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Perturbation analysis and condition numbers of rational Riccati equations

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In this paper, we consider the perturbation analyses of the discrete-time rational Riccati equations using the normwise, mixed and componentwise analyses, which arises from the stochastic H_∞ problems and the indefinite stochastic linear quadratic control problems. We derive sufficient conditions for the existence of stabilizing solutions of the perturbed rational Riccati equations. Moreover, we obtain the perturbation bounds for the relative errors with respect to the stabilizing solutions of the rational Riccati equations under three kinds of perturbation analyses. Numerical results are presented to illustrate sharper perturbation bounds under the normwise, mixed and componentwise perturbation analyses.

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1. Introduction

From [13, 15, 20], the discrete-time stochastic control system gives rise to the following discrete-time rational Riccati equation (DRRE):

$$\mathcal{D}(X) \equiv \mathcal{S}_A(X) + Q + \Pi_1(X) - G(X)H(X)^\dagger G(X)^* = 0, \tag{1}$$

with $\mathcal{S}_A(X) \equiv A^*XA - X$, $G(X) \equiv L + A^*XB + \Pi_{12}(X)$ and $H(X) \equiv R + B^*XB + \Pi_2(X)$. The stochastic disturbances $\Pi_1(X)$, $\Pi_2(X)$ and $\Pi_{12}(X)$ associated with A_0^i and B_0^i are defined as $\Pi_1(X) \equiv \sum_{i=1}^N A_0^{i*}XA_0^i$, $\Pi_2(X) \equiv \sum_{i=1}^N B_0^{i*}XB_0^i$, $\Pi_{12}(X) \equiv \sum_{i=1}^N A_0^{i*}XB_0^i$. Methods for solving DRREs (1) to get the maximal and stable solution X can be found in [13, 14, 15, 20, 30]. In this paper, we apply an efficient method called the generalized Smith method (GSM) [14] to the generalized Stein equation (GSE) from the DRREs (1) solved by NM

$$\mathcal{S}_{A-B\tilde{F}_{X_k}(X_k)}(X_{k+1}) + \Pi_{X_k}(X_{k+1}) + T_{X_k} = 0, \tag{2}$$

with

$$\tilde{F}_Y(X) \equiv (R + B^*YB + \Pi_2(X))^\dagger (L + A^*YB + \Pi_{12}(X))^*, \tag{3}$$

and

$$\Pi_Y(Z) \equiv \begin{bmatrix} I \\ -\tilde{F}_Y(Y) \end{bmatrix}^* \Pi(Z) \begin{bmatrix} I \\ -\tilde{F}_Y(Y) \end{bmatrix}, \tag{4}$$
$$T_Y \equiv \begin{bmatrix} I \\ -\tilde{F}_Y(Y) \end{bmatrix}^* T \begin{bmatrix} I \\ -\tilde{F}_Y(Y) \end{bmatrix}.$$

In order to show the efficient method GSM for solving GSEs, we summarize some points of the GSM:

Let $A_k \equiv A - B\tilde{F}_{X_k}(X_k)$ and $r_k \equiv T_{X_k}$, (2) can then be solved using the functional iteration as described below: (for $Y_0 = r_k$ and $j = 1, 2, \cdots$)

$$Y_{j+1} = A_k^\top Z A_k + \Pi_{X_k}(Z) + r_k.$$

To accelerate convergence, the GSM computes the sequence $Z_j = Y_{2^j-1}$ $(j \geq 0)$. Consequently, the GSM has the form

$$\widetilde{\mathcal{S}}_{j+1} = \widetilde{\mathcal{S}}_j^2, \quad Z_{j+1} = Z_j + \widetilde{\mathcal{S}}_j(Z_j) \quad (j \geq 0),$$

with $Z_0 = r_k$ and $\widetilde{\mathcal{S}}_0(Z) \equiv A_k^\top Z A_k + \Pi_{X_k}(Z)$.

We also need the null space requirement for the generalized inverse in (1) and (3)

$$\ker[H(X)] \subseteq \ker[G(X)], H(X) \geq 0.$$

The stabilizing solution plays an important role in some applications of control theory and we introduce the stability definitions. We first represent $\sigma(T) \subset \mathbb{C}$ for the spectrum of a linear operator T and $\rho(T) = \max\{|\lambda| | \lambda \in \sigma(T)\}$ for the spectral radius. An $n \times n$ matrix M is said to be c-stable if all of its eigenvalues lie in the open left-half complex plane s.t. $\sigma(M) \subset \mathbb{C}_-$, and M is said to be d-stable if its spectral radius satisfies $\rho(M) < 1$.

DRREs (1) can be reduced into discrete-time algebraic Riccati equations called DAREs

$$-X + A^\top X(I_n + GX)^{-1}A + H = 0,$$

with $G = BR^{-1}B^\top$ and $H = CT^{-1}C^\top$, where $A, X \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{n\times p}$, $R \in \mathbb{R}^{m\times m}$ and $T \in \mathbb{R}^{p\times p}$, which arise in linear-quadratic optimal control problems [23, 27]. In the past 40 years, many efficient methods such as disk function method [1], matrix sign function method [4], structure-preserving doubling algorithm (SDA) [8, 9], NM [19], Schur method [24], and many others were developed.

The condition number, which is a measure of the sensitivity, is important in the numerical computation. Moreover, perturbation analysis is to study the sensitivity of solutions to the small perturbation in the input data, and the perturbation bounds are usually discussed. There were a number of references about perturbation analyses and perturbation bounds in [3, 5, 12, 16, 21, 22, 28, 29, 31], but only relatively few references discussed those of the CAREs with stochastic disturbances. Chiang et al. [6, 7] discussed the residual bound of the continuous-time stochastic algebraic Riccati equation (SARE) with one-dimensional Wiener process of white noise in the stochastic disturbance, including the normwise local and non-local residual bounds, derived from the appropriate solution of the SARE by NM. Furthermore, they derived the relative error and the condition number of SARE and provided a tight perturbation bound of the stabilizing solution to SARE. This paper derives new perturbation bounds for the relative errors with respect to the stabilizing solutions of DRREs with multi-dimensional disturbances. The DRREs are more complex than SARE, but the derivations of new perturbation bounds under normwise, mixed and componentwise perturbation analyses are simple. Moreover, we apply an efficient method called GSM to solve the perturbed DRREs and get the unique and stabilizing solutions.

The rest of this paper is organized as follows. We discuss the solvable results of DRREs (1) in Section 2. In Section 3, the perturbation equation of DRREs (1) is derived. We provide some sufficient conditions to present the existence of the solution of the perturbed equation (7) in the Supplementary Material S1 http://intlpress.com/site/pub/files/_supp/amsa/2021/0006/0001/AMSA-2021-0006-0001-s001.pdf, then recall some lemmas of stability analysis of the linear operator to discuss the uniqueness of the stabilizing solution shown in the Supplementary Material S2. Then, we compute the relative errors with respect to the unique and stabilizing solution of DRREs (1). By dropping the second and high-order terms in the perturbed DRREs (7), we derive new perturbation bounds under normwise, mixed and componentwise perturbation analyses, originated from Gohberg and Koltracht (1993). The algorithm about perturbation analyses of DRREs is provided and we select two representative numerical examples to illustrate the sharpness of new perturbation bounds, corresponding to the relative errors of the stabilizing solutions of DRREs (1) in Section 4. Section 5 concludes the paper. Finally, we provide some proofs of several theorems in the Supplementary Material.

2. Solvable conditions

Before we discuss the perturbation analysis of DRREs, we introduce the concept of d-stability and some solvability results for stochastic control systems in the discrete-time case. This leads to the unique and stable solution of DRREs.

First we quote the theorem on d-stability.

Theorem 2.1. *Assume there exist Hermitian matrices $\hat{X}$ and X_0 such that $\mathcal{D}(\hat{X}) \geq 0$ and $X_0 \geq \hat{X}$, $\mathcal{D}(X_0) < 0$ and $A - B\tilde{F}_{X_0}(X_0)$ is d-stable. Then the matrix sequence $\{X_k\}$ defined by (2) satisfies*

- *(a) $X_k \geq \hat{X}$, $\mathcal{D}(X_k) < 0$, $k = 0, 1, 2, \ldots$;*
- *(b) $X_k \geq X_{k+1}$, $k = 0, 1, 2, \ldots$;*
- *(c) $A - B\tilde{F}_{X_k}(X_k)$ is d-stable for $k = 0, 1, 2, \ldots$;*
- *(d) $\lim_{k\to\infty} X_k = \tilde{X}$ is a solution of $\mathcal{D}(X) = 0$ with $\tilde{X} \geq \hat{X}$; Moreover, if $X_0 \geq X$ for all solutions X of $\mathcal{D}(X) = 0$, then $\tilde{X}$ is the maximal solution;*
- *(e) the eigenvalues of $A - B\tilde{F}_{\tilde{X}}(\tilde{X})$ lie in the closed unit disk; in addition, if $\mathcal{D}(\hat{X}) > 0$, then all eigenvalues of $A - B\tilde{F}_{\tilde{X}}(\tilde{X})$ lie in the open unit disk.*

The proof of Theorem 2.1 is referred to [15, 20]. We first define some properties and quote the stability theorem from [14, Theorem 1.3], originally from [15, Theorem 3.3].

Definition 2.2 (Damm and Hinrichsen [10])**.** *A linear operator $\mathcal{L}$ on $\mathcal{H}^n$ is called positive if $\mathcal{L}(A) \geq 0$ whenever $A \geq 0$. $\mathcal{L}$ is called inverse positive if $\mathcal{L}^{-1}$ exists and is positive. Moreover, $\mathcal{L}$ is called resolvent positive if the operator $\alpha\mathcal{I} - \mathcal{L}$ is inverse positive for all sufficiently large $\alpha > 0$.*

Theorem 2.3 (d-Stability and GSEs [15, Theorem 3.3])**.** *Let $A \in \mathbb{R}^{n\times n}$ and consider linear operators $W, \mathcal{S}_A, \Pi : \mathcal{S}^{n\times n} \to \mathcal{S}^{n\times n}$, where W and $\mathcal{S}_A$ are defined by $W(X) = A^*XA$ and $\mathcal{S}_A(X) = A^*XA - X$, respectively and Π is nonnegative defined in (4). The following are equivalent:*

- *(a) All eigenvalues of A lie in the open unit disk and $\rho(W^{-1}\Pi) < 1$.*
- *(b) $-(W + \Pi)$ is inverse positive.*
- *(c) There is some $X > 0$ such that $\mathcal{S}_A(X) + \Pi < 0$.*
- *(d) If $T_{X_k} > 0$ is defined in (4), then (2) has a unique solution $X > 0$.*
- *(e) $W + \Pi$ is d-stable.*

If any of these conditions is fulfilled then A is called d-stable relative to Π.

Then we present some associated definitions.

Definition 2.4 (d-Stability). *A pair (A, B) of matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ is said to be d-stabilizable relative to $\hat{\Pi}$ if there is a matrix $\hat{F} \in \mathbb{R}^{m\times n}$ such that $A - B\hat{F}$ is d-stable relative to $\hat{\Pi}$, where*

$$\hat{\Pi} \equiv \begin{bmatrix} I \\ -\hat{F} \end{bmatrix}^* \Pi(X) \begin{bmatrix} I \\ -\hat{F} \end{bmatrix}. \tag{5}$$

Definition 2.5 (Stabilizing Solution). *Let $\hat{X} \in D(\mathcal{D})$ (the domain of $\mathcal{D}$, in which the null space requirement is satisfied) be a solution of $\mathcal{D}(X) = 0$ if $\hat{F} = \hat{F}(\hat{X})$ denotes the corresponding feedback matrix and $\hat{\Pi}$ (defined in (5)), then $\hat{X}$ is called stabilizing (or almost stabilizing) if $\rho(\mathcal{S}_{A-B\hat{F}} + I + \hat{\Pi})$ is contained in the open (or closed) unit disk.*

Definition 2.6 (d-Dectectability). *A matrix pair (C, A) ($A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{l\times n}$) is said to be d-detectable relative to Π if there is a matrix $K \in \mathbb{R}^{n\times l}$ such that $A - KC$ is d-stable relative to Π.*

Next we quote some solvability results for DRREs in our notations:

Theorem 2.7 (Theorem 6.2 [15]). *Assume that (A, B) is d-stabilizable relative to Π and that there exists a solution $\hat{X}$ of $\mathcal{D}(X) \geq 0$ for which $ker[R + B^*\hat{X}B + \Pi_2(\hat{X})] \subseteq ker(B)$. Then there exists an almost stabilizing solution X_+ of $\mathcal{D}(X) = 0$, and we have $X_+ \geq X$ for all solutions of $\mathcal{D}(X) \geq 0$ with $ker[R + B^*XB + \Pi_2(\hat{X})] \subseteq ker(B)$.*

Corollary 2.8 (Corollary 6.3 [15]). *Assume that $ker(R) \subseteq ker(B)$, (A, B) is d-stabilizable relative to Π and $T \geq 0$. Then $\mathcal{D}(X) = 0$ has an almost stabilizing solution X_+, and $X_+ \geq 0$.*

Corollary 2.9 (Corollary 6.4 [15]). *Assume that (A, B) is d-stabilizable relative to Π and that there exists a solution $\hat{X}$ of $\mathcal{D}(X) > 0$ for which $ker[R + B^*\hat{X}B + \Pi_2(\hat{X})] \subseteq ker(B)$. Then there exists a stabilizing solution X_+ of $\mathcal{D}(X) = 0$, and $X_+ > \hat{X}$.*

Lemma 2.10 (Lemma 6.5 [15]). *If $\mathcal{D}(X) = 0$ has a stabilizing solution X_s, then $X_s \geq X$ for every solution X of $\mathcal{D}(X) \geq 0$. In particular, X_s is the maximal solution of $\mathcal{D}(X) = 0$.*

Lemma 2.11 (Lemma 6.6 [15]). *Assume that* $R > 0$, $T \geq 0$ *and that* $(Q - LR^{\dagger}L^*, A - BR^{\dagger}L^*)$ *is d-detectable relative to*

$$\check{\Pi} \equiv \begin{bmatrix} I \\ -R^{\dagger}L^* \end{bmatrix}^* \Pi \begin{bmatrix} I \\ -R^{\dagger}L^* \end{bmatrix}. \tag{6}$$

Then every positive semidefinite solution of $\mathcal{D}(X) = 0$ *is stabilizing.*

Lemma 2.12 (Lemma 6.8 [15]). *Assume that* $R \geq 0$, $ker(R) \subseteq ker(L)$, *and* $Q > LR^{\dagger}L^*$. *If* $X \geq 0$ *is a solution of* $\mathcal{D}(X) = 0$, *then* X *is stabilizing and positive definite.*

Theorem 2.13 (Theorem 6.9 [15]). *Assume that* (A, B) *is d-stabilizable relative to* Π, $R > 0$ *and* $T \geq 0$. *If* $(Q - LR^{\dagger}L^*, A - BR^{\dagger}L^*)$ *is d-detectable relative to* $\check{\Pi}$ *(defined in (6)), then* $\mathcal{D}(X) = 0$ *has a unique positive semidefinite solution* X_+. *Moreover,* X_+ *is stabilizing and maximal among all solution of* $\mathcal{D}(X) = 0$.

3. Perturbation analysis of DRREs

Consider the perturbed DRRE

$$\tilde{\mathcal{D}}(\tilde{X}) = \tilde{\mathcal{S}}_A(\tilde{X}) + \tilde{Q} + \tilde{\Pi}_1(\tilde{X}) - \tilde{G}(\tilde{X})\tilde{H}(\tilde{X})^{\dagger}\tilde{G}(\tilde{X})^* = 0, \tag{7}$$

with $\tilde{\mathcal{S}}_A(\tilde{X}) = \tilde{A}^*\tilde{X}\tilde{A} - \tilde{X}$, $\tilde{G}(\tilde{X}) = \tilde{L} + \tilde{A}^*\tilde{X}\tilde{B} + \tilde{\Pi}_{12}(\tilde{X})$, $\tilde{H}(\tilde{X}) = \tilde{R} + \tilde{B}^*\tilde{X}\tilde{B} + \tilde{\Pi}_2(\tilde{X})$. The perturbation matrices $\tilde{A}$, $\tilde{Q}$, $\tilde{L}$, $\tilde{B}$, $\tilde{R}$, $\tilde{\Pi}_1(\tilde{X})$, $\tilde{\Pi}_2(\tilde{X})$ and $\tilde{\Pi}_{12}(\tilde{X})$ are defined

$$\begin{aligned} \tilde{A} &= A + \Delta A, \tilde{Q} = Q + \Delta Q, \tilde{L} = L + \Delta L, \tilde{B} = B + \Delta B, \\ \tilde{R} &= R + \Delta R, \tilde{A}_0^i = A_0^i + \Delta A_0^i, \tilde{B}_0^i = B_0^i + \Delta B_0^i, \tilde{X} = X + \Delta X, \end{aligned}$$

for $i = 1, 2, \ldots, N$ and ΔA, ΔQ, ΔL, ΔB, ΔR, ΔA_0^i, ΔB_0^i are small perturbation matrices. To discuss the first-order perturbation matrices for computing the condition numbers of DRRE (1), we simplify $\tilde{G}(\tilde{X})$ and $\tilde{H}(\tilde{X})$ into

$$\begin{aligned} \tilde{G}(\tilde{X}) &= \tilde{G}(X) + G(\Delta X),\ \tilde{G}(X) = G(X) + \delta G, \\ \tilde{H}(\tilde{X}) &= \tilde{H}(X) + H(\Delta X),\ \tilde{H}(X) = H(X) + \delta H, \end{aligned}$$

with

$$\begin{aligned} G(\Delta X) &= G_1(\Delta X) + G_2(\Delta X),\ \delta G = \delta G_1 + \delta G_2, \\ H(\Delta X) &= H_1(\Delta X) + H_2(\Delta X),\ \delta H = \delta H_1 + \delta H_2, \end{aligned}$$

where

$$\begin{aligned} G_1(\Delta X) &= A^*\Delta X B + \sum_{i=1}^{N} A_0^{i*}\Delta X B_0^i,\ H_1(\Delta X) \\ &= B^*\Delta X B + \sum_{i=1}^{N} B_0^{i*}\Delta X B_0^i, \\ G_2(\Delta X) &= A^*\Delta X\Delta B + \Delta A^*\Delta X B + \Delta A^*\Delta X\Delta B + \sum_{i=1}^{N}(A_0^{i*}\Delta X\Delta B_0^i \\ &+ \Delta A_0^{i*}\Delta X B_0^i + \Delta A_0^{i*}\Delta X\Delta B_0^i), \\ H_2(\Delta X) &= B^*\Delta X\Delta B + \Delta B^*\Delta X B + \Delta B^*\Delta X\Delta B + \sum_{i=1}^{N}(B_0^{i*}\Delta X\Delta B_0^i \\ &+ \Delta B_0^{i*}\Delta X B_0^i + \Delta B_0^{i*}\Delta X\Delta B_0^i), \\ \delta G_1 &= \Delta L + A^* X\Delta B + \Delta A^* X B + \sum_{i=1}^{N}(A_0^{i*} X\Delta B_0^i + \Delta A_0^{i*} X B_0^i), \\ \delta H_1 &= \Delta R + B^* X\Delta B + \Delta B^* X B + \sum_{i=1}^{N}(B_0^{i*} X\Delta B_0^i + \Delta B_0^{i*} X B_0^i), \\ \delta G_2 &= \Delta A^* X\Delta B + \sum_{i=1}^{N}\Delta A_0^{i*} X\Delta B_0^i, \\ \delta H_2 &= \Delta B^* X\Delta B + \sum_{i=1}^{N}\Delta B_0^{i*} X\Delta B_0^i. \end{aligned}$$

$G(\Delta X)$ and $H(\Delta X)$ are linear functions of ΔX, $G_1(\Delta X)$ and $H_1(\Delta X)$ are to collect the first-order perturbation matrices, $G_2(\Delta X)$ and $H_2(\Delta X)$ are to collect the high-order perturbation matrices, δG_1 and δH_1 are the first-order perturbation matrices, δG_2 and δH_2 are the second-order perturbation matrices. Set $\Phi_D = A - BH(X)^\dagger G(X)^*$ and $\Psi_D^i = A_0^i - B_0^i H(X)^\dagger G(X)^*$, for $i = 1, 2, \ldots, N$, then we can get $\tilde{\Phi}_D = \tilde{A} - \tilde{B}\tilde{H}(X)^\dagger \tilde{G}(X)^*$ and $\tilde{\Psi}_D^i =$

$\tilde{A}_0^i - \tilde{B}_0^i \tilde{H}(X)^\dagger \tilde{G}(X)^*$, we consider the perturbed DRRE (7)

(8)

$$\tilde{\mathcal{D}}(\tilde{X}) - \mathcal{D}(X) = \tilde{\Phi}_D^* \Delta X \tilde{\Phi}_D + \sum_{i=1}^{N} \tilde{\Psi}_D^{i*} \Delta X \tilde{\Psi}_D^i - \Delta X - G_1 - G_2 - f_2(\Delta X) = 0,$$

using the generalization of the Sherman-Morrison-Woodbury formula (GSMWF) [11]

$$(A + UV^*)^\dagger = A^\dagger - A^\dagger U (I + V^* A^\dagger U)^{-1} V^* A^\dagger,$$

to

$$\begin{aligned}
\tilde{H}(\tilde{X})^\dagger &= (\tilde{H}(X) + H(\Delta X))^\dagger = \tilde{H}(X)^\dagger - \tilde{H}(X)^\dagger H(\Delta X) H_1 \tilde{H}(X)^\dagger, \\
\tilde{H}(X)^\dagger &= (H(X) + \delta H)^\dagger = H(X)^\dagger - H(X)^\dagger \delta H H_2 H(X)^\dagger, \\
H_1 &= (I_m + \tilde{H}(X)^\dagger H(\Delta X))^{-1} \\
&= I_m - \tilde{H}(X)^\dagger (I_m + H(\Delta X) \tilde{H}(X)^\dagger)^{-1} H(\Delta X), \\
H_2 &= (I_m + H(X)^\dagger \delta H)^{-1} = I_m - H(X)^\dagger (I_m + \delta H H(X)^\dagger)^{-1} \delta H,
\end{aligned}$$

with

$$\begin{aligned}
G_1 &= -(A^* X \Delta A + \Delta A^* X A + \Delta Q + \sum_{i=1}^{N} (A_0^{i*} X \Delta A_0^i + \Delta A_0^{i*} X A_0^i) \\
&+ G(X) H(X)^\dagger \delta H_1 H(X)^\dagger G(X)^* - \delta G_1 H(X)^\dagger G(X)^* \\
&- G(X) H(X)^\dagger \delta G_1^*), \\
G_2 &= -(\Delta A^* X \Delta A + \sum_{i=1}^{N} \Delta A_0^{i*} X \Delta A_0^i + G(X) H(X)^\dagger \delta H_2 H(X)^\dagger G(X)^* \\
&- G(X) H(X)^\dagger \delta H H(X)^\dagger H_2^* \delta H H(X)^\dagger G(X)^* - \delta G_2 H(X)^\dagger G(X)^* \\
&+ \delta G H(X)^\dagger \delta H H_2 H(X)^\dagger G(X)^* - G(X) H(X)^\dagger \delta G_2^* \\
&+ G(X) H(X)^\dagger \delta H H_2 H(X)^\dagger \delta G^* - \delta G H(X)^\dagger \delta G^* \\
&+ \delta G H(X)^\dagger \delta H H_2 H(X)^\dagger \delta G^*), \\
f_2(\Delta X) &= -(-\tilde{G}(X) \tilde{H}(X)^\dagger H(\Delta X) \tilde{H}(X)^\dagger H_1^* H(\Delta X) \tilde{H}(X)^\dagger \tilde{G}(X)^* \\
&+ \tilde{G}(X) \tilde{H}(X)^\dagger H(\Delta X) H_1 \tilde{H}(X)^\dagger G(\Delta X)^* \\
&+ G(\Delta X) \tilde{H}(X)^\dagger H(\Delta X) H_1 \tilde{H}(X)^\dagger \tilde{G}(X)^*
\end{aligned}$$

$$- \quad G(\Delta X)\tilde{H}(\tilde{X})^{\dagger}G(\Delta X)^{*}).$$

We rewrite (8) into

$$\text{(9)} \qquad \tilde{\Phi}_D^{*}\Delta X\tilde{\Phi}_D + \sum_{i=1}^{N}\tilde{\Psi}_D^{i*}\Delta X\tilde{\Psi}_D^{i} - \Delta X = G_1 + G_2 + f_2(\Delta X),$$

where

$$\begin{aligned}
\tilde{\Phi}_D &= (A+\Delta A) - (B+\Delta B)(H(X)^{\dagger} - H(X)^{\dagger}\delta H H_2 H(X)^{\dagger}) \\
&\quad \times (G(X)+\delta G)^{*} \\
&= \Phi_D + \Delta\Phi_D, \\
\tilde{\Psi}_D^{i} &= (A_0^i + \Delta A_0^i) - (B_0^i + \Delta B_0^i)(H(X)^{\dagger} - H(X)^{\dagger}\delta H H_2 H(X)^{\dagger}) \\
&\quad \times (G(X)+\delta G)^{*} \\
&= \Psi_D^i + \Delta\Psi_D^i,
\end{aligned}$$

with

$$\begin{aligned}
\Delta\Phi_D &= \Delta A + BH(X)^{\dagger}\delta H H_2 H(X)^{\dagger}G(X)^{*} - BH(X)^{\dagger}\delta G^{*} \\
&+ BH(X)^{\dagger}\delta H H_2 H(X)^{\dagger}\delta G^{*} - \Delta BH(X)^{\dagger}G(X)^{*} \\
&+ \Delta BH(X)^{\dagger}\delta H H_2 H(X)^{\dagger}G(X)^{*} - \Delta BH(X)^{\dagger}\delta G^{*} \\
&+ \Delta BH(X)^{\dagger}\delta H H_2 H(X)^{\dagger}\delta G^{*}, \\
\Delta\Psi_D^i &= \Delta A_0^i + B_0^i H(X)^{\dagger}\delta H H_2 H(X)^{\dagger}G(X)^{*} - B_0^i H(X)^{\dagger}\delta G^{*} \\
&+ B_0^i H(X)^{\dagger}\delta H H_2 H(X)^{\dagger}\delta G^{*} - \Delta B_0^i H(X)^{\dagger}G(X)^{*} \\
&+ \Delta B_0^i H(X)^{\dagger}\delta H H_2 H(X)^{\dagger}G(X)^{*} - \Delta B_0^i H(X)^{\dagger}\delta G^{*} \\
&+ \Delta B_0^i H(X)^{\dagger}\delta H H_2 H(X)^{\dagger}\delta G^{*}.
\end{aligned}$$

We express the left-hand side of (9) using $\Delta\Phi_D$ and $\Delta\Psi_D^i$ as follows

$$\begin{aligned}
\text{(10)} \qquad & \tilde{\Phi}_D^{*}\Delta X\tilde{\Phi}_D + \sum_{i=1}^{N}\tilde{\Psi}_D^{i*}\Delta X\tilde{\Psi}_D^{i} - \Delta X \\
&= \Phi_D^{*}\Delta X\Phi_D + \sum_{i=1}^{N}\Psi_D^{i*}\Delta X\Psi_D^{i} - \Delta X - f_1(\Delta X),
\end{aligned}$$

with

$$f_1(\Delta X) = -(\Phi_D^{*}\Delta X\Delta\Phi_D + \Delta\Phi_D^{*}\Delta X\Phi_D + \Delta\Phi_D^{*}\Delta X\Delta\Phi_D$$

$$+ \sum_{i=1}^{N}(\Psi_D^{i*}\Delta X \Delta\Psi_D^i + \Delta\Psi_D^{i*}\Delta X \Psi_D^i + \Delta\Psi_D^{i*}\Delta X \Delta\Psi_D^i)).$$

Combining (9) and (10), the perturbed DRRE can be represented

$$\Phi_D^*\Delta X \Phi_D + \sum_{i=1}^{N}\Psi_D^{i*}\Delta X \Psi_D^i - \Delta X = G_1 + G_2 + f_1(\Delta X) + f_2(\Delta X).$$

Lemma 3.1. *Let X be the stabilizing solution of the DRRE (1) and $\tilde{X}$ be a symmetric solution of the perturbed DRRE (7), then ΔX satisfies the equation*

$$\Phi_D^*\Delta X \Phi_D + \sum_{i=1}^{N}\Psi_D^{i*}\Delta X \Psi_D^i - \Delta X = G_1 + G_2 + f_1(\Delta X) + f_2(\Delta X), \tag{11}$$

where

$$\begin{aligned}
G_1 &= -(A^*X\Delta A + \Delta A^*XA + \Delta Q + \sum_{i=1}^{N}(A_0^{i*}X\Delta A_0^i + \Delta A_0^{i*}XA_0^i) \\
&\quad + G(X)H(X)^\dagger\delta H_1 H(X)^\dagger G(X)^* - \delta G_1 H(X)^\dagger G(X)^* \\
&\quad - G(X)H(X)^\dagger \delta G_1^*), \\
G_2 &= -(\Delta A^*X\Delta A + \sum_{i=1}^{N}\Delta A_0^{i*}X\Delta A_0^i + G(X)H(X)^\dagger\delta H_2 H(X)^\dagger G(X)^* \\
&\quad - G(X)H(X)^\dagger\delta H H(X)^\dagger H_2^*\delta H H(X)^\dagger G(X)^* - \delta G_2 H(X)^\dagger G(X)^* \\
&\quad + \delta G H(X)^\dagger \delta H H_2 H(X)^\dagger G(X)^* - G(X)H(X)^\dagger\delta G_2^* \\
&\quad + G(X)H(X)^\dagger\delta H H_2 H(X)^\dagger\delta G^* - \delta G H(X)^\dagger \delta G^* \\
&\quad + \delta G H(X)^\dagger\delta H H_2 H(X)^\dagger \delta G^*), \\
f_1(\Delta X) &= -(\Phi_D^*\Delta X\Delta\Phi_D + \Delta\Phi_D^*\Delta X\Phi_D + \Delta\Phi_D^*\Delta X\Delta\Phi_D \\
&\quad + \sum_{i=1}^{N}(\Psi_D^{i*}\Delta X\Delta\Psi_D^i + \Delta\Psi_D^{i*}\Delta X\Psi_D^i + \Delta\Psi_D^{i*}\Delta X\Delta\Psi_D^i)), \\
f_2(\Delta X) &= -(-\tilde{G}(X)\tilde{H}(X)^\dagger H(\Delta X)\tilde{H}(X)^\dagger H_1^* H(\Delta X)\tilde{H}(X)^\dagger\tilde{G}(X)^* \\
&\quad + \tilde{G}(X)\tilde{H}(X)^\dagger H(\Delta X) H_1\tilde{H}(X)^\dagger G(\Delta X)^* \\
&\quad + G(\Delta X)\tilde{H}(X)^\dagger H(\Delta X)H_1\tilde{H}(X)^\dagger\tilde{G}(X)^* \\
&\quad - G(\Delta X)\tilde{H}(\tilde{X})^\dagger G(\Delta X)^*).
\end{aligned}$$

G_1 and G_2 are first-order and high-order perturbation matrices respectively, $f_1(\Delta X)$ and $f_2(\Delta X)$ are the linear and high-order degree functions of ΔX, respectively.

3.1. Perturbation equation

We solve the equation (11) before discussing the condition numbers of DRREs (1). The following lemma describes the property of the nonsingularity of S_d defined in (12).

Lemma 3.2 (Freiling and Hochhaus [15])**.** *Given a matrix $A \in \mathbb{R}^{n\times n}$ and consider linear operators $\mathcal{S}_A$, $\Pi : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, where $\mathcal{S}_A$ is the discrete-time Lyapunov operator and Π is positive. The following statements are equivalent*

- *(a) All eigenvalues of A lie in the open unit disk and $\rho(-\mathcal{S}_A^{-1}\Pi) < 1$.*
- *(b) $-(\mathcal{S}_A + \Pi)$ is inverse positive.*
- *(c) There is some $X > 0$ such that $(\mathcal{S}_A + \Pi)(X) < 0$.*
- *(d) If $T_{X_k} > 0$, then (2) has a unique solution $X > 0$.*
- *(e) $\mathcal{S}_A + \Pi + \mathcal{I}$ is d-stable.*

Assume that X is the maximal and stabilizing solution to the DRREs (1), the eigenvalues of Φ_D and $S_d + \mathcal{I}$ are located in the open unit disk such that $\sigma(\Phi_D) \subset \mathbb{D}$ and $\sigma(S_d + \mathcal{I}) \subset \mathbb{D}$. The definitions of $\mathcal{S}_{\Phi_D}$ and Π_X suggest that $\mathcal{S}_{\Phi_D}(X) = \Phi_D^* X \Phi_D - X$ is the discrete-time Lyapunov operator and $\Pi_X(X) = \sum_{i=1}^{N} \Psi_D^{i*} X \Psi_D^i$ is positive. Then

$$S_d \equiv \mathcal{S}_{\Phi_D} + \Pi_X \tag{12}$$

is invertible such that S_d^{-1} exists, which is concluded by applying the Lemma 3.2. Next, we define a function $g(\Delta X)$:

$$g(\Delta X) = S_d^{-1} G_1 + S_d^{-1} G_2 + S_d^{-1} f_1(\Delta X) + S_d^{-1} f_2(\Delta X).$$

It can be regarded as a continuous mapping $g : \mathcal{S}^{n\times n} \to \mathcal{S}^{n\times n}$ and the fixed points of the mapping g are the solutions to the perturbed DRRE (11). We prove about the existence and uniqueness conditions of the fixed point, and get the existence and uniqueness of the stabilizing solution to the perturbed DRREs (7). For the details of proofs about the existence and uniqueness of the stabilizing solution to the perturbed DRREs (7) are referred in S1 and S2 in the Supplementary Material.

3.2. Normwise condition number

Consider the normwise condition number of DRRE (1), we remove high-order terms in (11)

$$
\begin{aligned}
(13)\qquad & \Phi_D^* \Delta X \Phi_D + \sum_{i=1}^{N} \Psi_D^{i*} \Delta X \Psi_D^i - \Delta X \\
& = -A^* X \Delta A - \Delta A^* X A - \Delta Q \\
& - \sum_{i=1}^{N} (A_0^{i*} X \Delta A_0^i + \Delta A_0^{i*} X A_0^i) \\
& - G(X) H(X)^\dagger \Delta R H(X)^\dagger G(X)^* \\
& - G(X) H(X)^\dagger B^* X \Delta B H(X)^\dagger G(X)^* \\
& - G(X) H(X)^\dagger \Delta B^* X B H(X)^\dagger G(X)^* \\
& - G(X) H(X)^\dagger \sum_{i=1}^{N} (B_0^{i*} X \Delta B_0^i \\
& + \Delta B_0^{i*} X B_0^i) H(X)^\dagger G(X)^* \\
& + \Delta L H(X)^\dagger G(X)^* \\
& + A^* X \Delta B H(X)^\dagger G(X)^* \\
& + \Delta A^* X B H(X)^\dagger G(X)^* \\
& + \sum_{i=1}^{N} (A_0^{i*} X \Delta B_0^i + \Delta A_0^{i*} X B_0^i) H(X)^\dagger G(X)^* \\
& + G(X) H(X)^\dagger \Delta L^* + G(X) H(X)^\dagger \Delta B^* X A \\
& + G(X) H(X)^\dagger B^* X \Delta A \\
& + G(X) H(X)^\dagger \sum_{i=1}^{N} (\Delta B_0^{i*} X A_0^i \\
& + B_0^{i*} X \Delta A_0^i).
\end{aligned}
$$

We recall some notations about vectorization. For a matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, we define $\mathrm{vec}(A) = [a_1^\top, a_2^\top, \ldots, a_n^\top]^\top \in \mathbb{R}^{n^2}$, where $A = [a_1, a_2, \ldots, a_n]$ with $a_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, n$. Some useful properties of Kronecker products found in [18] are listed below:

$$\|\mathrm{vec}(A)\|_2 = \|A\|_F,$$

$$\begin{array}{rcl} \|\text{vec}(A)\|_\infty & = & \|A\|_{\max}, \\ \text{vec}(AXB) & = & (B^\top \otimes A)\text{vec}(X), \end{array} \tag{14}$$

where $\|A\|_{\max} = \max_{i,j} |a_{ij}|$, for $i, j = 1, 2, \ldots, n$ and B, $X \in \mathbb{R}^{n\times n}$. We apply the property of Kronecker product (14) to discuss the condition numbers of DRRE (1) and derive their upper bounds under three kinds of perturbation analyses.

Set

$$\begin{array}{c} A^* X \equiv a_d,\ A_0^{j*} X \equiv a_d^j,\ G(X)H(X)^\dagger \equiv g_h, \\ G(X)H(X)^\dagger B^* X \equiv g_b,\ G(X)H(X)^\dagger B_0^{j*} X \equiv g_b^j, \end{array} \tag{15}$$

we apply the operator "vec" to both sides of the equation (13) and reset the index i into j such as A_0^j and B_0^j, then obtain

$$M\text{vec}(\Delta X) = Nr, \tag{16}$$

where

$$\begin{array}{rcl} M & = & \Phi_D^* \otimes \Phi_D^* + \sum_{i=1}^{N} \Psi_D^{i*} \otimes \Psi_D^{i*} - I_n \otimes I_n, \\ N & = & [N_1, N_2], N_2 = [N_2^1, N_2^2, \ldots, N_2^j], \\ r & = & [r_1, r_2]^\top, r_2 = [r_2^1, r_2^2, \ldots, r_2^j], \end{array}$$

with

$$\begin{aligned} N_1 = [&-(I_n \otimes a_d), -(a_d \otimes I_n), -(I_n \otimes I_n), -(g_h \otimes g_h), -(g_h \otimes g_b), \\ &- (g_b \otimes g_h), (g_h \otimes I_n), (g_h \otimes a_d), (g_b \otimes I_n), (I_n \otimes g_h), (a_d \otimes g_h), \\ &(I_n \otimes g_b)], \\ N_2^j = [&-(I_n \otimes a_d^j), -(a_d^j \otimes I_n), -(g_h \otimes g_b^j), -(g_b^j \otimes g_h), (g_h \otimes a_d^j), (g_b^j \otimes I_n), \\ &(a_d^j \otimes g_h), (I_n \otimes g_b^j)], \\ r_1 = [&\text{vec}(\Delta A)^\top, \text{vec}(\Delta A^*)^\top, \text{vec}(\Delta Q)^\top, \text{vec}(\Delta R)^\top, \text{vec}(\Delta B)^\top, \text{vec}(\Delta B^*)^\top, \\ &\text{vec}(\Delta L)^\top, \text{vec}(\Delta B)^\top, \text{vec}(\Delta A^*)^\top, \text{vec}(\Delta L^*)^\top, \text{vec}(\Delta B^*)^\top, \\ &\text{vec}(\Delta A)^\top], \\ r_2^j = [&\text{vec}(\Delta A_0^j)^\top, \text{vec}(\Delta A_0^{j*})^\top, \text{vec}(\Delta B_0^j)^\top, \text{vec}(\Delta B_0^{j*})^\top, \text{vec}(\Delta B_0^j)^\top, \\ &\text{vec}(\Delta A_0^{j*})^\top, \text{vec}(\Delta B_0^{j*})^\top, \text{vec}(\Delta A_0^j)^\top], \end{aligned}$$

for $j = 1, 2, \ldots, N$. Before we solve the equation (16) to derive the normwise condition number and its perturbation bound of DRRE (1), we discuss the property of the nonsingularity of the equation (16). Assume that X is the maximal and stabilizing solution of DRREs (1), we can conclude that the eigenvalues of $S_d + \mathcal{I}$ are located in the open unit disk s.t. $\sigma(S_d + \mathcal{I}) \subset \mathbb{D}$. The following statement has provided

$$(S_d + \mathcal{I})(I_n) = \Phi_D^* \Phi_D + \sum_{i=1}^{N} \Psi_D^{i*} \Psi_D^i \text{ and } \sigma(\Phi_D^* \Phi_D + \sum_{i=1}^{N} \Psi_D^{i*} \Psi_D^i) \subset \mathbb{D}.$$

By using the Lemma 3.2, we then obtain the following statement

$$M \text{ is nonsingular s.t. } \det(M) \neq 0.$$

Therefore, we can solve the equation (16)

$$\operatorname{vec}(\Delta X) = M^{-1} N r. \tag{17}$$

Define a mapping

$$\omega : (A, B, Q, L, R, A_0^j, B_0^j) \mapsto \operatorname{vec}(X),$$

where X is the stabilizing solution of DRRE (1). We discuss the normwise condition number defined by

$$n(\omega, v) = \lim_{\epsilon \to 0} \sup_{\mathcal{E}} \frac{\|\operatorname{vec}(\Delta X)\|}{\delta(v + \Delta v, v)\|X\|_F}, \tag{18}$$

where

$$\begin{aligned} \delta(v + \Delta v, v) &= \max_{\substack{i=1,2,\ldots,p \\ v_i \neq 0}} \left\{ \frac{|\Delta v_i|}{|v_i|} \right\}, \\ \mathcal{E} &= \{\Delta E | \|\Delta E\|_F \leq \epsilon \|E\|_F, E : A, B, Q, L, R, A_0^j, B_0^j, \epsilon > 0\}, \\ v &\equiv [v_i] \equiv (v_1, v_2)^\top, v_2 = [v_2^1, v_2^2, \ldots, v_2^j], \text{for } j = 1, 2, \ldots, N, \end{aligned}$$

with

$$\begin{aligned} v_1 &= (\operatorname{vec}(A)^\top, \operatorname{vec}(A^*)^\top, \operatorname{vec}(Q)^\top, \operatorname{vec}(R)^\top, \operatorname{vec}(B)^\top, \operatorname{vec}(B^*)^\top, \operatorname{vec}(L)^\top \\ &, \operatorname{vec}(B)^\top, \operatorname{vec}(A^*)^\top, \operatorname{vec}(L^*)^\top, \operatorname{vec}(B^*)^\top, \operatorname{vec}(A)^\top); \\ v_2^j &= (\operatorname{vec}(A_0^j)^\top, \operatorname{vec}(A_0^{j*})^\top, \operatorname{vec}(B_0^j)^\top, \operatorname{vec}(B_0^{j*})^\top, \operatorname{vec}(B_0^j)^\top, \operatorname{vec}(A_0^{j*})^\top \\ &, \operatorname{vec}(B_0^{j*})^\top, \operatorname{vec}(A_0^j)^\top). \end{aligned}$$

Theorem 3.3. *By using the notation given, the explicit expression and its upper bound of the normwise condition number of DRRE (1) are*

$$\begin{aligned} n(\omega, v) &= \|m\|/\|X\|_F, \\ n_v(\omega) &= \|M^{-1}\|n_1/\|X\|_F, \end{aligned} \tag{19}$$

where

$$\begin{aligned} m &= |M^{-1}(I_n \otimes a_d)|vec(|A|) + |M^{-1}(a_d \otimes I_n)|vec(|A^*|) \\ &+ |M^{-1}(I_n \otimes I_n)|vec(|Q|) + |M^{-1}(g_h \otimes g_h)|vec(|R|) \\ &+ |M^{-1}(g_h \otimes g_b)|vec(|B|) + |M^{-1}(g_b \otimes g_h)|vec(|B^*|) \\ &+ |M^{-1}(g_h \otimes I_n)|vec(|L|) + |M^{-1}(g_h \otimes a_d)|vec(|B|) \\ &+ |M^{-1}(g_b \otimes I_n)|vec(|A^*|) + |M^{-1}(I_n \otimes g_h)|vec(|L^*|) \\ &+ |M^{-1}(a_d \otimes g_h)|vec(|B^*|) + |M^{-1}(I_n \otimes g_b)|vec(|A|) \\ &+ \sum_{j=1}^{N}(|M^{-1}(I_n \otimes a_d^j)|vec(|A_0^j|) + |M^{-1}(a_d^j \otimes I_n)|vec(|A_0^{j*}|) \\ &+ |M^{-1}(g_h \otimes g_b^j)|vec(|B_0^j|) + |M^{-1}(g_b^j \otimes g_h)|vec(|B_0^{j*}|) \\ &+ |M^{-1}(g_h \otimes a_d^j)|vec(|B_0^j|) + |M^{-1}(g_b^j \otimes I_n)|vec(|A_0^{j*}|) \\ &+ |M^{-1}(a_d^j \otimes g_h)|vec(|B_0^{j*}|) + |M^{-1}(I_n \otimes g_b^j)|vec(|A_0^j|)); \\ n_1 &= 2\|a_{dx}\|_F\|A\|_F + \|Q\|_F + \|g_{hd}\|_F^2\|R\|_F + 2\|g_{bd}\|_F\|B\|_F\|g_{hd}\|_F \\ &+ 2\|L\|_F\|g_{hd}\|_F + 2\|a_{dx}\|_F\|B\|_F\|g_{hd}\|_F + 2\|A\|_F\|g_{bd}\|_F \\ &+ \sum_{j=1}^{N}(2\|a_{dx}^j\|_F\|A_0^j\|_F + 2\|g_{bd}^j\|_F\|B_0^j\|_F\|g_{hd}\|_F + 2\|a_{dx}^j\|_F\|B_0^j\|_F\|g_{hd}\|_F \\ &+ 2\|A_0^j\|_F\|g_{bd}^j\|_F). \end{aligned}$$

Proof. By applying the property of the absolute value to (15), we can obtain

$$\begin{aligned} &|a_d| \le |A^*||X| \equiv a_{dx},\ |a_d^j| \le |A_0^{j*}||X| \equiv a_{dx}^j,\ |g_h| \le |G(X)||H(X)^\dagger| \equiv g_{hd}, \\ &|g_b| \le |G(X)||H(X)^\dagger||B^*||X| \equiv g_{bd},\ |g_b^j| \le |G(X)||H(X)^\dagger||B_0^{j*}||X| \le g_{bd}^j. \end{aligned}$$

From the formula of the normwise condition number (18), we can simplify it into

$$n(\omega, v) = \frac{\||M^{-1}N||v|\|}{\|X\|_F};$$

$$
\begin{aligned}
&= \| |M^{-1}(I_n \otimes a_d)|\mathrm{vec}(|A|) + |M^{-1}(a_d \otimes I_n)|\mathrm{vec}(|A^*|) \\
&+ |M^{-1}(I_n \otimes I_n)|\mathrm{vec}(|Q|) + |M^{-1}(g_h \otimes g_h)|\mathrm{vec}(|R|) \\
&+ |M^{-1}(g_h \otimes g_b)|\mathrm{vec}(|B|) + |M^{-1}(g_b \otimes g_h)|\mathrm{vec}(|B^*|) \\
&+ |M^{-1}(g_h \otimes I_n)|\mathrm{vec}(|L|) + |M^{-1}(g_h \otimes a_d)|\mathrm{vec}(|B|) \\
&+ |M^{-1}(g_b \otimes I_n)|\mathrm{vec}(|A^*|) + |M^{-1}(I_n \otimes g_h)|\mathrm{vec}(|L^*|) \\
&+ |M^{-1}(a_d \otimes g_h)|\mathrm{vec}(|B^*|) + |M^{-1}(I_n \otimes g_b)|\mathrm{vec}(|A|) \\
&+ \sum_{j=1}^{N}(|M^{-1}(I_n \otimes a_d^j)|\mathrm{vec}(|A_0^j|) + |M^{-1}(a_d^j \otimes I_n)|\mathrm{vec}(|A_0^{j*}|) \\
&+ |M^{-1}(g_h \otimes g_b^j)|\mathrm{vec}(|B_0^j|) + |M^{-1}(g_b^j \otimes g_h)|\mathrm{vec}(|B_0^{j*}|) \\
&+ |M^{-1}(g_h \otimes a_d^j)|\mathrm{vec}(|B_0^j|) + |M^{-1}(g_b^j \otimes I_n)|\mathrm{vec}(|A_0^{j*}|) \\
&+ |M^{-1}(a_d^j \otimes g_h)|\mathrm{vec}(|B_0^{j*}|) + |M^{-1}(I_n \otimes g_b^j)|\mathrm{vec}(|A_0^j|))\|/\|X\|_F; \\
&\equiv \|m\|/\|X\|_F; \\
&\leq \|M^{-1}\| \| |a_d||A| + |A^*||a_d^*| + |Q| + |g_h||R||g_h^*| + |g_b||B||g_h^*| \\
&+ |g_h||B^*||g_b^*| + |L||g_h^*| + |a_d||B||g_h^*| + |A^*||g_b^*| + |g_h||L^*| \\
&+ |g_h||B^*||a_d^*| \\
&+ |g_b||A| + \sum_{j=1}^{N}(|a_d^j||A_0^j| + |A_0^{j*}||a_d^{j*}| + |g_b^j||B_0^j||g_h^*| + |g_h||B_0^{j*}||g_b^{j*}| \\
&+ |a_d^j||B_0^j||g_h^*| + |A_0^{j*}||g_b^{j*}| + |g_h||B_0^{j*}||a_d^{j*}| + |g_b^j||A_0^j|)\|_F/\|X\|_F; \\
&\leq \|M^{-1}\|(2\|a_{dx}\|_F\|A\|_F + \|Q\|_F + \|g_{hd}\|_F^2\|R\|_F \\
&+ 2\|g_{bd}\|_F\|B\|_F\|g_{hd}\|_F \\
&+ 2\|L\|_F\|g_{hd}\|_F + 2\|a_{dx}\|_F\|B\|_F\|g_{hd}\|_F + 2\|A\|_F\|g_{bd}\|_F \\
&+ \sum_{j=1}^{N}(2\|a_{dx}^j\|_F\|A_0^j\|_F + 2\|g_{bd}^j\|_F\|B_0^j\|_F\|g_{hd}\|_F \\
&+ 2\|a_{dx}^j\|_F\|B_0^j\|_F\|g_{hd}\|_F + 2\|A_0^j\|_F\|g_{bd}^j\|_F))/\|X\|_F; \\
&\equiv \|M^{-1}\|n_1/\|X\|_F.
\end{aligned}
$$

□

3.3. Mixed and componentwise condition numbers

By considering the matrix structure, we introduce a lemma about mixed and componentwise perturbation anlayses to compute condition numbers of DRREs (1) efficiently.

Lemma 3.4 (Gohberg and Koltracht [17]). *Let $F : \mathbb{R}^p \to \mathbb{R}^q$ be a continuous map defined on an open set $Dom(F) \subset \mathbb{R}^p$. For a given $a \neq 0 \in Dom(F)$, such that $F(a) \neq 0$, where $Dom(F)$ denotes the domain of the function F. Let $B^0(a,\epsilon) = \{x : |x_i - a_i| \le \epsilon |a_i|, i = 1,2,\dots,p\}$, where $a = (a_1, a_2, \dots, a_p)^\top, x = (x_1, x_2, \dots, x_p)^\top \in \mathbb{R}^p$ and $\epsilon > 0$.*

(a) The mixed condition number of the map F at the point a is defined

$$m(F,a) = \lim_{\epsilon \to 0} \sup_{\substack{x \in B^0(a,\epsilon) \\ x \neq a}} \frac{\|F(x) - F(a)\|_\infty}{\|F(a)\|_\infty} \frac{1}{\delta(x,a)},$$

where $\delta(x,a) = \max_{\substack{i=1,2,\dots,p \\ a_i \neq 0}} \left\{ \frac{|x_i - a_i|}{|a_i|} \right\}$.

(b) Suppose $F(a) = (f_1(a), f_2(a), \dots, f_q(a))$ such that $f_j(a) \neq 0$, for $j = 1,2,\dots,q$. The componentwise condition number of F at the point a is

$$c(F,a) = \lim_{\epsilon \to 0} \sup_{\substack{x \in B^0(a,\epsilon) \\ x \neq a}} \frac{\delta(F(x), F(a))}{\delta(x,a)},$$

where $\delta(F(x), F(a)) = \max_{j=1,2,\dots,q} \left\{ \frac{|f_j(x) - f_j(a)|}{|f_j(a)|} \right\}$.

Theorem 3.5. *Based on Lemma 3.4 and the definitions of ω and v, we express the mixed and componentwise condition numbers of DRRE (1) as*

$$\begin{aligned} m(\omega, v) &= \|m\|_\infty / \|X\|_{\max}, \\ c(\omega, v) &= \|m / vec(X)\|_\infty, \end{aligned}$$

and their upper bounds

$$m_v(\omega) = \|M^{-1}\|_\infty n_2 / \|X\|_{\max}, \tag{20}$$

$$c_v(\omega) = \|Diag^{-1}(vec(|X|)) M^{-1}\|_\infty n_2, \tag{21}$$

where

$$\begin{aligned} n_2 &= 2\|a_{dx}\|_{\max}\|A\|_{\max} + \|Q\|_{\max} + \|g_{hd}\|_{\max}^2 \|R\|_{\max} \\ &+ 2\|g_{bd}\|_{\max}\|B\|_{\max}\|g_{hd}\|_{\max} + 2\|L\|_{\max}\|g_{hd}\|_{\max} \\ &+ 2\|a_{dx}\|_{\max}\|B\|_{\max}\|g_{hd}\|_{\max} + 2\|A\|_{\max}\|g_{bd}\|_{\max} \\ &+ \sum_{j=1}^{N} (2\|a_{dx}^j\|_{\max}\|A_0^j\|_{\max} + 2\|g_{bd}^j\|_{\max}\|B_0^j\|_{\max}\|g_{hd}\|_{\max} \end{aligned}$$

$$+ \quad 2\|a^j_{dx}\|_{\max}\|B^j_0\|_{\max}\|g_{hd}\|_{\max} + 2\|A^j_0\|_{\max}\|g^j_{bd}\|_{\max}).$$

If $x = (x_1, x_2, \ldots, x_p)^\top \in \mathbb{R}^p$, then Diag$(x)$ denotes the $p\times p$ diagonal matrix with $x_1, x_2, \ldots, x_p$ on its diagonal.

Proof. Let $x \equiv v + \Delta v$, $a \equiv v$ and $F \equiv \omega$. Based on (a) of Lemma 3.4, the definition of the mapping ω and the properties of Kronecker products (14), we can obtain

$$\begin{aligned} \frac{\|F(x) - F(a)\|_\infty}{\|F(a)\|_\infty} &= \frac{\|(\omega(v + \Delta v) - \omega(v)\|_\infty}{\|\omega(v)\|_\infty} \\ &= \frac{\|\mathrm{vec}(X + \Delta X) - \mathrm{vec}(X)\|_\infty}{\|\mathrm{vec}(X)\|_\infty} \\ &= \frac{\|\mathrm{vec}(\Delta X)\|_\infty}{\|\mathrm{vec}(X)\|_\infty} = \frac{\|\mathrm{vec}(\Delta X)\|_\infty}{\|X\|_{\max}}. \end{aligned}$$

Next, we rewrite the formula of mixed condition number of DRRE (1) into

$$m(\omega, v) = \lim_{\epsilon\to 0} \sup_{\mathcal{F}} \frac{\|\mathrm{vec}(\Delta X)\|_\infty}{\delta(v + \Delta v, v)\|X\|_{\max}}, \tag{22}$$

where $\mathcal{F} = \{\Delta F||\Delta F| \le \epsilon|F|, F : A, B, Q, L, R, A^j_0, B^j_0, \epsilon > 0\}$. Similarly, the formula of componentwise condition number of DRRE (1) can be rewritten into

$$c(\omega, v) = \lim_{\epsilon\to 0} \sup_{\mathcal{F}} \frac{1}{\delta(v + \Delta v, v)} \left\| \frac{\mathrm{vec}(\Delta X)}{\mathrm{vec}(X)} \right\|_\infty. \tag{23}$$

Using (17) and (22), we can derive the explicit expression and its perturbation bound of the mixed condition number of DRRE (1)

$$\begin{aligned} m(\omega, v) &= \frac{\||M^{-1}N||v|\|_\infty}{\|X\|_{\max}} \\ &= \|m\|_\infty/\|X\|_{\max} \\ &\le \||M^{-1}||N||v|\|_\infty/\|X\|_{\max} \le \|M^{-1}\|_\infty\||N||v|\|_\infty/\|X\|_{\max} \\ &\le \|M^{-1}\|_\infty n_2/\|X\|_{\max}. \end{aligned}$$

Analogously, the explicit expression and its upper bound of the componentwise condition number can be obtained via (23)

$$c(\omega, v) = \left\| \frac{|M^{-1}N||v|}{\mathrm{vec}(X)} \right\|_\infty$$

$$
\begin{aligned}
&= \|m/\mathrm{vec}(X)\|_\infty \\
&\le \|\mathrm{Diag}^{-1}(\mathrm{vec}(|X|))M^{-1}\|_\infty \||N||v|\|_\infty \\
&\le \|\mathrm{Diag}^{-1}(\mathrm{vec}(|X|))M^{-1}\|_\infty n_2. \quad \square
\end{aligned}
$$

4. Numerical experiments

In this section, we have shown relative errors with respect to the solutions of DRREs (1) under normwise, mixed and componentwise perturbation analyses such as $\frac{\|\Delta X\|_F}{\|X\|_F}$, $\frac{\|\Delta X\|_{\max}}{\|X\|_{\max}}$ and $\left\|\frac{\Delta X}{X}\right\|_{\max}$ and their sharper upper bounds in (19), (20), (21). The numerical algorithm is described in Algorithm 1 for condition numbers of rational Riccati equations. We have chosen two representative examples:

(1) The DRREs in Example 4.1 are quoted from [2].
(2) The DRREs in Example 4.2 are quoted from [26].

All the numerical experiments were conducted using MATLAB [25] Version R2018a on a MacBook Pro with a 2.30 GHz Intel Core 8 Duo processor and 64 GB RAM, with machine accuracy $eps = 2.22 \times 10^{-16}$.

For the numerical results, we express r_n^{DRRE}, r_m^{DRRE}, r_c^{DRRE} for relative errors; n_v^{DRRE}, m_v^{DRRE}, c_v^{DRRE} for their upper bounds in the normwise, mixed and componentwise perturbation analyses.

Algorithm 1 (Condition numbers of rational Riccati equations)

Input: $A, Q, A_0^i \in \mathbb{K}^{n\times n}$, $B, L, B_0^i \in \mathbb{K}^{n\times m}$, and $R \in \mathbb{K}^{m\times m}$, for $i = 1, 2, \dots, N$;

Output: Normwise, mixed and componentwise condition numbers r_n^{DRRE}, r_m^{DRRE} and r_c^{DRRE}, upper bounds n_v^{DRRE}, m_v^{DRRE} and c_v^{DRRE};

Let ΔA, ΔQ, $\Delta A_0^i \in \mathbb{K}^{n\times n}$, ΔB, ΔL, $\Delta B_0^i \in \mathbb{K}^{n\times m}$, and $\Delta R \in \mathbb{K}^{m\times m}$ be selected from normal distribution and the weighted coefficients be 10^{-k} such as $\Delta E = 10^{-k} \times \mathrm{randn}(n, m)$, for $E \in \mathbb{K}^{n\times m}$;

Set $\tilde{A} = A + \Delta A$, $\tilde{Q} = Q + \Delta Q$, $\tilde{B} = B + \Delta B$, $\tilde{L} = L + \Delta L$, $\tilde{R} = R + \Delta R$, $\tilde{A}_0^i = A_0^i + \Delta A_0^i$ and $\tilde{B}_0^i = B_0^i + \Delta B_0^i$;

Solve the DRREs (1) and its perturbation equation (7) by the GSM and get the solutions X and $\tilde{X}$, respectively;

Compute the relative errors with respect to the solution X under the normwise, mixed and componentwise perturbation analyses, $r_n^{DRRE} = \frac{\|\tilde{X}-X\|_F}{\|X\|_F}$, $r_m^{DRRE} = \frac{\|\tilde{X}-X\|_{\max}}{\|X\|_{\max}}$, and $r_c^{DRRE} = \left\|\frac{\mathrm{vec}(\tilde{X}-X)}{\mathrm{vec}(X)}\right\|_\infty$, where $\frac{A}{B} = \frac{(a_{ij})}{(b_{ij})}$, with $A = (a_{ij})$ and $B = (b_{ij})$;

Estimate the upper bounds n_v^{DRRE}, m_v^{DRRE} and c_v^{DRRE} in (19), (20), and (21);

End Do

Example 1 (DRREs [2])

This example 1 is quoted from [2, Example 2] about a linear-quadratic control problem with the addition of stochastic disturbances in Π. Consider the DRREs (1) with $n = m = 2$ and $N = 1$,

$$A = \begin{bmatrix} 0.9512 & 0 \\ 0 & 0.9048 \end{bmatrix}, \quad B = \begin{bmatrix} 4.8770 & 4.8770 \\ -1.1895 & 3.5690 \end{bmatrix}, \quad R = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{bmatrix},$$

$$L = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.0028 & -0.0013 \\ -0.0013 & 0.0190 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.1 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 0.1 & 0.01 \\ 0.01 & 0.1 \end{bmatrix}.$$

X_0 is the initial solution of (2) to the DRREs (1) solved by NM. The perturbed coefficient matrices ΔA, ΔB, ΔR, ΔL, ΔQ, ΔA_0, ΔB_0 are generated by the multiplication of random matrix and weighted coefficient, and the random matrix is constructed by the normal distribution. Let $\tilde{A} = A + \Delta A$, $\tilde{B} = B + \Delta B$, $\tilde{R} = R + \Delta R$, $\tilde{L} = L + \Delta L$, $\tilde{Q} = Q + \Delta Q$, $\tilde{A}_0 = A_0 + \Delta A_0$, $\tilde{B}_0 = B_0 + \Delta B_0$ be the coefficient matrices of the perturbed DRREs (7) and the GSM is applied to the DRREs (1) and perturbed DRREs (7) respectively, then we can get the unique stabilizing solutions X and $\tilde{X}$. Therefore, the relative errors with respect to the unique stabilizing solution X, r_n^{DRRE}, r_m^{DRRE}, r_c^{DRRE} and their upper bounds n_v^{DRRE}, m_v^{DRRE}, c_v^{DRRE} under normwise, mixed and componentwise perturbation analyses are discussed, with different weighted coefficients 10^{-k}, for $k = 5, \dots, 10$.

Table 1: Example 4.1 (Condition Numbers of DRREs; $n = 2$, $m = 2$, $N = 1$)

k	r_n^{DRRE}	$\epsilon_n n_v^{DRRE}$	r_m^{DRRE}	$\epsilon_m m_v^{DRRE}$	r_c^{DRRE}	$\epsilon_m c_v^{DRRE}$
5	5.8085e−04	9.2735e−03	5.7952e−04	9.9097e−05	5.7952e−04	1.1999e−03
6	2.7954e−05	7.4664e−04	1.6658e−05	8.3417e−07	1.6658e−05	1.1000e−05
7	4.7849e−06	9.9172e−05	4.6919e−06	3.4406e−07	4.6919e−06	4.1658e−06
8	5.4381e−07	1.0133e−05	3.4325e−07	3.3215e−08	3.4325e−07	4.0216e−07
9	1.3981e−08	3.6270e−07	1.1938e−08	5.9963e−09	1.1938e−08	7.2602e−08
10	4.8175e−09	1.0262e−07	3.3346e−09	7.8087e−10	3.3346e−09	9.4546e−09

Table 1 shows that the relative errors under three kinds of perturbation analyses are closely related to their upper bounds such as $r_n^{DRRE} \lesssim \epsilon_n n_v^{DRRE}$, $r_m^{DRRE} \sim \epsilon_m m_v^{DRRE}$ and $r_c^{DRRE} \lesssim \epsilon_m c_v^{DRRE}$. As a result, perturbation bounds (19), (20) and (21) are tight.

Example 2 (DRREs [26])

The example 2 is quoted from [26, Example 3] with the addition of stochastic disturbances A_0 and B_0. Consider the DRREs (1) with $n = 100$, $m = 1$, $N = 1$,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & & & 0 \\ 0 & & & & & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad L = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

$$A_0 = 0.1 * I_{100}, \quad R = 1, \quad Q = I_{100}, \quad X_0 = I_{100}.$$

We generate one random sample (ΔA, ΔB, ΔR, ΔL, ΔQ, ΔA_0, ΔB_0) and obtain coefficient matrices of the perturbed DRREs (7), then the relative errors under the normwise, mixed and componentwise perturbation analysis are computed according to different weighted coefficients 10^{-k}, for $k = 9, 10, 11, 12, 13, 14$, respectively. Moreover, upper bounds under three kinds of perturbation analysis are following derived.

Table 2: Example 4.2 (Condition Numbers of DRREs; $n = 100$, $m = 1$, $N = 1$)

k	r_n^{DRRE}	$\epsilon_n n_v^{DRRE}$	r_m^{DRRE}	$\epsilon_m m_v^{DRRE}$	r_c^{DRRE}	$\epsilon_m c_v^{DRRE}$
9	7.2747e−08	2.0002e−04	1.8023e−08	1.7966e−07	1.8023e−08	1.7966e−07
10	6.2374e−09	2.0047e−05	1.7606e−09	3.3057e−08	1.7606e−09	3.3057e−08
11	6.1440e−10	1.9900e−06	1.8872e−10	6.8090e−10	1.8872e−10	6.8090e−10
12	7.1770e−11	2.0008e−07	1.6712e−11	4.1124e−10	1.6712e−11	4.1124e−10
13	7.5871e−12	2.0062e−08	1.8303e−12	1.6002e−11	1.8303e−12	1.6002e−11
14	6.6254e−13	2.0131e−09	1.7878e−13	4.0277e−12	1.7878e−13	4.0277e−12

It can be seen from Table 2 that the values of the relative errors are closely bounded by our perturbation bounds of (19), (20) and (21). In other words, (19), (20) and (21) provide a sharp upper bound of the relative errors of the stabilizing solution X. Table 2 shows that our estimates are tight.

5. Conclusions

In this paper, we discuss condition numbers of rational Riccati equations, in discrete-time systems. We provide efficient techniques to derive the relative errors and perturbation bounds under the normwise, mixed and componentwise perturbation analyses. Furthermore, some sufficient conditions for the existence and uniqueness of the stabilizing solution to the DRREs (1) and perturbed DRREs (7) are presented. We highlight and compare the practical performance of the derived condition numbers and perturbation bounds under the normwise, mixed and componentwise perturbation analyses through two numerical experiments. Our numerical results show that three kinds of perturbation bounds provide sharper bounds in small- and medium-sized problems about applications of linear-quadratic control problems [2] and [26], respectively. For the large-scale problem, we can also get the condition estimates and upper bounds by modifying our algorithm and only need to consider the numerically low-rank and sparsity structures in solving DRREs. This will be our future work. In summary, we introduce some efficient measurement tools for the perturbation analyses of DRREs (1).

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