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A CRITERIA FOR CLASSIFICATION OF WEIGHTED DUAL GRAPHS OF SINGULARITIES AND ITS APPLICATION*

STEPHEN S.-T. YAU†, QIWEI ZHU‡, AND HUAIQING ZUO§

Abstract. Let (V, p) be a normal surface singularity. Let $\pi : (M, E) \to (V, p)$ be a minimal good resolution of V, such that the irreducible components E_i of $E = \pi^{-1}(p)$ are nonsingular and have only normal crossings. There is a natural weighted dual graph Γ associated to E. Along with the genera of the E_i, Γ fully describes the topology and differentiable structure of the embedding of E in M. Intuitively, normal surface singularity has simplest topology if all the irreducible curves in the exceptional set are smooth rational curves with self-intersection number -2. It can be shown that these are necessary ADE-singularities. In our previous work we classify all the weighted dual graphs of $E = \cup_{i=1}^{n} E_i$ such that one of the curves E_i is -3 curve, and the rest all are -2 curves. This is a natural generalization of Artin's classification of rational triple points. However there is no general method to classify or examine all possible weighted dual graphs of $E = \cup_{i=1}^{n} E_i$. In this article, we introduce a new concept, component factor, which is useful and computable for classifying weighted dual graphs. Based on it, we present a criteria for verifying whether a graph is the weighted dual graph associated to E. As a result, we give a complete classification of weighted dual graphs consist of -2 curves and exactly one -4 curve.

Key words. normal singularities, topological classification, weighted dual graph.

Mathematics Subject Classification. 14B05, 32S25.

1. Introduction.

Let (V, p) be a normal surface singularity and $\pi \colon M \to V$ be a resolution of V such that the irreducible components E_i, $1 \leq i \leq n$, of $E = \pi^{-1}(p)$ are nonsingular and have only normal crossings. Associated to E is a weighted dual graph Γ (e.g., see [4] or [9]) which, along with the genera of the E_i, fully describes the topology and differentiable structure of E in M [15]. On a nonsingular surface M, a $-k$ curve means a nonsingular rational curve with self-intersection $-k$.

M. Artin [1] has studied the rational singularities (i.e., those for which $R^1\pi_*(\mathcal{O}) = 0$). He has shown that all weighted dual graphs of rational double points are one of the graphs: A_k, $k \geq 1$; D_k, $k \geq 4$; E_6, E_7 and E_8 which arise in the classification of simple Lie groups. He also shows that the existence of fundamental cycle (see Definiton 2.1) are equivalent to the negative definiteness of $(E_i \cdot E_j)$. Moreover, rational triple points are also classified into 9 classes according to the dual graphs in [1]. These 9 classes graphs consist of -2 curves and exactly one -3 curve. In our recent work [28], we classify all the weighted dual graphs of $E = \cup_{i=1}^{n} E_i$ such that one of the curves E_i is -3 curve, and the rest all are -2 curves. This is a natural generalization of Artin's classification of rational triple points.

The simple here usually means that only finitely many isomorphism classes occur in the versal deformation. By the rational double points and rational triple points are simple. Stevens [18] conjectures that the simple normal surface singularities are

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†Department of Mathematical Sciences, Tsinghua University, Beijing, 100084; and Yanqi Lake Beijing Institute of Mathematical Sciences and Applications, Huairou District, Beijing 101408, P. R. China (yau@uic.edu). Yau is supported by Tsinghua University Education Foundation fund (042202008) and NSFC Grant 11961141005.

‡Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P. R. China (zhuqw19@mails.tsinghua.edu.cn). Zhu is the co-first author of this paper.

§Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P. R. China (hqzuo@mail.tsinghua.edu.cn). Zuo is supported by NSFC Grants 12271280, 11961141005, and Tsinghua University Initiative Scientific Research Program.

exactly those rational singularities whose resolution graph can be obtained from the graph of a rational double point or rational triple point by making any number of vertex weights more negative. He shows that no other rational singularities can be simple. He proves simpleness for some special classes of singularities, namely rational quadruple points or sandwiched singularities in [18]. For the classification of certain classes of rational singularities, the interesting readers can refer the recent papers [20], [19], [23], [24].

In [12], Laufer investigates a class of elliptic singularities which satisfy a minimality condition. These minimally elliptic singularities have a theory much like the theory for rational singularities. Laufer [12] also lists all dual graphs which correspond to minimally elliptic hypersurface singularities. These singularities are exactly Gorenstein singularities with geometric genus equals to 1. Such a list is extremely useful for researchers in the field. For the classification of Gorenstein singularities with geometric genus greater than 1, the interesting reader can refer to the papers [16], [5], [25], [27], [22], [6]-[7], [8], [15]. Later, in [29] (resp. [2]), the authors generalize Laufer's list of dual graphs of minimally elliptic hypersurface singularities. They classify all weighted dual graphs of the simplest Gorenstein non-complete intersection (resp. complete intersection) singularities of dimension two. These singularities are exactly those minimal elliptic singularities with fundamental cycle self intersection number -5 (resp. -4).

In [13], Laufer classifies all possible taut singularities. Though his classification is complete, the last step for verifying the negative definiteness of weighted dual graph is not illustrated. In this paper, we solve this problem completely. We give an explicit criteria for verifying whether a graph is negative-definite. As a result, we give a complete classification of weighted dual graphs consist of -2 curves and exactly one -4 curve. This generalizes the work in [28].

Our article is organized as follows. In Section 3, we introduce a new concept, component factor, for tree graph (cf. Definition 3.6). Furthermore, we generalize this construction to loop and multiple edge graphs in Section 4 (cf. Definition 4.5 and Definition 4.8). The following criteria for negative definiteness can be concluded:

THEOREM 1.1. *Let Γ be a weighted dual graph. Let Γ_i 's be subgraphs connected to E_j such that Γ_i 's are negative-definite. Let $CF(\Gamma_i)$ be component factor of Γ_i. Then Γ is negative-definite if and only if*

$$E_j^2 + \sum_i CF(\Gamma_i) < 0.$$

As an application of Theorem 1.1, we give the complete classification of weighted dual graphs consist of -2 curves and exactly one -4 curve as follows.

THEOREM 1.2. *Let (V,p) be a normal surface singularity. Let $\pi : (M,E) \to (V,p)$ be a minimal good resolution of V, such that the irreducible components E_i of $E = \pi^{-1}(p)$ are nonsingular and have only normal crossings. Γ is the weighted dual graph associated to E. Assuming that all the exceptional curves E_i are -2 curves and except exactly one E_j is a -4 curve. Then the weighted dual graph Γ must be one of the three cases: Tree graph, Loop graph or Multiple edge graph (cf. Section 3 for tree case and Section 4 for the last two cases). The complete classifications of tree graphs are listed in Section 3 (cf. Theorem 3.5, and from Theorem 3.11 to Theorem 3.24), loop graphs and multiple edge graphs are listed in Section 4 (cf. Theorem 4.7 and Theorem 4.10).*

2. Preliminaries.

2.1. Riemann-Roch and fundamental cycle. Let $\pi\colon M \to V$ be a resolution of the normal two-dimensional Stein space V. We assume that p is the only singularity of V. Let $\pi^{-1}(p) = E = \cup E_i$, $1 \le i \le n$, be the decomposition of the exceptional set E into irreducible components.

A cycle $D = \Sigma d_i E_i$, $1 \le i \le n$ is an integral combination of the E_i, with d_i an integer. There is a natural partial ordering denoted by $\ge$, between cycles defined by comparing the coefficients: $\sum_i m_i E_i \ge \sum_i n_i E_i$ if $m_i \ge n_i$ for all i. If $D_1 \ge D_2$ but $D_1 \ne D_2$ then we write $D_1 > D_2$. We let supp $D = \cup E_i$, $d_i \ne 0$, denote the support of D.

Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on M. Let $\mathcal{O}(-D)$ be the sheaf of germs of holomorphic functions on M which vanish to order d_i on E_i. Let $\mathcal{O}_D$ denote $\mathcal{O}/\mathcal{O}(-D)$. Define

$$\chi(D) := \dim H^0(M, \mathcal{O}_D) - \dim H^1(M, \mathcal{O}_D). \tag{2.1}$$

The Riemann-Roch theorem [17, Proposition IV.4, p. 75] says

$$\chi(D) = -\frac{1}{2}(D^2 + D \cdot K), \tag{2.2}$$

where K is the canonical divisor on M and $D \cdot K$ is the intersection number of D and K. In fact, let g_i be the geometric genus of E_i, i.e., the genus of the desingularization of E_i. Then the adjunction formula [17, Proposition IV, 5, p. 75] says

$$A_i \cdot K = -A_i^2 + 2g_i - 2 + 2\delta_i \tag{2.3}$$

where δ_i is the "number" of nodes and cusps on A_i. Each singular point on E_i other than a node or cusp counts as at least two nodes. It follows immediately from (2.2) that if B and C are cycles, then

$$\chi(B + C) = \chi(B) + \chi(C) - B \cdot C. \tag{2.4}$$

Definition 2.1. Associated to π is a unique fundamental cycle Z [1, pp. 131-132] such that $Z > 0$, $E_i \cdot Z \le 0$ for all E_i and such that Z is minimal with respect to those two properties.

The fundamental cycle Z may be computed from the intersection as follows via a computation sequence for Z in the sense of Laufer [10, Proposition 4.1, p. 607].

$$\begin{aligned} Z_0 = 0, Z_1 = E_{i_1}, Z_2 = Z_1 + E_{i_2}, \ldots, Z_j = Z_{j-1} + E_{i_j}, \ldots, \\ Z_\ell = Z_{\ell-1} + E_{i_\ell} = Z \end{aligned}$$

where E_{i_1} is arbitrary and $E_{i_j} \cdot Z_{j-1} > 0$, $1 < j \le \ell$.

$\mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j)$ represents the sheaf of germs of sections of a line bundle over E_{i_j} of Chern class $-E_{i_j} \cdot Z_{j-1}$. So

$$H^0(M, \mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j)) = 0$$

for $j > 1$.

$$0 \to \mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j) \to \mathcal{O}_{Z_j} \to \mathcal{O}_{Z_{j-1}} \to 0 \tag{2.5}$$

is an exact sheaf sequence. From the long exact cohomology sequence for (2.5), it follows by induction that

$$H^0(M, \mathcal{O}_{Z_k}) = \mathbb{C}, \quad 1 \leq k \leq \ell \tag{2.6}$$

$$\dim H^1(M, \mathcal{O}_{Z_k}) = \sum_{1 \leq j \leq k} \dim H^1\big(M, \mathcal{O}(-Z_{j-1}))/\mathcal{O}(-Z_j)\big). \tag{2.7}$$

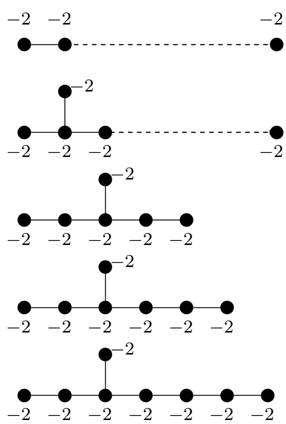
LEMMA 2.2 ([12]). *Let Z_k be part of a computation sequence for Z and such that $\chi(Z_k) = 0$. Then $\dim H^1(M, \mathcal{O}_D) \leq 1$ for all cycles D such that $0 \leq D \leq Z_k$. Also $\chi(D) \geq 0$.*

2.2. Classfication of weighted dual graphs. In this section, we recall two beautiful results given by Artin in [1]. Let (V, p) be a normal 2-dimensional singularity, $\pi \colon M \to V$ be the minimal resolution and Z be the fundamental cycle.

DEFINITION 2.3. The singularity (V, p) is said to be rational if $\chi(Z) = 1$.

If p is a rational singularity, then π is also a minimal good resolution, i.e., exceptional set with nonsingular E_i and normal crossings. Moreover each A_i is a rational curve and $E_i^2 = -2$.

THEOREM 2.4 ([1]). *If (V, p) is a hypersurface rational singularity, then (V, p) is a rational double point. Moreover the set of weighted dual graphs of hypersurface rational singularities consists of the following graphs:*



(1) $A_n, n \geq 1$ $\quad Z = 1\ 1 \ldots 1$

(2) $D_n, n \geq 4$ $\quad Z = \begin{matrix} & 1 & & & \\ 1 & 2 & 2 \ldots & 2 & 1 \end{matrix}$

(3) E_6 $\quad Z = \begin{matrix} & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{matrix}$

(4) E_7 $\quad Z = \begin{matrix} & & 2 & & & \\ 2 & 3 & 4 & 3 & 2 & 1 \end{matrix}$

(5) E_8 $\quad Z = \begin{matrix} & & 3 & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 \end{matrix}$

Theorem 2.4 completely classifies the weighted dual graphs with all $E_i^2 = -2$, which are called ADE graphs. In general, to classify the weighted dual graph we firstly need to classify corresponding negative definite matrices:

PROPOSITION 2.5 ([1]). *Let $\{E_i\}_{i=1,\cdots,n}$ be a connected bunch of complete curves on a regular two-dimensional scheme:*

(i) *Suppose that $\|(E_i \cdot E_j)\|$ is negative definite, then there exist positive cycles $Z = \sum r_i E_i$ such that $(Z \cdot E_i) \leq 0$ for all i.*

(ii) *Conversely, if there exists a positive cycle* $Z = \sum r_i E_i$ *such that* $(Z \cdot E_i) \le 0$ *for all* i, *then* $\|(E_i \cdot E_j)\|$ *is negative semi-definite. If in addition* $\left(Z^2\right) < 0$, *then* $\|(E_i \cdot E_j)\|$ *is negative definite.*

3. Classfication of tree graph based on component factor. In this section, we give a complete classification of the weighted dual graphs consist of -2 curves and exactly one -4 curve, i.e., all E_i's are nonsingular rational curves, $E_j^2 = -4$ for one j and $E_i^2 = -2$ for all i such that $i \neq j$. We use the notation $\bullet$ to denote those E_i with $E_i^2 = -2$ and $*$ denotes the E_j with $E_j^2 = -4$. All the exceptional curves are assumed to be rational.

By [3], we know the classification of weighted dual graphs which we want is equivalent to classification of all negative definite matrix $(E_i \cdot E_j)$.

By Theorem 2.4, if all E_i have $E_i^2 = -2$, then the graph must be ADE graphs. Recall that a tree graph is a connected graph without loops. ADE graphs are all tree graphs.

Notation. For a tree graph with a curve E, we denote the subgraphs connected to E as $\Gamma_1, ..., \Gamma_s$, the point connected to E as $F_1, ..., F_s$. The subgraphs connected to F_i are denoted as $G_{i,1}, ..., G_{i,r_i}$:

$$\begin{array}{ccccc} & & G_{i,r_i} & & \\ & & | & \ddots & \\ \Gamma' & \text{—}E\text{—} & \bullet & \text{——} & G_{i,1} \\ & & F_i & & \end{array} \qquad .$$

Theorem 3.1 (Tree determinant formula). *Let the weighted dual graph* Γ *be as above. Then*

$$det(\Gamma) = (\prod_{i=1}^{s} det(\Gamma_i))(E^2 + \sum_{j=1}^{s} \frac{(-1)\prod_{l=1}^{r_j} det(G_{j,l})}{det(\Gamma_j)})$$

Proof. We do it by induction. Assume the formula is proved when $s \le k-1$, now we prove it is true for k. Let the weighted dual graph be as in notation with the number of subgraphs connected to E is k, i.e. the weighted dual graph is:

$$\begin{array}{ccccc} & & G_{k,r_k} & & \\ & & | & \ddots & \\ \Gamma' & \text{—}E\text{—} & \bullet & \text{——} & G_{k,1} \\ & & E_k & & \end{array} \qquad .$$

Let n_i be the number of points of Γ_i. The intersection matrix can be represented as :

$$\begin{pmatrix} \Gamma' & 1 & 0 & 1 & \dots & 1 \\ 1 & E^2 & 1 & 0 & \dots & 0 \\ 0 & 1 & E_k^2 & 1 & \dots & 1 \\ 0 & 0 & 1 & G_{k,1} & \dots & 0 \\ 0 & 0 & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 1 & 0 & \dots & G_{k,r_k} \end{pmatrix}.$$

For simplicity here we use $\begin{pmatrix} \Gamma' & 1 \\ 1 & E^2 \end{pmatrix}$ to denote

$$\begin{pmatrix} E^2 & 1 & \dots & 1 \\ 1 & \Gamma_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & \Gamma_{k-1} \end{pmatrix}.$$

When doing Laplacian expansion on E, we get

$$det(\Gamma) = det(\begin{pmatrix} \Gamma' & 1 \\ 1 & E^2 \end{pmatrix})det(\Gamma_k) + (-1)det(\begin{pmatrix} \Gamma' & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & G_{k,1} & \dots & 0 \\ 0 & 0 & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & G_{k,r_k} \end{pmatrix})$$

$$= det(\begin{pmatrix} \Gamma' & 1 \\ 1 & E^2 \end{pmatrix})det(\Gamma_k) + (-1)det(\Gamma') \prod_{l=1}^{r_k} det(G_{k,l}).$$

By induction assumption:

$$det(\begin{pmatrix} \Gamma' & 1 \\ 1 & E^2 \end{pmatrix}) = (\prod_{i=1}^{k-1} det(\Gamma_i))((E^2) + \sum_{j=1}^{k-1} \frac{(-1)\prod_{l=1}^{r_j} det(G_{j,l})}{det(\Gamma_j)}).$$

Thus

$$\begin{aligned} det(\Gamma) &= (\prod_{i=1}^{k-1} det(\Gamma_i))((E^2) + \sum_{j=1}^{k-1} \frac{(-1)\prod_{l=1}^{r_j} det(G_{j,l})}{det(\Gamma_j)})det(\Gamma_k) \\ &\quad + (-1)\prod_{i=1}^{k-1} det(\Gamma_i) \prod_{l=1}^{r_k} det(G_{k,l}) \\ &= (\prod_{i=1}^{k} det(\Gamma_i))((E^2) + \sum_{j=1}^{s} \frac{(-1)\prod_{l=1}^{r_j} det(G_{j,l})}{det(\Gamma_j)}). \end{aligned}$$

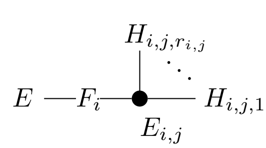
□

Lemma 3.2. *Assumptions as in notation. If*

$$(E^2) + \sum_{j=1}^{s} \frac{\prod_{k=1}^{r_j} |det(G_{j,k})|}{|det(\Gamma_j)|} < 0,$$

then there exists a rational cycle D with $D \cdot E < 0$ and $D \cdot E_i = 0$ for any exceptional curve $E_i \neq E$.

Proof. Denote points connected to F_i as $E_{i,j}$, and the subgraph connected to $E_{i,j}$ as $H_{i,j,k}$, i.e:



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We construct a rational cycle D supported on exceptional set by induction. Let the coefficient of D on E be 1, on F_i be $\prod_j |det(G_{i,j})|/|det\Gamma_i|$. Next, let the coefficient of F_i be $\prod_j |det(G_{i,j})|/|det\Gamma_i|$, the coefficient of $E_{i,j}$ be $(\prod_j |det(G_{i,j})|/|det\Gamma_i|) \cdot (\prod_k |det(H_{i,j,k})|/|det(G_{i,j})|)$. Repeat this procedure to get the coefficient of D on all the exceptional curves. Then we get

$$D \cdot E = E^2 + \sum_i \prod_j \frac{|det(G_{i,j})|}{|det\Gamma_i|} < 0,$$

$$D \cdot F_i = 1 + (\prod_j \frac{|det(G_{i,j})|}{|det\Gamma_i|})(F_i^2 + \sum_l \frac{\prod_k |det(H_{i,l,k})|}{|det(G_{i,l})|}).$$

Use Theorem 3.3 for Γ_i:

$$|det(\Gamma_i)| = \prod_j |det(G_{i,j})| \cdot |(F_i^2 + \sum_l |\frac{\prod_k det(H_{i,l,k})}{det(G_{k,l})}|)|.$$

Notice that Γ_i is negative-definite, hence

$$|(E_i^2 + \sum_l |\frac{\prod_k det(H_{i,l,k})}{det(G_{k,l})}|)| = -(F_i^2 + \sum_l |\frac{\prod_k det(H_{i,l,k})}{det(G_{k,l})}|).$$

Combining with the above three equations we get

$$D \cdot F_i = 0.$$

We get a rational cycle D, such that $D \cdot E < 0$, $D \cdot E_i = 0$ for any exceptional curve $E_i \neq E$. □

THEOREM 3.3 (Criteria for negative definiteness of tree graph). *Assumptions as in Notation. Assume furthermore that each Γ_i is negative definite for $i = 1, ..., s$. Then the weighted dual graph is negative definite if and only if*

$$(E^2) + \sum_{j=1}^{s} \frac{\prod_{k=1}^{r_j} |det(G_{j,k})|}{|det(\Gamma_j)|} < 0.$$

Proof. Let n_j be the number of points in Γ_j. By Γ_j is negative-definite, we have $det(\Gamma_j) = (-1)^{n_j}|det(\Gamma_j)|$, $\prod_{k=1}^{r_j} det(G_{j,k}) = (-1)^{n_j - 1} \prod_{k=1}^{r_j} |det(G_{j,k})|$. Combine this with Theorem 3.3 we get:

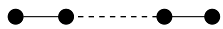
$$det(\Gamma) = (-1)^{\sum_{i=1}^{s} n_i}(|\prod_{i=1}^{s} det(\Gamma_i)|)((E^2) + \sum_{j=1}^{s} |\frac{\prod_{k=1}^{r_j} det(G_{j,k})}{det(\Gamma_j)}|).$$

If Γ is negative-definite then $(-1)^{(1+\sum_{i=1}^{s} n_i)} det(\Gamma) > 0$, thus

$$E^2 + \sum_{j=1}^{s} \frac{\prod_{k=1}^{r_j} |det(G_{j,k})|}{|det(\Gamma_j)|} < 0.$$

The converse is immediately by Lemma 3.2. □

Now we turn to the classfication. We first begin with tree weighted dual graphs. We abuse the notation of weighted dual graphs and the corresponding matrices in the following discussion, if without any confusion. Henceforth, whether A_k a weighted dual graph or a matrix should be clear from context. For example, A_n could either denote the weighted dual graph:



or the matrix:

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

We choose -4 curve to be E. The remaining connected graphs are denoted as $\Gamma_1, \dots, \Gamma_s$. In a dual graph, the $*$ represents the -4 curve. We call it -4 point or -4 cycle later. Others are the point corresponding -2 curve, we call it -2 point or -2 cycle later.

Lemma 3.4. *$s \leq 7$, and Γ_i must be ADE for any $1 \leq i \leq s$.*

Proof. It is easy to see that the matrix

$$\begin{pmatrix} -4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

has determinant 0, so the -4 curve can not connect with more than seven -2 curves, thus we have $s \leq 7$. As for Γ_i, notice that if we require Γ to be negative definite, then the fundamental cycle Z, when restricted to each Γ_i, satisfies $Z|_{\Gamma_i} \cdot E_j \leq 0, \forall E_j \in \Gamma_i$ (here it means that E_j is in the support of Γ_i). Denote E_{j_0} the cycle in Γ_i connected with -4 point, then $Z|_{\Gamma_i} \cdot E_{j_0} < 0$. Thus by Proposition 2.5, we conclude that Γ_i is negative definite, which must be ADE. ∎

We can classify weighted dual graph according to s. $s = 0$ is the simplist case. When $s = 1$, we need to illustrate the different connection way of Γ_i with -4 point. For $s \geq 1$, we should compute if the matrix is negative definite, which is based on a general formula.

Case 1. $s = 0$.

The weighted dual graph is just one -4 point:

$$*$$

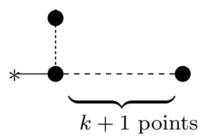
Case 2. $s = 1$.

Theorem 3.5. *When $s = 1$, Γ_1 must be one of the following:*

$$(1)\ k - A_n\text{:}\ \begin{cases} k = 0,1,2,3,\ \ n \geq 2k+1; \\ k = 4,\ \ 9 \leq n \leq 23; \\ k = 5,\ \ 11 \leq n \leq 16; \\ k = 6,\ \ 13 \leq n \leq 15. \end{cases}$$

(2) $k - D_n$: $k = 0, 1, 2$ *with* $n \geq k + 4$.
(3) D'_n: $5 \leq n \leq 15$.
(4) D''_n: $n = 4, 5$.
(5) E_6, E_7, E_8.
(6) E'_7.
(7) $1 - E_6$.
(8) E''_6, E''_7.

Here we use the notation $k - A_n$ to denote the following graph: $\Gamma_1 = A_n$ and Γ is

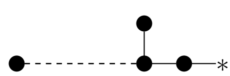


with $n \geq 2k+1$. $0 - A_n$ means that the -4 curve connects A_n at the left or right end point.

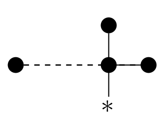
Similarly the notation $1 - D_n$ means $\Gamma_1 = D_n$, with $n \geq 5$ and Γ is

$0 - D_n$ means that the -4 curve connects longest branch of D_n.

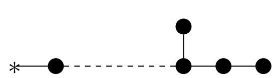
D'_n is



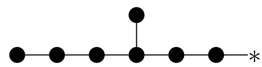
D''_n is



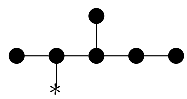
$E_k, k = 6, 7, 8$ are



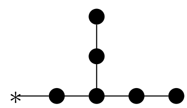
E_7' is



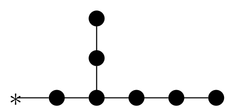
$1 - E_6$ is



E_6'' is



E_7'' is



Proof. Use the criteria, for $k - A_n$ case, we require:

$$-4 + \left|\frac{det(A_k)det(A_{n-k-1})}{det(A_n)}\right| < 0,$$

i.e.

$$-4 + \frac{(k+1)(n-k)}{n+1} < 0.$$

Thus we have

$$k - 3 < \frac{(k+1)^2}{n+1}.$$

For $k \leq 3$, this is right. For $k \geq 4$, we have

$$n + 1 < \frac{(k+1)^2}{k-3}.$$

However, when $\Gamma_1 = k - A_n$, we must require $n \geq 2k+1$. Thus

$$2k + 1 \leq n < \frac{(k+1)^2}{k-3} - 1,$$

which gives

$$\begin{cases} 9 \leq n \leq 23, \ k = 4. \\ 11 \leq n \leq 16, \ k = 5. \\ 13 \leq n \leq 15, \ k = 6. \end{cases}$$

For $k - D_n$ case, we require:

$$-4 + |\frac{det(D_{n-k-1})det(A_k)}{det(D_n)}| < 0,$$

i.e.

$$-4 + k + 1 < 0.$$

Thus $k = 0, 1, 2$, with $n \geq 4 + k$.

For D'_n case, we require:

$$-4 + |\frac{det(A_{n-1})}{det(D_n)}| < 0,$$

thus $4 \leq n \leq 15$.

For D''_n-case, we require:

$$-4 + |\frac{det(A_{n-3})ndet(A_1)det(A_1))}{det(D_n)}| < 0,$$

thus $n = 4, 5$. ☐

Definition 3.6 (Component factor). Assumptions as in Notation. The component factor of Γ_j is defined to be

$$CF(\Gamma_j) := \frac{\prod_{k=1}^{r_j} |det(G_{j,k})|}{|det(\Gamma_j)|}.$$

Remark 3.7. One should be careful that the component factor of Γ_j not only depends on the graph of Γ_j, but also depends on the connection way of Γ_j with the central curve.

The criteria for tree graph immediately implies:

Corollary 3.8. *Assumptions as in Notation. Assume each Γ_i is negative definite for $i = 1, ..., s$. Then the weighted dual graph is negative definite if and only if*

$$E^2 + \sum CF(\Gamma_i) < 0.$$

Lemma 3.9. *The component factor of graphs in $s = 1$ case (cf. Theorem 3.5) is as follows:*

(1) $k - A_n$: $k + 1 - \frac{(k+1)^2}{n+1}$.

(2) $k - D_n$: $k + 1$.

(3) D'_n: $\frac{n}{4}$.

(4) D''_n: $n-2$.

(5) $E_6 : \frac{4}{3}$, $E_7 : \frac{3}{2}$, $E_8 : 2$.

(6) $E'_7 : 2$.

(7) $1 - E_6 : \frac{10}{3}$.

(8) $E''_6 : 2$, $E''_7 : \frac{7}{2}$.

EXAMPLE 3.10. Let the weighted dual graph be $D_n + E_8$: Then by Corollary 3.8 we need to check

$$-4 + CF(D_n) + CF(E_8) = -4 + 1 + 2 < 0,$$

which is satisfied. Furthermore, by Corollary 3.9 (5)(6) and (8), we know that $D_n + E'_7$, $D_n + E''_6$ are negative-definite because $CF(E_8) = CF(E'_7) = CF(E''_6) = 2$.

To classify all the possible Γ_i we need to know the lower bound of each component factor. For $\Gamma_i = k - A_n$, the lower bound is taken when $n = 2k+1$:

$$CF(k - A_n) \geq k + 1 - \frac{k+1}{2} = \frac{k+1}{2}.$$

Thus the smallest number of component factor of $k - A_n$ is $1/2$, and is taken when $k = 0, n = 1$. The lower bound of other graphs are listed below:

$$CF(k - D_n) \geq 1, CF(D'_n) \geq \frac{5}{4}, CF(D''_n) \geq 2.$$

Case 3. $s = 2$.

THEOREM 3.11. *Let $s = 2$, when $\Gamma_1 = k_1 - A_{n_1}$, then $\Gamma_1 + \Gamma_2$ must be one of the following:*

(1) $(k_1 - A_{n_1}) + (k_2 - A_{n_2})$:

$$\frac{(k_1+1)^2}{n_1+1} + \frac{(k_2+1)^2}{n_2+1} > k_1 + k_2 - 2.$$

(2) $(k_1 - A_{n_1}) + (k_2 - D_{n_2})$:

$$\frac{(k_1+1)^2}{n_1+1} > k_1 + k_2 - 2.$$

(3) $(k_1 - A_{n_1}) + (D'_{n_2})$:

$$\frac{(k_1+1)^2}{n_1+1} > k_1 + \frac{n_2}{4} - 3.$$

(4) $(k_1 - A_{n_1}) + (D''_{n_2})$:
If $n_2 = 4, k_1 = 0, 1$, then n_1 can be arbitrary.
If $n_2 = 4, k_1 = 2$, then $n_1 = 5, 6, 7, 8$.
If $n_2 = 5$, then $k_1 = 0$ and n_1 arbitrary.

(5) $(k_1 - A_{n_1}) + (E_6)$:
If $k_1 = 0, 1,$ n_1 *can be arbitrary.*
If $k_1 = 2,$ *then* $5 \le n_1 \le 26.$

(6) $(k_1 - A_{n_1}) + (E_7)$:
If $k_1 = 0, 1,$ n_1 *can be arbitrary.*
If $k_1 = 2,$ *then* $5 \le n_1 \le 17.$

(7) $(k_1 - A_{n_1}) + (E_8, E_7', E_6'')$:
If $k_1 = 0, 1,$ n_1 *can be arbitrary.*
If $k_1 = 2,$ *then* $n_1 = 5, 6, 7, 8.$

(8) $(A_1) + (1 - E_6)$.

Proof. We have already known that $CF(A_1) = 1/2$, thus when $\Gamma_1 = k_1 - A_{n_1}$, we only require

$$CF(\Gamma_2) < 4 - CF(A_1) = 4 - 1/2 = 7/2,$$

which gives (1) to (8). (1) to (3) are simply component factor inequality. We discuss (4) to (8).
(4): If $n_2 = 4$, then it is same as (7) (See below). If $n_2 = 5$, then

$$\frac{(k_1+1)^2}{n_1+1} > k_1 + n_2 - 5 = k_1,$$

which holds only when $k_1 = 0$. And in this case, n_1 can be arbitrary.
(5): The component factor inequality shows

$$\frac{(k_1+1)^2}{n_1+1} > k_1 - \frac{5}{3}.$$

Thus if $k_1 = 0, 1$, n_1 can be arbitrary. If $k_1 = 2$, then

$$\frac{9}{n_1+1} > \frac{1}{3},$$

thus $5 \le n_1 < 27$.
(6): The component factor inequality shows

$$\frac{(k_1+1)^2}{n_1+1} > k_1 - \frac{3}{2}.$$

Thus if $k_1 = 0, 1$, n_1 can be arbitrary. If $k_1 = 2$, then

$$\frac{9}{n_1+1} > \frac{1}{2},$$

thus $5 \le n_1 < 18$.
(7): The component factor inequality shows

$$\frac{(k_1+1)^2}{n_1+1} > k_1 - 1.$$

Thus if $k_1 = 0, 1$, n_1 can be arbitrary. If $k_1 = 2$, then

$$\frac{9}{n_1+1} > 1,$$

thus $n_1 = 5, 6, 7, 8$.
(8): $CF(1 - E_6) = 10/3 > 3$, thus $k_1 = 0$. The component factor inequality shows

$$\frac{1}{n_1 + 1} > \frac{1}{3}.$$

So $n_1 = 1$. □

THEOREM 3.12. *Let $s = 2$, when $\Gamma_1 = k_1 - D_{n_1}$ and $\Gamma_2 \neq k_2 - A_{n_2}$, then $\Gamma_1 + \Gamma_2$ must be one of the following:*
(1) $(D_{n_1}) + (k_2 - D_{n_2}) : k_2 = 0, 1.$
(2) $(k_1 - D_{n_1}) + (D'_{n_2}) :$
$k_1 = 0,\ 5 \leq n_2 \leq 11.$
$k_1 = 1,\ 5 \leq n_2 \leq 7.$
(3) $(D_{n_1}) + (D''_4).$
(4) $(k_1 - D_{n_1}) + (E_6, E_7) : k_1 = 1, 2.$
(5) $(D_{n_1}) + (E_8, E'_7, E''_6).$

Proof. $CF(k_1 - D_{n_1}) = k_1 + 1$, thus the bound of k_1 is $4 - CF(\Gamma_2)$. And $CF(\Gamma_2) < 4 - CF(D_n) = 3$, which gives (1) to (5). □

THEOREM 3.13. *Let $s = 2$, when $\Gamma_1 = D'_{n_1}$ and $\Gamma_2 \neq k_2 - A_{n_2}$ or $k_2 - D_{n_2}$, then $\Gamma_1 + \Gamma_2$ must be one of the following:*
(1) $(D'_{n_1}) + (D'_{n_2}) :$

$$16 > n_1 + n_2.$$

(2) $(D'_{n_1}) + (E_6) : 5 \leq n_1 \leq 10.$
(3) $(D'_{n_1}) + (E_7) : 5 \leq n_1 \leq 9.$
(4) $(D'_{n_1}) + (D''_4, E_8, E'_7, E''_6) : n_1 = 5, 6, 7, 8.$

THEOREM 3.14. *Let $s = 2$, besides Theorem 3.11, Theorem 3.12 and Theorem 3.13, the rest $\Gamma_1 + \Gamma_2$ must be one of the following:*
(1) $(D''_4) + (E_6, E_7)$:
(2) $(E_6) + (E_6, E_7, E_8, E'_7, E''_6).$
(3) $(E_7) + (E_7, E_8, E'_7, E''_6)$.

Case 4. $s = 3$.

THEOREM 3.15. *Let $s = 3$, when $\Gamma_1 = k_1 - A_{n_1}$ and $\Gamma_2 = k_2 - A_{n_2}$, then $\Gamma_1 + \Gamma_2 + \Gamma_3$ must be one of the following:*
(1) $(k_1 - A_{n_1}) + (k_2 - A_{n_2}) + (k_3 - A_{n_3})$: $k_1 + k_2 + k_3 \leq 4$.
If $k_1 + k_2 + k_3 \leq 1$, then n_i can be arbitrary. If else, then n_i must satisfy

$$\sum_{i=1}^{3} \frac{(k_i + 1)^2}{n_i + 1} > -1 + \sum_{i=1}^{3} k_i.$$

(2) $(k_1 - A_{n_1}) + (k_2 - A_{n_2}) + (k_3 - D_{n_3})$: $k_1 + k_2 + 2k_3 \leq 3$.
If $k_1 + k_2 + k_3 \leq 1$, then n_i can be arbitrary. If else, then n_i must satisfy

$$\sum_{i=1}^{2} \frac{(k_i + 1)^2}{n_i + 1} > -1 + \sum_{i=1}^{3} k_i.$$

(3) $(k_1 - A_{n_1}) + (k_2 - A_{n_2}) + (D'_{n_3})$: $2k_1 + 2k_2 + n_3 \leq 11$.
If $4k_1 + 4k_2 + n_3 \leq 8$*, then* n_i *can be arbitrary. If else, then* n_i *must satisfy:*

$$\sum_{i=1}^{2} \frac{(k_i+1)^2}{n_i+1} > -2 + \frac{n_3}{4} + \sum_{i=1}^{2} k_i.$$

(4) $(k_1 - A_{n_1}) + (k_2 - A_{n_2}) + (D''_{n_3})$: $k_1 + k_2 + n_3 \leq 9$.. *If* $k_1 = k_2 = 0, n_3 = 4$*, then* n_i *can be arbitrary. If else, then* n_i *must satisfy*

$$\sum_{i=1}^{2} \frac{(k_i+1)^2}{n_i+1} > n_3 - 4 + \sum_{i=1}^{2} k_i.$$

(5) $(k_1 - A_{n_1}) + (k_2 - A_{n_2}) + (E_6)$: $k_1 + k_2 \leq 3$.
If $k_1 = k_2 = 0$*, then* n_i *can be arbitrary. If else, then* n_i *must satisfy*

$$\sum_{i=1}^{2} \frac{(k_i+1)^2}{n_i+1} > -\frac{2}{3} + \sum_{i=1}^{2} k_i.$$

(6) $(k_1 - A_{n_1}) + (k_2 - A_{n_2}) + (E_7)$: $k_1 + k_2 \leq 2$.
If $k_1 = k_2 = 0$*, then* n_i *can be arbitrary. If else, then* n_i *must satisfy*

$$\sum_{i=1}^{2} \frac{(k_i+1)^2}{n_i+1} > -\frac{1}{2} + \sum_{i=1}^{2} k_i.$$

(7) $(k_1 - A_{n_1}) + (k_2 - A_{n_2}) + (E_8, E'_7, E''_6)$: $k_1 + k_2 \leq 1$.
If $k_1 = k_2 = 0$*, then* n_i *can be arbitrary. If else, then* n_i *must satisfy*

$$\sum_{i=1}^{2} \frac{(k_i+1)^2}{n_i+1} > \sum_{i=1}^{2} k_i.$$

Proof. (1): By computing component factor we get:

$$\sum_{i=1}^{3} \frac{(k_i+1)^2}{n_i+1} > -1 + \sum_{i=1}^{3} k_i.$$

If $k_1 + k_2 + k_3 \leq 1$, then n_i can be arbitrary. The rest is to discuss the bound of k_i. The lower bound of $CF(k_i - A_{n_i})$ is $(k_i + 1)/2$ when $n_i = 2k_i + 1$. In this case, the inequality can be exchanged to

$$4 > \sum_{i=1}^{3} \frac{k_i+1}{2}.$$

Thus

$$\sum_{i=1}^{3} k_i \leq 4.$$

(2): Similar as (1), by computing component factor we get:

$$\sum_{i=1}^{2} \frac{(k_i+1)^2}{n_i+1} > -1 + \sum_{i=1}^{3} k_i.$$

if $k_1+k_2+k_3 \le 1$ then n_i can be arbitrary. Now consider the lower bound of $CF(k_i - A_{n_i})$ we get

$$4 > \frac{k_1+1}{2} + \frac{k_2+1}{2} + k_3 + 1,$$

i.e.

$$k_1+k_2+2k_3 \le 3.$$

(3): By computing component factor we get:

$$\sum_{i=1}^{2} \frac{(k_i+1)^2}{n_i+1} > -2 + \frac{n_3}{4} + \sum_{i=1}^{2} k_i.$$

Consider the lower bound of $CF(k_i - A_{n_i})$ we get

$$4 > \frac{k_1+1}{2} + \frac{k_2+1}{2} + \frac{n_3}{4},$$

i.e.

$$2k_1+2k_2+n_3 \le 11.$$

(4): By computing component factor we get: $(k_1 - A_{n_1}) + (k_2 - A_{n_2}) + (D''_{n_3})$:

$$\sum_{i=1}^{2} \frac{(k_i+1)^2}{n_i+1} > n_3 - 4 + \sum_{i=1}^{2} k_i.$$

Consider the lower bound of $CF(k_i - A_{n_i})$ we get

$$4 > \frac{k_1+1}{2} + \frac{k_2+1}{2} + n_3 - 2,$$

i.e.

$$k_1+k_2+n_3 \le 9.$$

(5): By computing component factor we get:

$$\sum_{i=1}^{2} \frac{(k_i+1)^2}{n_i+1} > -\frac{2}{3} + \sum_{i=1}^{2} k_i.$$

Consider the lower bound of $CF(k_i - A_{n_i})$ we get

$$4 > \frac{k_1+1}{2} + \frac{k_2+1}{2} + \frac{4}{3},$$

i.e.

$$k_1+k_2 \le 3.$$

(6): By computing component factor we get:

$$\sum_{i=1}^{2}\frac{(k_i+1)^2}{n_i+1} > -\frac{1}{2}+\sum_{i=1}^{2}k_i.$$

Consider the lower bound of $CF(k_i - A_{n_i})$ we get

$$4 > \frac{k_1+1}{2}+\frac{k_2+1}{2}+\frac{3}{2},$$

i.e.

$$k_1+k_2 \leq 2.$$

(7): By computing component factor we get:

$$\sum_{i=1}^{2}\frac{(k_i+1)^2}{n_i+1} > \sum_{i=1}^{2}k_i.$$

Consider the lower bound of $CF(k_i - A_{n_i})$ we get

$$4 > \frac{k_1+1}{2}+\frac{k_2+1}{2}+2,$$

i.e.

$$k_1+k_2 \leq 1.$$

$\square$

THEOREM 3.16. *Let $s=3$, when $\Gamma_1 = k_1 - A_{n_1}$ and $\Gamma_2 = k_2 - D_{n_2}$, then $\Gamma_1+\Gamma_2+\Gamma_3$ must be one of the following:*

(1) *$(k_1 - A_{n_1}) + (k_2 - D_{n_2}) + (k_3 - D_{n_3})$: $k_1 \leq 2, k_2 = k_3 = 0$ or $k_1 = k_3 = 0, k_2 = 1$.*
If $k_1 = 2$ then $n_1 = 5,6,7$, n_2, n_3 can be arbitrary.
If else, all n_i can be arbitrary.

(2) *$(k_1 - A_{n_1}) + (k_2 - D_{n_2}) + (D'_{n_3})$: $2k_1 + 4k_2 + n_3 \leq 9$, $n_3 \geq 5$. And*

$$\frac{(k_1+1)^2}{n_1+1} > -2+\frac{n_3}{4}+\sum_{i=1}^{2}k_i.$$

(3) *$(A_{n_1}) + (D_{n_2}) + (D''_4)$: n_1, n_2 can be arbitrary.*

(4) *$(k_1 - A_{n_1}) + (k_2 - D_{n_2}) + (E_6)$: $k_1 = 0,1,2$, $k_2 = 0$ or $k_1 = 0$, $k_2 = 1$. And*

$$\frac{(k_1+1)^2}{n_1+1} > -\frac{2}{3}+\sum_{i=1}^{2}k_i.$$

(5) *$(k_1 - A_{n_1}) + (k_2 - D_{n_2}) + (E_7)$: $k_1 = 0,1$, $k_2 = 0$. And*

$$\frac{(k_1+1)^2}{n_1+1} > -\frac{1}{2}+k_1.$$

(6) *$(A_{n_1}) + (D_{n_2}) + (E_8, E'_7, E''_6)$: n_1, n_2 can be arbitrary.*

Proof. (1): The lower bound of $CF(k_1 - A_{n_1})$ shows that

$$4 > \frac{k_1+1}{2} + k_2 + 1 + k_3 + 1,$$

i.e.

$$k_1 + 2k_2 + 2k_3 \leq 2.$$

Thus it holds only when $k_1 \leq 2, k_2 = k_3 = 0$ or $k_1 = k_3 = 0, k_2 = 1$. The component factor shows that

$$\frac{(k_1+1)^2}{n_1+1} > -1 + \sum_{i=1}^{3} k_i.$$

Note when $k_1 + k_2 + k_3 \leq 1$ then n_i can be arbitrary. When $k_1 = 2, k_2 = k_3 = 0$, then

$$\frac{9}{n_1+1} > 1,$$

i.e. $n_1 < 8$. Thus $5 = 2k_1 + 1 \leq n_1 \leq 7$.
(2): The lower bound shows

$$4 > \frac{k_1+1}{2} + k_2 + 1 + \frac{n_3}{4},$$

i.e.

$$9 \geq 2k_1 + 4k_2 + n_3.$$

The component factor gives

$$\frac{(k_1+1)^2}{n_1+1} > -2 + \frac{n_3}{4} + \sum_{i=1}^{2} k_i.$$

Note $n_3 \geq 5$, thus

$$2k_1 + 4k_2 \leq 4.$$

Only $k_1 = 0, 1, 2,\ k_2 = 0$ or $k_1 = 0,\ k_2 = 1$ is permitted.
(3): Consider $(k_1 - A_{n_1}) + (k_2 - D_{n_2}) + (D''_{n_3})$. The lower bound shows

$$4 > \frac{k_1+1}{2} + k_2 + 1 + n_3 - 2,$$

i.e.

$$9 > k_1 + 2k_2 + 2n_3$$

However, $n_3 \geq 4$, thus $n_3 = 4, k_1 = k_2 = 0$. The component factor gives

$$\frac{(k_1+1)^2}{n_1+1} > -2 + n_3 - 2 + \sum_{i=1}^{2} k_i,$$

i.e.

$$\frac{1}{n_1+1} > -2+2$$

which always holds.
(4): The lower bound shows

$$4 > \frac{k_1+1}{2} + k_2 + 1 + \frac{4}{3},$$

i.e.

$$\frac{7}{3} > k_1 + 2k_2.$$

Thus $k_1 = 0, 1, 2$, $k_2 = 0$ or $k_1 = 0$, $k_2 = 1$.
(5): The lower bound shows

$$4 > \frac{k_1+1}{2} + k_2 + 1 + \frac{3}{2},$$

i.e.

$$2 > k_1 + 2k_2.$$

Thus $k_1 = 0, 1$, $k_2 = 0$.
(6): This is same as (3). □

THEOREM 3.17. *Let $s = 3$, when $\Gamma_1 = k_1 - A_{n_1}$ and $\Gamma_2 = D'_{n_2}$, then $\Gamma_1 + \Gamma_2 + \Gamma_3$ must be one of the following:*

(1) $(k_1 - A_{n_1}) + (D'_{n_2}) + (D'_{n_3})$: $k_1 = 0, 1$.
$2k_1 + n_2 + n_3 \leq 13$, $n_2, n_3 \geq 5$. *And*

$$\frac{(k_1+1)^2}{n_1+1} > -3 + \frac{n_2+n_3}{4} + k_1.$$

(2) $(A_{n_1}) + (D'_5) + (D''_4)$: $n_1 = 1, 2$.
(3) $(k_1 - A_{n_1}) + (D'_{n_2}) + (E_6)$: $k_1 = 0, 1$.
$2k_1 + n_2 \leq 8$, $n_2 \geq 5$. *And*

$$\frac{(k_1+1)^2}{n_1+1} > \frac{n_2}{4} - \frac{8}{3} + k_1.$$

(4) $(k_1 - A_{n_1}) + (D'_{n_2}) + (E_7)$: $k_1 = 0, 1$.
$2k_1 + n_2 \leq 7$, $n_2 \geq 5$. *And*

$$\frac{(k_1+1)^2}{n_1+1} > \frac{n_2}{4} - \frac{3}{2} + k_1.$$

(5) $(A_{n_1}) + (D'_5) + (E_8, E'_7, E''_6)$: $n_1 = 1, 2$.

Proof. (1): The lower bound shows

$$4 > \frac{k_1+1}{2} + \frac{n_2+n_3}{4},$$

i.e.

$$14 > 2k_1 + n_2 + n_3.$$

$n_2, n_3 \geq 5$, thus $k_1 = 0, 1$.
(2): Consider $(k_1 - A_{n_1}) + (D'_{n_2}) + (D''_{n_3})$. The lower bound shows

$$4 > \frac{k_1+1}{2} + \frac{n_2}{4} + n_3 - 2,$$

$n_2 \geq 5, n_3 \geq 4$, thus $k_1 = 0$, $n_2 = 5$, $n_3 = 4$. Then component factor inequality shows that

$$\frac{1}{n_1+1} > -4 + 2 + \frac{5}{4} + 1,$$

thus $n_1 = 1, 2$.
(3): The lower bound shows

$$4 > \frac{k_1+1}{2} + \frac{n_2}{4} + \frac{4}{3},$$

i.e.

$$\frac{26}{3} > 2k_1 + n_2.$$

Then component factor inequality shows that

$$\frac{(k_1+1)^2}{n_1+1} > -3 + \frac{n_2}{4} + \frac{4}{3} + k_1.$$

(4): The lower bound shows

$$4 > \frac{k_1+1}{2} + \frac{n_2}{4} + \frac{3}{2},$$

i.e.

$$8 > 2k_1 + n_2.$$

Thus $k_1 = 0, 1$. Then component factor inequality shows that

$$\frac{(k_1+1)^2}{n_1+1} > -3 + \frac{n_2}{4} + \frac{3}{2} + k_1.$$

(5): The same as (2). □

THEOREM 3.18. *Let $s = 3$, when $\Gamma_1 = k_1 - A_{n_1}$, besides Theorem 3.15, Theorem 3.16 and Theorem 3.17, the rest $\Gamma_1 + \Gamma_2 + \Gamma_3$ must be one of the following:*
(1) $(A_1) + (E_6) + (D''_4, E_8, E'_7, E''_6)$.
(2) $(k_1 - A_{n_1}) + (E_6) + (E_6)$: $k_1 = 0, 1$.
If $k_1 = 0$, then n_1 can be arbitrary.
If $k_1 = 1$, then $n_1 = 3, 4, 5$.

(3) $(k_1 - A_{n_1}) + (E_6) + (E_7)$: $k_1 = 0, 1$.
If $k_1 = 0$, *then* n_1 *can be arbitrary.*
If $k_1 = 1$, *then* $n_1 = 3$.
(4) $(A_{n_1}) + (E_7) + (E_7)$: n_1 *can be arbitrary.*

Proof. (1): Consider $(k_1 - A_{n_1}) + (E_6) + (D_4'', E_8, E_7', E_6'')$. The lower bound shows that

$$4 > \frac{k_1+1}{2} + 2 + \frac{4}{3}.$$

Thus $k_1 = 0$. The component factor inequality gives that

$$\frac{(k_1+1)^2}{n_1+1} + 4 > 2 + \frac{4}{3} + k_1 + 1.$$

Thus $n_1 = 1$.
(2): The lower bound shows that

$$4 > \frac{k_1+1}{2} + \frac{8}{3}.$$

Thus $k_1 = 0, 1$. The component factor inequality gives that if $k_1 = 0$, then n_1 can be arbitrary, if $k_1 = 1$ then

$$\frac{4}{n_1+1} + 4 > \frac{8}{3} + 1 + 1.$$

$n_1 \geq 2k_1 + 1$, thus when $k_1 = 1$, $n_1 = 3, 4, 5$.
(3): The lower bound shows that

$$4 > \frac{k_1+1}{2} + \frac{4}{3} + \frac{3}{2}.$$

Thus $k_1 = 0, 1$. The component factor inequality gives that if $k_1 = 0$, then n_1 can be arbitrary, if $k_1 = 1$ then

$$\frac{4}{n_1+1} + 4 > \frac{4}{3} + \frac{3}{2} + 1 + 1.$$

$n_1 \geq 2k_1 + 1$, thus when $k_1 = 1$, $n_1 = 3$.
(4): Similar as (3). □

THEOREM 3.19. *Let* $s = 3$, *when* $\Gamma_1 = k_1 - D_{n_1}$, *and* $\Gamma_2 \neq k_2 - A_{n_2}, \Gamma_3 \neq k_3 - A_{n_3}$, *then* $\Gamma_1 + \Gamma_2 + \Gamma_3$ *must be one of the following:*
(1) $(D_{n_1}) + (D_{n_2}) + (D_{n_3})$.
(2) $(D_{n_1}) + (D_{n_2}) + (D'_{n_3})$: $n_3 = 5, 6, 7$.
(3) $(D_{n_1}) + (D_{n_2}) + (E_6, E_7)$.
(4) $(D_{n_1}) + (D'_5) + (D'_{n_3})$: $n_3 = 5, 6$.
(5) $(D_{n_1}) + (D'_{n_2}) + (E_6)$: $n_2 = 5, 6$.
(6) $(D_{n_1}) + (D'_5) + (E_7)$.
(7) $(D_{n_1}) + (E_6) + (E_6, E_7)$.

Proof. Firstly by computing component factor of $(k_1 - D_{n_1}) + (k_2 - D_{n_2}) + (k_3 - D_{n_3})$ we know

$$4 > k_1 + 1 + k_2 + 1 + k_3 + 1.$$

Thus $k_1 = k_2 = k_3 = 0$.

For $\Gamma_i \neq k_i - A_{n_i}$, we have $CF(\Gamma_i) \geq CF(D_{n_i}) = 1$. Thus for $\Gamma_1 + \Gamma_2 + \Gamma_3$ with $\Gamma_i \neq k_i - A_{n_i}$, if some $\Gamma_i = k_i - D_{n_i}$, then $k_i = 0$. The rest is to compute component factor, we omit here. □

THEOREM 3.20. *Let $s = 3$, when $\Gamma_1 = D'_{n_1}$, and $\Gamma_2 \neq k_2 - A_{n_2}$ or $k_2 - D_{n_2}$, then $\Gamma_1 + \Gamma_2 + \Gamma_3$ must be one of the following:*

(1) $(D'_5) + (D'_5) + (D'_5)$.

(2) $(D'_5) + (D'_5) + (E_6)$.

(3) $(D'_5) + (E_6) + (E_6)$.

Proof. The criteria shows that above three are permitted. One may consider $D'_5 + E_6 + E_7$, $D'_5 + D'_5 + E_7$, $D'_5 + D'_5 + D'_6$ which are not permitted. □

Later we will not emphasize on the inequality induced by component factor if there is no further conclusions.

Case 5. $s = 4$.

THEOREM 3.21. *Let $s = 4$, then there must be some i such that $\Gamma_i = k_1 - A_{n_i}$. Assume $\Gamma_1 = k_1 - A_{n_1}$, then $\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ must be one of the following:*

(1) $(A_{n_1}) + (k_2 - A_{n_2}) + (k_3 - A_{n_3}) + (k_4 - A_{n_4})$:

$$\frac{1}{n_1 + 1} + \sum_{i=2}^{4} \frac{(k_i + 1)^2}{n_i + 1} > \sum_{i=2}^{4} k_i.$$

(2) $(A_{n_1}) + (k_2 - A_{n_2}) + (k_3 - A_{n_3}) + (k_4 - D_{n_4})$:

$$\frac{1}{n_1 + 1} + \sum_{i=2}^{3} \frac{(k_i + 1)^2}{n_i + 1} > \sum_{i=2}^{4} k_i.$$

(3) $(A_{n_1}) + (k_2 - A_{n_2}) + (k_3 - A_{n_3}) + (D'_{n_4})$:

$$\frac{1}{n_1 + 1} + \sum_{i=2}^{3} \frac{(k_i + 1)^2}{n_i + 1} > \sum_{i=2}^{3} k_i + \frac{n_4}{4} - 1.$$

(4) $(A_{n_1}) + (k_2 - A_{n_2}) + (k_3 - A_{n_3}) + (E_6)$:

$$\frac{1}{n_1 + 1} + \sum_{i=2}^{3} \frac{(k_i + 1)^2}{n_i + 1} > \sum_{i=2}^{3} k_i + \frac{1}{3}.$$

(5) $(A_{n_1}) + (k_2 - A_{n_2}) + (k_3 - A_{n_3}) + (E_7)$:

$$\frac{1}{n_1 + 1} + \sum_{i=2}^{3} \frac{(k_i + 1)^2}{n_i + 1} > \sum_{i=2}^{3} k_i + \frac{1}{2}.$$

(6) $(A_{n_1}) + (A_{n_2}) + (A_{n_3}) + (D''_4, E_8, E'_7, E''_6)$:

$$\sum_{i=1}^{3} \frac{1}{n_i + 1} > 1.$$

(7) $(A_{n_1})+(k_2-A_{n_2})+(k_3-D_{n_3})+(D'_{n_4})$:

$$\frac{1}{n_1+1}+\frac{(k_2+1)^2}{n_2+1}>\sum_{i=2}^{3}k_i+\frac{n_4}{4}-1.$$

(8) $(A_{n_1})+(k_2-A_{n_2})+(k_3-D_{n_3})+(E_6)$:

$$\frac{1}{n_1+1}+\frac{(k_2+1)^2}{n_2+1}>\sum_{i=2}^{3}k_i+\frac{1}{3}.$$

(9) $(A_{n_1})+(A_{n_2})+(D_{n_3})+(E_7)$:

$$\sum_{i=1}^{2}\frac{1}{n_i+1}>\frac{1}{2}.$$

(10) $(A_{n_1})+(A_{n_2})+(D'_{n_3})+(D'_{n_4})$:

$$\sum_{i=1}^{2}\frac{1}{n_i+1}>k_2+\frac{n_3+n_4}{4}-2.$$

(11) $(A_{n_1})+(A_{n_2})+(D'_{n_3})+(E_6)$:

$$\sum_{i=1}^{2}\frac{1}{n_i+1}>\frac{n_3}{4}-\frac{2}{3}.$$

(12) $(A_{n_1})+(A_{n_2})+(D'_{n_3})+(E_7)$:

$$\sum_{i=1}^{2}\frac{1}{n_i+1}>\frac{n_3}{4}-\frac{1}{2}.$$

(13) $(A_{n_1})+(A_{n_2})+(E_6)+(E_6)$:

$$\sum_{i=1}^{2}\frac{1}{n_i+1}>\frac{2}{3}.$$

(14) $(A_{n_1})+(A_{n_2})+(E_6)+(E_7)$:

$$\sum_{i=1}^{2}\frac{1}{n_i+1}>\frac{5}{6}.$$

(15) $(A_{n_1})+(D_{n_2})+(D_{n_3})+(D_{n_4})$.

(16) $(A_{n_1})+(D_{n_2})+(D_{n_3})+(D'_{n_4})$:

$$\frac{1}{n_1+1}>\frac{n_4}{4}-1.$$

(17) $(A_{n_1})+(D_{n_2})+(D_{n_3})+(E_6)$: $n_1=1,2$.

Proof. If $\Gamma_i \neq A_{n_i}$ for all $i = 1, 2, 3, 4$. Then the lower bound of $CF(\Gamma_i)$ will give

$$\sum_{i=1}^{4} CF(\Gamma_i) > 1 + 1 + 1 + 1 = 4,$$

which means negative definiteness is not satisfied. Thus there must be at least one $\Gamma_i = A_{n_i}$. With out loss of generality let $\Gamma_1 = A_{n_1}$. Note the lower bound of $CF(A_{n_i}) = 1/2$, when $n_i = 1$. Thus the criteria tells us

$$CF(\Gamma_2) + CF(\Gamma_3) + CF(\Gamma_4) < \frac{7}{2}.$$

Thus, for example, $A_{n_1} + D_{n_2} + D'_{n_3} + D'_{n_4}$ is not permitted. Next, we consider $\Gamma_2 = k_2 - A_{n_2}$. If $k_2 = 1$, then we must require

$$CF(\Gamma_3) + CF(\Gamma_4) < 4 - CF(A_1) - CF(1 - A_3) = \frac{5}{2},$$

Thus when $CF(\Gamma_3) + CF(\Gamma_4) \geq 5/2$, $k_2 = 0$. This gives above all cases. □

Case 6. $s = 5$.

THEOREM 3.22. *Let $s = 5$, then $\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5$ must be one of the following*

(1) $(A_{n_1}) + (A_{n_2}) + (A_{n_3}) + (k_4 - A_{n_4}) + (k_5 - A_{n_5})$*:*

$$\sum_{i=1}^{3} \frac{1}{n_i + 1} + \sum_{i=4}^{5} \frac{(k_i + 1)^2}{n_i + 1} > k_4 + k_5 + 1.$$

(2) $(A_{n_1}) + (A_{n_2}) + (A_{n_3}) + (k_4 - A_{n_4}) + (k_5 - D_{n_5})$*:*

$$\sum_{i=1}^{3} \frac{1}{n_i + 1} + \frac{(k_4 + 1)^2}{n_4 + 1} > k_4 + k_5 + 1.$$

(3) $(A_{n_1}) + (A_{n_2}) + (A_{n_3}) + (k_4 - A_{n_4}) + (D'_{n_5})$*:*

$$\sum_{i=1}^{3} \frac{1}{n_i + 1} + \frac{(k_4 + 1)^2}{n_4 + 1} > k_4 + \frac{n_5}{4}.$$

(4) $(A_{n_1}) + (A_{n_2}) + (A_{n_3}) + (k_4 - A_{n_4}) + (E_6)$*:*

$$\sum_{i=1}^{3} \frac{1}{n_i + 1} + \frac{(k_4 + 1)^2}{n_4 + 1} > k_4 + \frac{4}{3}.$$

(5) $(A_{n_1}) + (A_{n_2}) + (A_{n_3}) + (k_4 - A_{n_4}) + (E_7)$*:*

$$\sum_{i=1}^{3} \frac{1}{n_i + 1} + \frac{(k_4 + 1)^2}{n_4 + 1} > k_4 + \frac{3}{2}.$$

(6) $(A_{n_1}) + (A_{n_2}) + (A_{n_3}) + (D_{n_4}) + (D_{n_5})$*:*

$$\sum_{i=1}^{3} \frac{1}{n_i + 1} > 1.$$

(7) $(A_{n_1}) + (A_{n_2}) + (A_{n_3}) + (D_{n_4}) + (D'_{n_5})$:

$$\sum_{i=1}^{3} \frac{1}{n_i+1} > \frac{n_5}{4}.$$

(8) $(A_{n_1}) + (A_{n_2}) + (A_{n_3}) + (D_{n_4}) + (E_6)$:

$$\sum_{i=1}^{3} \frac{1}{n_i+1} > \frac{4}{3}.$$

Proof. Similar as the discussion in $s = 4$ case, we can assume $\Gamma_1 = A_{n_1}, \Gamma_2 = A_{n_2}$, $\Gamma_3 = A_{n_3}$. The rest is to consider component factor, which we omit here. □

Case 7. $s = 6$.

THEOREM 3.23. *Let $s = 6$, then $\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5 + \Gamma_6$ must be one of the following:*

(1) $(A_{n_1}) + (A_{n_2}) + (A_{n_3}) + (A_{n_4}) + (A_{n_5}) + (k_6 - A_{n_6})$: $k_6 = 0, 1$.

$$\sum_{i=1}^{5} \frac{1}{n_i+1} + \frac{(k_6+1)^2}{n_6+1} > 2 + k_6.$$

(2) $(A_{n_1}) + (A_{n_2}) + (A_{n_3}) + (A_{n_4}) + (A_{n_5}) + (k_6 - D_{n_6})$:

$$\sum_{i=1}^{5} \frac{1}{n_i+1} > 2 + k_6.$$

(3) $(A_{n_1}) + (A_{n_2}) + (A_{n_3}) + (A_{n_4}) + (A_{n_5}) + (D'_{n_6})$:

$$\sum_{i=1}^{5} \frac{1}{n_i+1} > 1 + \frac{n_6}{4}.$$

(4) $(A_{n_1}) + (A_{n_2}) + (A_{n_3}) + (A_{n_4}) + (A_{n_5}) + (E_6)$:

$$\sum_{i=1}^{5} \frac{1}{n_i+1} > \frac{7}{3}.$$

Proof. Similar as above we can show there must be at least five A_{n_i} in $\Gamma_1, ..., \Gamma_6$. Thus

$$CF(\Gamma_6) < 1 - 5 \cdot CF(A_1) = \frac{5}{2},$$

which gives (1) to (4). □

Case 8. $s = 7$.

THEOREM 3.24. *When $s = 7$, then $\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5 + \Gamma_6 + \Gamma_7$ must be one of the following:*

$A_{n_1} + A_{n_2} + A_1 + A_1 + A_1 + A_1 + A_1$: *$n_2 = 1$,$n_1$ arbitrary or $n_2 = 2, n_1 = 2, 3, 4$.*

Proof. It is easy to show that all Γ_i must be A_{n_i}. The component factor inequality shows that

$$\sum_{i=1}^{7} \frac{1}{n_i+1} > 3.$$

Without loss of generality we can assume $n_1 \geq n_2 \geq ... \geq n_7$. Thus

$$\frac{7}{n_7+1} \geq 3.$$

This means $n_7 = 1$. Take $n_7 = 1$ into inequality we get

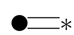
$$\sum_{i=1}^{6} \frac{1}{n_i+1} > \frac{5}{2}.$$

This means $n_6 = 1$. Repeat this argument we stops at

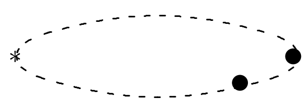
$$\frac{1}{n_1+1} + \frac{1}{n_2+1} > \frac{1}{2}.$$

Thus $n_2 = 1$, n_1 arbitrary or $n_2 = 2, n_1 = 2, 3, 4$. □

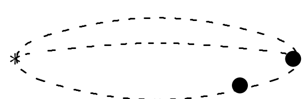
4. Component factor of non-tree graphs and classfication. In this section, we explain the definition of component factor for loop graph and multiple edge graphs. Then we generalize the criteria for tree graph and use it to classify all possible the non-tree graphs. Similarly as some $E_j^2 = -3$ presented in [28], the following two cases are allowed:

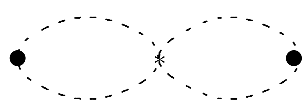


and



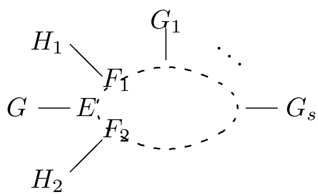
We first consider loop case, it is easy to show that



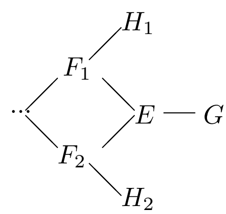


are not negative definite. Thus we only need to consider one loop case.

Notation. Denote the total graph as Γ. Let E be a point in loop. Denote the two points in loop connected to E as F_1, F_2. Denote the tree subgraphs connected to loop as $G, H_1.H_2, G_1, ..., G_s$, where G is connected to E and H_1, H_2 are connected to F_1, F_2 respectively. Removing point E and subgraph G to get a tree graph, we denote it as R_E. Removing point E and E_i together with G and H_i to get two treegraphs, we denote it as R_i, $i = 1, 2$. Denote $E - G$ as E_G.
Total graph Γ:



Total graph Γ: (We omit G_i, $i = 1, ..., s$.)



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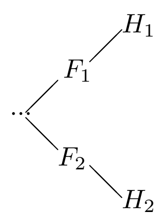
R_E:



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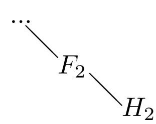
R_1:



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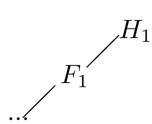
R_2:



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E_G:

$$E - G$$

.

THEOREM 4.1 (Loop determinant formula). *Assumptions as in Notation. Then*

$$\begin{aligned} det(\Gamma) =& det(R_E)det(E_G) - det(G)det(R_1)det(H_1) - det(G)det(R_2)det(H_2) \\ & - (-1)^L \cdot 2det(G)det(H_1)det(H_2)\prod_{i=1} det(G_i) \end{aligned}$$

where L is the number of points in the loop.

Proof. We begin with the simplist case that $G, G_i = \emptyset$, and F_1, F_2 are connected. Then

$$\Gamma = \begin{pmatrix} E^2 & 1 & 1 & 0 & 0 \\ 1 & F_1 & 1 & 1 & 0 \\ 1 & 1 & F_2 & 0 & 1 \\ 0 & 1 & 0 & H_1 & 0 \\ 0 & 0 & 1 & 0 & H_2 \end{pmatrix}.$$

Using Laplacian expansion we get

$$det(\Gamma) = E^2 det(R_E) - det \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & F_2 & 0 & 1 \\ 0 & 0 & H_1 & 0 \\ 0 & 1 & 0 & H_2 \end{pmatrix} + det \begin{pmatrix} 1 & F_1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & H_1 & 0 \\ 0 & 0 & 0 & H_2 \end{pmatrix}$$

While

$$det(\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & F_2 & 0 & 1 \\ 0 & 0 & H_1 & 0 \\ 0 & 1 & 0 & H_2 \end{pmatrix}) = det(H_1)det(\begin{pmatrix} 1 & 1 & 0 \\ 1 & F_2 & 1 \\ 0 & 1 & H_2 \end{pmatrix})$$

$$= det(R_1)det(H_1) - det(H_1)det(H_2).$$

Similarly, the third term equals

$$det(R_2)det(H_2) - det(H_1)det(H_2).$$

Replace E^2 by $det(E_G)$, the formula holds.
Now we turn to the case that F_1, F_2 are connected by one point, say A. And G_1 is connected to A outside the loop, i.e.

$$\Gamma = \begin{pmatrix} G & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & E^2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & F_1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & A & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & F_2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & H_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & H_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & G_1 \end{pmatrix}.$$

Use Laplacian expansion on E_G we get

$$\begin{aligned} det(\Gamma) =& det(E_G)det(R_E) \\ & - det\begin{pmatrix} G & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & A & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & F_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & H_1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & H_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & G_1 \end{pmatrix} - det\begin{pmatrix} G & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & A & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & F_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & H_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & H_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & G_1 \end{pmatrix}. \end{aligned}$$

The second term is

$$\begin{aligned} & det\begin{pmatrix} G & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & A & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & F_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & H_1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & H_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & G_1 \end{pmatrix} = det(G)det\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & A & 1 & 0 & 0 & 1 \\ 1 & 1 & F_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & H_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & H_2 & 0 \\ 0 & 1 & 0 & 0 & 0 & G_1 \end{pmatrix} \\ & = det(G)det(H_1)det\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & A & 1 & 0 & 1 \\ 1 & 1 & F_2 & 1 & 0 \\ 0 & 0 & 1 & H_2 & 0 \\ 0 & 1 & 0 & 0 & G_1 \end{pmatrix}. \end{aligned}$$

Expand on first colomn we get

$$\begin{aligned} & det\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & A & 1 & 0 & 1 \\ 1 & 1 & F_2 & 1 & 0 \\ 0 & 0 & 1 & H_2 & 0 \\ 0 & 1 & 0 & 0 & G_1 \end{pmatrix} = det\begin{pmatrix} A & 1 & 0 & 1 \\ 1 & F_2 & 1 & 0 \\ 0 & 1 & H_2 & 0 \\ 1 & 0 & 0 & G_1 \end{pmatrix} + det\begin{pmatrix} 1 & 0 & 0 & 0 \\ A & 1 & 0 & 1 \\ 0 & 1 & H_2 & 0 \\ 1 & 0 & 0 & G_1 \end{pmatrix} \\ & = det(R_1) + det(H_2)det(G_1). \end{aligned}$$

Take them into $det(\Gamma)$ we get the formula.
The last thing is to illustrate the $(-1)^L$. Note when F_1, F_2 is connected by $L-3$ points, then the expansion

$$\begin{aligned} & det\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & A & 1 & 0 & 1 \\ 1 & 1 & F_2 & 1 & 0 \\ 0 & 0 & 1 & H_2 & 0 \\ 0 & 1 & 0 & 0 & G_1 \end{pmatrix} = det\begin{pmatrix} A & 1 & 0 & 1 \\ 1 & F_2 & 1 & 0 \\ 0 & 1 & H_2 & 0 \\ 1 & 0 & 0 & G_1 \end{pmatrix} + det\begin{pmatrix} 1 & 0 & 0 & 0 \\ A & 1 & 0 & 1 \\ 0 & 1 & H_2 & 0 \\ 1 & 0 & 0 & G_1 \end{pmatrix} \\ & = det(R_1) + det(H_2)det(G_1) \end{aligned}$$

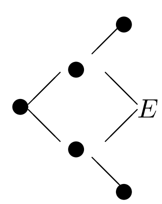
changes to

$$
\begin{aligned}
& det \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & A & 1 & 0 & 1 \\ 1 & 1 & F_2 & 1 & 0 \\ 0 & 0 & 1 & H_2 & 0 \\ 0 & 1 & 0 & 0 & G_1 \end{pmatrix} \\
&= det \begin{pmatrix} A & 1 & 0 & 1 \\ 1 & F_2 & 1 & 0 \\ 0 & 1 & H_2 & 0 \\ 1 & 0 & 0 & G_1 \end{pmatrix} + (-1)^{(L-2)} det \begin{pmatrix} 1 & 0 & 0 & 0 \\ A & 1 & 0 & 1 \\ 0 & 1 & H_2 & 0 \\ 1 & 0 & 0 & G_1 \end{pmatrix} \\
&= det(R_1) + (-1)^L det(H_2) det(G_1).
\end{aligned}
$$

When $s \geq 1$, i.e. $G_1, ..., G_s$ connected to A, it does not affect this expansion. Thus the formula is proved. □

REMARK 4.2. The determinant of $R_E, E_G, R_i, H_i, G, G_i$ can be computed by using tree graph determinant formula (Theorem 3.3). Compared to tree graph formula, there exists an extra term $2det(G)det(H_1)det(H_2)\prod_{i=1} det(G_i)$.

EXAMPLE 4.3. Consider the weighted dual graph:



,

where $E^2 = -4$. Then $H_1 = H_2 = A_1$, $R_E = A_5$, $R_1 = R_2 = A_3$. Thus

$$det(\Gamma) = (-4) \cdot (-6) - 2 \cdot (-2) \cdot (-4) - 2 \cdot -2 \cdot -2 = 0.$$

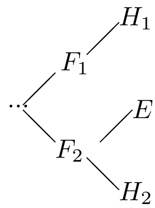
THEOREM 4.4 (Criteria for loop graph). *Assumptions as in Notation. Assume furthermore R_E and G are negative-definite. Then Γ is negative definite if and only if:*

$$E^2 + CF(G) + |\frac{det(R_1)det(H_1)}{det(R_E)}| + |\frac{det(R_2)det(H_2)}{det(R_E)}| + 2|\frac{det(H_1)det(H_2)\prod_{i=1} det(G_i)}{det(R_E)}| < 0.$$

Proof. Notice that $|det(E_G)| = |E^2 \cdot det(G) \cdot CF(G)|$. Thus dividing $|det(G)det(R_E)|$, we know the only if part holds.

For the if part, we construct a rational cycle D such that $D \cdot E < 0$ and $D \cdot E_i = 0$ for any $E_i \neq E$ exceptional curve.

First remove the connection between E and F_1 and together remove G, we get a tree graph:



.

Use the construction on tree graph (cf. Lemma 3.2) we get a rational cycle D_2 such that $D_2 \cdot E < 0$. $D_1 \cdot E_i = 0$. And the coefficient of D_2 on F_2 is

$$|\frac{det(R_2)det(H_2)}{det(R_E)}|.$$

We need to compute the coefficient of D_2 on F_1. If all G_i are empty then the induction of coefficient tells us the coefficient of D_2 on F_1 is

$$|\frac{det(H_1)det(H_2)}{det(R_E)}|.$$

Now if G_i is not empty then G_i will add a term on numerator, i.e. the coefficient of D_2 on F_1 is

$$|\frac{det(H_1)det(H_2)\prod_{i=1} det(G_i)}{det(R_E)}|.$$

Similarly we construct D_1 by removing G and the connection between E and F_2. The coefficient of D_1 on F_1 is

$$|\frac{det(R_1)det(H_1)}{det(R_E)}|,$$

on F_2 is

$$|\frac{det(H_1)det(H_2)\prod_{i=1} det(G_i)}{det(R_E)}|.$$

Let D_3 be the rational cycle constructed on E_G.
Let $D = D_1 + D_2 + D_3 - 2E$. $D \cdot E_i = 0$ for any $E_i \neq F_1, F_2, E$ because $D_1, D_2, D_3, E \cdot E_i = 0$ if $E_i \neq F_1, F_2, E$. Meanwhile,

$$D \cdot F_1 = (D_1 + D_2 + D_3 - 2E) \cdot F_1 = 0 + 1 + 0 - 1 = 0.$$

So is $D \cdot F_2$. At last

$$\begin{aligned} D \cdot E =& E^2 + CF(G) + |\frac{det(R_1)det(H_1)}{det(R_E)}| + |\frac{det(R_2)det(H_2)}{det(R_E)}| \\ &+ 2|\frac{det(H_1)det(H_2)\prod_{i=1} det(G_i)}{det(R_E)}| < 0. \end{aligned}$$

Thus Γ is negative-definite by Proposition 2.5. □

Definition 4.5 (Component factor for loop graph). Assumptions as in notation. The component factor of R_E is defined to be

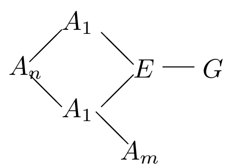
$$CF(R_E) = |\frac{det(R_1)det(H_1)}{det(R_E)}| + |\frac{det(R_2)det(H_2)}{det(R_E)}| + 2|\frac{det(H_1)det(H_2)\prod_{i=1} det(G_i)}{det(R_E)}|.$$

Notice that $CF(R_E)$ only depends on the graph and connection way of R_E with E.

COROLLARY 4.6. *Assume R_E and G are negative-definite. The loop graph is negative-definite if and only if*

$$E^2 + CF(R_E) + CF(G) < 0.$$

THEOREM 4.7. *The loop graph must be one of the following:*
(1) *R_E is A-type:*



$m = 4, 5,\ n = 0,\ G = \emptyset.$
$m = 3,\ n = 1, 2,\ G = \emptyset.$
$m = 2,\ G =$

$\emptyset;$
$A_{n_2},\ n = 0, n_2 = 1, 2, 3$ *or* $n = 1, 2,\ n_2 = 1.$

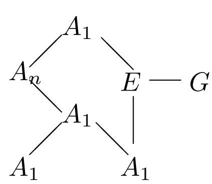
$m = 1,\ G =$

$\emptyset;$
$D_{n_2};\quad A_{n_2};\quad 1 - A_{n_2},$ *if* $n_2 = 4$ *then* $n = 0,$ *if* $n_2 = 3$ *then* $n \geq 0$ *can be arbitrary.*
$A_{n_2} + A_{n_3},\ n_2 = n_3 = 1,\ n \geq 0$ *or* $n_2 = 3,\ n_3 = 2,\ n = 0, 1.$

$m = 0,\ G =$

$\emptyset;$
$D_{n_2};\quad 1 - A_{n_2};\quad E_6, E_7;\quad 2 - A_{n_2}, n_2 \leq 7;\quad D_{n_2'}, n_2 \leq 7.$
$(A_{n_2}) + (A_{n_3});\quad (A_{n_2}) + (1 - A_3)/(D_{n_3});\quad (A_{n_2}) + (D_{n_3});\quad (A_1) + (E_6)/(E_7);$
$(1 - A_{n_2}) + (A_1), n_2 \leq 6;\quad (1 - A_4) + (A_{n_2}), n_2 \leq 3;\quad (D_5') + (A_{n_2}), n_2 \leq 2.$
$(A_1) + (A_1) + (A_{n_3});\quad (A_1) + (A_2) + (A_{n_2}),\ n_2 \leq 4.$

(2) *R_E is D-type:*
$G = \emptyset.$
$G = A_{n_2},\ n = 0, n_2 \geq 1$ *or* $n = 1, n_2 \leq 2.$



Proof. Let E be the -4 point. Then R_E must be ADE graph. We first discuss cases in the condition that $G = \emptyset$, i.e. $E_G = E$.

Step 1: R_E is A-type.
Consider following graph:

$$R_E = A_{n+2}, H_1 = H_2 = A_1, G_i = \emptyset.$$

i.e.

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & A_1 \\
 & A_1 \arrow[ur, dash] \arrow[dr, dash] & \\
A_n \arrow[ur, dash] \arrow[dr, dash] & & E \\
 & A_1 \arrow[ur, dash] \arrow[dr, dash] & \\
 & & A_1
\end{tikzcd}\quad .$$

Then $D := \sum 1 \cdot E_i$ is the fundamental cycle with $D \cdot E_i = 0$ for any exceptional curve E_i. Thus this graph has determinant 0, which is not negative-definite.
When R_E is A-type, next we consider the following:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & A_1 \arrow[dr, dash] & \\
A_n \arrow[ur, dash] \arrow[dr, dash] & & E \\
 & A_1 \arrow[ur, dash] \arrow[dr, dash] & \\
 & & A_m
\end{tikzcd}\quad .$$

Then

$$R_E = A_{m+n+2}, H_1 = \emptyset, H_2 = A_m, G = G_i = \emptyset, R_1 = A_{m+n+1}, R_2 = A_{n+1}.$$

If Γ is negative-definite, then if $m \geq 1$.

$$-4(m+n+3) + (m+n+2) + (n+2)(m+1) + 2(m+1) < 0,$$

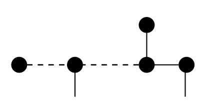
i.e.

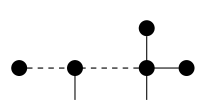
$$m - 2n + mn - 6 < 0.$$

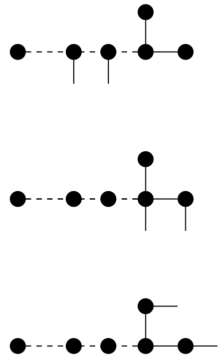
Thus

$$\begin{aligned}
&m = 0, 1, 2, \ n \geq 0. \\
&m = 3, \ n \leq 2. \\
&m = 4, 5, \ n = 0.
\end{aligned}$$

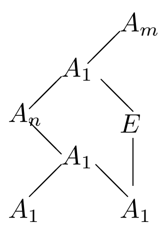
Step 2: R_E is D-type.
In general $R_E = D_n$ has 5 possibilities:







Where the lines imply the connection to -4 point.
In fact only the first one is permitted. Consider the following graph:



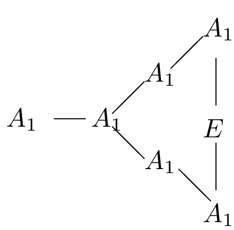
Then

$$R_E = D_{m+n+4}, R_1 = D_{n+3}, R_2 = A_{m+n+3}, H_1 = A_m, H_2 = A_1.$$

If Γ is negative-definite, then

$$-4 \cdot 4 + 4 \cdot (m+1) + (m+n+4) + 2 \cdot 2 \cdot (m+1) < 0.$$

Thus $m = 0, n \leq 3$. The rest cases are not permitted by computing similarly.
Step 3: R_E is E-type.
We use $R_E = E_6$ as an example, the rest are similar. Consider the following graph:

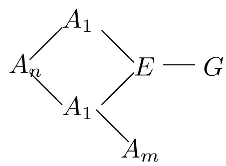


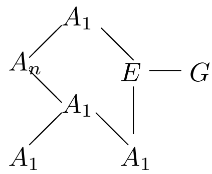
Then

$$det(\Gamma) = -4 \cdot det(E_6) - 2 \cdot det(D_5) - 2 \cdot det(A_1) = 0.$$

If we change the connection way of E with $R_E = E_6$, then $-2 \cdot det(D_5) - 2 \cdot det(A_1)$ will be exchanged for larger value terms. Thus $R_E = E_6$ is not permitted. Similarly

for $R_E = E_7$ and E_8.
Step 4: G is not empty.
By Step 1,2 and 3, we know that the only possibilities for G nonempty are the followings:





We use Corollary 4.6 to determine G. First we consider the case R_E is A-type.

$$|det(R_E)| = m+n+3, |det(R_1)| = m+n+2, |det(R_2)| = n+2, |det(H_2)| = m+1.$$

This shows

$$CF(R_E) = \frac{mn+5m+2n+6}{m+n+3} = \frac{mn+3m}{m+n+3} + 2 = \frac{-m^2}{m+n+3} + m + 2.$$

We can compute $CF(R_E)$ of cases listed in Step 1:

$$\begin{aligned}
&m=0,\ CF(R_E) = 2.\\
&m=1,\ CF(R_E) = 3 - \frac{1}{n+4}.\\
&m=2,\ CF(R_E) = 4 - \frac{4}{n+5}.\\
&m=3,\ n=1,\ CF(R_E) = 5 - \frac{9}{3+1+3} = 4 - \frac{2}{7}.\\
&m=3,\ n=2,\ CF(R_E) = 5 - \frac{9}{3+2+3} = 4 - \frac{1}{8}.\\
&m=4,\ n=0,\ CF(R_E) = 6 - \frac{16}{7} = 4 - \frac{2}{7}.\\
&m=5,\ n=0,\ CF(R_E) = 7 - \frac{25}{8} = 4 - \frac{1}{8}.
\end{aligned}$$

Note $CF(A_1) = 1/2$, thus $m = 3, 4, 5$ cannot connect more subgraphs.
When $m = 2$, then $G = A_{n_2}$ and

$$4 - \frac{4}{n+5} + 1 - \frac{1}{n_2+1} - 4 < 0,$$

i.e.

$$1 < \frac{4}{n+5} + \frac{1}{n_2+1}.$$

Thus $n = 0, n_2 = 1, 2, 3$ or $n = 1, 2,\ n_2 = 1$.
When $m = 1$, then $CF(G) < 1 + 1/(n + 4)$. Thus $G = D_{n_2}$ is satisfied.
When $G = k_2 - A_{n_2}$ then

$$k_2 + 1 - \frac{(k_2 + 1)^2}{n_2 + 1} < 1 + \frac{1}{n + 4}.$$

So $k_2 = 0$ is always satisfied. When $k_2 = 1$ then

$$1 < \frac{1}{n + 4} + \frac{4}{n_2 + 1}.$$

Note $n_2 \geq k_2 + 1 = 3$, thus $n = 0, n_2 = 4$ or $n_2 = 3, n \geq 0$. $k_2 \geq 2$ is not permitted by lower bound of $CF(k_2 - A_{n_2}) \geq 3/2 > 1 + 1/(n + 4)$.
When $G = A_{n_2} + A_{n_3}$ then

$$1 < \frac{1}{n + 4} + \frac{1}{n_2 + 1} + \frac{1}{n_3 + 1},$$

i.e. $n_2 = n_3 = 1,\ n \geq 0$ or $n_2 = 3,\ n_3 = 2,\ n = 0, 1$.
The case $m = 0$ is the same as tree graph with some $\Gamma_i = (E_6'', E_7', E_8)$, we list them in the theorem.
Next we consider the case R_E is D-type.

$$|det(R_E)| = 4, |det(R_1)| = 4, |det(R_2)| = n + 4, |det(G_1)| = 2.$$

This shows

$$CF(R_E) = \frac{4 + n + 4 + 2 \cdot 2}{4} = \frac{n}{4} + 3.$$

Thus $CF(G) < 1 - \frac{n}{4}$, G must be A_{n_2} and

$$\frac{1}{n_2 + 1} > \frac{n}{4}.$$

So

$$n = 0,\ n_2 \geq 1.$$
$$n = 1,\ n_2 \leq 2.$$

□

Next we consider multiple edge graph. The criteria for multiple edge graph is similar. To be more specific, we present it as follows:

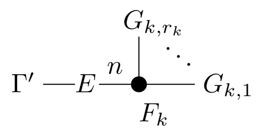
Definition 4.8 (Component factor for multiple edge). Let G be a subgraph connected to E with the multiplicity equals n. Denote this connection way as $G-^n$. Define

$$CF(G-^n) = n^2 CF(G).$$

Theorem 4.9 (General criteria for negative definiteness). *Let Γ be a weighted dual graph. Let Γ_i be subgraphs connected to E such that Γ_i is negative-definite. Then Γ is negative-definite if and only if*

$$E^2 + \sum_i CF(\Gamma_i) < 0.$$

Proof. We have already illustrate the cases when Γ_i is tree graph or loop graph in Corollary 3.8 and Corollary 4.6. Note component factor only depends on the connection way and the graph of Γ_i, thus we only need to show the case for multiple edge graph. This is a direct observation of Laplacian expansion.
Assume the weighted dual graph is:



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Where $-^n$ denotes the multiplicity is n, i.e. Γ_k connects E with multiplicity n. The intersection matrix can be represented as :

$$\begin{pmatrix} \Gamma' & 1 & 0 & 1 & \dots & 1 \\ 1 & E^2 & n & 0 & \dots & 0 \\ 0 & n & F_k^2 & 1 & \dots & 1 \\ 0 & 0 & 1 & G_{k,1} & \dots & 0 \\ 0 & 0 & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 1 & 0 & \dots & G_{k,r_k} \end{pmatrix}.$$

$$\begin{aligned} det(\Gamma) &= det(\begin{pmatrix} \Gamma' & 1 \\ 1 & E^2 \end{pmatrix})det(\Gamma_k) + (-1)^{(n_k)} \cdot n \cdot det(\begin{pmatrix} \Gamma' & 1 & 1 & \dots & 1 \\ 0 & n & 1 & \dots & 1 \\ 0 & 0 & G_{k,1} & \dots & 0 \\ 0 & 0 & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & G_{k,r_k} \end{pmatrix}) \\ &= det(\begin{pmatrix} \Gamma' & 1 \\ 1 & E^2 \end{pmatrix})det(\Gamma_k) + (-1)^{(n_k)} \cdot n^2 \cdot det(\Gamma') \prod_{l=1}^{r_k} det(G_{k,l}). \end{aligned}$$

Here n_k is the number of points in Γ_k. Compared with the determinant formula for tree graph, Γ_k contributes $n^2 CF(\Gamma_k)$. By definition, $CF(\Gamma_k -^n) = CF(\Gamma_k)$, thus the criteria holds. □

This criteria helps us for classify possible multiple edge graphs with one -4 point:

Theorem 4.10. *If Γ is multiple edge graph, then there exists only one multiple edge with multiplicity 2. Denote the subgraph connected to E with multiplicity 2 as Γ_1, then $\Gamma_1 = A_{n_1}$. And the rest Γ_i 's are the followings:*
$n_1 = 1$*:*

D_{n_2}; $1 - A_{n_2}$; E_6, E_7; $2 - A_{n_2}, n_2 \leq 7$; $D_{n_2'}, n_2 \leq 7$.
$(A_{n_2}) + (A_{n_3})$; $(A_{n_2}) + (1 - A_3)/(D_{n_3})$; $(A_{n_2}) + (D_{n_3})$; $(A_1) + (E_6)/(E_7)$;
$(1 - A_{n_2}) + (A_1), n_2 \leq 6$; $(1 - A_4) + (A_{n_2}), n_2 \leq 3$; $(D_5') + (A_{n_2}), n_2 \leq 2$.
$(A_1) + (A_1) + (A_{n_3})$; $(A_1) + (A_2) + (A_{n_2})$, $n_2 \leq 4$.

$n_1 = 2$*:*

$$1 - A_{n_2}, n_2 = 3, 4; \quad A_{n_2}; \quad D_{n_2}; \quad D_5'.$$

$6 \geq n_1 \geq 3$:

$$A_{n_2}$$

satisfying

$$\frac{4}{n_1+1} > 1 - \frac{1}{n_2+1}.$$

$n_1 \geq 7$, Γ_i 's $= \emptyset$.

Proof. We first discuss the multiplicity. By $CF(\Gamma_i{-}^n) = CF(\Gamma^n)$ and $CF(A_1) = 1/2$ we know $n \leq 2$.
Now let Γ_1 be the subgraph connect -4 point with multiplicity 2. Then

$$CF(\Gamma_1{-}^2) = 4CF(\Gamma_1) < 4,$$

i.e. $CF(\Gamma_1) < 1$. So $\Gamma_1 = A_{n_1}$.
The rest graph must satisfy

$$\sum_{i\geq 2} CF(\Gamma_i) < 4 - 4CF(A_{n_1}) = 4 - 4(1 - \frac{1}{n_1+1}) = \frac{4}{n_1+1}.$$

We discuss n_1.
$n_1 = 1$ then $\sum_{i\geq 2} CF(\Gamma_i) < 2$. Thus Γ_i 's are the following:

$$\begin{aligned}
&D_{n_2};\quad 1-A_{n_2};\quad E_6, E_7;\quad 2-A_{n_2}, n_2 \leq 7;\quad D_{n_2'}, n_2 \leq 7.\\
&(A_{n_2})+(A_{n_3});\quad (A_{n_2})+(1-A_3)/(D_{n_3});\quad (A_{n_2})+(D_{n_3});\quad (A_1)+(E_6)/(E_7);\\
&(1-A_{n_2})+(A_1), n_2 \leq 6;\quad (1-A_4)+(A_{n_2}), n_2 \leq 3;\quad (D_5')+(A_{n_2}), n_2 \leq 2.\\
&(A_1)+(A_1)+(A_{n_3});\quad (A_1)+(A_2)+(A_{n_2}),\ n_2 \leq 4.
\end{aligned}$$

$n_1 = 2$ then $\sum_{i\geq 2} CF(\Gamma_i) < 4/3$. Thus Γ_i 's are the following:

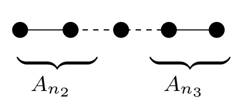
$$1-A_{n_2}, n_2 = 3, 4;\quad A_{n_2};\quad D_{n_2};\quad D_5'.$$

$6 \geq n_1 \geq 3$ then $\sum_{i\geq 2} CF(\Gamma_i) < 1$, thus when $n_1 \geq 3$, Γ_i 's can only be A_{n_2} satisfying

$$\frac{4}{n_1+1} > 1 - \frac{1}{n_2+1}.$$

When $n_1 \geq 7$, then Γ_i 's $= \emptyset$. □

Remark 4.11. In fact, ADE graphs help us to completely classify all possible Γ_i 's such that $\sum_i CF(\Gamma_i) < 2$. We can select any point in ADE graphs to be E and the subgraphs connected to E are the possible Γ_i 's. For example, if we remove a point in the chain of A_n (not the point on sides), then the rest graphs are Γ_i 's, which are $A_{n_2} + A_{n_3}$:



Remove all possible points in ADE graphs we get Γ_i 's are the following:

$$D_{n_2};\ \ 1-A_{n_2};\ \ E_6, E_7;\ \ 2-A_{n_2}, n_2 \leq 7;\ \ D_{n_2'}, n_2 \leq 7.$$
$$(A_{n_2})+(A_{n_3});\ \ (A_{n_2})+(1-A_3)/(D_{n_3});\ \ (A_{n_2})+(D_{n_3});\ \ (A_1)+(E_6)/(E_7);$$
$$(1-A_{n_2})+(A_1), n_2 \leq 6;\ \ (1-A_4)+(A_{n_2}), n_2 \leq 3;\ \ (D_5')+(A_{n_2}), n_2 \leq 2.$$
$$(A_1)+(A_1)+(A_{n_3});\ \ (A_1)+(A_2)+(A_{n_2}),\ n_2 \leq 4.$$

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