac{\omega_Y^p}{p!}\right)$$

is a moment map with respect to the action $\eta \circ f \circ \sigma^{-1}$, where

$$c_1 := c_1(f) = \frac{n!}{(n-1-p)!(p+1)!} \frac{\int_X \omega_X^{n-1-p} \wedge f^* \omega_Y^{p+1}}{\int_X \omega_1^n},$$

$$c_2 := c_2(f) = \frac{n!}{(n-p)!p!} \frac{\int_Y f_* \omega_X^{n-p} \wedge \omega_Y^p}{\int_Y \omega_Y^n}.$$

Proof. Recall that $Lie(\mathrm{Ham}(X, \omega_X)) \cong C_0^\infty(X, \mathbb{R}) := \{\varphi | \int_X \varphi \omega_X^n = 0\}$ such that for any $\varphi \in C^\infty(X, \mathbb{R})$,

$$d\varphi = \iota_{\xi_\varphi} \omega_X.$$

Hence we have $Lie(\mathrm{Ham}(X, \omega_X))^* \cong \Omega^n(X, \mathbb{R})$, the space of volume form of X.

We let $(\varphi, \psi) \in C^\infty(X) \times C^\infty(Y)$ and $H_{(\varphi,\psi)}(f) := \langle \mu_p(f), (\varphi, \psi) \rangle$. Then

$$H_{(\varphi,\psi)}(f) = \frac{n}{n-p}\left(-\int_X \varphi \frac{\omega_X^{n-1-p}}{(n-1-p)!} \wedge f^* \frac{\omega_Y^{p+1}}{(p+1)!} + c_1 \int_X \varphi \frac{\omega_X^n}{n!}\right.$$
$$\left. - \int_Y \psi \frac{f_* \omega_X^{n-p}}{(n-p)!} \wedge \frac{\omega_Y^p}{p!} + c_2 \int_Y \psi \frac{\omega_Y^n}{n!}\right).$$

Our goal is to show that

$$\iota_{X_{(\varphi,\psi)}} \Omega_p(v) = dH_{(\varphi,\psi)}(v).$$

Let f_t be a family of diffeomorphism. By defining $\eta_t = f_t \circ f^{-1}$, then $f_t = \eta_t \circ f$. we denote

$$v = \left.\frac{d}{dt}\right|_{t=0} \eta_t \in T_{id}(\mathrm{Map}(Y, Y; p)^+).$$

Notice that $f^* \omega_Y$ is a symplectic form which is closed, therefore,

$$\left.\frac{d}{dt}\right|_{t=0} f_t^* \omega_Y = \left.\frac{d}{dt}\right|_{t=0} f^* \eta_t^* \omega_Y = f^* \mathcal{L}_v \omega_Y = f^* d\iota_v \omega_Y = df^* \iota_v \omega_Y.$$

Similarly,

$$\left.\frac{d}{dt}\right|_{t=0} f_{t*} \omega_X = \left.\frac{d}{dt}\right|_{t=0} \eta_{t*} f_* \omega_X = d\iota_v f_* \omega_X.$$

Notice that these are exact forms. By the fact that for any compact manifold M^{2n}, and for any $2k-1$ form β and $2n-2k$ closed form α,

$$\int_M \alpha \wedge d\beta = -\int_M (d\alpha) \wedge \beta = 0.$$

It implies

$$\frac{d}{dt}\int_X \omega_X^{n-p} \wedge f_t^*\omega_Y^p = \frac{d}{dt}\int_Y f_{t*}\omega_X^{n-p} \wedge \omega_Y^p = 0.$$

We now identify $v' \in T\mathrm{Map}(X,Y)$ and $v \in T\mathrm{Map}(Y,Y)$ by

$$v|_{f(x)} = v'|_x.$$

Then

$$\begin{aligned}
&dH_{(\varphi,\psi)}(v') \\
=& \frac{-n}{(n-p)!(p)!}\int_X \varphi\omega_X^{n-1-p} \wedge f^*\omega_Y^p \wedge \left(\frac{d}{dt}f_t^*\omega_Y(\cdot,\cdot)\right) \\
&- \frac{n}{(n-p)!p!}\int_Y \psi\left(\frac{d}{dt}(f_t^{-1})^*\omega_X\right) \wedge f_*\omega_X^{n-p-1} \wedge \omega_Y^p \\
=& \frac{-n}{(n-p)!(p)!}\int_X \varphi\omega_X^{n-1-p} \wedge f^*\omega_Y^p \wedge df^*\iota_v\omega_Y \\
&+ \frac{n}{(n-p)!p!}\int_Y \psi d\iota_v f_*\omega_X \wedge f_*\omega_X^{n-p-1} \wedge \omega_Y^p \\
=& \frac{n}{(n-p)!(p)!}\int_X d\varphi \wedge f^*\iota_v\omega_Y \wedge \omega_X^{n-1-p} \wedge f^*\omega_Y^p \\
&- \frac{n}{(n-p)!p!}\int_Y d\psi \wedge \iota_v f_*\omega_X \wedge f_*\omega_X^{n-p-1} \wedge \omega_Y^p \\
=& \frac{n}{(n-p)!(p)!}\int_X \iota_{\xi_\varphi}\omega_X \wedge f^*\iota_v\omega_Y \wedge \alpha_f - \frac{n}{(n-p)!p!}\int_Y \iota_{\xi_\psi}\omega_Y \wedge \iota_v f_*\omega_X \wedge f_*\alpha_f,
\end{aligned}$$

where $\alpha_f = \omega_X^{n-p-1} \wedge f^*\omega_Y^p$. By Lemma 2.5, as $f_*\alpha_f = f_*\omega_X^{n-p-1} \wedge \omega_Y^p$,

$$\begin{aligned}
n\iota_{\xi_\psi}\omega_Y \wedge \iota_v f_*\omega_X \wedge \alpha_f &= -n\iota_v f_*\omega_X \wedge \iota_{\xi_\psi}\omega_Y \wedge \alpha_f \\
&= \omega_Y(v,\xi_\psi) f_*\omega_X \wedge \alpha_f \\
&= -\omega_Y(\xi_\psi, v) f_*\omega_X \wedge \alpha_f.
\end{aligned}$$

Moreover,

$$\begin{aligned}
n\iota_{\xi_\varphi}\omega_X \wedge f^*\iota_v\omega_Y \wedge \alpha_f|_x =& nf^*\left(\iota_{f_*\xi_\varphi} f_*\omega_X \wedge \iota_v\omega_Y \wedge f_*\alpha_f|_{f(x)}\right) \\
=& -\omega_Y(f_*\xi_\varphi|_x, v|_{f(x)}) f^*(f_*\omega_X \wedge f_*\alpha_f|_{f(x)}) \\
=& -\omega_Y(f_*\xi_\varphi, v')\omega_X \wedge \alpha_f.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&dH_{(\varphi,\psi)}(v') \\
=& \frac{1}{(n-p)!p!}\left(-\int_X \omega_Y(f_*\xi_\varphi, v')\omega_X \wedge \alpha_f + \int_Y \omega_Y(\xi_\psi, v) f_*\omega_X \wedge f_*\alpha_f\right) \\
=& \frac{1}{(n-p)!p!}\left(-\int_X \omega_Y(f_*\xi_\varphi, v')\omega_X \wedge \alpha_f + \int_X \omega_Y(\xi_\psi \circ f, v')\omega_X \wedge \alpha_f\right) \\
=& \frac{1}{(n-p)!p!}\left(\int_X \omega_Y(\xi_\psi \circ f, v')\omega_X \wedge \alpha_f - \int_X \omega_Y(f_*\xi_\varphi, v')\omega_X \wedge \alpha_f\right).
\end{aligned}$$

On the other hand, for the action $\eta \circ f \circ \sigma^{-1}$ with $(\varphi, \psi) \in C^\infty(X) \times C^\infty(Y)$, the induced vector field is given by

$$X_{(\varphi,\psi)} = \xi_\psi \circ f - f_* \cdot \xi_\varphi.$$

So

$$\iota_{X_{(\varphi,\psi)}} \Omega_p(v) = \frac{1}{(n-p)!p!} \int_X \omega_Y(\xi_\psi \circ f - f_* \cdot \xi_\varphi, v')\omega_X \wedge \alpha_f = dH_{(\varphi,\psi)}(v).$$

□

REMARK 2.7. Notice that $\mathrm{Map}(X, Y; p)^+$ is an open set in $\mathrm{Map}(X, Y; p)$, hence it is still a symplectic submanifold. Also, c_1, c_2 may not be constant, as $\mathrm{Diffeo}(X, Y; p)^+$ may not be connected. However, if f_1, f_2 are path connected, then $c_1(f_1) = c_1(f_2)$ and $c_2(f_1) = c_2(f_2)$.

DEFINITION 2.8. We call the above moment map to be the moment map p with respect to ω_X, ω_Y, denoted as $\mu_{p;\omega_X,\omega_Y}$, or simply μ_p if no confusion arises.

Finally, we define the "dual" moment map by the following.

DEFINITION 2.9. We define the dual moment map of μ_p to be $\mu_p^* : \mathrm{Map}(Y, X; n-p-1)^+ \to \mathrm{Ham}(Y, \omega_Y) \times \mathrm{Ham}(X, \omega_X)$, with

$$\mu_p^*(g) := \mu_{n-p-1,\omega_Y,\omega_X}(g) = \frac{n}{p+1}\left(c_1 \frac{\omega_Y^n}{n!} - \frac{\omega_Y^p \wedge g^* \omega_X^{n-p}}{p!(n-p)!}, c_2 \frac{\omega_X^n}{n!} - \frac{g_* \omega_Y^{p+1} \wedge \omega_X^{n-p-1}}{(p+1)!(n-p-1)!}\right).$$

Notice that $(\mu_p^*)^* = \mu_p$. Also, it is obvious that

LEMMA 2.10. *$f : X \to Y$ solves the coupled equation p (i.e., $\mu_p(f) = 0$) iff f^{-1} solve*

$$\mu_p^*(g) = 0.$$

Proof. $f^* = f_*^{-1}$ and $f_* = (f^{-1})^*$; and the result follows. □

The main difference between $\mu_p(f)$ and $\mu_p^*(f^{-1})$ is the following: if we put $g = f^{-1}$, and we reorder the domain into $Lie(\mathrm{Ham}(X, \omega_X) \times \mathrm{Ham}(Y, \omega_Y))^*$,

$$\mu_p^*(f^{-1}) = \frac{p+1}{n-p}\mu_p(f).$$

Hence, we can change the sign of the moment map without changing the action on $\mathrm{Map}(X, Y)$.

Also, we can change the sign by changing the action. For example, we may change the action to be $\eta^{-1} \circ f \circ \sigma^{-1}$, then the sign of the second part of the moment map will change.

3. moment map picture for coupled equations with curvature.

3.1. Combining moment maps. We now use this moment map to get some coupled equations related to Ricci curvature. Recall that we have the following fact:

LEMMA 3.1. *Let (M_1, α_1), (M_2, α_2) be two symplectic manifolds with hamiltonian group action G_1, G_2, and let their corresponding moment map be $\mu_i : M_i \to Lie(G_i)^*$. Let H be a subgroup of $G_1 \times G_2$ and M be a (even dimensional) submanifold of $M_1 \times M_2$ such that H is closed under M and $(\Omega_1 + \Omega_2)|_M$ is non degenerate (i.e, it is a symplectic form). Then the map $\mu : M \to Lie(H)^*$ corresponding to the symplectic form $\Omega_1 + \Omega_2$ defined by*

$$\mu_H = \mathrm{Proj}_{Lie(H)^*}(\mu_1, \mu_2)|_M$$

is a moment map, where $\mathrm{Proj}_{Lie(H)^*} : Lie(G_1)^* \times Lie(G_2)^* \to Lie(H)^*$ is the projection map.

Using this lemma, we can combine the moment map we defined above and the scalar curvature to get different coupled equations.

REMARK 3.2. If Ω is a Kähler form, and M is a complex submanifold, then $\Omega|_M$ is also a Kähler form. However, in general, for a submanifold M, $\Omega|_M$ may be degenerate. For example, we may take $M \subset L$, where L is the Lagrangian of $M_1 \times M_2$.

Let $Y = X$ with symplectic forms ω_i. Let $\mathcal{Z}_i := \mathrm{Map}((X, \omega_0), (X, \omega_i); p)^+$,

$$\Omega_{p,i}((\delta f)_1, (\delta f)_2) := \frac{1}{(n-p)!p!} \int_X \omega_i((\delta f)_1, (\delta f)_2)\omega_0^{n-p} \wedge f^*\omega_i^p.$$

Denote $\mathrm{Ham}(X, \omega_i) := H_i$, and we denote the corresponding moment map to be $\mu_{p,i}$, in which

$$\mu_{p,i}(f) := \frac{n}{n-p}\left(c_{1,i}\frac{\omega_0^n}{n!} - \frac{\omega_0^{n-1-p}}{(n-1-p)!} \wedge f^*\frac{\omega_i^{p+1}}{(p+1)!}, \frac{f_*\omega_0^{n-p}}{(n-p)!} \wedge \frac{\omega_i^p}{p!} - c_{2,i}\frac{\omega_i^n}{n!}\right),$$

by theorem 2.6. Then by considering the space $\mathcal{Z}_1 \times \mathcal{Z}_3 \times \cdots \mathcal{Z}_k$ with

$$\Omega = \sum_{i=1}^k \frac{n-p}{n}\pi_{i-1}^*\Omega_{p,i},$$

we have a moment map $\overline{\mu_p} : \mathcal{Z}_1 \times ...\mathcal{Z}_k \to \prod_{i=1}^k (H_0 \times H_i)$ defined by

$$\overline{\mu_p} = \frac{n-p}{n}(\mu_{p,1}, ..., \mu_{p,k}).$$

The next step is finding the suitable subgroup so that the image of moment map can be combined. To be precise, the embedding $\iota : H_1 \times ...H_k \to (H_0 \times H_1) \times \cdots \times (H_0 \times H_k)$ defined by

$$\iota(\sigma_0, ..., \sigma_k) = (\sigma_0, \sigma_1, \sigma_0, \sigma_2, ..., \sigma_0, \sigma_k)$$

induces a map $\iota : Lie(\prod_{i=1}^{k}(H_0 \times H_i))^* \to Lie(H_0 \times H_2 \times ... \times H_k)^*$. Hence the moment map $\mu_p := \iota^* \circ \overline{\mu_p|_{\mathcal{Z}}}$ is given by

$$\mu_p(f_1, ..., f_k) = \begin{cases} c_0 \dfrac{\omega_0^n}{n!} - \dfrac{f_1^*\omega_1^{p+1} + ... + f_k^*\omega_k^{p+1}}{(p+1)!} \wedge \dfrac{\omega_0^{n-p-1}}{(n-p-1)!} \\ \dfrac{f_{1*}\omega_0^{n-p}}{(n-p)!} \wedge \dfrac{\omega_1^p}{p!} - c_1 \dfrac{\omega_1^n}{n!} \\ \vdots \\ \dfrac{f_{k*}\omega_0^{n-p}}{(n-p)!} \wedge \dfrac{\omega_k^p}{p!} - c_k \dfrac{\omega_k^n}{n!}. \end{cases}$$

In general, for different i, we can choose different $0 \leq p_i \leq n-1$, hence we have the following:

LEMMA 3.3. *Let (X_i, ω_i, J_i) be Kähler manifolds, and X_0 is diffeomorphic to X_i for all $i = 0, 1, ..., k$. Denote $\vec{p} = (p_1, ..., p_k)$. Consider the space*

$$\mathcal{Z}_{\vec{p}} := \prod_{i=1}^{k} \mathrm{Map}(X_0, X_i)^+_{p_i},$$

where

$$\mathrm{Map}(X_0, X_i)^+_p := \{f \in \mathrm{Map}(X_0, X_i) | \omega_0^{n-p_i} \wedge f^*\omega_i^{p_i} > 0\}.$$

We define the symplectic form on $\mathcal{Z}(p_1, ..., p_k)$ by

$$\Omega_{\vec{p}}((v_1, ..., v_k), (w_1, ..., w_k)) := \sum_i \frac{n - p_i}{n} \int_X \omega_i(v_i, w_i)\omega_0^{n-p_i} \wedge \omega_i^{p_i}.$$

Then with the action of $\prod_{i=0}^{k} \mathrm{Ham}(X_i, \omega_i)$, the moment map is given by

$$\mu_{\vec{p}}(\vec{f}) := \begin{cases} c_0 \dfrac{\omega_0^n}{n!} - \displaystyle\sum_{i=1}^{k} \dfrac{\omega_0^{n-p_i-1}}{(n-p_i-1)!} \wedge \dfrac{f_i^*\omega_i^{p_i+1}}{(p_i+1)!} \\ \dfrac{f_{1*}\omega_0^{n-p_1}}{(n-p_1)!} \wedge \dfrac{\omega_1^{p_1}}{p_1!} - c_1 \dfrac{\omega_1^n}{n!} \\ \vdots \\ \dfrac{f_{k*}\omega_0^{n-p_k}}{(n-p_k)!} \wedge \dfrac{\omega_k^{p_k}}{p_k!} - c_k \dfrac{\omega_k^n}{n!} \end{cases}.$$

Also, by identifying $\mathcal{Z}_i$ and $\mathcal{Z}_i^*$, and considering

$$\Omega^* = \sum_{i=1}^{k} \frac{p+1}{n} \pi_{i-1}^*|\Omega_i,$$

we have

$$\mu^*_{\vec{p}}(\vec{f}) := \begin{cases} \sum_{i=1}^k \frac{\omega_0^{n-p_i-1}}{(n-p_i-1)!} \wedge \frac{f_i^*\omega_i^{p_i+1}}{(p_i+1)!} - c_0\frac{\omega_0^n}{n!} \\ c_1\frac{\omega_1^n}{n!} - \frac{f_{1*}\omega_0^{n-p_1}}{(n-p_1)!} \wedge \frac{\omega_1^{p_1}}{p_1!} \\ \vdots \\ c_k\frac{\omega_k^n}{n!} - \frac{f_{k*}\omega_0^{n-p_k}}{(n-p_k)!} \wedge \frac{\omega_k^{p_k}}{p_k!} \end{cases}.$$

Let $(\mathcal{J}(X,\omega_X),\Omega_J)$ to be the space of all integrable almost complex structure which are compatible to ω_X, and for all $A,B\in T_J\mathcal{J}(X,\omega_X)$,

$$\Omega_J(A,B) = \frac{1}{n!}\int_X \langle A,B\rangle_{g_J}\omega_X^n,$$

where $g_J(v,w)=\omega(v,Jw)$. Also, let the action $\mathrm{Ham}(X,\omega_X)$ acts on $\mathcal{J}(X,\omega_X)$ by

$$\sigma\cdot J = D\sigma^{-1}\cdot J\cdot D\sigma,$$

and denote

$$\wedge_0^n(X) := \{\alpha\in\wedge^n(X)|\int_X\alpha=0\}.$$

Then we have a moment map ([Don00],[Don01])

$$\mu_J : \mathcal{J}(X,\omega_X)\to Lie(\mathrm{Ham}(X,\omega_X))^*\cong\wedge_0^n(X)$$

which is given by

$$\mu_J(\sigma_\varphi) = \mathrm{Ric}(\omega_\varphi)\wedge\frac{\omega_\varphi^{n-1}}{(n-1)!} - \bar{S}\frac{\omega^n}{n!} = \left(S_\varphi-\overline{S}\right)\omega_{X,\varphi}^n,$$

where $\sigma_\varphi^*\omega_X=\omega_{X,\varphi}$. To sum up, we have the following lemma.

LEMMA 3.4. *Let $\mathcal{Z}_{\vec{p}}$ be as above and consider $\mathcal{J}(X,\omega_X)\times\mathcal{Z}_{\vec{p}},\Omega_{J,\vec{p}} := \pi_{\mathcal{J}}^*\Omega_J + \pi_{\mathcal{Z}}^*\Omega_{\vec{p}}$, then we have a moment map*

$$\widehat{\mu}_{\mathcal{J},\vec{p}} : \mathcal{J}(X,\omega_X)\times\mathcal{Z}_{\vec{p}}\to Lie(H_1\times H_1\times ...\times H_k)^*$$

defined by

$$\widehat{\mu}_{\mathcal{J},\vec{p}}(J,f_1,...,f_k) = (\mu_J,\mu_{\mathcal{Z}}^*).$$

Moreover, by considering the group action $\iota : H_0\times ...\times H_k\to H_0\times H_0\times H_1\times ...\times H_k$ by

$$\iota(\sigma_0,\sigma_1,...,\sigma_k) = (\sigma_0^{-1},\sigma_0,\sigma_1,...,\sigma_k)$$

we can restrict the moment map to be

$$\mu_{\mathcal{J},\vec{p}} : \mathcal{J}(X,\omega_X)\times\mathcal{Z}_{\vec{p}}\to Lie(H_0\times ...\times H_k)^*$$

which is given by

$$\mu_{\mathcal{J},\vec{p}}(J,f_1,...,f_k) = \begin{pmatrix} c_0\dfrac{\omega_0^n}{n!} - \displaystyle\sum_{i=0}^{k}\left(\dfrac{f_i^*\omega_i^{p_i+1}}{(p_i+1)!}\wedge\dfrac{\omega_0^{n-p_i-1}}{(n-p_i-1)!}\right) + \mathrm{Ric}(\omega_0,J)\wedge\dfrac{\omega_0^{n-1}}{(n-1)!} \\ c_1\dfrac{\omega_1^n}{n!} - \dfrac{(f_1)_*\omega_0^{n-p_1}}{(n-p_1)!}\wedge\dfrac{\omega_1^{p_1}}{p_1!} \\ \vdots \\ c_k\dfrac{\omega_k^n}{n!} - \dfrac{(f_k)_*\omega_0^{n-p_k}}{(n-p_k)!}\wedge\dfrac{\omega_k^{p_k}}{p_k!} \end{pmatrix}.$$

REMARK 3.5. We can consider the action on $\mathcal{Z}_i$ to be $(\sigma,\eta)\cdot f_i = (\eta^{-1}\circ f_i\circ\sigma^{-1})$, then we can change the sign of all the expression $c_i\omega_0^n - f_{i*}\omega_i^n$.

Notice that it is not the equation we aim to obtain yet. In the next section, we will define a suitable submanifold as the domain of the moment map, and discuss how to transform this moment map equation into the moment map equation we want.

3.2. Kähler structure on generalized ccscK. We now define the domain of the generalized ccscK $\mathcal{Y}_{\vec{p}}$, which hope to be the largest Kähler manifold which is closed in the group action, and

$$\mathcal{Y}_{\vec{p}} \subset \{(\omega_0,\cdots\omega_k)\in\Omega^2(X_0,\mathbb{R})\times\cdots\times\Omega^2(X_k,\mathbb{R})|\omega_i \text{ is Kähler}\}.$$

This space is important as it is useful to study the deformation of solutions. Also, with this Kähler manifold, any complex orbit is a Kähler manifold.

DEFINITION 3.6. Denote $J^f := DfJDf^{-1}$. Define $\mathcal{Y}_{\vec{p}}\subset\mathcal{J}(X_0,\omega_0)\times\mathcal{Z}_{\vec{p}}$ by

$$\mathcal{Y}_{\vec{p}} := \{(J,f_1,..,f_k)|J^{f_i}\in\mathcal{J}(X_i,\omega_i).\}$$

Our goal is to show that $\mathcal{Y}_{\vec{p}}$ is Kähler with respect to the symplectic form

$$\Omega_{\mathcal{J},p} := \Omega_J + \Omega_{\vec{p}}.$$

As a remark, in [DS02], the defintion of complex manifold is really the classical one; locally homeomorphic to the tangent space, and the change of coordinate maps is biholomorphic. Or in this case, the change of coordinate maps perserve the J.

Notice that we have a natural almost complex structure on $\mathcal{J}(X_0,\omega_0)\times\prod_{i=1}^{k}\mathrm{Map}(X_0,X_i)$, denote by $\hat{J}$, which

$$\hat{J}(\delta J,\delta f_1,...,\delta f_k) = (J\delta J, J^{f_1}\delta f_1,...,J^{f_k}\delta f_k).$$

On the other hand, let (X_i,ω_i) be Kähler manifolds diffeomorphic to each other.

DEFINITION 3.7. Let X be a compact smooth manifold. Then we define $\mathcal{J}(X)$ is the space of all almost complex structure, and $\mathcal{J}_{int}(X)$ be the space of all integrable almost complex structure. Moreover, suppose (X,ω) be a Kähler manifold. Then we denote

$$\mathcal{J}(X,\omega) := \{J\in\mathcal{J}_{int}(X)|\omega(J\bullet,J\bullet)=\omega(\bullet,\bullet),\omega(J\bullet,\bullet)>0\}.$$

There is a natural almost complex structure in $\mathcal{J}(X_0, \omega_0) \times \prod_{i=1}^k \mathcal{J}(X_i))$, where $\widetilde{J} \in End(TX) \times End(TX_1) \times \cdots \times End(TX_k)$ which is defined by

$$\hat{J}|_{(J_0,\cdots,J_k)}(A_0, \cdots, A_k) := (J_0 A_0, \cdots, J_k A_k).$$

By [DS02], $\widetilde{J}$ is indeed integrable, and it is a Kähler manifold. Moreover, the map

$$F : \mathcal{J}(X_0, \omega_0) \times \prod_{i=1}^k \mathrm{Map}(X_0, X_i)) \to \mathcal{J}(X_0, \omega_0) \times \prod_{i=1}^k \mathcal{J}(X_i)$$

defined by

$$F(J, f_1, ..., f_k) = (J, J^{f_1}, ..., J^{f_k}) = (J, Df_1 J Df_1^{-1}, ..., Df_k J Df_k^{-1})$$

is a smooth map satisfying

$$\widetilde{J}(DF(A, \vec{v})|_{(J,f_1,...,f_k)}) = (JA, J^{f_1} v_1, ..., J^{f_k} v_k) = (DF)(\hat{J}(A, \vec{v})|_{(J,f_1,...,f_k)}).$$

Hence $\mathcal{J}(X_0, \omega_0) \times \prod_{i=1}^k \mathrm{Map}(X_0, X_i))$ can be considered as a $\widetilde{J}$ closed submanifold of $\mathcal{J}(X_0, \omega_0) \times \displaystyle\prod_{i=1}^k \mathrm{Map}(X_0, X_i))$, and

$$\mathcal{Y}_{\vec{p}} = F^{-1}(\mathcal{J}(X_0, \omega_0) \times \prod_{i=1}^k \mathcal{J}(X_i, \omega_i)).$$

Therefore, $\hat{J}$ is integrable as $\hat{J} = F^* \widetilde{J}$, and hence F is biholomorphic. By theorem 4 of [DS02], $(\mathcal{J}(X_0, \omega_0) \times \prod_{i=1}^k \mathcal{J}(X_i, \omega_i))$ is a complex manifold, hence we have:

Lemma 3.8. *$\mathcal{Y}_{\vec{p}}$ is a complex manifold with integrable almost complex structure $\hat{J}$.*

As a consequence, we have the following result:

Theorem 3.9. *$(\mathcal{Y}_{\vec{p}}, \Omega_{J,\vec{p}} := \Omega_J + \Omega_{\vec{p}}, \hat{J})$ is a Kähler manifold which is closed under the action $\displaystyle\prod_{i=0}^k \mathrm{Ham}(X_i, \omega_i)$, in which*

$$\begin{aligned}(\sigma_0, ..., \sigma_i) \cdot (J, f_1, ..., f_k) =&(\sigma_0^{-1} \cdot J, \sigma_1 \circ f_1 \circ \sigma_0^{-1}, \cdots, \sigma_k \circ f_k \circ \sigma_0^{-1}) \\ =&(D\sigma_0 J D\sigma_0^{-1}, \sigma_1 \circ f_1 \circ \sigma_0^{-1}, \cdots, \sigma_k \circ f_k \circ \sigma_0^{-1}).\end{aligned}$$

Therefore, the moment map defined in Lemma 3.4 can be restricted in $\mathcal{Y}_{\vec{p}}$.

We denote this moment map as $\mu_{\mathcal{J},\vec{p}}$.

Proof. Let $(A, \varphi_1, ..., \varphi_k), (B, \psi_1, ..., \psi_k) \in T_{(J,f_1,...,f_k)} \mathcal{Y}_{\vec{p}}$. Then

$$\begin{aligned}&\Omega_{J,\vec{p}}((A, \varphi_1, ..., \varphi_k), (B, \psi_1, ..., \psi_k)) \\ :=& \langle A, B\rangle_{\omega_0} + \sum_{i=1}^k \frac{(n-p_i)}{(n-p_i)!p_i!} \int_{X_i} \omega_i(\varphi_i, \psi_i) \omega_0^{n-p_i} \wedge f_i^* \omega_i^{p_i}.\end{aligned}$$

Then

$$\begin{aligned}
&\Omega_{J,\vec{p}}(\hat{J}(A,\varphi_1,...,\varphi_k),\hat{J}(B,\psi_1,...,\psi_k))\\
&=\langle JA,JB\rangle_{\omega_0}+\sum_{i=1}^{k}\frac{(n-p_i)}{(n-p_i)!p_i!}\int_{X_i}\omega_i(J^{f_i}\varphi_i,J^{f_i}\psi_i)\omega_0^{n-p_i}\wedge f_i^*\omega_i^{p_i}\\
&=\langle A,B\rangle_{\omega_0}+\sum_{i=1}^{k}\frac{(n-p_i)}{(n-p_i)!p_i!}\int_{X_i}\omega_i(\varphi_i,\psi_i)\omega_0^{n-p_i}\wedge f_i^*\omega_i^{p_i}\\
&=\Omega_{J,\vec{p}}((A,\varphi_1,...,\varphi_k),(B,\psi_1,...,\psi_k))
\end{aligned}$$

as $J^{f_i}\in\mathcal{J}(X_i,\omega_i)$. Hence $\Omega_{J,\vec{p}}$ is J invariant, which implies it is a Kähler form.

For the action part, first,

$$\begin{aligned}
&(\sigma_0,...,\sigma_i)\cdot(J,J^{f_1},\cdots,J^{f_k})\\
&=(D\sigma_0JD\sigma_0^{-1},D\sigma_1Df_1D\sigma_0^{-1}D\sigma_0JD\sigma_0^{-1}D\sigma_0Df_1^{-1}D\sigma_1^{-1},\cdots,\\
&\quad D\sigma_kDf_kD\sigma_0^{-1}D\sigma_0JD\sigma_0^{-1}D\sigma_0Df_k^{-1}D\sigma_k^{-1})\\
&=(D\sigma_0JD\sigma_0^{-1},D\sigma_1Df_1JDf_1^{-1}D\sigma_1^{-1},\cdots,D\sigma_kDf_kJDf_k^{-1}D\sigma_k^{-1}).
\end{aligned}$$

As $J^{f_i}\in\mathcal{J}(X_i,\omega_i)$, $\omega_i(J^{f_i}\bullet,J^{f_i}\bullet)=\omega_i(\bullet,\bullet)$,

$$\begin{aligned}
\omega_i((D\sigma_iJ^{f_i}D\sigma_i^{-1})\bullet,(D\sigma_iJ^{f_i}D\sigma_i^{-1})\bullet)=&\sigma_i^*\omega_i(J^{f_i}D\sigma_i^{-1}\bullet,J^{f_i}D\sigma_i^{-1}\bullet)\\
=&\omega_i(J^{f_i}D\sigma_i^{-1}\bullet,J^{f_i}D\sigma_i^{-1}\bullet)\\
=&\omega_i(D\sigma_i^{-1}\bullet,D\sigma_i^{-1}\bullet)\\
=&\sigma_i^*\omega_i(D\sigma_i^{-1}\bullet,D\sigma_i^{-1}\bullet)\\
=&\omega_i(\bullet,\bullet).
\end{aligned}$$

Hence $(\sigma_0,...,\sigma_i)\cdot(J,J^{f_1},\cdots,J^{f_k})\in\mathcal{Y}_{\vec{p}}$. □

Recall that $\mathrm{Ham}_J^{\mathbb{C}}(X,\omega_X)$ is given as

$$\mathrm{Ham}_J^{\mathbb{C}}(X,\omega_X)=\{\sigma\in\mathrm{Map}(X,X)|\varphi^*\omega_X=\omega_X+\sqrt{-1}\partial_J\bar{\partial}_Jh_\sigma\}$$

for some Kähler potential h. Notice that the $\prod_{i=0}^{k}\mathrm{Ham}(X_i,\omega_i)$ action is closed in $\mathcal{Y}_{\vec{p}}$ and $\mathcal{Y}_{\vec{p}}$ is a complex manifold implies that the orbit space is given by

$$\mathcal{O}_{J,\vec{f}}:=\left(\prod_{i=0}^{k}\mathrm{Ham}_{J_i}^{\mathbb{C}}(X_i,\omega_i)\right)\cdot(J,f_1,...,f_k)$$

is in $\mathcal{Y}_{\vec{p}}$. Moreover,

$$\begin{aligned}
&F((\sigma_0,\cdots,\sigma_k)\cdot(J,f_1,\cdots,f_k))\\
&=F(D\sigma_0JD\sigma_0^{-1},\sigma_1\circ f_1\circ\sigma_0^{-1},\cdots,\sigma_k\circ f_k\circ\sigma_k^{-1})\\
&=(D\sigma_0JD\sigma_0^{-1},D(\sigma_1\circ f_1\circ\sigma_0^{-1})(D\sigma_0JD\sigma_0^{-1})D(\sigma_1\circ f_1\circ\sigma_0^{-1})^{-1},\cdots,\\
&\quad D(\sigma_k\circ f_k\circ\sigma_0^{-1})(D\sigma_0JD\sigma_0^{-1})D(\sigma_k\circ f_k\circ\sigma_0^{-1})^{-1})\\
&=(D\sigma_0JD\sigma_0^{-1},D(\sigma_1\circ f_1)JD(\sigma_1\circ f_1)^{-1},\cdots,D(\sigma_k\circ f_k)JD(\sigma_k\circ f_k)^{-1})\\
&=(\varphi_0,\cdots,\varphi_k)\cdot F(J,f_1,\cdots,f_k).
\end{aligned}$$

Therefore, we have

$$\mathcal{O}_{J,\vec{f}} = F^{-1}(\prod_{i=0}^{k}\Big(\mathrm{Ham}^{\mathbb{C}}_{J_i}(X,\omega_i)\Big),$$

hence $\mathcal{O}_{J,\vec{f}}$ is also a Kähler submanifold of $\mathcal{Y}_{\vec{p}}$.

REMARK 3.10. Notice that although $(\varphi_0,\cdots,\varphi_k)\cdot(J,f_1,...,f_k)$ is well defined, $\mathrm{Ham}^{\mathbb{C}}_J(X,\omega_X)$ is not a group. As a remark, we can consider the orbit space as a subset of the action coming from $\prod_{i=0}^{k}\mathrm{Diffeo}(X_i)$.

THEOREM 3.11. *Consider the moment map $\mu_{\mathcal{J},\vec{p}}:\mathcal{O}_{J,\vec{f}}\to Lie(H_0\times...\times H_k)^*$ defined by Theorem 3.9 restricted on $\mathcal{O}_{J,\vec{f}}$. Then $\mu_{\vec{p}}=0$ iff*

$$\begin{cases}\sum_{i=1}^{k}\left(\frac{f_i^*\omega_{i,\varphi_i}^{p_i+1}}{(p_i+1)!}\wedge\frac{\omega_{0,\varphi_0}^{n-p_i-1}}{(n-p_i-1)!}\right)-\mathrm{Ric}(\omega_{0,\varphi_0},J_0)\wedge\frac{\omega_{0,\varphi_0}^{n-1}}{(n-1)!}-c_0\frac{\omega_{0,\varphi_0}^{n}}{n!} & = & 0\\ \frac{\omega_{0,\varphi_0}^{n-p_1}}{(n-p_1)!}\wedge\frac{f_1^*\omega_{1,\varphi_1}^{p_1}}{p_1!}-c_1\frac{f_1^*\omega_{1,\varphi_1}^{n}}{n!} & = & 0\\ \vdots & & \\ \frac{\omega_{0,\varphi_0}^{n-p_k}}{(n-p_k)!}\wedge\frac{f_k^*\omega_{k,\varphi_k}^{p_k}}{p_k!}-c_k\frac{f_k^*\omega_{k,\varphi_k}^{n}}{n!} & = & 0.\end{cases}$$

In particular, if $X_0=\cdots=X_k$, $f_1=f_2=\cdots=f_k=id$, $\vec{p}=(0,...,0)$, then this is the ccscK equation with the classes fixed.

Proof.

$$\begin{cases}\sum_{i=0}^{k}\frac{(\varphi_0^{-1})^*f_i^*\varphi_i^*\omega_i^{p_i+1}\wedge\omega_0^{n-p_i-1}}{(n-p_i-1)!(p_i+1)!}-\mathrm{Ric}(\omega_0,J_0^{\varphi_0^{-1}})\wedge\frac{\omega_0^{n-1}}{(n-1)!}-c_0\frac{\omega_0^n}{n!} & = & 0\\ c_1\frac{\omega_1^n}{n!}-\frac{\varphi_{1*}(f_1)_*(\varphi_0^{-1})_*\omega_0^{n-p_1}\wedge\omega_1^{p_1}}{(n-p_1)!p_1!} & = & 0\\ \vdots & & \\ c_k\frac{\omega_k^n}{n!}-\frac{\varphi_{k*}(f_k)_*(\varphi_0^{-1})_*\omega_0^{n-p_k}\wedge\omega_k^{p_k}}{(n-p_k)!p_k!} & = & 0\end{cases}$$

$$\begin{cases}\sum_{i=0}^{k}\frac{f_i^*\omega_{i,\varphi_i}^{p_i+1}\wedge\omega_{0,\varphi_0}^{n-p_i-1}}{(p_i+1)!(n-p_i-1)!}-\varphi_0^*\mathrm{Ric}(\omega_0,J_0^{\varphi_0^{-1}})\wedge\frac{\varphi_0^*\omega_0^{n-1}}{(n-1)!}-c_0\frac{\omega_{0,\varphi_0}^n}{n!} & = & 0\\ \frac{\varphi_0^*\omega_0^{n-p_1}\wedge f_1^*\varphi_1^*\omega_1^{p_1}}{(n-p_1)!p_1!}-c_1\frac{f_1^*\varphi_1^*\omega_1^n}{n!} & = & 0\\ \vdots & & \\ \frac{\varphi_0^*\omega_0^{n-p_k}\wedge f_k^*\varphi_k^*\omega_k^{p_k}}{(n-p_k)!p_k!}-c_k\frac{f_k^*\varphi_k^*\omega_k^n}{n!} & = & 0\end{cases}$$

$$\begin{cases} \displaystyle\sum_{i=0}^{k} \frac{f_i^*\omega_{i,\varphi_i}^{p_i+1} \wedge \omega_{0,\varphi_0}^{n-p_i-1}}{(p_i+1)!(n-p_i-1)!} - \mathrm{Ric}(\omega_{0,\varphi_0}, J_0) \wedge \frac{\omega_{0,\varphi_0}^{n-1}}{(n-1)!} - c_0\omega_{0,\varphi_0}^n &= 0 \\ \dfrac{\omega_{0,\varphi_0}^{n-p_1} \wedge f_1^*\omega_{1,\varphi_1}^{p_1}}{(n-p_1)!p_1!} - c_1 \dfrac{f_1^*\omega_{1,\varphi_1}^n}{n!} &= 0 \\ \vdots \\ \dfrac{\omega_{0,\varphi_0}^{n-p_k} \wedge f_k^*\omega_{k,\varphi_k}^{p_k}}{(n-p_k)!p_k!} - c_k \dfrac{f_k^*\omega_{k,\varphi_k}^n}{n!} &= 0. \end{cases}$$

Finally, if $f_i = id$, and $p_i = 0$, then $f_i^*\omega_{i,\varphi_i} = \omega_{i,\varphi_i}$, the equations become

$$\begin{cases} \displaystyle\sum_{i=0}^{k} (\omega_{i,\varphi_i} - \mathrm{Ric}(\omega_{0,\varphi_0}), J_0) \wedge \omega_{0,\varphi_0}^{n-1} - c_0\omega_{0,\varphi_0}^n &= 0 \\ \omega_{0,\varphi_0}^n - c_1\omega_{1,\varphi_1}^n &= 0 \\ \vdots \\ \omega_{0,\varphi_0}^n - c_k\omega_{k,\varphi_k}^n &= 0, \end{cases}$$

which is the ccscK equation. □

As a remark, c_i are constants along the whole orbit. Also, we can replace ω_i by $a_i\omega_i$, so the equation becomes

$$\begin{cases} \displaystyle\sum_{i=1}^{k} a_i \left(\frac{\omega_{i,\varphi_i}^{p_i+1}}{(p_i+1)!} \wedge \frac{\omega_{0,\varphi_0}^{n-p_i-1}}{(n-p_i-1)!} \right) - \mathrm{Ric}(\omega_{0,\varphi_0}, J_0) \wedge \frac{\omega_{0,\varphi_0}^{n-1}}{(n-1)!} - b_0 \frac{\omega_{0,\varphi_0}^n}{n!} &= 0 \\ \dfrac{\omega_{0,\varphi_0}^{n-p_1}}{(n-p_1)!} \wedge \dfrac{\omega_{1,\varphi_1}^{p_1}}{p_1!} - b_1 \dfrac{\omega_{1,\varphi_1}^n}{n!} &= 0 \\ \vdots \\ \dfrac{\omega_{0,\varphi_0}^{n-p_k}}{(n-p_k)!} \wedge \dfrac{\omega_{k,\varphi_k}^{p_k}}{p_k!} - b_k \dfrac{\omega_{k,\varphi_k}^n}{n!} &= 0, \end{cases}$$

where b_i are the normalizing constants.

3.3. An alternate setup for a special case of the coupled Kähler Yang-Mills equation. We first construct the moment map equation described in [ACGFGP13] for $U(1)$ case. Notice that to solve the equation, we first need

$$\mu(J, f) = \alpha_0\mu_{\mathcal{J}}(J) + \alpha_1\mu_1^*(f) + \alpha_2\mu_0^*(f)$$

under a suitable subspace. We define

$$\Omega_{\mathcal{J},01;\alpha_0,\alpha_1,\alpha_2} = \alpha_0\Omega_{\mathcal{J}} - \alpha_1\Omega_0^* + \alpha_2\Omega_1^*,$$

that is, for $g = f^{-1} : Y \to X$

$$(\alpha_2\Omega_1^* - \alpha_1\Omega_0^*)(\delta g_1, \delta g_2) = \int_Y \omega_X(\delta g_1, \delta g_2)(\alpha_2\omega_Y^{[2]} \wedge g^*\omega_X^{[n-2]} - (\alpha_1)\omega_Y \wedge g^*\omega_X^{[n-1]}).$$

Notice that we take dual moment map as we need

$$\int_Y (\alpha_1\omega_Y \wedge \omega_X^{[n-1]} - \alpha_2\omega_X^{[n]}) = 0,$$

so we cannot choose this as the Kähler form.

Also, we take

$$\begin{aligned}\mathcal{YM}_{01} \subset \{(J, f, g) \in \mathcal{J}(X, \omega_X) \times \mathrm{Map}(Y, X; n-1)^+ \\ \times \mathrm{Map}(Y, X; n-2)^+ | f = g^{-1}, J^f \in \mathcal{J}(X, \omega_X)\}\end{aligned}$$

such that

$$\Omega_{\mathcal{J},01;\alpha_0,\alpha_1,\alpha_2} > 0\}.$$

Then we have the following proposition.

Proposition 3.12. *$\mathcal{YM}_{01}$ is Kähler and closed under the action. Moreover, if*

$$\alpha_0 \Omega_{\mathcal{J}} - \alpha_1 \Omega_0^* + \alpha_2 \Omega_1^* > 0,$$

then the map $\mu_{\mathcal{J},01} : \mathcal{YM}_{01} \to Lie(\mathrm{Ham}(X, \omega_X) \times \mathrm{Ham}(Y, \omega_Y))^*$

$$\mu(J, f) = \begin{pmatrix} \alpha_0 \dfrac{\mathrm{Ric}(X, \omega_X, J) \wedge \omega_X^{n-1}}{(n-1)!} - \alpha_1 \dfrac{\omega_X^{n-1}}{(n-1)!} \wedge f^* \omega_Y + \alpha_2 \dfrac{\omega_X^2 \wedge f^* \omega_Y^{n-2}}{(n-2)!2!} + z\omega_X^n \\ +\alpha_2 \dfrac{f_* \omega_X^{n-1}}{(n-1)!} \wedge \omega_Y - \alpha_1 \dfrac{f_* \omega_X^n}{n!} \end{pmatrix}$$

is a moment map, where $z = \dfrac{\bar{S}}{2} - c_{10}\alpha_1 - c_{11}\alpha_2$, and here we choose α_1 such that $\alpha_1 - \alpha_2 = 0$. As a corollary, $\mu_{\mathcal{J},01} = 0$ iff

$$\begin{cases} \alpha_0 \dfrac{\mathrm{Ric}(X, \omega_X, J) \wedge \omega_X^{n-1}}{(n-1)!} + \alpha_2 \dfrac{\omega_X^2 \wedge f^* \omega_Y^{n-2}}{(n-2)!2!} = c \dfrac{\omega_x^n}{n!} \\ \dfrac{f_* \omega_X^{n-1}}{(n-1)!} \wedge \omega_Y = d \dfrac{f_* \omega_X^n}{n!} \end{cases},$$

where $d = \dfrac{\alpha_1}{\alpha_2}$, $c = \alpha_1 d + z$.

Proof. Notice that $\mathcal{YM}_{01}$ is a submanifold of $\mathcal{Y}_0$, so the complex structure is defined directly by $\mathcal{Y}_0$. Then $\Omega_{\mathcal{J}}$ is J invariant and Ω_0 is J^f invariant. Also, if we define $inv : \mathrm{Map}(X, Y) \to \mathrm{Map}(Y, X)$, and we define J' on $\mathrm{Map}(Y, X)$ such that

$$inv_*(J^f \delta f) = J' inv_* \delta f,$$

then

$$J' D f^{-1} = D f^{-1} J^f = J D f^{-1}.$$

That means the map inv is a biholomorphism, and Ω^* is also Kähler if Ω is. As ω_X is J-invariant, Ω_1 is also J-invariant, hence

$$\Omega_{\mathcal{J},01;\alpha_0,\alpha_1,\alpha_2} = \alpha_0 \Omega_{\mathcal{J}} - \alpha_1 \Omega_0^* + \alpha_2 \Omega_1^*$$

is J-invariant, which implies $\mathcal{YM}_{01}$ is Kähler.

Moreover, the moment map is given by

$$\begin{aligned}
&\mu_{\mathcal{J},01}(J,f)\\
&=\begin{pmatrix}
\alpha_0 \dfrac{\mathrm{Ric}(X,\omega_X,J)\wedge\omega_X^{n-1}}{(n-1)!} - \alpha_1 \dfrac{\omega_X^{n-1}}{(n-1)!}\wedge f^*\omega_Y \\
\qquad +\alpha_2 \dfrac{\omega_X^2\wedge f_*^{-1}\omega_Y^{n-2}}{(n-2)!2!} - \left(\dfrac{\bar{S}}{2} - c_{10}\alpha_1 - c_{11}\alpha_2\right)\omega_X^n \\
\alpha_2 \dfrac{f^{-1*}\omega_X^{n-1}}{(n-1)!}\wedge\omega_Y - \alpha_1 \dfrac{f_*\omega_X^n}{n!} - (c_{20}\alpha_1 - c_{21}\alpha_2)\dfrac{\omega_Y^n}{n!}
\end{pmatrix}\\
&=\begin{pmatrix}
\alpha_0 \dfrac{\mathrm{Ric}(X,\omega_X,J)\wedge\omega_X^{n-1}}{(n-1)!} - d\dfrac{\omega_X^{n-1}}{(n-1)!}\wedge f^*\omega_Y + \alpha_2 \dfrac{\omega_X^2\wedge f^*\omega_Y^{n-2}}{(n-2)!2!} + z\omega_X^n \\
\alpha_2 \dfrac{f_*\omega_X^{n-1}}{(n-1)!}\wedge\omega_Y - d\dfrac{f_*\omega_X^n}{n!}
\end{pmatrix},
\end{aligned}$$

where $z = \dfrac{\bar{S}}{2} - c_{10}\alpha_1 - c_{11}\alpha_2$, and we choose $d = \alpha_1$ such that $c_{20}\alpha_1 - c_{21}\alpha_2 = 0$.

The last part is obvious. □

Hence we get the same moment map equation for the Kähler Yang-Mill's equation with $G = U(1)$ case (see [ACGFGP13] for general). As a remark, we can easily generalize it into $U(1)^n$ case. We can generalized this moment map by considering the following equation:

$$\Omega_{\mathcal{J},pq} := \alpha_0\Omega_{\mathcal{J}} - \frac{\alpha_1(n-p)}{n}\Omega_p^* + \frac{\alpha_2(q+1)}{n}\Omega_q^*.$$

Define

$$\begin{aligned}
\mathcal{YM}_{pq} \subset \{&(J,f,g)\in\mathcal{J}(X,\omega_0)\times\mathrm{Map}(Y,X;n-p)^+\\
&\times\mathrm{Map}(Y,X;n-q-1)^+ | g = f^{-1}, J^f\in\mathcal{J}(Y,\omega_Y)\}
\end{aligned}$$

such that $\Omega_{\mathcal{J},pq} > 0$. That is, $f\in\mathrm{Map}(X,Y;p)^+\cap\mathrm{Map}(X,Y;q)^+$. Then $\mathcal{YM}_{pq}$ is closed under the action of $\mathrm{Ham}(X,\omega_X)\times\mathrm{Ham}(Y,\omega_Y)$, and the map

$$\mu_{\mathcal{J},p,q,\alpha_0,\alpha_1,\alpha_2}(J,f) := \alpha_0\mu_{\mathcal{J}}(J) - \frac{\alpha_1(n-p)}{n}\mu_p(f) + \frac{\alpha_2(q+1)}{n}\mu_q(f)$$

is the moment map for

$$\begin{aligned}
&\mu_{\mathcal{J},pq}(J,f)\\
&=\begin{pmatrix}
\alpha_0 \dfrac{\mathrm{Ric}(X,\omega_X,J)\wedge\omega_X^{n-1}}{(n-1)!} + \alpha_1 \dfrac{\omega_X^{n-p-1}}{(n-p-1)!}\wedge\dfrac{f^*\omega_Y^{p+1}}{(p+1)!} - \alpha_2\dfrac{\omega_X^{n-q-1}\wedge f_*^{-1}\omega_Y^{q+1}}{(n-q-1)!(q+1)!} + z\omega_X^n \\
-\alpha_2 \dfrac{f^{-1*}\omega_X^{n-q}}{(n-q)!}\wedge\dfrac{\omega_Y^q}{q!} + \alpha_1\dfrac{f_*\omega_X^{n-p}}{(n-p)!}\wedge\dfrac{\omega_Y^p}{p!} - (c_{20}\alpha_1 - c_{21}\alpha_2)\dfrac{\omega_Y^n}{n!}
\end{pmatrix}.
\end{aligned}$$

Moreover, we can choose α_1 such that $c_{20}\alpha_1 - c_{21}\alpha_2 = 0$.

3.4. Coupled DHYM types equation. In [SS19], Schlitzer and Stoppa studied coupled Deformed Hermitian Yang-Mills equation using Extended Gauged group theory. We now using the theory in this note to recover the coupled DHYM equation.

Recall that the DHYM equation is given by the following: Let (X, ω, L) be a projective manifold, and $\alpha = \sqrt{-1}F(L)$. Then the DHYM is given by

$$Im(e^{\sqrt{-1}\theta}(\omega + \sqrt{-1}\alpha))^n = 0$$

with $Re(e^{\sqrt{-1}\theta}(\omega + \sqrt{-1}\alpha))^n > 0$. Here θ is some constant defined by the class of ω and α. Expend the expression $e^{\sqrt{-1}\theta}(\omega + \sqrt{-1}\alpha)^n$, we get

$$\text{Imarginay part} : \cos\theta \sum_{r=0}^{k}(-1)^r C^n_{2r+1}\omega^{n-2r-1} \wedge \alpha^{2r+1} + \sin\theta \sum_{r=0}^{k}(-1)^r C^n_{2r}\omega^{n-2r} \wedge \alpha^{2r} = 0;$$

$$\text{Real part} : \cos\theta \sum_{r=0}^{k}(-1)^r C^n_{2r}\omega^{n-2r} \wedge \alpha^{2r} - \sin\theta \sum_{r=0}^{k}(-1)^r C^n_{2r+1}\omega^{n-2r-1} \wedge \alpha^{2r+1} > 0.$$

Here k is the value such that the $2k = n-1$ or n.

Under the previous construction, consider

$$\sum_{r=0}^{k}(-1)^r \cos\theta C^n_{2r}\mu_{2r} - \sin\theta \sum_{s=0}^{l}(-1)^r C_{2r+1}\mu_{2r+1},$$

where k is chosen such that $2k \leq n-1$, $2l+1 \leq n-1$ under the domain

$$\mathcal{Y}_{dHYM} \subset \bigcap_{p=0}^{n-1} \mathrm{Map}(X, Y; p)^+$$

such that

$$\cos\theta \sum_{r=0}^{k}(-1)^r C^n_{2r}\omega^{n-2r} \wedge f^*\alpha^{2r} - \sin\theta \sum_{r=0}^{k}(-1)^r C^n_{2r+1}\omega^{n-2r-1} \wedge f^*\alpha^{2r+1} > 0.$$

Suppose this space is non empty, the equation is given by

$$\cos\theta \sum_{r=0}^{k}(-1)^r C^n_{2r}\omega^{n-2r-1} \wedge f^*\alpha^{2r+1} - \sin\theta \sum_{r=0}^{k-1}(-1)^r C^n_{2r+1}\omega^{n-2r-2} \wedge f^*\alpha^{2r+2} = c_1\omega^n$$

$$\cos\theta \sum_{r=0}^{k}(-1)^r C^n_{2r} f_*\omega^{n-2r} \wedge \alpha^{2r} - \sin\theta \sum_{r=0}^{k}(-1)^r C^n_{2r+1} f_*\omega^{n-2r-1} \wedge \alpha^{2r+1} = c_2\alpha^n.$$

Rewrite it, we get

$$\cos\theta \sum_{r=0}^{k}(-1)^r C^n_{2r}\omega^{n-2r-1} \wedge f^*\alpha^{2r+1} + \sin\theta \sum_{r=1}^{k}(-1)^r C^n_{2r+1}\omega^{n-2r} \wedge f^*\alpha^{2r} = c_1\omega^n$$

$$X\cos\theta \sum_{r=0}^{k}(-1)^r C^n_{2r} f_*\omega^{n-2r} \wedge \alpha^{2r} - \sin\theta \sum_{r=0}^{k}(-1)^r C^n_{2r+1} f_*\omega^{n-2r-1} \wedge \alpha^{2r+1} = c_2\alpha^n,$$

where k is the value such that the $2k = n-1$ or n. Notice that the 2 form

$$\Omega(\delta f_1, \delta f_2) = \int_X \alpha(\delta f_1, \delta f_2)\Bigg(\cos\theta \sum_{r=0}^{k}(-1)^r C^n_{2r} f_*\omega^{n-2r} \wedge \alpha^{2r} - \sin\theta \sum_{r=0}^{k}(-1)^r C^n_{2r+1} f_*\omega^{n-2r-1} \wedge \alpha^{2r+1}\Bigg)$$

define a positive symplectic form iff

$$\cos\theta\sum_{r=0}^{k}(-1)^r C^n_{2r}\omega^{n-2r}\wedge\alpha^{2r}-\sin\theta\sum_{r=0}^{k}(-1)^r C^n_{2r+1}\omega^{n-2r-1}\wedge\alpha^{2r+1}>0.$$

Hence, when this Ω, we get the domain of the moment map. If we also restrict the subgroup to be $\mathrm{Ham}(X,\omega)$, it is the DHYM equation. So we can recover a moment map set up in [CXY17]. However, we cannot recover the coupled DHYM using this moment map as we will couple the scalar curvature with the imginary part, not the real part.

To recover the setup of the coupled deformed HYM, we consider another setup, namely,

$$\mu_{\mathcal{J}}+\cos\theta\sum_{s=0}^{l}(-1)^s C^n_{2s+1}\mu_{2s+1}+\sin\theta\sum_{r=0}^{k}(-1)^r C^n_{2r}\omega^{n-2r}\mu_{2r};$$

where k,l is chosen such that $2k\le n-1$, $2l+1\le n-1$. Denote the space as $\mathcal{Y}'_{dHYM}$ similar to the definition of $\mathcal{Y}_{dHYM}$, and we can define

$$\mathcal{Y}_{cdHYM}\subset\mathcal{J}_{int}\times\mathcal{Y}'_{dHYM}$$

to be the largest Kähler submanifold which is closed under the orbit similar to the setup of gerenal ccscK. Then the resulting moment map equation is given by

$$\begin{aligned}
Ric(\omega,J)\wedge\omega^{n-1}+\cos\theta\sum_{r=0}^{k}(-1)^r C^n_{2r}\omega^{n-2r}\wedge f^*\alpha^{2r}&\\
-\sin\theta\textstyle\sum_{r=0}^{k}(-1)^r C^n_{2r+1}\omega^{n-2r-1}\wedge f^*\alpha^{2r+1}&= c_1\omega^n\\
\cos\theta\sum_{r=0}^{k}(-1)^r C^n_{2r}f_*\omega^{n-2r-1}\wedge\alpha^{2r+1}+\sin\theta\sum_{r=1}^{k}(-1)^r C^n_{2r+1}f_*\omega^{n-2r}\wedge\alpha^{2r}&= c_2\alpha^n
\end{aligned}$$

if c_2 is positive. In particular, if we consider the orbit space

$$(\mathrm{Ham}(X,\omega)\times\mathrm{Ham}(X,\alpha))^{\mathbb{C}}\cdot\{f_0=id\},$$

then the equation can be reformulated as

$$\begin{aligned}
Ric(\omega_\varphi)\wedge\omega_\varphi^{n-1}+\cos\theta\sum_{r=0}^{k}(-1)^r C^n_{2r}\omega_\varphi^{n-2r}\wedge\alpha_\psi^{2r}&\\
-\sin\theta\textstyle\sum_{r=0}^{k}(-1)^r C^n_{2r+1}\omega_\varphi^{n-2r-1}\wedge\alpha_\psi^{2r+1}&= c_1\omega_\varphi^n.\\
\cos\theta\sum_{r=0}^{k}(-1)^r C^n_{2r}\omega_\varphi^{n-2r-1}\wedge\alpha_\psi^{2r+1}+\sin\theta\sum_{r=1}^{k}(-1)^r C^n_{2r+1}\omega_\varphi^{n-2r}\wedge\alpha_\psi^{2r}&= c_2\alpha_\psi^n
\end{aligned}$$

Finally, to avoid the sign problem, we may replace all $-\mu_r$ into μ_r^*, then we can make sure the Ω is positive.

4. Application.

4.1. Obstructions on solving generalized ccscK. For moment maps on the complexified orbit, there are some standard results (see [Wan04]). For example, we can define the Futaki invariant, Calabi functional and Mabuchi functional that can provide some obstructions of the moment map equation $\mu = 0$ (see [Don01], [Don02], [PS10] for cscK, [DP20] for ccscK, and [ACGFGP13] for Kähler Yang Mill). We will consider the generalized ccscK equation

$$\mu_{\mathcal{J},\vec{p}} : \mathcal{O}_{\mathcal{J},id} \to Lie\left(\prod_{i=0}^{k} \mathrm{Ham}(X_i,\omega_i)\right)^{*}.$$

For fix f_i, we can define a map $f_i^* : \mathrm{Diffeo}(X_i, X_i) \to \mathrm{Diffeo}(X_0, X_0)$ by

$$f_i^*\varphi := f_i^{-1} \circ \varphi \circ f_i.$$

We also denote $(f_i)_* = (f_i^{-1})^*$. Then we can define

$$G_0^{\vec{f}} := Aut(X_0, L_0) \cap \cap_{i=1}^{k} f_i^* Aut(X_i, L_i),$$

and

$$G_j^{\vec{f}} = (f_j)_* G_0^{\vec{f}}.$$

Lemma 4.1. *$G_j^{\vec{f}}$ are subgroup of $Aut(X_j, L_j)$. Moreover, the embedding map*

$$\iota : G_0^{\vec{f}} \to \prod_{i=0}^{k} Aut(X_i, L_i)$$

defined by

$$\iota(\varphi) = (\varphi, (f_1)_*\varphi, \cdots, (f_k)_*\varphi)$$

is an homomorphism, and $G_0^{\vec{f}}$ is the stabilizer of $(J, \vec{f})$ as a subgroup of $\prod_{i=0}^{k} Aut(X_i, L_i)$.

Proof. Let $\varphi, \psi \in f_i^* Aut(X_i, L_i)$ Then $f_i \circ \varphi \circ f_i^{-1}, f_i \circ \psi \circ f_i^{-1} \in Aut(X_i, L_i)$. Then

$$(f_i \circ \varphi \circ f_i^{-1}) \circ (f_i \circ \psi \circ f_i^{-1})^{-1} = f_i \circ \varphi \circ \psi^{-1} \circ f_i^{-1} \in Aut(X_i, L_i),$$

hence $\varphi \circ \psi^{-1} \in f_i^* Aut(X_i, L_i)$.

For the second part, first,

$$\pi_i(\iota(\varphi)\iota(\psi)^{-1}) = (f_i \circ \varphi \circ f_i^{-1}) \circ (f_i \circ \psi \circ f_i^{-1})^{-1} = f_i \circ \varphi \circ \psi^{-1} \circ f_i^{-1} = \pi_i(\iota(\varphi \circ \psi^{-1})).$$

It is well known that we can identify $Aut(X_0, L_0)$ with G_J. We can identify $f \in \mathfrak{g}_J$ with $\xi_f \in \mathfrak{aut}(X_0, L_0)$ defined by

$$\mathfrak{g}_J := \{f \in \mathfrak{g}_J^{\mathbb{C}} | \bar{\partial}\xi_f = 0, \iota_{\xi_f}\omega = df.\}$$

Also, for $\varphi \in G_0^{\vec{f}}$,

$$((f_i)_*\varphi) \cdot f_i = f_i \circ \varphi \circ f_i^{-1} \circ f_i \circ \varphi^{-1} = f_i.$$

Finally, if $(\varphi_0, \cdots, \varphi_k) \in \prod_{i=0}^{k} Aut(X_i, L_i)$ such that $\varphi_i \circ f_i \circ \varphi_0^{-1} = f_i$, then

$$\varphi_i = f_i \circ \varphi_0 \circ f_i^{-1},$$

which implies $\varphi_0 \in G_0^{\vec{f}}$. ◻

Corollary 4.2. *Suppose (X_i, L_i) is a projective manifold with line bundles with their respective curvatures $\omega_0, ..., \omega_k$. Suppose generalized ccscK has a solution, then $\bigcap_{i=0}^{k} Aut(X_i, L_i)$ is reductive.*

Proof. We use the result in [Wan04], corollary 12. Suppose $\mu(J_X^{\varphi_0^{-1}}, \varphi_0, \varphi_1, ..., \varphi_k) = 0$ has a solution. Then $G^{\mathbb{C}}_{\varphi_0,\varphi_1,...,\varphi_k}$ is reductive. By assuming $(\omega_{0,h_0}, \cdots, \omega_{k,h_k})$ be the solution, we have $f_i = id$ and

$$G_0^{\vec{f}} = G_0^{id} = \bigcap_{i=0}^{k} Aut(X_i, L_i)$$

is reductive. ◻

We can also define the Calabi functional, Futaki invariant and Mabuchi functional as follow.

Definition 4.3. Let $\xi = (\xi_0, ..., \xi_k)$ be an $\mathbb{C}^*$ action on $\prod_{i=0}^{k} Aut(X_i, L_i)$, where

$$\iota_{\xi_i}\omega_i = dh_i, \bar{\partial}_{J_i}\xi_i(t) = 0.$$

Then the Futaki invariant for the moment map defined in Theorem 3.11 is defined by

$$\begin{aligned} F_{\mathcal{J},\vec{p}}(\xi) :=& \langle \mu_{\mathcal{J},\vec{p}}(f), \xi \rangle \\ =& \int_{X_0} h_0 \left(\sum_{i=1}^{k} \frac{f_{i,0}^* \omega_i^{p_i+1} \wedge \omega_0^{n-p_i-1}}{(p_i+1)!(n-p_i-1)!} - \mathrm{Ric}(\omega_0) \wedge \frac{\omega_0^{n-1}}{(n-1)!} - c_0 \frac{\omega_0^n}{n!} \right) \\ &+ \sum_{i=1}^{k} \int_{X_i} h_i \left(\frac{\omega_i^{p_i} \wedge f_{i,0*}\omega_0^{n-p_i}}{p_i!(n-p_i)!} - c_i \frac{\omega_i^n}{n!} \right). \end{aligned}$$

The Futaki invariant of the ccscK equation is the case $f_{i,0} = id$ and $p_i = 0$ for all $i = 1, ..., k$.

Again, by the standard result (for example, see proposition 6 in [Wan04], theorem 3.9 in [LSW22] for the independence; or see [Fut83] for the KE case), we have

Corollary 4.4. *Futaki invariant is independent of the choice of ω_i with the given class. Moreover, if the Futaki invariant is non zero for some holomorphic vector field, then this moment map equation has no solution in the given Kähler classes.*

Besides, we can define the Calabi functional, which is $||\mu||^2$.

By [Wan04], corollary 13, we have the following:

COROLLARY 4.5. *We define the extremal metric corresponding to $\mu_{\mathcal{J},\vec{p}}$ to be the critical point of $\mathcal{C}_{\mathcal{J},\vec{p}}$. Then the extremal metric solves $\mu_{\mathcal{J},\vec{p}} = 0$ (in the domain $\mathcal{O}_{J,id}$) iff the Futaki invariaant are zero for all holomorphic vector field.*

Let K be a Lie group, and $K^{\mathbb{C}}$ be the complexify orbit. Suppose K acts on a space X with a hamilitonian group action, and $\mu : X \to Lie(K)^*$, we can define a K invariant one form on $(K^{\mathbb{C}}/K)$, defined by the following: for any $v \in Lie(K)$,

$$\alpha\left(\frac{d}{dt}e^{-\sqrt{-1}tv}\cdot g\right) := \langle \mu(g\cdot z), v\rangle.$$

It is well-defined and independent of the choice of $g' \in K \cdot g$ as

$$\langle \mu(kg\cdot z), \mathrm{Ad}_k\xi\rangle = \langle \mathrm{Ad}_k\mu(g\cdot z), Ad_k\xi\rangle = \langle \mu(g\cdot z), \xi\rangle.$$

LEMMA 4.6. *α is closed. Therefore, it is an exact form, and hence, there is a functional $\mathcal{M} : K^{\mathbb{C}}/K \to \mathbb{R}$ defined by*

$$\mathcal{M}(g) := \int_0^1 \alpha(g_t)dt,$$

where g_t is any curve connecting a fix point g_0 and g.

Proof. Assume $[\xi, \eta] = 0$, then by identifying $Lie(K)$ and $Lie(K)^*$,

$$\begin{aligned} d\alpha(\xi,\eta) =& \left\langle \left.\frac{d}{dt}\right|_{t=0} \mu\left(e^{-\sqrt{-1}t\xi}\cdot z\right), \eta\right\rangle - \left.\frac{d}{dt}\right|_{t=0}\left\langle \mu\left(e^{\sqrt{-1}t\eta}\cdot z\right), \xi\right\rangle \\ =& \langle d\mu(JX_\xi), \eta\rangle - \langle d\mu(JX_\eta), \xi\rangle \\ =& \omega(JX_\xi, X_\eta) - \omega(JX_\eta, X_\xi) \\ =& 0, \end{aligned}$$

where X_ξ is the vector field $\left.\frac{d}{dt}\right|_{t=0} e^{-\sqrt{-1}t\xi}\cdot z$. Therefore it is closed. □

As a result, given a moment map, we can define the Mabuchi functional by

$$\mathcal{M}_\mu(g) := \int_0^1 \alpha(\dot{g}_t)dt,$$

where $g_0 = id$.

In our case, we can define $K^{\mathbb{C}}$ as a complex manifold (the orbit space). Notice that

$$K^{\mathbb{C}}/K\cdot id \cong \prod_{i=0}^{k} PSH(X_i, \omega_i),$$

so we can define the Calabi functional and Mabuchi functional by the following.

DEFINITION 4.7. Let (X_i, ω_i) be Kähler manifold, then we denote

$$H_j^{i,p}(h_i, h_j) := \frac{n!}{(n-p)!p!} \frac{\omega_{i,h_i}^{n-p} \wedge \omega_{j,h_j}^{p}}{\omega_{j,h_j}^{n}},$$

and the mean is defined by

$$\underline{H_j^{i,p}}(h_i, h_j) := \int_X H_j^{i,p}(h_i, h_j) \frac{\omega_j^n}{n!}.$$

As

$$Lie(\mathrm{Ham}(X_i, \omega_i)) \cong \{\varphi_i \in C^\infty(X) | d\varphi = \iota_{X_\varphi} \omega_i\}/\mathbb{R},$$

and

$$\begin{aligned}\int_{X_j} \varphi_j \frac{\omega_j^{n-p} \wedge f^* \omega_i^p}{(n-p)!(p)!} &= \int_{\sigma_j(X_j)} \varphi_j(\sigma_j(x)) \left. \frac{\omega_{j,h_j}^{n-p} \wedge \omega_{i,\sigma_i}^p}{(n-p)!!} \right|_{\sigma_j(x)} \\ &= \int_{X_j} \varphi_i H_0^{i,p_i+1}(h_0, h_i) \frac{\omega_{0,h_0}^n}{n!},\end{aligned}$$

the explicit formula of Calabi and Mabuchi functional is given by the following:

DEFINITION 4.8. The Calabi functional $\mathcal{C}_{\mathcal{J},\vec{p}} : \prod_{i=0}^k \mathrm{PSH}(X_i, \omega_i) \to \mathbb{R}$ is defined by the formula

$$\begin{aligned}\mathcal{C}_{\mathcal{J},\vec{p}}(\vec{h}) =& ||\mu_{\mathcal{J},\vec{p}}(\vec{\sigma_h})||^2 \\ =& \int_{X_0} \left| \sum_{i=1}^k H_0^{i,p_i+1}(h_0, h_i) - S_{h_0} - \sum_{i=1}^k \underline{H_0^{i,p_i+1}(h_0, h_i)} + \underline{S_{h_0}} \right|^2 \frac{\omega_{0,h_0}^n}{n!} \\ &+ \sum_{j=1}^k \int_{X_j} |H_j^{0,n-p_i}(h_0, h_i) - \underline{H_j^{0,n-p_i}}(h_0, h_i)|^2 \frac{\omega_{j,h_j}^n}{n!}.\end{aligned}$$

The Mabuchi functional corresponding to $\mu_{\mathcal{J},\vec{p}}$ is given by

$$\mathcal{M}_{\mathcal{J},\vec{p}} : \prod_{i=0}^k \mathrm{PSH}(X_i, \omega_i) \to \mathbb{R}$$

such that the variational formula is

$$\begin{aligned}& d\mathcal{M}_{\mathcal{J},\vec{p}}|_{h_0,\ldots,h_k}(\vec{\varphi}) \\ :=& \langle \mu_{\mathcal{J},\vec{p}}((\sigma_0, \cdots, \sigma_k) \cdot (J, \vec{f})), \vec{\xi_\varphi} \rangle \\ =& \int_{X_0} \varphi_0 \left(\sum_{i=1}^k H_0^{i,p_i+1}(h_0, h_i) - S_{h_0} + \underline{S} - \sum_{i=1}^k \underline{H_0^{i,p_i+1}(h_0, h_i)} \right) \frac{\omega_{0,h_0}^n}{n!} \\ &+ \sum_{j=1}^k \int_{X_j} \varphi_j \left(H_j^{0,n-p_i}(h_0, h_i) - \underline{H_j^{0,n-p_i}}(h_0, h_i) \right) \frac{\omega_{j,h_j}^n}{n!},\end{aligned}$$

where $\varphi \in C^\infty(X, \mathbb{R})$.

Following the standard result of moment map on the comlex orbit (for example, see [Don02],[Wan04]), as the geodesic is given by $e^{-\sqrt{-1}t\xi} \cdot g$,

$$\mathcal{M}'(t) = \langle \mu(e^{-\sqrt{-1}t\xi} \cdot z), \xi \rangle,$$

$$\mathcal{M}''(t) = \omega(-J\xi, \xi) = ||\xi||^2 > 0.$$

We have the following corollary.

Corollary 4.9. *$\mathcal{M}_{\mathcal{J},\vec{p}}$ is convex along smooth geodesics. Hence the solution of the generalized ccscK is the minimum of $\mathcal{M}_{\mathcal{J},\vec{p}}$.*

Notice that by [Dar14], not any two Kähler potential can be connected by the smooth geodesic in general, not even the limit of a sequence of smooth geodesic. Therefore, in general, the convexity of Mabuchi functional for smooth geodesic cannot imply the critical point is unique. However, under some special case, we will still have uniqueness result directly.

Let $(X, L_0, \cdots, L_k)$ to be a polarized toric manifold and the curvature of the toric equivariant line bundle L_i is ω_i which are positive. Let P_i be the moment polytopes corresponding to L_i. We also denoteP_i is defined by the equations

$$\cap_\alpha \{l_i^\alpha(x) \geq 0\},$$

where $l_i^\alpha(x)$ are affine functions. Recall that (See [Gua99] [Don02], [Gue14]), the space of the $(S^1)^n$ invariant Kähler form with the Kähler class $[\omega_i]$

$$\{\varphi \in C^\infty(X, \mathbb{R}) | \omega_i + \sqrt{-1}\partial\bar{\partial}\varphi > 0, \varphi(\theta \cdot x) = \varphi(x), \theta \in (S^1)^n\}$$

is isometric to the space

$$\mathcal{H}_i := \{u \in C^\infty(P_i^0) | u \text{ is convex}, u_i = \sum_\alpha (l_i^\alpha(x) \log(l_i^\alpha(x))), u - u_i \in C^\infty(P_i)\},$$

with the geodesic is given by $u + tv$, $v \in C^\infty(P_i, \mathbb{R})$. Therefore, the orbit space is isometric to the space

$$\mathcal{H}_0 \times \cdots \times \mathcal{H}_k$$

which is geodesicly convex. Therefore, as a direct consequence of 4.9, if we have two minimum point, we can connect them by a stricly convex geodesic, which lead a contradiction. Therefore, we have

Corollary 4.10. *Let $(X, L_0, \cdots, L_k)$ to be a polarized toric manifold and the curvature of the toric equivariant line bundle L_i is ω_i. Then the $(S^1)^n$ invariant solution of the equation*

$$\begin{cases} \sum_{i=1}^k \left(\frac{f_i^* \omega_{i,\varphi_i}^{p_i+1}}{(p_i+1)!} \wedge \frac{\omega_{0,\varphi_0}^{n-p_i-1}}{(n-p_i-1)!} \right) - \mathrm{Ric}(\omega_{0,\varphi_0}, J_0) \wedge \frac{\omega_{0,\varphi_0}^{n-1}}{(n-1)!} - c_0 \frac{\omega_{0,\varphi_0}^n}{n!} & = 0 \\ \frac{\omega_{0,\varphi_0}^{n-p_1}}{(n-p_1)!} \wedge \frac{f_1^* \omega_{1,\varphi_1}^{p_1}}{p_1!} - c_1 \frac{f_1^* \omega_{1,\varphi_1}^n}{n!} & = 0 \\ \vdots & \\ \frac{\omega_{0,\varphi_0}^{n-p_k}}{(n-p_k)!} \wedge \frac{f_k^* \omega_{k,\varphi_k}^{p_k}}{p_k!} - c_k \frac{f_k^* \omega_{k,\varphi_k}^n}{n!} & = 0 \end{cases}$$

is unique (if exists).

5. Kähler construction for coupled equation p. In this section, we will try to find a suitable space for the coupled equation p which is a Kähler manifold. However, the case is much more subtle then the pervious case. The problem is, unlike $\mathcal{J} \times \mathrm{Map}(X, Y; p)^+$, it is not easy to find a good complex submanifold inside $\mathrm{Map}(X, Y; p)^+$ such that both $f^*\omega_Y$ and ω_X form of X and $f_*\omega_X$ and ω_Y are Kahler forms of Y. Before we go on our main disscusion, notice that if we restrict the group to be either $\mathrm{Ham}(X, \omega_X)$ or $\mathrm{Ham}(Y, \omega_Y)$ we do have a good complex submanifold. For the first subgroup, the coupled moment map equation become

$$\omega_X^{n-p-1} \wedge f^*\omega_Y^{p+1} = c_1\omega_X^n.$$

The second subgroup gives

$$f_*\omega_X^{n-p} \wedge \omega_Y^p = c_2\omega_Y^n.$$

We have seen this trick when we re-construct the deformed HYM.

The method we suggest is the following: we consider $F_f : \mathrm{Map}(X, X) \times \mathrm{Map}(Y, Y) \to \mathrm{Map}(X, Y)$ by

$$F_f(\sigma, \eta) = \eta \circ f \circ \sigma^{-1},$$

and consider the pull back image $F_f^{-1}(\mathrm{Map}(X, Y; p)^+)$. Notice that $F_f^{-1}(\Omega_p)$ is not a symplectic form as it may be degenerated. Then we can find a "lagrest complex submanifold" $\mathcal{X}_p^+$, and the orbit space $\mathrm{Ham}^{\mathbb{C}}(X, \omega_X) \times \mathrm{Ham}^{\mathbb{C}}(Y, \omega_Y)$ inside $F^{-1}(\mathrm{Map}(X, Y; p)^+)$. And we will show that if $F_f(\mathcal{X}_p^+)$ and $F_f(\mathrm{Ham}^{\mathbb{C}}(X, \omega_X) \times \mathrm{Ham}^{\mathbb{C}}(Y, \omega_Y))$ are complex manifold, then these are the spaces for the moment map picture for moment map p.

Let J_X to be an integrable almost complex structure of X. Then for any diffeomorphism $g : X \to Y$, we can define an almost complex structure of Y by

$$J_Y := J_X^g := DgJ_XDg^{-1}.$$

This is integrable as the complex local coordinate of Y can be defined by X and g, namely, if

$$\{U_i, \varphi_i : U_i \to \Omega_i \subset \mathbb{C}^n\}$$

are complex local coordinate of X, then $\{g(U_i), \psi_i := \varphi_i \circ g^{-1} : g(U_i) \to \Omega_i\}$ with transition map

$$\psi_j \circ \psi_i^{-1}|_{\psi_i(g^{-1}(U_i \cap U_j))} = \varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_i \cap U_j)}$$

defines the complex local coordinate of Y.

However, let $(X, \omega_X, J_X), (Y, \omega_Y, J_Y)$ be two Kähler manifold. Notice that J_X is compatible with ω_X doesn't implies J_X^g is compatible with ω_Y.

Definition 5.1. Let (X, ω_X, J_X), (Y, ω_Y, J_Y) be compact Kähler manifolds. Define

$$\mathcal{J}(X, \omega_X) := \{J \in \mathcal{J}_{int}(X) | \omega_X(J\cdot, J\cdot) = \omega_X(\cdot, \cdot), \omega_X(J\cdot, \cdot) > 0\}.$$

Define $F_f : \mathrm{Map}(X, X) \times \mathrm{Map}(Y, Y) \to \mathrm{Map}(X, Y)$ to be

$$F_f(\varphi, \psi) := \psi \circ f \circ \varphi^{-1}.$$

We also denote $J_X^{\varphi} := D\varphi \circ J_X D\varphi^{-1}$ for any $\varphi \in \mathrm{Map}(X,X)$ (and similarly for J_Y^{ψ}.) Notice that

$$D\varphi^{-1}J_X^{\varphi} = J_X D\varphi^{-1}.$$

Then we define the following:

DEFINITION 5.2. Let (X, ω_X, J_X), (Y, ω_Y, J_Y) be compact Kähler manifolds. Then we define

$$\mathrm{KMap}_{\omega_Y}(X,Y;J_X) := \{f \in \mathrm{Diffeo}(X,Y) | J_X^f = DfJ_XDf^{-1} \in \mathcal{J}(Y,\omega_Y)\}.$$

We denote $\mathrm{KMap}(X,Y) = \mathrm{KMap}_{\omega_Y}(X,Y;J_X)$ if there is no confusion on the Kähler form.

As a remark, we can also define $\mathrm{KMap}(X,Y)$ by fixing J_Y and moving J_X.

LEMMA 5.3. *The manifold* $(\mathrm{Map}(X,Y), J)$, *where* $J\delta f := DfJ_XDf^{-1}\delta f = J_X^f$, *is a complex manifold.*

Proof. It is obvious that $(\mathrm{Map}(Y,X), J_X)$ is a complex manifold as J_X is integrable. Notice that the map $inv : \mathrm{Map}(X,Y) \to \mathrm{Map}(Y,X)$ defined by

$$inv(f) = f^{-1}$$

is a diffeomorphism, hence we can consider J on $\mathrm{KMap}(X,Y)$ by

$$inv_* J_X = D\,inv^{-1} J_X D\,inv.$$

For any $v \in T_f\mathrm{KMap}(X,Y)$, $D(inv(v)) = Df^{-1}v \circ f^{-1} \in T_{f^{-1}}\mathrm{KMap}(Y,X)$. Therefore, for any $w \in T_f^{-1}\mathrm{KMap}(Y,X)$,

$$inv_* J(v) = Df(J_XDf^{-1}v \circ f^{-1}) \circ f = DfJ_XDf^{-1}v.$$

Therfore, $(\mathrm{KMap}(X,Y), J)$ is a complex manifold with integrable almost complex structure J. ☐

LEMMA 5.4. $\mathrm{KMap}(X,Y)$ *and* $\mathrm{KMap}(X,Y;p)^+$ *is a complex submanifold.*

Proof. Consider $\mathrm{Map}(X,Y) \times \mathcal{J}(Y,\omega_Y)$ with the product complex structure $J(\sigma, A) = (J_X^f\sigma, J_YA)$ for all $(\sigma, A) \in T_{f,J_Y}\mathrm{Map}(X,Y) \times \mathcal{J}(Y,\omega_Y)$. Then we have a subvariety

$$\mathcal{W} := \{(f, J_Y) | Df(J_X^0)Df^{-1} = J_Y\},$$

here J_X^0 is fixed. We can rewrite the relation as $f_*J_Y - J_X = 0$ as an endmorphism. By this, we can consider the map $F : \mathrm{Map}(X,Y) \times \mathcal{J}(Y,\omega_Y) \to End(TX)$ by

$$F(f, J_Y) = f_*J_Y - J_X.$$

Then $\mathcal{W} = \{F(f, J_Y) = 0\}$. When we differenate with respect to J_Y direction, say A, then

$$\delta_A F(f, J_Y) = \left.\frac{d}{dt}\right|_{t=0} (Df^{-1}J_Y^tDf) = Df^{-1}ADf$$

which is bounded and indeed $c|A| < |\delta_A F(f, J_Y)| < C|A|$ for some c, C, and for any norm. So $\mathcal{W}$ is locally a graph, which gives the smooth structure of $\mathcal{W}$.

We now show $\mathcal{W}$ is a complex subvariety. Let $\frac{d}{dt} f_t = \sigma$ and $\frac{d}{dt} J_Y^t = A$ at $t = 0$. Then the condition on tangent space is given by

$$-Df^{-1} D\sigma Df^{-1} J_Y Df + Df^{-1} A Df + Df^{-1} A D\sigma = 0.$$

We now see if the vector $(J_X^f \sigma, J_Y A)$ with $J_X^f = J_Y$ satisfies this relation. Notice that $J_X^f = J_Y$ implies

$$J_X Df^{-1} = Df^{-1} J_Y.$$

So

$$\begin{aligned}
&-Df^{-1} D J_X^f \sigma Df^{-1} J_Y Df + Df^{-1} J_Y A Df + Df^{-1} J_Y D J_X^f \sigma \\
&= -J_X Df^{-1} D\sigma Df^{-1} J_Y Df + J_X Df^{-1} A Df + Df^{-1} J_X^f J_Y D\sigma \\
&= -J_X Df^{-1} D\sigma Df^{-1} J_Y Df + J_X Df^{-1} A Df + J_X Df^{-1} J_Y D\sigma \\
&= J_X(-Df^{-1} D\sigma Df^{-1} J_Y Df + Df^{-1} A Df + Df^{-1} A D\sigma) \\
&= 0.
\end{aligned}$$

Also, we need to show that the map $\pi : \mathcal{W} \to \mathrm{Map}(X, Y)$ is injective, holomorphic and the image is $\mathrm{KMap}(X, Y)$. The injectivity is obvious as if $\pi(f, J_Y) = \pi(f', J_Y')$, then $f = f'$. When $f = f'$, $J_X^f = J_X^{f'}$. By the definition, $J_Y \in \mathcal{J}(X, \omega_X)$, hence it is $\mathrm{KMap}_{\omega_X}(X, Y)$. Finally it is holomorphic as this is the restriction of the projection map $\pi : \mathrm{Map}(X, Y) \times \mathcal{J}(Y, \omega_Y) \to \mathrm{Map}(X, Y)$ which is holomorphic.

Notice that $\mathrm{KMap}(X, Y; p)^+$ is an open subset of KMap. As $\omega_X^{n-p} \wedge \omega_Y^p$ is J invariant for $(J_X, J_Y) \in \mathcal{J}(X, \omega_X) \times \mathcal{J}(Y, \omega_Y)$, so this is a complex submanifold. □

Remark 5.5. Using the same argument, we can prove that $\mathrm{KMap}(X, Y)$ is a complex submanifold of $(\mathrm{Map}(X, Y), J_Y)$ as well.

Definition 5.6. Let (X, ω_X, J_X), (Y, ω_Y, J_Y) be compact Kähler manifolds. Then we define

$$\mathrm{KMap}_{\omega_Y}(X, Y; J_X) := \{f \in \mathrm{Diffeo}(X, Y) | J_X^f = Df J_X Df^{-1} \in \mathcal{J}(Y, \omega_Y)\}.$$

We denote $\mathrm{KMap}(X, X) = \mathrm{KMap}_{\omega_X}(X, X, J_X)$ if there is no confusion on the choice of the Kähler form. Then we define

$$\mathcal{X}_{f,p}^+ := (\mathrm{KMap}(X, X) \times \mathrm{KMap}(Y, Y)) \cap F_f^{-1}(\mathrm{Map}(X, Y; p)^+).$$

Moreover, let $(v, w) \in T_{(\varphi,\psi)}(\mathrm{Map}(X, X) \times \mathrm{Map}(Y, Y))$ we also define an almost complex structure on $\mathrm{Map}(X, X) \times \mathrm{Map}(Y, Y)$ by

$$J_{\mathrm{Map}}(v, w) := (J_X^\varphi v, J_Y^\psi w).$$

We now prove the main proposition in this section:

Proposition 5.7. *Let $f : X \to Y$ is a biholomorphism, and define an action* $\mathrm{Ham}(X, \omega_X) \times \mathrm{Ham}(Y, \omega_Y)$ *on* $\mathrm{Map}(X, X) \times \mathrm{Map}(Y, Y)$ *which is given by*

$$(\sigma, \eta) \cdot (\varphi, \psi) := (\sigma \circ \varphi, \eta \circ \psi).$$

Then

1) *F_f commutes with the group action.*
2) *$\mathcal{X}^+_{f,p}$ is a complex submanifold,*
3) *The action* $\mathrm{Ham}(X,\omega_X)\times\mathrm{Ham}(Y,\omega_Y)$ *is closed in $\mathcal{X}^+_{f,p}$.*
4) *$F_f^*\Omega_p$ is J_{Map} invariant.*

Proof.
1) $F_f(\sigma\circ\varphi,\eta\circ\psi)=\eta\circ\psi\circ f\circ\varphi^{-1}\sigma^{-1}=(\sigma,\eta)\cdot F_f(\varphi,\psi)$.
2) To show $\mathcal{X}^+_{f,p}$ is a smooth manifold, we only need to show $\mathrm{KMap}(X,X)$ is smooth. As F_f is continuous, so it implies $\mathcal{X}^+_{f,p}$ is an open subset, hence it is smooth.

   Define $G:\mathcal{J}(X,\omega_X)\times\mathrm{KMap}(X,X)\to\mathrm{End}(\Gamma(TX))$ by

$$G(J,\varphi)=J_X^{\varphi}-J.$$

   Then $\mathrm{KMap}(X,X)\cong\{(J,\varphi)|G(J,\varphi)=0.\}$ Also, let $A\in T_{J,\varphi}\mathcal{J}(X,\omega_X)$, then

$$DG_{J,\varphi}(A,0)=-Id,$$

   hence the implicit function theorem implies that there exists $H:U\subset\mathrm{KMap}(X,X)\to V\subset\mathcal{J}(X,\omega_X)$ which for $G:U\times V\to\mathrm{End}(\Gamma(TX))$, we have

$$G(J,\varphi)=G(H(\varphi),\varphi)=0.$$

   Therefore, (U,H) gives a local coordinate, which implies $\mathrm{KMap}(X,X)$ is smooth. Hence $\mathrm{KMap}(Y,Y)$ is also smooth, and thus $\mathcal{X}^+_{f,p}$ is smooth.

   We now show $\mathcal{X}^+_{f,p}$ is J invariant. Again, as $F_f^{-1}(\mathrm{Map}(X,Y;p))$ is open, and $\mathrm{KMap}(X,X)$ and $\mathrm{KMap}(Y,Y)$ have the same defining function, we only need to show $\mathrm{KMap}(X,X)$ is J_X^{φ} invariant. Then the argument can be used as showing $\mathrm{KMap}(Y,Y)$ is also J_Y^{ψ} invariant. Suppose $\sigma\in T_\varphi\mathrm{KMap}(X,X)$. The equation we have is the following: for all $(v,w)\in T_xX$

$$\varphi^*\omega_X(J_Xv,J_Xw)=\varphi^*\omega_X(v,w).$$

   Differentiating it along v, we get

$$\begin{aligned}&\omega_X(D\sigma J_Xv,D\varphi J_Xw)+\omega_X(D\varphi J_Xv,D\sigma J_Xw)\\=&\ \omega_X(D\sigma v,D\varphi w)+\omega_X(D\varphi v,D\sigma w).\end{aligned}$$

   Now

$$\begin{aligned}&\omega_X(J_X^{\varphi}D\sigma J_Xv,D\varphi J_Xw)+\omega_X(D\varphi J_Xv,J_X^{\varphi}D\sigma J_Xw)\\&\quad-\omega_X(J_X^{\varphi}D\sigma v,D\varphi w)-\omega_X(D\varphi v,J_X^{\varphi}D\sigma w)\\&=\omega_X(D\sigma J_Xv,J_X^{\varphi}D\varphi J_Xw)+\omega_X(J_X^{\varphi}D\varphi J_Xv,D\sigma J_Xw)\\&\quad-\omega_X(D\sigma v,J_X^{\varphi}D\varphi w)-\omega_X(J_X^{\varphi}D\varphi v,D\sigma w)\\&=\omega_X(D\sigma J_Xv,D\varphi J_XJ_Xw)+\omega_X(D\varphi J_XJ_Xv,D\sigma J_Xw)\\&\quad-\omega_X(D\sigma v,D\varphi J_Xw)-\omega_X(D\varphi J_Xv,D\sigma w).\end{aligned}$$

We let $u = J_X w$, then $w = -J_X u$, hence the expression becomes

$$\begin{aligned}
&\omega_X(D\sigma J_X v, D\varphi J_X u) - \omega_X(D\varphi v, D\sigma u)\\
&\quad - \omega_X(D\sigma v, D\varphi u) + \omega_X(D\varphi J_X v, D\sigma J_X u)\\
&= \omega_X(D\sigma J_X v, D\varphi J_X u) + \omega_X(D\varphi J_X v, D\sigma J_X u)\\
&\quad -\omega_X(D\varphi v, D\sigma u) - \omega_X(D\sigma v, D\varphi u)\\
&= 0.
\end{aligned}$$

3) As F_f preserves the group action, we only need to show $\mathrm{KMap}(X,X) \times \mathrm{KMap}(Y,Y)$ is closed under the action. Let $(\sigma,\eta) \in \mathrm{Ham}(X,\omega_X) \times \mathrm{Ham}(Y,\omega_Y)$. Then

$$(\sigma \circ \varphi)^* \omega_X = \varphi^* \sigma^* \omega_X = \varphi^* \omega_X,$$

hence $(\sigma\circ\varphi)^*\omega_X$ is J_X-invariant. Similarly, $(\eta\circ\psi)^*\omega_Y$ is J_Y-invariant.

4) For $(v,w),(v',w') \in T_{\varphi,\psi}\mathrm{KMap}(X,X) \times \mathrm{KMap}(Y,Y)$,

$$DF_{p_{\varphi,\psi}}(v,w)|_x = w|_{f\circ\varphi^{-1}(x)} - D\psi Df D\varphi^{-1} v|_{\varphi^{-1}(x)},$$

so

$$\begin{aligned}
&F_p^*\Omega_p(J_{\mathrm{Map}}(v,w), J_{\mathrm{Map}}(v',w'))\\
&= F_p^*\Omega_p(J_X^\varphi v, J_Y^\psi w), (J_X^\varphi v', J_Y^\psi w')\\
&= \Omega_p(J_Y^\psi w - D\psi Df D\varphi^{-1} J_X^\varphi v, J_Y^\psi w' - D\psi Df D\varphi^{-1} J_X^\varphi v')\\
&= \Omega_p(J_Y^\psi w - D\psi Df J_X D\varphi^{-1} v, J_Y^\psi w' - D\psi Df J_X D\varphi^{-1} v')\\
&\quad (\because D\varphi^{-1} J_X^\varphi = J_X D\varphi)\\
&= \Omega_p(J_Y^\psi w - D\psi J_Y Df D\varphi^{-1} v, J_Y^\psi w' - D\psi J_Y Df D\varphi^{-1} v')\\
&\quad (\because J_Y Df = Df J_X)\\
&= \Omega_p(J_Y^\psi (w - D\psi Df D\varphi^{-1} v), J_Y^\psi (w' - D\psi Df D\varphi^{-1} v'))\\
&\quad (\because D\psi J_Y = J_Y^\psi D\psi)\\
&= \int_X \omega_Y(J_Y^\psi (w - D\psi Df D\varphi^{-1} v), J_Y^\psi (w' - D\psi Df D\varphi^{-1} v'))\omega_X^{n-p} \wedge \omega_Y^p\\
&= \int_X \omega_Y((w - D\psi Df D\varphi^{-1} v), (w' - D\psi Df D\varphi^{-1} v'))\omega_X^{n-p} \wedge \omega_Y^p\\
&\quad (\because J_Y^\psi \in \mathcal{J}(Y,\omega_Y))\\
&= F_p^*\Omega_p((v,w),(v',w')).
\end{aligned}$$

Hence it is J_{Map}-invariant.

∎

As $\mathcal{X}_{f,p}^+$ is a complex manifold, we observe that if $(v,w) \in T_{(\varphi,\psi)}(\mathcal{X}_{f,p}^+)$, then $J_{\mathrm{Map}}(v,w) = (J_X^\varphi v, J_Y^\psi w) \in T_{(\varphi,\psi)}(\mathcal{X}_{f,p}^+)$. As $\omega_Y(J_Y^\psi u, u) > 0$ if $u \neq 0$, so for any (v,w), if

$$w - D\psi Df D\varphi^{-1} v \neq 0,$$

then we can choose

$$(v',w') = -J_{\mathrm{Map}}(v,w).$$

However, if $w = D\psi Df D\varphi^{-1}v$, then it is degenerate. Indeed, the problem is F_f may not be injective. Indeed, if we consider $X = Y$, $\omega_X = \omega_Y$, then $f = id$ solve the problems, but for any $\sigma \in \mathrm{Ham}^{\mathbb{C}}(X)$, (σ, σ) will solve the equation as well. Therefore, we cannot apply the theory directly.

As $\mathcal{X}^+_{f,p}$ is closed under the action of $\mathrm{Ham}(X, \omega_X) \times \mathrm{Ham}(Y, \omega_Y)$ and F_f preserves the action, we can still consider the orbit space

$$\mathcal{O}_f := \mathrm{Ham}^{\mathbb{C}}(X, \omega_X) \times \mathrm{Ham}^{\mathbb{C}}(Y, \omega_Y) \cdot (id, id) \subset (\mathcal{X}^+_{f,p})$$

(as we mentioned before, $\mathrm{Ham}^{\mathbb{C}}(X, \omega_X) \times \mathrm{Ham}^{\mathbb{C}}(Y, \omega_Y)$ is not a group). Notice that it may not be a manifold, but only a complex variety.

Indeed, suppose there exists $\sigma \in \mathrm{Ham}^{\mathbb{C}}(X, \omega_X) \cap \mathrm{Ham}^{\mathbb{C}}(Y, \omega_Y)$, and ω_X, ω_Y solved coupled equation p, then $\sigma^*\omega_X$, $\sigma^* f^* \omega_Y$ also solved equation p. This example exists, say,

Example 5.8. Consider (X, ω_0, ω_1) with

$$[\omega_0] = [\omega_1].$$

Then by definition, there exists $\sigma \in \mathrm{Ham}^{\mathbb{C}}(X, \omega_0)$ such that

$$\sigma^*\omega_1 = \omega_0.$$

Then

$$\omega_0^{n-p-1} \wedge \sigma^* \omega_1^{p+1} = \omega_0^n;\ \sigma_* \omega_0^{n-p} \wedge \omega_1^p = \omega_1^n,$$

that is $(id, \sigma) \in \mathcal{O}_{id}$ solves the equation. Moreover, for any $\eta \in \mathrm{Ham}^{\mathbb{C}}(X, \omega_0) = \sigma^* \mathrm{Ham}^{\mathbb{C}}(X, \omega_1)$, $(\eta, \sigma\eta) \in \mathcal{O}_{id}$ and

$$\sigma \circ \eta \circ \eta^{-1} = \sigma$$

implies it also solves the same moment map equation.

Notice that we can consider the equivalent class, namely,

$$(\sigma, \eta) \sim (\sigma', \eta') \text{ if } F_f((\sigma, \eta)) = F_f((\sigma', \eta')),$$

that is,

$$\eta' \circ f \circ \sigma'^{-1} = \eta \circ f \circ \sigma^{-1}.$$

Notice that we may simply consider $[\mathcal{O}_f] \subset \mathrm{Map}(X, Y; p)^+$. Hence we can restrict the moment map into $[\mathcal{O}_f]$ if it is a manifold.

Corollary 5.9. *Let (X, ω_X) and (Y, ω_Y) be two Kähler manifolds with two Kähler forms, and f is a biholomorphism. Suppose $[\mathcal{O}_f]$ is a manifold, then μ_p : $[\mathcal{O}_f] \to Lie\left(\prod_{i=0}^{k} \mathrm{Ham}(X_i, \omega_i)\right)^*$ is a moment map. In particular, if $\mathrm{Ham}(X, \omega_X) \cap f^*\mathrm{Ham}(Y, \omega_Y) = id$, then μ_p is well defined.*

Proof. Notice that $[\mathcal{O}_f] \subset \mathrm{Map}(X, Y; p)^+$, and it is closed under the action. Hence $\mu_p|_{[\mathcal{O}_f]}$ is well defined. □

6. moment map for embedding. In the previous theory, we always assume $X = Y$ as a same Kähler manifold, and $f_0 = id$. We now provide a case that X and Y are not diffeomorphic.

Let $(X, \omega_X), (Y, \omega_Y)$ be two symplectic manifolds with dimensions n, m, where $n \leq m$. Define $\mathrm{EMap}(X, Y)$ to be the space of embedding maps and

$$\mathrm{EMap}(X, Y; p)^+ := \{f \in \mathrm{EMap}(X, Y) | \omega_X^{n-p} \wedge f^*\omega_Y^p > 0\}.$$

Notice that f^{-1} is well defined on $f(X)$ and for this case, $f_* = f^{-1}$ on $f(X)$. Let $Z \subset Y$ be a k dimensional submanifold. Then we denote δ_Z be the $m-k$ current on Y, which

$$\int_Y \delta_Z \wedge \alpha := \int_Z \alpha$$

for all k forms α on Y.

Lemma 6.1. *Let $(X, \omega_X), (Y, \omega_Y)$ are two symplectic manifolds with finite volume with respect to ω_X, ω_Y, and let $0 \leq p \leq n-1$. Then the moment map $\mu_p : \mathrm{EMap}(X, Y; p)^+ \to Lie(\mathrm{Ham}(X, \omega_X) \times \mathrm{Ham}(Y, \omega_Y))^*$ is given by*

$$\mu_p(f) = \left(\frac{n}{n-p}\left(c_1\frac{\omega_X^n}{n!} - \frac{\omega_X^{n-p-1} \wedge f^*\omega_Y^{p+1}}{(n-p-1)!(p+1)!}\right),\right.$$
$$\left.\frac{m}{m-p}\left(c_2\frac{\omega_Y^m}{m!} - \delta_{f(X)} \wedge \left(f_*(\omega_X^{n-p} \wedge f^*\omega_Y^p)\right)\right)\right).$$

Proof. The proof is basically the same as the proof of theorem 2.6. The main difference is that $\mathrm{EMap}(X, Y)$ and $\mathrm{Map}(Y, Y)$ is not a one-one correspondence. However, given $v' \in T_f\mathrm{EMap}(X, Y)$, $v'|_x \in T_{f(x)}Y$. Hence, we can still identify it as $v \in T_g\mathrm{Map}(f(X), Y)$, where $g_t(y) := f_t \circ f^{-1}(y)$. After that, we extend this g_t to $\hat{g}_t : Y \to Y$. Then the same proof can be applied.

$$\begin{aligned}
&\frac{d}{dt}\bigg|_{t=0}\int_{f_t(X)} \psi(y)\frac{f_{t*}(\omega_X^{n-p} \wedge f_t^*\omega_Y^p)}{(n-p)!p!} \\
=&\frac{d}{dt}\bigg|_{t=0}\int_X \psi(f_t(x))\frac{\omega_X^{n-p} \wedge f_t^*\iota_v\omega_Y^p}{(n-p)!p!} \\
=&\int_X d\psi(v \circ f)\frac{\omega_X^{n-p} \wedge f_t^*\iota_v\omega_Y^p}{(n-p)!p!} + p\int_X \psi(f(x))\frac{df^*\iota_v\omega_Y \wedge \omega_X^{n-p} \wedge f_t^*\iota_v\omega_Y^{p-1}}{(n-p)!(p)!} \\
=&\int_X \omega_Y(\xi_\psi, (v \circ f))\frac{\omega_X^{n-p} \wedge f_t^*\iota_v\omega_Y^p}{(n-p)!p!} - \frac{p}{m}\int_{f(X)} \omega_Y(\xi_\psi, (v))\frac{\omega_X^{n-p} \wedge f_t^*\iota_v\omega_Y^p}{(n-p)!p!} \\
=&\frac{m-p}{m}\int_X \omega_Y(\xi_\psi, (v \circ f))\frac{\omega_X^{n-p} \wedge f_t^*\iota_v\omega_Y^p}{(n-p)!p!}.
\end{aligned}$$

Notice that this is independent of the choice of extension of g as the term $f_t^*\iota_v\omega_Y$ only depends on v and f_t, but not the extension $\hat{v} := \dot{\hat{g}}_t|_{t=0}$. □

Remark 6.2. Notice that as p is fixed, we can take $\Omega'_X := \dfrac{n}{(n-p)}\Omega_X$, $\Omega'_Y = \dfrac{m}{m-p}\Omega_Y$ to remove the leading coefficient. We will denote this moment map as μ_p from now on.

REMARK 6.3. When Y is compact, given any function ψ,

$$\psi - \frac{1}{Vol(Y)}\int_Y \psi \frac{\omega_Y^m}{m!} \in Lie(\mathrm{Ham}(Y, \omega_Y)).$$

However, for the case where Y is non compact, and $\displaystyle\int_Y \frac{\omega_Y^m}{m!} = \infty$, we cannot normalized ψ. So we need to assume $\displaystyle\int_Y \psi\omega_Y^m = 0$.

REMARK 6.4. For $p = n$, the map $\mu_n(f) : \mathrm{EMap}(X, Y; p)^+ \to Lie(\mathrm{Ham}(Y, \omega_Y); n)^*$ is a moment map. Therefore, we can still get a non-trivial moment map for $p = n$ if this is an embedding.

Appendix A. Analytic compuation of convexity of $\mathcal{M}_{\mathcal{J},p}$. In this section, we will show that the Mabuchi functional $\mathcal{M}_{\mathcal{J},p}$ is stictly convex along the smooth geodesic $(h_{0,t}, \cdots, h_{k,t})$, where the geodesic equation is given by

$$\ddot{h}_{i,t} - |\nabla \dot{h}_{i,t}|^2_{\omega_{i,h_i}} = 0$$

for all $0 \le i \le k$. As the standard Mabuchi functional $\mathcal{M}_{\mathcal{J}_p}$ is well known to be convex, it suffices to consider $\mathcal{M}_p = \mathcal{M}_{\mathcal{J},p} - \mathcal{M}_{\mathcal{J}}$. For simplicity, we will only consider the case $k = 1$. We also denote $\varphi_{i,t} = \dot{h}_{i,t}$, and $\omega^{[k]} := \dfrac{\omega^k}{k!}$. As

$$|\nabla\varphi_i|^2_{\omega_i} = \frac{\omega_i(X_{\varphi_i}, JX_{\varphi_i})}{\omega_i^{[n]}},$$

The geodesic equation with Lemma 2.5 implies that

$$|\nabla\varphi_i|^2_{\omega_i}\omega_i^{n-p} \wedge \alpha^p = \sqrt{-1}n\partial\varphi \wedge \bar{\partial}\varphi_i \wedge \omega_i^{n-p-1} \wedge \alpha^p.$$

Hence

$$\frac{n-p}{n}\int_X |\nabla\varphi_i|^2_{\omega_i}\omega_i^{[n-p]} \wedge \alpha^{[p]} = -\int_X \varphi \wedge \sqrt{-1}\partial\bar{\partial}\varphi_i \wedge \omega_i^{[n-p-1]} \wedge \alpha^{[p]}.$$

As

$$d\mathcal{M}_p(\varphi_{0,t}, \varphi_{1,t}) := \int_X \varphi_{0,t}(\omega_{0,t}^{[n-p-1]} \wedge \omega_{1,t}^{[p+1]} - c_1\omega_{0,t}^{[n]}) + \int_X \varphi_{1,t}(\omega_{0,t}^{[n-p]} \wedge \omega_{1,t}^{[p]} - c_2\omega_{1,t}^{[n]}),$$

$$\begin{aligned}
&\frac{d^2}{dt^2}\mathcal{M}_p(h_{0,t}, h_{1,t}) \\
= &\int_X \dot{\varphi}_{0,t}(\omega_{0,t}^{[n-p-1]} \wedge \omega_{1,t}^{[p+1]} - c_1\omega_{0,t}^{[n]}) + \int_X \dot{\varphi}_{1,t}(\omega_{0,t}^{[n-p]} \wedge \omega_{1,t}^{[p]} - c_2\omega_{1,t}^{[n]}) \\
&+ \int_X \varphi_{0,t}(\sqrt{-1}\partial\bar{\partial}\varphi_{0,t} \wedge \omega_{0,t}^{[n-p-2]} \wedge \omega_{1,t}^{[p+1]} - c_1\sqrt{-1}\partial\bar{\partial}\varphi_{0,t} \wedge \omega_{0,t}^{[n-1]}) \\
&+ \int_X \varphi_{1,t}(\sqrt{-1}\partial\bar{\partial}\varphi_{0,t} \wedge \omega_{0,t}^{[n-p-1]} \wedge \omega_{1,t}^{[p]}) + \int_X \varphi_{0,t}(\sqrt{-1}\partial\bar{\partial}\varphi_{1,t} \wedge \omega_{0,t}^{[n-p-1]} \wedge \omega_{1,t}^{[p]}) \\
&+ \int_X \varphi_{1,t}(\sqrt{-1}\partial\bar{\partial}\varphi_{1,t} \wedge \omega_{0,t}^{[n-p]} \wedge \omega_{1,t}^{[p-1]} - c_2\sqrt{-1}\partial\bar{\partial}\varphi_{1,t} \wedge \omega_{1,t}^{[n-1]})
\end{aligned}$$

$$
\begin{aligned}
&= \int_X \dot{\varphi}_{0,t}(\omega_{0,t}^{[n-p-1]} \wedge \omega_{1,t}^{[p+1]} - c_1\omega_{0,t}^{[n]}) + \int_X \dot{\varphi}_{1,t}(\omega_{0,t}^{[n-p]} \wedge \omega_{1,t}^{[p]} - c_2\omega_{1,t}^{[n]}) \\
&\quad - \int_X |\nabla\varphi_{0,t}|^2_{\omega_{0,t}}(c_1\omega_{0,t}^{[n]} - \frac{n-p-1}{n}\omega_{0,t}^{[n-p-1]} \wedge \omega_{1,t}^{[p+1]}) \\
&\quad - \int_X |\nabla\varphi_{1,t}|^2_{\omega_{1,t}}(\frac{p}{n}\omega_{0,t}^{[n-p]} \wedge \omega_{1,t}^{[p]} - c_2\omega_{1,t}^{[n]}) \\
&\quad + \int_X \varphi_{1,t}(\sqrt{-1}\partial\bar{\partial}\varphi_{0,t} \wedge \omega_{0,t}^{[n-p-1]} \wedge \omega_{1,t}^{[p]}) + \int_X \varphi_{0,t}(\sqrt{-1}\partial\bar{\partial}\varphi_{1,t} \wedge \omega_{0,t}^{[n-p-1]} \wedge \omega_{1,t}^{[p]}) \\
&= \frac{p+1}{n}\int_X |\nabla\varphi_{0,t}|^2_{\omega_{0,t}}\omega_{0,t}^{[n-p-1]} \wedge \omega_{1,t}^{[p+1]} + \frac{n-p}{n}\int_X |\nabla\varphi_{1,t}|^2_{\omega_{1,t}}\omega_{0,t}^{[n-p]} \wedge \omega_{1,t}^{[p]} \\
&\quad + \int_X \varphi_{1,t}(\sqrt{-1}\partial\bar{\partial}\varphi_{0,t} \wedge \omega_{0,t}^{[n-p-1]} \wedge \omega_{1,t}^{[p]}) + \int_X \varphi_{0,t}(\sqrt{-1}\partial\bar{\partial}\varphi_{1,t} \wedge \omega_{0,t}^{[n-p-1]} \wedge \omega_{1,t}^{[p]}) \\
&= \frac{(p+1)\sqrt{-1}}{n-p-1}\int_X \partial\varphi_{0,t} \wedge \bar{\partial}\varphi_{0,t} \wedge \omega_{0,t}^{[n-p-2]} \wedge \omega_{1,t}^{[p+1]} \\
&\quad + \frac{\sqrt{-1}(n-p)}{p}\int_X \partial\varphi_{1,t} \wedge \bar{\partial}\varphi_{1,t} \wedge \omega_{0,t}^{[n-p]} \wedge \omega_{1,t}^{[p-1]} \\
&\quad - \sqrt{-1}(\int_X \partial\varphi_{1,t} \wedge \bar{\partial}\varphi_{0,t} \wedge \omega_{0,t}^{[n-p-1]} \wedge \omega_{1,t}^{[p]} + \int_X \partial\varphi_{0,t} \wedge \bar{\partial}\varphi_{1,t} \wedge \omega_{0,t}^{[n-p-1]} \wedge \omega_{1,t}^{[p]})
\end{aligned}
$$

Using the same proof as in Lemma 2.5, and

$$\sqrt{-1}\partial\varphi_{i,t} \wedge \bar{\partial}\varphi_{j,t} = d\varphi_{i,t} \wedge d^c\varphi_{j,t} = \omega_{i,t}(X_{\varphi_{i,t}}, \bullet) \wedge \omega_{j,t}(-JX_{\varphi_{j,t}}, \bullet),$$

the expression becomes

$$
\begin{aligned}
&\frac{1}{(n-p-1)!p!}\int_X \omega_{0,t}(X_{\varphi_{0,t}}, \bullet) \wedge \omega_{0,t}(-JX_{\varphi_{0,t}}, \bullet) \wedge \omega_{0,t}^{n-p-2} \wedge \omega_{1,t}^{p+1} \\
&+ \frac{1}{(n-p-1)!p!}\int_X \omega_{1,t}(X_{\varphi_{1,t}}, \bullet) \wedge \omega_{1,t}(-JX_{\varphi_{1,t}}, \bullet) \wedge \omega_{0,t}^{n-p} \wedge \omega_{1,t}^{p-1} \\
&- \frac{2}{(n-p-1)!p!}\int_X \omega_{1,t}(X_{\varphi_{1,t}}, \bullet) \wedge \omega_{0,t}(-JX_{\varphi_{0,t}}, \bullet) \wedge \omega_{0,t}^{n-p-1} \wedge \omega_{1,t}^{p} \\
= &\frac{1}{(n-p-1)!p!}\int_X \omega_{0,t}(X_{\varphi_{0,t}}, JX_{\varphi_{0,t}}) \wedge \omega_{1,t} \wedge \wedge\omega_{0,t}^{n-p-1} \wedge \omega_{1,t}^{p} \\
&+ \frac{1}{(n-p-1)!p!}\int_X \omega_{1,t}(X_{\varphi_{1,t}}, JX_{\varphi_{1,t}}) \wedge \omega_{0,t}^{n-p-1} \wedge \omega_{1,t}^{p} \\
&- \frac{2}{(n-p-1)!p!}\int_X \omega_{1,t}(X_{\varphi_{1,t}}, JX_{\varphi_{0,t}}) \wedge \omega_{0,t}^{n-p} \wedge \omega_{1,t}^{p}.
\end{aligned}
$$

We claim that if α, β are two forms, then

$$\iota_v\iota_w\alpha \wedge \beta = \alpha \wedge \iota_v\iota_w\beta = \iota_v\iota_w\beta \wedge \alpha.$$

With this claim, and

$$\omega_{1,t}(X_{\varphi_{1,t}}, JX_{\varphi_{0,t}}) = \omega_{1,t}(X_{\varphi_{0,t}}, JX_{\varphi_{1,t}}),$$

the expression becomes

$$\begin{aligned}
&\frac{1}{(n-p-1)!p!}\int_X \omega_{1,t}(X_{\varphi_{0,t}}, JX_{\varphi_{0,t}}) \wedge \omega_{0,t} \wedge \wedge\omega_{0,t}^{n-p-1} \wedge \omega_{1,t}^p \\
&+\frac{1}{(n-p-1)!p!}\int_X \omega_{1,t}(X_{\varphi_{1,t}}, JX_{\varphi_{1,t}}) \wedge \omega_{0,t}^{n-p-1} \wedge \omega_{1,t}^p \\
&-\frac{2}{(n-p-1)!p!}\int_X \omega_{1,t}(X_{\varphi_{1,t}}, \bullet) \wedge \omega_{0,t}(-JX_{\varphi_{0,t}}, \bullet) \wedge \omega_{0,t}^{n-p-1} \wedge \omega_{1,t}^p \\
=&\frac{1}{(n-p-1)!p!}\int_X \omega_{1,t}(X_{\varphi_{0,t}} - X_{\varphi_{1,t}}, J(X_{\varphi_{0,t}} - X_{\varphi_{1,t}}))\omega_{0,t}^{n-p} \wedge \omega_{1,t}^p.
\end{aligned}$$

We finally show the claim. To show that, observe that

$$(\iota_v\iota_w\alpha)\wedge\beta = \iota_v(\iota_w\alpha\wedge\beta) + \iota_w\alpha\wedge\iota_v\beta = \iota_v((\iota_w\alpha)\wedge\beta) + \iota_w(\alpha\wedge(\iota_v\beta)) - \alpha\wedge\iota_w\iota_v\beta.$$

Therefore,

$$(\iota_v\iota_w\alpha)\wedge\beta - \alpha\wedge\iota_v\iota_w\beta = \iota_v((\iota_w\alpha)\wedge\beta) + \iota_w(\alpha\wedge(\iota_v\beta)).$$

Also,

$$(\iota_v\iota_w\alpha)\wedge\beta - \alpha\wedge\iota_v\iota_w\beta = (\iota_w\iota_v\beta)\wedge\alpha - \beta\wedge\iota_w\iota_v\alpha = \iota_w((\iota_v\beta)\wedge\alpha) + \iota_v(\beta\wedge(\iota_w\alpha)).$$

as

$$\iota_w((\iota_v\beta)\wedge\alpha) + \iota_v(\beta\wedge(\iota_w\alpha)) = -\iota_w(\alpha\wedge(\iota_v\beta)) - \iota_v((\iota_w\alpha)\wedge\beta),$$

$$2((\iota_v\iota_w\alpha)\wedge\beta - \alpha\wedge\iota_v\iota_w\beta) = ((\iota_v\iota_w\alpha)\wedge\beta - \alpha\wedge\iota_v\iota_w\beta) + (\iota_w\iota_v\beta)\wedge\alpha - \beta\wedge\iota_w\iota_v\alpha = 0.$$

REMARK A.1. We could point out that from this definition, we can see that $\mathcal{M}_p$ is not strictly convex when $X_\varphi = X_\phi$. Hence we can see that μ_p is not a moment map unless we mod out this relation.

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