Annals of Mathematical Sciences and Applications
Volume 6, Number 2, 121–144, 2021

Computational solution of fractional pantograph equation with varying delay term

M. Khalid*, S. K. Fareeha, and S. Mariam

Delay Differential Equations DDEs have great importance in real life phenomena. Among them is a special type of equation known as Pantograph Delay Differential Equation PDDE. Such kind of equations cannot be solved using ordinary methods, and hence, it becomes a challenge when the complexity increases, especially if one wants to study Fractional Pantograph Delay Differential Equation (FPDDE). In this work, FPDDEs with a general Delay term is solved numerically by an iteration method called Perturbation Iteration Algorithm (PIA). It is based on the Taylor series and eliminates the non-linear terms easily. Iterative results are discussed in detail in both tabular and graphical forms. A graphical interpretation of the variability of the Delay term is also provided for a deeper understanding of its range.

AMS 2000 subject classifications: Primary 34K37, 65L03; secondary 815xx, 97m50, 93C23.
Keywords and phrases: Perturbation Iteration Algorithm, Fractional Delay Differential Equation, Series Solution, Fractional Calculus.

1. Introduction

There is a class of differential equations known as Delay Differential Equations (DDEs), in which, for a certain time, the derivative of an unknown function is provided in the form of previous functional values. DDEs are important in applied form as:

- It is well understood that engineers want their prototypes and simulations to respond very much like the real processes, along with increasing demands of dynamic efficiency. Most systems have a delay effect phenomenon in their inner dynamics. Furthermore, such delays are introduced by actuators, communication networks, and sensors now involved in feedback control loops. Furthermore, in contrast to real

*Corresponding author.

delays, time lags are often used to simplify models of very high order. This way, interest in DDEs continues to grow in all scientific fields, particularly in control engineering.

- Delay systems are always sensitive to many basic controllers. If one might try to replace them with some finite-dimensional approximations, which would be the easiest solution, sadly, neglecting results represented adequately by DDEs is not a particularly acceptable solution. It contributes towards the same level of certainty in the control model in the best possible situation (even for constant and known delays). In the worst case scenario, it is potentially devastating in terms of stabilization and oscillations (for example, time-varying delays).
- Delay characteristics are, however, remarkable as numerous researches have shown that the insertion of delays on a voluntary basis can also benefit control.
- However, given their complexities, DDEs often emerge as simple arbitrary infinite-dimensional models in differential equations.

General form of a time-delaying differential equation is given as

$$\frac{d}{dt}u(t) = f(t, u(kt), u(t)) \quad \text{where } u(t) \in \Re^n$$

Among them is a special kind of delay differential equation known as the Pantograph Delay Differential equation

$$u'(t) = u(kt) \quad \text{where } 0 < k < 1$$

Due to the presence of k, even the linear form of pantograph equation cannot be solved by using the basic methods available for solving such differential equations. Also its initial value problems do not provide a particular solution. Existence and uniqueness of solutions for the initial value problems of the linear pantograph equation have notable differences for the varying choices of initial points.

Pantograph equations were first derived in 1940 as part of the Number Theory by Mahler [1]. But the term pantograph originates from Ockendon and Taylor's work on the existing range using the electric locomotive's catenary system. Their work was to assess the motion of a pantograph head on an electric locomotive that receives current from a trolley overhead wire by forming an equation [6]. Later, Kato & Mcleod [5] discussed the asymptotic properties and stability of solution for the pantograph equation of the type $u'(t) = au(kt) + bu(t)$. Later, Hale and Bellman & Cooke wrote two remarkable and comprehensive informative texts referring to the theories of these DDEs, which included existence and uniqueness of their solutions, continuous dependence, continuity of their solutions, differentiability of their solutions, backward continuation, caratheodory conditions, asymptotic behaviour and stability of solutions [9, 8]. Some other researchers namely Derfel [10], Kuang & Feldstein [11] and Morris [12] studied pantograph equation for its stability properties, existence, and the uniqueness, continuous dependence, continuity, differentiability of its solutions in complex domain $\mathbb{C}$. Later, Iserles [7] was the one who presented the pantograph equation in generalized form i.e

$$u'(t) = au(kt) + bu(t) + cu'(kt)$$

and analyzed that there was no comprehensive explanation for the behaviour of u on the stability boundary. Iserles also established that u was almost periodic along a significant portion of that boundary, and if k was kept rational, it was almost rotationally symmetrical.

This fascinating equation has a broad range of applications in the physical world and various fields of science as well. Some of these examples are the graph theory [21], light absorption by interstellar structures [13], population dynamics [20], analytic number theory [1], stochastic games [19], non linear dynamical systems [14], queues and risk theory [18], probability theory of algebraic structures [15], the theory of dielectric materials [17] and continuum mechanics [16]. For such widely used DDEs, solving them analytically is not always easy because of its delay term therefore numerical methods have been applied to evaluate DDEs behaviour.

Wanjin & Hui [22] solved pantograph equation by utilizing higher-order derivative methods with variable step size. Another researcher in [23] provided optimal tests of local super convergence for the differential pantograph delay equation as special case of their general analysis. Numerical methods that have been used to find the numerical solution of pantograph equation are Differential Transform Method [26], Adomian Decomposition Method [27], Homotopy Perturbation Method [28], Taylor Polynomials Method [29], Bessel Functions Method [30], quadrature and interpolation procedures [24], pseudo-spectral methods [25] etc. Another work was published in which Perturbation Iteration Algorithm (PIA) was utilized for analyzing the numerical solution of pantograph equation [2] and their results were approximately better when compared with other numerical methods. PIA has already proven to be, among other contemporary numerical methods, the easiest and uncomplicated numerical method. It can numerically solve any kind of equation and it provides detailed and accurate results, for details, see [31, 32, 33, 34, 35, 36]. However, all of the aforementioned methods have only been used to solve the pantograph equation. Fractional pantograph equation becomes more abstract and therefore requires enormous effort and uncertainty in calculations.

Although the fractional calculus theory is as old as the integer calculus, researchers are becoming more interested in it because of its comprehensive and in-depth understanding of the real-world phenomena represented by mathematical models. Fractional calculus can indeed be considered as a generalization of classical calculus; it examines the complex number as well as the real number as a derivative order. Infectious disease models, signal processing, electrical networks, fluid mechanics, process of diffusion reaction, HIV / Aids model, dengue fever and other physical phenomena have recently been studied using fractional derivatives by a number of researchers [3, 4].

By using concepts of fractional calculus, various researchers studied fractional pantograph equation. Among them Isah et al. [37] used a collocation method on Genocchi Operational Matrix to obtain a solution for generalized fractional pantograph equation. Fractional pantograph equation was also solved using M̈untz-Legendre wavelet operational matrix of fractional-order integration by Rahimkhani et al. see [41]. Spectral collocation method was also utilized for finding the solution of fractional pantograph equation see [42]. Another interesting technique was applied based on Bernoulli wavelets for the numerical solution of pantograph equation see [40]. By using preliminary concepts of fractional calculus and some fixed point theorems, K. Balachandran et al. [43] studied the existence of solutions of non linear fractional pantograph equation. Some analytical work is also been done by some researchers including D. Vivek et al. [38] and A. Anguraj et al. [39]. By considering fractional order derivatives and some related fractional calculus principles, they studied the existence and uniqueness properties and stability of fractional pantograph equation in a very interesting way.

Another numerical scheme of spectral collocation method was introduced for solving this fractional pantograph equation with variable coefficients on a semi-infinite domain. Spectral collocation method is based on the generalized Laguerre polynomials and Gauss quadrature integration, which then reduces to a system of algebraic equations for solving the generalized fractional pantograph equation [44].

The main aim of this study is to numerically solve the fractional pantograph equation with delay term ‘k’, then review the results and evaluate the numerical differences caused by the delay term and fractional order derivative.

This paper is divided into the following sections: Section 1 is about the historical background of pantograph equation with a brief overview of its applications and numerical methods. Section 2 is about the preliminary concepts of fractional calculus used in PIA for solving FPDDEs. Section 3 is related to the mathematical formulation of PIA for solving FPDDEs. Finally, Section 4 is based on four numerical examples that prove the efficacy of this numerical method. Section 5 is the conclusion of this work.

2. Preliminary concepts of fractional calculus

Basic definitions of fractional calculus that will be used for solving FPDDEs in this paper are as follows:

Definition 2.1. First of all, a famous and mostly used definition of Caputo fractional derivative for fractional order α is defined as

$$D_t^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} u^{(n)}(\tau)\, d\tau, \tag{1}$$

where $n-1 < \alpha \le n,\ n \in N,\ t > 0$. For $\alpha = 1$, Caputo sense derivative becomes

$$D_t^{\alpha} u(t) = \frac{du(t)}{dt}. \tag{2}$$

2.1. Properties of Caputo fractional derivative

Some important properties of Caputo fractional derivative are given below:

(a) $D_t^{\alpha} t^{\gamma} = \dfrac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}$, $\gamma > 0$,

(b) $D_t^{\alpha}\Big(cu(t)\Big) = cD_t^{\alpha}u(t)$,

(c) $D_t^{\alpha}\Big(au(t) + bv(t)\Big) = aD_t^{\alpha}u(t) + bD_t^{\alpha}v(t)$,

(d) $D_t^{\alpha}c = 0$, where a, b and c are constants.

Definition 2.2. Another important definition is of Riemann-Liouville integral

$$J^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-\tau)^{\alpha-1} u(\tau)\, d\tau, \quad \alpha > 0\ t > 0. \tag{3}$$

Another way to define Riemann-Liouville integral is as follows:

$$ {}_aD_t^{-\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} u(t)(t-\tau)^{\alpha-1}d\tau, \tag{4} $$

t_0 is arbitrary but have fixed base point. Riemann-Liouville integral is well defined and locally integrable function.

2.2. Properties of Riemann-Liouville integral

Some properties of Riemann-Liouville integral used in this paper are given below:

$$ \begin{aligned} &J^0u(t) = u(t),\\ &\frac{d}{dt}J^{\alpha+1}u(t) = J^{\alpha}u(t),\\ &J^{\alpha}t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)}t^{\alpha+\gamma},\\ &J^{\alpha}(J^{\beta}u) = J^{\alpha+\beta}u. \end{aligned} \tag{5} $$

3. Mathematical formulation

Perturbation iteration algorithm is a method based on the Taylor series which requires a small perturbation parameter to remove non linear terms by formulating an iterative expansion. Therefore the equation obtained after iterative formula is the fractional delay differential equation that can be solved by using any fractional definitions for solving a simple fractional differential equation. The steps of perturbation iteration algorithm for solving fractional pantograph delay differential equation are given below.

Consider the general form of FPDDE be written as

(6)

$$ P = F_t^{\alpha}u(t) - g\Big(t, u(k_\circ t), F_t u(k_1 t), \cdots, F_t^{j}u(k_j t)\Big) = 0, \quad j = 0,1,2,\cdots, $$

with initial conditions $F^r u(0) = C_r(t)$ for $r = 0,1,2,\cdots$. Also $p < \alpha \le p+1$ where $p \in R$ that describes the order of time fractional derivative and t is an independent variable. Here $k_j \in (0,1)$ for $j \in R$ and $u(t)$ is the unknown function.

PIA(1,1) denotes the expansion of Taylor series and correction terms up to first order derivatives only. Assume an approximate solution of the system

$$u_{1,m+1} = u_{1,m} + \epsilon u^c_{1,m}. \tag{7}$$

In Eq. (7) $u^c_{1,m}$ is the only one correction term in the perturbation expansion within the neighbourhood of $\epsilon = 0$ and can be expanded or approximated by Taylor series, where subscript m describes the number of iteration as m^{th} iteration over this approximate solution

$$P = \sum_{m=0}^{M} \frac{1}{m!}\Big[\Big(\frac{d}{d\epsilon}\Big)^m P\Big]_{\epsilon=0}\epsilon^m, \tag{8}$$

where $\frac{d}{d\epsilon}$ is defined as

$$\frac{d}{d\epsilon} = \frac{\partial u}{\partial \epsilon}\frac{\partial}{\partial u} + \sum_{j=1}^{q}\Big(\frac{\partial F^j_t u(k_j t)}{\partial \epsilon}\frac{\partial}{\partial F^j_t u(k_j t)}\Big) + \frac{\partial}{\partial \epsilon}. \tag{9}$$

Combining Eq. (8) and Eq. (9), an iteration equation is obtained

$$P = \sum_{m=0}^{M} \frac{1}{m!}\Big[\Big(\frac{\partial u}{\partial \epsilon}\frac{\partial}{\partial u} + \sum_{j=1}^{q}\Big(\frac{\partial F^j_t u(k_j t)}{\partial \epsilon}\frac{\partial}{\partial F^j_t u(k_j t)}\Big) + \frac{\partial}{\partial \epsilon}\Big]_{\epsilon=0}\epsilon^m. \tag{10}$$

Eq. (6) is defined for the $m+1^{th}$ iterative equation by using Eq. (10) as

$$F^\alpha_t u(t) - g\Big(t, u_{m+1}(k_\circ t), F^{m+1}_t u(k_1 t), \cdots, F^{j,(m+1)}_t u(k_j t)\Big) = 0, \tag{11}$$

where $j = 0, 1, 2, \cdots$. By using initial condition on Eq. (11) it may simply reduce to the correction terms only.

$$\big(F^\alpha_t u(t)\big)^c = L\Big(g\Big(t, u(k_\circ t), F_t u(k_1 t), \cdots, F^j_t u(k_j t)\Big)\Big). \tag{12}$$

Eq. (12) can be integrated some initial guess, in most cases this initial guess is the initial condition of respective problem. Therefore the first solution of iterative process is obtained as $u_{1,0}(t) = C_r(t)$. Similarly by using Eq. (7) further iterations i.e. $u_{1,1}(t), u_{1,2}(t), u_{1,3}(t), \cdots$ can be obtained up to n iterations. These iterations can be terminated after getting a satisfactory result.

For more general algorithm in PIA(n,m) more corrected terms can be added in Eq. (7), since by increasing the number of corrected terms more algebra and higher calculations will be involved so for this paper only one correction term i.e. PIA(1,1) will be used.

4. Numerical examples

Now to show the efficacy of PIA for finding the numerical solution of FPDDEs, four examples are considered

4.1. Example 1

First example [2] is

$$D_t^{\alpha} u(t) = 1 - u^2(kt), 0 \leq t \leq 1,\ 0 < \alpha \leq 1, \tag{13}$$

with initial values $u(t) = 0$ and exact solution $u(t) = Sin(t)$. For applying PIA, put ϵ with the nonlinear terms

$$D_t^{\alpha} u(t) = 1 - 2\epsilon u^2(kt).$$

Using iteration formula of Eq. (10) and taking $\epsilon = 1$, we have

$$(D_t^{\alpha} u(t))^c = -D_t^{\alpha} u(t) + 1 - 2u^2(kt).$$

By applying the iteration formula with initial guess as $u_0(t) = 0$ and solving for PIA(1,1) following iterations have been obtained

$$\begin{aligned}
u_1(t) &= \frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
u_2(t) &= \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{2^{1-2\alpha}k^{2\alpha}t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)},\\
u_3(t) &= \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{2^{1-2\alpha}k^{2\alpha}t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} + \frac{2^3k^{6\alpha}t^{5\alpha}\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} - \\
&\quad \frac{2^3k^{10\alpha}t^{7\alpha}\Gamma(2\alpha+!1)^2\Gamma(6\alpha+!1)}{\Gamma(\alpha+1)^4\Gamma(3\alpha+1)^2\Gamma(7\alpha+1)},\\
u_4(t) &= \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{2^{1-2\alpha}k^{2\alpha}t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} + \\
&\quad \frac{2^3k^{6\alpha}t^{5\alpha}\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} - \frac{2^3k^{10\alpha}t^{7\alpha}\Gamma(2\alpha+1)^2\Gamma(6\alpha+1)}{\Gamma(\alpha+1)^4\Gamma(3\alpha+1)^2\Gamma(7\alpha+1)} +
\end{aligned}$$

$$\frac{2^5 k^{12\alpha} t^{7\alpha}\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)}{\Gamma(\alpha+1)^4\Gamma(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)}+$$

$$\frac{2^6 k^{16\alpha} t^{9\alpha}\Gamma(2\alpha+1)^2\Gamma(4\alpha+1)\Gamma(8\alpha+1)}{\Gamma(\alpha+1)^4\Gamma(3\alpha+1)^2\Gamma(5\alpha+1)\Gamma(9\alpha+1)}+\,+\cdots. \tag{14}$$

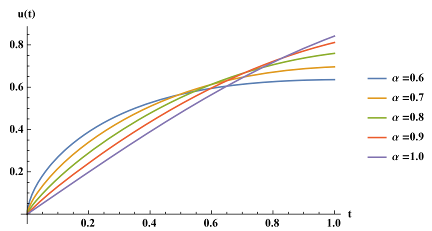
Iterations for a specific case of $k = 1/2$

$$\begin{aligned}
u_1(t) &= \frac{t^\alpha}{\Gamma(\alpha+1)},\\
u_2(t) &= \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{2^{1-2\alpha}t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)},\\
u_3(t) &= \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2^{1-2\alpha}t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} - \frac{2^{2-2\alpha}t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}+\\
&\quad \frac{2^{3-6\alpha}t^{5\alpha}\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\Gamma(5\alpha+1)} - \frac{2^{3-10\alpha}t^{7\alpha}\Gamma(2\alpha+1)^2\Gamma(6\alpha+1)}{\Gamma(\alpha+1)^4\Gamma(3\alpha+1)^2\Gamma(7\alpha+1)},\\
u_4(t) &= \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2^{1-2\alpha}t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} - \frac{2^{2-2\alpha}t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}+\\
&\quad \frac{2^{3-6\alpha}t^{5\alpha}\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\Gamma(5\alpha+1)}-\\
&\quad \frac{2^{3-10\alpha}t^{7\alpha}\Gamma(2\alpha+1)^2\Gamma(6\alpha+1)}{\Gamma(\alpha+1)^4\Gamma(3\alpha+1)^2\Gamma(7\alpha+1)}-\\
&\quad \frac{2^{5-12\alpha}t^{7\alpha}\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)}{\Gamma(\alpha+1)^4\Gamma(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)}+\\
&\quad \frac{2^{6-16\alpha}t^{9\alpha}\Gamma(2\alpha+1)^2\Gamma(4\alpha+1)\Gamma(8\alpha+1)}{\Gamma(\alpha+1)^5\Gamma(3\alpha+1)^2\Gamma(5\alpha+1)\Gamma(9\alpha+1)}+\\
&\quad \frac{2^{5-18\alpha}t^{9\alpha}\Gamma(2\alpha+1)^2\Gamma(6\alpha+1)\Gamma(8\alpha+1)}{\Gamma(\alpha+1)^5\Gamma(3\alpha+1)^2\Gamma(7\alpha+1)\Gamma(9\alpha+1)}+\cdots.
\end{aligned} \tag{15}$$

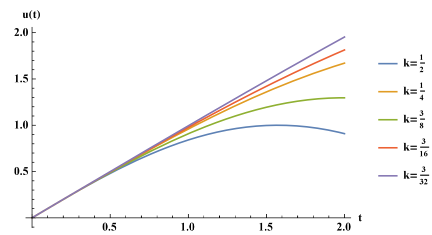
Now, in order to check whether this numerical solution is accurate, compare the iterative numerical solution at $\alpha = 1$ and $k = \frac{1}{2}$ with the exact solution in tabular and graphical form. In Table 1 the comparison of series solution of Example 4.1 obtained by PIA and the exact solution clearly shows the accuracy of PIA for FPDDE. In Figure 1d the convergence of iterative solutions is shown with the exact solution for interval $t \in (0, 10)$, it can be clearly observed that as the number of iterations increases, the series solution converges. Hence the accuracy of PIA is proved for FPDDE. Figure 1c also indicates the accuracy by the propagation of absolute error in fourth

Table 1: Comparison of series solution of FPDDE in Example 4.1 obtained through PIA at $\alpha = 1$ and $k = \frac{1}{2}$ with its exact solution and its absolute error

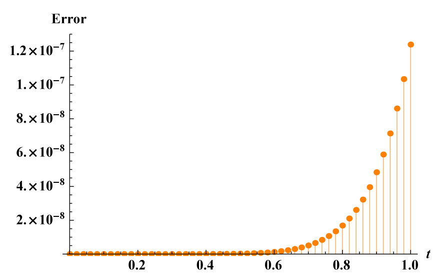
t	Exact	PIA	Abs Error
0.0	0.0000000000	0.0000000000	0.000000000E+00
0.1	0.0998334166	0.0998334166	1.249000903E-16
0.2	0.1986693308	0.1986693308	6.600275881E-14
0.3	0.2955202067	0.2955202067	2.532973831E-12
0.4	0.3894183423	0.3894183423	3.363703760E-11
0.5	0.4794255386	0.4794255384	2.496770013E-10
0.6	0.5646424734	0.5646424721	1.282366990E-09
0.7	0.6442176872	0.6442176821	5.107050005E-09
0.8	0.7173560909	0.7173560740	1.687976903E-08
0.9	0.7833269096	0.7833268613	4.837695300E-08
1.0	0.8414709848	0.8414708609	1.238749689E-07



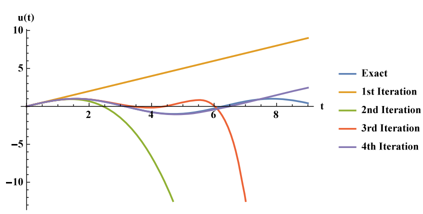
(a) PIA solution for different value of α



(b) Comparison of solution with different values of k at $\alpha = 1$ for Example 4.1



(c) Propagation of absolute error in 4th iteration at $\alpha = 1$ and $k = 1/2$



(d) Convergence of solution for different iteration with the exact solution for $t \in (0, 10)$.

Figure 1: A complete graphical description of numerical solution of Example 4.1 obtained by PIA.

iteration of PIA at $\alpha = 1$ and $k = \frac{1}{2}$. Figure 1b is very interesting as it shows the behaviour of numerical solution at $\alpha = 1$ for different values of k. It is observed that as the value of k gets closer to zero, the function stretches and deviates downwards; but as the value of k gets bigger, the graph stretches upwards. Figure 1a shows the fractional behaviour of PIA series solution obtained for FPDDE at $k = \frac{1}{2}$. Figure 1 gives a complete numerical analysis of FPDDE in Example 4.1.

4.2. Example 2

Consider the linear fractional Pantograph differential equation

$$D_t^\alpha u(t) = \frac{3}{4}u(t) + u(kt) - t^2 + 2, \quad 1 < \alpha \leq 2,\ 0 \leq t \leq 1, \tag{16}$$

with initial condition $u(0) = 0$, $u'(0) = 0$ and exact solution $u(t) = t^2$. Now apply ϵ with the nonlinear terms

$$D_t^\alpha u(t) = \frac{3}{4}u(t) + \epsilon u(kt) - t^2 + 2.$$

Using iteration formula of Eq. (10) and consider $\epsilon = 1$, we have

$$(D_t^\alpha u(t))^c = -D_t^\alpha u(t) + \frac{3}{4}u(t) + u(kt) - t^2 + 2.$$

By applying the iteration formula and solving for PIA(1,1) with initial guess as $u_0(t) = t^2$ following iterations have been obtained

$$\begin{aligned}
u_1(t) &= \frac{2t^\alpha}{\Gamma(\alpha+1)} - \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)},\\
u_2(t) &= \frac{2t^\alpha}{\Gamma(\alpha+1)} - \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{(3+4k^\alpha)t^{2\alpha}}{2\Gamma(2\alpha+1)} - \frac{(3+4k^{\alpha+2})t^{2\alpha+2}}{2\Gamma(2\alpha+3)},\\
u_3(t) &= \frac{2t^\alpha}{\Gamma(\alpha+1)} - \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{(3+4k^\alpha)t^{2\alpha}}{2\Gamma(2\alpha+1)} - \frac{(3+4k^{\alpha+2})t^{2\alpha+2}}{2\Gamma(2\alpha+3)} +\\
&\quad \frac{2k^{3\alpha}t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{3(4k^{2\alpha}+4k^\alpha+3)t^{3\alpha}}{8\Gamma(3\alpha+1)} +\\
&\quad \frac{(16k^{3\alpha+4}+12k^{2\alpha+2}+12k^{\alpha+2}-9)t^{3\alpha+2}}{8\Gamma(3\alpha+3)},\\
u_4(t) &= \frac{2t^\alpha}{\Gamma(\alpha+1)} - \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{(3+4k^\alpha)t^{2\alpha}}{2\Gamma(2\alpha+1)} - \frac{(3+4k^{\alpha+2})t^{2\alpha+2}}{2\Gamma(2\alpha+3)} +
\end{aligned}$$

$$\frac{2k^{3\alpha}t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{3(4k^{2\alpha}+4k^{\alpha}+3)t^{3\alpha}}{8\Gamma(3\alpha+1)}+$$
$$\frac{(16k^{3\alpha+4}+12k^{2\alpha+2}+12k^{\alpha+2}-9)t^{3\alpha+2}}{8\Gamma(3\alpha+3)}-$$
$$\frac{(64k^{6\alpha}+48k^{5\alpha}+48k^{4\alpha}+84k^{3\alpha}+36k^{2\alpha}+36k^{\alpha}+27)t^{4\alpha}}{32\Gamma(4\alpha+1)}+$$
$$\frac{(64k^{6\alpha+6}+48k^{4}(k^{5\alpha}+k^{4\alpha}+k^{3\alpha}))t^{4\alpha+2}}{32\Gamma(4\alpha+3)}+$$
$$\frac{(36k^{2}(k^{3\alpha}+k^{2\alpha}+k^{\alpha})-27)t^{4\alpha+2}}{32\Gamma(4\alpha+3)}+\cdots \tag{17}$$

iterations after assuming $k = 1/2$

$$u_1(t)=\frac{2t^{\alpha}}{\Gamma(\alpha+1)}-\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)},$$

$$u_2(t)=\frac{2t^{\alpha}}{\Gamma(\alpha+1)}-\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}+\frac{3t^{2\alpha}}{2\Gamma(2\alpha+1)}+\frac{2^{1-\alpha}t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{3t^{2\alpha+2}}{2\Gamma(2\alpha+3)}-\frac{2^{-\alpha-1}t^{2\alpha+2}}{\Gamma(2\alpha+3)},$$

$$u_3(t)=\frac{2t^{\alpha}}{\Gamma(\alpha+1)}-\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}+\frac{3t^{2\alpha}}{2\Gamma(2\alpha+1)}+\frac{2^{1-\alpha}t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{3t^{2\alpha+2}}{2\Gamma(2\alpha+3)}-$$
$$\frac{2^{-\alpha-1}t^{2\alpha+2}}{\Gamma(2\alpha+3)}+\frac{3^{2}\cdot t^{3\alpha}}{2^{3}\Gamma(3\alpha+1)}+\frac{2^{-3\alpha+1}2t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{3\cdot 2^{-2\alpha-1}t^{3\alpha}}{\Gamma(3\alpha+1)}+$$
$$\frac{3\cdot 2^{-\alpha-1}t^{3\alpha}}{\Gamma(\alpha+1)}-\frac{3^{2}t^{3\alpha+2}}{2^{3}\Gamma(3\alpha+3)}-\frac{3\cdot 2^{-2\alpha-3}t^{3\alpha+2}}{\Gamma(3\alpha+3)}-\frac{3\cdot 2^{-\alpha-1}t^{3\alpha+2}}{\Gamma(3\alpha+3)}-$$
$$\frac{2^{-3\alpha-3}t^{3\alpha+2}}{\Gamma(3\alpha+3)},$$

$$u_4(t)=\frac{2t^{\alpha}}{\Gamma(\alpha+1)}-\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}+\frac{3t^{2\alpha}}{2\Gamma(2\alpha+1)}+\frac{2^{1-\alpha}t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{3t^{2\alpha+2}}{2\Gamma(2\alpha+3)}-$$
$$\frac{2^{-\alpha-1}t^{2\alpha+2}}{\Gamma(2\alpha+3)}+\frac{3^{2}\cdot t^{3\alpha}}{2^{3}\Gamma(3\alpha+1)}+\frac{2^{-3\alpha+1}2t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{3\cdot 2^{-2\alpha-1}t^{3\alpha}}{\Gamma(3\alpha+1)}+$$
$$\frac{3\cdot 2^{-\alpha-1}t^{3\alpha}}{\Gamma(\alpha+1)}-\frac{3^{2}t^{3\alpha+2}}{2^{3}\Gamma(3\alpha+3)}-\frac{2^{-3\alpha-3}t^{3\alpha+2}}{\Gamma(3\alpha+3)}-\frac{3\cdot 2^{-2\alpha-3}t^{3\alpha+2}}{\Gamma(3\alpha+3)}-$$
$$\frac{3\cdot 2^{-\alpha-3}t^{3\alpha+2}}{2\Gamma(2\alpha+3)}+\frac{3^{3}t^{4\alpha}}{2^{5}\Gamma(4\alpha+1)}+\frac{2^{-6\alpha+1}t^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{3\cdot 2^{-5\alpha-1}t^{4\alpha}}{\Gamma(4\alpha+1)}+$$
$$\frac{3\cdot 2^{-4\alpha-1}t^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{21\cdot 2^{-3\alpha-3}t^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{3^{2}\cdot 2^{-2\alpha-3}t^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{3^{2}\cdot 2^{-\alpha-3}t^{4\alpha}}{\Gamma(4\alpha+1)}-$$

Table 2: Comparison of series solution of FPDDE in Example 4.2 obtained through PIA at $\alpha = 2$ and $k = \frac{1}{2}$ with its exact solution and its absolute error

t	Exact	PIA	Abs Error
0.0	0.0000000000	0.0000000000	0.0000000000E+00
0.1	0.0100000000	0.0100000000	0.0000000000E+00
0.2	0.0400000000	0.0400000000	6.9388939039E-18
0.3	0.0900000000	0.0900000000	7.7715611724E-16
0.4	0.1600000000	0.1600000000	2.4674706722E-14
0.5	0.2500000000	0.2500000000	3.5901837059E-13
0.6	0.3600000000	0.3600000000	3.2010505358E-12
0.7	0.4900000000	0.4900000000	2.0354162800E-11
0.8	0.6400000000	0.6399999999	1.0105460913E-10
0.9	0.8100000000	0.8099999996	4.1532333128E-10
1.0	1.0000000000	0.9999999985	1.4705379137E-09

$$(18) \qquad \frac{3^3 t^{4\alpha+2}}{2^5\Gamma(4\alpha+3)} - \cdots .$$

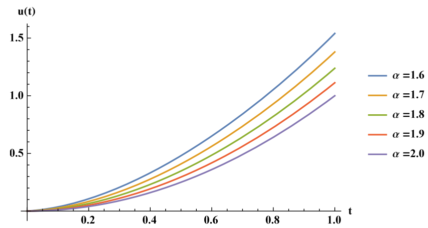
Now to check the accuracy, compare the iterative numerical solution at $\alpha = 2$ and $k = \frac{1}{2}$ with the exact solution in tabular and graphical form. In Table 2, the comparison of series solution of Example 4.2 obtained by PIA and exact solution clearly shows the accuracy of PIA for FPDDE. In Figure 2d, it can be clearly observed that as the number of iterations increase the series solution converges to exact solution. Figure 2c also indicates the accuracy by propagation of absolute error in fourth iteration of PIA at $\alpha = 2$ and $k = \frac{1}{2}$. Figure 2b is very interesting as it shows the behaviour of FPDDE for different values of k at $\alpha = 2$. It is observed that as the value of k gets closer to zero, the function stretches and deviates downwards. As the value of k gets bigger, the graph stretches upwards. Figure 2a shows the fractional behaviour of PIA series solution obtained for FPDDE at $k = \frac{1}{2}$. Figure 2 gives a complete numerical analysis of Example 4.2.

4.3. Example 3

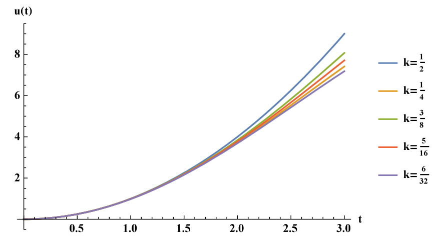
Consider non linear variable coefficient fractional differential equation with pantograph delay [2]

$$(19) \qquad D_t^{\alpha} u(t) = u(t) - \frac{8}{t^2} u(kt)^2, \quad t \geq 0,\ 1 < \alpha \leq 2,$$

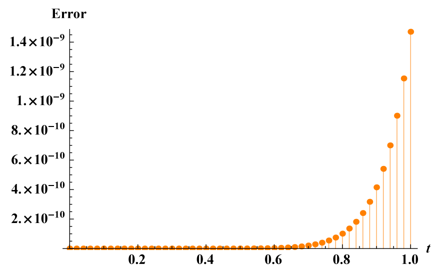
with initial condition $u(0) = 0$, $u^{'}(0) = 1$, and $u(t) = te^{-t}$.



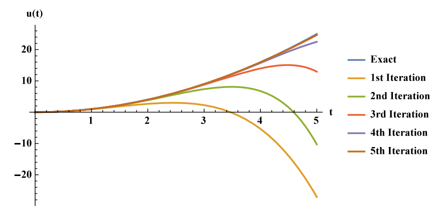
(a) PIA solution for different values of α



(b) Comparison of solution with different values of k at $\alpha = 2$ for Example.4.2



(c) Propagation of absolute error in 4th iteration at $\alpha = 2$ and $k = 1/2$



(d) Convergence of solution for different iteration with the exact solution for $t \in (0, 5)$.

Figure 2: A complete graphical description of numerical solution of Example 4.2 obtained by perturbation iteration algorithm describing the accuracy of this method.

Apply ϵ with the non linear terms

$$D_t^\alpha u(t) = u(t) - \epsilon \frac{8}{t^2} u(kt)^2.$$

Using iteration formula of Eq. (10) and consider $\epsilon = 1$, we have

$$(D_t^\alpha u(t))^c = -D_t^\alpha u(t) + u(t) - \frac{8}{t^2} u(kt)^2.$$

By applying the iteration formula and solving for PIA(1,1) with initial guess as $u_0(t) = t$

$$u_1(t) = t - \frac{8k^2 t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)},$$

$$
\begin{aligned}
u_2(t) = {}& t - \frac{8k^2t^\alpha}{\Gamma(\alpha+1)} + \frac{128k^{\alpha+3}t^{2\alpha-1}\Gamma(\alpha)}{\Gamma(2\alpha)\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \\
& \frac{128k^{2\alpha+3}t^{3\alpha-1}\Gamma(2\alpha)}{\Gamma(3\alpha)\Gamma(\alpha+1)\Gamma(\alpha+2)} - \frac{8k^2t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{16k^{\alpha+2}t^{2\alpha}\Gamma(\alpha+1)}{\Gamma(\alpha+2)\Gamma(2\alpha+1)} + \\
& \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{512k^{2\alpha+4}t^{3\alpha-2}\Gamma(2\alpha-1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha-1)} - \frac{8k^{2\alpha+2}t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(\alpha+2)^2\Gamma(3\alpha+1)}, \\
u_3(t) = {}& t - \frac{8k^2t^\alpha}{\Gamma(\alpha+1)} + \frac{128k^{\alpha+3}t^{2\alpha-1}\Gamma(\alpha)}{\Gamma(2\alpha)\Gamma(\alpha+1)} + \frac{128k^{\alpha+3}t^{3\alpha-1}\Gamma(\alpha)}{\Gamma(3\alpha)\Gamma(\alpha+1)} + \\
& \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{128k^{2\alpha+3}t^{3\alpha-1}\Gamma(2\alpha)}{\Gamma(3\alpha)\Gamma(\alpha+1)\Gamma(\alpha+2)} - \frac{128k^{2\alpha+3}t^{4\alpha-1}\Gamma(2\alpha)}{\Gamma(4\alpha)\Gamma(\alpha+1)\Gamma(\alpha+2)} + \\
& \frac{8k^2t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{128k^{2\alpha+3}t^{3\alpha-1}\Gamma(2\alpha)}{\Gamma(3\alpha)\Gamma(2\alpha+1)} + \frac{256k^{4\alpha+3}t^{4\alpha-1}\Gamma(3\alpha)\Gamma(\alpha+1)}{\Gamma(4\alpha)\Gamma(\alpha+1)^2\Gamma(2\alpha+1)} + \\
& \frac{128k^{3\alpha+3}t^{4\alpha-1}\Gamma(3\alpha)}{\Gamma(4\alpha)\Gamma(\alpha+2)\Gamma(2\alpha+1)} - \frac{16k^{\alpha+2}t^{2\alpha}\Gamma(\alpha+1)}{\Gamma(\alpha+2)\Gamma(2\alpha+1)} + \\
& \frac{256k^{3\alpha+3}t^{3\alpha-1}\Gamma(2\alpha)\Gamma(\alpha+1)}{\Gamma(4\alpha)\Gamma(\alpha+2)\Gamma(2\alpha+1)} + \cdots
\end{aligned} \tag{20}
$$

iterations for $k = 1/2$

$$
\begin{aligned}
u_1(t) = {}& t - \frac{2t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \\
u_2(t) = {}& t - \frac{2t^\alpha}{\Gamma(\alpha+1)} + \frac{2^{4-\alpha}t^{2\alpha-1}\Gamma(\alpha)}{\Gamma(2\alpha)\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \\
& \frac{2^{4-2\alpha}t^{3\alpha-1}\Gamma(2\alpha)}{\Gamma(3\alpha)\Gamma(\alpha+1)\Gamma(\alpha+2)} - \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{(2^{2-\alpha}t^{2\alpha}\Gamma(\alpha+1)}{\Gamma(\alpha+2)\Gamma(2\alpha+1)} + \\
& \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{(2^{5-2\alpha}t^{3\alpha-2}\Gamma(2\alpha-1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha-1)} - \frac{(2^{1-2\alpha}t^{2\alpha-1}\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}, \\
u_3(t) = {}& t - \frac{2t^\alpha}{\Gamma(\alpha+1)} - \frac{2^{4-\alpha}t^{2\alpha-1}\Gamma(\alpha)}{\Gamma(2\alpha)\Gamma(\alpha+1)} + \frac{2^{5-\alpha}t^{2\alpha-1}\Gamma(\alpha)}{\Gamma(2\alpha)\Gamma(\alpha+1)} + \\
& \frac{2^{4-\alpha}t^{3\alpha-1}\Gamma(\alpha)}{\Gamma(3\alpha)\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{2^{4-2\alpha}t^{3\alpha-1}\Gamma(2\alpha)}{\Gamma(3\alpha)\Gamma(\alpha+1)\Gamma(\alpha+2)} + \\
& \frac{2^{4-2\alpha}t^{4\alpha-1}\Gamma(2\alpha)}{\Gamma(4\alpha)\Gamma(\alpha+1)\Gamma(\alpha+2)} - \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2^{4-2\alpha}t^{3\alpha-1}\Gamma(2\alpha)}{\Gamma(3\alpha)\Gamma(2\alpha+1)} + \\
& \frac{2^{5-4\alpha}t^{4\alpha-1}\Gamma(3\alpha)\Gamma(\alpha+1)}{\Gamma(4\alpha)\Gamma(\alpha+2)^2\Gamma(2\alpha+1)} + \frac{2^{4-3\alpha}t^{4\alpha-1}\Gamma(3\alpha)}{\Gamma(4\alpha)\Gamma(\alpha+1)\Gamma(2\alpha+2)} + \cdots .
\end{aligned} \tag{21}
$$

Table 3: Comparison of series solution of FPDDE in Example 4.3 obtained through PIA at $\alpha = 2$ and $k = \frac{1}{2}$ with its exact solution and its absolute error

t	Exact	PIA	Abs Error
0.0	0.0000000000	0.0000000000	0.0000000000E+00
0.1	0.0904837418	0.0904837362	5.6269247733E-09
0.2	0.1637461506	0.1637460076	1.4298986870E-07
0.3	0.2222454662	0.2222446256	8.4055811256E-07
0.4	0.2681280184	0.2681253667	2.6516651139E-06
0.5	0.3032653299	0.3032595460	5.7838779841E-06
0.6	0.3292869817	0.3292773811	9.6005419847E-06
0.7	0.3476097127	0.3475973945	1.2318176999E-05
0.8	0.3594631713	0.3594520921	1.1079200653E-05
0.9	0.3659126938	0.3659101533	2.5404711865E-06
1.0	0.3678794412	0.3678953574	1.5916191025E-05

Comparison in Table 3 and graphical figure as Figure 3 is given between numerical solution obtained at $\alpha = 2$ and $k = \frac{1}{2}$ with its exact solution. In Figure 3d convergence of iterative solutions is shown with the exact solution, as the number of iterations increases, the series solution converges to exact solution. Figure 3c also indicates the accuracy by the propagation of absolute error in fourth iteration of PIA at $\alpha = 2$ and $k = \frac{1}{2}$. Figure 3b at $\alpha = 2$ gives the behaviour of FPDDE for different values of k. It is observed that as the value of k gets closer to zero, the function stretches and deviates downwards; as the value of k gets bigger, the graph stretches upwards. Figure 3a shows the fractional behaviour of PIA series solution obtained for FPDDE at $k = \frac{1}{2}$.

4.4. Example 4

Consider non linear variable coefficient fractional differential equation with pantograph delay [45]

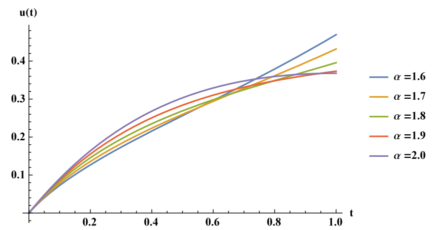
$$D_t^{\alpha} u(t) = 2u(kt) - u(t) - t^2 - 1, \quad 0 \leq t \leq 1,\ 2 < \alpha \leq 3, \tag{22}$$

with initial condition, $u(0) = 1$, $u'(0) = -4$ and $u''(0) = 0$. The exact solution is $u(t) = 1 - 2t^2$. Apply ϵ with the nonlinear terms

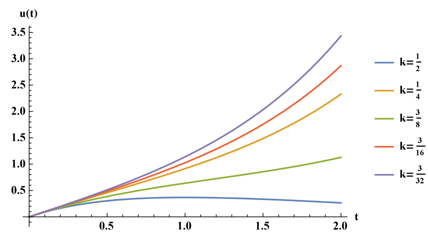
$$D_t^{\alpha} u(t) = \epsilon 2u(kt) - u(t) - \epsilon t^2 - 1. \tag{23}$$

By using iteration formula in Eq. (10) and taking $\epsilon = 1$ Eq. (23) becomes

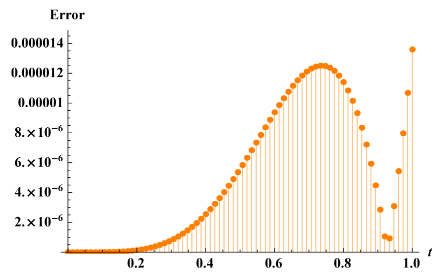
$$(D_t^{\alpha} u(t))^c = -D_t^{\alpha} u(t) + 2u(kt) - u(t) - t^2 - 1. \tag{24}$$



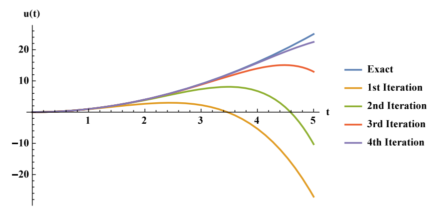
(a) PIA solution for different values of α



(b) Comparison of solution with different values of k and $\alpha = 2$ for Example.4.3



(c) Propagation of absolute error in 4th iteration at $\alpha = 2$ and $k = 1/2$



(d) Convergence of solution for different iteration with the exact solution for $t \in (0, 5)$.

Figure 3: Numerical solution of Example 4.3 obtained by PIA is given with complete analysis.

By applying the iteration formula and solving with initial guess as $u_0(t) = -4t + 1$ following iterations have been obtained

$$
\begin{aligned}
u_1(t) &= 1 - 2t^2 - \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}, \\
u_2(t) &= 1 - 2t^2 - \frac{2(4k^2-1)t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{(2k^\alpha - 1)t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{2(2k^{\alpha+2})t^{2\alpha+2}}{\Gamma(2\alpha+3)}, \\
u_3(t) &= 1 - 2t^2 - \frac{2(4k^2-1)t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{2(8k^{\alpha+4} - 2k^{\alpha+2} - 4k^2 - 1)t^{2\alpha+2}}{\Gamma(2\alpha+3)} + \\
&\quad \frac{(4k^{3\alpha} - 2k^{2\alpha} - 2k^\alpha - 1)t^{3\alpha}}{\Gamma(3\alpha+1)} + \\
&\quad \frac{2(4k^{3\alpha+4} - 2k^{2\alpha+2} - 2k^{\alpha+2} - 1)t^{3\alpha+2}}{\Gamma(3\alpha+3)},
\end{aligned}
$$

$$u_4(t) = 1 - 2t^2 - \frac{2(4k^2-1)t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{2(8k^{\alpha+4} - 2k^{\alpha+2} - 4k^2 - 1)t^{2\alpha+2}}{\Gamma(2\alpha+3)} - \frac{2(4k^2-1)(4k^{3\alpha+4} - 2k^{2\alpha+2} - 2k^{\alpha+2} + 1)t^{3\alpha+2}}{\Gamma(3\alpha+3)} - \frac{(8k^{6\alpha} - 4k^{5\alpha} - 4k^{4\alpha} - 2k^{3\alpha} + 2k^{2\alpha} + 2k^{\alpha} - 1)t^{4\alpha}}{\Gamma(4\alpha+1)} - \cdots . \tag{25}$$

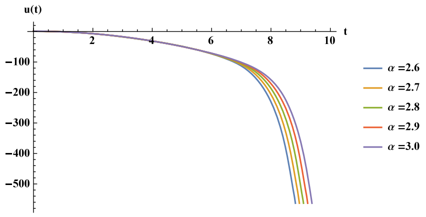
Iterations for $k = \frac{1}{2}$ are

$$u_1(t) = 1 - 2t^2 - \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)},$$
$$u_2(t) = 1 - 2t^2 + \frac{(2^\alpha - 2)t^{2\alpha}}{2^\alpha\Gamma(2\alpha+1)} + \frac{(2^{\alpha+1}-1)t^{2\alpha+2}}{2^\alpha\Gamma(\alpha+3)},$$
$$u_3(t) = 1 - 2t^2 - \frac{(2^{3\alpha} + 2^{2\alpha+1} - 4)t^{3\alpha}}{2^{3\alpha}\Gamma(3\alpha+1)} - \frac{(2^{3\alpha+2} + 2^{2\alpha+1} + 2^{\alpha+1} - 1)t^{3\alpha+2}}{2^{3\alpha+1}\Gamma(3\alpha+3)},$$
$$u_4(t) = 1 - 2t^2 - \frac{(2^{6\alpha} + 2^{2\alpha+2} - 2^{5\alpha+1} - 2^{4\alpha+1} + 2^{3\alpha+1} + 2^{\alpha+2} - 8)t^{4\alpha}}{2^{6\alpha}\Gamma(4\alpha+1)} + \frac{(2^{12\alpha+2} - 2^{4\alpha+2} - 2^{5\alpha+2} - 2^{3\alpha+1} + 2^{2\alpha+1} + 2^{\alpha+1} - 1)t^{4\alpha+2}}{2^{6\alpha+2}\Gamma(4\alpha+3)} + \cdots . \tag{26}$$

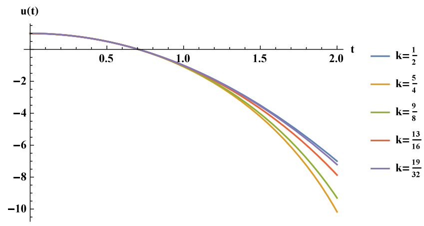
In order to confirm the accuracy of Eq. (26), it is compared with exact solution at $\alpha = 3$ in tabular and graphical form. Figure 4 gives the complete numerical analysis of Eq. (25) and Eq. (26) and result observations are approximately the same as in above examples. In Figure 4d shows that as the number of iterations increase the series solution converges to exact solution. Figure 4c also indicates the accuracy by the propagation of absolute error in fourth iteration of PIA. Figure 4b shows that as the value of k gets closer to zero, the function stretches and deviates downwards; as the value gets bigger, the graph stretches upwards. Figure 4a shows the fractional behaviour of Eq. (26).

5. Conclusion

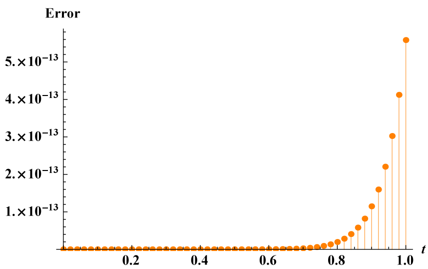
To sum up, an easy, fast converging numerical method is constructed in this paper for solving the pantograph type fractional delay differential equation. The proposed algorithm is based on the perturbation parameter ϵ. The final results are approximately equal to exact solutions. In this work, the role of the delay term k in the fractional equation is observed. This delay term,



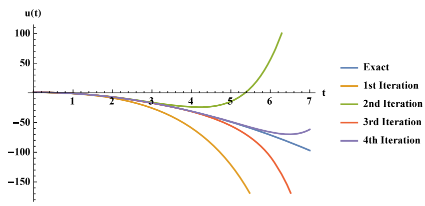
(a) PIA solution for different values of α



(b) Comparison of solution with different values of k and $\alpha = 3$ for Example.4.4



(c) Propagation of absolute error in 4th iteration at $\alpha = 3$ and $k = 1/2$



(d) Convergence of solution for different iteration with the exact solution for $t \in (0, 7)$.

Figure 4: Numerical analysis of series solution for Example 4.4 obtained by PIA.

when present in solution, can stretch the solution of FPDDE upward and downwards. When k is approaching zero, the solution stretches downwards; whereas if the value of k gets farther away from zero, the solution deviates upwards. In this work, not only is the efficacy of PIA for FPDDE proven, the fractional behaviour for fixed delay term and variation in solutions due to delay term is also shown.

Acknowledgements

Authors acknowledge the comments of anonymous referees which improved their manuscript qualitatively and are indebted to the editor for his illuminating advice and valuable discussions.

References

[1] K. Mahler, On a special functional equation. *Journal of London Mathematical Society* **1**(2), (1940), 115–123. MR0002921

[2] M.M. Bahşi and M. Çevik, Numerical solution of pantograph-type delay differential equations using perturbation-iteration algorithms. *Journal of Applied Mathematics* **2015**, (2015) Article ID 139821. MR3433598

[3] K. Diethelm, *Analysis of Fractional Differential Equations*. Springer-Verlag, Berlin, 2010. MR2680847

[4] I. Podlubny, *Fractional Differential Equations*. Academic Press, New York, 1999. MR1658022

[5] T. Kato and J.B. McLeod, The functional-differential equation $y(x) = ay(x)+by(x)$. *Bulletin of the American Mathematical Society* **77** (1971) 891–937. MR0283338

[6] J.R. Ockendon and A.B. Taylor, The dynamics of current collection system for an electric locomotive. *Proceedings of Royal Society A* **322** (1971) 447–486.

[7] A. Iserles, On the generalized pantograph functional–differential equation. *European Journal of Applied Mathematics* **4** (1993) 1–38. MR1208418

[8] J. Hale, *Theory of Functional Differential Equations*. Springer-Verlag, 1977. MR0508721

[9] R. Bellman and K.L. Cooke, *Differential-Difference Equations*. Academic Press, New York & London, 1963. MR0147745

[10] G.A. Derfel, On the asymptotics of solutions of one class of functional-differential equations. In: *Asymptotic Behaviour of Solutions of Differential Difference Equations*. Inst. Math. Akad. Nauk UkrSSR Press, 58–66, 1978. MR0558018

[11] Y. Kuang, and A. Feldstein, Monotonic and oscillatory solutions of a linear neutral delay equation with infinite lag. *J. Math. Anal.*, **21**(6) (1990) 1633–1641. MR1075595

[12] G.R. Morris, A. Feldstein and E.W. Bowen, The Phragmen-Lindelof principle and a class of functional differential equations. In: *Ordinary Differential Equations*. Academic Press, 513–540, 1972. MR0427771

[13] V.A. Ambartsumian, On the fluctuation of the brightness of the Milky Way. *Doklady Akad. Nauk USSR* **44** (1994) 223–226. MR0011992

[14] G.A. Derfel, Kato problem for functional-differential equations and difference Schrodinger operators. *Operator Theory: Advances and Applications* **46** (1990) 319–321. MR1124676

[15] E.Y. Romanenko and A.N. Sharkovski, Asymptotic solutions of differential-functional equations. In: *Asymptotic Behaviour of Solutions of Differential Difference Equations*. Inst. Math. Akad. Nauk UkrSSR Press, 5–39, 1978.

[16] R. Rodeman, D.B. Longcope and L.F. Shampine, Response of a string to an accelerating mass. *J. Appl. Mech.* **98** (1976) 675–680.

[17] M.D. Buhmann and A. Iserles, On the dynamics of a discretized neutral equation. *IMA J. Num. Anal.* **12**(3) (1992) 339–363. MR1181255

[18] D. Gaver Jr., An absorption probability problem. *J. Math. Anal. Appl.* **9** (1964) 384–393. MR0170376

[19] T.S. Ferguson, Lose a dollar or double your fortune. In: *Proceedings of the 6th Berkeley Symposium on Mathematical Statistics and Probability*, vol. III. University of California Press, Berkeley, 657–666, 1972. MR0407962

[20] A. Hall and G. Wake, A functional differential equation arising in the modeling of cell growth. *J. Austral. Math. Soc. Ser. B* **30** (1989) 424–435. MR0982622

[21] R.W. Robinson, Counting labeled acyclic digraphs. In: *New Directions in the Theory of Graphs*. Academic Press, New York, 239–279, 1973. MR0363994

[22] W. Lv and H. Li, Numerical stability of higher-order derivative methods for the pantograph equation. *Appl. Math. Comput.* **218** (2012) 5739–5745. MR2870089

[23] H. Brunner and Q. Hu, Optimal superconvergence results for delay integro-differential equations of pantograph type. *SIAM J. Numer. Anal.* **45**(3) (2007) 986–1004. MR2318798

[24] A.M. Bica, M. Curila and S. Curila, About a numerical method of successive interpolations for two-point boundary value problems with deviating argument. *Appl. Math. Comput.* **217** (2011) 7772–7789. MR2799789

[25] I. Ali, H. Brunner and T. Tang, A spectral method for pantograph-type delay differential equations and its convergence analysis. *J. Comput. Math.* **27**(2–3) (2009) 254–265. MR2495059

[26] S. Widatalla, Extension of Zhou's method to neutral functional-differential equation with proportional delays. *ISRN Appl. Math.* **2012** (2012) Article ID 518361. MR2957714

[27] D.J. Evans and K.R. Raslan, The Adomian decomposition method for solving delay differential equation. *Int. J. Comput. Math.* **82**(1) (2005) 49–54. MR2159285

[28] H. Koçak and A. Yildirim, Series solution for a delay differential equation arising in electrodynamics. *Commun. Numer. Methods Eng.* **25** (2009) 1084–1096. MR2589406

[29] M. Sezer, S. Yalçinbas and M. Gülsu, A Taylor polynomial approach for solving generalized pantograph equations with nonhomogenous term. *Int. J. Comput. Math.* **85**(7) (2008) 1055–1063. MR2431785

[30] S. Yüzbasi, An efficient algorithm for solving multi-pantograph equation systems. *Comput. Math. Appl.* **64**(4) (2012) 589–603. doi:10.1016/j.camwa.2011.12.062. MR2948605

[31] M. Khalid, S. Mariam, Z. Faheem and S.K. Fareeha, Solving polluted lakes system by using perturbation iteration method. *Int J Comput Appl.* **114**(4) (2015) 1–7.

[32] M. Khalid, S. Mariam and S.K. Fareeha, Numerical solution of SIR model of Dengue fever. *Int J Comput Appl.* **118**(21) (2015) 1–4.

[33] M. Khalid, S. Mariam, Z. Faheem and S.K. Fareeha, A numerical solution of a model for HIV infection CD 4+ T-Cell. *Int. J. Innov. Sci. Res.* **16**(1) (2015) 79–85.

[34] M. Khalid, S.K. Fareeha and I. Azhar, Perturbation-Iteration Algorithm to Solve Fractional Giving Up Smoking Mathematical Model. *Int J Comput Appl.* **142**(9) (2016) 1–6.

[35] M. Khalid, S.K. Fareeha, Z. Husna and M. Shoaib, A highly accurate numerical method for solving time fractional partial differential equation. *Prog Fract Differ Appl.* **2**(3) (2016) 227–232.

[36] M. Khalid, S.K. Fareeha and S. Mariam, A highly accurate numerical method for solving nonlinear time-fractional differential difference equation. *Math Meth Appl Sci.* (2019) 1–11. doi:10.1002/mma.5883. MR4271073

[37] A. Isah, C. Phang and P. Phang, Collocation Method Based on Genocchi Operational Matrix for Solving Generalized Fractional Pantograph Equations. *International Journal of Differential Equations* **2017** (2017) Article ID 2097317. MR3666267

[38] D. Vivek, K. Kanagarajan and S. Harikrishnan, Existence and uniqueness results for pantograph equations with generalized fractional derivative. *Journal Nonlinear Analysis and Application* **2017**(2) (2017) 105–112.

[39] A. Anguraj, A. Vinodkumar and K. Malar, Existence and Stability Results for Random Impulsive Fractional Pantograph Equations, *Filomat* **30**(14) (2016) 3839–3854. MR3593754

[40] P. Rahimkhani, Y. Ordokhani and E. Babolian, Numerical solution of fractional pantograph differential equations by using generalized fractional-order Bernoulli wavelet. *Journal of Computational and Applied Mathematics* **309**(1) (2017) 493–510. MR3539800

[41] P. Rahimkhani, Y. Ordokhani and E. Babolian, M̈untz-Legendre wavelet operational matrix of fractional-order integration and its applications for solving the fractional pantograph differential equations. *Numerical Algorithms* **77** (2018) 1283–1305. doi:10.1007/s11075-017-0363-4. MR3779088

[42] Y. Yin and H. Yunqing, Spectral-collocation methods for fractional pantograph delay-integrodifferential equations. *Advanced Topics in Fractional Dynamics*, **2013** (2013) Article ID 821327. doi:10.1155/2013/821327. MR3132689

[43] K. Balachandran, S. Kiruthika and J.J. Trujillo, Existence of solutions of nonlinear fractional pantograph equations. *Acta Mathematica Scientia* **33**(3) (2013) 712–720. MR3048171

[44] A.H. Bhrawy, A.A. Al-Zahrani, Y.A. Alhamed and B. Dumitru, A new generalized Laguerre-Gauss collocation scheme for numerical solution of generalized fractional pantograph equations. *Rom. Journ. Phys.* **59**(7-8) (2014) 646–657.

[45] D. Baleanu, B. Agheli and R. Darzi, An optimal method for approximating the delay differential equations of noninteger order. *Advances in Difference Equations* **2018** (2018) Article ID 284. doi:10.1186/s13662-018-1717-5. MR3844652

M. Khalid
Department of Mathematical Sciences
Federal Urdu University of Arts, Science & Technology
Karachi-75300
Pakistan
E-mail address: khalidsiddiqui@fuuast.edu.pk

S. K. Fareeha
Department of Mathematical Sciences
Federal Urdu University of Arts, Science & Technology
Karachi-75300
Pakistan
E-mail address: fareeha.samikhan@fuuast.edu.pk

S. Mariam
Department of Mathematical Sciences
Federal Urdu University of Arts, Science & Technology
Karachi-75300
Pakistan
E-mail address: marium.sultana@fuuast.edu.pk

Received March 12, 2021