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SPECTRAL CONVERGENCE IN GEOMETRIC QUANTIZATION ON $K3$ SURFACES*

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Abstract. We study the geometric quantization on $K3$ surfaces from the viewpoint of the spectral convergence. We take a special Lagrangian fibrations on the $K3$ surfaces and a family of hyper-Kähler structures tending to large complex structure limit, and show a spectral convergence of the $\overline{\partial}$-Laplacians on the prequantum line bundle to the spectral structure related to the set of Bohr-Sommerfeld fibers.

Key words. Geometric quantization, K3 surface, Bohr-Sommerfeld fiber, measured Gromov-Hausdorff convergence.

Mathematics Subject Classification. 53D50, 58C40.

1. Introduction. In this paper we study the geometric quantization on the $K3$ surfaces from the viewpoint of the spectral convergence of the $\overline{\partial}$-Laplacian acting on sections of the prequantum line bundle.

The prequantum line bundle on a symplectic manifold (X, ω) is a triple (L, h, ∇) of a complex line bundle $\pi\colon L \to X$ equipped with a hermitian metric h and a hermitian connection ∇ whose curvature form F^∇ is equal to $-\sqrt{-1}\omega$. The geometric quantization is the procedure to derive the quantum Hilbert space consisting of the regular sections of L in the appropriate sense. To derive it, we consider the Kähler quantization coming from the integrable complex structures and the real quantization coming from the Lagrangian fibrations in this paper.

Let J be an integrable complex structure on X and suppose that it is ω-compatible. Then ω is a Kähler form on the complex manifold $X_J := (X, J)$ and L is a holomorphic line bundle over X_J. The quantum Hilbert space coming from J is defined by

$$V_J := H^0(X_J, L).$$

Next we take a Lagrangian fibration $\mu\colon X \to B$. We suppose that B is a smooth manifold of dimension $\dim X/2$, μ is almost everywhere submersion and $\omega|_{\mu^{-1}(b)} \equiv 0$ for every regular value $b \in B$. By the Lagrangian condition, the restriction of (L, ∇) to every fiber $\mu^{-1}(b)$ is a flat bundle. The fiber $\mu^{-1}(b)$ is called a *Bohr-Sommerfeld fiber* if $(L|_{\mu^{-1}(b)}, \nabla|_{\mu^{-1}(b)})$ has a nontrivial parallel section. We can also define this notion even if b is a critical value. Here, we put

$$BS := \{b \in B \,|\, \mu^{-1}(b) \text{ is a Bohr-Sommerfeld fiber}\},$$
$$V_\mu := \mathbb{C}^{\#BS}.$$

The ω-compatible complex structures and the Lagrangian fibrations can be treated uniformly by the notion of polarizations. Now, suppose that a family of ω-compatible complex structures $\{J_s\}_{s>0}$ is given and it converges to a Lagrangian fibration μ as $s \to 0$ in the sense of polarizations. The aim of this paper is to show the convergence of the quantum Hilbert spaces $V_{J_s} \to V_\mu$ as $s \to 0$. Such a phenomenon has already been observed in the several examples. In [2], Baier, Mourão and Nunes showed such

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convergence on the smooth abelian varieties, and in [1], Baier, Florentino, Mourão and Nunes showed it on the smooth toric varieties. In [11], Hamilton and Konno showed it on the flag manifolds. In these results, they constructed a family of complex structures J_s and the basis $\{\vartheta_{1,s},\dots,\vartheta_{N,s}\}$ of V_{J_s} explicitly then showed that the sections $\vartheta_{i,s}$ converge to the delta function section of L supported by the Bohr-Sommerfeld fibers as $s\to 0$. In [21], Yoshida studied the convergence of the holomorphic sections on the neighborhood of nonsingular fibers of μ by only using the local description of the almost complex structures.

In [14], Yamashita and the author introduced the new approach to this problem. We identified the holomorphic sections of (X_{J_s}, L) with the eigenfunctions of the Laplace operators on some Riemannian manifolds related to J_s and ∇, and then showed the spectral convergence as $s\to 0$ in the sense of Kuwae and Shioya [17]. In [14], we considered the case of μ has only nonsingular fibers and in [15] we considered the case of the smooth toric varieties, then obtained the another proof of the results in [2], [1], respectively.

In this paper we show the convergence $V_{J_s}\to V_\mu$ in the case of the $K3$ surface, where J_s come from the family of hyper-Kähler structures tending to a large complex structure limit in the sense of [10], and μ comes from the elliptic fibration. One of the difficulty to work on the $K3$ surfaces is that we cannot describe the complex structures J_s and the holomorphic sections explicitly, since the hyper-Kähler structures on the $K3$ surfaces are determined by the solutions of the Monge-Ampère equation. However, the method developed in [14] does not require the explicit description of V_{J_s}. We mention that $\dim V_J = \dim V_\mu$ has been proved by Tyurin in the case of the $K3$ surfaces in [19]. Moreover, Chan and Suen constructed the canonical isomorphism $V_J\cong V_\mu$ via the SYZ transforms in the case of the semi-flat Lagrangian torus fibrations over the compact complete special integral affine manifolds and compact toric manifolds [3].

Next we explain the main result. Let X be a smooth manifold of dimension 4 diffeomorphic to the $K3$ surfaces, (g,J_1,J_2,J_3) be a hyper-Kähler structure on X and put $\omega_i := g(J_i\cdot,\cdot)$, then we regard (X,ω_1) as a symplectic manifold. We assume $[\omega_1]\in 2\pi H^2(X,\mathbb{Z})$ and take a prequantum line bundle (L,h,∇) on (X,ω_1). Next we take a family of Kähler forms $(\omega_{3,s})_{s>0}$ on X_{J_3} such that $\omega_{3,s}^2=\omega_1^2=\omega_2^2$. We call $\mu\colon X\to\mathbb{P}^1$ a special Lagrangian fibration if $\mu^{-1}(b)$ is smooth and $\omega_1|_{\mu^{-1}b}=\omega_2|_{\mu^{-1}b}=0$ for every regular value $b\in\mathbb{P}^1$. We assume that μ comes from the elliptic fibration $X_{J_3}\to\mathbb{P}^1$ whose singular fibers are of Kodaira type I_1 and $\lim_{s\to 0}\int_{\mu^{-1}(b)}\omega_{3,s}=0$.

We define $\Delta^1_{\mathbb{R}^2}$ by

$$\Delta^1_{\mathbb{R}^2}\varphi := \sum_{i=1}^2\left(-\frac{\partial^2\varphi}{\partial\xi_i^2}+2\xi_i\frac{\partial\varphi}{\partial\xi_i}\right)$$

for $\varphi\colon\mathbb{R}^2\to\mathbb{C}$.

THEOREM 1.1. *Let $(X,\omega_1,\omega_2,\omega_{3,s})$, $\mu\colon X\to\mathbb{P}^1$ and (L,h,∇) be as above. Then we have the compact convergence of the spectral structures*

$$\left(L^2\left(X,\frac{\omega_1^2}{2s},L,h\right),\,\Delta_{\overline{\partial}_{J_{1,s}}}\right)\to\bigoplus_{b\in BS}\left(L^2\left(\mathbb{R}^2,e^{-\|\xi\|^2}d\xi_1 d\xi_2\right)\otimes\mathbb{C},\,\frac{\Delta^1_{\mathbb{R}^2}}{2}\right)$$

as $s \to 0$ in the sense of Definition 3.3. Moreover, if we denote by

$$P_s\colon L^2\left(X, \frac{\omega_1^2}{2s}, L, h\right) \to H^0(X_{J_{1,s}}, L),$$
$$P_0\colon \bigoplus_{b\in BS} L^2\left(\mathbb{R}^2, e^{-\|\xi\|^2} d\xi_1 d\xi_2\right) \to \bigoplus_{b\in BS} \mathbb{C}$$

the orthogonal projections to the 0*-eigenspaces, then we have the compact convergence of the bounded operators*

$$P_s \to P_0$$

as $s \to 0$ in the sense of Definition 3.5.

By the Kodaira Vanishing Theorem, Theorem 1.1 implies the next corollary, which has been obtained by Tyurin [19].

Corollary 1.2. *Let (X, g, J_1, J_2, J_3) be a $K3$ surface equipped with a hyper-Kähler structure, $\mu\colon X \to \mathbb{P}^1$ be a special Lagrangian fibration coming from the elliptic fibration $X_{J_3} \to \mathbb{P}^1$ with 24 singular fibers of Kodaira type I_1. Let $[\omega_1]/2\pi \in H^2(X, \mathbb{Z})$ and (L, h, ∇) be a prequantum line bundle on (X, ω_1). Then we have*

$$\dim H^0(X_{J_1}, L) = \#BS.$$

This paper is organized as follows. In Section 2, we review fundamentals of the hyper-Kähler structures on the $K3$ surfaces and describe the setting of this paper. Moreover, we see that the holomorphic sections on L can be identified with some eigenfunctions on the frame bundle of (L, h) equipped with some Riemannian metrics. In Section 3, we review the convergence of the spectral convergence following [17] and the notion of the S^1-equivariant pointed measured Gromov-Hausdorff topology following [13]. In Section 4 we describe the main results of this paper and the outline of the proof. In Subsection 4.3, we explain how to construct the approximation map between the frame bundle of (L, h) and the limit spaces. In Section 5, we study the family of hyper-Kähler structures on the $K3$ surfaces tending to the large complex structure limit. It is known by Gross and Wilson [10] that such structures are approximated by gluing the standard semi-flat metrics and the Ooguri-Vafa metrics. We modify their argument to apply to our situation, then we may reduce the problem to the local argument on the standard semi-flat metrics and the Ooguri-Vafa metrics. In Section 6 we study the detail of the former metric and in Section 7 we consider the latter one, then we obtain the strong convergence of the spectral structures. To show the compact convergence of the spectral structures in Theorem 1.1, we need further argument for the localization of the functions on $\mathbb{S}$, which is discussed in Section 8 following [14]. In Section 9, we show the convergence of the quantum Hilbert spaces and obtain the latter half of Theorem 1.1.

Notations.

- For a Riemannian manifold (X, g), denote by d_g the Riemannian distance and denote by ν_g the Riemannian measure. For a piecewise smooth path $c\colon [0, 1] \to X$, denote by $\mathfrak{L}_g(c)$ the length of c with respect to g.

- For a metric space (X, d) or a Riemannian manifold (X, g), denote by $B(p, r)$ the open metric ball of radius $r > 0$ centered at $p \in X$. If we need to emphasize the dependence on d or g, we also write $B_d(p, r)$ or $B_g(p, r)$. For a subset $A \subset X$, we denote the diameter of A by

$$\mathrm{diam}(A) := \sup\{d(p, q);\, p, q \in A\}.$$

  We write $\mathrm{diam}_d(A)$ or $\mathrm{diam}_g(A)$ when we emphasize the dependence on d or g.
- For sets A, B and points $a \in A$, $b \in B$, denote by $f\colon (A, a) \to (B, b)$ the map $f\colon A \to B$ such that $f(a) = b$.

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2. Geometric quantization on the $K3$ surfaces.

2.1. Hyper-Kähler structures.

Definition 2.1. Let X be a smooth manifold of dimension $4d$. *A hyper-Kähler structure on* X is a quadruple (g, J_1, J_2, J_3) of a Riemannian metric g and integrable complex structures J_i with

$$J_1J_2 = J_3, \quad J_2J_3 = J_1, \quad J_3J_1 = J_2, \quad g(J_i\cdot, J_i\cdot) = g,$$

such that every 2-form $\omega_i := g(J_i\cdot, \cdot)$ is closed. Then (X, g, J_1, J_2, J_3) is called the hyper-Kähler manifold and g is called the hyper-Kähler metric.

Remark 2.2. If (X, g, J_1, J_2, J_3) is a hyper-Kähler manifold, then ω_i is a Kähler form on the complex manifold

$$X_{J_i} := (X, J_i).$$

Moreover, if $(i, j, k) = (1, 2, 3), (2, 3, 1)$ or $(3, 1, 2)$, then $\omega_j + \sqrt{-1}\omega_k$ is a holomorphic volume form, i.e., nondegenerate holomorphic 2-form on X_{J_i}.

Remark 2.3. If (X, g, J_1, J_2, J_3) is a hyper-Kähler manifold of dimension 4, then we have

$$\omega_i \wedge \omega_j = \delta_{ij} vol \tag{1}$$

for some nowhere vanishing 4-form vol on X. Conversely, if 2-forms $\omega_1, \omega_2, \omega_3$ on smooth 4-manifold satisfy (1), then it recovers the hyper-Kähler structure on X after reordering the forms. For this reason the triple $(\omega_1, \omega_2, \omega_3)$ is also called the hyper-Kähler structure on X.

2.2. Holomorphic sections and eigenfunctions. Let (X, ω) be a symplectic manifold. A *prequantum line bundle* $(\pi\colon L \to X, h, \nabla)$ *on* (X, ω) is a complex line bundle L over X with a hermitian metric h and a hermitian connection ∇ whose curvature form F^∇ is equal to $-\sqrt{-1}\omega$. If we consider the prequantum line bundle on a hyper-Kähler manifold $(X, \omega_1, \omega_2, \omega_3)$, then we always suppose F^∇ is equal to $-\sqrt{-1}\omega_1$ in this paper. Then L is a holomorphic line bundle over X_{J_1}. The aim of this paper is to analyze the behavior of the vector space $H^0(X_{J_1}, L)$ consisting of the holomorphic sections under fixing ω_1 and varying J_1. To achieve it, we use the correspondence between holomorphic sections and some eigenfunctions on a Riemannian manifold constructed by (X, g) and (L, h, ∇), which was also considered in [13], [14] and [15].

Let $(X, \omega_1, \omega_2, \omega_3)$ be a hyper-Kähler manifold of dimension 4. First of all put

$$\mathbb{S} = \mathbb{S}(L, h) := \{u \in L;\ h(u, u) = 1\}.$$

Notice that $\mathbb{S}$ is a principal S^1-bundle over X and the connection ∇ induces the horizontal distribution $H \subset T\mathbb{S}$. Denote by $\sqrt{-1}\Gamma^\nabla \in \Omega^1(\mathbb{S}, \sqrt{-1}\mathbb{R})$ the connection form on $\mathbb{S}$ corresponding to ∇. Then we have a S^1-invariant Riemannian metric $\hat{g}$ on $\mathbb{S}$ such that

$$\hat{g} := (\Gamma^\nabla)^2 + (d\pi|_H)^* g. \tag{2}$$

Denote by $C^\infty(X, L)$ the set of smooth sections of L. There is the natural identification

$$\begin{aligned} C^\infty(X, L^k) &\cong (C^\infty(\mathbb{S}) \otimes \mathbb{C})^{\rho_k} \\ &:= \left\{f \in C^\infty(\mathbb{S}) \otimes \mathbb{C};\ f(u\lambda) = \lambda^{-k} f(u) \text{ for all } u \in \mathbb{S},\ \lambda \in S^1\right\}, \end{aligned} \tag{3}$$

where $\rho_k \colon S^1 \to S^1$ is the unitary representation defined by $\rho_k(\lambda) = \lambda^k$. Let $\Delta_{\overline{\partial}_{J_1}} = \nabla^*_{\overline{\partial}_{J_1}} \nabla_{\overline{\partial}_{J_1}}$ is the $\overline{\partial}$-Laplacian acting on $C^\infty(L)$. We can also define the $\overline{\partial}$-Laplacian $\Delta_{k,\overline{\partial}_{J_1}}$ acting on $C^\infty(L^k)$. Denote by $\Delta_{\hat{g}}$ the Laplacian of $\hat{g}$ acting on $C^\infty(\mathbb{S})$, then it extends to the operator on $C^\infty(\mathbb{S}) \otimes \mathbb{C}$ $\mathbb{C}$-linearly. Since S^1 acts on $(\mathbb{S}, \hat{g})$ isometrically, $\Delta_{\hat{g}}$ induces

$$\Delta_{\hat{g}}^{\rho_k} \colon (C^\infty(\mathbb{S}) \otimes \mathbb{C})^{\rho_k} \to (C^\infty(\mathbb{S}) \otimes \mathbb{C})^{\rho_k}.$$

By [13], we can see

$$2\Delta_{k,\overline{\partial}_{J_1}} = \Delta_{\hat{g}}^{\rho_k} - \left(k^2 + 2k\right) \tag{4}$$

under the identification (3). Consequently, if X is compact, we have the following isomorphism

$$H^0(X_{J_1}, L^k) \cong \left\{f \in (C^\infty(\mathbb{S}) \otimes \mathbb{C})^{\rho_k};\ \Delta_{\hat{g}} f = \left(k^2 + 2k\right) f\right\}.$$

2.3. Special Lagrangian fibrations. Let (X, ω) be a symplectic manifold. In this paper we say that $\mu \colon X \to B$ is a *Lagrangian fibration* if μ is a surjective smooth proper map from X to a smooth manifold B of dimension $(\dim X)/2$ such that $\mu^{-1}(b)$ are Lagrangian submanifolds, namely, $\omega|_{\mu^{-1}(b)} = 0$, for all regular values $b \in B$, and we also suppose all of the fibers are connected. Then by the Liouville-Arnold theorem, for all regular values b, the fibers $\mu^{-1}(b)$ are diffeomorphic to the torus.

Let $(X, \omega_1, \omega_2, \omega_3)$ be a hyper-Kähler manifold of dimension 4. In this paper $\mu \colon X \to B$ is said to be a *special Lagrangian fibration* if it is the Lagrangian fibration with respect to both of ω_1 and ω_2.

Since the condition $\omega_1|_{\mu^{-1}(b)} = \omega_2|_{\mu^{-1}(b)} = 0$ is equivalent to that $\mu^{-1}(b)$ is complex submanifold of X_{J_3}, hence the special Lagrangian fibration on X is the elliptic fibration on X_{J_3}.

Remark 2.4. Harvey and Lawson showed in [12] that the special Lagrangian submanifold minimizes the volume in its homology class, therefore, the volume of $\mu^{-1}(b)$ is independent of b. By the above argument, $\mu^{-1}(b)$ is a complex submanifold of X_{J_3}, hence the volume of $\mu^{-1}(b)$ is given by $\int_{\mu^{-1}(b)} \omega_3$ by choosing the orientation appropriately.

The inverse image of a critical value of μ is called a singular fiber. The singular fibers of elliptic fibrations are classified by Kodaira. In particular, we suppose in this paper that all of the singular fibers are of Kodaira type I_1, which is the irreducible rational curve with a double point.

Definition 2.5. Let $\mu\colon X \to B$ be a special Lagrangian fibrations on the 4-dimensional hyper-Kähler manifold X. In this paper we say μ *is of Kodaira type* I_1 if all of the singular fibers of the corresponding elliptic fibration on X_{J_3} are of Kodaira type I_1.

Next we define the Bohr-Sommerfeld fibers.

Definition 2.6. Let (X, ω) be a symplectic manifold with a prequantum line bundle (L, h, ∇) and a Lagrangian fibration $\mu\colon X \to B$.

(i) $\mu^{-1}(b)$ is called a *Bohr-Sommerfeld fiber* if the holonomy group of the connection $\nabla|_{\mu^{-1}(b)}$ on $L|_{\mu^{-1}(b)}$ is trivial. Moreover we call b a *Bohr-Sommerfeld point.*

(ii) Let m be a positive integer. We also denote by ∇ the connection on $L^m := L^{\otimes m}$ naturally induced by ∇ on L. $\mu^{-1}(b)$ is called a *Bohr-Sommerfeld fiber of level* m if the holonomy group of the connection $\nabla|_{\mu^{-1}(b)}$ on $L^m|_{\mu^{-1}(b)}$ is trivial. We put

$$BS_m := \left\{ b \in B;\ \mu^{-1}(b) \text{ is a Bohr-Sommerfeld fiber of level } m \right\},$$
$$BS_m^{\mathrm{str}} := BS_m \setminus \left(\bigcup_{l=1}^{m-1} BS_l \right).$$

Remark 2.7. Notice that we can define the holonomy group not only for the smooth fibers, but also for the singular fibers.

2.4. $K3$ surfaces.

Definition 2.8. *A* $K3$ *surface* is a compact simply-connected hyper-Kähler manifold of dimension 4.

If the $K3$ surface admits an elliptic fibration $\mu\colon X \to B$, then it is known that B is the complex projective line $\mathbb{P}^1$. Moreover, if all of the singular fibers of μ are of Kodaira type I_1, then the number of singular fibers is equal to 24, which is the Euler characteristic of the $K3$ surface.

Definition 2.9. Let $s_0 > 0$, X be a $K3$ surface and $(\omega_1, \omega_2, \omega_{3,s})$ be a family of hyper-Kähler structures on X for every $0 < s \le s_0$. Suppose a special Lagrangian fibration $\mu\colon X \to \mathbb{P}^1$ of Kodaira type I_1 is given. Then $(\omega_1, \omega_2, \omega_{3,s})$ is *tending to a large complex structure limit* if the volume of the fibers of μ converges to 0 as $s \to 0$.

Remark 2.10. In Definition 2.9, we do not assume that $\{\omega_{3,s}\}$ continuously depends on s.

Here we show an example of a family of hyper-Kähler structures on the $K3$ surface tending to a large complex structure limit.

PROPOSITION 2.11. *Let $(\omega_1, \omega_2, \omega_3)$ be a hyper-Kähler structure on the $K3$ surface X, $\mu\colon X \to \mathbb{P}^1$ be a special Lagrangian fibration of Kodaira type I_1, and ω_{FS} be the Fubini-Study form on $\mathbb{P}^1$ normalized such that $\int_X \omega_3 \wedge \mu^*\omega_{FS} = 1$. Put $v_X = \int_X \omega_3^2$. Define the cohomology class $\alpha_s \in H^2(X, \mathbb{R})$ by*

$$\alpha_s := \left[s\left(\omega_3 - \frac{v_X}{2}\mu^*\omega_{FS}\right) + \frac{1}{s}\frac{v_X}{2}\mu^*\omega_{FS} \right].$$

Then there exists a unique Kähler form $\omega_{3,s} \in \alpha_s$ such that $(\omega_1, \omega_2, \omega_{3,s})$ form hyper-Kähler structures tending to a large complex structure limit.

Proof. Let (g, J_1, J_2, J_3) be the hyper-Kähler structure corresponding to $(\omega_1, \omega_2, \omega_3)$. Since α_s is represented by positive $(1,1)$ form on X_{J_3} if $0 < s \le 1$, then there is a Kähler form $\omega_{3,s} \in \Omega^{1,1}(X_{J_3})$ such that $\omega_{3,s}^2 = c\omega_1^2 = c\omega_2^2$ for some $c > 0$ by Yau's Theorem [20]. Since

$$\alpha_s^2 = [\omega_3]^2 = [\omega_1]^2 = [\omega_2]^2,$$

we have $c = 1$, hence $\omega_1, \omega_2, \omega_{3,s}$ form hyper-Kähler structures on X. Since

$$\int_{\mu^{-1}(b)} \omega_{3,s} = s \int_{\mu^{-1}(b)} \omega_3 \to 0$$

as $s \to 0$, we have the assertion. □

3. Convergence of spectral structures. In this paper we consider the convergence of $H^0(X_{J_{1,s}}, L^k)$ to some Hilbert spaces in an appropriate sense. By (4), $H^0(X_{J_{1,s}}, L^k)$ can be identified with the $(k^2 + 2k)$-eigenspace of the operator $\Delta^{\rho_k}_{\hat{g}_s}$. To consider the convergence of eigenspaces, we use the notion of the convergence of spectral structures introduced by Kuwae and Shioya in [17] to our situation.

3.1. Spectral structures. A *spectral structure* $\Sigma = (H, A)$ is a pair of a Hilbert space H and a self-adjoint positive linear operator $A\colon \mathcal{D}(A) \to H$, where $\mathcal{D}(A)$ is a subspace of H, such that the quadratic form $\mathcal{E}(f) := \langle Af, f\rangle_H$ is *closed*, i.e., the norm

$$\|f\|_A := \sqrt{\|f\|_H^2 + \mathcal{E}(f)}$$

can be extended to a dense subspace $\mathcal{D}(\mathcal{E}) \subset H$ continuously and $\mathcal{D}(\mathcal{E})$ is complete with respect to the norm $\|\cdot\|_A$.

Let (X, g) be a compact Riemannian manifold and Δ_g be the Laplacian acting on $C^\infty(X)$. Then

$$\Sigma(X, g) := \left(L^2\left(X, \nu_g\right), \Delta_g\right)$$

is a typical example of the spectral structures.

Next we review the definition of the Laplacian on a metric measure space appeared as the measured Gromov-Hausdorff limit of a sequence of Riemannian manifolds with a lower bound of the Ricci curvatures following [4].

Let (X, d, ν, p) be a pointed metric measure space, that is, d is a metric on X, ν is a Radon measure on X and $p \in X$. We assume that there are constant $\kappa \in \mathbb{R}$ and a sequence of Riemannian manifolds $\{(X_i, g_i)\}_i$ of dimension N such that

$$\mathrm{Ric}_{g_i} \ge \kappa g_i,$$

and (X, d, ν, p) is the pointed measured Gromov-Hausdorff limit of

$$\left(X_i, d_{g_i}, \frac{\nu_{g_i}}{\nu_{g_i}(B(p_i,1))}, p_i\right).$$

Denote by $\mathrm{Lip}_c(X)$ the set of compactly supported Lipschitz functions on X. Then a bilinear form on $\mathrm{Lip}_c(X)$, denoted by

$$\int_X \langle df_1, df_2\rangle d\nu \quad (f_1, f_2 \in \mathrm{Lip}_c(X))$$

can be defined so that we have

$$\int_X \langle df, df\rangle d\nu = \int_X \mathrm{Lip}(f)^2 d\nu =: \mathcal{E}(f),$$
$$\mathrm{Lip}(f)(x) := \inf_{r>0} \sup_{y\in B(x,r)\setminus\{x\}} \frac{|f(x)-f(y)|}{d(x,y)}.$$

Let $H^{1,2}(X, d, \nu)$ be the closure of $\mathrm{Lip}_c(X)$ with respect to the norm $\|f\|^2_{H^{1,2}} := \|f\|^2_{L^2} + \mathcal{E}(f)$. Denote by $\mathcal{D}(\Delta_{d,\nu})$ the subspace of $H^{1,2}(X, d, \nu)$ consisting of the functions f such that there is $h \in L^2(X, \nu)$ satisfying

$$\int_X h\varphi d\nu = \int_X \langle df, d\varphi\rangle d\nu \quad (\forall \varphi \in \mathrm{Lip}_c(X)).$$

We define a self-adjoint operator $\Delta_{d,\nu} \colon \mathcal{D}(\Delta_{d,\nu}) \to L^2(X, \nu)$ by $\Delta_{d,\nu} f := h$, then we obtain a spectral structure

$$\Sigma(X, d, \nu) := (L^2(X, \nu), \Delta_{d,\nu}).$$

If X is a smooth manifold and there are a Riemannian metric g on X and a function $\psi \in C^\infty(X)$ such that $d = d_g$ and $d\nu = e^\psi d\nu_g$, then we have

$$\Delta_{d,\nu} f = \Delta_g f - \langle d\psi, df\rangle_g.$$

For the brevity, we often write $L^2(X) = L^2(X, \nu)$, $H^{1,2}(X) = H^{1,2}(X, d, \nu)$ or $\Delta_X = \Delta_{d,\nu}$ if there is no fear of confusion.

3.2. Convergence of spectral structures. In this subsection we review the notion of convergence of the spectral structures, following [17]. Here, we take a one parameter family of Hilbert spaces $\{H_s\}_{s>0}$, unbounded self-adjoint operators $\{A_s\}_{s>0}$ and consider the convergence of them as $s \to 0$. The following notions can be also defined for sequences.

Let $\{H_s\}_{s\geq 0}$ be a family of Hilbert spaces over $\mathbb{C}$. Suppose a dense linear subspace $\mathcal{C} \subset H_0$ and linear maps $\Phi_s \colon \mathcal{C} \to H_s$ are given. We say the family $\{H_s\}_{s>0}$ *converges to* H_0 *as* $s \to 0$ if

$$\lim_{s\to 0} \|\Phi_s(f)\|_{H_s} = \|f\|_{H_0}$$

for all $f \in \mathcal{C}$. Although this convergence may depend on the choice of $\mathcal{C}$ or Φ_s, We often write $H_s \to H_0$ for the simplicity.

Next we define the convergence of $\{f_s\}_s$, where $f_s \in H_s$.

Definition 3.1. Let $H_s \to H_0$ as $s \to 0$, and take $f_s \in H_s$ for every $s \geq 0$.

(i) $\{f_s\}_{s>0}$ *converges to* f_0 *strongly* if there exists a sequence $\{\tilde{f}_k\}_{k=0}^{\infty}$ of $\mathcal{C}$ converging to f_0 such that

$$\lim_{k\to\infty}\limsup_{s\to 0}\|\Phi_s(\tilde{f}_k) - f_s\|_{H_s} = 0.$$

(ii) $\{f_s\}_{s>0}$ *converges to* f_0 *weakly* if

$$\lim_{s\to 0}\langle f_s, f_s'\rangle_{H_s} = \langle f_0, f_0'\rangle_{H_0}$$

for all $\{f_s'\}_{s\geq 0}$ with $f_s' \to f_0'$ strongly.

Next we consider a family of spectral structures $\Sigma_s = (H_s, A_s)$ for every $s \geq 0$. Denote by $\mathcal{E}_s$ the closed quadratic forms defined by A_s.

Definition 3.2. Let $H_s \to H_0$ as $s \to 0$. In the followings, we suppose $f_s, f_s' \in H_s$.

(i) $\{\mathcal{E}_s\}_{s>0}$ *Mosco converges to* $\mathcal{E}_0$ if

$$\mathcal{E}_0(f_0) \leq \liminf_{s\to 0}\mathcal{E}_s(f_s)$$

for any family $\{f_s\}_{s\geq 0}$ with $f_s \to f_0$ weakly, and for any $f_0' \in H_0$ there exists a family $\{f_s'\}_{s>0}$ such that $f_s' \to f_0'$ strongly and

$$\limsup_{s\to 0}\mathcal{E}_s(f_s') \leq \mathcal{E}_0(f_0').$$

(ii) The family $\{\mathcal{E}_s\}_{s>0}$ is *asymptotically compact* if for any $\{f_s\}_{s\geq 0}$ such that

$$\limsup_{s\to 0}\left(\|f_s\|_{H_s}^2 + \mathcal{E}_s(f_s)\right) < \infty,$$

there exists a sequence $s_i > 0$ with $\lim_{i\to\infty} s_i = 0$ such that $f_{s_i} \to f_0$ strongly as $i \to \infty$.

Definition 3.3. Let $\Sigma_s = (H_s, A_s)$ be a spectral structure for every $s \geq 0$ and suppose $H_s \to H_0$ as $s \to 0$.

(i) $\{\Sigma_s\}_{s>0}$ *converges to* Σ_0 *strongly* if $\mathcal{E}_s$ Mosco converges to $\mathcal{E}_0$.
(ii) $\{\Sigma_s\}_{s>0}$ *converges to* Σ_0 *compactly* if $\Sigma_s \to \Sigma_0$ strongly as $s \to 0$ and $\{\mathcal{E}_s\}_{s\geq 0}$ is asymptotically compact.

Remark 3.4. There are several conditions equivalent to the strong (resp. compact) convergence of $\{\Sigma_s\}_{s>0}$. See [17, Theorem 2.4].

Example. Let $\{(X_s, g_s)\}_{s>0}$ be a family of complete Riemannian manifolds and suppose there is $\kappa \in \mathbb{R}$ such that $\mathrm{Ric}_{g_s} \geq \kappa g_s$ for all s. Moreover we take $p_s \in X_s$ and assume that $(X_s, d_{g_s}, \nu_{g_s}/\nu_{g_s}(B(p_s, 1)), p_s)$ converges to some metric measure space (X, d, ν, p) in the sense of pointed measured Gromov-Hausdorff topology. Cheeger and Colding showed that if all of X_s are compact and $\sup_s \mathrm{diam}(X_s) < \infty$, then $\Sigma(X_s, g_s) \to \Sigma(X, d, \nu)$ compactly as $s \to 0$ in [4]. In the case of X_s are noncompact or $\mathrm{diam}(X_s) \to \infty$, then Kuwae and Shioya showed that $\Sigma(X_s, g_s) \to \Sigma(X, d, \nu)$ strongly as $s \to 0$ in [17].

Definition 3.5. Let $H_s \to H_0$ as $s \to 0$ and $B_s\colon H_s \to H_s$ be a bounded operator for every $s \geq 0$. We say that $B_s \to B_0$ compactly as $s \to 0$ if

$$\lim_{s\to 0}\langle B_s f_s, f_s'\rangle_{H_s} = \langle B_0 f_0, f_0'\rangle_{H_0}$$

for any $f_s, f_s' \in H_s$ with $f_s \to f_0$, $f_s' \to f_0'$ weakly as $s \to 0$.

3.3. S^1-equivariant convergence of metric measure spaces. From now on we consider metric measure spaces with S^1-actions. We say an action is isomorphic if it preserves both the metric and the measure. If $(\mathbb{S}, d, \nu)$ is a metric measure space with an isomorphic S^1-action, we denote by $\pi\colon \mathbb{S} \to \mathbb{S}/S^1$ the quotient map and put $X := \mathbb{S}/S^1$, $\bar{u} := \pi(u)$ for $u \in \mathbb{S}$. X has the natural metric defined by

$$\bar{d}(\bar{u}, \bar{u}') := \inf_{e^{\sqrt{-1}t} \in S^1} d\left(u \cdot e^{\sqrt{-1}t}, u'\right).$$

For example, if $\mathbb{S} = \mathbb{S}(L, h)$ and $\hat{g}$ is the metric defined by (2), then we have

$$\bar{d}_{\hat{g}} = d_g.$$

Definition 3.6.

(i) Let $(\mathbb{S}, d, \nu)$ and $(\mathbb{S}_0, d_0, \nu_0)$ be metric measure spaces with isomorphic S^1-action. An S^1-equivariant Borel map $\phi\colon \mathbb{S} \to \mathbb{S}_0$ is said to be *S^1-equivariant Borel ε-isometry* if $|d_0(\phi(u), \phi(u')) - d(u, u')| < \varepsilon$ for all $u, u' \in \mathbb{S}$ and $\mathbb{S}_0 \subset B(\phi(\mathbb{S}), \varepsilon)$.

(ii) For every $s \geq 0$, let $(\mathbb{S}_s, d_s, \nu_s)$ be metric measure space with isomorphic S^1-action and $p_s \in \mathbb{S}_s$. Denote by $\pi_s\colon \mathbb{S}_s \to \mathbb{S}_s/S^1$ the quotient map. The family $(\mathbb{S}_s, d_s, \nu_s, p_s)_{s>0}$ *converges to* $(\mathbb{S}_0, d_0, \nu_0, p_0)$ *in the sense of S^1-equivariant pointed measured Gromov-Hausdorff topology*, or we also write

$$(\mathbb{S}_s, d_s, \nu_s, p_s) \overset{S^1\text{-pmGH}}{\longrightarrow} (\mathbb{S}_0, d_0, \nu_0, p_0),$$

if for any $s > 0$ there are $\varepsilon_s, R_s, R'_s > 0$ and S^1-equivariant Borel ε_s-isometry

$$\phi_s\colon \left(\pi_s^{-1}(B(\bar{p}_s, R'_s)), p_s\right) \to \left(\pi_0^{-1}(B(\bar{p}_0, R_s)), p_0\right)$$

such that $\lim_{s\to 0} \varepsilon_s = 0$, $\lim_{s\to 0} R'_s = \lim_{s\to 0} R_s = \infty$ and

$$\lim_{s\to 0} \int_{\mathbb{S}_s} f \circ \phi_s d\nu_s = \int_{\mathbb{S}_0} f d\nu_0$$

for any $f \in C_c(\mathbb{S}_0)$.

Remark 3.7. The above convergence was already introduced by Fukaya and Yamaguchi in more general setting. They did not assume that the approximation map is S^1-equivariant. They assume that it is almost equivariant instead. See [7, Definition 4.1].

Let $(\mathbb{S}, d, \nu)$ be a metric measure space with isomorphic S^1-action and assume that the Laplacian $\Delta_{\mathbb{S}}$ can be defined. Then since $\Delta_{\mathbb{S}}$ is S^1-equivariant, it induces a self-adjoint operator on $(L^2(\mathbb{S}) \otimes \mathbb{C})^{\rho_k}$, which we denote by $\Delta_{\mathbb{S}}^{\rho_k}$. Here, recall that ρ_k is the 1-dimensional unitary representation of S^1 defined by $\rho_k(\lambda) = \lambda^k$. Then we have the spectral structure

$$\Sigma(\mathbb{S}, d, \nu)^{\rho_k} := \left((L^2(\mathbb{S}) \otimes \mathbb{C})^{\rho_k}, \Delta_{\mathbb{S}}^{\rho_k}\right)$$

for each $k \in \mathbb{Z}$.

Let $(X, \omega_1, \omega_2, \omega_{3,s})$ be a family of hyper-Kähler structures on the $K3$ surface for $s > 0$, $K(s) > 0$ be constants depending on s and (L, h, ∇) be a prequantum bundle on (X, ω_1). Let $\mathbb{S} = \mathbb{S}(L, h)$ and $\hat{g}_s$ be the Riemannian metric defined by g_s, ∇ and (2). Put

$$\mathbb{S}_s := \left(\mathbb{S}, d_{\hat{g}_s}, \frac{\nu_{\hat{g}_s}}{K(s)}\right).$$

Now we take points $p^b \in \mathbb{S}$ for $b = 1, \ldots, N$. We assume

$$\lim_{s\to 0} d_{g_s}(\bar{p}^b, \bar{p}^{b'}) = \infty \quad (\text{if } b \neq b'), \tag{5}$$

$$\left(\mathbb{S}_s, p^b\right) \overset{S^1\text{-pmGH}}{\longrightarrow} \left(\mathbb{S}_0^b, p_0^b\right), \tag{6}$$

and put

$$\begin{aligned} H_s &:= L^2(\mathbb{S}_s) \otimes \mathbb{C}, \\ H_s^{\rho_k} &:= \left(L^2(\mathbb{S}_s) \otimes \mathbb{C}\right)^{\rho_k}, \\ H_0 &:= \bigoplus_{b=1}^N L^2(\mathbb{S}_0^b) \otimes \mathbb{C}, \\ H_0^{\rho_k} &:= \bigoplus_{b=1}^N \left(L^2(\mathbb{S}_0^b) \otimes \mathbb{C}\right)^{\rho_k}. \end{aligned}$$

Then we have the convergence $H_s^{\rho_k} \to H_0^{\rho_k}$ as $s \to 0$ in an obvious way. We put

$$\bigoplus_{b=1}^N \Sigma(\mathbb{S}_0^b)^{\rho_k} := \left(H_0^{\rho_k}, \bigoplus_{b=1}^N \Delta_{\mathbb{S}_0^b}^{\rho_k}\right).$$

Now, since g_s is the hyper-Kähler metric, $\mathrm{Ric}_{g_s} \equiv 0$. Then by [14, Proposition 3.15], we have $\mathrm{Ric}_{\hat{g}_s} \geq -(1/2)\hat{g}_s$. Therefore, we obtain the following.

Fact 3.8 ([14, Propositions 3.14, 3.15]). *Let $(X, \omega_1, \omega_2, \omega_{3,s}, g_s, L, h, \nabla)$ be as above. Assume* (5) *and* (6). *Then $\Sigma(\mathbb{S}_s)^{\rho_k}$ converges to $\bigoplus_{b=1}^N \Sigma(\mathbb{S}_0^b)^{\rho_k}$ strongly.*

4. Main results and outline of the proof.

4.1. Main results. In this subsection we describe the main theorems of this paper and explain the outline of the proof.

Let $\{(X, \omega_1, \omega_2, \omega_{3,s})\}_{0<s\leq s_0}$ be a family of hyper-Kähler structures on a $K3$ surface, $\mu\colon X \to \mathbb{P}^1$ be a special Lagrangian fibration of Kodaira type I_1. Suppose the family tending to a large complex structure limit as $s \to 0$. We may suppose

$$s = \int_{\mu^{-1}(b)} \omega_{3,s} > 0$$

without loss of generality. Here, recall that the above integral is independent of the choice of b by Remark 2.4. Moreover we assume that the cohomology class $[\omega_1]$ is in $2\pi H^2(X, \mathbb{Z})$. Then there is a prequantum line bundle (L, h, ∇) on (X, ω_1). Moreover it is unique up to rescaling and gauge transformations since the $K3$ surfaces are simply-connected by [16, Theorem 2.2.1]. Denote by

$$(g_s, J_{1,s}, J_{2,s}, J_{3,s})$$

the hyper-Kähler structure given by $(\omega_1, \omega_2, \omega_{3,s})$. Then $J_{3,s}$ is independent of s, so we write $J_3 = J_{3,s}$. Put $\mathbb{S} = \mathbb{S}(L, h)$ and let $\hat{g}_s$ be the Riemannian metric on $\mathbb{S}$ defined by (2). Let $\pi\colon \mathbb{S} \to X$ be the restriction of $\pi\colon L \to X$ to $\mathbb{S} \subset L$.

Denote the coordinates on $\mathbb{R}^2$ and S^1 by $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $e^{\sqrt{-1}t} \in S^1$. Define a Riemannian metric $\hat{g}_{0,m}$ and a measure $\hat{\nu}_0$ on $S^1 \times \mathbb{R}^2$ by

$$\begin{aligned} \hat{g}_{0,m} &:= \frac{(dt)^2}{m^2(1+\|\xi\|^2)} + (d\xi_1)^2 + (d\xi_2)^2 \quad (m \in \mathbb{Z}_{>0}), \\ d\hat{\nu}_0 &:= d\xi_1 d\xi_2 dt, \end{aligned}$$

where $\|\xi\|^2 = \xi_1^2 + \xi_2^2$, and put

$$\mathbb{S}_{0,m} := \left(S^1 \times \mathbb{R}^2,\, d_{\hat{g}_{0,m}},\, \hat{\nu}_0\right).$$

We define an isomorphic S^1-action on $\mathbb{S}_{0,m}$ by

$$\left(e^{\sqrt{-1}t}, \xi\right) \cdot e^{\sqrt{-1}\tau} := \left(e^{\sqrt{-1}(t+m\tau)}, \xi\right)$$

for $e^{\sqrt{-1}\tau} \in S^1$.

The next theorem is the first main result in this paper.

Theorem 4.1. *Let $b \in \mathbb{P}^1$ and $p^b \in (\mu \circ \pi)^{-1}(b)$. If $b \in BS_m^{\mathrm{str}}$, then*

$$\left(\mathbb{S}, d_{\hat{g}_s}, \frac{\nu_{\hat{g}_s}}{s}, p^b\right) \overset{S^1\text{-pmGH}}{\longrightarrow} (\mathbb{S}_{0,m}, (1_{S^1}, \mathbf{0}_{\mathbb{R}^2}))$$

as $s \to 0$. Moreover, if $b, b' \in BS_k$ and $b \neq b'$, then $\lim_{s\to 0} d_{g_s}(\bar{p}^b, \bar{p}^{b'}) = \infty$.

By assuming Theorem 4.1, we can show the next lemma.

Lemma 4.2. *For $b \in BS_m^{\mathrm{str}}$, put $m(b) := m$. $\Sigma(\mathbb{S}_s)^{\rho_k}$ converges to $\bigoplus_{b\in BS_k} \Sigma(\mathbb{S}_{0,m(b)})^{\rho_k}$ strongly.*

Proof. It follows from Fact 3.8 and Theorem 4.1. □

The next theorem is the second main result.

Theorem 4.3. *Let $\mathcal{E}_s^{\rho_k}$ be the closed quadratic form associated with $\Sigma(\mathbb{S}_s)^{\rho_k}$. Then the family $\{\mathcal{E}_s^{\rho_k}\}_{s>0}$ is asymptotically compact with respect to the strong convergence $\Sigma(\mathbb{S}_s)^{\rho_k} \to \bigoplus_{b\in BS_k} \Sigma(\mathbb{S}_{0,m(b)})^{\rho_k}$.*

Combining Lemma 4.2 and Theorem 4.3, we have the following results.

Theorem 4.4. *$\Sigma(\mathbb{S}_s)^{\rho_k}$ converges to $\bigoplus_{b\in BS_k} \Sigma(\mathbb{S}_{0,m(b)})^{\rho_k}$ compactly.*

The spectral structure of $\mathbb{S}_{0,m}$ was already known by [13] as follows. For a positive integer k, let

$$\begin{aligned} H^k_{\mathbb{R}^2} &:= L^2(\mathbb{R}^2, e^{-k\|\xi\|^2} d\xi_1 d\xi_2) \otimes \mathbb{C}, \\ \Delta^k_{\mathbb{R}^2} f &:= -\left(\frac{\partial^2 f}{\partial \xi_1^2} + \frac{\partial^2 f}{\partial \xi_2^2}\right) + 2k\left(\xi_1 \frac{\partial f}{\partial \xi_1} + \xi_2 \frac{\partial f}{\partial \xi_2}\right) \end{aligned}$$

for $f = f(\xi_1, \xi_2)$. Here, $\Delta^k_{\mathbb{R}^2}$ is the Laplacian on the Gaussian space $(\mathbb{R}^2, d\xi_1^2 + d\xi_2^2, e^{-k\|\xi\|^2} d\xi_1 d\xi_2)$. For a spectral structure $\Sigma = (H, A)$ and constants $a_1 > 0$, $a_2 \in \mathbb{R}$, we put $a_1\Sigma + a_2 := (H, a_1 A + a_2 \cdot \mathrm{id}_H)$. If $H = \{0\}$, then we write $\Sigma = 0$.

FACT 4.5 ([13, Section 8]). *Let k be a positive integer. If $k \in m\mathbb{Z}$, we have*

$$\Sigma(\mathbb{S}_{0,m})^{\rho_k} - (k^2 + 2k) \cong (H^k_{\mathbb{R}^2}, \Delta^k_{\mathbb{R}^2}).$$

If $k \notin m\mathbb{Z}$, we have $\Sigma(\mathbb{S}_{0,m})^{\rho_k} = 0$.

Now, we put

$$\Sigma(X_{J_{1,s}}, L^k) := \left(L^2 \left(X, \frac{\omega_1^2}{2s}, L, h \right), \Delta_{k,\overline{\partial}_{J_{1,s}}} \right).$$

Here, the norm of the Hilbert space $L^2\left(X, \omega_1^2/2s, L, h\right)$ is given by $\|\varphi\|^2_{L^2} := \int_X \frac{|\varphi|^2_h}{2s}\omega_1^2$ for a section $\varphi\colon X \to L^k$. Note that we have $d\nu_{g_s} = \omega_1^2/2$ for a hyper-Kähler metric. By the identification (4), we have

$$\Sigma(\mathbb{S}_s)^{\rho_k} \cong 2\Sigma(X_{J_{1,s}}, L^k) + k^2 + 2k.$$

Then by Fact 4.5 and Theorem 4.4, we have the following.

THEOREM 4.6. *$\Sigma(X_{J_{1,s}}, L^k)$ converges to $\bigoplus_{b\in BS_k}(H^k_{\mathbb{R}^2}, \Delta^k_{\mathbb{R}^2}/2)$ compactly.*

So our goal is to prove Theorems 4.1 and 4.3.

4.2. Approximation of metrics. Let $(\mathbb{S}, \hat{g}_s)$ be as in Subsection 4.1. To show Theorems 4.1 and 4.3, we need to study the asymptotic behavior of the metrics $\hat{g}_s$ on $\mathbb{S} = \mathbb{S}(L, h)$ as $s \to 0$. The metrics g_s are obtained by solving the Monge-Ampère equation, and the solutions cannot be described explicitly. Instead of describing the metrics explicitly, we construct another family of explicit metrics denoted by g'_s, which approximates $\{g_s\}_s$. This strategy is justified by the following argument.

Denote by $\hat{g}'_s$ the Riemannian metrics defined by g'_s, ∇ and (2).

LEMMA 4.7. *Let (X, g_s) be as above and g'_s be another family of Riemannian metrics on X. Assume that there are constants $C_s \geq 1$ with $\lim_{s\to 0} C_s = 1$ such that $C_s^{-1} g'_s \leq g_s \leq C_s g'_s$ on X. Let $(\mathbb{S}_0, d_0, \nu_0, p_0)$ be a pointed metric measure space with isomorphic S^1-action and $K(s) > 0$ be constants depending only on s such that*

$$\left(\mathbb{S}, d_{\hat{g}'_s}, \frac{\nu_{\hat{g}'_s}}{K(s)}, p \right) \overset{S^1\text{-pmGH}}{\longrightarrow} (\mathbb{S}_0, d_0, \nu_0, p_0)$$

as $s \to 0$ for some $p \in \mathbb{S}$. Then

$$\left(\mathbb{S}, d_{\hat{g}_s}, \frac{\nu_{\hat{g}_s}}{K(s)}, p \right) \overset{S^1\text{-pmGH}}{\longrightarrow} (\mathbb{S}_0, d_0, \nu_0, p_0)$$

as $s \to 0$.

Proof. By the definition of $\hat{g}_s$ and $\hat{g}'_s$, we have

$$\hat{g}_s = (\Gamma^\nabla)^2 + g_s, \quad \hat{g}'_s = (\Gamma^\nabla)^2 + g'_s,$$

on $\mathbb{S}$, hence we have

$$C_s^{-1}\hat{g}'_s \leq \hat{g}_s \leq C_s \hat{g}'_s.$$

Therefore, by the definition of Riemannian distance, we obtain

$$C_s^{-1/2} d_{\hat{g}'_s}(u_0, u_1) \leq d_{\hat{g}_s}(u_0, u_1) \leq C_s^{1/2} d_{\hat{g}'_s}(u_0, u_1).$$

for $u_0, u_1 \in \mathbb{S}$. Since $\lim_{s\to 0} C_s^{1/2} = 1$, then we have the convergence of the metric structures. The vague convergence of the measure follows from

$$C_s^{-5/2}\nu_{\hat{g}_s} \leq \nu_{\hat{g}'_s} \leq C_s^{5/2}\nu_{\hat{g}_s}.$$

□

4.3. The metric on the frame bundles. In Definition 3.6 (ii), we call ϕ_s the approximation maps and $(\mathbb{S}_0, d_0, \nu_0)$ the limit space. To show Theorem 4.1, we need to construct the approximation map from $(\mathbb{S}, \hat{g}_s, \nu_{\hat{g}_s}/s)$ to the limit space. In this subsection, we discuss how to construct the approximation maps under some assumptions.

First of all we describe the setting and the assumptions in this subsection. Let (X, ω) be a symplectic manifold of dimension 4 with a prequantum line bundle $(\pi\colon L \to X, h, \nabla)$, and g be a Riemannian metric on X. Put $\mathbb{S} = \mathbb{S}(L, h)$ and define the metric $\hat{g}$ on $\mathbb{S}$ by (2). Let B be a smooth manifold of dimension 2 and $\mu\colon X \to B$ be a proper smooth map such that all of the fibers $\mu^{-1}(b)$ are connected. We suppose there is an open subset $B^{\mathrm{rg}} \subset B$ such that $\#(B \setminus B^{\mathrm{rg}}) < \infty$, all $b \in B^{\mathrm{rg}}$ are regular values of μ and $\mu^{-1}(b)$ are Lagrangian submanifolds for all $b \in B^{\mathrm{rg}}$. We set $\nu_B := \mu_*\nu_g$.

Let $q \in \mu^{-1}(B^{\mathrm{rg}})$ and put $(V_f)_q := \mathrm{Ker}(d\mu_q)$. Denote by $(V_f^\perp)_q \subset T_qX$ the orthogonal complement of $(V_f)_q$ with respect to g_q, then we have the orthogonal decomposition $TX|_{\mu^{-1}(B^{\mathrm{rg}})} = V_f \oplus V_f^\perp$. By putting $g_f := g|_{V_f}$, $g_\perp := g|_{V_f^\perp}$, we may write $g = g_f + g_\perp$. Similarly, for a 1-form $\gamma \in \Omega^1(X)$ we put $\gamma_f = \gamma|_{V_f}$, $\gamma_\perp = \gamma|_{V_f^\perp}$ and we write $\gamma|_{\mu^{-1}(B^{\mathrm{rg}})} = \gamma_f + \gamma_\perp$.

Next we describe the limit space. Let $(\mathbb{R}^2, g_0)$ be the Euclidean space of dimension 2 and $\mathbf{r}\colon \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be defined by $\mathbf{r}(\xi) = \|\xi\| = \sqrt{\xi_1^2 + \xi_2^2}$ for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. Denote by $\mathbf{0}_{\mathbb{R}^2} \in \mathbb{R}^2$ the origin.

Let $\hat{g}_0 = (dt)^2/(1 + \mathbf{r}^2) + g_0$ be a Riemannian metric on $S^1 \times \mathbb{R}^2$. If we put $c(\tau) = (e^{\sqrt{-1}c_1(\tau)}, c_2(\tau)) \in S^1 \times \mathbb{R}^2$, then the length of c with respect to $\hat{g}_0$ is given by

$$\mathfrak{L}_{\hat{g}_0}(c) = \int_0^1 \sqrt{\frac{|c_1'(\tau)|^2}{1 + \mathbf{r}(c(\tau))^2} + \|c_2'(\tau)\|^2}\, d\tau.$$

Let $\mathcal{B}(R) := \{\xi \in \mathbb{R}^2;\, \mathbf{r}(\xi) < R\}$ for $R > 0$.

Now, let $b \in B$, $W \subset B$ be an open neighborhood of b such that $W \setminus \{b\} \subset B^{\mathrm{rg}}$, $U := \mu^{-1}(W)$, $\gamma \in \Omega^1(U)$ be a 1-form with $\omega|_U = d\gamma$, $\zeta\colon W \to \mathbb{R}^2$ be a continuous map such that $\zeta(b) = \mathbf{0}_{\mathbb{R}^2}$ and $\zeta|_{W\setminus\{b\}}$ is an open embedding, and σ, R, δ, K be positive constants with $\delta < 1, \sigma < R$. For the following tuple

$$(g, b, W, R, \gamma, \zeta, \sigma, \delta, K),$$

we consider the next conditions.

($\star$1) Let $X_w := \mu^{-1}(w)$ and $\iota_w\colon X_w \to U$ be the inclusion map for every $w \in W$. $\iota_b^*\colon H^1(U, \mathbb{Z}) \to H^1(X_b, \mathbb{R})$ is an isomorphism.

($\star$2) $L|_U$ is trivial as a complex line bundle.

($\star$3) There are 1-cycles $e_{i,w}$ in X_w for each $i = 1, 2$ and each $w \in W$ such that $\{e_{1,w}, e_{2,w}\}$ generates $H_1(X_w, \mathbb{Z})$ and, for each $i = 1, 2$, $(\iota_w)_*(e_{i,w}) \in H_1(U, \mathbb{Z})$ is independent of $w \in W$. Moreover, the functions $\Psi_i\colon W \to \mathbb{R}$ defined by

$\Psi_i(w) := \int_{e_{i,w}} \gamma$ are continuous, $\Psi_i(b) = 0$ for each i and b is isolated in the subset

$$\{w \in W;\, \Psi_i(w) = 0 \text{ for all } i = 1, 2\}.$$

($\star$4) We have

$$|\gamma_\perp|_g \leq \delta, \quad |\gamma_\perp|_{(\zeta\circ\mu)^* g_0} \leq \delta,$$

on $(\zeta\circ\mu)^{-1}(\mathcal{B}(3R)\backslash\{\mathbf{0}_{\mathbb{R}^2}\})$,

$$(1+\delta)^{-1}(\zeta\circ\mu)^* g_0 \leq g_\perp \leq (1+\delta)(\zeta\circ\mu)^* g_0,$$
$$(1+\delta)^{-1}(\zeta\circ\mu)^* \mathbf{r}^2 \leq |\gamma_f|_g^2 \leq (1+\delta)(\zeta\circ\mu)^* \mathbf{r}^2$$

on $(\zeta\circ\mu)^{-1}(\mathcal{B}(3R)\setminus\mathcal{B}(\sigma))$ and

$$|\gamma_f|_g^2 \leq \delta$$

on $(\zeta\circ\mu)^{-1}(\overline{\mathcal{B}(\sigma)})$.
($\star$5) $\overline{\mathcal{B}(3R)} \subset \zeta(W)$.
($\star$6)

$$\sup_{w\in\zeta^{-1}(\mathcal{B}(3R))} \mathrm{diam}_{g|_{X_w}}(X_w) < \delta,$$
$$\mathrm{diam}_{g|_{(\zeta\circ\mu)^{-1}(\overline{\mathcal{B}(\sigma)})}}\left((\zeta\circ\mu)^{-1}(\overline{\mathcal{B}(\sigma)})\right) < \delta.$$

($\star$7) We have $(1+\delta)^{-1}\nu_{g_0} \leq K\cdot\zeta_*\nu_B \leq (1+\delta)\nu_{g_0}$ on $\mathcal{B}(R)$.

Remark 4.8. ($\star 1, 2, 3$) is the topological assumption for μ on the neighborhood of $\mu^{-1}(b)$. By Liouville-Arnold theorem (see [6, Theorem 1.1]), if $b \in B^{\mathrm{rg}}$, then we can see that every fiber of μ is 2-torus, hence ($\star 1, 2, 3$) are satisfied for some (W, γ).

Remark 4.9. In the above conditions, we often suppose that R is large and δ, σ are small. The condition ($\star$4) implies that $g_\perp$ and $|\gamma_f|$ can be controlled by g_0 and $\mathbf{r}$ on the complement of $(\zeta\circ\mu)^{-1}(\overline{\mathcal{B}(\sigma)})$, which is a neighborhood of $\mu^{-1}(b)$. The condition ($\star$6) implies the diameters of fibers and $(\zeta\circ\mu)^{-1}(\overline{\mathcal{B}(\sigma)})$ are small. In the setting of this paper, if b is the critical value of the special Lagrangian fibration on the $K3$ surfaces, we cannot obtain the good estimate for the metric g on the neighborhood of $\mu^{-1}(b)$, however, we may show that the diameter of such neighborhood is sufficiently small.

By ($\star$2) we can take a smooth section $\mathbf{E}_1 \in \Gamma(L|_U)$ such that $h(\mathbf{E}_1, \mathbf{E}_1) = 1$. Then there is $\gamma_1 \in \Omega^1(U)$ such that $\nabla\mathbf{E}_1 = -\sqrt{-1}\gamma_1 \otimes \mathbf{E}_1$. The holonomy group of $(L|_{X_b}, \nabla|_{X_b})$ is given by

$$\left\{\exp\left(\sqrt{-1}\int_C \gamma_1\right);\, C \in H_1(X_b, \mathbb{Z})\right\}.$$

Lemma 4.10. *Suppose that the triple (b, W, γ) satisfies ($\star 1, 2, 3$). Then b is not an accumulation point of $BS_m \cap W$.*

Proof. Let γ_1 be as above. Since $d\gamma_1 = d\gamma = \omega|_U$, hence $\gamma_1 - \gamma$ is a closed 1-form on U, then there are constants $c_i \in \mathbb{R}$ such that $\Psi_i'(w) := \int_{e_{i,w}} \gamma_1 = \Psi_i(w) + c_i$ by $(\star 3)$.

Assume $b \in BS_m$. Then $\Psi_i'(b) \in (2\pi/m)\mathbb{Z}$ for any $i = 1, 2$. By the continuity of Ψ_i', if there exists $w_n \in BS_m$ such that $w_n \to b$ as $n \to \infty$ then $\lim_{n\to\infty} \Psi_i'(w_n) = \Psi_i'(b)$. Since b is isolated in $\{w;\, \Psi_i'(w) = \Psi_i'(b) \text{ for all } i\}$ by $(\star 3)$, hence $w_n = b$ for sufficiently large n.

Next we suppose $b \notin BS_m$. Then $\Psi_i'(b) \notin (2\pi/m)\mathbb{Z}$ for some i. By the continuity of Ψ_i', there is an open neighborhood $W' \subset B$ of b such that $\Psi_i'(W') \cap (2\pi/m)\mathbb{Z} = \emptyset$, hence $W' \cap BS_m = \emptyset$. ◻

Next we fix $b \in BS_m^{\mathrm{str}}$ and describe $\hat{g}$ on the neighborhood of $(\mu \circ \pi)^{-1}(b)$, then construct an approximation map. The following argument is quite technical, therefore we assume $b \in BS_1$ for the simplicity, and it is enough to explain the essence of this subsection. The argument for general $b \in BS_m^{\mathrm{str}}$ is written in the last of this subsection. See also [13, Subsection 7.3].

LEMMA 4.11. *Let $b \in BS_1$, i.e., the holonomy group of $(L|_{X_b}, \nabla|_{X_b})$ is trivial. Suppose that there are an open neighborhood W of b and $\gamma \in \Omega^1(U)$ with $\omega|_U = d\gamma$ such that the triple (b, W, γ) satisfies $(\star 1, 2)$, where $U := \mu^{-1}(W)$. Then there exists a trivialization of principal S^1-bundles $S^1 \times U \cong \mathbb{S}(L|_U, h)$ such that*

$$\hat{g} = (dt - \gamma)^2 + g.$$

Proof. Let $\mathbf{E}_1$ and γ_1 be as above. By the assumption for the holonomy groups, we can choose $\mathbf{E}_1$ such that $\int_C \gamma_1 = 0$ for all $C \in H_1(X_b, \mathbb{Z})$. Then by $(\star 1)$, there is $\varphi \in C^\infty(U)$ such that $\gamma = \gamma_1 + d\varphi$, hence we may choose $\mathbf{E}_1$ such that $\gamma_1 = \gamma$. Then by the definition of $\hat{g}$, we obtain the result. ◻

From now on, let $b \in BS_1$ and we assume that

$$(g, b, W, R, \gamma, \zeta, \sigma, \delta, K)$$

satisfies $(\star$1-7$)$. Then we may suppose

$$\mathbb{S}(L|_U, h) = S^1 \times U, \quad \hat{g} = (dt - \gamma)^2 + g$$

by Lemma 4.11. Now we put

$$U(r) := (\zeta \circ \mu)^{-1}(\mathcal{B}(r)),$$

for $r > 0$ and we study the distance functions $d_g, d_{\hat{g}}$ restricting to $B_g(q, R)$, $\pi^{-1}(B_g(q, R))$, respectively. To study them, we need to consider the length of paths, however, we should remind that a path c connecting points in $B_g(q, R)$ may not be included in U in general. It is inconvenient to apply $(\star 4)$, therefore, we need the next lemma.

LEMMA 4.12. *Let $q \in \mu^{-1}(b)$. Then $B_g(q, R)$ is contained in $U(\sqrt{1+\delta}R + \sigma)$. Moreover, if $\sigma \le (3 - 2\sqrt{2})R/(2\sqrt{2})$, then for any piecewise smooth path $c\colon [0,1] \to X$ connecting $x_0, x_1 \in U(\sqrt{1+\delta}R + \sigma)$ with $\mathfrak{L}_g(c) < (3/\sqrt{2})R$, $c([0,1])$ is contained in $U(3R)$.*

Proof. Let $x \in B_g(q,R)$ and c be a path connecting $c(0) = q$ and $c(1) = x$. Assume that $c([0,1])$ is not included in $U(\sqrt{1+\delta}R+\sigma)$. Then by ($\star$5), there are $\tau_0, \tau_1 \in [0,1]$ such that

$$c([\tau_0,\tau_1)) \subset U(\sqrt{1+\delta}R+\sigma) \setminus U(\sigma),$$
$$\mathbf{r}\circ\zeta\circ\mu\circ c(\tau_0) = \sigma,$$
$$\mathbf{r}\circ\zeta\circ\mu\circ c(\tau_1) = \sqrt{1+\delta}R+\sigma.$$

By ($\star$4), we have

$$\mathfrak{L}_g(c) \geq \frac{1}{\sqrt{1+\delta}}\mathfrak{L}_{g_0}(\zeta\circ\mu\circ c|_{[\tau_0,\tau_1]}) \geq R,$$

hence we can show that if $\mu\circ c(0) = b$ and $\mathfrak{L}_g(c) < R$, then $c([0,1]) \subset U(\sqrt{1+\delta}R+\sigma)$, therefore $x = c(1) \in U(\sqrt{1+\delta}R+\sigma)$.

Next we take $x_0, x_1 \in U(\sqrt{1+\delta}R+\sigma)$ and a path c connecting x_0 and x_1. Suppose that the image of c is not contained in $U(3R)$. Then by ($\star$5), there are $\tau_0, \tau_1, \tau_2, \tau_3 \in [0,1]$ such that

$$c([\tau_0,\tau_1)), c((\tau_2,\tau_3]) \subset (\zeta\circ\mu)^{-1}(\mathcal{B}(3R)),$$
$$\mathbf{r}\circ\zeta\circ\mu\circ c(\tau_0) = \mathbf{r}\circ\zeta\circ\mu\circ c(\tau_3) = \sqrt{1+\delta}R+\sigma,$$
$$\mathbf{r}\circ\zeta\circ\mu\circ c(\tau_1) = \mathbf{r}\circ\zeta\circ\mu\circ c(\tau_2) = 3R.$$

Then by the similar argument and by $0 < \delta \leq 1$, we have

$$\mathfrak{L}_g(c) \geq 2\left(\frac{3-\sqrt{2}}{\sqrt{2}}\cdot R - \sigma\right).$$

Since $\sigma \leq (3-2\sqrt{2})R/(2\sqrt{2})$, we have $\mathfrak{L}_g(c) \geq 3R/\sqrt{2}$. Therefore, $\mathfrak{L}_g(c) < (3/\sqrt{2})R$ implies that the image of c is included in $\mathcal{B}(3R)$. □

Let $u_0, u_1 \in S^1 \times U(\sqrt{1+\delta}R+\sigma)$ and $c\colon [0,1] \to S^1 \times U$ be a piecewise smooth path connecting u_0 and u_1. Put $c = (e^{\sqrt{-1}c_1}, c_2)$, then we have $\mathfrak{L}_{\hat{g}}(c) \geq \mathfrak{L}_g(c_2)$. By applying Lemma 4.12, we also obtain the next corollary.

Corollary 4.13. *Let $b \in BS_1$ and $c\colon [0,1] \to S^1 \times U$ be a piecewise smooth path connecting $u_0, u_1 \in S^1 \times U(\sqrt{1+\delta}R+\sigma)$ such that $\mathfrak{L}_{\hat{g}}(c) < (3/\sqrt{2})R$. If $\sigma \leq (3-2\sqrt{2})R/(2\sqrt{2})$, then $c([0,1]) \subset S^1 \times U(3R)$.*

Now, we define the approximation map $\phi\colon S^1 \times U \to S^1 \times \mathbb{R}^2$ by

$$\phi(e^{\sqrt{-1}t}, x) := \left(e^{\sqrt{-1}t}, \zeta\circ\mu(x)\right).$$

The next aim is to show that $|d_{\hat{g}}(u_0,u_1) - d_{\hat{g}_0}(\phi(u_0),\phi(u_1))|$ is small if δ, σ is small. To show it, we need to estimate the difference between $\mathfrak{L}_{\hat{g}}(c)$ and $\mathfrak{L}_{\hat{g}_0}(\phi\circ c)$ for a path c in $S^1 \times U(3R)$ and the diameter of the fibers $\phi^{-1}(u)$ for $u \in S^1 \times \mathcal{B}(3R)$. We estimate it in the case of $\mathrm{Im}(c) \subset S^1 \times (U(3R) \setminus U(\sigma))$ and $\mathrm{Im}(c) \subset S^1 \times \overline{U(\sigma)}$.

(I) Estimates on $S^1 \times (U(3R) \setminus U(\sigma))$.

We describe $\hat{g}$ on $W^{\mathrm{rg}} := W \cap B^{\mathrm{rg}}$. On $\mu^{-1}(W^{\mathrm{rg}})$, we have the decomposition $g = g_f + g_\perp$ and $\gamma = \gamma_f + \gamma_\perp$. On $S^1 \times \mu^{-1}(W^{\mathrm{rg}})$, we have

$$\hat{g} = (dt - \gamma_\perp)^2 + g_\perp + (\gamma_f)^2 + g_f - 2\gamma_f \cdot (dt - \gamma_\perp).$$

Fix any point $x \in \mu^{-1}(W^{\mathrm{rg}})$ and let $\mathbf{e}^1, \dots, \mathbf{e}^n \in (V_f)^*|_x$ be an orthonormal basis with respect to $g_f|_x$, then we may write $g_f|_x = \delta_{ij}\mathbf{e}^i \cdot \mathbf{e}^j$. The basis can be chosen such that $\gamma_f|_x = k\mathbf{e}^1$ for some $k \in \mathbb{R}$. Then we have

$$\begin{aligned}\hat{g} = &\frac{1}{1+|\gamma_f|_g^2}(dt-\gamma_\perp)^2 + g_\perp \\ &+ \left(\sqrt{1+|\gamma_f|_g^2}\,\mathbf{e}^1 - \frac{k}{\sqrt{1+|\gamma_f|_g^2}}(dt-\gamma_\perp)\right)^2 + \sum_{i=2}^n (\mathbf{e}^i)^2.\end{aligned} \tag{7}$$

Define a subspace $\mathcal{W}_p \subset T_p(S^1 \times U) = \mathbb{R}\frac{\partial}{\partial t} \oplus T_x U$ by

$$\mathcal{W}_p := \mathrm{Ker}\left(\sqrt{1+|\gamma_f|_g^2}\,\mathbf{e}^1 - \frac{k}{\sqrt{1+|\gamma_f|_g^2}}(dt-\gamma_\perp)\right) \cap \left(\bigcap_{i=2}^n \mathrm{Ker}(\mathbf{e}^i)\right).$$

We say the piecewise smooth path $c\colon [0,1] \to S^1 \times U$ is *horizontal with respect to* ϕ if the image of $\mu \circ \pi \circ c$ is contained in W^{rg} and $c'(\tau) \in \mathcal{W}_{c(\tau)}$ for every τ.

Next we compare $\mathfrak{L}_{\hat{g}}(c)$ and $\mathfrak{L}_{\hat{g}_0}(\phi \circ c)$ for a path c, however, it is difficult to compare them directly. Now we define $\mathfrak{L}_{\hat{g}_0^\sigma}(c)$ as follows such that $\mathfrak{L}_{\hat{g}_0^\sigma}(c) \le \mathfrak{L}_{\hat{g}_0}(c)$ and compare $\mathfrak{L}_{\hat{g}}(c)$ and $\mathfrak{L}_{\hat{g}_0^\sigma}(\phi \circ c)$ instead.

Let g_0^σ be a noncontinuous Riemannian metric on $\mathbb{R}^2$ defined by

$$\begin{aligned}(g_0^\sigma)_\xi &:= (g_0)_\xi \quad (\xi \notin \mathcal{B}(\sigma)),\\ (g_0^\sigma)_\xi &:= 0 \quad (\xi \in \mathcal{B}(\sigma))\end{aligned}$$

Then $d_{g_0^\sigma}$ is a pseudodistance function on $\mathbb{R}^2$. Next we put

$$\hat{g}_0^\sigma := \frac{dt^2}{1+\mathbf{r}^2} + g_0^\sigma.$$

By the definition we have $g_0^\sigma \le g_0$ and $\hat{g}_0^\sigma \le \hat{g}_0$.

Proposition 4.14. *Let $b \in BS_1$. For any piecewise smooth path c in $S^1 \times U(3R)$, we have*

$$\mathfrak{L}_{\hat{g}}(c) \ge \sqrt{\frac{1-\delta}{1+\delta}}\,\mathfrak{L}_{\hat{g}_0^\sigma}(\phi \circ c).$$

Moreover, if c is horizontal with respect to ϕ and $\mathrm{Im}(c) \subset S^1 \times U(3R) \setminus U(\sigma)$, *then*

$$\mathfrak{L}_{\hat{g}}(c) \le \sqrt{\frac{1+\delta}{1-\delta}}\,\mathfrak{L}_{\hat{g}_0^\sigma}(\phi \circ c).$$

To show Proposition 4.14, we need the next lemma.

LEMMA 4.15. *Let $b \in BS_1$, $(e^{\sqrt{-1}t}, x) \in S^1 \times U(3R)$ and $w \in \mathbb{R} \cong T_{e^{\sqrt{-1}t}} S^1$, $\tilde{v} \in T_x U$ and put $\hat{g}_0 = (dt)^2/(1+\mathbf{r}^2)+g_0$ on $S^1 \times (\mathbb{R}^2 \setminus \{\mathbf{0}_{\mathbb{R}^2}\})$. Assume that $\mu(x) \in B^{\mathrm{rg}}$. Then we have*

$$|(w,v)|^2_{\hat{g}} \geq \frac{1-\delta}{1+|\gamma_f|^2_g} |w|^2 + (1-\delta)|v_\perp|^2_{g_\perp},$$
$$|d\phi(w,v)|^2_{\hat{g}_0} \geq \frac{1-\delta}{1+\mathbf{r}^2} |w - \gamma_\perp(v_\perp)|^2 + (1-\delta)|d(\zeta \circ \mu)(v)|^2_{g_0},$$

where $v_\perp$ is the $V_f^\perp$-component of v.

Proof. By (7), we have

$$|(w,v)|^2_{\hat{g}} \geq \frac{|w - \gamma_\perp(v_\perp)|^2}{1+|\gamma_f|^2_g} + |v_\perp|^2_{g_\perp}$$

Moreover, by ($\star$4), we have $|\gamma_\perp(v_\perp)| \leq \delta |v_\perp|_{g_\perp}$, hence

$$|(w,v)|^2_{\hat{g}} \geq \frac{\left||w| - \delta|v_\perp|_{g_\perp}\right|^2}{1+|\gamma_f|^2_g} + |v_\perp|^2_{g_\perp}.$$

Since we have $(a-\delta b)^2 \geq (1-\delta)a^2 - \delta(1-\delta)b^2$ for $a, b \in \mathbb{R}$ and $0 < \delta \leq 1$, we can see that

$$\begin{aligned} |(w,v)|^2_{\hat{g}} &\geq \frac{(1-\delta)|w|^2 - \delta(1-\delta)|v_\perp|^2_{g_\perp}}{1+|\gamma_f|^2_g} + |v_\perp|^2_{g_\perp} \\ &\geq \frac{(1-\delta)|w|^2}{1+|\gamma_f|^2_g} + (1-\delta)|v_\perp|^2_{g_\perp}. \end{aligned}$$

Since $d\mu(v_\perp) = d\mu(v)$, we have the first inequality. Next we consider the second inequality. By $d\phi(w,v) = (w, d(\zeta \circ \mu)(v))$, we have

$$|d\phi(w,v)|^2_{\hat{g}_0} = \frac{|w|^2}{1+\mathbf{r}^2} + |d(\zeta \circ \mu)(v)|^2_{g_0}.$$

Then by the similar argument we also have the second inequality. □

Proof of Proposition 4.14. Let $c = (e^{\sqrt{-1}c_1}, c_2)\colon [0,1] \to S^1 \times U(3R)$. By Lemma 4.15, we have

$$\mathfrak{L}_{\hat{g}}(c) \geq \sqrt{1-\delta} \int_0^1 \sqrt{\frac{|c_1'|^2}{1+|\gamma_f|^2} + |v_\perp|^2_{g_\perp}}\, d\tau,$$

where $v_\perp$ is the $V_f^\perp$-component of c_2'. By ($\star$4), we have $(1+\delta)^{-1}(\zeta \circ \mu)^* g_0^\sigma \leq g_\perp$ and

$$\frac{(1+\delta)^{-1}}{1+(\mathbf{r} \circ \zeta \circ \mu)^2} \leq \frac{1}{1+|\gamma_f|^2}$$

on $U(3R)$, hence $\mathfrak{L}_{\hat{g}}(c) \geq \sqrt{(1-\delta)/(1+\delta)}\, \mathfrak{L}_{\hat{g}_0^\sigma}(\phi \circ c)$.

Next we assume c is horizontal with respect to ϕ and $\mathrm{Im}(c) \subset S^1 \times U(3R) \setminus U(\sigma)$. Then we have

$$\mathfrak{L}_{\hat{g}}(c) = \int_0^1 \sqrt{\frac{|c_1' - \gamma_\perp(v_\perp)|^2}{1+|\gamma_f|^2} + |v_\perp|^2_{g_\perp}}\, d\tau.$$

Since ($\star$4) gives $1/\{1+(\mathbf{r}\circ\zeta\circ\mu)^2\} \geq (1+\delta)^{-1}/(1+|\gamma_f|^2)$ on $U(3R)\setminus U(\sigma)$, then by Lemma 4.15 we have

$$\mathfrak{L}_{\hat{g}_0^\sigma}(\phi\circ c) \geq \sqrt{\frac{1-\delta}{1+\delta}}\,\mathfrak{L}_{\hat{g}}(c).$$

□

(II) Estimates on $S^1\times\overline{U(\sigma)}$.

PROPOSITION 4.16. *Let $b\in BS_1$. For any piecewise smooth path $c\colon[0,1]\to S^1\times\overline{\mathcal{B}(\sigma)}$, we have*

$$d_{\hat{g}}(u_0,u_1) \leq \sqrt{1+\sigma^2}\mathfrak{L}_{\hat{g}_0^\sigma}(c)+2\delta,$$
$$d_{\hat{g}_0}(c(0),c(1)) \leq \sqrt{1+\sigma^2}\mathfrak{L}_{\hat{g}_0^\sigma}(c)+2\sigma$$

for any $u_0\in\phi^{-1}(c(0))$ and $u_1\in\phi^{-1}(c(1))$.

Proof. Let $c=(e^{\sqrt{-1}c_1},c_2)\colon[0,1]\to S^1\times\overline{\mathcal{B}(\sigma)}$ be a piecewise smooth path. Since $|c'|_{\hat{g}_0^\sigma}\geq|c_1'|/\sqrt{1+\sigma^2}$, we have

$$\mathfrak{L}_{\hat{g}_0^\sigma}(c) \geq \frac{|c_1(1)-c_1(0)|}{\sqrt{1+\sigma^2}}. \tag{8}$$

If we take $u_i\in\phi^{-1}(c(i))$ for $i=0,1$, then we may put $u_i=(e^{\sqrt{-1}t_i},x_i)$ for some $t_i\in\mathbb{R}$ and $x_i\in\overline{U(\sigma)}$ such that $c_1(i)=t_i$. Let $\eta_2\colon[0,1]\to\overline{U(\sigma)}$ be a piecewise smooth path connecting x_0,x_1 and let $\eta(\tau):=(e^{\sqrt{-1}c_1(0)},\eta_2(\tau))$. Then we have $\eta(0)=u_0$, $\eta(1)=(e^{\sqrt{-1}c_1(0)},x_1)$ and

$$d_{\hat{g}}(\eta(0),\eta(1)) \leq \int_0^1\sqrt{|\gamma(\eta_2')|^2+|\eta_2'|_g^2}\,d\tau = \int_0^1\sqrt{1+|\gamma|^2}|\eta_2'|_g\,d\tau,$$
$$d_{\hat{g}}(\eta(1),u_1) \leq |c_1(1)-c_1(0)|.$$

By ($\star$4) and $\delta\leq 1$, we have $|\gamma|_g^2\leq 2$ on $\overline{\mathcal{B}(\sigma)}$. Then we have

$$d_{\hat{g}}(u_0,u_1) \leq 2\,\mathfrak{L}_g(\eta_2)+|c_1(1)-c_1(0)|.$$

By ($\star$6) and (8), we have the first inequality.

Next we consider the second inequality. Since

$$\begin{aligned} d_{\hat{g}_0}(c(0),c(1)) &\leq d_{g_0}(c_2(0),c_2(1))+|c_1(1)-c_1(0)| \\ &\leq 2\sigma+|c_1(1)-c_1(0)|, \end{aligned}$$

then (8) implies

$$d_{\hat{g}_0}(c(0),c(1)) \leq 2\sigma+\sqrt{1+\sigma^2}\mathfrak{L}_{\hat{g}_0^\sigma}(c).$$

□

(III) The diameter of a fiber of ϕ.

PROPOSITION 4.17. *Let $b \in BS_1$. Then we have*

$$\operatorname{diam}_{\hat{g}}(\phi^{-1}(e^{\sqrt{-1}t}, \xi)) \leq \sqrt{2 + 18R^2}\, \delta$$

for $\xi \in \mathcal{B}(3R)$.

Proof. Let $u_0, u_1 \in S^1 \times U$ and assume that $\phi(u_0) = \phi(u_1) = (e^{\sqrt{-1}t}, \xi)$. Put $u_i = (e^{\sqrt{-1}t}, x_i)$ for $i = 0, 1$, then $\zeta \circ \mu(x_0) = \zeta \circ \mu(x_1) = \xi$. For any $\varepsilon > 0$ there is a piecewise smooth path $c\colon [0,1] \to (\zeta \circ \mu)^{-1}(\xi)$ connecting x_0 and x_1 such that $\mathfrak{L}_g(c) < d_g(x_0, x_1) + \varepsilon$. We define a path $\hat{c}\colon [0,1] \to \phi^{-1}(e^{\sqrt{-1}t}, \xi)$ connecting u_0 and u_1 by $\hat{c}(\tau) := \left(e^{\sqrt{-1}t}, c(\tau)\right)$. Then we can see that

$$\mathfrak{L}_{\hat{g}}(\hat{c}) = \int_0^1 \sqrt{\{\gamma(c')\}^2 + |c'|_g^2} d\tau.$$

Since $c' \in \operatorname{Ker}(d\mu)$, we have $\gamma_\perp(c') = 0$. By ($\star$4), if $\xi \notin \mathcal{B}(\sigma)$ then we have $\{\gamma(c')\}^2 \leq (1+\delta)\mathbf{r}^2|c'|_g^2$, and if $\xi \in \mathcal{B}(\sigma)$ then $\{\gamma(c')\}^2 \leq \delta|c'|_g^2$. Therefore, we obtain

$$\begin{aligned} \mathfrak{L}_{\hat{g}}(\hat{c}) &\leq \int_0^1 \sqrt{1 + \max\{(1+\delta)\mathbf{r}^2, \delta\}} \cdot |c'|_g d\tau \\ &< \sqrt{1 + \delta + 9(1+\delta)R^2}\, \{d_g(x_0, x_1) + \varepsilon\}. \end{aligned}$$

Since we can take $\varepsilon \to 0$ and we have supposed $\delta \leq 1$, then

$$\mathfrak{L}_{\hat{g}}(\hat{c}) \leq \sqrt{2 + 18R^2} d_g(x_0, x_1).$$

Hence we have the result by ($\star$6). □

Next we compare $d_{\hat{g}}, d_{\hat{g}_0^\sigma}$ and compare $d_{\hat{g}_0}, d_{\hat{g}_0^\sigma}$ by applying the results in (I,II,III).

PROPOSITION 4.18. *Let $R \geq 4\sqrt{2}(3-2\sqrt{2})^{-1}$, $\delta \leq (4-\pi)/2$, $b \in BS_1$, $q \in \mu^{-1}(b)$, $u_0, u_1 \in \pi^{-1}(B_g(q, R))$ and $\sigma \leq (3 - 2\sqrt{2})R/(2\sqrt{2})$. Then*

$$\sqrt{\frac{1-\delta}{1+\delta}}\, d_{\hat{g}_0^\sigma}(\phi(u_0), \phi(u_1)) \leq d_{\hat{g}}(u_0, u_1).$$

Proof. Fix a sufficiently small $\varepsilon > 0$. Let $\hat{c}\colon [0,1] \to S^1 \times U$ be a piecewise smooth path connecting $u_0, u_1 \in \pi^{-1}(B_g(q, R))$ such that $\mathfrak{L}_{\hat{g}}(\hat{c}) < d_{\hat{g}}(u_0, u_1) + \varepsilon$. If $\operatorname{Im}(\hat{c}) \subset S^1 \times U(3R)$, then by the first inequality of Proposition 4.14, we have the result. We show $\operatorname{Im}(\hat{c}) \subset S^1 \times U(3R)$. Since we have

$$d_{\hat{g}}(u_0, u_1) \leq d_g\left(\pi(u_0), \pi(u_1)\right) + \pi, \tag{9}$$

then $\mathfrak{L}_{\hat{g}}(\hat{c}) \leq d_g(\pi(u_0), \pi(u_1)) + \pi + \varepsilon$. Take ε such that $\varepsilon \leq (4 - \pi)/2$. Since $R \geq 4\sqrt{2}(3 - 2\sqrt{2})^{-1}$, we obtain

$$\mathfrak{L}_{\hat{g}}(\hat{c}) < 2R + 4 \leq \frac{3R}{\sqrt{2}},$$

hence $\operatorname{Im}(\hat{c}) \subset S^1 \times U(3R)$ by Corollary 4.13. □

Next we give the opposite direction of the estimate in Proposition 4.18. Let $c = (e^{\sqrt{-1}c_1}, c_2)\colon [0,1] \to S^1 \times \mathbb{R}^2$ be a piecewise smooth path. Now, we apply Proposition 4.16 to every connected component of $c|_{c_2^{-1}(\mathcal{B}(\sigma))}$, however, there are a lot of connected components in general, hence the error terms of the estimates in Proposition 4.16 may become large. To prevent it, we should show that we can replace c by another $\hat{c}$ such that $\mathfrak{L}_{\hat{g}_0^\sigma}(\hat{c}) \le \mathfrak{L}_{\hat{g}_0^\sigma}(c)$ and the number of the connected components of $c_2^{-1}(\mathcal{B}(\sigma))$ is small. We discuss it in the next two lemmas.

Lemma 4.19. *Let $\xi \in \mathbf{r}^{-1}(\sigma)$ and $t_0, t_1 \in \mathbb{R}$. Then there is a smooth minimizing geodesic $c\colon [0,1] \to S^1 \times (\mathcal{B}(4+\sigma) \setminus \mathcal{B}(\sigma))$ with respect to $d_{\hat{g}_0^\sigma}$ such that $c(0) = (e^{\sqrt{-1}t_0}, \xi)$ and $c(1) = (e^{\sqrt{-1}t_1}, \xi)$.*

Proof. Put $u_0 = (e^{\sqrt{-1}t_0}, \xi)$, $u_1 = (e^{\sqrt{-1}t_1}, \xi)$,

$$c(\tau) = \left(e^{\sqrt{-1}t(\tau)}, \rho(\tau)\cos(x(\tau)), \rho(\tau)\sin(x(\tau))\right),$$

where $t(\tau) \in \mathbb{R}$, $\rho(\tau) \ge 0$, and $x(\tau) \in \mathbb{R}$. Moreover, we suppose $\rho(0) = \rho(1) = \sigma$, $\xi = (\sigma\cos(x_0), \sigma\sin(x_0)) \in \mathbb{R}^2$ for some $x_0 \in \mathbb{R}$ and $x(0) = x(1) = x_0$. Let $c^*\colon [0,1] \to S^1 \times \mathcal{B}(\tilde{R})$ be a path defined by $c^*(\tau) := (e^{\sqrt{-1}t(\tau)}, \rho(\tau)\cos(x_0), \rho(\tau)\sin(x_0))$, then it connects u_0 and u_1. It is easy to see $\mathfrak{L}_{\hat{g}_0^\sigma}(c^*) \le \mathfrak{L}_{\hat{g}_0^\sigma}(c)$ and $\mathfrak{L}_{\hat{g}_0}(c^*) \le \mathfrak{L}_{\hat{g}_0}(c)$.

Since $\rho^{-1}([0,\sigma))$ is open in $[0,1]$, it is the union of countable open intervals. Let (τ_-, τ_+) be one of them, where $0 < \tau_- < \tau_+ < 1$. On (τ_-, τ_+), replace $c^*|_{(\tau_-,\tau_+)}$ with the path $\tau \mapsto (e^{\sqrt{-1}t(\tau)}, \sigma\cos(x_0), \sigma\sin(x_0))$, which is shorter than $c^*|_{(\tau_-,\tau_+)}$ with respect to both of $\mathfrak{L}_{\hat{g}_0^\sigma}, \mathfrak{L}_{\hat{g}_0}$. Therefore, if $c\colon [0,1] \to S^1 \times \mathbb{R}^2$ is the minimizing geodesic connecting u_0 and u_1, then its image is contained in $S^1 \times (\mathbb{R}^2 \setminus \mathcal{B}(\sigma))$. Since $\hat{g}_0^\sigma = \hat{g}_0$ on $S^1 \times (\mathbb{R}^2 \setminus \mathcal{B}(\sigma))$, hence $\hat{c}$ is minimizing geodesic with respect to $d_{\hat{g}_0^\sigma}$ iff it is minimizing geodesic with respect to $d_{\hat{g}_0}$.

Now, one can easily check that the geodesic ball $B_{d_{\hat{g}_0}}((1_{S^1}, \mathbf{0}_{\mathbb{R}^2}), \tilde{R})$ is contained in $S^1 \times \mathcal{B}(\tilde{R})$ for any $\tilde{R} > 0$, consequently, all of the bounded sets in $(S^1 \times \mathbb{R}^2, d_{\hat{g}_0})$ are precompact. Then by the Hopf-Rinow Theorem there is a minimizing geodesic c with respect to $d_{\hat{g}_0}$ connecting u_0 and u_1. By the above argument, it is also minimizing geodesic with respect to $d_{\hat{g}_0^\sigma}$ and its image is contained in $S^1 \times (\mathbb{R}^2 \setminus \mathcal{B}(\sigma))$. Since c is the geodesic in the smooth Riemannian manifold, it is smooth.

Finally, we show $\mathrm{Im}(c) \subset S^1 \times \mathcal{B}(4+\sigma)$. By considering the path $\tau \mapsto (e^{\sqrt{-1}\tau}, \xi)$ for $\tau \in [t_0, t_1]$, we can see $d_{\hat{g}_0}(u_0, u_1) \le \pi/\sqrt{1+\sigma^2}$. If $\mathrm{Im}(c)$ is not contained in $S^1 \times \mathcal{B}(4+\sigma)$, then we can see $d_{\hat{g}_0}(u_0, u_1) \ge 4$, which is the contradiction. □

Lemma 4.20. *Let $\sigma > 0$, $\tilde{R} \ge 4 + \sigma$ and $c\colon [0,1] \to S^1 \times \mathcal{B}(\tilde{R})$ be a piecewise smooth path. Then we have $\mathrm{Im}(c) \subset S^1 \times (\mathcal{B}(\tilde{R}) \setminus \mathcal{B}(\sigma))$, $\mathrm{Im}(c) \subset S^1 \times \overline{\mathcal{B}(\sigma)}$ or there is a piecewise smooth path $\hat{c}\colon [0,1] \to S^1 \times \mathcal{B}(\tilde{R})$ such that $c(0) = \hat{c}(0)$, $c(1) = \hat{c}(1)$, $\mathfrak{L}_{\hat{g}_0^\sigma}(\hat{c}) \le \mathfrak{L}_{\hat{g}_0^\sigma}(c)$ and one of the following holds.*

(i) *There are $0 \le \tau_- < \tau_+ \le 1$ such that $\hat{c}([0,\tau_-] \sqcup [\tau_+, 1]) \subset S^1 \times (\mathcal{B}(\tilde{R}) \setminus \mathcal{B}(\sigma))$ and $\hat{c}([\tau_-, \tau_+]) \subset S^1 \times \overline{\mathcal{B}(\sigma)}$.*

(ii) *There are $0 \le \tau_- < \tau_+ \le 1$ such that $\hat{c}([0,\tau_-] \sqcup [\tau_+, 1]) \subset S^1 \times \overline{\mathcal{B}(\sigma)}$ and $\hat{c}([\tau_-, \tau_+]) \subset S^1 \times (\mathcal{B}(\tilde{R}) \setminus \mathcal{B}(\sigma))$.*

(iii) *There are $0 \le \tau_* \le 1$ such that $\hat{c}([0,\tau_*]) \subset S^1 \times \overline{\mathcal{B}(\sigma)}$ and $\hat{c}([\tau_*, 1]) \subset S^1 \times (\mathcal{B}(\tilde{R}) \setminus \mathcal{B}(\sigma))$.*

(iv) *There are $0 \le \tau_* \le 1$ such that $\hat{c}([\tau_*, 1]) \subset S^1 \times \overline{\mathcal{B}(\sigma)}$ and $\hat{c}([0,\tau_*]) \subset S^1 \times (\mathcal{B}(\tilde{R}) \setminus \mathcal{B}(\sigma))$.*

Proof. Put $c(\tau) = (e^{\sqrt{-1}t(\tau)}, \rho(\tau)\cos(x(\tau)), \rho(\tau)\sin(x(\tau)))$, where $t(\tau) \in \mathbb{R}$, $\rho(\tau) \geq 0$, and $x(\tau) \in \mathbb{R}$. We assume that neither $\mathrm{Im}(c) \subset S^1 \times (\mathcal{B}(\tilde{R}) \setminus \mathcal{B}(\sigma))$ nor $\mathrm{Im}(c) \subset S^1 \times \overline{\mathcal{B}(\sigma)}$. Then we can see that $\rho^{-1}(\sigma) \subset [0,1]$ is nonempty. Let $\tau_0 := \inf \rho^{-1}(\sigma)$ and $\tau_1 := \sup \rho^{-1}(\sigma)$.

Let c^* be the minimizing geodesic connecting

$$c(\tau_0), \quad \left(e^{\sqrt{-1}t(\tau_1)}, \sigma\cos(x(\tau_0)), \sigma\sin(x(\tau_0))\right),$$

obtained by Lemma 4.19. Define $c^\dagger, c^\ddagger \colon [0,\sigma] \to S^1 \times \overline{\mathcal{B}(\sigma)}$ by

$$c^\dagger(\tau) := \left(e^{\sqrt{-1}t(\tau_1)}, (\sigma-\tau)\cos(x(\tau_0)), (\sigma-\tau)\sin(x(\tau_0))\right),$$
$$c^\ddagger(\tau) := \left(e^{\sqrt{-1}t(\tau_1)}, \tau\cos(x(\tau_1)), \tau\sin(x(\tau_1))\right),$$

then $\mathfrak{L}_{\hat{g}_0^\sigma}(c^\dagger) = \mathfrak{L}_{\hat{g}_0^\sigma}(c^\ddagger) = 0$. Let $\hat{c}$ be the path constructed by joining $c|_{[0,\tau_0]}, c^*, c^\dagger, c^\ddagger, c|_{[\tau_1,1]}$. Then we have the result. □

PROPOSITION 4.21. *Let $R \geq 4\sqrt{2}(3-2\sqrt{2})^{-1}$, $b \in BS_1$, $q \in \mu^{-1}(b)$, $u_0, u_1 \in \pi^{-1}(B_g(q,R))$ and $0 < \sigma \leq (3-2\sqrt{2})R/(2\sqrt{2})$. Then*

$$d_{\hat{g}}(u_0,u_1) \leq \max\left\{\sqrt{\frac{1+\delta}{1-\delta}}, \sqrt{1+\sigma^2}\right\} d_{\hat{g}_0^\sigma}(\phi(u_0),\phi(u_1)) + \sqrt{2+18R^2}\delta + 4\delta.$$

Proof. Fix a small $\varepsilon > 0$ and let $c = (e^{\sqrt{-1}c_1}, c_2)\colon [0,1] \to S^1 \times \mathbb{R}^2$ be a path connecting $\phi(u_0), \phi(u_1)$ such that $\mathfrak{L}_{\hat{g}_0^\sigma}(c) < d_{\hat{g}_0^\sigma}(\phi(u_0),\phi(u_1)) + \varepsilon$. By Lemma 4.12, $\phi(u_0), \phi(u_1)$ are contained in $S^1 \times \mathcal{B}(\sqrt{1+\delta}R + \sigma)$. By the similar argument in the proof of Lemma 4.12 and the assumptions $R \geq 4\sqrt{2}(3-2\sqrt{2})^{-1}$, $\sigma \leq (3-2\sqrt{2})R/(2\sqrt{2})$, we have $\mathrm{Im}(c_2) \subset \mathcal{B}(3R)$ by taking $\varepsilon > 0$ sufficiently small.

Next we apply Lemma 4.20. By the assumption $R \geq 4\sqrt{2}(3-2\sqrt{2})^{-1}$ and $\sigma \leq (3-2\sqrt{2})R/(2\sqrt{2})$, we can see $4+\sigma \leq 3R$. Then we can apply Lemma 4.20 to c, hence we may assume that $\mathrm{Im}(c) \subset S^1 \times (\mathcal{B}(\tilde{R}) \setminus \mathcal{B}(\sigma))$, $\mathrm{Im}(c) \subset S^1 \times \overline{\mathcal{B}(\sigma)}$ or $c = \hat{c}$ satisfies one of (i)-(iv). If we assume (ii) in Lemma 4.20, then we denote by $\tilde{c}$ the horizontal lift of $c|_{[\tau_-,\tau_+]}$ with respect to ϕ. Then we have

$$\begin{aligned} d_{\hat{g}}(u_0,u_1) &\leq d_{\hat{g}}(u_0,\tilde{c}(\tau_-)) + \sqrt{\frac{1+\delta}{1-\delta}}\mathfrak{L}_{\hat{g}_0^\sigma}(c|_{[\tau_-,\tau_+]}) + d_{\hat{g}}(u_1,\tilde{c}(\tau_+)) \\ &\leq 4\delta + \max\left\{\sqrt{\frac{1+\delta}{1-\delta}}\mathfrak{L}_{\hat{g}_0^\sigma}(c), \sqrt{1+\sigma^2}\mathfrak{L}_{\hat{g}_0^\sigma}(c)\right\}. \end{aligned}$$

If $\mathrm{Im}(c) \subset S^1 \times (\mathcal{B}(3R) \setminus \mathcal{B}(\sigma))$, then let $\tilde{c}$ be the horizontal lift of c with respect to ϕ such that $\tilde{c}(0) = u_0$. By Proposition 4.17, we have $d_{\hat{g}}(\tilde{c}(1),u_1) < \sqrt{2+18R^2}\delta$. Therefore, we have

$$d_{\hat{g}}(u_0,u_1) \leq \sqrt{\frac{1+\delta}{1-\delta}}\mathfrak{L}_{\hat{g}_0^\sigma}(c) + \sqrt{2+18R^2}\delta.$$

In the other cases, we also have the result by the similar way. □

Proposition 4.22. *For any $u_0, u_1 \in S^1 \times \mathbb{R}^2$, we have*

$$d_{\hat{g}_0}(u_0, u_1) \le \sqrt{1+\sigma^2} d_{\hat{g}_0^\sigma}(u_0, u_1) + 4\sigma.$$

Proof. The proof is similar to that of Proposition 4.21. For any $\varepsilon > 0$, there is a piecewise smooth path $c\colon [0,1] \to S^1 \times \mathbb{R}^2$ connecting u_0 and u_1 such that $\mathfrak{L}_{\hat{g}_0^\sigma}(c) < d_{\hat{g}_0^\sigma}(u_0, u_1) + \varepsilon$. We apply Lemma 4.20 to c. For example, assume that $c = \hat{c}$ satisfies (ii) in Lemma 4.20. Then we have

$$d_{\hat{g}_0}(c(0), c(1)) \le d_{\hat{g}_0}(c(0), c(\tau_-)) + \mathfrak{L}_{\hat{g}_0}(c|_{[\tau_-,\tau_+]}) + d_{\hat{g}_0}(c(1), c(\tau_+))$$

By the second inequality of Proposition 4.16, we have

$$d_{\hat{g}_0}(c(0), c(1)) \le \sqrt{1+\sigma^2} \mathfrak{L}_{\hat{g}_0^\sigma}(c) + 4\sigma,$$

which gives the result. In the other cases, we have the result by the similar argument. ∎

The next proposition implies that ϕ is an almost isometry.

Proposition 4.23. *Let $b \in BS_1$. For any $R \ge 4\sqrt{2}(3-2\sqrt{2})^{-1}$ and $\varepsilon > 0$ there is a constant $\delta_{R,\varepsilon}, \sigma_{R,\varepsilon} > 0$ depending only on $R, \varepsilon > 0$ such that for any $q \in \mu^{-1}(b)$, $u_0, u_1 \in \pi^{-1}(B_g(q, R))$, if $0 < \delta \le \delta_{R,\varepsilon}$ and $0 < \sigma \le \sigma_{R,\varepsilon}$,*

$$|d_{\hat{g}}(u_0, u_1) - d_{\hat{g}_0}(\phi(u_0), \phi(u_1))| \le \varepsilon.$$

Proof. First of all, put

$$C := \max\left\{ \sqrt{\frac{1+\delta}{1-\delta}},\ \sqrt{1+\sigma^2} \right\} > 1,$$
$$\delta' := \max\left\{ \left(\sqrt{2+18R^2}+4\right)\delta,\ 4\sigma \right\} > 0,$$

then $\lim_{\delta,\sigma\to 0} C = 1$ and $\lim_{\delta,\sigma\to 0} \delta' = 0$. By Proposition 4.21 and by $d_{\hat{g}_0^\sigma} \le d_{\hat{g}_0}$, we have

$$d_{\hat{g}}(u_0, u_1) - d_{\hat{g}_0}(\phi(u_0), \phi(u_1)) \le (C-1) d_{\hat{g}_0^\sigma}(\phi(u_0), \phi(u_1)) + \delta'.$$

Then by Proposition 4.18 and (9), we obtain

$$\begin{aligned} d_{\hat{g}}(u_0, u_1) - d_{\hat{g}_0}(\phi(u_0), \phi(u_1)) &\le C(C-1) d_{\hat{g}}(u_0, u_1) + \delta' \\ &\le C(C-1)(2R+\pi) + \delta'. \end{aligned}$$

By Propositions 4.18 and 4.22, we have

$$\begin{aligned} C^{-2} d_{\hat{g}_0}(\phi(u_0), \phi(u_1)) - C^{-2}\delta' &\le d_{\hat{g}}(u_0, u_1) \\ &\le C^{-2} d_{\hat{g}}(u_0, u_1) + (1-C^{-2})(2R+\pi), \end{aligned}$$

hence we obtain

$$-(C^2-1)(2R+\pi) - \delta' \le d_{\hat{g}}(u_0, u_1) - d_{\hat{g}_0}(\phi(u_0), \phi(u_1)).$$

Since

$$(C^2-1)(2R+\pi)+\delta' \to 0, \quad C(C-1)(2R+\pi)+\delta' \to 0$$

as $\delta, \sigma \to 0$, we have the result. □

The next proposition implies the almost surjectivity of $\phi\colon \pi^{-1}(B_g(q,R)) \to S^1 \times \mathcal{B}(\sqrt{1+\delta}R+\sigma)$.

Proposition 4.24. *Let $b \in BS_1$. For any $R, \varepsilon > 0$ there are $\delta_{R,\varepsilon} > 0$ such that if $q \in \mu^{-1}(b)$, $0 < \delta < \delta_{R,\varepsilon}$ and $\sigma > 0$, we have*

$$\begin{aligned} S^1 \times \mathcal{B}\left(\frac{R-\delta}{\sqrt{1+\delta}}+\sigma\right) &\subset \phi(\pi^{-1}(B_g(q,R))) \\ &\subset S^1 \times \mathcal{B}(\sqrt{1+\delta}R+\sigma). \end{aligned}$$

Proof. First of all, one can see

$$\phi\left(\pi^{-1}(B_g(q,R))\right) \subset S^1 \times \mathcal{B}(\sqrt{1+\delta}R+\sigma)$$

by Lemma 4.12. Next we show

$$S^1 \times \mathcal{B}\left(\frac{R-\delta}{\sqrt{1+\delta}}+\sigma\right) \subset \phi\left(\pi^{-1}(B_g(q,R))\right).$$

Let $(e^{\sqrt{-1}t}, \xi) \in S^1 \times \mathcal{B}((1+\delta)^{-1/2}(R-\delta)+\sigma)$. By ($\star$5), there is $x \in U$ such that $\phi(e^{\sqrt{-1}t}, x) = (e^{\sqrt{-1}t}, \xi)$. Denote by $c\colon [0,1] \to \mathbb{R}^2$ the minimizing geodesic such that $c(0) = \mathbf{0}_{\mathbb{R}^2}$ and $c(1) = \xi$. Then there is a smooth path $\tilde{c}\colon (0,1] \to U$ such that $\zeta \circ \mu \circ \tilde{c} = c|_{(0,1]}$ and $\tilde{c}'(\tau) \in V_f^\perp$ for all $\tau \in (0,1]$ and $\tilde{c}(1) = x$. Assume $\mathbf{r}(\xi) \geq \sigma$. Since c is a geodesic departing from the $\mathbf{0}_{\mathbb{R}^2}$, there is a unique $\tau_0 \in [0,1]$ such that $\mathbf{r}(c(\tau_0)) = \sigma$. Now, we have

$$d_g(q,x) \leq \mathfrak{L}_g(\tilde{c}|_{[\tau_0,1]}) + \mathrm{diam}_g(\overline{U(\sigma)}) < \mathfrak{L}_g(\tilde{c}|_{[\tau_0,1]}) + \delta.$$

By ($\star$4), we have

$$\mathfrak{L}_g(\tilde{c}|_{[\tau_0,1]}) \leq \sqrt{1+\delta}\,\mathfrak{L}_{g_0}(c|_{[\tau_0,1]}) = \sqrt{1+\delta}\,(\mathbf{r}(\xi)-\sigma),$$

hence $d_g(q,x) < R$. Thus we obtain $(e^{\sqrt{-1}t}, x) \in \pi^{-1}(B_g(q,R))$. If $\mathbf{r}(\xi) < \sigma$, then we can see

$$d_g(q,x) \leq \mathrm{diam}_g(\overline{U(\sigma)}) < \delta.$$

By taking $\delta_{R,\varepsilon} \leq R$, we have $(e^{\sqrt{-1}t}, x) \in \pi^{-1}(B_g(q,R))$. □

Theorem 4.25. *Let $b \in BS_1$, $q \in \mu^{-1}(b)$, $p \in \pi^{-1}(q)$. For any $R \geq 4\sqrt{2}(3-2\sqrt{2})^{-1}$ and $\varepsilon > 0$ there is a constant $\delta_{R,\varepsilon}, \sigma_{R,\varepsilon} > 0$ depending only on $R, \varepsilon > 0$ such that if $0 < \delta \leq \delta_{R,\varepsilon}$ and $0 < \sigma \leq \sigma_{R,\varepsilon}$, then*

$$\phi\colon \left(\pi^{-1}(B_g(q,R)), p\right) \to \left(S^1 \times \mathcal{B}(\sqrt{1+\delta}R+\sigma), (1_{S^1}, \mathbf{0}_{\mathbb{R}^2})\right)$$

is an S^1-equivariant Borel ε-isometry.

Proof. It is easy to check $S^1 \times \mathcal{B}(r_1 + r_2) \subset B_{d_{\hat{g}_0}}(S^1 \times \mathcal{B}(r_1), r_2)$ for $r_1, r_2 > 0$. Then by Proposition 4.24, we have

$$S^1 \times \mathcal{B}(\sqrt{1+\delta}R + \sigma) \subset B_{d_{\hat{g}_0}}\left(\phi(\pi^{-1}(B_g(q,R))), \frac{\delta(R+1)}{\sqrt{1+\delta}}\right).$$

Since $\lim_{\delta\to 0} \delta(R+1)/\sqrt{1+\delta} = 0$, hence we have the result by combining with Proposition 4.23. □

Theorem 4.26. *Let $b \in BS_1$. We have*

$$\left| K \int_{\mathbb{S}|_U} f \circ \phi \, d\nu_{\hat{g}} - \int_{S^1 \times \mathbb{R}^2} f \, dt d\nu_{g_0} \right| \le 2\pi\delta \sup|f| \, \nu_{g_0}(\mathcal{B}(R))$$

for $f \in C(S^1 \times \mathbb{R}^2)$ with $\mathrm{supp}(f) \subset S^1 \times \mathcal{B}(R)$.

Proof. Since $d\nu_{\hat{g}} = dt d\nu_g$, we have

$$\int_{S^1 \times U} f \circ \phi \, d\nu_{\hat{g}} = \int_{S^1 \times \mathbb{R}^2} f \, dt d\zeta_* \nu_B. \tag{10}$$

Next we put $f_+ := \min\{f, 0\}$, $f_- := \min\{-f, 0\}$ and write $f = f_+ - f_-$. By ($\star$7), we have

$$\begin{aligned}\int_{S^1 \times \mathbb{R}^2} \left(\frac{f_+}{1+\delta} - (1+\delta) f_-\right) dt d\nu_{g_0} &\le K \int_{S^1 \times \mathbb{R}^2} f \, dt d\zeta_* \nu_B \\ &\le \int_{S^1 \times \mathbb{R}^2} \left((1+\delta) f_+ - \frac{f_-}{1+\delta}\right) dt d\nu_{g_0},\end{aligned}$$

hence we obtain

$$\begin{aligned}\left| K \int_{S^1 \times \mathbb{R}^2} f \, dt d\zeta_* \nu_B - \int_{S^1 \times \mathbb{R}^2} f \, dt d\nu_{g_0} \right| &\le \delta \int_{S^1 \times \mathbb{R}^2} (f_+ + f_-) \, dt d\nu_{g_0} \\ &\le \delta \sup|f| \cdot 2\pi \nu_{g_0}(\mathcal{B}(R)).\end{aligned}$$

Combining with (10), we obtain the result. □

For general positive integer m and $b \in BS_m^{\mathrm{str}}$, we can show the generalization of Lemma 4.11 as follows.

Lemma 4.27. *Let $b \in BS_m^{\mathrm{str}}$, i.e., the holonomy group of $(L|_{X_b}, \nabla|_{X_b})$ is given by $\{e^{2\pi\sqrt{-1}l/m}; l = 0, 1, \dots, m-1\}$. Suppose that there are an open neighborhood W of b and $\gamma \in \Omega^1(U)$ with $\omega|_U = d\gamma$ such that the triple (b, W, γ) satisfies ($\star$1, 2, 3), where $U := \mu^{-1}(W)$. Then there exist covering maps $p_m \colon \tilde{U}_m \to U$ and $\hat{p}_m \colon S^1 \times \tilde{U}_m \to \mathbb{S}(L|_U, h)$ such that $\pi \circ \hat{p}_m = p_m \circ \tilde{\pi}_m$ and we have the following, where $\tilde{\pi}_m \colon S^1 \times \tilde{U}_m \to \tilde{U}_m$ is the projection to the second component.*

(i) $\hat{p}_m^* \hat{g} = (dt - p_m^* \gamma)^2 + p_m^* g$.

(ii) *The group of the Deck transformations of p_m is $\mathbb{Z}/m\mathbb{Z}$.*

(iii) *Denote by $\beta \colon \mathbb{Z}/m\mathbb{Z} \to \mathrm{Diff}(\tilde{U}_m)$ the deck transformation of p_m. Then the map $\hat{\beta} \colon \mathbb{Z}/m\mathbb{Z} \to \mathrm{Diff}(S^1 \times \tilde{U}_m)$ defined by*

$$\hat{\beta}(e^{2\pi\sqrt{-1}l/m})\left(e^{\sqrt{-1}t}, x\right) = \left(e^{\sqrt{-1}(t - 2\pi l/m)}, \beta(e^{2\pi\sqrt{-1}l/m})x\right)$$

is the deck transformation of $\hat{p}_m$.

(iv) $\hat{p}_m(e^{\sqrt{-1}(t+t')}, x) = \hat{p}_m(e^{\sqrt{-1}t}, x) \cdot e^{\sqrt{-1}t'}$.

Proof. Let $\mathbf{E}_1$ and γ_1 be as above. Since ι_b^* is an isomorphism, there is a closed one form γ' on U such that $\int_C \gamma' = \int_C \gamma_1$ for all $C \in H_1(X_b, \mathbb{Z})$. Then $\gamma - \gamma_1 + \gamma'$ is a closed 1-form on U such that $\int_C (\gamma - \gamma_1 + \gamma') = 0$ for all $C \in H_1(X_b, \mathbb{Z})$ by ($\star$3). By ($\star$1), there is a function $\varphi_1 \in C^\infty(U)$ such that $\gamma - \gamma_1 + \gamma' = d\varphi_1$.

Denote by $p\colon \tilde{U} \to U$ the universal cover of U. Then there is $\varphi_2 \in C^\infty(\tilde{U})$ such that $p^*\gamma' = d\varphi_2$. If we denote by $\beta'\colon \pi_1(U) \to \mathrm{Diff}(\tilde{U})$ the deck transformation of p, then there exists a group homomorphism $F\colon \pi_1(U) \to \mathbb{R}$ with $\varphi_2(\beta'(h)(\tilde{x})) = \varphi_2(\tilde{x}) + F(h)$. Moreover, by the assumption for the holonomy groups, we can see that $\{\int_C \gamma' \in \mathbb{R};\, C \in H_1(U, \mathbb{Z})\} = (2\pi/m)\mathbb{Z}$, hence the image of F is equal to $(2\pi/m)\mathbb{Z}$. Now, let $H \subset \pi_1(U)$ be the subgroup defined by $H = \{h \in \pi_1(U);\, F(h) \in 2\pi\mathbb{Z}\}$ and put $\tilde{U}_m := \tilde{U}/H$, then we obtain an m-fold covering $p_m\colon \tilde{U}_m \to U$. Since we have

$$\pi_1(U)/H \cong \mathbb{Z}/m\mathbb{Z} = \left\{ e^{2\pi\sqrt{-1}l/m};\, l = 0, 1, \ldots, m-1 \right\},$$

β' induces the deck transformation $\beta\colon \mathbb{Z}/m\mathbb{Z} \to \mathrm{Diff}(\tilde{U}_m)$ of p_m.

Define a $\mathbb{Z}/m\mathbb{Z}$-action on $S^1 \times \tilde{U}_m$ by

$$e^{2\pi\sqrt{-1}l/m} \cdot (e^{\sqrt{-1}t}, x) := (e^{\sqrt{-1}(t - 2\pi l/m)},\, \beta(e^{2\pi\sqrt{-1}l/m})(x)),$$

and a smooth map $\hat{p}_m\colon S^1 \times \tilde{U}_m \to \mathbb{S}(L|_U, h)$ by

$$\left(e^{\sqrt{-1}t}, \tilde{x} \bmod H\right) \mapsto e^{\sqrt{-1}(t - \varphi_1(p(\tilde{x})) + \varphi_2(\tilde{x}))} \left(\mathbf{E}_1\right)_{p(\tilde{x})}$$

for $\tilde{x} \in \tilde{U}$ and $e^{\sqrt{-1}t} \in S^1$. Here, φ_2 descends to the function on $\tilde{U}_m$. Since $\hat{p}_m$ is $\mathbb{Z}/m\mathbb{Z}$-invariant, it induces the diffeomorphism $(S^1 \times \tilde{U}_m)/(\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{S}(L|_U, h)$. By the definition of $\hat{p}_m$, we can see

$$\hat{p}_m^* \hat{g} = \hat{p}_m^* \left((dt - \gamma_1)^2 + g \right) = (dt - p_m^*\gamma)^2 + p_m^* g.$$

□

If $b \in BS_m^{\mathrm{str}}$, we follow the argument in this subsection for $(S^1 \times \tilde{U}_m, \hat{p}_m^*\hat{g})$ instead of $(\mathbb{S}|_U, \hat{g})$. Then we can construct the approximation map between $(S^1 \times \tilde{U}_m, \hat{p}_m^*\hat{g})$ and $(S^1 \times \mathbb{R}^2, \hat{g}_0)$ which is S^1-equivariant and $\mathbb{Z}/m\mathbb{Z}$-equivariant. Here, the $\mathbb{Z}/m\mathbb{Z}$-action on on $S^1 \times \mathbb{R}^2$ is defined by

$$\left(e^{\sqrt{-1}t}, \xi\right) \cdot e^{2\pi l\sqrt{-1}/m} := \left(e^{\sqrt{-1}(t - 2\pi l/m)}, \xi\right).$$

then the limit space should be the quotient space $S^1 \times \mathbb{R}^2/(\mathbb{Z}/m\mathbb{Z})$ with the metric naturally induced by $\hat{g}_0$. This space is isometric to $(S^1 \times \mathbb{R}^2, d_{\hat{g}_{0,m}})$, where $\hat{g}_{0,m}$ is the metric as in Subsection 4.1. Then we obtain the generalization of Theorems 4.25 and 4.26 as follows.

THEOREM 4.28. *Let $b \in BS_m^{\mathrm{str}}$, $q \in \mu^{-1}(b)$, $p \in \pi^{-1}(q)$. For any $R \geq 4\sqrt{2}(3 - 2\sqrt{2})^{-1}$ and $\varepsilon > 0$ there is a constant $\delta_{R,\varepsilon}, \sigma_{R,\varepsilon} > 0$ depending only on $R, \varepsilon > 0$ such that if $0 < \delta \leq \delta_{R,\varepsilon}$ and $0 < \sigma \leq \sigma_{R,\varepsilon}$, then*

$$\phi\colon \left(\pi^{-1}(B_g(q, R)), p\right) \to \left(S^1 \times \mathcal{B}(\sqrt{1+\delta}R + \sigma), (1_{S^1}, \mathbf{0}_{\mathbb{R}^2})\right)$$

is an S^1-equivariant Borel ε-isometry with respect to the distance functions $d_{\hat{g}}$ and $d_{\hat{g}_{0,m}}$.

THEOREM 4.29. *Let $b \in BS_m^{\mathrm{str}}$. We have*

$$\left| K \int_{\mathbb{S}|_U} f \circ \phi \, d\nu_{\hat{g}} - \int_{S^1 \times \mathbb{R}^2} f \, dt d\nu_{g_0} \right| \leq 2\pi\delta \sup |f| \, \nu_{g_0}(\mathcal{B}(R))$$

for $f \in C(S^1 \times \mathbb{R}^2)$ with $\mathrm{supp}(f) \subset S^1 \times \mathcal{B}(R)$.

4.4. Convergence. Let $(X, \omega, L, h, \nabla)$ and $\mu\colon X \to B$ be as in the previous subsection and let $\{g_s\}_{s>0}$ be a family of Riemannian metrics on X. Define $\hat{g}_s$ by g_s, ∇ as in (2).

DEFINITION 4.30. Let $b \in B$ and W be an open neighborhood of b such that $W \setminus \{b\} \subset B^{\mathrm{rg}}$. Let $K_s > 0$ and put $U := \mu^{-1}(W)$. We write

$$(g_s, K_s, b, W) \overset{s \to 0}{\to} (\mathbb{R}^2, g_0)$$

if there are $R_0 > 0$ and $s_R > 0$ for every $R \geq R_0$ such that for any $0 < s \leq s_R$ there are $\zeta_{s,R}\colon W \to \mathbb{R}^2$, $\gamma_{s,R} \in \Omega^1(U)$, and $\sigma_{s,R}, \delta_{s,R} > 0$ with $\lim_{s \to 0} \sigma_{s,R} = \lim_{s \to 0} \delta_{s,R} = 0$ such that the following tuple

$$(g_s, b, W, R, \gamma_{s,R}, \zeta_{s,R}, \sigma_{s,R}, \delta_{s,R}, K_s)$$

satisfies ($\star$1-7) for all $R \geq R_0$ and $0 < s \leq s_R$.

THEOREM 4.31. *Let $b \in B$, W be an open neighborhood of b such that $W \setminus \{b\} \subset B^{\mathrm{rg}}$ and $U := \mu^{-1}(W)$. Fix $q \in \mu^{-1}(b)$. Assume that there are constants $K_s > 0$ such that $(g_s, K_s, b, W) \to (\mathbb{R}^2, g_0)$ as $s \to 0$. Then for any $R > 0$ there is $s_R > 0$ such that $B_{g_s}(q, R) \subset U$ for all $0 < s \leq s_R$, and b is not an accumulation point of $BS_m \cap W$. Moreover, if $b \in BS_m^{\mathrm{str}}$, then for some $p \in \pi^{-1}(q)$ we have*

$$(\mathbb{S}, d_{\hat{g}_s}, K_s \nu_{\hat{g}_s}, p) \overset{S^1\text{-pmGH}}{\longrightarrow} (S^1 \times \mathbb{R}^2, d_{\hat{g}_{0,m}}, dt d\nu_{g_0}, (1_{S^1}, \mathbf{0}_{\mathbb{R}^2}))$$

as $s \to 0$.

Proof. Take s_R as in Definition 4.30 and replace by the smaller one if necessary such that $\sqrt{1 + \delta_{s,R}} R + \sigma_{s,R} \leq 3R$ for all $0 < s \leq s_R$. Then by Lemma 4.12 and ($\star$5), we have $B_{g_s}(q, R) \subset U(3R) \subset U$ for $0 < s \leq s_R$. By Lemma 4.10, b is not an accumulation point of $BS_m \cap W$.

Let $\sigma_{R,\varepsilon}, \delta_{R,\varepsilon}$ be as in Theorem 4.28. Fix a positive integer k, then take $0 < s_k \leq s_R$ such that $\sigma_s \leq \sigma_{R_0 + k, k^{-1}}$ and $\delta_s \leq \delta_{R_0 + k, k^{-1}}$ for any $0 < s \leq s_k$. We determine s_k inductively such that

$$s_{k+1} \leq \frac{s_k}{2}.$$

If we put

$$\varepsilon_s := k^{-1}, \quad R_s := R_0 + k, \quad R'_s := \sqrt{1 + \delta_s}(R_0 + k) + \sigma_s$$

for $s_{k+1} \le s < s_k$, then

$$\phi\colon \left(\pi^{-1}(B_{g_s}(q,R_s)),p\right) \to \left(S^1 \times \mathcal{B}(R'_s),(1_{S^1},\mathbf{0}_{\mathbb{R}^2})\right)$$

is an S^1-equivariant Borel ε_s-isometry and $\lim_{s\to 0}\varepsilon_s = 0$, $\lim_{s\to 0} R_s = \lim_{s\to 0} R'_s = \infty$.

Next we take $f \in C(S^1 \times \mathbb{R}^2)$ whose support is compact. Take $R \ge R_0$ such that $\mathrm{supp}(f) \subset S^1 \times \mathcal{B}(R)$. Then by Theorem 4.29, we have

$$\lim_{s\to 0}\left| K_s \int_{\mathbb{S}} f\circ\phi \, d\nu_{\hat{g}_s} - \int_{S^1\times\mathbb{R}^2} f\, dt\, d\nu_{g_0}\right| \le \lim_{s\to 0} 2\pi\delta_s \sup|f|\,\nu_{g_0}(\mathcal{B}(R)) = 0.$$

□

Now, we show some results which is needed in Section 8.

Lemma 4.32. *Let $q \in \mu^{-1}(b)$ and $p \in \pi^{-1}(q)$. Then we have*

$$\pi^{-1}(U(r)) \subset B_{\hat{g}}(p, \sqrt{1+\delta}r + \delta + \pi)$$

for any $0 < r \le 3R$.

Proof. Let $u \in \pi^{-1}(U(r))$ and take the minimizing geodesic $c\colon [0,1] \to \mathbb{R}^2$ with $c(0) = \mathbf{0}_{\mathbb{R}^2}$ and $c(1) = \zeta\circ\mu\circ\pi(u)$. Suppose $\mathbf{r}(c(1)) \ge \sigma$. Then there exists $0 < \tau_0 \le 1$ such that $c(\tau_0) = \sigma$. Let $\tilde{c} : [\tau_0, 1] \to X$ be a smooth path such that $\tilde{c}(1) = \pi(u)$, $\zeta\circ\mu\circ\tilde{c} = c|_{[\tau_0,1]}$ and $d\mu(\tilde{c}'(\tau)) = 0$. Then we have

$$d_g(\tilde{c}(\tau_0),\pi(u)) \le \mathfrak{L}_g(\tilde{c}) \le \sqrt{1+\delta}\,\mathfrak{L}_{g_0}(c|_{[\tau_0,1]}) < \sqrt{1+\delta}(r-\sigma)$$

by ($\star$4). Moreover, by ($\star$6), we have

$$d_g(q,\tilde{c}(\tau_0)) < \delta.$$

Therefore, we obtain

$$d_g(q,\pi(u)) \le \delta + \mathfrak{L}_g(\tilde{c}) < \sqrt{1+\delta}r + \delta.$$

If $\mathbf{r}(c(1)) < \sigma$, we have $d_g(q,\pi(u)) < \delta$. By (9), we have

$$d_{\hat{g}}(p,u) \le \sqrt{1+\delta}r + \delta + \pi.$$

□

Proposition 4.33. *Let $b \in B$, W be an open neighborhood of b such that $W \setminus \{b\} \subset B^{\mathrm{rg}}$ and $U := \mu^{-1}(W)$. Fix $q \in \mu^{-1}(b)$ and $p \in \pi^{-1}(q)$. Assume that there are constants $K_s > 0$ such that $(g_s, K_s, b, W) \to (\mathbb{R}^2, g_0)$ as $s \to 0$. Let $\zeta_{s,R}\colon W \to \mathbb{R}^2$ be as in Definition 4.30. Then there is $s_R > 0$ for every $R \ge 7$ such that*

$$(\zeta_{s,R}\circ\mu\circ\pi)^{-1}(\mathcal{B}(R/2)) \subset B_{\hat{g}_s}(p,R)$$

for any $0 < s \le s_R$.

Proof. Let s_R and $\delta_{s,R}$ be as in Definition 4.30. By Lemma 4.32, we have

$$(\zeta_{s,R}\circ\mu\circ\pi)^{-1}(\mathcal{B}(r)) \subset B_{\hat{g}_s}(p, \sqrt{1+\delta_{s,R}}r + \delta_{s,R} + \pi)$$

for $0 < r \le 3R$. Since $R/2 \ge 7/2 > \pi$, we can replace s_R smaller such that we have $\sqrt{1+\delta_{s,R}}R/2 + \delta_{s,R} + \pi \le R$ for every $0 < s \le s_R$. Then we have the result by putting $r = R/2$. □

5. The approximation of hyper-Kähler metrics. In this section we review a construction of a family of Riemannian metric on a $K3$ surface, which is a good approximation of hyper-Kähler metrics $(g_s)_s$ tending to a large complex structure limit based on [10]. See also [5].

Let $(X, \omega_1, \omega_2, \omega_3)$ be a hyper-Kähler manifold. As we have already mentioned in Subsection 2.3, the special Lagrangian fibrations on X is equivalent to the elliptic fibrations on X_{J_3}. Moreover, $\Theta := \omega_1 + \sqrt{-1}\omega_2$ is a holomorphic volume form on X_{J_3} by Remark 2.2. Throughout this section we consider complex surfaces equipped with holomorphic volume forms and elliptic fibrations.

To construct the approximating family of metrics, we need two families of hyper-Kähler metrics. One is the *semi-flat metric* defined on the elliptic surface with no singular fibers, and the other is the *Ooguri-Vafa metric* defined on the neighborhood of the singular fibers of Kodaira type I_1. Gluing them by cut-off functions, we obtain the approximating family.

5.1. Semi-flat metrics. In this subsection we explain the construction of semi-flat metrics following [10]. The semi-flat metrics are Ricci-flat Kähler metrics on the elliptic surfaces, which were first constructed by Greene, Shapere, Vafa and Yau in [8].

Let X be a complex surface, not necessarily compact, with a holomorphic volume form $\Theta \in \Omega^{2,0}(X)$, B be a 1-dimensional complex manifold and $\mu\colon X \to B$ be a nonsingular elliptic fibration, that is, a holomorphic surjective map such that each $b \in B$ is a regular value of μ and $\mu^{-1}(b)$ is an elliptic curve.

Examples of such X can be constructed as follows. Denote by $\mathcal{T}_B^*$ the holomorphic cotangent bundle of B. A subset $\Lambda \subset \mathcal{T}_B^*$ is a *holomorphically varying family of lattices* if there are an open cover $B = \bigcup_i U_i$ and holomorphic functions $\tau_{i,1}, \tau_{i,2}$ defined on U_i such that $\mathrm{Im}(\overline{\tau_{i,1}(y)}\tau_{i,2}(y)) \neq 0$ and $\Lambda_y := \Lambda \cap \mathcal{T}_B^*|_y$ is given by

$$\Lambda_y = \{m_1\tau_{i,1}(y)dy + m_2\tau_{i,2}(y)dy;\ m_1, m_2 \in \mathbb{Z}\}$$

for any $y \in U_i$. Let $\Theta_{\mathrm{can}} = dx \wedge dy$ be the canonical holomorphic 2-form on $\mathcal{T}_B^*$, where (x, y) is a coordinate on $\mathcal{T}_B^*$ defined by $xdy \in \mathcal{T}_B^*$. Then Θ_{can} descends to $X = \mathcal{T}_B^*/\Lambda$ and the projection map $\mu_{\mathrm{can}}\colon X \to B$ determines an elliptic fibration. Obviously, the zero section of $\mathcal{T}_B^*$ induces a holomorphic section of $X \to B$. Conversely, every nonsingular elliptic fibration with a holomorphic 2-form and a holomorphic section can be obtained by the above process.

Let $\mathbf{a} \in \Omega^2(B, \mathbb{C})$. Another complex structure on $\mathcal{T}_B^*/\Lambda$ is defined so that the closed 2-form $\Theta := \Theta_{\mathrm{can}} + \mu_{\mathrm{can}}^*\mathbf{a}$ is holomorphic. Then $\mu_{\mathrm{can}}\colon \mathcal{T}_B^*/\Lambda \to B$ is also holomorphic with respect to this complex structure. In this case μ_{can} does not need to have holomorphic sections.

Let

$$\eta := \frac{\sqrt{-1}}{2}\left\{\mathbf{W}(dx + \mathbf{b}dy) \wedge \overline{(dx + \mathbf{b}dy)} + \mathbf{W}^{-1}dy \wedge d\bar{y}\right\},$$

where $\mathbf{W} \in C^\infty(\mathcal{T}_B^*/\Lambda, \mathbb{R})$ is positive valued and $\mathbf{b} \in C^\infty(\mathcal{T}_B^*/\Lambda, \mathbb{C})$. Then one can see that

$$\eta^2 = \mathrm{Re}(\Theta_{\mathrm{can}})^2 = \mathrm{Im}(\Theta_{\mathrm{can}})^2, \quad \eta \wedge \Theta_{\mathrm{can}} = 0.$$

η is called a *semi-flat metric* on $\mathcal{T}_B^*/\Lambda$ if it is Kähler. η is Kähler iff

$$\frac{\partial \mathbf{W}}{\partial y} = \frac{\partial(\mathbf{W}\mathbf{b})}{\partial x}, \tag{11}$$

$$\frac{\partial(\mathbf{W}\bar{\mathbf{b}})}{\partial y} = \frac{\partial}{\partial x}\left\{\mathbf{W}(\mathbf{W}^{-2}+|\mathbf{b}|^2)\right\}. \tag{12}$$

Now, take an oriented $\mathbb{Z}$-basis $\{\tau_{i,1},\tau_{i,2}\}$ of $\Lambda|_{U_i}$ such that $\mathrm{Im}(\bar{\tau}_{i,1}\tau_{i,2})>0$. If we put

$$\mathbf{W} = \frac{s}{\mathrm{Im}(\bar{\tau}_{i,1}\tau_{i,2})},$$
$$\mathbf{b} = -\frac{\mathbf{W}}{s}\left\{\mathrm{Im}(\tau_{i,2}\bar{x})\frac{\partial\tau_{i,1}}{\partial y} + \mathrm{Im}(\bar{\tau}_{i,1}x)\frac{\partial\tau_{i,2}}{\partial y}\right\},$$

then we have (11) and (12) for any positive constant s, and they are independent of the local coordinate. Hence we obtain the Ricci-flat Kähler metric

$$\eta_s^{\mathrm{SF}} = \eta$$

defined on $\mathcal{T}_B^*/\Lambda$ and we call it the *standard semi-flat metric.* The triple $(\eta_s^{\mathrm{SF}}, \mathrm{Re}(\Theta_{\mathrm{can}}), \mathrm{Im}(\Theta_{\mathrm{can}}))$ forms a hyper-Kähler structure on $\mathcal{T}_B^*/\Lambda$. We have

$$s = \int_{\mu_{\mathrm{can}}^{-1}(b)} \eta_s^{\mathrm{SF}}.$$

Let $\mu\colon X\to B$ be an elliptic $K3$ surface with a holomorphic section, $\mathrm{Crt}\subset B$ be the subset consisting of the critical values of μ and put $X^{\mathrm{rg}} = X\setminus\mu^{-1}(\mathrm{Crt})$, $B^{\mathrm{rg}} = B\setminus\mathrm{Crt}$. Since $\mu\colon X^{\mathrm{rg}}\to B^{\mathrm{rg}}$ has a holomorphic section, there exist a holomorphically varying family of lattices $\Lambda\subset\mathcal{T}_{B^{\mathrm{rg}}}^*$ and a biholomorphic map $X^{\mathrm{rg}}\to\mathcal{T}_{B^{\mathrm{rg}}}^*/\Lambda$ which identifies Θ and Θ_{can}. Therefore, X^{rg} admits the standard semi-flat metric η_s^{SF}.

5.2. Ooguri-Vafa metrics. Here we explain the construction of Ooguri-Vafa metrics following [10]. The Ooguri-Vafa metrics were first constructed by Ooguri and Vafa in [18]. Let $r,s>0$, $D(r):=\{z\in\mathbb{C};\,|z|<r\}$ and

$$\mathcal{U}(r,s) := D(r)\times\mathbb{R}\setminus\{(0,sn)\in D(r)\times\mathbb{R};\, n\in\mathbb{Z}\}.$$

Put

$$V_s^0(u) = \frac{1}{4\pi}\sum_{n\in\mathbb{Z}^\times}\left(\frac{1}{\sqrt{u_1^2+u_2^2+(u_3-sn)^2}} - \frac{1}{s|n|}\right) + \frac{1}{4\pi|u|}.$$

Then V_s^0 is a harmonic function on $\mathcal{U}(r,s)$, hence the 2-form $\star dV_s^0$ represents the cohomology class in $H^2(\mathcal{U}(r,s),\mathbb{R})$. Here, $\star$ is the Hodge star operator of the Euclidean metric on $\mathbb{R}^3$. Let $u^\sharp\colon\tilde{X}_{\mathrm{OV}}^\sharp\to\mathcal{U}(r,s)$ be the principal S^1-bundle over $\mathcal{U}(r,s)$ whose first Chern class is equal to $[\star dV_s^0]\in H^2(\mathcal{U}(r,s),\mathbb{Z})$. Then there is an S^1-connection $\sqrt{-1}\alpha\in\Omega^1(\mathcal{U}(r,s),\sqrt{-1}\mathbb{R})$ such that $d\alpha/2\pi = (u^\sharp)^*(\star dV_s^0)$. Now, using the standard coordinate on $\mathbb{C}\times\mathbb{R}\cong\mathbb{R}^3$, put $u^\sharp=(u_1,u_2,u_3)$. Then the following 2-forms

$$\omega_{1,s} = du_1\wedge\frac{\alpha}{2\pi} + V_s^0 du_2\wedge du_3,$$
$$\omega_{2,s} = du_2\wedge\frac{\alpha}{2\pi} + V_s^0 du_3\wedge du_1,$$
$$\omega_{3,s} = du_3\wedge\frac{\alpha}{2\pi} + V_s^0 du_1\wedge du_2$$

satisfy $\omega_{i,s}\wedge\omega_{j,s}=0$ for $i\neq j$ and $\omega_{1,s}^2=\omega_{2,s}^2=\omega_{3,s}^2$. In the above expressions, we suppose that V_s^0 is the pullback $(u^\sharp)^*V_s^0$, however, we omit $u^\sharp$ for the simplicity of the notations. Taking $r>0$ sufficiently small, we may suppose $\omega_{1,s}^2$ is nowhere vanishing, then they form a hyper-Kähler structure on $\tilde{X}_{\mathrm{OV}}^\sharp$. Here, by replacing V_s^0 with $V_s^0+h(u_1,u_2)$ for some harmonic function $h(u_1,u_2)$ on $D(r)$, we obtain other hyper-Kähler structures.

Moreover, there exist a smooth 4-manifold $\tilde{X}_{\mathrm{OV}}$, open embedding $\tilde{X}_{\mathrm{OV}}^\sharp\subset\tilde{X}_{\mathrm{OV}}$ and smooth map $u\colon\tilde{X}_{\mathrm{OV}}\to D(r)\times\mathbb{R}$ such that $u|_{\tilde{X}_{\mathrm{OV}}^\sharp}=u^\sharp$, $\tilde{X}_{\mathrm{OV}}\backslash\tilde{X}_{\mathrm{OV}}^\sharp=\{p_n;\,n\in\mathbb{Z}\}$ and $u(p_n)=(0,sn)$. Then one can see that $\omega_{i,s}$ extends to the smooth 2-form on $\tilde{X}_{\mathrm{OV}}$, which we denote by $\omega_{i,s}$ again. Thus we obtain a hyper-Kähler manifold $(\tilde{X}_{\mathrm{OV}},\omega_{1,s},\omega_{2,s},\omega_{3,s})$.

There is a free $\mathbb{Z}$-action on $\tilde{X}_{\mathrm{OV}}$ preserving $\omega_{i,s}$, u_1, u_2, α and satisfies $u_3(p\cdot n)=u_3(p)+sn$ for $n\in\mathbb{Z}$. Then we can see the action also preserves V_s^0 and $\omega_{i,s}$. Hence $\omega_{i,s}$ descend to 2-forms on the quotient space $X_{\mathrm{OV}}:=\tilde{X}_{\mathrm{OV}}/\mathbb{Z}$ which we denote by $\omega_{i,s}$ again. The hyper-Kähler manifold $(X_{\mathrm{OV}},\omega_{1,s},\omega_{2,s},\omega_{3,s})$ is called the Ooguri-Vafa metric.

Here, we regard X_{OV} as a complex manifold such that $\omega_{1,s}+\sqrt{-1}\omega_{2,s}$ is a holomorphic 2-form. Put $\mu_{\mathrm{OV}}=u_1+\sqrt{-1}u_2\colon X_{\mathrm{OV}}\to D(r)$. Then μ_{OV} is an elliptic fibration over $D(r)$. The fiber $\mu_{\mathrm{OV}}^{-1}(b)$ is nonsingular if $b\neq 0$ and $\mu_{\mathrm{OV}}^{-1}(0)$ is the singular fiber of Kodaira type I_1, with the critical point $0_{\mathrm{OV}}:=p_0 \mod \mathbb{Z}$. Here, we have

$$s=\int_{\mu_{\mathrm{OV}}^{-1}(b)}\omega_{3,s}.$$

5.3. Almost Ricci-flat Kähler metric. In this subsection let X be a $K3$ surface with an elliptic fibration $\mu\colon X\to\mathbb{P}^1$ over the complex projective line and a holomorphic volume form Θ, and suppose that all of the singular fibers of μ are of Kodaira type I_1, hence there are exactly 24 singular fibers. We denote by $\mathrm{Crt}:=\{b_1,\dots,b_{24}\}\subset\mathbb{P}^1$ the set of critical values.

For $\mathbf{q}=1,\dots,24$, let $X_{\mathbf{q}}=X_{\mathrm{OV}}$ be 24 copies of the underlying manifold on which the Ooguri-Vafa metric is defined. Put

$$\begin{aligned}V_{s,\mathbf{q}}(u)&:=\frac{1}{4\pi}\sum_{n\in\mathbb{Z}^\times}\left(\frac{1}{\sqrt{u_1^2+u_2^2+(u_3-sn)^2}}-\frac{1}{s|n|}\right)+\frac{1}{4\pi|u|}\\&\qquad+a_s+\frac{h_{\mathbf{q}}(u_1,u_2)}{s},\\ a_s&:=\frac{\lim_{n\to\infty}(\sum_{k=1}^n 1/k-\log n)-\log(2s)}{2\pi s}\end{aligned}$$

for some harmonic function $h_{\mathbf{q}}$, and define the hyper-Kähler structure on $X_{\mathbf{q}}$ by

$$\begin{aligned}\omega_{1,s,\mathbf{q}}&=du_1\wedge\alpha+V_{s,\mathbf{q}}du_2\wedge du_3,\\ \omega_{2,s,\mathbf{q}}&=du_2\wedge\alpha+V_{s,\mathbf{q}}du_3\wedge du_1,\\ \omega_{3,s,\mathbf{q}}&=du_3\wedge\alpha+V_{s,\mathbf{q}}du_1\wedge du_2\end{aligned}$$

Although these are defined on the universal covering space of X_{OV}, they descend to X_{OV}. The constant a_s normalizes $V_{s,\mathbf{q}}$ so that we have

$$\int_0^s V_{s,\mathbf{q}}(u_1,u_2,t)dt=-\frac{1}{2\pi}\log\sqrt{u_1^2+u_2^2}+h_{\mathbf{q}}(u_1,u_2).$$

Here, we regard $X_{\mathbf{q}}$ as a complex manifold such that $\omega_{1,s,\mathbf{q}} + \sqrt{-1}\omega_{2,s,\mathbf{q}}$ is a holomorphic 2-form. Put $\mu_{\mathbf{q}} = u_1 + \sqrt{-1}u_2 \colon X_{\mathbf{q}} \to \mathbb{C}$. By taking $r_2^{\mathbf{q}} > 0$ sufficiently small so that $-(\log\sqrt{u_1^2+u_2^2})/2\pi + h_{\mathbf{q}} > 0$ on $D(r_2^{\mathbf{q}})$, we may suppose $V_{s,\mathbf{q}}(u)$ is positive on $\mu_{\mathbf{q}}^{-1}(D(r_2^{\mathbf{q}}))$ for sufficiently small $s > 0$. Therefore, we can take $s_0 > 0$ such that $V_{s,\mathbf{q}}(u)$ is positive on $\mu_{\mathbf{q}}^{-1}(D(r_2^{\mathbf{q}}))$ for any $0 < s \le s_0$ and $\mathbf{q}$. Now, since $\mu_{\mathbf{q}} \colon \mu_{\mathbf{q}}^{-1}(D(r_2^{\mathbf{q}}) \setminus \{0\}) \to D(r_2^{\mathbf{q}}) \setminus \{0\}$ is a nonsingular elliptic fibration with a holomorphic section, we can identify it with $\mu_{\mathrm{can}} \colon \mathcal{T}_{D(r_2^{\mathbf{q}})\setminus\{0\}}/\Lambda \to D(r_2^{\mathbf{q}}) \setminus \{0\}$ for some Λ. By [10, Proposition 3.2], a $\mathbb{Z}$-basis of Λ is given by the following holomorphic functions

$$\tau_1(y) = 1, \quad \tau_2(y) = \frac{1}{2\pi\sqrt{-1}}\log y + \sqrt{-1}\hat{h}_{\mathbf{q}}, \tag{13}$$

where $\hat{h}_{\mathbf{q}}$ is one of the holomorphic functions on $D(r_2^{\mathbf{q}})$ such that $\mathrm{Re}(\hat{h}_{\mathbf{q}}) = h_{\mathbf{q}}$.

Next we fix $b_{\mathbf{q}} \in \mathrm{Crt}$ and a sufficiently small neighborhood $W_2^{\mathbf{q}} \subset \mathbb{P}^1$ of $b_{\mathbf{q}}$ such that $\mu \colon \mu^{-1}(W_2^{\mathbf{q}}) \to W_2^{\mathbf{q}}$ has a holomorphic section. Then we have an isomorphism

$$\begin{array}{ccc} \mu^{-1}(W_2^{\mathbf{q}}) & \xrightarrow{\cong} & \mathcal{T}^*_{W_2^{\mathbf{q}}}/\Lambda \\ {\scriptstyle\mu}\big\downarrow & & \big\downarrow{\scriptstyle\mu_{\mathrm{can}}} \\ W_2^{\mathbf{q}} & = & W_2^{\mathbf{q}} \end{array}$$

for some $\Lambda \subset \mathcal{T}^*_{W_2^{\mathbf{q}}}$. Since $\mu^{-1}(b_{\mathbf{q}})$ is of Kodaira type I_1, we can choose the holomorphic coordinate y on $W_2^{\mathbf{q}}$ such that Λ is generated by

$$dy, \quad \left(\frac{1}{2\pi\sqrt{-1}}\log y + \sqrt{-1}F_{\mathbf{q}}\right)dy,$$

for some holomorphic function $F_{\mathbf{q}}$ on $W_2^{\mathbf{q}}$. Therefor, by putting $h_{\mathbf{q}} = \mathrm{Re}(F_{\mathbf{q}})$, we have the holomorphic embeddings

$$\iota_{\mathbf{q}} \colon X_{\mathbf{q}} \hookrightarrow X, \quad \iota'_{\mathbf{q}} \colon D(r_2^{\mathbf{q}}) \hookrightarrow \mathbb{C}P^1,$$

harmonic functions $f_{\mathbf{q}} \colon D(r_2^{\mathbf{q}}) \to \mathbb{R}$ and $0 < r_1^{\mathbf{q}} < r_2^{\mathbf{q}}$ such that we have the following properties.

(i) $\iota'_{\mathbf{q}}(0) = b_{\mathbf{q}}$ and

$$\iota_{\mathbf{q}}(X_{\mathbf{q}}) \cap \iota_{\mathbf{p}}(X_{\mathbf{p}}) = \emptyset, \quad \iota'_{\mathbf{q}}(D(r_2^{\mathbf{q}})) \cap \iota'_{\mathbf{p}}(D(r_2^{\mathbf{p}})) = \emptyset$$

for any $\mathbf{q} \ne \mathbf{p}$.

(ii) $\Theta|_{\mu_{\mathbf{q}}^{-1}(D(r_2^{\mathbf{q}}))} = \omega_{1,s,\mathbf{q}} + \sqrt{-1}\omega_{2,s,\mathbf{q}}$.

(iii) $\mu \circ \iota_{\mathbf{q}} = \iota'_{\mathbf{q}} \circ \mu_{\mathbf{q}}$.

By taking $W_2^{\mathbf{q}}$ or $r_2^{\mathbf{q}}$ smaller if necessary, we may suppose $W_2^{\mathbf{q}} = \iota'_{\mathbf{q}}(D(r_2^{\mathbf{q}}))$. Moreover, we fix $0 < r_1^{\mathbf{q}} < r_2^{\mathbf{q}}$ arbitrarily, then put $W_1^{\mathbf{q}} = \iota'_{\mathbf{q}}(D(r_1^{\mathbf{q}}))$. To simplify the notations, we often write $W_i^{\mathbf{q}} = D(r_i^{\mathbf{q}})$ or $\iota_{\mathbf{q}}(X_{\mathbf{q}}) = X_{\mathbf{q}}$ if there is no fear of confusion.

Now, note that $\mu \colon X \to \mathbb{P}^1$, may have no holomorphic sections. There exists the unique elliptic surface $\mathbf{j} \colon \mathcal{J} \to \mathbb{P}^1$ which is locally isomorphic to μ and has a holomorphic section. We call $\mathbf{j}$ the Jacobian of $\mu \colon X \to \mathbb{P}^1$. Then $\mathcal{J} = X$ and $\mathbf{j} = \mu$ as smooth manifolds and smooth maps respectively, and the complex structure of $\mathcal{J}$ is given by $\Theta_{\mathcal{J}} := \Theta + \mu^*\mathbf{a}$ for some $\mathbf{a} \in \Omega^2(\mathbb{P}^1) \otimes \mathbb{C}$ by [9, Proposition 7.2].

For an open subset $W \subset \mathbb{P}^1$ and a 1-form $\beta \in \Omega^1(W)$, a diffeomorphism $T_\beta\colon \mu^{-1}(W) \to \mu^{-1}(W)$ is defined in [9, Section 2] as follows. Denote by $u_\beta \in \mathcal{X}(X)$ the vector field defined by $\iota_{u_\beta}(\mathrm{Re}(\Theta)) = \mu^*\beta$, and denote by $\phi_t \in \mathrm{Diff}(\mu^{-1}(W))$ the flow generated by u_β. Then define $T_\beta := \phi_1$ and call it the *translation by the* 1*-form* β. By [9], the translation acts on $\mu^{-1}(W)$ preserving the fibers of μ.

FACT 5.1 ([10, Theorem 4.5]). *Let $\mu\colon X \to \mathbb{P}^1$ be an elliptic $K3$ surface with 24 singular fibers of Kodaira type I_1 with holomorphic volume form Θ, $\mathrm{Crt} = \{b_1, \ldots, b_{24}\}$ be critical values of μ. Let $\mathbf{j}\colon \mathcal{J} \to \mathbb{P}^1$ be the Jacobian of $\mu\colon X \to \mathbb{P}^1$. Then there are sufficiently small positive numbers $r_1^{\mathbf{q}} < r_2^{\mathbf{q}}$, an open cover $\mathbb{P}^1 = \bigcup_a W_a$ such that for any $s < s_0$ and for each Kähler class $[\eta_s] \in H^{1,1}(X)$ with $\langle[\eta_s], \mu^{-1}(b)\rangle = s$ and $[\eta_s]^2 = [\mathrm{Re}(\Theta)]^2 = [\mathrm{Im}(\Theta)]^2$, there is a Kähler form η_s representing $[\eta_s]$ and translations $T_a\colon \mu^{-1}(W_a) \to \mathbf{j}^{-1}(W_a)$ by some 1-forms with respect to $\mathrm{Re}(\Theta_{\mathcal{J}})$ which satisfy the followings.*

(i) *We have $\#(W_a \cap \mathrm{Crt}) \le 1$. If $W_a \cap \mathrm{Crt} = \emptyset$, then $W_a \cap (\bigcup_{\mathbf{q}} W_2^{\mathbf{q}}) = \emptyset$. If $b_{\mathbf{q}} \in W_a$, then $\overline{W_2^{\mathbf{q}}} \subset W_a$.*

(ii) *We have*

$$\eta_s|_{\mu^{-1}(W_a \setminus (\bigcup_{\mathbf{q}} W_2^{\mathbf{q}}))} = T_a^*\left(\eta_s^{\mathrm{SF}}|_{\mathbf{j}^{-1}(W_a \setminus (\bigcup_{\mathbf{q}} W_2^{\mathbf{q}}))}\right),$$
$$\eta_s|_{\mu^{-1}(W_1^{\mathbf{q}})} = T_a^*\left(\omega_{3,s,\mathbf{q}}|_{\mathbf{j}^{-1}(W_1^{\mathbf{q}})}\right),$$
$$\Theta|_{\mu^{-1}(W_a)} = T_a^*\left(\Theta_{\mathcal{J}}|_{\mathbf{j}^{-1}(W_a)}\right).$$

(iii) *$\langle[\eta_s], \mu^{-1}(b)\rangle = s$ and $[\eta_s]^2 = [\mathrm{Re}(\Theta)]^2 = [\mathrm{Im}(\Theta)]^2$.*

Next we analyze the behavior of η_s obtained by Fact 5.1 on $W_2^{\mathbf{q}} \setminus W_1^{\mathbf{q}}$.

LEMMA 5.2. *There is a constant $C_s \ge 1$ for every $s > 0$ such that $\lim_{s\to 0} C_s = 1$ and*

$$C_s^{-1} T_a^*\omega_{3,s,\mathbf{q}} \le \eta_s|_{\mu^{-1}(W_2^{\mathbf{q}} \setminus W_1^{\mathbf{q}})} \le C_s T_a^*\omega_{3,s,\mathbf{q}},$$
$$C_s^{-1} T_a^*\eta_s^{\mathrm{SF}} \le \eta_s|_{\mu^{-1}(W_2^{\mathbf{q}} \setminus W_1^{\mathbf{q}})} \le C_s T_a^*\eta_s^{\mathrm{SF}}$$

for any pair of $\mathbf{q}, a$ with $b_{\mathbf{q}} \in W_a$.

Proof. The estimates are essentially obtained by the proof of [10, Theorem 4.4]. Now we recall the construction of η_s more precisely. Put $X_{\mathbf{q}}^* := \mu^{-1}(W_2^{\mathbf{q}} \setminus W_1^{\mathbf{q}})$. By the proof of [10, Theorem 4.4], there is a function $\varphi \in C^\infty(X_{\mathbf{q}}^*)$ such that

$$\eta_s^{\mathrm{SF}} = \omega_{3,s,\mathbf{q}} + \sqrt{-1}\partial\overline{\partial}\varphi$$

on $X_{\mathbf{q}}^*$. By the assumption that $b_{\mathbf{q}} \in W_a$ and by (i) of Fact 5.1, we have $\overline{W_2^{\mathbf{q}}} \subset W_a$. Let $0 \le \psi \le 1$ be some cut-off function defined on the neighborhood of $\overline{W_2^{\mathbf{q}}}$ such that $\psi \equiv 1$ on $\overline{W_1^{\mathbf{q}}}$ and $\psi \equiv 0$ on the complement of $\overline{W_2^{\mathbf{q}}}$. On $\mu^{-1}(W_a)$, η_s is given by

$$(T_a^{-1})^*\eta_s|_{X_{\mathbf{q}}^*} = \eta_s^{\mathrm{SF}} - \sqrt{-1}\partial\overline{\partial}(\mu^*\psi \cdot \varphi) + \mu^* A$$

for some $A \in \Omega^2(W_a)$, hence we have

$$\begin{aligned}(T_a^{-1})^*\eta_s|_{X_{\mathbf{q}}^*} - \eta_s^{\mathrm{SF}} &= -\sqrt{-1}\varphi\partial\overline{\partial}\mu^*\psi - \sqrt{-1}\partial\mu^*\psi \wedge \overline{\partial}\varphi - \sqrt{-1}\partial\varphi \wedge \overline{\partial}\mu^*\psi \\ &\quad - \sqrt{-1}\mu^*\psi\partial\overline{\partial}\varphi + \mu^* A,\\ (T_a^{-1})^*\eta_s|_{X_{\mathbf{q}}^*} - \omega_{3,s,\mathbf{q}} &= -\sqrt{-1}\varphi\partial\overline{\partial}\mu^*\psi - \sqrt{-1}\partial\mu^*\psi \wedge \overline{\partial}\varphi - \sqrt{-1}\partial\varphi \wedge \overline{\partial}\mu^*\psi \\ &\quad + \sqrt{-1}(1 - \mu^*\psi)\partial\overline{\partial}\varphi + \mu^* A.\end{aligned}$$

We estimate the norm of the right hand side of the above equations with respect to the metric η_s^{SF}. Since ψ is independent of s, we can see

$$|\partial\mu^*\psi| = \left|\overline{\partial}\mu^*\psi\right| = O(\sqrt{s}), \quad \left|\partial\overline{\partial}\mu^*\psi\right| = O(s).$$

By the proof of [10, Theorem 4.4], $|A| = O(e^{-C/s})$ for some constant $C > 0$, with respect to some metric on $\mathbb{P}^1$. Then we can see $|\mu^* A| = O(se^{-C/s})$ with respect to η_s^{SF}. The proof of [10, Theorem 4.4] also gives

$$\left|\partial\overline{\partial}\varphi\right| = \left|\eta_s^{\mathrm{SF}} - \omega_{3,s,\mathbf{q}}\right| = O(se^{-C/s}),$$

then [10, Lemma 4.1] implies

$$|\varphi| + |\partial\varphi| + \left|\overline{\partial}\varphi\right| = O(s^{-1}e^{-C/s}).$$

Consequently, we obtain

$$\begin{aligned}\left|(T_a^{-1})^*\eta_s|_{X_{\mathbf{q}}^*} - \eta_s^{\mathrm{SF}}\right| &= O(s^{-1/2}e^{-C/s}),\\ \left|(T_a^{-1})^*\eta_s|_{X_{\mathbf{q}}^*} - \omega_{3,s,\mathbf{q}}\right| &= O(s^{-1/2}e^{-C/s}).\end{aligned}$$

Since $\lim_{s\to 0} s^{-1/2}e^{-C/s} = 0$, we have the assertion. □

5.4. C^2 estimate. Let η_s be the Kähler forms on X obtained by Fact 5.1. Denote by ρ_{η_s} the Ricci form of η_s. If we put

$$\mathcal{F}_s := \log\left(\frac{\Theta\wedge\overline{\Theta}/2}{\eta_s^2}\right),$$

then $\rho_{\eta_s} = \sqrt{-1}\partial\overline{\partial}\mathcal{F}_s$.

FACT 5.3 ([10, Theorem 4.5]). *There are positive constants $D_1, \dots, D_6$ and s_0 such that*

$$\begin{aligned}\|\mathcal{F}_s\|_{C^0(X)} &\le D_1 e^{-D_2/s},\\ \|\Box\mathcal{F}_s\|_{C^0(X)} &\le D_1 e^{-D_2/s},\\ \rho_{\eta_s} &\ge -D_3 e^{-D_4/s}\eta_s,\\ \mathrm{diam}_{\eta_s}(X) &\le D_5 s^{-1/2},\\ \|R_{\eta_s}\|_{C^0(X)} &\le D_6 s^{-1}\log s^{-1}\end{aligned}$$

for all $0 < s < s_0$, where $\Box = \overline{\partial}^*\overline{\partial}$ is the $\overline{\partial}$-Laplacian of η_s acting on $C^\infty(X)$, diam_{η_s} and R_{η_s} are the diameter and the curvature tensor of η_s, respectively.

Now we consider the Monge Ampère equation with the normalization

$$\left(\eta_s + \sqrt{-1}\partial\overline{\partial}u_s\right)^2 = e^{\mathcal{F}_s}\eta_s^2, \tag{14}$$

$$\int_X u_s\eta^n = 0. \tag{15}$$

Fact 5.4 ([10, Lemma 5.2]). *Let D_2 and s_0 be as in Fact 5.3. There is a constant $D_7 > 0$ such that any $0 < s < s_0$ and any solution $u_s \in C^\infty(X)$ to (14)(15) satisfies*

$$\|u_s\|_{L^\infty(X)} \leq D_7 s^{-5} e^{-D_2/s}.$$

Next we improve [10, Lemma 5.3].

Lemma 5.5. *There are constants $C_s \geq 1$ such that $\lim_{s\to 0} C_s = 1$ and for any solution $u_s \in C^\infty(X)$ of (14)(15) satisfy*

$$C_s^{-1}\eta_s \leq \eta_s + \sqrt{-1}\partial\overline{\partial}u_s \leq C_s\eta_s.$$

Proof. The outline of the proof is similar to that of [10, Lemma 5.3], however, we need some modifications. Let $N_s > 0$ be a sufficiently large positive constant such that $N_s + \inf_{i\neq j} R_{i\bar{i}j\bar{j}}(x) > 0$ for any $x \in X$, where $R_{i\bar{i}j\bar{j}}$ is the holomorphic sectional curvature of η_s.

Let $\hat{\Box}$ be the $\overline{\partial}$-Laplacian with respect to $\eta_s + \sqrt{-1}\partial\overline{\partial}u_s$. By [20, (2.22)], we have

$$\begin{aligned} e^{N_s u_s}\hat{\Box}\left(e^{-N_s u_s}(2-\Box u_s)\right) \leq \Box\mathcal{F}_s + 4\inf_{i\neq j} R_{i\bar{i}j\bar{j}}(x) + 2N_s(2-\Box u_s) \\ -\left(N_s + \inf_{i\neq j} R_{i\bar{i}j\bar{j}}(x)\right) e^{-\mathcal{F}_s}(2-\Box u_s)^2. \end{aligned}$$

Notice that the Laplace operators in this paper are positive. Next we assume that $e^{-N_s u_s}(2-\Box u_s)$ attains its maximum at $x_{\max} \in X$. Then by the same argument in the proof of [10, Lemma 5.3] we obtain

$$\left|(2-\Box u_s) - \frac{2e^{\mathcal{F}_s}}{2-k_s}\right| \leq \left|\left(\frac{2e^{\mathcal{F}_s}}{2-k_s}\right)^2 + \frac{e^{\mathcal{F}_s}(2\Box\mathcal{F}_s - N_s k_s)}{(2-k_s)N_s}\right| \tag{16}$$

at $x_{\max}$, where $k_s := -\frac{2\inf_{i\neq j} R_{i\bar{i}j\bar{j}}(x_{\max})}{N_s}$.

Now, we fix

$$N_s := \max\left\{\frac{\left|\inf_{i\neq j} R_{i\bar{i}j\bar{j}}(x_{\max})\right|}{s}, 1\right\},$$

then we have $|k_s| \leq 2s$. Note that N_s is different from that is taken in the proof of [10, Lemma 5.3]. Since we have

$$\lim_{s\to 0} e^{\mathcal{F}_s} = 1, \quad \lim_{s\to 0} \Box\mathcal{F}_s = 0$$

by Fact 5.3, we obtain

$$\begin{aligned} \lim_{s\to 0}\frac{2e^{\mathcal{F}_s}}{2-k_s} &= 1, \\ \lim_{s\to 0}\left|\frac{e^{\mathcal{F}_s}(2\Box\mathcal{F}_s - N_s k_s)}{(2-k_s)N_s}\right| &\leq \lim_{s\to 0}\left(\frac{e^{\mathcal{F}_s}\left|\Box\mathcal{F}_s\right|}{2-2s} + \frac{2e^{\mathcal{F}_s}s}{2-2s}\right) \\ &= 0 \end{aligned}$$

at $x_{\max}$. Then (16) gives

$$\limsup_{s\to 0}(2-\Box u_s(x_{\max}))\le 1+1=2. \tag{17}$$

If we take $x\in X$ arbitrarily, then we have

$$e^{-N_s u_s(x)}(2-\Box u_s(x))\le e^{-N_s u_s(x_{\max})}(2-\Box u_s(x_{\max})),$$

consequently we have

$$\begin{aligned}(2-\Box u_s(x)) &\le e^{N_s(u_s(x)-u_s(x_{\max}))}(2-\Box u_s(x_{\max}))\\ &\le e^{2N_s\|u_s\|_{L^\infty}}(2-\Box u_s(x_{\max})).\end{aligned} \tag{18}$$

By Facts 5.3, 5.4 and by the definition of N_s we have

$$\begin{aligned}\lim_{s\to 0} N_s\|u_s\|_{L^\infty} &\le \lim_{s\to 0} D_7\max\left\{\frac{\|R_{\eta_s}\|_{C^0}}{s},1\right\}s^{-5}e^{-D_2/s}\\ &\le \lim_{s\to 0} D_7\max\left\{D_6 s^{-2}\log s^{-1},1\right\}s^{-5}e^{-D_2/s}\\ &\le \lim_{s\to 0} D_7 D_6 s^{-7}\log s^{-1}e^{-D_2/s}=0.\end{aligned}$$

Therefore, combining (17) with (18), we have

$$\limsup_{s\to 0}\left\{\sup_X(2-\Box u_s)\right\}\le 2.$$

Next we fix a point $x\in X$ and take a coordinate z^1,z^2 around x such that

$$\eta_s|_x=\sqrt{-1}\sum_{i,j}\delta_{ij}dz^i_x\wedge d\bar{z}^j_x$$

and we put

$$\left(\eta_s+\sqrt{-1}\partial\bar\partial u_s\right)|_x=\sqrt{-1}\sum_{i,j}A_{ij}dz^i_x\wedge d\bar{z}^j_x,\quad A=(A_{ij})_{i,j}.$$

Then we can see

$$\mathrm{tr}(A)=2-\Box u_s(x),\quad \det(A)=e^{\mathcal{F}_s(x)}$$

and

$$0\le\limsup_{s\to 0}\left\{\sup_X\mathrm{tr}(A)\right\}\le 2,\quad \limsup_{s\to 0}\sup_X|\det(A)-1|=0.$$

Let λ_1 and λ_2 be the eigenvalues of A. Since $\mathrm{tr}(A)>0$, we have

$$\begin{aligned}\limsup_{s\to 0}\sup_X|\lambda_1-\lambda_2|^2 &= \limsup_{s\to 0}\left\{\sup_X\left\{(\lambda_1+\lambda_2)^2-4\lambda_1\lambda_2\right\}\right\}\\ &\le \limsup_{s\to 0}\sup_X\left\{\mathrm{tr}(A)^2-4\right\}\\ &\qquad +4\limsup_{s\to 0}\left\{\sup_X|\det(A)-1|\right\}\\ &\le 0,\end{aligned}$$

hence

$$\limsup_{s\to 0}\sup_X|\lambda_1-1|=\limsup_{s\to 0}\sup_X|\lambda_2-1|=1,$$

thus we have the assertion. □

5.5. Proof of Theorem 4.1. Here we prove Theorem 4.1 by assuming some results on the standard semi-flat metrics and the Ooguri-Vafa metrics. Let $(X, \omega_1, \omega_2, \omega_{3,s}, g_s)$, $\mu\colon X \to \mathbb{P}^1$ and (L, h, ∇) be as in Subsection 4.1.

Let $\eta_s \in [\omega_{3,s}]$ be the Kähler form obtained by Fact 5.1 and denote by g_s' the Kähler metric of η_s. By Yau's theorem, there is a solution u_s of (14) and (15), and we can see $\omega_{3,s} = \eta_s + \sqrt{-1}\partial\overline{\partial}u_s$ by the uniqueness of the solution. Therefore, by Lemma 5.5, there are constants $C_s \geq 1$ with $\lim_{s\to 0} C_s = 1$ such that

$$C_s^{-1} g_s' \leq g_s \leq C_s g_s'. \tag{19}$$

Lemma 5.6. *For any $b_{\mathbf{q}} \in \mathrm{Crt}$, there are an open neighborhood W and $\gamma \in \Omega^1(\mu^{-1}(W))$ such that the triple $(b = b_{\mathbf{q}}, W, \gamma)$ satisfies ($\star$1-3).*

The above lemma will be shown in Section 7.

Lemma 5.7. *For every positive integer k, $BS_k \subset \mathbb{P}^1$ is a finite set.*

Proof. Note that no points in $\mathbb{P}^1 \setminus \mathrm{Crt}$ are accumulation points of BS_k. Therefore, by Lemmas 4.10 and 5.6, none of $b \in \mathbb{P}^1$ is an accumulation point of BS_k. Since $\mathbb{P}^1$ is compact, BS_k is finite. □

Fix k and let $\mathbb{P}^1 = \bigcup_a W_a$ be an open cover as in Fact 5.1. Now suppose that we have the assumption of Lemma 5.7, then BS_k is finite. By taking the refinement of $\{W_a\}_a$ if necessary, we may suppose that there is a map $b \mapsto a_b$ for $b \in BS_k$ such that $b \in W_{a_b}$, $W_{a_b} \cap W_{a_{b'}} = \emptyset$ if $b \neq b'$, $W_{a_b} \cap \bigcup_{\mathbf{q}} W_2^{\mathbf{q}} = \emptyset$ if $b \notin \mathrm{Crt}$ and $W_{a_b} \subset W_1^{\mathbf{q}}$ if $b = b_{\mathbf{q}}$. Then by Fact 5.1, $g_s'|_{\mu^{-1}(W_{a_b})}$ is isometric to either the standard semi-flat metric or the Ooguri-Vafa metric. If $b \in BS_k$, then there is the unique positive integer m such that $k/m \in \mathbb{Z}$ and $b \in BS_m^{\mathrm{str}}$.

Lemma 4.7 and the next proposition give Theorem 4.1.

Proposition 5.8. *Let $b \in BS_k$ and W_{a_b} be as above. Let (L, h, ∇) be a prequantum line bundle on (X, ω) and put $\mathbb{S} = \mathbb{S}(L, h)$. Let $q \in \mu^{-1}(b)$, $p \in \pi^{-1}(q)$, m be the positive integer such that $b \in BS_m^{\mathrm{str}}$ and denote by $\hat{g}_s$ be the metric on $\mathbb{S}$ defined by (2). Then for any $R > 0$ there is $s_R > 0$ such that $B_{g_s'}(q, R) \subset \mu^{-1}(W_{a_b})$ for any $0 < s \leq s_R$ and*

$$\left(\mathbb{S}, d_{\hat{g}_s'}, \frac{\nu_{\hat{g}_s'}}{s}, p\right) \overset{S^1\text{-pmGH}}{\longrightarrow} \left(S^1 \times \mathbb{R}^2, \hat{d}_{0,m}, dt d\nu_{g_0}, (1_{S^1}, \mathbf{0}_{\mathbb{R}^2})\right)$$

as $s \to 0$.

Thus, to prove Theorem 4.1, it suffices to show Lemma 5.6 and Proposition 5.8.

6. Neighborhood of nonsingular fibers. In this section we prove Proposition 5.8 for $b \in BS_m^{\mathrm{str}} \setminus \mathrm{Crt}$. To show it, we can reduce the argument to the local model. Let $B \subset \mathbb{C}$ be an open neighborhood of the origin $0 \in \mathbb{C}$ and $\Lambda \subset \mathcal{T}_B^*$ be a holomorphically varying family of lattices. We take B sufficiently small so that Λ is given by

$$\Lambda_y = \mathrm{span}_{\mathbb{Z}}\{\tau_1(y)dy, \tau_2(y)dy\}$$

for some holomorphic functions τ_1, τ_2 on B. By changing the holomorphic coordinate, we may suppose $\tau_1 \equiv 1$ and $\mathrm{Im}(\tau_2) > 0$. Let η_s^{SF} be the standard semi-flat Kähler form. Now,

$$\omega_1^{\mathrm{SF}} := \mathrm{Re}(\Theta_{\mathrm{can}}), \quad \omega_2^{\mathrm{SF}} := \mathrm{Im}(\Theta_{\mathrm{can}}), \quad \omega_{3,s}^{\mathrm{SF}} := \eta_s^{\mathrm{SF}}$$

form a hyper-Kähler structure on $X_{\mathrm{SF}} := \mathcal{T}_B^*/\Lambda$. Denote by $(g_s^{\mathrm{SF}}, J_{1,s}^{\mathrm{SF}}, J_{2,s}^{\mathrm{SF}}, J_3^{\mathrm{SF}})$ the induced hyper-Kähler structures. Let $(\pi\colon L \to X_{\mathrm{SF}}, h, \nabla)$ be a prequantum line bundle on $(X_{\mathrm{SF}}, \omega_1^{\mathrm{SF}})$ such that $0 \in BS_m^{\mathrm{str}}$. We identify $\mu^{-1}(W)$ with X_{SF} and identify b with $0 \in B$. Now, we apply [13, Theorem 1.1]. To apply it, we check that our situation satisfies the following assumptions in the theorem;

(i) Ric_{g_s} have the uniform lower bound,

(ii) the family $\{J_{1,s}^{\mathrm{SF}}\}_s$ satisfies ♠ in [13].

The condition (i) is automatically satisfied since g_s are Ricci-flat metrics.

Next we check (ii). Let $Y(y)$ be a holomorphic function on B such that $\frac{\partial Y}{\partial y} = \tau_2(y)$ and put $Y = Y_1 + \sqrt{-1}Y_2$, $y = y_1 + \sqrt{-1}y_2$ for some real valued functions Y_1, Y_2, y_1, y_2. Moreover, define real valued functions v_1, v_2 by $xdy = -(v_1 + v_2\tau_2(y))dy \in \mathcal{T}_B^*$. Then we have

$$\mathrm{Re}(\Theta_{\mathrm{can}}) = dy_1 \wedge dv_1 + dY_1 \wedge dv_2.$$

By using the action-angle coordinate $(y_1, Y_1, v_1.v_2)$, we describe a frame of $(1,0)$-forms with respect to $J_{1,s}^{\mathrm{SF}}$. If $dy_1 + A_{11}dv_1 + A_{12}dv_2$ and $dY_1 + A_{21}dv_1 + A_{22}dv_2$ are $(1,0)$-forms, then we have (ii) iff the matrix

$$\left.\frac{d}{ds}\right|_{s=0} \mathrm{Im}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

is positive definite.

Let $x, y, \mathbf{W}, \mathbf{b}$ be as in Subsection 5.1. Put

$$\mathbf{a}_{\mathrm{v}} := \sqrt{\mathbf{W}}(dx + \mathbf{b}dy), \quad \mathbf{a}_{\mathrm{h}} := \sqrt{\mathbf{W}}^{-1}dy.$$

Then we have

$$\eta_s^{\mathrm{SF}} = \frac{\sqrt{-1}}{2}\left(\mathbf{a}_{\mathrm{v}} \wedge \bar{\mathbf{a}}_{\mathrm{v}} + \mathbf{a}_{\mathrm{h}} \wedge \bar{\mathbf{a}}_{\mathrm{h}}\right), \quad \Theta_{\mathrm{can}} = \mathbf{a}_{\mathrm{v}} \wedge \mathbf{a}_{\mathrm{h}},$$

hence

$$\begin{aligned}\mathrm{Im}(\Theta_{\mathrm{can}}) + \sqrt{-1}\eta_s^{\mathrm{SF}} &= \frac{1}{2\sqrt{-1}}\left(\mathbf{a}_{\mathrm{v}} \wedge \mathbf{a}_{\mathrm{h}} - \bar{\mathbf{a}}_{\mathrm{v}} \wedge \bar{\mathbf{a}}_{\mathrm{h}}\right) - \frac{1}{2}\left(\mathbf{a}_{\mathrm{v}} \wedge \bar{\mathbf{a}}_{\mathrm{v}} + \mathbf{a}_{\mathrm{h}} \wedge \bar{\mathbf{a}}_{\mathrm{h}}\right) \\ &= \frac{-1}{2}\left(\mathbf{a}_{\mathrm{v}} + \sqrt{-1}\bar{\mathbf{a}}_{\mathrm{h}}\right) \wedge \left(\bar{\mathbf{a}}_{\mathrm{v}} + \sqrt{-1}\mathbf{a}_{\mathrm{h}}\right).\end{aligned}$$

It implies that $\mathbf{a}_{\mathrm{v}} + \sqrt{-1}\bar{\mathbf{a}}_{\mathrm{h}}, \bar{\mathbf{a}}_{\mathrm{v}} + \sqrt{-1}\mathbf{a}_{\mathrm{h}}$ form a frame of $\Omega^{1,0}_{J_{1,s}^{\mathrm{SF}}}(X_{\mathrm{SF}})$.

Since

$$\mathbf{W} = \frac{s}{\mathrm{Im}(\tau_2)}, \quad \mathbf{b} = -\frac{\mathrm{Im}(x)}{\mathrm{Im}(\tau_2)}\frac{\partial \tau_2}{\partial y},$$

we have

$$\begin{aligned}\mathbf{a}_{\mathrm{v}} &= -\sqrt{\mathbf{W}}\left(dv_1 + \tau_2 dv_2\right), \\ \mathbf{a}_{\mathrm{h}} &= \sqrt{\frac{-1}{s\mathrm{Im}(\tau_2)}}\left(\bar{\tau}_2 dy_1 - dY_1\right)\end{aligned}$$

and

$$\mathbf{a}_{\mathrm{v}} + \sqrt{-1}\bar{\mathbf{a}}_{\mathrm{h}} = -\sqrt{\frac{s}{\mathrm{Im}(\tau_2)}}\left(dv_1 + \tau_2 dv_2\right) + \sqrt{\frac{1}{s\mathrm{Im}(\tau_2)}}\left(\tau_2 dy_1 - dY_1\right),$$

$$\bar{\mathbf{a}}_{\mathrm{v}} + \sqrt{-1}\mathbf{a}_{\mathrm{h}} = -\sqrt{\frac{s}{\mathrm{Im}(\tau_2)}}\left(dv_1 + \bar{\tau}_2 dv_2\right) + \sqrt{\frac{1}{s\mathrm{Im}(\tau_2)}}\left(-\bar{\tau}_2 dy_1 + dY_1\right).$$

Therefore, we can take

$$dy_1 + \frac{\sqrt{-1}s}{\mathrm{Im}(\tau_2)}\left(dv_1 + \mathrm{Re}(\tau_2)dv_2\right),$$
$$dY_1 + \frac{\sqrt{-1}s}{\mathrm{Im}(\tau_2)}\left(\mathrm{Re}(\tau_2)dv_1 + |\tau_2|^2 dv_2\right),$$

as a frame of $\Omega^{1,0}_{J^{\mathrm{SF}}_{1,s}}(X_{\mathrm{SF}})$. Since the following symmetric matrix

$$\begin{pmatrix} 1 & \mathrm{Re}(\tau_2) \\ \mathrm{Re}(\tau_2) & |\tau_2|^2 \end{pmatrix}$$

is positive definite, hence we have (ii). Thus we obtained Proposition 5.8 for $b \notin \mathrm{Crt}$.

Remark 6.1. We can show Proposition 5.8 also by proving

$$\left(g_s^{\mathrm{SF}}, \frac{1}{s}, 0, B\right) \to (\mathbb{R}^2, g_0)$$

as $s \to 0$, where g_s^{SF} is the Kähler metric of η_s^{SF}.

7. Neighborhood of singular fibers of Kodaira type I_1. In this section we show Lemma 5.6 and Proposition 5.8 for $b \in BS_m^{\mathrm{str}} \cap \mathrm{Crt}$. Let $s > 0$, X_{OV} be as in Subsection 5.2, and put

$$V_s(u) := \frac{1}{4\pi}\sum_{n\in\mathbb{Z}^\times}\left(\frac{1}{\sqrt{u_1^2+u_2^2+(u_3-sn)^2}} - \frac{1}{s|n|}\right) + \frac{1}{4\pi|u|} + a_s + \frac{h(u_1,u_2)}{s},$$

for some harmonic function h. Here, a_s is the constant defined in Subsection 5.3. We fix a sufficiently small positive constant $\delta_0 > 0$ so that

$$V_s > 0 \text{ on } \mathcal{U}(\delta_0, s), \tag{20}$$

$$\max_{D(\delta_0)} |h(u_1,u_2)| \le \frac{1}{10\pi}\log\delta_0^{-1}, \tag{21}$$

$$\delta_0 \le \frac{1}{2}. \tag{22}$$

Define the hyper-Kähler structure on X_{OV} by

$$\omega_{1,s} = du_1 \wedge \frac{\alpha}{2\pi} + V_s du_2 \wedge du_3,$$
$$\omega_{2,s} = du_2 \wedge \frac{\alpha}{2\pi} + V_s du_3 \wedge du_1,$$
$$\omega_{3,s} = du_3 \wedge \frac{\alpha}{2\pi} + V_s du_1 \wedge du_2.$$

Let μ_{OV} and 0_{OV} be as in Subsection 5.2. Denote by g_s^{OV} the hyper-Kähler metric associated with $(\omega_{1,s}, \omega_{2,s}, \omega_{3,s})$.

To prove Lemma 5.6 and Proposition 5.8 for $b \in BS_m^{\mathrm{str}} \cap \mathrm{Crt}$, it suffices to show that

$$\left(g_s^{\mathrm{OV}}, \frac{1}{s}, 0, D(\delta_0)\right) \overset{s\to 0}{\to} (\mathbb{R}^2, g_0)$$

in the sense of Definition 4.30, where g_0 is the Euclidean metric. Here, we identify b with the origin $0 \in \mathbb{C}$ and we regard $D(\delta_0)$ the neighborhood of b.

First of all we determine the prequantum line bundle on X_{OV}. Let $(\pi \colon L \to X_{\mathrm{OV}}, h, \nabla)$ be a prequantum line bundle on $(X_{\mathrm{OV}}, \omega_{1,s})$. Since there is a deformation retraction X_{OV} onto $\mu_{\mathrm{OV}}^{-1}(0)$, the inclusion map $\mu_{\mathrm{OV}}^{-1}(0) \subset X_{\mathrm{OV}}$ induces an isomorphism $H^2(X_{\mathrm{OV}}, \mathbb{Z}) \cong H^2(\mu_{\mathrm{OV}}^{-1}(0), \mathbb{Z})$. Since $\mu_{\mathrm{OV}}^{-1}(0)$ is Lagrangian with respect to $\omega_{1,s}$, we can see $[\omega_{1,s}] = 0 \in H^2(X_{\mathrm{OV}}, \mathbb{R})$. Since $H^2(X_{\mathrm{OV}}, \mathbb{Z})$ is torsion-free, one can see

$$c_1(L) = \frac{\sqrt{-1}}{2\pi}[F^\nabla] = \frac{1}{2\pi}[\omega_{1,s}] = 0 \in H^2(X_{\mathrm{OV}}, \mathbb{Z}),$$

hence L is a trivial bundle.

Next we determine $\gamma_s \in \Omega^1(X_{\mathrm{OV}})$ such that $d\gamma_s = \omega_{1,s}$ and $(\star 3)$ of Subsection 4.3 is satisfied. Let $J_{1,s}, J_{2,s}, J_{3,s}$ be complex strucutures on X_{OV} associated with the hyper-Kähler structure $(\omega_{1,s}, \omega_{2,s}, \omega_{3,s})$.

Lemma 7.1. *Let*

$$\phi(u_1, u_2, u_3) = -\int_0^{u_1} tV_s(t, u_2, u_3)dt + \psi(u_2, u_3)$$

for a function ψ with $\frac{\partial^2\psi}{\partial u_2^2} + \frac{\partial^2\psi}{\partial u_3^2} = -V_s(0, u_2, u_3)$. Then we have $\omega_{1,s} = dJ_{1,s}d\phi$. Here, we write $\phi = u^*\phi$ for the brevity, if there is no fear of confusion.

Proof. By the definition of ϕ, we have

$$\frac{\partial\phi}{\partial u_1} = -u_1V_s(x),$$

$$\Delta_{\mathbb{R}^3}\phi = -\sum_{i=1}^3 \frac{\partial^2\phi}{\partial u_i^2} = 2V_s.$$

Since we have

$$J_{1,s}\left(\frac{\alpha}{2\pi}\right) = V_s du_1, \quad J_{1,s}(du_1) = -\frac{V_s^{-1}\alpha}{2\pi},$$
$$J_{1,s}(du_2) = -du_3, \quad J_{1,s}(du_3) = du_2,$$

then $dJ_{1,s}d\phi = \omega_{1,s}$. □

Put

$$\psi_s(u_2, u_3) := -\frac{1}{4\pi}\sum_{n\in\mathbb{Z}^\times}\left(\sqrt{u_2^2 + (u_3 - sn)^2} - \frac{u_2^2}{2s|n|} + \frac{|n|}{n}(u_3 - sn)\right)$$
$$-\frac{\sqrt{u_2^2 + u_3^2}}{4\pi} - \frac{a_s u_2^2}{2} + \frac{u_3^2 - u_2^2}{4\pi s} - \int_0^{u_2}\left(\int_0^{\tilde{t}} \frac{h(0,t)}{s}dt\right)d\tilde{t}$$

and

$$\phi_s(u_1,u_2,u_3) := -\int_0^{u_1} tV_s(t,u_2,u_3)dt + \psi_s(u_2,u_3)$$

for $s>0$. Now we can check that

- the series $\sum_{n\in\mathbb{Z}^\times}\left(\sqrt{u_2^2+(u_3-sn)^2}-\frac{u_2^2}{2s|n|}+\frac{|n|}{n}(u_3-sn)\right)$ converges absolutely,
- $\frac{\partial^2\psi_s}{\partial u_2^2}+\frac{\partial^2\psi_s}{\partial u_3^2}=-V_s(0,u_2,u_3)$,
- $u^*\phi_s$ is smooth on $\tilde{X}_{\mathrm{OV}}$,
- $\phi_s(u_1,u_2,u_3-s)=\phi_s(u_1,u_2,u_3)$.

Then

$$\gamma_s := J_{1,s}d\phi_s.$$

descends to a smooth 1-form on X_{OV}.

Next we give generators of $H_1(\mu_{\mathrm{OV}}^{-1}(y),\mathbb{Z})$. To give it, we observe the $\mathbb{Z}$-action on the covering space $\tilde{X}_{\mathrm{OV}}$. Denote by v the vector field on $\tilde{X}_{\mathrm{OV}}$ defined by $v_q := \frac{d}{dt}|_{t=0}q\cdot e^{\sqrt{-1}t}$. For $q\in\tilde{X}_{\mathrm{OV}}$ and $z_1,z_2\in\mathbb{R}$, we define the $\mathbb{C}^\times$-action on $\tilde{X}_{\mathrm{OV}}$ by

$$q\cdot e^{-z_2+\sqrt{-1}z_1} := \exp\left(z_1v+z_2J_{3,s}v\right)(q), \tag{23}$$

and we can see the action preserves u_1,u_2. Since the period of the elliptic fibration $\mu_{\mathrm{OV}}\colon X_{\mathrm{OV}}\to D(\delta_0)$ is given by (13), we have $q\cdot e^{2\pi\mathcal{V}(y)}$ and q are in the same orbit of $\mathbb{Z}$-action if $u_1(q)+\sqrt{-1}u_2(q)=y\neq 0$, where

$$\mathcal{V}(y) := \frac{\log y^{-1}}{2\pi}+\hat{h}(y)$$

and $\hat{h}$ is a harmonic function on $D(\delta_0)$ with $\mathrm{Re}\hat{h}=h$.

For $y=u_1+\sqrt{-1}u_2\in D(\delta_0)$, let $e_{1,y}$ be a 1-cycle in $\mu_{\mathrm{OV}}^{-1}(y)$ given as the S^1-orbit. If $y=0$, then $e_{1,y}=0$.

Next we construct a path $e_{2,y}\colon[0,s]\to\tilde{X}_{\mathrm{OV}}$ which generates $H_1(U,\mathbb{Z})$. First of all we construct the following paths $e_{2,y}^{(1)}$ and $e_{2,y}^{(2)}$, then we obtain $e_{2,y}$ by connecting them. Let $e_{2,y}^{(1)}\colon[0,s]\to\tilde{X}_{\mathrm{OV}}$ be the integral path of $-2\pi V_sJ_{3,s}(v)$ such that $e_{2,y}^{(1)}(0)=q$ for some q with $u(q)=(u_1,u_2,0)$. Then $u(e_{2,y}^{(1)}(s))=(u_1,u_2,s)$. Since we have

$$\log|\lambda| = 2\pi\int_{u_3(q)}^{u_3(q\cdot\lambda)} V_s(u_1(q),u_2(q),\tau)d\tau$$

for $\lambda\in\mathbb{C}^\times$, then $e_{2,y}^{(1)}(s)=q\cdot e^{2\pi\mathrm{Re}(\mathcal{V}(y))}$. Define $e_{2,y}^{(2)}$ by

$$e_{2,y}^{(2)}(t) := q\cdot e^{2\pi(\mathrm{Re}(\mathcal{V}(y))+\sqrt{-1}t\mathrm{Im}(\mathcal{V}(y)))}$$

for $0\le t\le 1$. This is the S^1-orbit containing $q\cdot e^{2\pi\mathrm{Re}(\mathcal{V}(y))}$ and $q\cdot e^{2\pi\mathcal{V}(y)}$. Here, $e_{2,y}^{(2)}$ depends on the choice of the value of $\mathrm{Im}(\mathcal{V}(y))$. Here, we suppose $\pi/2\le\mathrm{Im}(\log y)<5\pi/2$, then $u_1\mathrm{Im}(\log y)$ is continuous.

We can see that $\{e_{1,y},e_{2,y}\}_y$ satisfies the first half of ($\star$3). We can also see that $H_1(X_b,\mathbb{Z})\cong H_1(U,\mathbb{Z})\cong\mathbb{Z}$ is generated by $e_{2,b}$, hence we have ($\star$1). The next lemma completes the proof of Lemma 5.6.

Lemma 7.2. *Let $y = (u_1 + \sqrt{-1}u_2)$. We have*

$$\int_{e_{1,y}} \gamma_s = u_1, \quad \int_{e_{2,y}} \gamma_s = \mathcal{H}(u_1, u_2),$$

where

$$\mathcal{H}(u_1, u_2) := -\frac{u_2 \log \sqrt{u_1^2 + u_2^2}}{2\pi} + \frac{u_2}{2\pi} + \int_0^{u_1} t \frac{\partial h}{\partial u_2}(t, u_2) dt + \int_0^{u_2} h(0,t) dt$$
$$+ u_1 \mathrm{Im}(\mathcal{V}(y))$$

for $y \neq 0$, and $\mathcal{H}(0,0) := 0$. Moreover, the function $\mathcal{H}$ is continuous on $D(\delta_0)$ and the origin $0 \in D(\delta_0)$ is isolated in

$$\left\{ u_1 + \sqrt{-1}u_2 \in D(\delta_0);\ u_1 = 0,\ \mathcal{H}(u_1, u_2) = 0 \right\}.$$

Proof. First of all we have

$$\gamma_s = J_{1,s} d\phi_s = -V_s^{-1} \frac{\partial \phi}{\partial u_1} \frac{\alpha}{2\pi} - \frac{\partial \phi}{\partial u_2} du_3 + \frac{\partial \phi}{\partial u_3} du_2.$$

Then we obtain

$$\int_{e_{1,y}} J_{1,s} d\phi_s = -\frac{1}{2\pi} \int_{c_0} V_s^{-1} \frac{\partial \phi_s}{\partial u_1} \alpha = u_1.$$

By

$$\frac{\partial \phi_s}{\partial u_2} = -u_2 \left(V_s(x) - \frac{h}{s} + \frac{1}{2\pi s} \right)$$
$$+ \frac{1}{s} \left(-\int_0^{u_1} t \frac{\partial h}{\partial u_2}(t, u_2) dt - \int_0^{u_2} h(0,t) dt \right)$$

and $\int_0^s V_s(u_1, u_2, t) dt = -(\log \sqrt{u_1^2 + u_2^2})/2\pi + h$, we can show $\int_{e_{2,y}} J_{1,s} d\phi_s = \mathcal{H}(u_1, u_2)$. If $y = 0$, then $\frac{\partial \phi_s}{\partial u_2} = 0$, hence $\int_{e_{2,0}} J_{1,s} d\phi_s = 0$.

Although $\mathrm{Im}(\mathcal{V})$ is not continuous at a point in $\{u_1 = 0\}$, it is bounded on the neighborhood of $\{u_1 = 0\}$, hence $u_1 \mathrm{Im}(\mathcal{V}(y))$ is continuous. Therefore, $\mathcal{H}$ is continuous.

Suppose that $u_1 + \sqrt{-1}u_2$ is sufficiently close to the origin and $u_1 = \mathcal{H}(u_1, u_2) = 0$, hence $\mathcal{H}(0, u_2) = 0$. Since the function $t \mapsto \mathcal{H}(0,t)$ is strictly increasing on the neighborhood of $t = 0$, accordingly we have $\mathcal{H}(0, u_2) = 0$ only if $u_2 = 0$ for sufficiently small u_2. Therefore, $0 \in D(\delta_0)$ is isolated in

$$\left\{ u_1 + \sqrt{-1}u_2 \in D(\delta_0);\ u_1 = 0,\ \mathcal{H}(u_1, u_2) = 0 \right\}.$$

∎

The hyper-Kähler metric g_s^{OV} given by $\omega_{1,s}, \omega_{2,s}, \omega_{3,s}$ can be written as

$$g_s^{\mathrm{OV}} = V_s^{-1} \left(\frac{\alpha}{2\pi} \right)^2 + V_s \left(du_1^2 + du_2^2 + du_3^2 \right).$$

On $X_{\mathrm{OV}} \setminus \mu_{\mathrm{OV}}^{-1}(0)$, we have the decompositions $g_s^{\mathrm{OV}} = g_f + g_\perp$ and $\gamma_s = \gamma_f + \gamma_\perp$ as in Subsection 4.3. Then we may write

$$\begin{aligned} g_\perp &= V_s \left(du_1^2 + du_2^2\right), \\ \gamma_f &= -V_s^{-1}\frac{\partial \phi_s}{\partial u_1}\frac{\alpha}{2\pi} - \frac{\partial \phi_s}{\partial u_2} du_3, \\ \gamma_\perp &= \frac{\partial \phi_s}{\partial u_3} du_2. \end{aligned}$$

The aim of this section is to obtain the estimates in ($\star$4-7) of Subsection 4.3.

From now on we put $y := u_1 + \sqrt{-1}u_2 \in D(\delta_0)$, $|y| = \sqrt{u_1^2 + u_2^2}$ and

$$V_s^{\mathrm{sf}}(y) := \frac{1}{s}\int_0^s V_s(u_1, u_2, t)dt = -\frac{1}{2\pi s}\log|y| + \frac{h(y)}{s}.$$

By (21), we have

$$V_s^{\mathrm{sf}} \geq \frac{2\log|y|^{-1}}{5\pi s} \geq \frac{2\log \delta_0^{-1}}{5\pi s} \tag{24}$$

on $\overline{D(\delta_0)}$.

Fact 7.3 ([10, Lemma 3.1(c) and its proof]). *There is a constant $C > 0$ such that if $0 < s \leq \pi|y|$ then*

$$\left|V_s - V_s^{\mathrm{sf}}\right| \leq \frac{C}{s} e^{-2\pi|y|/s}.$$

Lemma 7.4. *Let $0 < r \leq \delta_0$. There is $s_r > 0$ for every r such that for any $0 < s \leq s_r$ we have $V_s \geq \log r^{-1}/(10\pi s)$ on $\overline{\mathcal{U}(r,s)}$. In particular, There is $s_0 > 0$ such that for any $0 < s \leq s_0$ we have $V_s \geq \log \delta_0^{-1}/(10\pi s)$ on $\overline{\mathcal{U}(\delta_0, s)}$.*

Proof. By Fact 7.3 and (24), if $0 < s \leq \pi r$ and $r = |y|$, then we have

$$V_s \geq V_s^{\mathrm{sf}} - \frac{Ce^{-2\pi r/s}}{s} \geq \frac{2\log r^{-1}}{5\pi s} - \frac{Ce^{-2\pi r/s}}{s}.$$

Put $h_M := \sup_{D(\delta_0)} h < +\infty$ and $h_m := \inf_{D(\delta_0)} h > -\infty$. Now, take $s_r > 0$ such that $s_r \leq \pi r$ and $Ce^{-2\pi\delta_0 r/s_r} \leq \log r^{-1}/(10\pi)$, then we can see

$$V_s \geq \frac{3\log r^{-1}}{10\pi s} \tag{25}$$

for $0 < s \leq s_r$. Since

$$\frac{1}{\sqrt{u_1^2 + u_2^2 + (u_3 - sn)^2}} \geq \frac{1}{\sqrt{r^2 + (u_3 - sn)^2}}$$

for $u \in \mathcal{U}(r,s)$, then

$$V_s(u_1, u_2, u_3) \geq V_s(r, 0, u_3) - \frac{h_M - h_m}{s}.$$

By (21), we have $h_M - h_m \leq \log \delta_0^{-1}/(5\pi)$. Therefore, by (25),

$$V_s(u_1, u_2, u_3) \geq \frac{3 \log r^{-1}}{10\pi s} - \frac{\log \delta_0^{-1}}{5\pi s} \geq \frac{\log r^{-1}}{10\pi s}$$

if $|y| \leq r$. □

Lemma 7.5. *There are constants* $s_0 > 0$ *and* $C_s \geq 1$ *for every* $0 < s \leq s_0$ *with* $\lim_{s\to 0} C_s = 1$ *such that if* $0 < s \leq \pi|y|$ *then we have*

$$C_s^{-1} V_s^{\mathrm{sf}} \leq V_s \leq C_s V_s^{\mathrm{sf}}.$$

Proof. By Fact 7.3, if $s \leq \pi|y|$ we have

$$V_s^{\mathrm{sf}} \left(1 - \frac{Ce^{-2\pi|y|/s}}{sV_s^{\mathrm{sf}}}\right) \leq V_s \leq V_s^{\mathrm{sf}} \left(1 + \frac{Ce^{-2\pi|y|/s}}{sV_s^{\mathrm{sf}}}\right),$$

therefore it suffices to show

$$\sup_{y\in D(\delta_0)} \frac{e^{-2\pi|y|/s}}{sV_s^{\mathrm{sf}}(y)} \to 0$$

as $s \to 0$. If $|y| \leq \sqrt{s}$, then we can see $e^{-2\pi|y|/s} \leq 1$, hence by (24) we have

$$\frac{e^{-2\pi|y|/s}}{sV_s^{\mathrm{sf}}(y)} \leq \frac{5\pi}{2\log|y|^{-1}} \leq \frac{5\pi}{\log s^{-1}} \xrightarrow{s\to 0} 0.$$

If $|y| \geq \sqrt{s}$, then we can see $e^{-2\pi|y|/s} \leq e^{-2\pi/\sqrt{s}}$. Therefore, we obtain

$$\frac{e^{-2\pi|y|/s}}{sV_s^{\mathrm{sf}}(y)} \leq \frac{5\pi e^{-2\pi/\sqrt{s}}}{2\log \delta_0^{-1}} \xrightarrow{s\to 0} 0.$$

□

Lemma 7.6. *We have*

$$V_s \leq V_s^{\mathrm{sf}} + \frac{1}{2\pi\sqrt{u_1^2 + u_2^2}}.$$

Proof. By the periodicity of V_s, we may suppose $0 \leq u_3 \leq s$. Since

$$\frac{1}{\sqrt{u_1^2 + u_2^2 + (u_3 - sn)^2}} \leq \frac{1}{\sqrt{u_1^2 + u_2^2 + s^2(n-1)^2}} \quad (n > 0),$$
$$\frac{1}{\sqrt{u_1^2 + u_2^2 + (u_3 - sn)^2}} \leq \frac{1}{\sqrt{u_1^2 + u_2^2 + s^2 n^2}} \quad (n \leq 0),$$

we have

$$V_s \leq V_s(u_1, u_2, 0) + \frac{1}{4\pi\sqrt{u_1^2 + u_2^2}}. \tag{26}$$

Similarly, we can also see

$$V_s(u_1,u_2,0) - \frac{1}{4\pi\sqrt{u_1^2+u_2^2}} \le V_s. \tag{27}$$

By integrating (27), we obtain

$$V_s(u_1,u_2,0) \le V_s^{\mathrm{sf}} + \frac{1}{4\pi\sqrt{u_1^2+u_2^2}}, \tag{28}$$

then by (26)(28) we have the result. □

Let $\chi = \chi(t)$ be the inverse function of $\tau \mapsto (\tau^2 \log \tau^{-1})/2\pi$ for $\tau \in [0,1/2]$. Then χ is an increasing function such that $\chi(0) = 0$ and $\chi((\log 2)/(8\pi)) = 1/2$. For a given $R > 0$, $\chi(sR^2) \le \delta_0$ iff $s \le \delta_0^2 \log \delta_0^{-1}/(2\pi R^2)$.

Lemma 7.7. *Take $s_0 > 0$ as in Lemmas 7.4 and 7.5. There is a positive constant C such that for any $0 < s \le s_0$ we have*

$$|\gamma_f|^2_{g_s^{\mathrm{OV}}} \le C\left(V_s^{\mathrm{sf}}|y|^2 + \frac{|y|}{2\pi}\right).$$

For any $R > 0$ there is $0 < s_R \le \min\{s_0, \delta_0^2 \log \delta_0^{-1}/(18\pi R^2)\}$ such that the following holds. For every $0 < s \le s_R$, there are constants $C_{s,R} \ge 1$ with $\lim_{s\to 0} C_{s,R} = 1$ such that if $y \in D(\chi(9sR^2)) \setminus D(s/\pi)$ then we have

$$C_{s,R}^{-1} V_s^{\mathrm{sf}}|y|^2 \le |\gamma_f|^2_{g_s^{\mathrm{OV}}} \le C_{s,R} V_s^{\mathrm{sf}}|y|^2.$$

Proof. First of all we have

$$\begin{aligned} |\gamma_f|^2_{g_s^{\mathrm{OV}}} &= V_s^{-1}\left(\frac{\partial \phi_s}{\partial u_1}\right)^2 + V_s^{-1}\left(\frac{\partial \phi_s}{\partial u_2}\right)^2 \\ &= V_s|y|^2\left(1 - \frac{2u_2F(u_1,u_2)}{sV_s|y|^2} + \frac{F(u_1,u_2)^2}{s^2V_s^2|y|^2}\right), \end{aligned}$$

where

$$F(u_1,u_2) := hu_2 - \frac{u_2}{2\pi} - \int_0^{u_1} t\frac{\partial h}{\partial u_2}(t,u_2)dt - \int_0^{u_2} h(0,t)dt.$$

Since F is C^∞ and $F(0,0) = 0$, there is a constant $A_1 > 0$ such that $F(u_1,u_2) \le A_1|y|$ for all $y \in D(\delta_0)$. Then we have $|F(u_1,u_2)|/(sV_s|y|) \le A_1/(sV_s)$ and $2|u_2F(u_1,u_2)|/(sV_s|y|^2) \le 2A_1/(sV_s)$, therefore, we can see

$$\left(1 - \frac{A_1}{sV_s}\right)^2 \le \frac{|\gamma_f|^2_{g_s^{\mathrm{OV}}}}{V_s|y|^2} \le \left(1 + \frac{A_1}{sV_s}\right)^2. \tag{29}$$

By Lemma 7.4, we have $1 + A_1/(sV_s) \le 1 + A_1 \log \delta_0^{-1}/(10\pi)$. Then by Lemma 7.6, there is a constant $C > 0$ such that

$$|\gamma_f|^2_{g_s^{\mathrm{OV}}} \le CV_s|y|^2 \le C\left(V_s^{\mathrm{sf}}|y|^2 + \frac{|y|}{2\pi}\right).$$

Next we assume $s/\pi \le \pi|y| < \chi(9sR^2)$, then by Lemma 7.5 we have

$$C_s^{-1} V_s^{\mathrm{sf}} |y^2| \le V_s |y^2| \le C_s V_s^{\mathrm{sf}} |y^2|, \quad \frac{A_1}{sV_s} \le \frac{C_s A_1}{sV_s^{\mathrm{sf}}},$$

hence (29) implies

$$\left(1 - \frac{C_s A_1}{sV_s^{\mathrm{sf}}}\right)^2 V_s^{\mathrm{sf}} |y|^2 \le |\gamma_f|^2_{g_s^{\mathrm{OV}}} \le \left(1 + \frac{C_s A_1}{sV_s^{\mathrm{sf}}}\right)^2 V_s^{\mathrm{sf}} |y|^2$$

Since $1/(sV_s^{\mathrm{sf}}) \le 5\pi/(2\log|y|^{-1}) \le 5\pi/(2\log\chi(9sR^2)^{-1})$ and $\lim_{s\to 0}\chi(9sR^2) = 0$, we have the second estimates. □

To give the estimate for $\gamma_\perp$, we need to compute $\frac{\partial \phi_s}{\partial u_3}$. We have

$$\begin{aligned} \frac{\partial \phi_s}{\partial u_3} &= -\int_0^{u_1} t \frac{\partial V_s}{\partial u_3}(t, u_2, u_3) dt + \frac{\partial \psi_s}{\partial u_3}(u_2, u_3) \\ &= -\frac{1}{4\pi} \sum_{n \in \mathbb{Z}^\times} \left\{ \frac{u_3 - sn}{\sqrt{u_1^2 + u_2^2 + (u_3 - sn)^2}} + \frac{|n|}{n} \right\} \\ &\quad - \frac{1}{4\pi} \frac{u_3}{\sqrt{u_1^2 + u_2^2 + u_3^2}} + \frac{u_3}{2\pi s}. \end{aligned}$$

Lemma 7.8. *We have* $|\frac{\partial \phi_s}{\partial u_3}| \le 1/2\pi$.

Proof. If we put

$$F(x,t) := -\frac{1}{4\pi} \sum_{n \in \mathbb{Z}^\times} \left\{ \frac{t-n}{\sqrt{x^2 + (t-n)^2}} + \frac{|n|}{n} \right\} - \frac{1}{4\pi} \frac{t}{\sqrt{x^2 + t^2}} + \frac{t}{2\pi},$$

then we may write

$$\frac{\partial \phi_s}{\partial u_3}(u_1, u_2, u_3) = F\left(\frac{|y|}{s}, \frac{u_3}{s}\right).$$

We show that $|F| \le 1/2\pi$. Since $F(x, t+n) = F(x,t)$ for $n \in \mathbb{Z}$, we may suppose $0 \le t \le 1$. Since the function $t \mapsto t/\sqrt{x^2 + t^2}$ is nondecreasing, we have

$$\frac{1}{4\pi} \frac{n-1}{\sqrt{x^2 + (n-1)^2}} \le -\frac{1}{4\pi} \frac{t-n}{\sqrt{x^2 + (t-n)^2}} \le \frac{1}{4\pi} \frac{n}{\sqrt{x^2 + n^2}}$$

for every $n \in \mathbb{Z}$. By using these inequalities, we can show $-1/2\pi \le F(x,t) \le 1/2\pi$. □

Corollary 7.9. *Let* $g_B := V_s^{\mathrm{sf}}(du_1^2 + du_2^2)$. *Then*

$$|\gamma_\perp|^2_{g_s^{\mathrm{OV}}} \le \frac{5s}{2\pi \log \delta_0^{-1}}, \quad |\gamma_\perp|^2_{\mu_{\mathrm{OV}}^* g_B} \le \frac{5s}{8\pi \log \delta_0^{-1}}.$$

Proof. Since we have

$$|\gamma_\perp|^2_{g_s^{\mathrm{OV}}} = V_s^{-1} \left(\frac{\partial \phi_s}{\partial u_3}\right)^2, \quad |\gamma_\perp|^2_{\mu_{\mathrm{OV}}^* g_B} = \left(V_s^{\mathrm{sf}}\right)^{-1} \left(\frac{\partial \phi_s}{\partial u_3}\right)^2,$$

then we have the result by Lemma 7.4 and (24). □

Next we define a map $\zeta_s\colon D(\delta_0)\to\mathbb{R}^2$ by

$$\zeta_s(y) := \sqrt{\frac{\log|y|^{-1}}{2\pi s}}\cdot y.$$

Then we have

$$|y| = \chi(s|\zeta_s(y)|^2) \tag{30}$$

for any $y\in D(\delta_0)$. Recall that we have put $\mathcal{B}(r) := \{\xi\in\mathbb{R}^2;\ \|\xi\|<r\}$, then we have $D(\chi(sr^2)) = \zeta_s^{-1}(\mathcal{B}(r))$. Hence we have the following.

Proposition 7.10. *We have $\zeta_s(D(\delta_0)) = \mathcal{B}(\delta_0\sqrt{\log\delta_0^{-1}/(2\pi s)})$. In particular, $\mathcal{B}(3R)\subset\zeta_s(D(\delta_0))$ iff $s\le\delta_0\sqrt{\log\delta_0^{-1}/(18\pi R^2)}$.*

Lemma 7.11. *Let $\xi=(\xi_1,\xi_2)\in\mathbb{R}^2$ be the standard coordinate and denote by $|d\xi|^2 := d\xi_1^2+d\xi_2^2$ be the Euclidean metric and s_0 be as in Lemma 7.5. Let $R>0$ and $0<s\le\min\{\delta_0^2\log\delta_0^{-1}/(18\pi R^2), s_0\}$. Then there are constants $C_{s,R}\ge 1$ such that $\lim_{s\to 0}C_{s,R}=1$ and*

$$C_{s,R}^{-1}\zeta_s^*|d\xi|^2 \le V_s^{\mathrm{sf}}|dy|^2 \le C_{s,R}\zeta_s^*|d\xi|^2$$

for $y\in\zeta_s^{-1}(\mathcal{B}(3R))$.

Proof. Let $y=r_y e^{\sqrt{-1}\theta}$ and $\xi=r_\xi e^{\sqrt{-1}\theta}$ be the polar coordinates. If $\xi=\zeta_s(y)$, then $r_y=\chi(sr_\xi^2)$ by (30). Then we have

$$\begin{aligned}(\zeta_s^{-1})^*\left\{V_s^{\mathrm{sf}}|dy|^2\right\} &= \left(\frac{\log\chi(sr_\xi^2)^{-1}}{2\pi s}+\frac{h(\zeta_s^{-1}(\xi))}{s}\right)(2s\chi' r_\xi)^2dr_\xi^2\\ &\quad + r_\xi^2\left(1+\frac{2\pi h(\zeta_s^{-1}(\xi))}{\log\chi(sr_\xi^2)^{-1}}\right)d\theta^2.\end{aligned}$$

Since $\chi' = 2\pi/(2\chi\log\chi^{-1}-\chi)$ and $r_\xi^2=\chi^2\log\chi^{-1}/(2\pi s)$, we have

$$(2s\chi' r_\xi)^2 = \frac{2\pi s}{\log\chi^{-1}}\left(1-\frac{1}{2\log\chi^{-1}}\right)^{-2},$$

hence we obtain

$$(\zeta_s^{-1})^*\left\{V_s^{\mathrm{sf}}|dy|^2\right\} = \left(1+\frac{2\pi h}{\log\chi^{-1}}\right)\left(\left(1-\frac{1}{2\log\chi^{-1}}\right)^{-2}dr_\xi^2+r_\xi^2d\theta^2\right).$$

Since h is bounded on $D(\delta_0)$ and we have $\lim_{s\to 0}1/(\log\chi(sr_\xi^2)^{-1})=0$, then there are constants $C_{s,R}\ge 1$ such that $\lim_{s\to 0}C_{s,R}=1$ and $C_{s,R}^{-1}\zeta_s^*|d\xi|^2\le V_s^{\mathrm{sf}}|dy|^2\le C_{s,R}\zeta_s^*|d\xi|^2$. □

Proposition 7.12. *For every $R>0$, there is a constant $s_R>0$ such that the following holds. For every $R>0$ and $0<s\le s_R$ there are positive constants $C_{s,R},\sigma_s,\delta_{s,R}$ with*

$$\lim_{s\to 0}C_{s,R}=1,\quad \lim_{s\to 0}\delta_{s,R}=\lim_{s\to 0}\sigma_s=0,$$

such that if $y \in \zeta_s^{-1}(\mathcal{B}(3R))$*, then*

$$|\gamma_\perp|_{g_s^{\mathrm{OV}}} \le \delta_{s,R}, \quad |\gamma_\perp|_{(\zeta_s\circ\mu_{\mathrm{OV}})^*|d\xi|^2} \le \delta_{s,R},$$

and if $y \in \zeta_s^{-1}(\mathcal{B}(3R)\setminus\mathcal{B}(\sigma_s))$*, then*

$$C_{s,R}^{-1}(\zeta_s\circ\mu_{\mathrm{OV}})^*|d\xi|^2 \le g_\perp \le C_{s,R}(\zeta_s\circ\mu_{\mathrm{OV}})^*|d\xi|^2,$$
$$C_{s,R}^{-1}(\zeta_s\circ\mu_{\mathrm{OV}})^*\mathbf{r}^2 \le |\gamma_f|^2_{g_s^{\mathrm{OV}}} \le C_{s,R}(\zeta_s\circ\mu_{\mathrm{OV}})^*\mathbf{r}^2.$$

Proof. Put

$$\sigma_s := \sqrt{\frac{s(\log s^{-1}+\log\pi)}{2\pi^3}}.$$

Note that $\sigma_s \le |\zeta_s(y)| < 3R$ iff $s/\pi \le |y| < \chi(9sR^2)$. If $s/\pi \le |y| < \chi(9sR^2)$ and s is sufficiently small, then Lemmas 7.5 and 7.11 give

$$C_{s,R}^{-1}(\zeta_s\circ\mu_{\mathrm{OV}})^*|d\xi|^2 \le g_\perp \le C_{s,R}(\zeta_s\circ\mu_{\mathrm{OV}})^*|d\xi|^2.$$

for some constant $C_{s,R} \ge 1$ with $\lim_{s\to 0} C_{s,R} = 1$. Combining Corollary 7.9 with Lemma 7.11, we have

$$|\gamma_\perp|^2_{g_s^{\mathrm{OV}}} \le \frac{5s}{2\pi\log\delta_0^{-1}}, \quad |\gamma_\perp|^2_{(\zeta_s\circ\mu_{\mathrm{OV}})^*|d\xi|^2} \le \frac{5C_{s,R}\cdot s}{8\pi\log\delta_0^{-1}}.$$

By putting $\delta_{s,R} := 5s/(2\pi\log\delta_0^{-1})\max\{1, C_{s,R}/4\}$, we obtain the estimates for $\gamma_\perp$.

Since

$$\frac{V_s^{\mathrm{sf}}|y|^2}{|\zeta_s(y)|^2} = \frac{2\pi s V_s^{\mathrm{sf}}}{\log|y|^{-1}} \to 1 \tag{31}$$

as $s \to 0$, then we obtain the inequalities for $|\gamma_f|_{g_s^{\mathrm{OV}}}$ by Lemma 7.7. □

Proposition 7.13. *Let* $s_0 > 0$ *be as in Lemma 7.7 and* $\sigma_s > 0$ *be as in Proposition 7.12. Then there are constants* $\delta_s > 0$ *for every* $0 < s \le s_0$ *with* $\lim_{s\to 0}\delta_s = 0$ *such that if* $0 < s \le s_0$ *and* $y \in \overline{\zeta_s^{-1}(\mathcal{B}(\sigma_s))}$*, then* $|\gamma_f|^2_{g_s^{\mathrm{OV}}} \le \delta_s$.

Proof. By Lemma 7.7 we have

$$|\gamma_f|^2_{g_s^{\mathrm{OV}}} \le C\left(V_s^{\mathrm{sf}}|y|^2 + \frac{|y|}{2\pi}\right)$$

for some constant $C > 0$. Then by (31), it suffices to show that $|\zeta_s(y)|^2 \to 0$ and $|y| \to 0$ as $s \to 0$. Since $|\zeta_s(y)|^2 \le \sigma_s^2 \to 0$ and $|y| \le \chi(s\sigma_s^2) \to 0$ as $s \to 0$, we have the result. □

Fact 7.14 ([10, Proposition 3.5]). *There is a constant* $C > 0$ *such that*

$$\mathrm{diam}_{g_s^{\mathrm{OV}}|_{\mu_{\mathrm{OV}}^{-1}(y)}}(\mu_{\mathrm{OV}}^{-1}(y)) \le C\sqrt{s\log s^{-1}}$$

for every $y \in D(\delta_0)$.

Proposition 7.15 ([10, Corollary 3.7]). *Let σ_s be as in Proposition 7.12. There is a constant $C > 0$ such that*

$$\operatorname{diam}_{g_s^{\mathrm{OV}}|_{\mu_{\mathrm{OV}}^{-1}\left(\zeta_s^{-1}(\mathcal{B}(\sigma_s))\right)}}\left(\mu_{\mathrm{OV}}^{-1}\left(\overline{\zeta_s^{-1}(\mathcal{B}(\sigma_s))}\right)\right) \le C\sqrt{s\log s^{-1}}.$$

Proof. The proof was essentially obtained in the proof of [10, Corollary 3.7]. Note that $D(s/\pi) = \zeta_s^{-1}(\mathcal{B}(\sigma_s))$. Take a point $p \in X_{\mathrm{OV}}$ with $u(p) = (s\cos\theta, s\sin\theta, s/2)$. Then the infimum of the distance between p and the singular fiber $\mu_{\mathrm{OV}}^{-1}(0)$ is bounded from the above by

$$\int_0^s \sqrt{V_s(r\cos\theta, r\sin\theta, s/2)}\, dr.$$

By the proof of [10, Corollary 3.7], there is a constant $C > 0$ such that the above integral is not more than $Cs\log s^{-1}$. Since $D(s/\pi) \subset D(s)$, by combining Fact 7.14, we have the result. □

Next we consider the measure. Define a measure ν_B on $D(\delta_0)$ by $\nu_B := (\mu_{\mathrm{OV}})_*\nu_{g_s^{\mathrm{OV}}}$. Since $\nu_{g_s^{\mathrm{OV}}} = V_s(\alpha/2\pi) \wedge du_3 \wedge du_1 \wedge du_2$, we have

$$\nu_B = sV_s^{\mathrm{sf}} du_1 du_2.$$

Proposition 7.16. *There are constants $C_{s,R} \ge 1$ with $\lim_{s\to 0} C_{s,R} = 1$ such that*

$$C_{s,R}^{-1} d\xi_1 d\xi_2 \le \frac{(\zeta_s)_*\nu_B}{s} \le C_{s,R} d\xi_1 d\xi_2$$

if $|\xi| < R$.

Proof. Let $y = r_y e^{\sqrt{-1}\theta}$ and $\xi = r_\xi e^{\sqrt{-1}\theta}$. Since $du_1 du_2 = r_y dr_y d\theta$, therefore, by the computation in the proof of Lemma 7.11,

$$\begin{aligned}(\zeta_s)_*\nu_B &= sV_s^{\mathrm{sf}}(\zeta_s^{-1}(\xi))\chi(sr_\xi^2)\cdot 2s\chi' r_\xi dr_\xi d\theta\\ &= s\left(1 + \frac{2\pi h}{\log\chi^{-1}}\right)\left(1 - \frac{1}{2\log\chi^{-1}}\right)^{-1} r_\xi dr_\xi d\theta.\end{aligned}$$

If $r_\xi < R$, we have

$$\left(1 + \frac{2\pi h(\zeta_s^{-1}(\xi))}{\log\chi^{-1}(sr_\xi^2)}\right)\left(1 - \frac{1}{2\log\chi^{-1}(sr_\xi^2)}\right)^{-1} \to 0$$

as $s \to 0$, hence we obtain the result. □

By Propositions 7.12, 7.13, 7.15, 7.16 and Fact 7.14, we have shown

$$\left(g_s^{\mathrm{OV}}, \frac{1}{s}, 0, D(\delta_0)\right) \to (\mathbb{R}^2, g_0)$$

as $s \to 0$ for sufficiently small δ_0. Thus we complete the proof of Theorem 4.1.

8. Compact convergence. The aim of this section is to prove Theorem 4.3. In this section let (X, g_s), $\mu\colon X \to \mathbb{P}^1$, (L, h, ∇) be as in Subsection 4.1 and let g'_s be as in Subsection 5.5. We fix a positive integer k.

8.1. Preparation. Here, we review [14] for the preparation for the following subsections. Let $B \subset \mathbb{R}^2$ be an open set, $X_0 := B \times (\mathbb{R}^2/2\pi\mathbb{Z}^2)$, $x = (x_1, x_2) \in \mathbb{R}^2$ and $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2/2\pi\mathbb{Z}^2$ be the standard coordinates. Put $\omega := dx_1 \wedge d\theta_1 + dx_2 \wedge d\theta_2$ and let $L_0 := X_0 \times \mathbb{C}$. Denote by h_0 the hermitian metric on L_0 such that $h_0((x, 1), (x, 1)) \equiv 1$, and ∇_0 be the hermitian connection defined by $\nabla_0 = d - \sqrt{-1}\sum_{i=1}^2 x_i \wedge d\theta_i$. Here, we have $BS_k = (1/k)\mathbb{Z}^2 \cap B$.

Let g be a Riemannian metric on X_0 such that

$$\frac{(1+\delta)^{-1}\omega^2}{2} \le d\nu_g \le \frac{(1+\delta)\omega^2}{2}$$

for a constant $\delta \ge 0$. Note that if g is the Kähler metric of ω with respect to an ω-compatible complex structure, then we can take $\delta = 0$. In the following subsections we will take $g = g'_s$. In this case we can take $\delta = \delta_s$ such that $\lim_{s\to 0}\delta_s = 0$ by (19).

Next we consider the induced metric on every fiber

$$g|_{\{x\}\times(\mathbb{R}^2/2\pi\mathbb{Z}^2)} = \sum_{i,j=1}^2 g_{ij}(x,\theta)d\theta_i d\theta_j.$$

Let $\mathbb{S}_0 = \mathbb{S}(L_0, h_0)$ and $\hat{g}$ be defined by (2). The next lemma is the generalization of [14, Proposition 4.3].

LEMMA 8.1. *Let k be a positive integer and $\overline{g}(x) = \sum_{i,j=1}^2 \overline{g}_{ij}(x)d\theta_i d\theta_j$ be a family of Riemannian metrics on $\mathbb{R}^2/2\pi\mathbb{Z}^2$ such that $g|_{\{x\}\times(\mathbb{R}^2/2\pi\mathbb{Z}^2)} \le \overline{g}(x)$ for all $x \in B$. Denote by $(\overline{g}^{ij}(x))_{i,j}$ the inverse matrix of $(\overline{g}_{ij}(x))_{i,j}$. Then we have*

$$\int_{\mathbb{S}_0} |df|^2_{\hat{g}}\, d\nu_{\hat{g}} \ge 2\pi\frac{k^2+K}{(1+\delta)^2}\int_{\mathbb{S}_0}|f|^2 d\nu_{\hat{g}}$$

for $f \in (H^{1,2}(\mathbb{S}_0, d_{\hat{g}}, \nu_{\hat{g}}) \otimes \mathbb{C})^{\rho_k}$, where

$$K := k^2 \inf_{x\in B}\inf\left\{\|x+l\|^2_{\overline{g}(x)};\, l \in \frac{1}{k}\mathbb{Z}^2\right\},$$

$$\|x+l\|_{\overline{g}(x)} := \sqrt{\sum_{i,j=1}^2 (x_i+l_i)(x_j+l_j)\overline{g}^{ij}(x)}.$$

Proof. The proof is same as that of [14, Propositions 4.2, 4.3]. Here we explain the outline. First of all we have

$$\int_{\mathbb{S}_0} |df|^2_{\hat{g}}\, d\nu_{\hat{g}} \ge \int_{\mathbb{S}_0} \left|df|_{\mathbb{S}_0|_x}\right|^2_{\hat{g}_x} d\nu_{\hat{g}},$$

where $\mathbb{S}_0|_x = S^1 \times \{x\} \times \mathbb{R}^2/2\pi\mathbb{Z}^2$ and $\hat{g}_x := (dt - \sum_i x_i d\theta_i)^2 + g|_{\{x\}\times\mathbb{R}^2/2\pi\mathbb{Z}^2}$.

Since $d\nu_{\hat{g}} = dt \cdot d\nu_g$ and $\omega^2/2 = dx_1 dx_2 d\theta_1 d\theta_2$, we have

$$\int_{\mathbb{S}_0} \left|df|_{\mathbb{S}_0|_x}\right|^2_{\hat{g}_x} d\nu_{\hat{g}} \geq (1+\delta)^{-1} \int_B \left(\int_{\mathbb{S}_0|_x} \left|df|_{\mathbb{S}_0|_x}\right|^2_{\hat{g}_x} dt d\theta \right) dx,$$

$$\int_B \left(\int_{\mathbb{S}_0|_x} |f|^2 \, dt d\theta \right) dx \geq (1+\delta)^{-1} \int_{\mathbb{S}_0} |f|^2 d\nu_{\hat{g}}.$$

Since $f \in (H^{1,2}(\mathbb{S}_0, d_{\hat{g}}, \nu_{\hat{g}}) \otimes \mathbb{C})^{\rho_k}$, we may put $f|_{\mathbb{S}_0|_x} = e^{-\sqrt{-1}kt}\varphi(\theta)$ for some $\varphi\colon \mathbb{R}^2/2\pi\mathbb{Z}^2 \to \mathbb{C}$. Then we have

$$\left|df|_{\mathbb{S}_0|_x}\right|^2_{\hat{g}_x} \geq k^2|\varphi|^2 + \sum_{i,j} \left(\frac{\partial \varphi}{\partial \theta_i} + \sqrt{-1}x_i\varphi \right) \left(\frac{\partial \bar{\varphi}}{\partial \theta_j} - \sqrt{-1}x_j\bar{\varphi} \right) \overline{g}^{ij}(x).$$

If we put $\varphi(\theta) = e^{\sqrt{-1}\sum_i l_i\theta_i}$ for $l_1, l_2 \in \mathbb{Z}$, then

$$\begin{aligned} &k^2|\varphi|^2 + \sum_{i,j} \left(\frac{\partial \varphi}{\partial \theta_i} + \sqrt{-1}x_i\varphi \right) \left(\frac{\partial \bar{\varphi}}{\partial \theta_j} - \sqrt{-1}x_j\bar{\varphi} \right) \overline{g}^{ij}(x) \\ &= k^2 + (kx_i + l_i)(kx_j + l_j)\,\overline{g}^{ij}(x), \end{aligned}$$

hence we have the result. □

Denote by $N_x(\theta)$ the maximum eigenvalue of the symmetric positive matrix $(g_{ij}(x,\theta))_{i,j}$. Put

$$N_x := \sup_{\theta \in \mathbb{R}^2/2\pi\mathbb{Z}^2} N_x(\theta),$$

$$\lambda(k,x) := k^2 \inf \left\{ \sum_{i=1}^{2} (l_i + x_i)^2;\ l_1, l_2 \in \frac{1}{k}\mathbb{Z} \right\}$$

for $x \in B$. Then we have

$$K \geq \inf_{x\in B} \frac{\lambda(k,x)}{N_x}.$$

8.2. Estimates on the nonsingular fibers. Let $W_1^{\mathbf{q}} \subset W_2^{\mathbf{q}} \subset \mathbb{P}^1$ be as in Fact 5.1. Since $\mathbb{P}^1$ and $\mathcal{K} := \mathbb{P}^1 \setminus (\bigsqcup_{\mathbf{q}} W_1^{\mathbf{q}})$ are compact and all of the points in $\mathcal{K}$ are regular values of μ, then by Liouville-Arnold Theorem, there are open sets $W_a'' \subset W_a' \subset \mathbb{P}^1 \setminus \mathrm{Crt}$ for $a = 1, \ldots, N_0$ such that the following holds.

(i) On every W_a' there is an action-angle coordinate $x_{a,1}, x_{a,2}, \theta_{a,1}, \theta_{a,2}$ with

$$\omega_1|_{\mu^{-1}(W_a')} = dx_{a,1} \wedge d\theta_{a,1} + dx_{a,2} \wedge d\theta_{a,2}.$$

(ii) $\mathcal{K} \subset \bigcup_a W_a''$ and $\overline{W}_a'' \subset W_a'$,

(iii) $BS_k \cap \partial W_a'' = \emptyset$ and $x_a(\overline{W}_a'') \subset \mathbb{R}^2$ is bounded for all a.

Put $U_a' := \mu^{-1}(W_a')$. Here, $x_a = (x_{a,1}, x_{a,2})$ is a coordinate on $\mathbb{P}^1$ and $\theta_a = (\theta_{a,1}, \theta_{a,2})$ is the coordinate on the fibers $\mu^{-1}(x_a) \cong \mathbb{R}^2/2\pi\mathbb{Z}^2$. By [14, Proposition 2.4], we can choose a trivialization $L|_{U_a'} = U_a' \times \mathbb{C}$ and the action-angle coordinate such that $\nabla|_{U_a'} = d - \sqrt{-1}\sum_{i=1}^2 x_{a,i} d\theta_{a,i}$. Now, we may suppose $BS_k \cap W_2^{\mathbf{q}} \subset \{b_{\mathbf{q}}\}$.

Next we apply Lemma 8.1. To apply it, we estimate N_b and $\lambda_{k,b}$.

If $b \in \overline{W}''_a \setminus (\bigsqcup_{\mathbf{q}} W_2^{\mathbf{q}})$, then g'_s is isometric to the standard semi-flat metric. Denote by $g'_{s,b} = g'_s|_{\mu^{-1}(b)}$ the fiberwise metric. By the explicit description of η_s^{SF}, we have

$$g'_{s,b} = s g'_{1,b},$$

consequently we have $N_b = sN_{b,1}$ for some constant $N_{b,1} > 0$ depending only on b. If $b \in \overline{W}''_a \cap (W_2^{\mathbf{q}} \setminus W_1^{\mathbf{q}})$, then by Lemma 5.2, we also have $N_b \le sN_{b,1}$ for some $N_{b,1} > 0$ depending only on b. Here, we may suppose that $N_{b,1}$ is depending on b continuously on $\overline{W}''_a \cap \mathcal{K}$. Therefore, there is a constant $C_{1,a} > 0$ such that $N_b \le sC_{1,a}$ for all $b \in \overline{W}''_a \cap \mathcal{K}$, hence $N_b \le sC_1$ for all $b \in \mathcal{K}$, where $C_1 = \max_a C_{1,a}$.

Next we put

$$\mathcal{K}(r) := \mathcal{K} \setminus \left(\bigcup_{a=1}^{N_0} \left(\bigcup_{b \in W''_a \cap BS_k} B(a;b,r) \right) \right),$$
$$B(a;b,r) := \left\{ y \in W'_a;\ |x_a(y) - x_a(b)| < r \right\}.$$

Note that $x_a(W'_a \cap BS_k) = x_a(W'_a) \cap (1/k)\mathbb{Z}^2$. By (iii), there is $r_0 > 0$ such that if $0 < r \le r_0$ then

$$\left\{ y \in \overline{W}''_a;\ |x_a(y) - l| \ge r \text{ for all } l \in \frac{1}{k}\mathbb{Z}^2 \right\} = \overline{W}''_a \setminus \left(\bigcup_{b \in W''_a \cap BS_k} B(a;b,r) \right).$$

Therefore, we have $\lambda(k,b) \ge k^2 r^2$ for any $b \in \mathcal{K}(r)$.

Now, we take a Borel set $U \subset (\mu \circ \pi)^{-1}(\mathcal{K}(r))$ and let $U = \bigsqcup_{a'} U(a')$ such that $U(a')$ are Borel sets and every $U(a')$ is contained in $(\mu \circ \pi)^{-1}(W'_a)$ for some a. By applying Lemma 8.1 to each $W(a')$, we have

$$\int_{\mathbb{S}|_{\mu^{-1}(W)}} |df|^2_{\hat{g}'_s}\, d\nu_{\hat{g}'_s} \ge \frac{2\pi k^2}{(1+\delta_s)^2} \left(1 + \frac{r^2}{sC_1} \right) \int_{\mathbb{S}|_{\mu^{-1}(W)}} |f|^2 d\nu_{\hat{g}'_s}$$

for some $\delta_s > 0$ with $\lim_{s\to 0} \delta_s = 0$. For every $b \in BS_k \cap W'_a$, fix $q^b \in \mu^{-1}(b)$. By [13, Proposition 7.12 (iii)], there are $\delta_b > 0$, $R_0 > 0$ and $s_R > 0$ such that

$$\mu^{-1}\left(B(a;b,\delta_b\sqrt{s}R) \right) \subset B_{g'_s}(q^b, R)$$

for $R \ge R_0$ and $0 < s \le s_R$. Moreover, we also have

$$\pi^{-1}\left(B_{g'_s}(q^b, R) \right) \subset B_{\hat{g}'_s}(p^b, R+\pi)$$

for $p^b \in \pi^{-1}(q^b)$ by (9). If we put $\delta = \min_{b \in BS_k, a} \delta_b$, then we have

$$\mathbb{S}_{\mathcal{K},R} := (\mu\circ\pi)^{-1}(\mathcal{K}) \setminus \left(\bigcup_{b \in BS_k \setminus \mathrm{Crt}} B_{\hat{g}'_s}(p^b, R) \right)$$
$$\subset (\mu\circ\pi)^{-1}(\mathcal{K}(\delta\sqrt{s}(R-\pi))),$$

hence we obtain

$$\int_{\mathbb{S}_{\mathcal{K},R}} |df|^2_{\hat{g}'_s}\, d\nu_{\hat{g}'_s} \ge \frac{2\pi k^2}{(1+\delta_s)^2} \left(1 + \frac{\delta^2(R-\pi)^2}{C_1} \right) \int_{\mathbb{S}_{\mathcal{K},R}} |f|^2 d\nu_{\hat{g}'_s} \tag{32}$$

for $R \ge R_0$ and $0 < s \le s_R$.

8.3. Estimates on the neighborhood of the singular fibers. In this subsection we fix one of the critical points $b_{\mathbf{q}} \in \mathrm{Crt}$ of μ and consider the restriction of g'_s on $\mu^{-1}(W_1^{\mathbf{q}})$, which is isometric to the Ooguri-Vafa metric g_s^{OV} by Fact 5.1. Accordingly, we put $\mu^{-1}(W_2^{\mathbf{q}}) = X_{\mathrm{OV}}$, $r_2^{\mathbf{q}} \le \delta_0$ and we go back to the setting in Section 7. As we have seen in Section 7, $L|_{X_{\mathrm{OV}}}$ is a trivial bundle, hence we may put $L|_{X_{\mathrm{OV}}} = X_{\mathrm{OV}} \times \mathbb{C}$, $h((x,1),(x,1)) \equiv 1$ for $x \in X_{\mathrm{OV}}$, $\nabla|_{X_{\mathrm{OV}}} = d - \sqrt{-1}\gamma_1$ for some $\gamma_1 \in \Omega^1(X_{\mathrm{OV}})$ such that $\omega_{1,s} = d\gamma_1$.

First of all we describe $\omega_{1,s}$ by the action-angle coordinate. Recall that we have defined $\mathbb{C}^\times$-action on $\tilde{X}_{\mathrm{OV}}$ by (23). Let $y = u_1 + \sqrt{-1}u_2$ and $z = z_1 + \sqrt{-1}z_2$ be a holomorphic coordinate with respect to $J_{3,s}$. Then we have

$$\omega_{1,s} + \sqrt{-1}\omega_{2,s} = dy \wedge \left(\frac{\alpha}{2\pi} - \sqrt{-1}V_s du_3\right) = \frac{1}{2\pi} dy \wedge dz.$$

Define another coordinate $\theta = (\theta_1, \theta_2)$ on fibers by $\theta \mapsto q \cdot e^{\sqrt{-1}\theta_1 - \theta_2 \mathcal{V}(\mu_{\mathrm{OV}(q)})}$, where $\mathcal{V}(y) = (\log y^{-1})/2\pi + \hat{h}(y)$ and $\hat{h}$ is a holomorphic function such that $\mathrm{Re}(\hat{h}) = h$. Here we assume

$$0 \le \mathrm{Im}(\log y^{-1}) < 2\pi.$$

Since we have

$$\begin{aligned} dz &= d\left(\theta_1 + \sqrt{-1}\theta_2 \mathcal{V}(y)\right) \\ &= d\theta_1 + \sqrt{-1}\mathcal{V}(y)d\theta_2 + \sqrt{-1}\theta_2 \frac{\partial \mathcal{V}}{\partial y} dy, \end{aligned}$$

we obtain

$$dy \wedge dz = dy \wedge d\theta_1 + \sqrt{-1}\mathcal{V}(y) dy \wedge d\theta_2.$$

If we denote by $\hat{\mathcal{H}}(y)$ the holomorphic function such that $\frac{\partial \hat{\mathcal{H}}}{\partial y} = \mathcal{V}$, then we have $dy \wedge dz = dy \wedge d\theta_1 + \sqrt{-1} d\hat{\mathcal{H}} \wedge d\theta_2$, hence

$$\omega_{1,s} = \frac{1}{2\pi}(du_1 \wedge d\theta_1 - d(\mathrm{Im}\hat{\mathcal{H}}) \wedge d\theta_2).$$

Since the integral path of $\frac{\partial}{\partial \theta_2}$ represents the homology class $-e_{2,y}$ defined in Lemma 7.2, hence we can see that $-d\mathrm{Im}\hat{\mathcal{H}} = d\mathcal{H}$. Here, we define $x = (x_1, x_2)$ by

$$x_1 = \int_{e_{1,y}} \gamma_1, \quad x_2 = \int_{e_{2,y}} \gamma_1,$$

where $e_{1,y}, e_{2,y}$ are as in Lemma 7.2. Since $\omega_{1,s} = \sum_{i=1}^2 dx_i \wedge d\theta_i$, we have

$$x = (x_1, x_2) = (u_1 + a_1,\, \mathcal{H}(y) + a_2)$$

for some constants $a_1, a_2 \in \mathbb{R}$. Here, the origin $0 \in D(\delta_0)$ is in BS_k iff $(a_1, a_2) \in (1/k)\mathbb{Z}^2$.

By the definition of the coordinate θ, we have

$$\begin{aligned} \frac{\partial}{\partial \theta_1} &= v, \\ \frac{\partial}{\partial \theta_2} &= \mathrm{Im}(\mathcal{V}(y))v + \mathrm{Re}(\mathcal{V}(y)) J_{3,s} v. \end{aligned}$$

Consequently, we have

$$g_s^{\mathrm{OV}}\left(\frac{\partial}{\partial\theta_1},\frac{\partial}{\partial\theta_1}\right)=\frac{V_s^{-1}}{4\pi^2},$$
$$g_s^{\mathrm{OV}}\left(\frac{\partial}{\partial\theta_1},\frac{\partial}{\partial\theta_2}\right)=\frac{\mathrm{Im}(\mathcal{V}(y))V_s^{-1}}{4\pi^2},$$
$$g_s^{\mathrm{OV}}\left(\frac{\partial}{\partial\theta_2},\frac{\partial}{\partial\theta_2}\right)=\frac{|\mathcal{V}(y)|^2V_s^{-1}}{4\pi^2},$$

therefore we have

$$\begin{aligned} g_s^{\mathrm{OV}}|_{\mu_{\mathrm{OV}}^{-1}(y)} &= \frac{V_s^{-1}}{4\pi^2}\left(d\theta_1^2+2\mathrm{Im}(\mathcal{V})d\theta_1 d\theta_2+|\mathcal{V}(y)|^2 d\theta_2^2\right)\\ &= \frac{V_s^{-1}}{4\pi^2}\left\{(d\theta_1+\mathrm{Im}(\mathcal{V})d\theta_2)^2+\mathrm{Re}(\mathcal{V})^2 d\theta_2^2\right\}. \end{aligned}$$

To apply Lemma 8.1, we estimate K. By Lemma 7.4, we have

$$g_s^{\mathrm{OV}}|_{\mu_{\mathrm{OV}}^{-1}(y)} \le \frac{5s}{6\pi\log|y|^{-1}}\left\{(d\theta_1+\mathrm{Im}(\mathcal{V})d\theta_2)^2+\mathrm{Re}(\mathcal{V})^2 d\theta_2^2\right\}.$$

Now, $\mathrm{Im}(\mathcal{V})$ is multivalued, however, we can take the branch of it on every neighborhood such that it is bounded. Moreover, Since $\log|y|^{-1}\to\infty$ as $|y|\to 0$, there is $0<\delta_1\le\delta_0$ and $C>0$ such that

$$g_s^{\mathrm{OV}}|_{\mu_{\mathrm{OV}}^{-1}(y)} \le \frac{Cs}{\log|y|^{-1}}\left\{d\theta_1^2+(\log|y|^{-1})^2 d\theta_2^2\right\} =: \overline{g}_y$$

for $y\in D(\delta_1)$. For $\xi=(\xi_1,\xi_2)\in\mathbb{R}^2$, we put

$$\|\xi\|_{\overline{g}_y} := \sqrt{\frac{\log|y|^{-1}}{Cs}\xi_1^2+\frac{1}{Cs\log|y|^{-1}}\xi_2^2}.$$

Lemma 8.2. *There are positive constants $\delta_0,\delta_1>0$ such that*

$$\inf_{l\in(1/k)\mathbb{Z}^2,\, l\neq -a}\|x(y)+l\|_{\overline{g}_y}^2 \ge \frac{\delta_1}{s\log|y|^{-1}}$$

for any $y\in D(\delta_0)$ and δ_0 satisfies (20)(21)(22).

Proof. We have

$$\|x-a\|_{\overline{g}_y}^2=\frac{u_1^2\log|y|^{-1}}{Cs}+\frac{1}{Cs\log|y|^{-1}}\mathcal{H}(y)^2.$$

There is $C_1>0$ such that $|\mathcal{H}(y)-u_2\log|y|^{-1}/2\pi|\le C_1|y|$ on $D(\delta_0)$. Since $|y|<\delta_0$ and $\log|y|^{-1}\ge\log\delta_0^{-1}>0$, by taking C_1 larger if necessary, we have $|\mathcal{H}(y)|\le C_1|y|\log|y|^{-1}$. Therefore, there is $C_2>0$ such that

$$\|x-a\|_{\overline{g}_y}^2\le\frac{C_2|y|^2\log|y|^{-1}}{s}.$$

Moreover, there is $\delta_1' > 0$ such that for any $\xi \in \mathbb{R}^2$ we have $\|\xi\|^2_{\overline{g}_y} \geq \delta_1'(s\log|y|^{-1})^{-1}|\xi|^2$, where $|\xi|^2 = \xi_1^2 + \xi_2^2$. Consequently, if we take $l \in (1/k)\mathbb{Z}^2$, then

$$\begin{aligned}\|x+l\|_{\overline{g}_y} &\geq \|a+l\|_{\overline{g}_y} - \|x-a\|_{\overline{g}_y}\\ &\geq \sqrt{\frac{\delta_1'}{s\log|y|^{-1}}}|a+l| - \sqrt{\frac{C_2\log|y|^{-1}}{s}}|y|\\ &= \frac{\sqrt{\delta_1'}|a+l| - \sqrt{C_2}|y|\log|y|^{-1}}{\sqrt{s\log|y|^{-1}}}.\end{aligned}$$

Now,

$$\delta_2' := \inf_{l\in(1/k)\mathbb{Z}^2,\, l\neq -a} |a+l|$$

is a positive number depending only on the critical value $b_{\mathbf{q}} \in \mathrm{Crt}$. Since $|y|\log|y|^{-1} \to 0$ as $y \to 0$, we can take δ_0 sufficiently small such that we have (20)(21)(22) and

$$\sqrt{\delta_1'}|a+l| - \sqrt{C_2}|y|\log|y|^{-1} \geq \frac{\sqrt{\delta_1'}\delta_2'}{2}$$

for every $y \in D(\delta_0)$. □

LEMMA 8.3. *There are constants $\delta_0, \delta_2 > 0$ such that δ_0 satisfies* (20)(21)(22) *and the following holds. Let $R > 0$ and take $s_R > 0$ such that $\chi(s_R R^2/4) \leq \delta_0$. For any $0 < s \leq s_R$ and $y \in D(\delta_0)\setminus D(\chi(sR^2/4))$, we have*

$$\inf_{l\in(1/k)\mathbb{Z}^2} \|x(y)+l\|^2_{\overline{g}_y} \geq \delta_2 R^2.$$

Proof. First of all, we give the lower bound of $\|x(y)-a\|^2_{\overline{g}_y}$ where $a \in (1/k)\mathbb{Z}^2$. Note that there is a constant $C_1 > 0$ such that

$$\left|\mathcal{H}(y) - \frac{u_2\log|y|^{-1}}{2\pi}\right| \leq C_1|y|$$

for $y \in D(\delta_0)$. If $2|u_1| \geq |u_2|$, then we have

$$\|x(y)\|^2_{\overline{g}_y} \geq \frac{u_1^2\log|y|^{-1}}{Cs} \geq \frac{(u_1^2/5 + 4u_1^2/5)\log|y|^{-1}}{Cs} \geq \frac{|y|^2\log|y|^{-1}}{5Cs}.$$

Since we have $|\mathcal{H}(y)| \geq |u_2|\log|y|^{-1}/2\pi - C_1|y|$ for a constant $C_1 > 0$, if we assume $2|u_2| \geq |u_1|$ then $|\mathcal{H}(y)| \geq |u_2|\log|y|^{-1}/2\pi - C_2|u_2|$ for a constant $C_2 > 0$. By taking δ_0 sufficiently small, we may suppose $C_2 \leq (\log|y|^{-1})/2$ for $y \in D(\delta_0)$. Then we have

$$\|x(y)\|^2_{\overline{g}_y} \geq \frac{u_1^2\log|y|^{-1}}{Cs} + \frac{u_2^2\log|y|^{-1}}{4Cs} \geq \frac{|y|^2\log|y|^{-1}}{4Cs}.$$

In both cases, we have

$$\|x(y)\|^2_{\overline{g}_y} \geq \frac{|y|^2\log|y|^{-1}}{5Cs}.$$

Since $y \notin D(\chi(sR^2/4))$ iff $|y|^2 \log |y|^{-1}/(2\pi) \geq sR^2/4$, hence we have

$$\|x(y)\|_{\bar{g}_y}^2 \geq \frac{|y|^2 \log |y|^{-1}}{5Cs} \geq \frac{\pi}{10C} R^2.$$

Combining with Lemma 8.2, we have

$$\inf_{l \in (1/k)\mathbb{Z}^2} \|x(y) + l\|_{\bar{g}_y}^2 \geq \inf \left\{ \frac{\delta_1}{s \log |y|^{-1}}, \frac{\pi}{10C} R^2 \right\}.$$

Since $|y| \geq \chi(sR^2/4)$, we have

$$\frac{\delta_1}{s \log |y|^{-1}} \geq \frac{\delta_1}{s \log \chi^{-1}} = \frac{\delta_1 R^2/4}{(sR^2/4) \log \chi^{-1}} = \frac{\delta_1 R^2}{4\chi^2 (\log \chi^{-1})^2}.$$

By the assumption $\chi(sR^2/4) \leq \delta_0$, we can see

$$\frac{\delta_1}{s \log |y|^{-1}} \geq \frac{\delta_1 R^2}{4\delta_0^2 (\log \delta_0^{-1})^2},$$

hence we have the result. □

Now, let $\mathbb{S} := \mathbb{S}(L|_{X_{\mathrm{OV}}}, h)$ and denote by $\pi \colon \mathbb{S} \to X_{\mathrm{OV}}$ the projection. By Proposition 4.33, we have the followings.

Lemma 8.4. *Let $p \in \pi^{-1}(0_{\mathrm{OV}})$. For every $R \geq 7$ there is $s_R > 0$ such that we have*

$$(\mu_{\mathrm{OV}} \circ \pi)^{-1} \left(D(\chi(sR^2/4)) \right) \subset B_{\hat{g}_s}(p, R)$$

for any $0 < s \leq s_R$.

Proof. Take $s_R > 0$ as in Proposition 4.33. By (30), we have $y \in D(\chi(sr^2))$ iff $|\zeta_s(y)| < r$. By Proposition 4.33, we have

$$(\mu_{\mathrm{OV}} \circ \pi)^{-1} \left(\mathcal{B}(R/2) \right) \subset B_{\hat{g}_s}(p, R)$$

for $0 < s \leq s_R$, hence we have the result. □

By Lemmas 8.3 and 8.4, we have the next proposition.

Proposition 8.5. *For every $b_{\mathbf{q}} \in \mathrm{Crt}$ and $p_{\mathbf{q}} \in (\mu \circ \pi)^{-1}(b_{\mathbf{q}})$ there are $r_1^{\mathbf{q}} > 0$, $C_{\mathbf{q}} > 0$, $R_0 > 0$ and $s_R > 0$ for every $R \geq R_0$ such that if $R \geq R_0$ and $0 < s \leq s_R$ then*

$$\int_{\mathbb{S}|_{\mu^{-1}(W_1^{\mathbf{q}})} \setminus B_{\hat{g}'_s}(p_{\mathbf{q}}, R)} |df|_{\hat{g}'_s}^2 d\nu_{\hat{g}'_s} \geq 2\pi k^2 (1 + CR^2) \int_{\mathbb{S}|_{\mu^{-1}(W_1^{\mathbf{q}})} \setminus B_{\hat{g}'_s}(p_{\mathbf{q}}, R)} |f|^2 d\nu_{\hat{g}'_s}$$

for any $f \in (H^{1,2}(\mathbb{S}, d_{\hat{g}'_s}, \nu_{\hat{g}'_s}) \otimes \mathbb{C})^{\rho_k}$.

8.4. Proof of Theorem 4.3. Let $(X, \omega_1, \omega_2, \omega_{3,s})$, $\mu\colon X \to \mathbb{P}^1$ and (L, h, ∇) be as in Subsection 4.1. Denote by $(g_s, J_{1,s}, J_{2,s}, J_3)$ be the associated hyper-Kähler structure of $(\omega_1, \omega_2, \omega_{3,s})$ and let g'_s be as in Subsection 5.5. For every $b \in BS_k$, we fix $p^b \in (\mu \circ \pi)^{-1}(b)$. By Proposition 8.5 and (32), we have the following.

Proposition 8.6. *Let k be a positive integer. There are constants $C > 0$, $R_0 > 0$ independent of s, R, f and $s_R > 0$ for every $R \geq R_0$ such that if $R \geq R_0$ and $0 < s \leq s_R$ then*

$$\int_{\mathbb{S}\setminus \bigcup_{b\in BS_k} B_{\hat{g}'_s}(p^b,R)} |df|^2_{\hat{g}'_s} d\nu_{\hat{g}'_s} \geq CR^2 \int_{\mathbb{S}\setminus \bigcup_{b\in BS_k} B_{\hat{g}'_s}(p^b,R)} |f|^2 d\nu_{\hat{g}'_s}$$

for any $f \in (H^{1,2}(\mathbb{S}, d_{\hat{g}'_s}, \nu_{\hat{g}'_s}) \otimes \mathbb{C})^{\rho_k}$.

Moreover, by (19), we have the following corollary by taking the constant C in the above proposition smaller.

Corollary 8.7. *Let k be a positive integer. There are constants $C > 0$, $R_0 > 0$ independent of s, R, f and $s_R > 0$ for every $R \geq R_0$ such that if $R \geq R_0$ and $0 < s \leq s_R$ then*

$$\int_{\mathbb{S}\setminus (\bigcup_{b\in BS_k} B_{\hat{g}_s}(p^b,R))} |df|^2_{\hat{g}_s} d\nu_{\hat{g}_s} \geq CR^2 \int_{\mathbb{S}\setminus (\bigcup_{b\in BS_k} B_{\hat{g}_s}(p^b,R))} |f|^2 d\nu_{\hat{g}_s}$$

for any $f \in (H^{1,2}(\mathbb{S}, d_{\hat{g}_s}, \nu_{\hat{g}_s}) \otimes \mathbb{C})^{\rho_k}$.

Let

$$\mathbb{S}_s := \left(\mathbb{S}, d_{\hat{g}_s}, \frac{\nu_{\hat{g}_s}}{s}\right).$$

The next proposition was essentially shown in [14, Proposition 4.4].

Proposition 8.8. *For any $\varepsilon, A > 0$ there is $R_{\varepsilon,A} > 0$ and $s_{\varepsilon,A} > 0$ such that the following holds. For any family $f_s \in (H^{1,2}(\mathbb{S}_s) \otimes \mathbb{C})^{\rho_k}$ such that $\|f_s\|_{L^2(\mathbb{S}_s)} = 1$ and $\sup_{s>0} \|df\|_{L^2(\mathbb{S}_s)} \leq A$, we have*

$$\int_{\mathbb{S}\setminus (\bigcup_{b\in BS_k} B_{\hat{g}_s}(p^b,R_\varepsilon))} \frac{|f|^2 d\nu_{\hat{g}_s}}{s} \geq 1 - \varepsilon$$

for any $0 < s \leq s_\varepsilon$.

Proof. Put $\mathbf{B}(R) := \bigcup_{b\in BS_k} B_{\hat{g}_s}(p^b, R)$. By Corollary 8.7, there is $s_R > 0$ such that

$$\begin{aligned} 1 = \int_{\mathbb{S}} \frac{|f|^2 d\nu_{\hat{g}_s}}{s} &= \int_{\mathbf{B}(R)} \frac{|f|^2 d\nu_{\hat{g}_s}}{s} + \int_{\mathbb{S}\setminus \mathbf{B}(R)} \frac{|f|^2 d\nu_{\hat{g}_s}}{s} \\ &\leq \int_{\mathbf{B}(R)} \frac{|f|^2 d\nu_{\hat{g}_s}}{s} + \frac{A^2}{CR^2} \end{aligned}$$

for $0 < s \leq s_R$. Therefore, we have the result by putting $R_\varepsilon = A/\sqrt{C\varepsilon}$ and $s_\varepsilon = s_{R_\varepsilon}$. □

Proof of Theorem 4.3. Let $f_s \in (H^{1,2}(\mathbb{S}_s) \otimes \mathbb{C})^{\rho_k}$ such that $\sup_s(\|f\|^2_{L^2(\mathbb{S}_s)} + \mathcal{E}^{\rho_k}_s(f_s)) < \infty$. Now, since g_s are Ricci-flat, the Ricci curvatures of $\hat{g}_s$ have the uniform lower bound by [14, Proposition 3.15]. Then by Proposition 8.8, we can apply [14, Proposition 4.7] to this situation, then we obtain the strongly converging subsequence $\{f_{s_i}\}_i \subset \{f_s\}_s$. □

9. Convergence of the quantum Hilbert spaces. Let $(X, \omega_1, \omega_2, \omega_{3,s}, L, h, \nabla, \mu)$ be as in the previous section. Denote by $H_s := L^2(X, g_s, L^k, h)$ the Hilbert space consisting of L^2-sections of the complex line bundle $L^k \to X$ and let

$$P_{k,s} \colon H_s \to H^0(X_{J_{1,s}}, L^k), \quad P_{k,0} \colon H^k_{\mathbb{R}^2} \to \mathrm{Ker}(\Delta^k_{\mathbb{R}^2})$$

be the orthogonal projections. Since the Ricci curvature of g_s is zero, we may apply the argument in [14, Section 5] to our situation, hence the analogous statement with [14, Theorem 5.1] can be obtained as follows.

THEOREM 9.1. *Let k be a positive integer. We have a compact convergence*

$$P_{k,s} \to \bigoplus_{b \in BS_k} P_{k,0}$$

in the sense of Definition 3.5 *as $s \to 0$.*

By Theorem 9.1 and Kodaira Vanishing Theorem, we have

$$\dim H^0(X_{J_{1,s}}, L^k) = \# BS_k \tag{33}$$

for any $s > 0$ and $k > 0$.

Now, let $(\omega_1, \omega_2, \omega_3)$ be a hyper-Kähler structure on X, (L, h, ∇) be a prequantum line bundle on (X, ω_1) and $\mu \colon X \to \mathbb{P}^1$ be a special Lagrangian fibration coming from the elliptic fibration $X_{J_3} \to \mathbb{P}^1$ with 24 singular fibers of Kodaira type I_1. Then by Proposition 2.11, there is a family of hyper-Kähler structures $(\omega_1, \omega_2, \omega_{3,s})$ tending to a large complex structure limit and $\omega_{3,1} = \omega_3$. Therefore, we obtain Corollary 1.2 by (33).

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