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The classification of symplectic matrices and pairs

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In this paper we classify symplectic matrices and regular symplectic pairs. By applying a complementary bases theorem (Lemma 2.2), the classification is indexed by symplectic swap matrices. Both symplectic matrices and regular symplectic pairs are classified into 2^n categories. We show that the classification is minimal and each category forms an open set.

Keywords and phrases: Symplectic matrices, symplectic pairs, complementary bases theorem, minimal classification.

1. Introduction

Symplectic matrices and pairs play an important role in classical mechanics and Hamiltonian dynamical systems [1, 3]. Moreover, they appear in the linear control theory for discrete-time systems [6, 12] and the computation for Riccati type equations [4, 5, 7, 10, 11, 12, 13]. A standard symplectic matrix $\mathcal{J}_n \in \mathbb{C}^{2n\times 2n}$ is a matrix in the partitioned form

$$\mathcal{J}_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix. For convenience, we use $\mathcal{J}$ for $\mathcal{J}_n$ by dropping the subscript "n" if the order of $\mathcal{J}_n$ is clear in the context. In the following text, $2n\times 2n$ matrices are always partitioned in four $n\times n$ submatrices if not otherwise stated. That is, in the representation $\mathcal{S} = \left[\begin{array}{c|c} S_{11} & S_{12} \\ \hline S_{21} & S_{21} \end{array}\right]$, S_{11}, S_{12}, S_{21}, and S_{22} are all in $\mathbb{C}^{n\times n}$. Before state our motivation for the study of this paper, we first define symplectic matrices, symplectic pairs and terminologies as following.

Definition 1.1. A matrix $\mathcal{S} \in \mathbb{C}^{2n\times 2n}$ is symplectic if $\mathcal{S}\mathcal{J}\mathcal{S}^* = \mathcal{J}$, where $\mathcal{S}^*$ is the conjugate transpose of $\mathcal{S}$. Denote by $\mathbb{S}_n$ the multiplicative group of all $2n \times 2n$ symplectic matrices. A matrix pair $(\mathcal{A}, \mathcal{B})$ with $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{2n\times 2n}$ is called a symplectic pair if $\mathcal{A}\mathcal{J}\mathcal{A}^* = \mathcal{B}\mathcal{J}\mathcal{B}^*$. The matrix pairs (A_1, B_1) and

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$(A_2, B_2) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}$ are said to be left equivalent, denoted by

$$(A_1, B_1) \overset{\text{l.e.}}{\sim} (A_2, B_2)$$

if $A_1 = QA_2$, $B_1 = QB_2$ for some invertible matrix Q. A matrix pair (A, B) is said to be *regular* if $\det(A - \lambda B) \neq 0$ for some $\lambda \in \mathbb{C}$. $\mathbb{SP}_n$ is denoted the set of all $2n \times 2n$ regular symplectic pairs.

The Structure-Preserving Doubling Algorithms (SDAs) [4, 7, 11] are usually employed for solving the stabilizing solutions of Riccati type equations [5, 10, 12, 13], in which two types of symplectic pairs are introduced as follows: the *first standard symplectic form* (SSF-1)

$$(\mathcal{A}, \mathcal{B}) = \left(\begin{bmatrix} C & 0 \\ H & I \end{bmatrix}, \begin{bmatrix} I & G \\ 0 & C^* \end{bmatrix} \right);$$

and *the second standard symplectic form* (SSF-2)

$$(\mathcal{A}, \mathcal{B}) = \left(\begin{bmatrix} C & 0 \\ H & -I \end{bmatrix}, \begin{bmatrix} G & I \\ C^* & 0 \end{bmatrix} \right)$$

where the matrices $C, G, H \in \mathbb{C}^{n\times n}$ and G, H are Hermitian. Here we see that a symplectic pair either of SSF-1 or of SSF-2 involves a generic Hermitian matrix $\begin{bmatrix} G & C \\ C^* & H \end{bmatrix}$. It is of our interest to ask whether or not a regular symplectic pair is left equivalent to a pair of SSF-1 or SSF-2. However, this is not true. For example, the regular symplectic pair

$$(\mathcal{A}, \mathcal{B}) = \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

is of this case. To see this, if we multiply an arbitrary invertible matrix Q on the left side of both $\mathcal{A}$ and $\mathcal{B}$ and denote the resulting matrices by $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$, respectively. The third column of $\tilde{\mathcal{A}}$, the second column of $\tilde{\mathcal{B}}$ and the fourth column of $\tilde{\mathcal{B}}$ are zero vectors. Therefore, $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ is neither of SSF-1 nor of SSF-2. This example motivates us to study the classification of symplectic matrices and regular symplectic pairs. In this paper we are going to classify them with "symplectic swap matrices". Let $\mathbf{v} \in \{0, 1\}^n$, we define a *symplectic swap matrix* as an orthogonal symplectic matrix given by

$$\Pi_{\mathbf{v}} = \begin{bmatrix} \operatorname{diag}(\mathbf{v}) & \operatorname{diag}(\mathbf{v}^c) \\ -\operatorname{diag}(\mathbf{v}^c) & \operatorname{diag}(\mathbf{v}) \end{bmatrix},$$

where $\mathbf{v}^c$ is the vector with $\mathbf{v}^c(i) = 1 - \mathbf{v}(i)$ and $\mathbf{v}(i)$ is the i-th component of the vector $\mathbf{v}$. We denote by $\mathfrak{S}^{2n}$ the set of all $2n \times 2n$ symplectic swap matrices. It is easily seen that symplectic swap matrices are symplectic matrices, i.e., $\mathfrak{S}^{2n} \subseteq \mathbb{S}_n$. We also denote by $\mathbb{H}^{2n}$ the additive group of all $2n \times 2n$ Hermitian matrices.

Definition 1.2. Let Π, Π_A, and $\Pi_B \in \mathfrak{S}^{2n}$. We define the categories

$$\mathbb{S}_{\Pi} := \left\{ \mathcal{S} \in \mathbb{S}_n \;\middle|\; \mathcal{S}\Pi = \begin{bmatrix} I & X_{11} \\ 0 & X_{12}^* \end{bmatrix}^{-1} \begin{bmatrix} X_{12} & 0 \\ X_{22} & I \end{bmatrix}, \right.$$
$$\left. \text{for some } \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \in \mathbb{H}^{2n} \right\},$$

$$\mathbb{SP}_{\Pi_A,\Pi_B} := \left\{ (\mathcal{A},\mathcal{B}) \in \mathbb{SP}_n \;\middle|\; (\mathcal{A},\mathcal{B}) \overset{\text{l.e.}}{\sim} \left(\begin{bmatrix} X_{12} & 0 \\ X_{22} & I \end{bmatrix} \Pi_A, \begin{bmatrix} I & X_{11} \\ 0 & X_{12}^* \end{bmatrix} \Pi_B \right), \right.$$
$$\left. \text{for some } \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \in \mathbb{H}^{2n} \right\},$$

and $\mathbb{SP}_{\cdot,\Pi_B} = \bigcup_{\Pi \in \mathfrak{S}^{2n}} \mathbb{SP}_{\Pi,\Pi_B}$.

Therefore, SSF-1 and SSF-2 can be rewritten as $\mathbb{SP}_{I,I}$ and $\mathbb{SP}_{I,\mathcal{J}}$, respectively. Our main results in this paper show both symplectic matrices and regular symplectic pairs are classified into 2^n categories indexed by $\mathfrak{S}^{2n}$. We show that the classification is "minimal" and each category forms an open set. Here, the "minimal" classification means that if one of these 2^n categories is absent, then the other $2^n - 1$ categories can't represent all symplectic matrices. We state them as the two following theorems.

Theorem 1.1 (Classification of Symplectic Matrices)**.**

(*i*) $\mathbb{S}_n = \bigcup_{\Pi \in \mathfrak{S}^{2n}} \mathbb{S}_{\Pi}$.
(*ii*) *For each* $\Pi \in \mathfrak{S}^{2n}$, *there exists* $\mathcal{S} \in \mathbb{S}_{\Pi}$ *such that* $\mathcal{S} \notin \mathbb{S}_{\Pi'}$ *for all* $\Pi' \neq \Pi$.
(*iii*) *For each* $\Pi \in \mathfrak{S}^{2n}$, $\mathbb{S}_{\Pi}$ *is an open set relative to* $\mathbb{S}_n$.

Theorem 1.2 (Classification of Regular Symplectic Pairs)**.**

(*i*) *For each regular symplectic pair* $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_n$, *there exist symplectic swap matrices* $\Pi_A, \Pi_B \in \mathfrak{S}^{2n}$ *such that* $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_{\Pi_A,\Pi_B}$. *Consequently,* $\mathbb{SP}_n = \bigcup_{\Pi_B \in \mathfrak{S}^{2n}} \mathbb{SP}_{\cdot,\Pi_B}$.

(*ii*) *For each* $\Pi_B \in \mathfrak{S}^{2n}$, *there exists* $(\mathcal{A}, \mathcal{B}) \in \mathbb{SP}_n$ *such that* $(\mathcal{A}, \mathcal{B}) \notin \mathbb{SP}_{\cdot,\Pi'_B}$ *for all* $\Pi'_B \neq \Pi_B$.

(*iii*) *For each* $\Pi_A, \Pi_B \in \mathfrak{S}^{2n}$, $\mathbb{SP}_{\Pi_A,\Pi_B}$ *is an open set relative to* $\mathbb{SP}_n$. *Consequently,* $\mathbb{SP}_{\cdot,\Pi_B}$ *is open.*

This paper is organized as follows. In section 2 we introduce notation, preliminary properties, and the complementary bases theorem. In section 3, based on the complementary bases theorem, we prove Theorem 1.1. In section 4, we prove Theorem 1.2 and give some more detailed classification for $\mathbb{SP}_n$. Finally, conclusions and problems are presented in section 5.

2. Preliminaries

In this section, we denote some notation and give some preliminary results. Throughout this paper, we use the bold face letter to denote the vector. We denote by $\mathbf{e}_k$ the k-th column of the identity matrix and by $\mathbf{e}$ the vector whose elements are all ones. Denote $[n] = \{1, 2, \cdots, n\}$. For a matrix $A \in \mathbb{C}^{n\times n}$, A^* is the conjugate transpose A. For $\alpha \subseteq [n]$, $A(:, \alpha)$ (or $A(\alpha, :)$) denotes the matrix with columns (or rows) of A indexed by α. The notation $|\alpha|$ denotes the cardinality of a set α and $\alpha^c \subseteq [n]$ represents the subset of all indices that do not belong to α. For a vector $\mathbf{v} \in \mathbb{C}^n$, $\mathbf{v}(i)$ denotes the ith component of the vector $\mathbf{v}$. Two subspaces $\mathbb{U}$ and $\mathbb{V}$ of $\mathbb{C}^{2n}$ are called $\mathcal{J}$-orthogonal if $\mathbf{u}^*\mathcal{J}\mathbf{v} = 0$ for each $\mathbf{u} \in \mathbb{U}$ and $\mathbf{v} \in \mathbb{V}$. A subspace $\mathbb{U}$ of $\mathbb{C}^{2n}$ is called isotropic if $\mathbf{x}^*\mathcal{J}\mathbf{y} = 0$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{U}$. An n-dimensional isotropic subspace is called a Lagrangian subspace.

Proposition 2.1. *If* $\mathcal{S}$ *is symplectic, then so are* $\mathcal{S}^*$ *and* $\mathcal{S}^{-1}$.

Proof. To show that $\mathcal{S}^*$ and $\mathcal{S}^{-1}$ are symplectic, we simply need to verify if these two matrices satisfy the definition given in Definition 1.1. First note that

$$\begin{aligned}
\mathcal{J} = \mathcal{S}\mathcal{J}\mathcal{S}^* &\iff \mathcal{S}^{-1}\mathcal{J}\mathcal{S}^{*-1} = \mathcal{J} \\
&\iff \mathcal{S}^{-1}(-\mathcal{J})\mathcal{S}^{*-1} = -\mathcal{J} \\
&\iff \mathcal{S}^{-1}\mathcal{J}^{-1}\mathcal{S}^{*-1} = \mathcal{J}^{-1} \\
&\iff (\mathcal{S}^*\mathcal{J}\mathcal{S})^{-1} = \mathcal{J}^{-1} \\
&\iff \mathcal{S}^*\mathcal{J}\mathcal{S} = \mathcal{J}.
\end{aligned}$$

The equivalence above shows that $\mathcal{S}^*$ is symplectic. Similarly, we also have

$$\mathcal{S}^*\mathcal{J}\mathcal{S} = \mathcal{J} \iff \mathcal{S}^*(-\mathcal{J})\mathcal{S} = -\mathcal{J}$$

$$\begin{aligned} &\iff \mathcal{S}^*\mathcal{J}^{-1}\mathcal{S} = \mathcal{J}^{-1} \\ &\iff (\mathcal{S}^{-1}\mathcal{J}\mathcal{S}^{*-1})^{-1} = \mathcal{J}^{-1} \\ &\iff \mathcal{S}^{-1}\mathcal{J}(\mathcal{S}^{-1})^* = \mathcal{J}. \end{aligned}$$

It follows that $\mathcal{S}$ is symplectic if and only if $\mathcal{S}^*$ or $\mathcal{S}^{-1}$ is symplectic. □

Let $\mathcal{S} = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \in \mathbb{C}^{2n\times 2n}$ be symplectic, where $S_1, S_2 \in \mathbb{C}^{n\times 2n}$. It is easy to see that the row spaces of S_1 and of S_2 are Lagrangian subspaces, i.e., $S_i\mathcal{J}S_i^* = 0$ and $\operatorname{rank}(S_i) = n$ for $i = 1, 2$. The following lemma gives a useful property for a matrix whose row space is isotropic which improves Theorem 3.1 in [2].

Lemma 2.2 (Complementary Bases Theorem)**.** *Let X, $Y \in \mathbb{C}^{\ell\times n}$ and rank($[X, Y]$) $= \ell$. Suppose that $\alpha \subseteq [n]$ with $|\alpha| = \operatorname{rank}(Y)$ such that $\operatorname{rank}(Y(:, \alpha)) = \operatorname{rank}(Y)$. If the row space of $[X, Y]$ is isotropic, then $\ell \leqslant n$ and rank($[Y(:, \alpha), X(:, \alpha^c)]$) $= \ell$.*

Proof. Suppose that $\operatorname{rank}(Y) = k \leqslant \ell$. There exist a permutation matrix $P \in \mathbb{R}^{n\times n}$ and an invertible matrix $Q \in \mathbb{C}^{\ell\times\ell}$ such that $QYP = \begin{bmatrix} I_k & Z \\ 0 & 0 \end{bmatrix}$. Then there is $\alpha \subseteq [n]$ with $|\alpha| = \operatorname{rank}(Y)$ such that $QY(:, \alpha) = \begin{bmatrix} I_k \\ 0 \end{bmatrix}$. Denote $QXP = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$, we have $[Y(:, \alpha), X(:, \alpha^c)] = \begin{bmatrix} I_k & X_{12} \\ 0 & X_{22} \end{bmatrix}$. To complete the proof, it suffices to show that $\operatorname{rank}(X_{22}) = \ell - k$. Since the row space of $[X, Y]$ is isotropic, we have

$$\begin{aligned} 0 = [X, Y]\mathcal{J}\begin{bmatrix} X^* \\ Y^* \end{bmatrix} &= Q[X, Y]\begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}\mathcal{J}\begin{bmatrix} P^* & 0 \\ 0 & P^* \end{bmatrix}\begin{bmatrix} X^* \\ Y^* \end{bmatrix}Q^* \\ &= \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}\begin{bmatrix} I_k & 0 \\ Z^* & 0 \end{bmatrix} - \begin{bmatrix} I_k & Z \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_{11}^* & X_{21}^* \\ X_{12}^* & X_{22}^* \end{bmatrix} \\ &= \begin{bmatrix} X_{11} + X_{12}Z^* - X_{11}^* - ZX_{12}^* & -X_{21}^* - ZX_{22}^* \\ X_{21} + X_{22}Z^* & 0 \end{bmatrix}. \end{aligned}$$

Hence, we obtain $X_{21} = -X_{22}Z^*$. This implies that

$$Q[X, Y]\begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} = \left[\begin{array}{cc|cc} X_{11} & X_{12} & I_k & Z \\ -X_{22}Z^* & X_{22} & 0 & 0 \end{array}\right].$$

Since $\operatorname{rank}([X, Y]) = \ell$, we then have $\operatorname{rank}(X_{22}) = \ell - k$ and hance $\operatorname{rank}([Y(:, \alpha), X(:, \alpha^c)]) = \ell$. □

3. The classification of symplectic matrices

In this section we classify the symplectic matrices with symplectic swap matrices. First we show that for each $\mathcal{S} \in \mathbb{S}_n$, there exists a symplectic swap matrix $\Pi \in \mathfrak{S}^{2n}$ such that $\mathcal{S}\Pi = \mathcal{B}^{-1}\mathcal{A}$ where $\mathcal{B} \in \mathbb{C}^{2n\times 2n}$ is an upper triangle matrix and $\mathcal{A} \in \mathbb{C}^{2n\times 2n}$ is a lower triangle matrix. Based on this we classify the symplectic matrices. That is, $\mathbb{S}_n = \bigcup_{\Pi\in\mathfrak{S}^{2n}} \mathbb{S}_\Pi$. According to the definition of symplectic swap matrix, we have 2^n classes of symplectic matrices. We furthermore prove that this classification is minimal and that each $\mathbb{S}_\Pi$ is an open set in $\mathbb{S}_n$ in the end.

Lemma 3.1. *Let* $\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \in \mathbb{S}_n$. *Then* S_{22} *is nonsingular if and only if there exists* $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \in \mathbb{H}^{2n}$ *with* X_{12} *being nonsingular such that*

$$\mathcal{S} = \begin{bmatrix} I & X_{11} \\ 0 & X_{12}^* \end{bmatrix}^{-1} \begin{bmatrix} X_{12} & 0 \\ X_{22} & I \end{bmatrix}.$$

Proof. Suppose that $\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \in \mathbb{S}_n$ with S_{22} being nonsingular. Then

$$\begin{bmatrix} I & -S_{12}S_{22}^{-1} \\ 0 & S_{22}^{-1} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} S_{11} - S_{12}S_{22}^{-1}S_{21} & 0 \\ S_{22}^{-1}S_{21} & I \end{bmatrix}.$$

To prove this lemma, it suffices to show that the matrix

$$\begin{bmatrix} -S_{12}S_{22}^{-1} & S_{11} - S_{12}S_{22}^{-1}S_{21} \\ S_{22}^{-1} & S_{22}^{-1}S_{21} \end{bmatrix}$$

is Hermitian. Since $\mathcal{S}$ is symplectic, we have

(3.1)
$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = \mathcal{J} = \mathcal{S}\mathcal{J}\mathcal{S}^* = \begin{bmatrix} -S_{12}S_{11}^* + S_{11}S_{12}^* & -S_{12}S_{21}^* + S_{11}S_{22}^* \\ -S_{22}S_{11}^* + S_{21}S_{12}^* & -S_{22}S_{21}^* + S_{21}S_{22}^* \end{bmatrix}.$$

Noticing the (2,2)-block of (3.1), it turns out that $S_{21}S_{22}^* = S_{22}S_{21}^*$. Since S_{22} is nonsingular, we have

$$S_{22}^{-1}S_{21} = S_{21}^*S_{22}^{*-1} = \left(S_{22}^{-1}S_{21}\right)^*. \tag{3.2}$$

It follows from Proposition 2.1 that $\mathcal{S}^*$ is symplectic. Hence, we have

(3.3)
$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = \mathcal{J} = \mathcal{S}^* \mathcal{J} \mathcal{S} = \begin{bmatrix} -S_{21}^* S_{11} + S_{11}^* S_{21} & -S_{21}^* S_{12} + S_{11}^* S_{22} \\ -S_{22}^* S_{11} + S_{12}^* S_{21} & -S_{22}^* S_{12} + S_{12}^* S_{22} \end{bmatrix}.$$

From (2,2)-block of (3.3), we obtain

$$S_{12} S_{22}^{-1} = S_{22}^{*-1} S_{12}^* = \left(S_{12} S_{22}^{-1}\right)^*. \tag{3.4}$$

Moreover, from the (1,2)-block of (3.3) we obtain that $-S_{21}^* S_{12} S_{22}^{-1} + S_{11}^* = S_{22}^{-1}$ by multiplying S_{22}^{-1} from the right hand side. Combining this with (3.4) we obtain

(3.5)
$$(S_{22}^{-1})^* = (S_{11}^* - S_{21}^* S_{12} S_{22}^{-1})^* = S_{11} - (S_{12} S_{22}^{-1})^* S_{21} = S_{11} - S_{12} S_{22}^{-1} S_{21}.$$

From (3.2), (3.4), and (3.5), we conclude that

$$\begin{bmatrix} -S_{12} S_{22}^{-1} & S_{11} - S_{12} S_{22}^{-1} S_{21} \\ S_{22}^{-1} & S_{22}^{-1} S_{21} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}$$

is Hermitian and $X_{12} = S_{22}^{-1*}$ is nonsingular.

The converse statement of this lemma is straightforward. This completes the proof. □

From Lemma 3.1, we see that $\mathbb{S}_\Pi$ in Definition 1.2 can be rewritten as

$$\mathbb{S}_\Pi = \{\mathcal{S} \in \mathbb{S}_n | \ \mathcal{S}' \equiv \mathcal{S}\Pi = [S'_{ij}], \ S'_{22} \text{ is nonsingular } \}.$$

Lemma 3.2. *If $\mathcal{S} \in \mathbb{S}_n$, then there exists $\Pi \in \mathfrak{S}^{2n}$ such that $\mathcal{S}\Pi = \begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix}$ and S'_{22} is nonsingular. That is, $\mathcal{S} \in \mathbb{S}_\Pi$.*

Proof. Partition $\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \in \mathbb{S}_n$. Suppose $\operatorname{rank}(S_{22}(:,\alpha)) = \operatorname{rank}(S_{22})$, where $\alpha \subset [n]$. By Lemma 2.2, the matrix $[S_{22}(:,\alpha), S_{12}(:,\alpha^c)] \in \mathbb{C}^{n\times n}$ is nonsingular. Now choose $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{v}(j) = 1$ for $j \in \alpha$ and $\mathbf{v}(j) = 0$ for $j \in \alpha^c$. Let $\Pi = \begin{bmatrix} \operatorname{diag}(\mathbf{v}) & \operatorname{diag}(\mathbf{v}^c) \\ -\operatorname{diag}(\mathbf{v}^c) & \operatorname{diag}(\mathbf{v}) \end{bmatrix} \in \mathfrak{S}^{2n}$, where $\mathbf{v}^c = \mathbf{e} - \mathbf{v}$. Denote $\mathcal{S}\Pi = \begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix}$. Since $[S_{22}(:,\alpha), S_{12}(:,\alpha^c)] \in \mathbb{C}^{n\times n}$ is invertible, we obtain that S'_{22} is nonsingular. □

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We first prove assertion (i). It suffices to show that $\mathbb{S}_n \subset \bigcup_{\Pi\in\mathfrak{S}^{2n}} \mathbb{S}_\Pi$. Let $\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \in \mathbb{S}_n$. By Lemma 3.2, there exists a symplectic swap matrix $\Pi \in \mathfrak{S}^{2n}$ such that $\mathcal{S}\Pi = \begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix}$ and S'_{22} is nonsingular. Then by Lemma 3.1, there exists $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \in \mathbb{H}^{2n}$ such that $\mathcal{S}\Pi = \begin{bmatrix} I & X_{11} \\ 0 & X_{12}^* \end{bmatrix}^{-1} \begin{bmatrix} X_{12} & 0 \\ X_{22} & I \end{bmatrix}$. Assertion (i) follows.

For assertion (ii), let $\Pi \in \mathfrak{S}^{2n}$. Then there is a vector $\mathbf{v} \in \{0,1\}^n$ such that

$$\Pi \equiv \Pi_{\mathbf{v}} = \begin{bmatrix} \mathrm{diag}(\mathbf{v}) & \mathrm{diag}(\mathbf{v}^c) \\ -\mathrm{diag}(\mathbf{v}^c) & \mathrm{diag}(\mathbf{v}) \end{bmatrix},$$

where $\mathbf{v}^c = \mathbf{e} - \mathbf{v}$. Since $\Pi_{\mathbf{v}}\Pi = \begin{bmatrix} \mathrm{diag}(\mathbf{v}) - \mathrm{diag}(\mathbf{v}^c) & 0 \\ 0 & \mathrm{diag}(\mathbf{v}) - \mathrm{diag}(\mathbf{v}^c) \end{bmatrix}$ whose (2,2)-block $\mathrm{diag}(\mathbf{v}) - \mathrm{diag}(\mathbf{v}^c)$ is nonsingular, we see that $\Pi \in \mathbb{S}_\Pi$. On the other hand, let $\Pi' \in \mathfrak{S}^{2n}$ with $\Pi' \neq \Pi$. Then there exists $\mathbf{v}' \in \{0,1\}^n$ such that $\mathbf{v}' \neq \mathbf{v}$ and $\Pi' = \Pi_{\mathbf{v}'}$. Note that $\Pi_{\tilde{\mathbf{v}}} \equiv \Pi\Pi' = \Pi_{\mathbf{v}}\Pi_{\mathbf{v}'} \in \mathfrak{S}^{2n}$ is also a symplectic swap matrix. Since $\mathbf{v}' \neq \mathbf{v}$, we have the (2,2)-block of $\Pi_{\tilde{\mathbf{v}}}$ is singular. Hence, $\Pi \notin \mathbb{S}_{\Pi'}$ by Lemma 3.1.

Now, we prove assertion (iii). For any given $\mathcal{S} \in \mathbb{S}_\Pi$, we have S'_{22} is invertible, where $\mathcal{S}\Pi = \begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix}$. Using the fact that S'_{22} is invertible, there exists $\delta > 0$ such that $\widetilde{S}_{22} \in \mathbb{C}^{n\times n}$ is invertible for each $\widetilde{S}_{22}$ satisfing $\|\widetilde{S}_{22} - S'_{22}\|_F < \delta$. Then for each $\widetilde{\mathcal{S}} \in \mathbb{S}_n$ with $\|\widetilde{\mathcal{S}} - \mathcal{S}\|_F < \delta$, we obtain that $\|\widetilde{S}_{22} - S'_{22}\|_F \leqslant \|\widetilde{\mathcal{S}}\Pi - \mathcal{S}\Pi\|_F = \|\widetilde{\mathcal{S}} - \mathcal{S}\|_F < \delta$, where $\widetilde{\mathcal{S}}\Pi = \begin{bmatrix} \widetilde{S}_{11} & \widetilde{S}_{12} \\ \widetilde{S}_{21} & \widetilde{S}_{22} \end{bmatrix}$. Then $\widetilde{S}_{22}$ is invertible and hence $\widetilde{\mathcal{S}} \in \mathbb{S}_\Pi$. Therefore $\mathbb{S}_\Pi$ is open. □

Remark 3.3. *From Theorem 1.1 (ii), we have proved that for each class of $\mathbb{S}_\Pi$, Π is the very element which lies only in $\mathbb{S}_\Pi$ and does not lie in any other class. Hence this classification is a minimal classification of $\mathbb{S}_n$. Since there are 2^n distinct vectors in $\{0,1\}^n$, there are 2^n corresponding symplectic swap matrices. So $\mathbb{S}_n$ is classified into 2^n categories.*

4. The classification of symplectic pairs

Now we are going to classify regular symplectic pairs. For succinct statements, we first give the following theorems.

Theorem 4.1. *Suppose that* $(\mathcal{A}, \mathcal{B})$ *is a regular symplectic pair, i.e.,* $(\mathcal{A}, \mathcal{B}) \in \mathbb{SP}_n$. *If nullity*$(\mathcal{B}) = \ell$, *then*

(*i*) $\operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{B}) = 2n - \ell$ *and* $\ell \leqslant n$;
(*ii*) $\operatorname{rank}(\mathcal{B}\mathcal{J}\mathcal{B}^*) = 2n - 2\ell$.

Proof. Suppose that $\operatorname{rank}(\mathcal{B}) = 2n - \ell$. There exists an invertible matrix Q such that

$$\widehat{\mathcal{B}} \equiv Q\mathcal{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \widehat{\mathcal{A}} \equiv Q\mathcal{A} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \tag{4.1}$$

where $B_1 \in \mathbb{C}^{(2n-\ell)\times 2n}$ is of full row rank $A_1 \in \mathbb{C}^{(2n-\ell)\times 2n}$ and $A_2 \in \mathbb{C}^{\ell\times 2n}$.

Now we first prove assertion (*i*). Since $(\mathcal{A}, \mathcal{B})$ is a regular symplectic pair, so is $(\widehat{\mathcal{A}}, \widehat{\mathcal{B}})$. Hence, $\operatorname{rank}(A_2) = \ell$. Using the fact that $A_2\mathcal{J}\widehat{\mathcal{A}}^* = 0\mathcal{J}\widehat{\mathcal{B}}^* = 0$, we have $\widehat{\mathcal{A}}(\mathcal{J}^* A_2^*) = 0$, and then, $\operatorname{nullity}(\widehat{\mathcal{A}}) \geqslant \ell$. Thus,

$$\operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\widehat{\mathcal{A}}) = 2n - \operatorname{nullity}(\widehat{\mathcal{A}}) \leqslant 2n - \ell = \operatorname{rank}(\mathcal{B}).$$

Since the roles of $\mathcal{A}$ and $\mathcal{B}$ in the argument can be reserved, we conclude that $\operatorname{rank}(\mathcal{B}) \leqslant \operatorname{rank}(\mathcal{A})$. Hence, $\operatorname{rank}(\mathcal{A}) = 2n - \ell$. Since $(\mathcal{A}, \mathcal{B})$ is regular, $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{2n\times 2n}$ and $\operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{B}) = 2n - \ell$, we have $\ell \leqslant n$.

(*ii*). Since $\operatorname{rank}(\mathcal{B}) = 2n - \ell$ and $\mathcal{J} \in \mathbb{R}^{2n\times 2n}$ is invertible, we have

$$2n - 2\ell = 2n - 2\operatorname{nullity}(\mathcal{B}) \leqslant \operatorname{rank}(\mathcal{B}\mathcal{J}\mathcal{B}^*) \leqslant \operatorname{rank}(\mathcal{B}) = 2n - \ell.$$

Now, we show that $\operatorname{rank}(\mathcal{B}\mathcal{J}\mathcal{B}^*) = 2n - 2\ell$. Suppose that $\operatorname{rank}(\mathcal{B}\mathcal{J}\mathcal{B}^*) > 2n - 2\ell$. From (4.1), we have

$$\operatorname{nullity}(\widehat{\mathcal{A}}\mathcal{J}\widehat{\mathcal{A}}^*) = \operatorname{nullity}(\widehat{\mathcal{B}}\mathcal{J}\widehat{\mathcal{B}}^*) = 2n - \operatorname{rank}(\mathcal{B}\mathcal{J}\mathcal{B}^*) < 2\ell. \tag{4.2}$$

From (*i*), we know that $\operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\widehat{\mathcal{A}}) = 2n - \ell$. Suppose that the row vectors of $Z \in \mathbb{C}^{\ell\times 2n}$ form a basis of left null space of $\widehat{\mathcal{A}}$, i.e., $Z\widehat{\mathcal{A}} = 0$ and $\operatorname{rank}(Z) = \ell$. From (4.1) and using the fact that $(\widehat{\mathcal{A}}, \widehat{\mathcal{B}})$ is symplectic, we obtain that

$$\left[\frac{Z}{0, I_\ell}\right] \widehat{\mathcal{A}}\mathcal{J}\widehat{\mathcal{A}}^* = \left[\frac{0}{A_2\mathcal{J}\widehat{\mathcal{A}}^*}\right] = \left[\frac{0}{0\mathcal{J}\widehat{\mathcal{B}}^*}\right] = 0$$

It follows from (4.2) that the row vectors of the matrix $\left[\frac{Z}{0, I_\ell}\right] \in \mathbb{C}^{2\ell\times 2n}$ are linearly dependent. Since $\operatorname{rank}(Z) = \operatorname{rank}([0, I_\ell]) = \ell$, there are nonzero vectors $\mathbf{b}, \mathbf{w} \in \mathbb{C}^\ell$ such that $[0, \mathbf{b}^*] = \mathbf{w}^* Z$. Then $[0, \mathbf{b}^*]\widehat{\mathcal{B}} = 0$ and $[0, \mathbf{b}^*]\widehat{\mathcal{A}} = \mathbf{w}^* Z\widehat{\mathcal{A}} = 0$. This is a contradiction because $(\widehat{\mathcal{A}}, \widehat{\mathcal{B}})$ is regular. □

Lemma 4.2. *Suppose that* $(\mathcal{A}, \mathcal{B}) = \left(\begin{bmatrix} X_{12} & 0 \\ X_{22} & I_n \end{bmatrix} \Pi_A, \begin{bmatrix} I_n & X_{11} \\ 0 & X_{21} \end{bmatrix} \Pi_B \right) \in \mathbb{SP}_n$*, where* $\Pi_A, \Pi_B \in \mathfrak{S}^{2n}$*. Then we have* $X_{11} = X_{11}^*$*,* $X_{22} = X_{22}^*$ *and* $X_{12} = X_{21}^*$*.*

Proof. Since $(\mathcal{A}, \mathcal{B})$ is a symplectic pair and $\Pi_A \mathcal{J} \Pi_A^* = \Pi_B \mathcal{J} \Pi_B^* = \mathcal{J}$, we have

$$\begin{bmatrix} 0 & X_{12} \\ -X_{21}^* & X_{22} - X_{22}^* \end{bmatrix} = \begin{bmatrix} -X_{11} + X_{11}^* & X_{21}^* \\ -X_{21} & 0 \end{bmatrix}.$$

Hence, we obtain that $X_{11} = X_{11}^*$, $X_{22} = X_{22}^*$ and $X_{12} = X_{21}^*$. □

Let $\Pi_A \in \mathfrak{S}^{2n}$ be a given symplectic swap matrix. For each $(\mathcal{A}, \mathcal{B}) \in \mathbb{SP}_n$, the following theorem gives a necessary and sufficient condition for $(\mathcal{A}, \mathcal{B}) \in \mathbb{SP}_{\Pi_A, I}$.

Theorem 4.3. *Let* $(\mathcal{A}, \mathcal{B}) = \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \right) \in \mathbb{SP}_n$*. Then* $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$ *is of full column rank if and only if there exists* $\Pi_A \in \mathfrak{S}^{2n}$ *such that* $(\mathcal{A}, \mathcal{B}) \in \mathbb{SP}_{\Pi_A, I}$*.*

Proof. We first prove the sufficiency. Suppose that $(\mathcal{A}, \mathcal{B}) \in \mathbb{SP}_{\Pi_A, I}$ for some $\Pi_A \in \mathfrak{S}^{2n}$. Then there is an invertible matrix Q such that $Q\mathcal{B} = \begin{bmatrix} I_n & X_{11} \\ 0 & X_{21} \end{bmatrix}$. Hence, $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = Q^{-1} \begin{bmatrix} I_n \\ 0 \end{bmatrix}$ is of full column rank.

Now, we prove the necessity. Suppose that $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$ is of full column rank, there is an invertible matrix Q_1 such that $Q_1 \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} I_n \\ 0 \end{bmatrix}$. Denote

$$\widehat{\mathcal{A}} \equiv Q_1 \mathcal{A} = \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{21} & \widehat{A}_{22} \end{bmatrix}, \ \widehat{\mathcal{B}} \equiv Q_1 \mathcal{B} = \begin{bmatrix} I_n & \widehat{B}_{12} \\ 0 & \widehat{B}_{22} \end{bmatrix}.$$

Then $(\mathcal{A}, \mathcal{B}) \overset{\text{l.e.}}{\sim} (\widehat{\mathcal{A}}, \widehat{\mathcal{B}})$ and $(\widehat{\mathcal{A}}, \widehat{\mathcal{B}}) \in \mathbb{SP}_n$. Using the fact that $\widehat{\mathcal{A}} \mathcal{J} \widehat{\mathcal{A}}^* = \widehat{\mathcal{B}} \mathcal{J} \widehat{\mathcal{B}}^*$, we have

(4.3)
$$\begin{bmatrix} -\widehat{A}_{12}\widehat{A}_{11}^* + \widehat{A}_{11}\widehat{A}_{12}^* & -\widehat{A}_{12}\widehat{A}_{21}^* + \widehat{A}_{11}\widehat{A}_{22}^* \\ -\widehat{A}_{22}\widehat{A}_{11}^* + \widehat{A}_{21}\widehat{A}_{12}^* & -\widehat{A}_{22}\widehat{A}_{21}^* + \widehat{A}_{21}\widehat{A}_{22}^* \end{bmatrix} = \begin{bmatrix} -\widehat{B}_{12} + \widehat{B}_{12}^* & \widehat{B}_{22}^* \\ -\widehat{B}_{22} & 0 \end{bmatrix}.$$

Now, we show that $\operatorname{rank}([\widehat{A}_{21}, \widehat{A}_{22}]) = n$. We prove this by a contradiction argument. Suppose that there exists a nonzero vector $\mathbf{u} \in \mathbb{C}^n$ such that $\mathbf{u}^*[\widehat{A}_{21}, \widehat{A}_{22}] = 0$. It follows from (4.3) that

$$\mathbf{u}^*\widehat{B}_{22} = \mathbf{u}^*(\widehat{A}_{22}\widehat{A}_{11}^* - \widehat{A}_{21}\widehat{A}_{12}^*) = 0.$$

Then we have $[0, \mathbf{u}^*]\widehat{\mathcal{A}} = [0, \mathbf{u}^*]\widehat{\mathcal{B}} = 0$. This is a contradiction because $(\widehat{\mathcal{A}}, \widehat{\mathcal{B}})$ is a regular pair.

From (4.3), we obtain $[\widehat{A}_{21}, \widehat{A}_{22}]\mathcal{J}\begin{bmatrix} \widehat{A}_{21}^* \\ \widehat{A}_{22}^* \end{bmatrix} = 0$. Applying Lemma 2.2 by setting $X = \widehat{A}_{21}$ and $Y = \widehat{A}_{22}$, there exists a symplectic swap matrix $\Pi_A \in \mathfrak{S}^{2n}$ such that $\widetilde{A}_{22}$ is invertible where $\widetilde{\mathcal{A}} \equiv \widehat{\mathcal{A}}\Pi_A^{-1} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}$. Then

$$\begin{aligned}
(\mathcal{A}, \mathcal{B}) \overset{\text{l.e.}}{\sim} (\widehat{\mathcal{A}}, \widehat{\mathcal{B}}) &= \left(\begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix} \Pi_A, \begin{bmatrix} I_n & \widehat{B}_{12} \\ 0 & \widehat{B}_{22} \end{bmatrix} \right) \\
&\overset{\text{l.e.}}{\sim} \left(\begin{bmatrix} \widetilde{A}_{11} - \widetilde{A}_{12}\widetilde{A}_{22}^{-1}\widetilde{A}_{21} & 0 \\ \widetilde{A}_{22}^{-1}\widetilde{A}_{21} & I_n \end{bmatrix} \Pi_A, \begin{bmatrix} I_n & \widehat{B}_{12} - \widetilde{A}_{12}\widetilde{A}_{22}^{-1}\widehat{B}_{22} \\ 0 & \widetilde{A}_{22}^{-1}\widehat{B}_{22} \end{bmatrix} \right) \\
&\equiv \left(\begin{bmatrix} \widetilde{X}_{12} & 0 \\ \widetilde{X}_{22} & I_n \end{bmatrix} \Pi_A, \begin{bmatrix} I_n & \widetilde{X}_{11} \\ 0 & \widetilde{X}_{21} \end{bmatrix} \right).
\end{aligned}$$

Using the fact that $\left(\begin{bmatrix} \widetilde{X}_{12} & 0 \\ \widetilde{X}_{22} & I_n \end{bmatrix} \Pi_A, \begin{bmatrix} I_n & \widetilde{X}_{11} \\ 0 & \widetilde{X}_{21} \end{bmatrix} \right)$ is symplectic pair, it follows from Lemma 4.2 that $\widetilde{X}_{11}^* = \widetilde{X}_{11}$, $\widetilde{X}_{22}^* = \widetilde{X}_{22}$ and $\widetilde{X}_{12}^* = \widetilde{X}_{21}$. □

Suppose that $(\mathcal{A}, \mathcal{B})$ is a regular symplectic pair. Theorem 4.1 (i) shows that $\operatorname{rank}(\mathcal{B}) \geqslant n$. In the following lemma, we show that there is $\Pi_B \in \mathfrak{S}^{2n}$ such that the first n columns of $\mathcal{B}\Pi_B$ are linearly independent.

Lemma 4.4. *Let $(\mathcal{A}, \mathcal{B}) \in \mathbb{SP}_n$. There exists $\Pi_B \in \mathfrak{S}^{2n}$ such that $\widehat{B}_1 \in \mathbb{C}^{2n\times n}$ is of full rank, where $\widehat{\mathcal{B}} \equiv \mathcal{B}\Pi_B = [\widehat{B}_1, \widehat{B}_2]$.*

Proof. Suppose that $\operatorname{nullity}(\mathcal{B}) = \ell$. Let $Q \in \mathbb{C}^{2n\times 2n}$ be an invertible matrix such that $Q\mathcal{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$, where $B_1 \in \mathbb{C}^{(2n-\ell)\times 2n}$ is of full row rank. Now, we show that there exists an invertible matrix $U \in \mathbb{C}^{(2n-\ell)\times(2n-\ell)}$ such that the following equality holds:

$$(4.4)\qquad UB_1\mathcal{J}B_1^*U^* = \begin{bmatrix} 0 & I_{n-\ell} & 0 \\ -I_{n-\ell} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{C}^{(2n-\ell)\times(2n-\ell)}.$$

Note that $B_1\mathcal{J}B_1^* \in \mathbb{C}^{(2n-\ell)\times(2n-\ell)}$ is skew-Hermitian. From Theorem 4.1, we see that $\ell \leqslant n$ and $\operatorname{rank}(B_1\mathcal{J}B_1^*) = 2n-2\ell$. There is an invertible matrix U_1 such that

$$(4.5)\qquad U_1B_1\mathcal{J}B_1^*U_1^* = \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix},$$

where $K \in \mathbb{C}^{(2n-2\ell)\times(2n-2\ell)}$ is an invertible skew-Hermitian matrix. Partition U_1B_1 as the block forms $U_1B_1 = \begin{bmatrix} B_1^1 \\ B_2^1 \end{bmatrix}$, where $B_1^1 \in \mathbb{C}^{(2n-2\ell)\times n}$ and $B_2^1 \in \mathbb{C}^{\ell\times n}$. Then we have $B_1^1\mathcal{J}B_1^{1^*} = K$, $B_2^1\mathcal{J}B_1^{1^*} = 0$ and $B_2^1\mathcal{J}B_2^{1^*} = 0$. Let $\check{B} = \begin{bmatrix} B_1^1 \\ B_2^1 \\ B_2^1\mathcal{J} \end{bmatrix} \in \mathbb{C}^{2n\times 2n}$. Using the fact that $\mathcal{J}^{-1} = \mathcal{J}^*$, (4.5) and $(B_2^1\mathcal{J})\mathcal{J}(B_2^1\mathcal{J})^* = 0$, we have

$$(4.6)\qquad \check{B}\mathcal{J}\check{B}^* = \begin{bmatrix} K & 0 & K_{13} \\ 0 & 0 & K_{23} \\ -K_{13}^* & -K_{23}^* & 0 \end{bmatrix} \in \mathbb{C}^{2n\times 2n},$$

where $K_{13} = B_1^1B_2^{1^*}$ and $K_{23} = B_2^1B_2^{1^*}$ is nonsingular. Since K_{23} and $\check{B}\mathcal{J}\check{B}^*$ are nonsingular, we obtain that $\check{B}$ is invertible and there is an invertible matrix $U_2 \in \mathbb{C}^{2n\times 2n}$ such that $U_2\check{B}\mathcal{J}_n\check{B}^*U_2^* = \begin{bmatrix} K & 0 \\ 0 & \mathcal{J}_\ell \end{bmatrix}$. From the congruence transformation of (4.6), it is easily seen that Hermitian matrices iK and $i\mathcal{J}_{n-\ell}$ have the same inertia. From (4.5), we obtain (4.4).

Let $\widetilde{B}_1 = [0, I_n]UB_1 \in \mathbb{C}^{n\times 2n}$. Then $\operatorname{rank}(\widetilde{B}_1) = n$. From (4.4), we have $\widetilde{B}_1\mathcal{J}\widetilde{B}_1^* = 0$. It follows from Lemma 2.2 that there exists a symplectic swap matrix $\Pi_B \in \mathfrak{S}^{2n}$ such that $\widetilde{B}_{11} \in \mathbb{C}^{n\times n}$ is invertible, where $\widetilde{B}_1\Pi_B = [\widetilde{B}_{11}, \widetilde{B}_{12}]$. Let $\widehat{\mathcal{B}} \equiv \mathcal{B}\Pi_B = [\widehat{B}_1, \widehat{B}_2]$. Then $\widehat{B}_1 \in \mathbb{C}^{2n\times n}$ is of full column rank. □

For a given regular symplectic pair $(\mathcal{A}, \mathcal{B})$, from Lemma 4.4 there exists a symplectic swap matrix $\Pi_B \in \mathfrak{S}^{2n}$ such that $(\widehat{\mathcal{A}}, \widehat{\mathcal{B}}) \equiv (\mathcal{A}\Pi_B, \mathcal{B}\Pi_B)$ satisfies the assumption of Theorem 4.3. Hence, we can prove the first assertion of Theorem 1.2.

Theorem 4.5. *Let* $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_n$*. There exist* $\Pi_A, \Pi_B \in \mathfrak{S}^{2n}$ *such that* $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_{\Pi_A,\Pi_B}$*.*

We also have the following immediate consequence from Lemma 4.4.

Theorem 4.6. *Let* $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_n$ *with* $\mathcal{B}$ *being invertible. There exists* $\Pi_A \in \mathfrak{S}^{2n}$ *such that* $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_{\Pi_A,I}$*.*

Remark 4.7. *Suppose that* $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_n$ *with* $\mathcal{B}$ *being invertible. Then there is a symplectic matrix* $\mathcal{S} = \mathcal{B}^{-1}\mathcal{A} \in \mathbb{S}_n$ *such that* $(\mathcal{A},\mathcal{B}) \overset{l.e.}{\sim} (\mathcal{S}, I_{2n})$*. That is, the set* $\{(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_n \mid \mathcal{B}$ *is invertible*$\}$ *can be regarded as the set of symplectic matrices* $\mathbb{S}_n$*. Theorem 4.6 shows that* $\{(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_n \mid \mathcal{B}$ *is invertible*$\} \subseteq \bigcup_{\Pi\in\mathfrak{S}^{2n}} \mathbb{SP}_{\Pi,I}$*. This coincides with Theorem 1.1 (*i*).*

Now we are in a position to prove Theorem 1.2. In the following, we shall show that $\mathbb{SP}_n$ is covered by 2^n categories, each of them is an open set, and none of them can be completely covered by the rest categories, i.e., the classification is minimal.

Proof of Theorem 1.2. Assertion (i) follows from Theorem 4.5 directly. Now we prove assertion (ii). For each $\Pi_B \in \mathfrak{S}^{2n}$, let $\mathcal{A} = \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \Pi_B$ and $\mathcal{B} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \Pi_B$. Then $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_n$ is a regular symplectic pair. It follows from Theorem 4.3 that $(\mathcal{A},\mathcal{B}) \notin \mathbb{SP}_{\cdot,\Pi'_B}$ for all $\Pi'_B \neq \Pi_B$.

For assertion (iii), let $\Pi_A, \Pi_B \in \mathfrak{S}^{2n}$, we show that $\mathbb{SP}_{\Pi_A,\Pi_B}$ is an open set. For each $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_{\Pi_A,\Pi_B}$, there exists an invertible matrix Q and Hermitian $X = [X_{ij}] \in \mathbb{H}^{2n}$ such that $Q\mathcal{A} = \begin{bmatrix} X_{21}^* & 0 \\ X_{22} & I_n \end{bmatrix} \Pi_A$ and $Q\mathcal{B} = \begin{bmatrix} I_n & X_{11} \\ 0 & X_{21} \end{bmatrix} \Pi_B$. Let $\delta = \frac{1}{2\|Q\|_F} > 0$. Suppose that $(\widetilde{\mathcal{A}},\widetilde{\mathcal{B}}) \in \mathbb{SP}_n$, $\mathcal{E}^A = [\mathcal{E}^A_{ij}] = \widetilde{\mathcal{A}} - \mathcal{A}$ and $\mathcal{E}^B = [\mathcal{E}^B_{ij}] = \widetilde{\mathcal{B}} - \mathcal{B}$ such that $\max\{\|\mathcal{E}^A\|_F, \|\mathcal{E}^B\|_F\} < \delta$. It suffices to show that $(\widetilde{\mathcal{A}},\widetilde{\mathcal{B}}) \in \mathbb{SP}_{\Pi_A,\Pi_B}$. Denote $\widetilde{\mathcal{E}}^A \equiv Q\mathcal{E}^A\Pi_A^{-1} = [\widetilde{\mathcal{E}}^A_{ij}]$ and $\widetilde{\mathcal{E}}^B \equiv Q\mathcal{E}^B\Pi_B^{-1} = [\widetilde{\mathcal{E}}^B_{ij}]$. Then

$$Q\widetilde{\mathcal{A}} = \left(\begin{bmatrix} X_{21}^* & 0 \\ X_{22} & I_n \end{bmatrix} + \begin{bmatrix} \widetilde{\mathcal{E}}^A_{11} & \widetilde{\mathcal{E}}^A_{12} \\ \widetilde{\mathcal{E}}^A_{21} & \widetilde{\mathcal{E}}^A_{22} \end{bmatrix} \right) \Pi_A,$$

$$Q\widetilde{\mathcal{B}} = \left(\begin{bmatrix} I_n & X_{11} \\ 0 & X_{21} \end{bmatrix} + \begin{bmatrix} \widetilde{\mathcal{E}}^B_{11} & \widetilde{\mathcal{E}}^B_{12} \\ \widetilde{\mathcal{E}}^B_{21} & \widetilde{\mathcal{E}}^B_{22} \end{bmatrix} \right) \Pi_B$$

where $\|\widetilde{\mathcal{E}}^A_{ij}\|_F \leqslant \|Q\|_F\|\mathcal{E}^A\|_F < \frac{1}{2}$ and $\|\widetilde{\mathcal{E}}^B_{ij}\|_F \leqslant \|Q\|_F\|\mathcal{E}^B\|_F < \frac{1}{2}$ for $i,j = 1,2$. Since $\|\widetilde{\mathcal{E}}^B_{11}\|_F < \frac{1}{2}$, the matrix $I_n + \widetilde{\mathcal{E}}^B_{11}$ is invertible and $(I_n + \widetilde{\mathcal{E}}^B_{11})^{-1} = \sum_{k=0}^{\infty}(-\widetilde{\mathcal{E}}^B_{11})^k$. Hence $\|(I_n + \widetilde{\mathcal{E}}^B_{11})^{-1}\|_F < \sum_{k=0}^{\infty}(\frac{1}{2})^k = 2$. Let $Q_B = \begin{bmatrix} (I_n + \widetilde{\mathcal{E}}^B_{11})^{-1} & 0 \\ -\widetilde{\mathcal{E}}^B_{21}(I_n + \widetilde{\mathcal{E}}^B_{11})^{-1} & I_n \end{bmatrix}$. Then

$$Q_B Q\widetilde{\mathcal{B}} = \begin{bmatrix} I_n & \star \\ 0 & \star \end{bmatrix}\Pi_B,$$

$$Q_B Q\widetilde{\mathcal{A}} = \begin{bmatrix} \star & \star \\ \star & I_n + \widetilde{\mathcal{E}}^A_{22} - \widetilde{\mathcal{E}}^B_{21}(I_n + \widetilde{\mathcal{E}}^B_{11})^{-1}\widetilde{\mathcal{E}}^A_{12} \end{bmatrix}\Pi_A.$$

Since $\|\widetilde{\mathcal{E}}^A_{22} - \widetilde{\mathcal{E}}^B_{21}(I_n+\widetilde{\mathcal{E}}^B_{11})^{-1}\widetilde{\mathcal{E}}^A_{12}\|_F \leqslant \|\widetilde{\mathcal{E}}^A_{22}\|_F + \|\widetilde{\mathcal{E}}^B_{21}\|_F\|(I_n+\widetilde{\mathcal{E}}^B_{11})^{-1}\|_F\|\widetilde{\mathcal{E}}^A_{12}\|_F < \frac{1}{2} + \frac{2}{4} = 1$, the matrix $I_n + \widetilde{\mathcal{E}}^A_{22} - \widetilde{\mathcal{E}}^B_{21}(I_n + \widetilde{\mathcal{E}}^B_{11})^{-1}\widetilde{\mathcal{E}}^A_{12}$ is invertible. Hence, $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}) \in \mathbb{SP}_{\Pi_A,\Pi_B}$. □

Here we note that the collection $\{\mathbb{SP}_{\Pi_A,\Pi_B} | \Pi_A, \Pi_B \in \mathfrak{S}^{2n}\}$ does not form a minimal classification for $\mathbb{SP}_n$. To see this, we consider a special case of regular symplectic pair. Suppose that $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_n$ and there exists $\Pi_B \in \mathfrak{S}^{2n}$ such that

$$(\mathcal{A},\mathcal{B}) \notin \mathbb{SP}_{\cdot,\Pi} \text{ for each } \Pi \in \mathfrak{S}^{2n} \text{ and } \Pi \neq \Pi_B, \tag{4.7}$$

where $\mathbb{SP}_{\cdot,\Pi}$ is defined in Definition 1.2.

Lemma 4.8. *Let $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_n$. Suppose that (4.7) holds for some symplectic swap matrix $\Pi_B \in \mathfrak{S}^{2n}$. Then $\mathcal{B}\Pi_B^{-1} = [\widehat{B}_1, 0]$, where $\widehat{B}_1 \in \mathbb{C}^{2n\times n}$ with rank$(\widehat{B}_1) = n$.*

Proof. From Theorem 4.5, we know that there exists a symplectic swap matrix $\widehat{\Pi} \in \mathfrak{S}^{2n}$ such that $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_{\cdot,\widehat{\Pi}}$. The condition (4.7) shows that $\widehat{\Pi} = \Pi_B$. That is, Π_B is unique symplectic swap matrix such that $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_{\Pi_A,\Pi_B}$ for some symplectic swap matrix $\Pi_A \in \mathfrak{S}^{2n}$. Let $\widehat{\mathcal{B}} \equiv \mathcal{B}\Pi_B^{-1} = [\widehat{B}_1, \widehat{B}_2]$. It follows from Theorem 4.3 that rank$(\widehat{\mathcal{B}}) = $ rank$(\widehat{B}_1) = n$. Then we have $\widehat{\mathcal{B}}\mathcal{J}\widehat{\mathcal{B}}^* = 0$ by Theorem 4.1.

Now, we show that $\widehat{B}_2 = 0$. Suppose that $\widehat{B}_2 \neq 0$. Without loss of generality, assume the first column of $\widehat{B}_2$ is nonzero. Let $\Pi_{\mathbf{v}_1} = \begin{bmatrix} \text{diag}(\mathbf{v}_1) & \text{diag}(\mathbf{v}_1^c) \\ -\text{diag}(\mathbf{v}_1^c) & \text{diag}(\mathbf{v}_1) \end{bmatrix} \in \mathfrak{S}^{2n}$, where $\mathbf{v}_1^c = \mathbf{e}_1$ and $\mathbf{v}_1 = \mathbf{e} - \mathbf{v}_1^c$. Denote

$$\widehat{\mathcal{B}}^{(1)} \equiv \widehat{\mathcal{B}}\Pi_{\mathbf{v}_1} = [\widehat{B}_1^{(1)}, \widehat{B}_2^{(1)}]. \tag{4.8}$$

We have $\widehat{\mathcal{B}}^{(1)}\mathcal{J}(\widehat{\mathcal{B}}^{(1)})^* = \widehat{\mathcal{B}}\mathcal{J}\widehat{\mathcal{B}}^* = 0$ and $\text{rank}(\widehat{B}_1^{(1)}) = n-1$. Sine $\widehat{B}_1^{(1)}(:,1) = \widehat{B}_2(:,1) \neq 0$, there exists an integer i with $2 \leqslant i \leqslant n$ such that $\text{rank}([\widehat{B}_1^{(1)}(:,1:i-1), \widehat{B}_1^{(1)}(:,i+1:n)]) = n-1$. Let $\Pi_{\mathbf{v}_2} = \begin{bmatrix} \text{diag}(\mathbf{v}_2) & \text{diag}(\mathbf{v}_2^c) \\ -\text{diag}(\mathbf{v}_2^c) & \text{diag}(\mathbf{v}_2) \end{bmatrix} \in \mathfrak{S}^{2n}$, where $\mathbf{v}_2^c = \mathbf{e}_i$ and $\mathbf{v}_2 = \mathbf{e} - \mathbf{v}_2^c$. Denote $\widehat{\mathcal{B}}^{(2)} \equiv \widehat{\mathcal{B}}^{(1)}\Pi_{\mathbf{v}_2} = [\widehat{B}_1^{(2)}, \widehat{B}_2^{(2)}]$, where $\widehat{\mathcal{B}}^{(1)}$ is defined in (4.8). Using the fact that $\widehat{\mathcal{B}}^{(2)}\mathcal{J}(\widehat{\mathcal{B}}^{(2)})^* = \widehat{\mathcal{B}}\mathcal{J}\widehat{\mathcal{B}}^* = 0$, from Lemma 2.2, we obtain that $\widehat{B}_1^{(2)} \in \mathbb{R}^{2n\times n}$ is of full column rank. Let $\widehat{\Pi}_B = (\Pi_B^{-1}\Pi_{\mathbf{v}_1}\Pi_{\mathbf{v}_2})^{-1} \in \mathfrak{S}^{2n}$. We have $\widehat{\mathcal{B}}^{(2)} = \mathcal{B}\widehat{\Pi}_B^{-1}$ and $\widehat{\Pi}_B \neq \Pi_B$ because $\Pi_{\mathbf{v}_1}\Pi_{\mathbf{v}_2} \neq I_{2n}$. Since $\text{rank}(\widehat{B}_1^{(2)}) = n$, if follows from Theorem 4.3 that $(\mathcal{A}\widehat{\Pi}_B^{-1}, \mathcal{B}\widehat{\Pi}_B^{-1}) \in \mathbb{SP}_{\cdot,I}$. Hence, $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_{\cdot,\widehat{\Pi}_B}$. This contradicts the condition (4.7), because $\widehat{\Pi}_B \neq \Pi_B$. □

Then we have the following results.

Theorem 4.9. *Let $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_n$. Suppose that* (4.7) *holds for some symplectic swap matrix $\Pi_B \in \mathfrak{S}^{2n}$. Then $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_{\Pi_B,\Pi_B}$.*

Proof. Denote $\widehat{\mathcal{A}} \equiv \mathcal{A}\Pi_B^{-1} = [\widehat{A}_1, \widehat{A}_2]$ and $\widehat{\mathcal{B}} \equiv \mathcal{B}\Pi_B^{-1} = [\widehat{B}_1, \widehat{B}_2]$. $(\mathcal{A},\mathcal{B})$ is a regular symplectic pair, so is $(\widehat{\mathcal{A}},\widehat{\mathcal{B}})$. Lemma 4.8 shows that $\text{rank}(\widehat{B}_1) = n$ and $\widehat{B}_2 = 0$. From Theorem 4.3, there exists a symplectic swap matrix $\Pi_A \in \mathfrak{S}^{2n}$ such that

$$(\widehat{\mathcal{A}},\widehat{\mathcal{B}}) \overset{\text{l.e.}}{\sim} \left(\begin{bmatrix} 0 & 0 \\ X_{22} & I_n \end{bmatrix} \Pi_A, \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \right). \tag{4.9}$$

Denote $[X_{22}, I_n]\Pi_A = [Y_1, Y_2]$. Since $(\widehat{\mathcal{A}},\widehat{\mathcal{B}})$ is regular, Y_2 is invertible. Let $\widehat{X}_{22} = Y_2^{-1}Y_1$. From (4.9), we have $(\widehat{\mathcal{A}},\widehat{\mathcal{B}}) \overset{\text{l.e.}}{\sim} \left(\begin{bmatrix} 0 & 0 \\ \widehat{X}_{22} & I_n \end{bmatrix}, \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \right)$. Using the fact that $(\mathcal{A},\mathcal{B}) = (\widehat{\mathcal{A}}\Pi_B, \widehat{\mathcal{B}}\Pi_B)$, we obtain

$$(\mathcal{A},\mathcal{B}) \overset{\text{l.e.}}{\sim} \left(\begin{bmatrix} 0 & 0 \\ \widehat{X}_{22} & I_n \end{bmatrix} \Pi_B, \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \Pi_B \right).$$

Hence, $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_{\Pi_B,\Pi_B}$. □

Remark 4.10. *Now we brief show that the collection $\{\mathbb{SP}_{\Pi_A,\Pi_B} | \Pi_A, \Pi_B \in \mathfrak{S}^{2n}\}$ does not form a minimal classification for $\mathbb{SP}_n$. On the contrary, suppose it does. We pick a category $\mathbb{SP}_{\Pi_A,\Pi_B}$ with $\Pi_A \neq \Pi_B$ such that there exists a regular symplectic pair $(\mathcal{A},\mathcal{B}) \in \mathbb{SP}_{\Pi_A,\Pi_B}$ satisfying $(\mathcal{A},\mathcal{B}) \notin \mathbb{SP}_{\Pi'_A,\Pi'_B}$*

for $(\Pi_A, \Pi_B) \neq (\Pi'_A, \Pi'_B)$. *Therefore,* $(\mathcal{A}, \mathcal{B}) \in \mathbb{SP}_n$ *satisfies* (4.7). *On the other hand, Theorem 4.9 implies* $(\mathcal{A}, \mathcal{B}) \in \mathbb{SP}_{\Pi_B,\Pi_B}$. *However,* $\Pi_A \neq \Pi_B$, *this is a contradiction.*

However, the collection $\{\mathbb{SP}_{\Pi,\Pi} \mid \Pi \in \mathfrak{S}^{2n}\}$ does not form a classification of $\mathbb{SP}_n$. In fact, $\bigcup_{\Pi\in\mathfrak{S}^{2n}} \mathbb{SP}_{\Pi,\Pi} \subsetneq \mathbb{SP}_n$. One can see the following example.

Example 4.1. Let $\mathcal{A} = \begin{bmatrix} 0 & 0 \\ Y_1 & Y_2 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} I_n & X_{11} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{6\times 6}$, where

$$Y_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -2 & 1 \end{bmatrix},\ Y_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } X_{11} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}. \tag{4.10}$$

Then $\mathcal{AJA}^* = 0 = \mathcal{BJB}^*$, hence $(\mathcal{A}, \mathcal{B})$ is a symplectic pair. Furthermore, $(\mathcal{A}, \mathcal{B})$ is regular because

$$Y_2 - Y_1 X_{11} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & -2 & 3 \end{bmatrix}$$

is invertible. Now, we show that there is no symplectic swap matrix $\Pi \in \mathfrak{S}^{2n}$ such that $(\mathcal{A}, \mathcal{B}) \in \mathbb{SP}_{\Pi,\Pi}$. We prove this by contradiction. Suppose that there is a symplectic swap matrix $\Pi \in \mathfrak{S}^{2n}$ such that $(\mathcal{A}, \mathcal{B}) \in \mathbb{SP}_{\Pi,\Pi}$, where

$$\Pi = \Pi_{\mathbf{v}} = \begin{bmatrix} \mathrm{diag}(\mathbf{v}) & \mathrm{diag}(\mathbf{v}^c) \\ -\mathrm{diag}(\mathbf{v}^c) & \mathrm{diag}(\mathbf{v}) \end{bmatrix},$$

$\mathbf{v} \in \{0, 1\}^3$ and $\mathbf{v}^c = \mathbf{e} - \mathbf{v}$. It is easily seen that $\Pi \neq I$ because Y_2 is singular, i.e., $\mathbf{v} \neq \mathbf{e}$. Since $(\mathcal{A}, \mathcal{B}) \in \mathbb{SP}_{\Pi,\Pi}$ and $\Pi^{-1} = \Pi^*$, we obtain that the (1,1)-block of $\mathcal{B}\Pi^{-1} = [\widehat{B}_{ij}]$ and the (2,2)-block of $\mathcal{A}\Pi^{-1} = [\widehat{A}_{ij}]$ are invertible, that is,

$$\widehat{B}_{11} = \mathrm{diag}(\mathbf{v}) + X_{11}\mathrm{diag}(\mathbf{v}^c) \text{ and } \widehat{A}_{22} = -Y_1\mathrm{diag}(\mathbf{v}^c) + Y_2\mathrm{diag}(\mathbf{v})$$

are invertible. From (4.10) and using the fact that $\mathbf{v} \neq \mathbf{e}$ and $\widehat{B}_{11}$ is invertible, we have $\mathbf{e}^*\mathbf{v} = 1$. Since $\widehat{A}_{22}$ is invertible and Y_1 is a rank-1 matrix, we have $\mathbf{e}^*\mathbf{v}^c = 1$. This is a contradiction. Hence, $(\mathcal{A}, \mathcal{B}) \notin \mathbb{SP}_{\Pi,\Pi}$ for each $\Pi \in \mathfrak{S}^{2n}$.

5. Concluding remarks

In this paper, we aims to study the classification of $\mathbb{S}_n$ and $\mathbb{SP}_n$. Both $\mathbb{S}_n$ and $\mathbb{SP}_n$ are classified by 2^n categories in which the classifications are minimal and each category is an open set. Those classifications can be applied in the doubling algorithms with permuted bases [13] and constructing the structure preserving flow [8, 9]. From Remark 4.10, we see that the classification $\{\mathbb{SP}_{\Pi_A,\Pi_B} | \ \Pi_A, \Pi_B \in \mathfrak{S}^{2n}\}$ does not form a minimal classification of $\mathbb{SP}_n$. It is of our current interest in the future that how can we drop some categories in $\{\mathbb{SP}_{\Pi_A,\Pi_B} | \ \Pi_A, \Pi_B \in \mathfrak{S}^{2n}\}$ such that the resulting collection forms a minimal classification of $\mathbb{SP}_n$.

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