Annals of Mathematical Sciences and Applications
Volume 7, Number 1, 3–19, 2022

A structure-preserving algorithm for the linear lossless dissipative Hamiltonian eigenvalue problem

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In this paper, we propose a structure-preserving algorithm for computing all eigenvalues of the generalized eigenvalue problem $BA\mathbf{x} = \lambda E\mathbf{x}$ that arises in linear lossless dissipative Hamiltonian descriptor systems, with B being skew-symmetric and $A^\top E = E^\top A$. We rewrite the problem as $BAE^{-1}\mathbf{y} = \lambda\mathbf{y}$ to preserve the symmetry of $A^\top E$ and convert the problem into the equivalent $\top$-Hamiltonian eigenvalue problem $\mathscr{H}\mathbf{z} = \lambda\mathbf{z}$. Furthermore, $\top$-symplectic URV decomposition and a corresponding periodic QR (PQR) method are proposed to compute all eigenvalues of $\mathscr{H}$. The structure-preserving property ensures that the computed eigenvalues appear pairwise, in the form $(\lambda, -\lambda)$, as they should. Numerical experiments show that the computed eigenvalues are more accurate and strictly paired than those of the classical QZ method, while the residuals of the eigenpairs are comparable.

Keywords and phrases: Structure-preserving algorithm, $\top$-Hamiltonian eigenvalue problem, $\top$-symplectic URV decomposition, periodic QR.

1. Introduction

In recent years, the energy based modeling of dynamic systems gains great attention, in which the port-Hamiltonian (PH) system [7, 8, 14, 18, 19, 23, 24] characterize models from variational principles. As a special case of PH descriptor system, the linear time-invariant dissipative Hamiltonian (DH) descriptor systems [2, 15, 16, 21], generally expressed as

$$E\dot{x} = (B - R)Ax, \tag{1}$$

where $B \in \mathbb{R}^{2n\times 2n}$ is nonsingular and skew-symmetric and $A^\top E = E^\top A \in \mathbb{R}^{2n\times 2n}$, have been widely considered in recent years. The matrix B is the structure matrix, reflecting the energy flux among energy storage elements; R is the dissipation matrix, describing the energy dissipation (due to dampers, viscosity, resistors, etc.); and $A^\top E = E^\top A$ guarantees the Hamiltonian nature of the system.

In this paper, for linear dissipative Hamiltonian descriptor systems, we assume that $R = 0$ [20, 22], which corresponds to a lossless (energy-conserving) model and is often adopted for RLC circuits and mass-spring-damper (MSD) systems. In the lossless case, the DH eigenvalue problem (DHEP) convert to a generalized eigenvalue problem (GEP)

$$BA\mathbf{x} = \lambda E\mathbf{x}, \tag{2}$$

of which the eigenpairs can be easily computed by QR-type eigensolvers, *e.g.*, the QZ method [17, 25]; however, the symmetry of $A^\top E$ is ignored, and consequently, the paired structure of the eigenvalues $(\lambda, -\lambda)$ of (2), which is an essential property of the original system, may be lost. Nevertheless, under the assumption that E is invertible, (2) can be rewritten in the following form:

$$BAE^{-1}\mathbf{y} = \lambda\mathbf{y}. \tag{3}$$

We can consider $B \in \mathbb{C}^{2n\times 2n}$ to be nonsingular and skew-symmetric, and $A^\top E = E^\top A \in \mathbb{C}^{2n\times 2n}$; this constitutes one complex extension of the real case. Notably, this complex case does not correspond to a complex lossless dissipative Hamiltonian descriptor system [15], and further applications may yet be discovered by researchers pursuing related work. However, the algorithms proposed in this paper can perfectly fit a real system with only minor revisions.

The main contributions of this paper are to prove that the product eigenvalue problem given in (3) can be solved by instead computing one $\top$-Hamiltonian eigenvalue problem ($\top$HEP) [12], and to propose a structure-preserving algorithm that guarantees both the pairing property of the eigenvalues and the accuracy of the eigenpairs. In fact, only half of the eigenvalues need to be computed during the implementation procedure.

The basic theories and algorithms for Hamiltonian eigenvalue problems have been discussed in recent years; see, *e.g.*, [1, 3, 6, 11]. Symplectic URV decomposition [3, 4, 9] followed by the periodic QR method [5, 9, 10, 13] is an efficient technique for computing all eigenvalues. Some trivial propositions regarding $\top$-Hamiltonian matrices or unitary $\top$-symplectic matrices are applied in this paper without further explanation (refer to [3, 11] for details).

Notations: Bold letters denote vectors; $\mathbf{e}_k$ is the k-th column of an identity matrix I_n; $\bar{A}$, $A^\top$ and A^* denote the conjugate, transpose and conjugate transpose of A, respectively; $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$; $\mathbb{H}_\top$ is the set of complex

$\top$-Hamiltonian matrices; $\mathbb{U}_{\top\mathrm{S}}$ is the set of unitary $\top$-symplectic matrices; and $\mathbb{U}$ is the set of unitary matrices.

2. Preliminaries

2.1. Similarity to a $\top$-Hamiltonian matrix

In this section, we prove that the matrix product BAE^{-1} is similar to a $\top$-Hamiltonian matrix and yields a structure-preserving algorithm for computing all eigenvalues of (3).

Theorem 1. *There exist $Q_B \in \mathbb{U}^{2n\times 2n}$ and a lower bidiagonal matrix $L_B \in \mathbb{C}^{n\times n}$ s.t. $Q_B B Q_B^\top = \begin{bmatrix} 0 & L_B \\ -L_B^\top & 0 \end{bmatrix}$.*

Proof. Since B is skew-symmetric, there exists a unitary matrix $\tilde{Q}_B$ *s.t.* $\tilde{B} \equiv \tilde{Q}_B B \tilde{Q}_B^\top$ is an upper Hessenberg matrix; as a result, $\tilde{B}$ is skew-symmetric and, thus, a tridiagonal matrix with all values on its main diagonal being zero. Moreover, we let $p = [1, n+1, 2, n+2, \ldots, n, 2n]$, and $\tilde{B}(p(:), p(:))$ becomes $\begin{bmatrix} 0 & L_B \\ -L_B^\top & 0 \end{bmatrix}$. $\square$

The detailed derivations of Q_B and L_B can be found in Algorithm 1.

Algorithm 1 Tridiagonal reduction of a complex skew-symmetric matrix

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Input: A complex skew-symmetric matrix $B \in \mathbb{C}^{2n\times 2n}$.
Output: $Q_B \in \mathbb{U}^{2n\times 2n}$ and a lower bidiagonal matrix $L_B \in \mathbb{C}^{n\times n}$ *s.t.*
$Q_B B Q_B^\top = \begin{bmatrix} 0 & L_B \\ -L_B^\top & 0 \end{bmatrix}$.

1: $Q_B \leftarrow I_{2n}$, $P_B \leftarrow I_{2n}$;
2: **for** $i = 1 : 2n-2$ **do**
3: Compute $Q = \mathbf{house}(B\mathbf{e}_i, i+1, 2n)$, $B \leftarrow QBQ^\top$, $Q_B \leftarrow QQ_B$;
4: **end for**
5: Set $p = \left[1, n+1, 2, n+2, \ldots, n, 2n\right]^\top$ and rearrange all columns of P_B as in p;
6: $B \leftarrow P_B B P_B^\top$, $Q_B \leftarrow P_B Q_B$ and $L_B \leftarrow B(1:n, n+1:2n)$.

Now, by combining the block structure of $Q_B B Q_B^\top$ with the symmetry of AE^{-1}, we can prove that BAE^{-1} is similar to a $\top$-Hamiltonian matrix.

Theorem 2. *Let* $R = \begin{bmatrix} 0 & I \\ -L_B^\top & 0 \end{bmatrix}$, $\tilde{A} = \bar{Q}_B A$, $\tilde{E} = Q_B E$, $T = \begin{bmatrix} L_B & 0 \\ 0 & I \end{bmatrix}$, *and*

$$\mathscr{H} = R\tilde{A}\tilde{E}^{-1}T \in \mathbb{C}^{2n\times 2n}. \tag{4}$$

Then, BAE^{-1} *is similar to the* $\top$*-Hamiltonian matrix* $\mathscr{H}$*.*

Proof. Consider that

$$\begin{aligned} Q_B(BAE^{-1})Q_B^* &= (Q_B B Q_B^\top)(\bar{Q}_B AE^{-1}Q_B^*) = \begin{bmatrix} 0 & L_B \\ -L_B^\top & 0 \end{bmatrix} (\bar{Q}_B A)(Q_B E)^{-1} \\ &= \begin{bmatrix} L_B & 0 \\ 0 & I \end{bmatrix} \left(\begin{bmatrix} 0 & I \\ -L_B^\top & 0 \end{bmatrix} (Q_B A)(Q_B E)^{-1} \begin{bmatrix} L_B & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} L_B & 0 \\ 0 & I \end{bmatrix}^{-1}. \end{aligned}$$

Let $\mathcal{Y} = AE^{-1}$. Since $A^\top E = E^\top A$ is symmetric, we know that $\mathcal{Y}$ is symmetric. Let $\mathcal{Y} \equiv \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$. In combination with the above definitions, we find that

$$\begin{aligned} \mathscr{H} &= R\tilde{A}\tilde{E}^{-1}T = R\mathcal{Y}T \\ &= \begin{bmatrix} 0 & I \\ -L_B^\top & 0 \end{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} L_B & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} Y_{21}L_B & Y_{22} \\ -L_B^\top Y_{11} L_B & -L_B^\top Y_{12} \end{bmatrix} \end{aligned}$$

is a complex $\top$-Hamiltonian matrix. □

Therefore, because of the similarity between BAE^{-1} and $\mathscr{H}$, the eigenvalue problem expressed in (3) can be converted into the following complex $\top$HEP:

$$\mathscr{H}\mathbf{z} = R\tilde{A}\tilde{E}^{-1}T\mathbf{z} = \lambda\mathbf{z}. \tag{5}$$

Accordingly, we can design a corresponding $\top$-symplectic URV decomposition and periodic QR (PQR) method, as introduced in Section 3, to exploit the matrix product structure and preserve the $\top$-Hamiltonian structure of $\mathscr{H}$.

Since $\mathscr{H}$ is a $\top$-Hamiltonian matrix, there exists a unitary $\top$-symplectic URV decomposition of $\mathscr{H}$; moreover, there exist unitary matrices W_1, W_2, and W_3 *s.t.*

$$
\begin{aligned}
U\mathscr{H}V^* &= UR\tilde{A}\tilde{E}^{-1}TV^* = (URW_1^*)(W_1\tilde{A}W_2^*)(W_3\tilde{E}W_2^*)^{-1}(W_3TV^*) \\
&\equiv R_M\tilde{A}_M\tilde{E}_M^{-1}T_M \equiv \begin{bmatrix} R_t & R_r \\ 0 & R_b \end{bmatrix}\begin{bmatrix} \tilde{A}_t & \tilde{A}_r \\ 0 & \tilde{A}_b \end{bmatrix}\begin{bmatrix} \tilde{E}_t & \tilde{E}_r \\ 0 & \tilde{E}_b \end{bmatrix}^{-1}\begin{bmatrix} T_t & T_r \\ 0 & T_b \end{bmatrix} \\
&= \begin{bmatrix} R_t\tilde{A}_t\tilde{E}_t^{-1}T_t & * \\ 0 & R_b\tilde{A}_b\tilde{E}_b^{-1}T_b \end{bmatrix},
\end{aligned} \tag{6}
$$

where $T_b^\top$ is an upper Hessenberg matrix and R_t, $R_b^\top$, $\tilde{A}_t$, $\tilde{A}_b^\top$, $\tilde{E}_t$, $\tilde{E}_b^\top$ and T_t are upper triangular matrices. In this paper, we refer to the matrix structure of R_M, $\tilde{A}_M$, and $\tilde{E}_M$ as the block-triangular form (BTF) and the structure of T_M as the triangular-Hessenberg form (THF).

2.2. Elementary unitary/unitary $\top$-symplectic transformations

The decomposition of $\mathscr{H}$ in (6) can be realized by means of a series of unitary $\top$-symplectic transformations and standard unitary transformations. Accordingly, the corresponding elementary transformations should be fully utilized.

- We refer to $H \in \mathbb{C}^{n\times n}$ as a Householder transformation matrix if

  $$H = I_n - 2\mathbf{w}\mathbf{w}^*, \tag{7}$$

  with $\mathbf{w} \in \mathbb{C}^n$ and $\|\mathbf{w}\|_2 = 1$. Additionally, if $\mathbf{x} = [x_1, \dots, x_n]^\top$, we write $H = \mathbf{house}(\mathbf{x}, i, j)$ if H satisfies (7) and

  $$H\mathbf{x} = [x_1, \dots, x_{i-1}, \hat{x}_i, 0, \dots, 0, x_{j+1}, \dots, x_n]^\top,$$

  where $\hat{x}_i = \|[x_i, x_{i+1}, \dots, x_j]^\top\|_2$. The corresponding $\top$-symplectic Householder matrix $\mathcal{H} \in \mathbb{U}_{\top\mathrm{S}}^{2n\times 2n}$ is a direct sum of two Householder matrices, $\mathcal{H} = \begin{bmatrix} H & 0 \\ 0 & \bar{H} \end{bmatrix} = \boldsymbol{\mathcal{S}}\textbf{-house}(\mathbf{x}, i, j)$.
- We refer to $G \in \mathbb{C}^{n\times n}$ as a Givens rotation matrix if

  $$G = I_n + s\mathbf{e}_i\mathbf{e}_j^\top - \bar{s}\mathbf{e}_j\mathbf{e}_i^\top + (c-1)(\mathbf{e}_i\mathbf{e}_i^\top + \mathbf{e}_j\mathbf{e}_j^\top), \tag{8}$$

  with $c = \cos(\theta)$ and $s = \sin(\theta)$. Additionally, we write $G = \mathbf{givens}(\mathbf{x}, i, j)$ if G satisfies (8) and

  $$\begin{bmatrix} c & s \\ -\bar{s} & c \end{bmatrix}\begin{bmatrix} x_i \\ x_j \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix},$$

where $r = \|[x_i, x_j]^\top\|_2$. The corresponding $\top$-symplectic Givens matrix $\mathcal{G} \in \mathbb{U}_{\top\mathrm{S}}^{2n\times 2n}$ is a direct sum of two Givens matrices, $\mathcal{G} = \begin{bmatrix} G & 0 \\ 0 & \bar{G} \end{bmatrix} = \mathcal{S}\text{-}\mathbf{givens}(\mathbf{x}, i, j)$.

3. A stable algorithm for computing the eigenvalues of $\mathscr{H}$

3.1. $\top$-symplectic URV decomposition of $\mathscr{H}$

We divide the $\top$-symplectic URV decomposition of $\mathscr{H}$ into four main steps; subsequently, two unitary $\top$-symplectic transformations and three unitary transformations are computed. In the following steps, $R^{(i)}$, $\tilde{A}^{(i)}$ and $\tilde{E}^{(i)}$ are matrices of the BTF, $T^{(i)}$ is a matrix of the THF, as shown in (6); $U^{(i)}, V^{(i)} \in \mathbb{U}_{\top\mathrm{S}}^{2n\times 2n}$ and $W_j^{(i)} \in \mathbb{U}^{2n\times 2n}$, for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$.

Step 1: Find $U^{(1)}$ and $W_1^{(1)}$ *s.t.* $R^{(1)} = U^{(1)} R W_1^{(1)\top}$ and $\tilde{A}^{(1)} = W_1^{(1)} \tilde{A}$. The purpose of this step is to reduce R to the BTF. It can be straightforwardly verified that $U^{(1)} = -J$ and $W_1^{(1)} = \begin{bmatrix} I & \\ & -I \end{bmatrix}$ due to the special structure of R.

Step 2: Find $U^{(2)}$, $W_1^{(2)}$ and $W_2^{(2)}$ *s.t.* $R^{(2)} = U^{(2)} R^{(1)} W_1^{(2)\top}$, $\tilde{A}^{(2)} = W_1^{(2)} \tilde{A}^{(1)} W_2^{(2)\top}$ and $\tilde{E}^{(2)} = E W_2^{(2)\top}$. The purpose of this step is to reduce $\tilde{A}^{(1)}$ to the BTF while also preserving the BTF of $R^{(1)}$. The elimination of nonzero elements in $\tilde{A}^{(1)}$ is similar to the standard $\top$-symplectic URV decomposition of a Hamiltonian matrix [3], but the upper left corner of $\tilde{A}^{(2)}$ should be upper triangular. In particular, each row transformation generates one new nonzero element in $R^{(1)}$; therefore, one unitary $\top$-symplectic Givens rotation and one unitary Givens rotation are needed to preserve the BTF of $R^{(2)}$. Additionally, the column Householder transformations of $\tilde{A}^{(1)}$ lead to column transformations of $\tilde{E}^{(1)}$.

Step 3: Find $U^{(3)}$, $W_1^{(3)}$, $W_2^{(3)}$ and $W_3^{(3)}$ *s.t.* $R^{(3)} = U^{(3)} R^{(2)} W_1^{(3)\top}$, $\tilde{A}^{(3)} = W_1^{(3)} \tilde{A}^{(2)} W_2^{(3)\top}$, $\tilde{E}^{(3)} = W_3^{(3)} \tilde{E}^{(2)} W_2^{(3)\top}$ and $T^{(3)} = W_3^{(3)} T$. The purpose of this step is to reduce $\tilde{E}^{(2)}$ to the BTF while preserving the BTFs of both $R^{(2)}$ and $\tilde{A}^{(2)}$. This step is similar to Step 2 but is more cautious. In this step, the detailed reduction procedure for $\tilde{E}^{(2)}$ follows the opposite order relative to that in Step 2. For example, the nonzero elements in the $(n+i)$-th row of $\tilde{E}^{(2)}$ need to be eliminated first, followed by the nonzero elements in the i-th column.

Step 4: Find $U^{(4)}$, $V^{(4)}$, $W_1^{(4)}$, $W_2^{(4)}$ and $W_3^{(4)}$ *s.t.* $R^{(4)} = U^{(4)}R^{(3)}W_1^{(4)\top}$, $\tilde{A}^{(4)} = W_1^{(4)}\tilde{A}^{(3)}W_2^{(4)\top}$, $\tilde{E}^{(4)} = W_3^{(4)}\tilde{E}^{(3)}W_2^{(4)\top}$ and $T^{(4)} = W_3^{(4)}T^{(3)}V^{(4)}$, where $T^{(4)}$ is a THF matrix. The purpose of this step is to reduce $T^{(3)}$ to the THF while preserving the BTFs of $R^{(3)}$, $\tilde{A}^{(3)}$ and $\tilde{E}^{(3)}$. This step is also similar to Step 2. Note that $T^{(3)}$ is a block diagonal matrix and that each block has dimensions of $n \times n$; accordingly, the computational costs of row and column transformations for $T^{(3)}$ can be reduced.

Algorithm 2 Step 2 of $\top$-symplectic URV decomposition for $\mathscr{H}$

Input: $R^{(1)} \in \mathbb{C}^{2n\times 2n}$ is of BTF, $\tilde{A}^{(1)} \in \mathbb{C}^{2n\times 2n}$ and $\tilde{E}^{(1)} \in \mathbb{C}^{2n\times 2n}$.

Output: $U^{(2)} \in \mathbb{U}_{\top\mathrm{S}}^{2n\times 2n}$, $W_1^{(2)}, W_2^{(2)} \in \mathbb{U}^{2n\times 2n}$ *s.t.* $R^{(2)} = U^{(2)}R^{(1)}W_1^{(2)\top}$ and $\tilde{A}^{(2)} = W_1^{(2)}\tilde{A}^{(1)}W_2^{(2)\top}$ are of BTF.

$U^{(2)} \leftarrow I_{2n}$, $W_1^{(2)} \leftarrow I_{2n}$, $W_2^{(2)} \leftarrow I_{2n}$, $R^{(2)} \leftarrow R^{(1)}$, $\tilde{A}^{(2)} \leftarrow \tilde{A}^{(1)}$, $\tilde{E}^{(2)} \leftarrow E$.
for $j = 1 : n$ **do**
  for $i = j : n-1$ **do**
    Apply **givens-trans**$(\tilde{A}^{(2)}\mathbf{e}_j, n+i+1, n+i, R^{(2)}, \tilde{A}^{(2)}, W_1^{(2)})$;
    Apply **givens-trans**$(R^{(2)\top}\mathbf{e}_{i+1}, i+1, i, R^{(2)}, \tilde{A}^{(2)}, W_1^{(2)})$;
  end for
  Apply **givens-trans**$(\tilde{A}^{(2)}\mathbf{e}_j, n, 2n, R^{(2)}, \tilde{A}^{(2)}, W_1^{(2)})$;
  Apply $\mathcal{S}$-**givens-trans**$(R^{(2)}\mathbf{e}_n, n, 2n, R^{(2)}, U^{(2)})$;
  for $i = n-1 : j$ **do**
    Apply **givens-trans**$(\tilde{A}^{(2)}\mathbf{e}_j, i, i+1, R^{(2)}, \tilde{A}^{(2)}, W_1^{(2)})$;
    Apply $\mathcal{S}$-**givens-trans**$(R^{(2)}\mathbf{e}_i, i, i+1, R^{(2)}, U^{(2)})$;
    Apply **givens-trans**$(R^{(2)}\mathbf{e}_{2n}, n+i, n+i+1, R^{(2)}, \tilde{A}^{(2)}, W_1^{(2)})$;
    if $j < n$ **then**
      Apply **house-trans**$(\tilde{A}^{(2)\top}\mathbf{e}_{n+j}, j+1, n, \tilde{A}^{(2)}, \tilde{E}^{(2)\top}, W_2^{(2)})$;
      Apply **givens-trans**$(\tilde{A}^{(2)\top}\mathbf{e}_{n+j}, n+j+1, j+1, \tilde{A}^{(2)}, \tilde{E}^{(2)\top}, W_2^{(2)})$;
      Apply **house-trans**$(\tilde{A}^{(2)\top}\mathbf{e}_{n+j}, n+j, 2n, \tilde{A}^{(2)}, \tilde{E}^{(2)\top}, W_2^{(2)})$;
    end if
  end for
end for

We denote some notations to save illustrative words here, since there are too many similar matrix operations during the URV decomposition of $\mathscr{H}$.

- **house-trans**$(\mathbf{x}, i, j, X, Y, Q)$: determine $H = \mathbf{house}(\mathbf{x}, i, j)$ and set $X = XH^\top$, $Y = HY$ and $Q = QH^\top$;
- $\mathcal{S}$-**house-trans**$(\mathbf{x}, i, j, X, Q)$: determine $\mathcal{H} = \mathcal{S}$-**house**$(\mathbf{x}, i, j)$ and set $X = \mathcal{H}X$ and $Q = \mathcal{H}Q$;

- **givens-trans**$(\mathbf{x}, i, j, X, Y, Q)$: determine $G = \textbf{givens}(\mathbf{x}, i, j)$ and set $X = XG^\top$, $Y = GY$ and $Q = QG^\top$;
- $\mathcal{S}$**-givens-trans**$(\mathbf{x}, i, j, X, Q)$: determine $\mathcal{G} = \mathcal{S}\textbf{-givens}(\mathbf{x}, i, j)$ and set $X = \mathcal{G}X$ and $Q = \mathcal{G}Q$;

We summarize Step 2, Step 3 and Step 4 into Algorithm 2, 3 and 4, respectively, for better showing the implementation details of the $\top$-symplectic URV decomposition of $\mathscr{H}$.

As a result of the $\top$-symplectic URV decomposition procedure, the $\top$-

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Algorithm 3 Step 3 of $\top$-symplectic URV decomposition for $\mathscr{H}$

Input: $R^{(2)}, \tilde{A}^{(2)} \in \mathbb{C}^{2n\times 2n}$ are of BTF and $\tilde{E}^{(2)} \in \mathbb{C}^{2n\times 2n}$.

Output: $U^{(3)} \in \mathbb{U}_{\top\mathrm{S}}^{2n\times 2n}$, $W_1^{(3)}, W_2^{(3)}, W_3^{(3)} \in \mathbb{U}^{2n\times 2n}$ *s.t.* $R^{(3)} = U^{(3)}R^{(2)}W_1^{(3)\top}$, $\tilde{A}^{(3)} = W_1^{(3)}\tilde{A}^{(2)}W_2^{(3)\top}$ and $\tilde{E}^{(3)} = W_2^{(3)}\tilde{E}^{(2)}W_3^{(3)\top}$ are of BTF.

$U^{(3)} \leftarrow I_{2n}$, $W_1^{(3)} \leftarrow I_{2n}$, $W_2^{(3)} \leftarrow I_{2n}$, $V^{(3)} \leftarrow I_{2n}$, $R^{(3)} \leftarrow R^{(2)}$, $\tilde{A}^{(3)} \leftarrow \tilde{A}^{(2)}$, $\tilde{E}^{(3)} \leftarrow \tilde{E}^{(2)}$, $T^{(3)} \leftarrow T$.
for $j = 1 : n$ **do**
  for $i = j : n-1$ **do**
    Apply **givens-trans**$(\tilde{E}^{(3)\top}\mathbf{e}_{n+j}, i+1, i, \tilde{E}^{(3)}, \tilde{A}^{(3)\top}, W_2^{(3)})$;
    Apply **givens-trans**$(\tilde{A}^{(3)}\mathbf{e}_i, i, i+1, R^{(3)}, \tilde{A}^{(3)}, W_1^{(3)})$;
    Apply $\mathcal{S}$**-givens-trans**$(R^{(3)}\mathbf{e}_i, i, i+1), R^{(3)}, U^{(3)}$;
    Apply **givens-trans**$(R^{(3)\top}\mathbf{e}_{n+i}, n+i, n+i+1, R^{(3)}, \tilde{A}^{(3)}, W_1^{(3)})$;
    Apply **givens-trans**$(\tilde{A}^{(3)\top}\mathbf{e}_{n+i}, n+i, n+i+1, \tilde{A}^{(3)}, \tilde{E}^{(3)\top}, W_2^{(3)})$;
  end for
  Apply **givens-trans**$(\tilde{E}^{(3)\top}\mathbf{e}_{n+j}, 2n, n, \tilde{A}^{(3)}, \tilde{E}^{(3)\top}, W_2^{(3)})$;
  Apply **givens-trans**$(\tilde{A}^{(3)}\mathbf{e}_n, n, 2n, R^{(3)}, \tilde{A}^{(3)}, W_1^{(3)})$;
  Apply $\mathcal{S}$**-givens-trans**$(R^{(3)}\mathbf{e}_n, n, 2n, R^{(3)}, U^{(3)})$;
  for $i = n-1 : j$ **do**
    Apply **givens-trans**$(\tilde{E}^{(3)\top}\mathbf{e}_{n+j}, n+i, n+i+1, \tilde{A}^{(3)}, \tilde{E}^{(3)\top}, W_2^{(3)})$;
    Apply **givens-trans**$(\tilde{A}^{(3)}\mathbf{e}_{n+i+1}, n+i+1, n+i, R^{(3)}, \tilde{A}^{(3)}, W_1^{(3)})$;
    Apply $\mathcal{S}$**-givens-trans**$(R^{(3)}\mathbf{e}_{n+i+1}, n+i+1, n+i, R^{(3)}, U^{(3)})$;
    Apply **givens-trans**$(R^{(3)\top}\mathbf{e}_{i+1}, i+1, i, R^{(3)}, \tilde{A}^{(3)}, W_1^{(3)})$;
    Apply **givens-trans**$(\tilde{A}^{(3)\top}\mathbf{e}_{i+1}, i+1, i, \tilde{A}^{(3)}, \tilde{E}^{(3)\top}, W_2^{(3)})$;
    if $j < n$ **then**
      Apply **house-trans**$(\tilde{E}^{(3)}\mathbf{e}_j, n+j+1, 2n, \tilde{E}^{(3)\top}, T^{(3)\top}, W_3^{(3)\top})$;
      Apply **givens-trans**$(\tilde{E}^{(3)}\mathbf{e}_j, j+1, n+j+1, \tilde{E}^{(3)\top}, T^{(3)\top}, W_3^{(3)\top})$;
      Apply **house-trans**$(\tilde{E}^{(3)}\mathbf{e}_j, j, n, \tilde{E}^{(3)\top}, T^{(3)\top}, W_3^{(3)\top})$;
    end if
  end for
end for

Algorithm 4 Step 4 of $\top$-symplectic URV decomposition for $\mathscr{H}$

Input: $R^{(3)}, \tilde{A}^{(3)}, \tilde{E}^{(3)} \in \mathbb{C}^{2n\times 2n}$ are of BTF, $T^{(3)} \in \mathbb{C}^{2n\times 2n}$.

Output: $U^{(4)}, V^{(4)} \in \mathbb{U}_{\top\mathrm{S}}^{2n\times 2n}$, $W_1^{(4)}, W_2^{(4)}, W_3^{(4)} \in \mathbb{U}^{2n\times 2n}$ *s.t.* $R^{(4)} = U^{(4)} R^{(3)} W_1^{(4)\top}$, $\tilde{A}^{(4)} = W_1^{(4)} \tilde{A}^{(3)} W_2^{(4)\top}$ and $\tilde{E}^{(4)} = W_3^{(4)} \tilde{E}^{(3)} W_2^{(4)\top}$ are of BTF, and $T^{(4)} = W_3^{(4)} T^{(3)} V^{(4)}$ is of THF.

$U^{(4)} \leftarrow I_{2n}$, $W_1^{(4)} \leftarrow I_{2n}$, $W_2^{(4)} \leftarrow I_{2n}$, $W_3^{(4)} \leftarrow I_{2n}$, $V^{(4)} \leftarrow I_{2n}$, $R^{(4)} \leftarrow R^{(3)}$, $\tilde{A}^{(4)} \leftarrow \tilde{A}^{(3)}$, $\tilde{E}^{(4)} \leftarrow \tilde{E}^{(3)}$ and $T^{(4)} \leftarrow T^{(3)}$.
for $j = 1, \dots, n$ **do**
  for $i = j, \dots, n-1$ **do**
    Apply **givens-trans**($T^{(4)}\mathbf{e}_j, n+i+1, n+i, \tilde{E}^{(4)\top}, T^{(4)\top}, W_3^{(4)\top}$);
    Apply **givens-trans**($\tilde{E}^{(4)\top}\mathbf{e}_{n+i}, n+i, n+i+1, \tilde{A}^{(4)}, \tilde{E}^{(4)\top}, W_2^{(4)}$);
    Apply **givens-trans**($\tilde{A}^{(4)}\mathbf{e}_{n+i+1}, n+i+1, n+i, R^{(4)}, \tilde{A}^{(4)}, W_1^{(4)}$);
    Apply $\mathcal{S}$-**givens-trans**($R^{(4)}\mathbf{e}_{n+i+1}, n+i+1, n+i, R^{(4)}, U^{(4)}$);
    Apply **givens-trans**($R^{(4)\top}\mathbf{e}_{i+1}, i+1, i, R^{(4)}, \tilde{A}^{(4)}, W_1^{(4)}$);
    Apply **givens-trans**($\tilde{A}^{(4)}\mathbf{e}_{i+1}, i+1, i, \tilde{A}^{(4)}, \tilde{E}^{(4)\top}, W_2^{(4)}$);
    Apply **givens-trans**($\tilde{E}^{(4)}\mathbf{e}_{i}, i, i+1, \tilde{E}^{(4)\top}, T^{(4)\top}, W_3^{(4)\top}$);
  end for
  Apply **givens-trans**($T^{(4)}\mathbf{e}_j, n, 2n, \tilde{E}^{(4)\top}, T^{(4)\top}, W_3^{(4)\top}$);
  Apply **givens-trans**($\tilde{E}^{(4)\top}\mathbf{e}_{2n}, 2n, n, \tilde{A}^{(4)}, \tilde{E}^{(4)\top}, W_2^{(4)}$);
  Apply **givens**($\tilde{A}^{(4)}\mathbf{e}_{n}, n, 2n, R^{(4)}, \tilde{A}^{(4)}, W_1^{(4)}$);
  Apply $\mathcal{S}$-**givens-trans**($R^{(4)}\mathbf{e}_{n}, n, 2n, R^{(4)}, U^{(4)}$);
  for $i = n-1, \dots, j$ **do**
    Apply **givens-trans**($T^{(4)}\mathbf{e}_j, i, i+1, \tilde{E}^{(4)\top}, T^{(4)\top}, W_3^{(4)\top}$);
    Apply **givens-trans**($\tilde{E}^{(4)\top}\mathbf{e}_{i+1}, i+1, i, \tilde{A}^{(4)}, \tilde{E}^{(4)\top}, W_2^{(4)}$);
    Apply **givens-trans**($\tilde{A}^{(4)}\mathbf{e}_{i}, i, i+1, R^{(4)}, \tilde{A}^{(4)}, W_1^{(4)}$);
    Apply $\mathcal{S}$-**givens-trans**($R^{(4)}\mathbf{e}_{i}, i, i+1, R^{(4)}, U^{(4)}$);
    Apply **givens-trans**($R^{(4)\top}\mathbf{e}_{n+i}, n+i, n+i+1, R^{(4)}, \tilde{A}^{(4)}, W_1^{(4)}$);
    Apply **givens-trans**($\tilde{A}^{(4)\top}\mathbf{e}_{n+i}, n+i, n+i+1, tAd, \tilde{E}^{(4)\top}, W_2^{(4)}$);
    Apply **givens-trans**($\tilde{E}^{(4)\top}\mathbf{e}_{n+i+1}, n+i+1, n+i, \tilde{E}^{(4)\top}, T^{(4)\top}, W_3^{(4)\top}$);
    if $j < n$ **then**
      Apply $\mathcal{S}$-**house-trans**($T^{(4)\top}\mathbf{e}_{n+j}, j+1, n, T^{(4)\top}, V^{(4)\top}$);
      Set $\mathbf{x} = T^{(4)\top}\mathbf{e}_{n+j}$ and determine $G =$ **givens**($\mathbf{x}, n+j+1, j+1$);
      Set $T^{(4)} = T^{(4)}G^\top$ and $V^{(4)} = V^{(4)}G^\top$;
      Set $\mathbf{x} = T^{(4)\top}\mathbf{e}_{n+j}$ and determine $\mathcal{H} = \mathcal{S}$-**house**($\mathbf{x}, n+j+1, 2n$);
      Set $T^{(4)} = T^{(4)}\mathcal{H}^\top$ and $V^{(4)} = T^{(4)}\mathcal{H}^\top$;
    end if
  end for
end for

Hamiltonian matrix $\mathscr{H}$ can be reduced to the specific form given in (6) with the help of the unitary transformations and unitary $\top$-symplectic transformations described in Section 2.2.

3.2. Periodic QR method

After computing the URV decomposition of $\mathscr{H}$, it is natural to computer the eigenpairs of the corresponding matrix product using the periodic QR method. Let $M = -T_b^\top \tilde{E}_b^{-\top} \tilde{A}_b^\top R_b^\top R_t \tilde{A}_t \tilde{E}_t^{-1} T_t \in \mathbb{C}^{n\times n}$, where R_t, R_b, $\tilde{A}_t$, $\tilde{A}_b$, $\tilde{E}_t$, $\tilde{E}_b, T_t$ and T_b are computed through the $\top$-symplectic URV decomposition of $\mathscr{H}$. It follows that

(9)

$$\begin{aligned}
V\mathscr{H}^2V^* &= (V\mathscr{H}U^*)JJ^\top(U\mathscr{H}V^*) = V\mathscr{H}JU^\top J^\top(U\mathscr{H}V^*)\\
&= -VJ\mathscr{H}^\top U^\top J(U\mathscr{H}V^*) = -J(\bar{V}\mathscr{H}^\top U^\top)J(U\mathscr{H}V^*)\\
&= \begin{bmatrix} -T_b^\top \tilde{E}_b^{-\top} \tilde{A}_b^\top R_b^\top & * \\ 0 & -T_t^\top \tilde{E}_t^{-\top} \tilde{A}_t^\top R_t^\top \end{bmatrix} \begin{bmatrix} R_t \tilde{A}_t \tilde{E}_t^{-1} T_t & * \\ 0 & R_b \tilde{A}_b \tilde{E}_b T_b \end{bmatrix}\\
&= \begin{bmatrix} -T_b^\top \tilde{E}_b^{-\top} \tilde{A}_b^\top R_b^\top R_t \tilde{A}_t \tilde{E}_t^{-1} T_t & * \\ 0 & -T_t^\top \tilde{E}_t^{-\top} \tilde{A}_t^\top R_t^\top R_b \tilde{A}_b \tilde{E}_b T_b \end{bmatrix}\\
&= \begin{bmatrix} M & * \\ 0 & M^\top \end{bmatrix}.
\end{aligned}$$

According to (9), the eigenvalues of M are exactly half of those of $\mathscr{H}^2$. By additionally considering the fact that the eigenvalues of the $\top$-Hamiltonian matrix $\mathscr{H}$ appear in pairs of the form $(\lambda, -\lambda)$ and the similarity between the GEP in (2) and the $\top$HEP in (5), we can summarize these relations into the following corollary.

Corollary 3. *If μ^* is an eigenvalue of M, then $\pm\lambda^*$ are eigenvalues of the GEP expressed in (2), where $\lambda^* = \sqrt{\mu^*}$.*

Therefore, it is necessary to stably compute all eigenvalues of M; for this purpose, the PQR algorithm is a sound approach. The detailed procedure for one single-shift step of the PQR method is introduced in Algorithm 5. We do not need to store the unitary matrices in Algorithm 5 since they are not necessary when computing the eigenvectors of M.

For clarity, let us first consider the three relevant cases of the single-shift PQR step for
$M = -T_b^\top \tilde{E}_b^{-\top} \tilde{A}_b^\top R_b^\top R_t \tilde{A}_t \tilde{E}_t^{-1} T_t \equiv A_1 A_2^{-1} A_3 A_4 A_5 A_6 A_7^{-1} A_8$ when eliminating nonzero elements:

1. For $A_i^{-1}A_{i+1}$, update A_i and A_{i+1} via $A_{i+1} \leftarrow Q_j A_{i+1}$ and $A_i \leftarrow Q_j A_i$, with $Q_1 = \mathbf{givens}(A_{i+1}\mathbf{e}_k, k, k+1)$ and $Q_2 = \mathbf{givens}(A_{i+1}\mathbf{e}_k, k, k+1)$, respectively, denoted by $\mathbf{PQR\text{-}a}(A_i, A_{i+1}, k)$.
2. For $A_i A_{i+1}^{-1}$, update A_i and A_{i+1} via $A_{i+1} \leftarrow A_{i+1}Q_j^*$ and $A_i \leftarrow A_i Q_j^*$, with $Q_1 = \mathbf{givens}(A_{i+1}^\top \mathbf{e}_{k+1}, k+1, k)$ and $Q_2 = \mathbf{givens}(A_{i+1}^\top \mathbf{e}_{k+1}, k+1, k)$, respectively, denoted by $\mathbf{PQR\text{-}b}(A_i, A_{i+1}, k)$.
3. For $A_i A_{i+1}$, update A_i and A_{i+1} via $A_{i+1} \leftarrow Q_j A_{i+1}$ and $A_i \leftarrow A_i Q_j^*$, with $Q_1 = \mathbf{givens}(A_{i+1}\mathbf{e}_k, k, k+1)$ and $Q_2 = \mathbf{givens}(A_{i+1}\mathbf{e}_k, k, k+1)$, respectively, denoted by $\mathbf{PQR\text{-}c}(A_i, A_{i+1}, k)$.

Below, we present Algorithm 5 for computing a single-shift PQR step for $M = A_1 A_2^{-1} A_3 A_4 A_5 A_6 A_7^{-1} A_8$.

Algorithm 5 One single-shift PQR step for $M = A_1 A_2^{-1} A_3 A_4 A_5 A_6 A_7^{-1} A_8$

Input: $A_i \in \mathbb{C}^{n\times n}$, where A_1 is an upper Hessenberg matrix and A_i is an upper triangular matrix for $i = 2, \ldots, 8$.

Output: $A_i \in \mathbb{C}^{n\times n}$ after one single-shift PQR step for M, where A_1 is an upper Hessenberg matrix and A_i is an upper triangular matrix for $i = 2, \ldots, 8$.

1: Determine the Francis shift μ of M, and $Q = \mathbf{givens}((M - \mu I_n)\mathbf{e}_1, 1, 2)$.
2: Set $A_1 = QA_1$ and $A_8 = A_8 Q^*$.
3: **for** $i = 1 : n-2$ **do**
4: Apply $\mathbf{PQR\text{-}a}(A_7, A_8, i)$, and $\mathbf{PQR\text{-}b}(A_6, A_7, i)$;
5: Apply $\mathbf{PQR\text{-}c}(A_k, A_{k+1}, i)$ with $k = 5, 4, 3$;
6: Apply $\mathbf{PQR\text{-}a}(A_2, A_3, i)$ and $\mathbf{PQR\text{-}b}(A_1, A_2, i)$;
7: **if** $i < n-2$ **then**
8: Update A_1 and A_8 via $A_1 \leftarrow QA_1$ and $A_8 \leftarrow A_8 Q^*$ with $Q = \mathbf{givens}(A\mathbf{e}_i, i+1, i+2)$;
9: **end if**
10: **end for**
11: Update A_1 and A_8 via $A_1 \leftarrow QA_1$ and $A_8 \leftarrow A_8 Q^*$ with $Q = \mathbf{givens}(A_1\mathbf{e}_{n-2}, n-1, n)$;
12: Update A_8 and A_7 via $A_8 \leftarrow QA_8$ and $A_7 \leftarrow QA_7$ with $Q = \mathbf{givens}(A_8\mathbf{e}_{n-1}, n-1, n)$;
13: Update A_7 and A_6 via $A_7 \leftarrow A_7 Q^*$ and $A_6 \leftarrow A_6 Q^*$ with $Q = \mathbf{givens}(A_7^\top \mathbf{e}_n, n, n-1)$;
14: Repeat step 11 three times in sequence while replacing A_1 with A_{k+1} and A_8 with A_k for $k = 5, 4, 3$;
15: Repeat steps 12 and 13 with A_8 replaced with A_3, A_7 replaced with A_2 and A_6 replaced with A_1.

According to Theorem 5.1 in [3], analogously, the $\top$-symplectic URV decomposition and PQR method proposed in this paper for computing all eigenvalues of $\mathscr{H} = R\tilde{A}\tilde{E}T$ are backward stable.

4. Numerical experiments

In this section, we present two numerical examples to demonstrate the computational precision and time consumption of our algorithm. For comparison, we implemented both our algorithm and the classical QZ method on the same examples. All numerical computations were carried out in MATLAB R2020a on a workstation with a 3.80 GHz Intel Core i7 processor and 64 GB of RAM, with a machine roundoff of $eps = 2.2204 \times 10^{-16}$.

In the following experiments, we constructed the matrices A, B, and E as follows.

1. Determine two unitary matrices U and V using the *svd* command for a random complex matrix, and determine three random vectors $\boldsymbol{b} \in \mathbb{C}^n$ and $\boldsymbol{a}, \boldsymbol{e} \in \mathbb{C}^{2n}$ using the *rand* command in MATLAB.
2. Let Σ_B be a tridiagonal matrix; set $[b_1, 0, b_2, 0, \cdots, b_n]^\top$ and $[-b_1, 0, -b_2, 0, \cdots, -b_n]^\top$ as its superdiagonal and subdiagonal, respectively, and let $B = U\Sigma_B U^\top$.
3. Let $\Sigma_A = \mathrm{diag}(\boldsymbol{a})$ and $\Sigma_E = \mathrm{diag}(\boldsymbol{e})$, and set $A = \bar{U}\Sigma_A V^*$ and $E = U\Sigma_E V^*$.

Note that this method of constructing A, B and E ensures that B is skew-symmetric and that $A^\top E = E^\top A$ is symmetric. Moreover, we can analytically calculate the eigenvalues of (2), $(\lambda_1^{exact}, \lambda_2^{exact}, \cdots, \lambda_n^{exact})$, with $\lambda_j^{exact} \equiv \pm b_j \sqrt{-a_{2j-1}a_{2j}e_{2j-1}^{-1}e_{2j}^{-1}}$ being the j-th eigenvalue of (2), thus providing accurate solutions for the numerical experiments.

We use $(\lambda, \mathbf{x})$ and $(\lambda^{qz}, \mathbf{x}^{qz})$ to denote the eigenpairs computed by our algorithm and the single-shift QZ method, respectively, while $t(\lambda)$ and $t(\lambda^{qz})$ are the corresponding amounts of time consumed. Additionally, we calculate the relative errors $e(\lambda)$ of the eigenvalues and the relative normalized residual norms $r(\lambda)$ as follows:

$$
\begin{aligned}
e(p) &= \max_{j=1,\cdots,n} \frac{|p_j - \lambda_j^{exact}|}{|\lambda_j^{exact}|}, \quad p = \lambda \text{ or } \lambda^{qz}, \\
r(p) &= \max_{j=1,\cdots,n} \frac{\|BA\mathbf{q}_j - p_j E\mathbf{q}_j\|_1}{(\|BA\|_1 + |p_j|\|E\|_1)\|\mathbf{q}_j\|_1}, \quad \mathbf{q} = \mathbf{x} \text{ or } \mathbf{x}^{qz}.
\end{aligned}
$$

We plot the amounts of time consumed $t(\lambda)$ and $t(\lambda^{qz})$ for random cases with dimensionalities n ranging from 10 to 1000 and show the detailed time consumption of the $\top$-symplectic URV decomposition and the PQR method in our algorithm and that of Hessenberg-triangular reduction and the QZ

Table 1: The FLOP counts for different problems

Problem	Phase	FLOPs
$\top$HEP (5)	$\top$-symplectic URV decomposition	$192n^3$
	one single-shift iteration of the PQR method	$48n^2$
GEP (2)	Hessenberg-triangular reduction	$128n^3$
	one single-shift iteration of the QZ method	$48n^2$

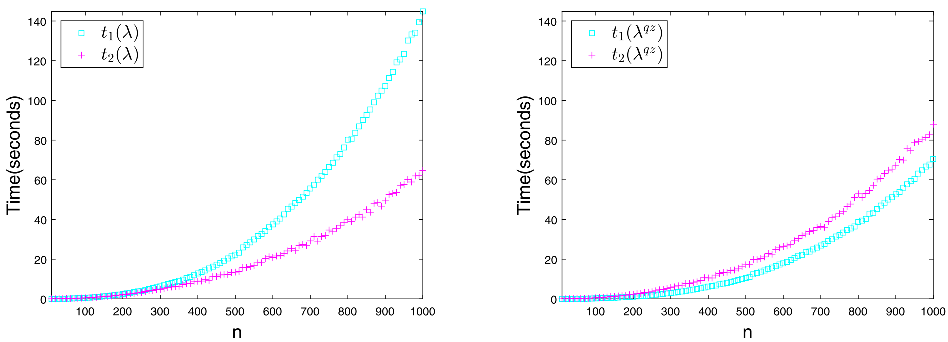

Figure 1: Time consumption for random cases with dimensionalities n ranging from 10 to 1000. $t_1(\lambda)$, $t_2(\lambda)$, $t_1(\lambda^{qz})$ and $t_2(\lambda^{qz})$ denote the time costs for $\top$-symplectic URV decomposition, PQR iterations, Hessenberg-triangular reduction and QZ iterations, respectively.

method in Figure 1(a) and Figure 1(b); the results are consistent with the computational cost estimates shown in Table 1. The overall time consumption of the proposed method is higher than that of the shifted QZ method because of the higher computational cost of the $\top$-symplectic URV decomposition of $\mathscr{H}$ compared to that of the Hessenberg-triangular reduction of (BA, E). Additionally, the floating point operation (FLOP) count of a single-shift iteration of the PQR method for $M = -T_b^\top \tilde{E}_b^{-\top} \tilde{A}_b^\top R_b^\top R_t \tilde{A}_t \tilde{E}_t^{-1} T_t$ is the same as that of a single-shift iteration of the QZ method for (BA, E), and the total time consumption for PQR iterations is less than that for QZ iterations in the numerical experiments.

In this example, the matrices A, B and E were designed using the above construction strategy with $n = 1000$ to ensure that the eigenvalues of (2) would be $\pm 1, \pm 2, \cdots, \pm 500$. We present a comparison with the eigenvalues computed using the URV+PQR and QZ methods, where the accuracies are denoted by $e(\lambda)$ and $e(\lambda^{qz})$, respectively, in Figure 2(a). The relative errors of the eigenvalues computed using our method are comparable in value to and more stable than those of the QZ method for all eigenvalues. Addition-

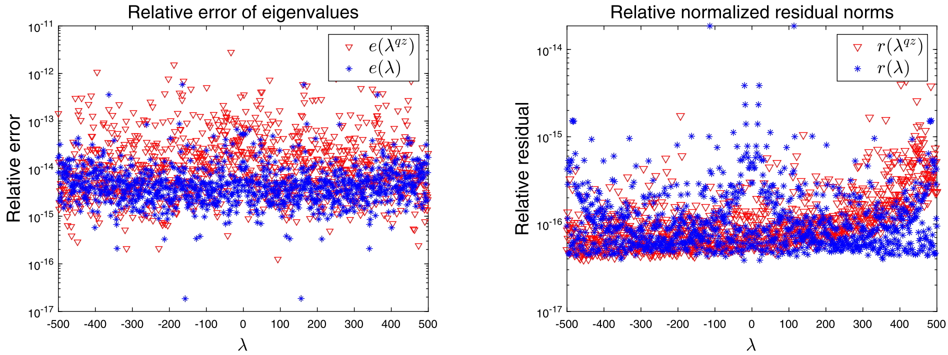

Figure 2: (a) Relative error comparison for eigenvalues of $\pm 1, \cdots, \pm 500$. (b) Relative residual comparison.

ally, the relative residuals computed for the two algorithms are shown in Figure 2(b), with the eigenvectors both being restored through one step of the inverse power method for (BA, E). To summarize, the accuracies of the eigenvalues or eigenpairs found using our method are comparable to those found with the QZ method, while the pairing property is implicitly satisfied.

5. Conclusions

In this paper, to address lossless dissipative Hamiltonian systems of the form expressed in (1), we have proposed a structure-preserving algorithm for computing all eigenvalues of (BA, E), with B being complex skew-symmetric and $A^\top E = E^\top A$. To utilize these matrix structures, instead of computing the original GEP (2), we computed the equivalent $\top$HEP $\mathscr{H}\mathbf{z} = \lambda \mathbf{z}$ (5), which can be reduced via a corresponding $\top$-symplectic URV decomposition procedure to a product eigenvalue problem of half as many dimensions. As a result, a PQR method can be applied to compute all eigenvalues of M (9), from which the eigenvalues of $\mathscr{H}$ can be computed immediately using Corollary 3, such that the paired form $(\lambda, -\lambda)$ of the eigenvalues is strictly maintained. Two numerical experiments revealed that our algorithm can preserve the pairing property of the eigenvalues and achieve a level of accuracy for the eigenpairs that is comparable to that of the QZ method.

Acknowledgments

The author was supported in part by the National Natural Science Foundation of China (NSFC) 11971105.

References

[1] Gregory Ammar and Volker Mehrmann. On hamiltonian and symplectic hessenberg forms. *Linear Algebra and its Applications*, 149:55–72, April 1991. MR1092869

[2] Christopher Beattie, Volker Mehrmann, Hongguo Xu, and Hans Zwart. Linear port-hamiltonian descriptor systems. *Mathematics of Control, Signals, and Systems*, 30(4), October 2018. MR3866574

[3] Peter Benner, Volker Mehrmann, and Hongguo Xu. A numerically stable, structure preserving method for computing the eigenvalues of real hamiltonian or symplectic pencils. *Numerische Mathematik*, 78(3):329–358, January 1998. MR1603338

[4] Adam W. Bojanczyk, Gene H. Golub, and Paul Van Dooren. Periodic schur decomposition: algorithms and applications. In Franklin T. Luk, editor, *SPIE Proceedings*. SPIE, November 1992.

[5] Adam W. Bojanczyk, Gene H. Golub, and Paul Van Dooren. Periodic schur decomposition: algorithms and applications. In Franklin T. Luk, editor, *Advanced Signal Processing Algorithms, Architectures, and Implementations III*. SPIE, November 1992.

[6] D. Chu, X. Liu, and V. Mehrmann. A numerical method for computing the Hamiltonian Schur form. *Numer. Math.*, 105(3):375–412, 2006. MR2266831

[7] Morten Dalsmo and Arjan van der Schaft. On representations and integrability of mathematical structures in energy-conserving physical systems. *SIAM Journal on Control and Optimization*, 37(1):54–91, January 1998. MR1645424

[8] Goran Golo, Arjan van der Schaft, Pieter Cornelis Breedveld, and Bernhard Maschke. Hamiltonian formulation of bond graphs. 2003.

[9] J. J. Hench and A. J. Laub. Numerical solution of the discrete-time periodic riccati equation. *IEEE Transactions on Automatic Control*, 39(6):1197–1210, June 1994. MR1283693

[10] Daniel Kressner. The periodic QR algorithm is a disguised QR algorithm. *Linear Algebra and its Applications*, 417(2-3):423–433, September 2006. MR2250322

[11] Wen-Wei Lin, Volker Mehrmann, and Hongguo Xu. Canonical forms for hamiltonian and symplectic matrices and pencils. *Linear Algebra and its Applications*, 302-303:469–533, December 1999. MR1733547

[12] Nanna Holmgaard List, Sonia Coriani, Ove Christiansen, and Jacob Kongsted. Identifying the hamiltonian structure in linear response theory. *The Journal of Chemical Physics*, 140(22):224103, June 2014.

[13] Charles F. Van Loan. A general matrix eigenvalue algorithm. *SIAM Journal on Numerical Analysis*, 12(6):819–834, December 1975. MR0408222

[14] B. M. Maschke, A. J. Van Der Schaft, and P.C. Breedveld. An intrinsic hamiltonian formulation of network dynamics: non-standard poisson structures and gyrators. *Journal of the Franklin Institute*, 329(5):923–966, September 1992. MR1180398

[15] C. Mehl, Volker Mehrmann, and M. Wojtylak. Linear algebra properties of dissipative hamiltonian descriptor systems. *SIAM Journal on Matrix Analysis and Applications*, 39(3):1489–1519, January 2018. MR3858806

[16] Volker Mehrmann and Riccardo Morandin. Structure-preserving discretization for port-hamiltonian descriptor systems. *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 6863–6868, 2019.

[17] C. B. Moler and G. W. Stewart. An algorithm for generalized matrix eigenvalue problems. *SIAM Journal on Numerical Analysis*, 10(2):241–256, April 1973. MR0345399

[18] R. Ortega, A. J. Van Der Schaft, I. Mareels, and B. Maschke. Putting energy back in control. *IEEE Control Systems*, 21(2):18–33, April 2001.

[19] Romeo Ortega, Arjan van der Schaft, Bernhard Maschke, and Gerardo Escobar. Interconnection and damping assignment passivity-based control of port-controlled hamiltonian systems. *Automatica*, 38(4):585–596, April 2002. MR2131469

[20] Hector Ramirez, Yann Le Gorrec, and Yongxin Wu. Introduction to control of port-hamiltonian systems. March 2015.

[21] A. J. Schaft. Port-hamiltonian systems: Network modeling and control of nonlinear physical systems. In *Advanced Dynamics and Control of Structures and Machines*, pages 127–167. Springer Vienna, 2004.

[22] A. J. Schaft, G. Escobar, S. Stramigioli, and A. Macchelli. Beyond passivity: port-hamiltonian systems. 2007.

[23] A. J. van der Schaft and B. M. Maschke. Port-hamiltonian systems on graphs. *SIAM Journal on Control and Optimization*, 51(2):906–937, January 2013. MR3032900

[24] A. J. van der Schaft and B.M. Maschke. Hamiltonian formulation of distributed-parameter systems with boundary energy flow. *Journal of Geometry and Physics*, 42(1-2):166–194, May 2002. MR1894081

[25] Robert C. Ward. The combination shift QZ algorithm. *SIAM Journal on Numerical Analysis*, 12(6):835–853, December 1975. MR0408223

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Received January 12, 2022